

Surds, and other roots

Roots and powers are closely related, but only some roots can be written as whole numbers. Surds are roots which cannot be written in this way. Nevertheless, it is possible to manipulate surds, and to simplify formulæ involving them.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- understand the relationship between negative powers and positive powers;
- understand the relationship between fractional powers and whole-number powers;
- replace formulæ involving roots with formulæ involving fractional powers;
- understand the difference between surds and whole-number roots;
- simplify expressions involving surds;
- rationalise fractions with surds in the denominator.

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1. Introduction

In this unit we are going to explore numbers written as powers, and perform some calculations involving them. In particular, we are going to look at square roots of whole numbers which produce irrational numbers — that is, numbers which cannot be written as fractions. These are called surds.

2. Powers and roots

We know that 2 cubed is $2 \times 2 \times 2$, and we say that we have 2 raised to the power 3, or to the index 3. An easy way of writing this repeated multiplication is by using a ‘superscript’, so that we would write 2^3 :

$$2^3 = 2 \times 2 \times 2 = 8.$$

Similarly, 4 cubed is $4 \times 4 \times 4$, and equals 64. So we write

$$4^3 = 4 \times 4 \times 4 = 64.$$

But what if we have negative powers? What would be the value of 4^{-3} ?

To find out, we shall look at what we know already:

$$\begin{aligned}4^3 &= 4 \times 4 \times 4 = 64, \\4^2 &= 4 \times 4 = 16, \\4^1 &= 4 = 4, \\ \text{and so } 4^0 &= 4 \div 4 = 1\end{aligned}$$

(because to get the answer you divide the previous one by 4).

Now let’s continue the pattern:

$$\begin{aligned}4^{-1} &= 1 \div 4 = \frac{1}{4}, \\4^{-2} &= \frac{1}{4} \div 4 = \frac{1}{16}, \\4^{-3} &= \frac{1}{16} \div 4 = \frac{1}{64}\end{aligned}$$

and $\frac{1}{64} = 1/4^3$. So a negative power gives the reciprocal of the number — that is, 1 over the number. Thus $4^{-2} = 1/4^2 = \frac{1}{16}$, and $4^{-1} = 1/4^1 = \frac{1}{4}$. Similarly,

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

and

$$5^{-3} = \frac{1}{5^3} = \frac{1}{125}.$$

A common misconception is that since the power is negative, the result must be negative: as you can see, this is not so.

Now we know that $4^0 = 1$ and $4^1 = 4$, but what is $4^{1/2}$?

Using the rules of indices, we know that $4^{1/2} \times 4^{1/2} = 4^1 = 4$ because $\frac{1}{2} + \frac{1}{2} = 1$. So $4^{1/2}$ equals 2, as $2 \times 2 = 4$. Therefore $4^{1/2}$ is the square root of 4. It is written as $\sqrt{4}$ and equals 2:

$$4^{1/2} = \sqrt{4} = 2.$$

Similarly,

$$9^{1/2} = \sqrt{9} = 3.$$

And in general, any number a raised to the power $\frac{1}{2}$ equals the square root of a :

$$a^{1/2} = \sqrt{a}.$$

So the power, or index, associated with square roots is $\frac{1}{2}$. Also, in the same way that the index $\frac{1}{2}$ represents the square root, other fractions can be used to represent other roots. The cube root of the number 4 is written as

$$4^{1/3} = \sqrt[3]{4}$$

where $\frac{1}{3}$ is the index representing cube root. Similarly, the fourth root of 5 may be written as $5^{1/4} = \sqrt[4]{5}$, and so on. The n -th root is represented by the index $1/n$, and the n -th root of a is written as

$$a^{1/n} = \sqrt[n]{a}.$$

So, for example, if we have $\sqrt[3]{64}$ then this equals 64 to the power $\frac{1}{3}$; and then

$$\begin{aligned} \sqrt[3]{64} &= 64^{1/3} \\ &= (4 \times 4 \times 4)^{1/3} \\ &= 4. \end{aligned}$$

There are some important points about roots, or fractional powers, that we need to remember. First, we can write

$$\begin{aligned} 4^{1/2} \times 4^{1/2} &= 4 \\ \sqrt{4} \times \sqrt{4} &= 4 \\ (\sqrt{4})^2 &= 4 \end{aligned}$$

so that the square root of 4, squared, gives you 4 back again. In fact the square root of any number, squared, gives you that number back again.

Next, if we have a very simple quadratic equation to solve, such as $x^2 = 4$, then the solutions are $x = +2$ or $x = -2$. There are two roots, as $(+2) \times (+2) = 4$ and also $(-2) \times (-2) = 4$. We can write the roots as ± 2 . So not all roots are unique. But in a lot of circumstances we only need the positive root, and you do not have to put a plus sign in front of the square root for the positive root. By convention, if there is no sign in front of the square root then the root is taken to be positive.

On the other hand, suppose we were given $\sqrt{-9}$. Could we work this out and get a real answer? Now

$$\sqrt{-9} = (-9)^{1/2},$$

and so we are looking for a number which multiplied by itself gives -9 . But there is no such number, because $3 \times 3 = 9$ and also $(-3) \times (-3) = 9$. So you cannot find the square root of a negative number and get a real answer.



Key Point

The square root of a is written as $a^{1/2}$ and is equal to \sqrt{a} . The n -th root of a is written as $a^{1/n}$ and is equal to $\sqrt[n]{a}$.

The formula

$$\sqrt{a} \times \sqrt{a} = a$$

can be used to simplify expressions involving square roots.

Not all roots are unique, for example the square root of 4 is 2 or -2 , sometimes written as ± 2 . But when written without a sign in front, the square root represents the positive root. You cannot find the square root of a negative number.

3. Surds and irrational numbers

We shall now look at some square roots in more detail. Take, for example, $\sqrt{25}$: its value is 5. And the value of $\sqrt{\frac{9}{4}}$ is $\frac{3}{2}$, or $1\frac{1}{2}$. So some square roots can be evaluated as whole numbers or as fractions, in other words as *rational numbers*. But what about $\sqrt{2}$ or $\sqrt{3}$? The roots to these are not whole numbers or fractions, and so they have irrational values. They are usually written as decimals to a given approximation. For example

$$\begin{aligned}\sqrt{2} &= 1.414 && \text{to 3 decimal places,} \\ \sqrt{3} &= 1.732 && \text{to 3 decimal places.}\end{aligned}$$

When we have square roots which give irrational numbers we call them *surds*. So $\sqrt{2}$ and $\sqrt{3}$ are surds. Other surds are

$$\sqrt{5}, \quad \sqrt{6}, \quad \sqrt{7}, \quad \sqrt{8}, \quad \sqrt{10} \quad \text{and so on.}$$

Surds are often found when using Pythagoras' Theorem, and in trigonometry. So, where possible, it is useful to be able to simplify expressions involving surds. Take, for example, $\sqrt{8}$. This can be written as $\sqrt{4 \times 2}$, which we can rewrite as $\sqrt{4} \times \sqrt{2}$, in other words as $2\sqrt{2}$:

$$\begin{aligned}\sqrt{8} &= \sqrt{4 \times 2} \\ &= \sqrt{4} \times \sqrt{2} \\ &= 2\sqrt{2}.\end{aligned}$$

In general, the square root of a product is the product of the square roots, and vice versa. This is useful to know when simplifying surd expressions.

Now suppose we have been given $\sqrt{5} \times \sqrt{15}$. At first glance this cannot be simplified. But we can rewrite the expression as the square root of 5 times 15, so it is the square root of 75, and 75 can be written as 25 times 5. But 25 is a perfect square, we can use this to simplify the expression.

$$\begin{aligned}\sqrt{5} \times \sqrt{15} &= \sqrt{5 \times 15} \\ &= \sqrt{75} \\ &= \sqrt{25 \times 3} \\ &= \sqrt{25} \times \sqrt{3} \\ &= 5\sqrt{3}.\end{aligned}$$

But watch out if you are given $\sqrt{4+9}$, which is the square root of 13. This does not equal $\sqrt{4} + \sqrt{9}$, which is $2 + 3 = 5$. Now 5 cannot be the answer as that is the square root 25, not the square root of 13.



Key Point

If a positive whole number is not a perfect square, then its square root is called a *surd*. A surd cannot be written as a fraction, and is an example of an irrational number.

4. Simplifying expressions involving surds

Knowing the common square numbers like 4, 9, 16, 25, 36 and so on up to 100 is very helpful when simplifying surd expressions, because you know their square roots straight away, and you can use them to simplify more complicated expressions. Suppose we were asked to simplify the expression $\sqrt{400} \times \sqrt{90}$:

$$\begin{aligned}\sqrt{400} \times \sqrt{90} &= \sqrt{4 \times 100} \times \sqrt{9 \times 10} \\ &= \sqrt{4} \times \sqrt{100} \times \sqrt{9} \times \sqrt{10} \\ &= 2 \times 10 \times 3 \times \sqrt{10} \\ &= 60\sqrt{10},\end{aligned}$$

which cannot be simplified any further.

We can also simplify the expression $\sqrt{2000}/\sqrt{50}$. We get

$$\begin{aligned}\frac{\sqrt{2000}}{\sqrt{50}} &= \sqrt{\frac{2000}{50}} \\ &= \sqrt{40} \\ &= \sqrt{4 \times 10} \\ &= \sqrt{4} \times \sqrt{10} \\ &= 2\sqrt{10}.\end{aligned}$$



Key Point

The following formulæ may be used to simplify expressions involving surds:

$$\begin{aligned}\sqrt{ab} &= \sqrt{a} \times \sqrt{b} \\ \sqrt{\frac{a}{b}} &= \frac{\sqrt{a}}{\sqrt{b}}, \end{aligned}$$

If we have a product of brackets involving surds, for example $(1 + \sqrt{3})(2 - \sqrt{2})$, we can expand out the brackets in the usual way:

$$\begin{aligned}(1 + \sqrt{3})(2 - \sqrt{2}) &= 2 - \sqrt{2} + 2\sqrt{3} - \sqrt{3}\sqrt{2} \\ &= 2 - \sqrt{2} + 2\sqrt{3} - \sqrt{6}.\end{aligned}$$

But what if we have this expression, $(1 + \sqrt{3})(1 - \sqrt{3})$? If we expand this out and simplify the answer, we get

$$(1 + \sqrt{3})(1 - \sqrt{3}) = 1 - \sqrt{3} + \sqrt{3} - \sqrt{3}\sqrt{3} = 1 - \sqrt{9} = 1 - 3 = -2.$$

So the product $(1 + \sqrt{3})(1 - \sqrt{3})$ does not involve surds at all. This is an example of a general result known as the *difference of two squares*. This general result may be written as

$$a^2 - b^2 = (a + b)(a - b)$$

for any numbers a and b . In our example $a = 1$ and $b = \sqrt{3}$, so

$$(1 + \sqrt{3})(1 - \sqrt{3}) = 1^2 - (\sqrt{3})^2 = 1 - 3 = -2.$$

The expansion of the difference of two squares is another useful fact to know and remember.



Key Point

The formula for the difference of two squares is

$$a^2 - b^2 = (a + b)(a - b).$$

5. Rationalising expressions containing surds

Sometimes in calculations we obtain surds as denominators, for example $1/\sqrt{13}$. It is best not to give surd answers in this way. Instead, we use a technique called *rationalisation*. This changes the surd denominator, which is irrational, into a whole number.

To see how to do this, take our example $1/\sqrt{13}$. To rationalise this, we multiply by the fraction $\sqrt{13}/\sqrt{13}$ which is equal to 1. When we multiply by this fraction we do not change the value of our original expression. We obtain

$$\begin{aligned}\frac{1}{\sqrt{13}} &= \frac{1}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}} \\ &= \frac{\sqrt{13}}{13}.\end{aligned}$$

We can also use the expansion of the difference of two squares to rationalise more complicated expressions involving surds.

Example

Rationalise the expression

$$\frac{1}{1 + \sqrt{2}}.$$

Solution

If we multiply this expression by this fraction $(1 - \sqrt{2})/(1 - \sqrt{2})$ we do not change its value, as the new fraction is equal to 1. When we do this, we use the formula for the difference of two squares to work out

$$(1 + \sqrt{2}) \times (1 - \sqrt{2}),$$

and that gives us $1^2 - (\sqrt{2})^2$, which is $1 - 2 = -1$. So now we have a whole number in the denominator of our fraction, and we can divide through. We get

$$\begin{aligned}\frac{1}{1 + \sqrt{2}} &= \frac{1}{1 + \sqrt{2}} \times \frac{1 - \sqrt{2}}{1 - \sqrt{2}} \\ &= \frac{1 - \sqrt{2}}{1^2 - (\sqrt{2})^2} \\ &= \frac{1 - \sqrt{2}}{-1} \\ &= -1 + \sqrt{2} \\ &= \sqrt{2} - 1.\end{aligned}$$

Example

Rationalise the expression

$$\frac{1}{\sqrt{5} - \sqrt{3}}$$

Solution

To rationalise this, we must multiply it by a fraction which equals 1. To choose a suitable fraction, we think of the difference of two squares. As our expression has $\sqrt{5} - \sqrt{3}$ in the denominator, the fraction to use is $(\sqrt{5} + \sqrt{3})$ over $(\sqrt{5} + \sqrt{3})$:

$$\begin{aligned}\frac{1}{\sqrt{5} - \sqrt{3}} &= \frac{1}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}} \\ &= \frac{\sqrt{5} + \sqrt{3}}{(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})} \\ &= \frac{\sqrt{5} + \sqrt{3}}{(\sqrt{5})^2 - (\sqrt{3})^2} \\ &= \frac{\sqrt{5} + \sqrt{3}}{2} \\ &= \frac{\sqrt{5}}{2} + \frac{\sqrt{3}}{2}.\end{aligned}$$



Key Point

A surd expression in the form

$$\frac{1}{(\sqrt{a} + \sqrt{b})} \quad \text{or} \quad \frac{1}{(\sqrt{a} - \sqrt{b})}$$

can be written as $c\sqrt{a} + d\sqrt{b}$ by using the formula for the difference of two squares.

Exercises

1. Simplify the following expressions:

- (a) $\sqrt{40} \times \sqrt{200}$ (b) $\sqrt{90} \times \sqrt{600000}$ (c) $\sqrt{1000}/\sqrt{40}$ (d) $(1 - \sqrt{5})(1 + \sqrt{5})$
(e) $(\sqrt{7} + 3)(\sqrt{7} - 3)$

2. Rationalise the following expressions:

- (a) $1/\sqrt{11}$ (b) $30/\sqrt{3}$ (c) $1/(1 - \sqrt{3})$ (d) $1/(\sqrt{7} + \sqrt{2})$ (e) $2/(\sqrt{5} - \sqrt{7})$

Answers

1. (a) $40\sqrt{5}$ (b) $3000\sqrt{6}$ (c) 5 (d) -4 (e) -2

2. (a) $\sqrt{11}/11$ (b) $10\sqrt{3}$ (c) $-\frac{1}{2} - \sqrt{3}/2$ (d) $\sqrt{7}/5 - \sqrt{2}/5$ (e) $-\sqrt{5} - \sqrt{7}$