AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

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The Improving Mathematics Education in Schools (TIMES) Project

RHOMBUSES, KITES AND TRAPEZIA

A guide for teachers - Years 9–10

MEASUREMENT AND GEOMETRY : Module 21

June 2011

YEARS 9 10

Rhombuses, kites and trapezia

(Measurement and Geometry: Module 21)

For teachers of Primary and Secondary Mathematics

510

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MEASUREMENT AND GEOMETRY : Module 21

June 2011

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YEARS 9 10

RHOMBUSES, KITES AND TRAPEZIA

ASSUMED KNOWLEDGE

The material in this module is a continuation of the module, *Parallelograms and Rectangles*, which is assumed knowledge for the present module. Thus the present module assumes:

- Confidence in writing logical argument in geometry written as a sequence of steps, each justified by a reason.
- Ruler-and-compasses constructions.
- The four standard congruence tests and their application to:
  - proving properties of and tests for isosceles and equilateral triangles,
  - proving properties of and tests for parallelograms and rectangles.
- Informal experience with rhombuses, kites, squares and trapezia.

MOTIVATION

Logical argument, precise definitions and clear proofs are essential if one is to understand mathematics. These analytic skills can be transferred to many areas in commerce, engineering, science and medicine but most of us first meet them in high school mathematics.

Apart from some number theory results such as the existence of an infinite number of primes and the Fundamental Theorem of Arithmetic, most of the theorems students meet are in geometry starting with Pythagoras' theorem.

Many of the key methods of proof such as proof by contradiction and the difference between a theorem and its converse arise in elementary geometry.

As in the module, *Parallelograms and Rectangles*, this module first stresses precise definitions of each special quadrilateral, then develops some of its properties, and then reverses the process, examining whether these properties can be used as tests for that particular special quadrilateral. We have seen that a *test* for a special quadrilateral is usually the *converse of a property*. For example, a typical property–test pair from the previous module is the pair of converse statements:

- If a quadrilateral is a parallelogram, then its diagonals bisect each other.
- If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Congruence is again the basis of most arguments concerning rhombuses, squares, kites and trapezia, because the diagonals dissect each figure into triangles.

A number of the theorems proved in this module rely on one or more of the previous theorems in the module. This means that the reader must understand a whole 'sequence of theorems' to achieve some results. This is typical of more advanced mathematics.

In addition, two other matters are covered in these notes.

- The reflection and rotation symmetries of triangles and special quadrilaterals are identified and related to congruence.
- The tests for the kite also allow several important standard constructions to be explained very simply as constructions of a kite.

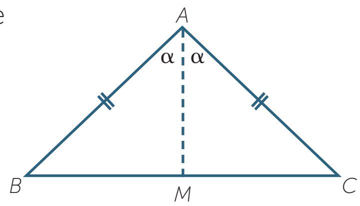
CONTENT

SYMMETRIES OF TRIANGLES, PARALLELOGRAMS AND RECTANGLES

We begin by relating the reflection and rotation symmetries of isosceles triangles, parallelograms and rectangles to the results that we proved in the previous module, *Rectangles and Parallelograms*.

The axis of symmetry of an isosceles triangle

In the module, *Congruence*, congruence was used to prove that the base angles of an isosceles triangle are equal. To prove that $\angle B = \angle C$ in the diagram opposite, we constructed the angle-bisector AM of the apex A, then used the SAS congruence test to prove that



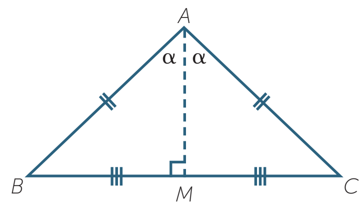
$$\triangle ABM \equiv \triangle ACM$$

This congruence result, however, establishes much more than the equality of the base angles. It also establishes that the angle bisector AM is the perpendicular bisector of the base BC. Moreover, this fact means that AM is an **axis of symmetry** of the isosceles triangle.

These basic facts of about isosceles triangles will be used later in this module and in the module, *Circle Geometry*:

Theorem

In an isosceles triangle, the following four lines coincide:



- The angle bisector of the apex angle.
- The line joining the apex and the midpoint of the base.
- The line through the apex perpendicular to the base.
- The perpendicular bisector of the base.

This line is an axis of symmetry of the isosceles triangle. It has, as a consequence, the interesting property that the centroid, the incentre, the circumcentre and the orthocentre of $\triangle ABC$ all tie on the line AM. In general, they are four different points. See the module, *Construction* for details of this.

Extension – Some further tests for a triangle to be isosceles

The theorem above suggests three possible tests for a triangle to be isosceles. The first two are easy to prove, but the third is rather difficult because simple congruence cannot be used in this 'non-included angle' situation.

EXERCISE 1

Use congruence to prove that $\triangle ABC$ is isosceles with $AB = AC$ if:

a the perpendicular bisector of BC passes through A, or

b the line through A perpendicular to BC bisects $\angle A$, or

c the angle bisector of $\angle A$ passes through the midpoint M of BC.

[Hint: For part **c**, let $\angle BAM = \angle CAM = \alpha$, and let $\angle C = \gamma$.

Suppose by way of contradiction that $AC < AB$.

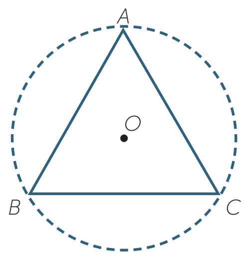
Choose P on the interval AB so that $AP = AC$, and join PM.

The symmetries of an equilateral triangle

An equilateral triangle is an isosceles triangle in three different ways, so the three vertex angle bisectors form *three axes of symmetry meeting each other at 60°*. In an equilateral triangle, each vertex angle bisector is the perpendicular bisector of the opposite side – we proved in the previous module that in any triangle, these three perpendicular bisectors are concurrent. They meet at a point which is the centre of a circle through all three vertices. The point is called the circumcentre and the circle is called the circumcircle of the triangle.

An equilateral triangle is also congruent to itself in two other orientations:

$$\triangle ABC \equiv \triangle BCA \equiv \triangle CAB \qquad \text{(SSS)},$$



corresponding to the fact that it has *rotation symmetry of order 3*. The centre of this rotation symmetry is the circumcentre O described above, because the vertices are equidistant from it.

Other triangles do not have reflection or rotation symmetry

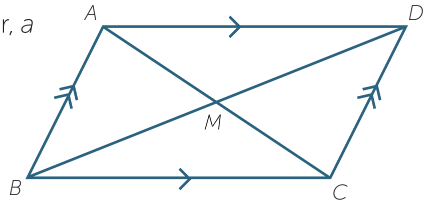
In a non-trivial rotation symmetry, one side of a triangle is mapped to a second side, and the second side mapped to the third side, so the triangle must be equilateral.

In a reflection symmetry, two sides are swapped, so the triangle must be isosceles.

Thus a triangle that is not isosceles has neither reflection nor rotation symmetry. Such a triangle is called **scalene**.

Rotation symmetry of a parallelogram

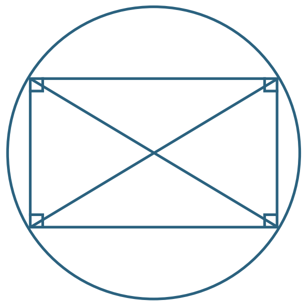
Since the diagonals of a parallelogram bisect each other, *a parallelogram has rotation symmetry of order 2 about the intersection of its diagonals*. Joining the diagonal AC of a parallelogram $ABCD$ produces two congruent triangles,



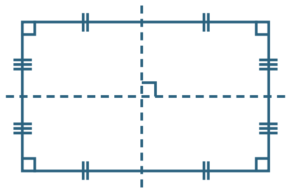
$\triangle ABC \equiv \triangle CDA$ (AAS);

Reflection symmetry of a rectangle

A rectangle is a parallelogram, so it has *rotationsymmetry of order 2 about the intersection of its diagonals*. This is even clearer in a rectangle than in a general parallelogram because the diagonals have equal length, so their intersection is the **circumcentre** of the **circumcircle** passing through all four vertices.

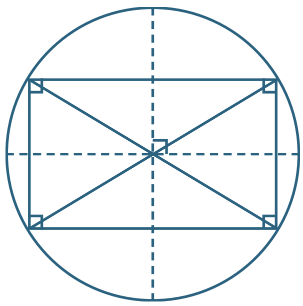


The line through the midpoints of two opposite sides of a rectangle dissects the rectangle into two rectangles that are congruent to each other, and are in fact reflections of each other in the constructed line.



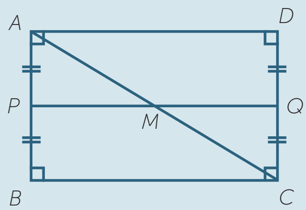
There are two such lines in a rectangle, so *a rectangle has two axes of symmetry meeting right angles*.

It may seem obvious to the eye that the intersection of these two axes of symmetry is the circumcentre of the rectangle, which is intersection of the two diagonals. This is illustrated in the diagram to the right, but it needs to be proven.



EXERCISE 2

Use the diagram to the right to prove that the line through the midpoints of opposite sides of a rectangle bisects each diagonal.



Axes of symmetry of triangles, parallelograms and rectangles

- An **isosceles triangle** has an axis of symmetry – this line is the bisector of the apex angle, it is the altitude from the vertex to the base, and it is the line joining the apex to the midpoint of the base.
- An **equilateral triangle** has three axes of symmetry, which are concurrent in the circumcentre of the circumcircle through its three vertices. It also has rotation symmetry of order three about its circumcentre.
- A triangle that is not isosceles has no axes of symmetry and no rotation symmetry.
- A **parallelogram** has rotation symmetry of order two about the intersection of its diagonals.
- A **rectangle** has rotation symmetry of order two about the intersection of its diagonals, and two axes of symmetry through the midpoints of opposite sides.

RHOMBUSES

The Greeks took the word *rhombos* from the shape of a piece of wood that was whirled about the head like a bullroarer in religious ceremonies. This derivation does not imply a definition, unlike the words 'parallelogram' and 'rectangle', but we shall take their classical definition of the rhombus as our definition because it is the one most usually adopted by modern authors.

Definition of a rhombus

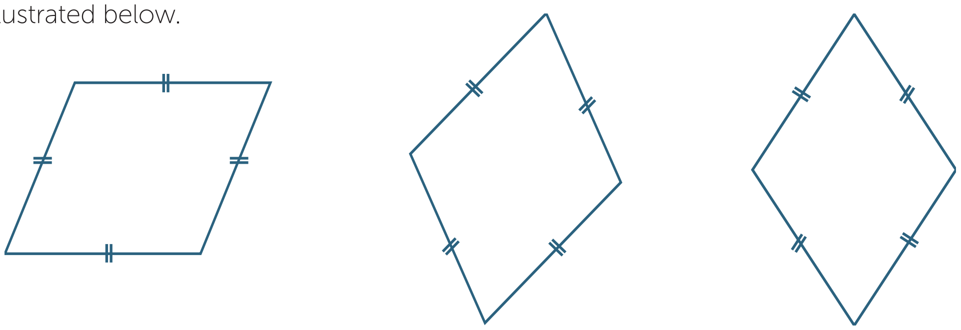
A **rhombus** is a quadrilateral with all sides equal.

First property of a rhombus – A rhombus is a parallelogram

Since its opposite sides are equal, a rhombus is a parallelogram – this was our second test for a parallelogram in the previous module. A rhombus thus has all the properties of a parallelogram:

- Its opposite sides are parallel.
- Its opposite angles are equal.
- Its diagonals bisect each other.
- It has rotation symmetry of order two about the intersection of its diagonals.

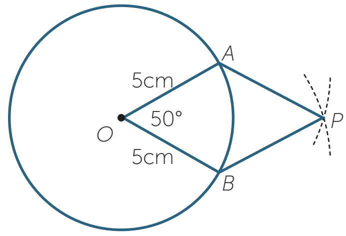
When drawing a rhombus, there are two helpful orientations that we can use, as illustrated below.



The rhombus on the left looks like a 'pushed-over square', and has the orientation we usually use for a parallelogram. The rhombus on the right has been rotated so that it looks like the diamond in a pack of cards. It is often useful to think of this as the standard shape of a rhombus.

Constructing a rhombus using the definition

It is very straightforward to construct a rhombus using the definition of a rhombus. Suppose that we want to construct a rhombus with side lengths 5cm and acute vertex angle 50°.



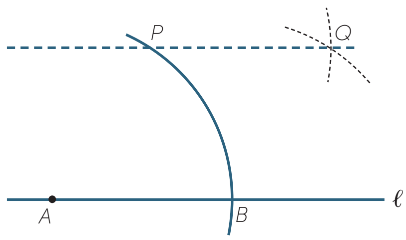
- Draw a circle with radius 5cm.
- Draw two radii *OA* and *OB* meeting at 50° at the centre *O*.
- Draw arcs with the same radius 5cm and centres *A* and *B*, and let *P* be their point of intersection.

The figure *OAPB* is a rhombus because all its sides are 5cm.

EXERCISE 3

Use the cosine rule (or drop a perpendicular and use simple trigonometry) to find the lengths n of the diagonals of the rhombus *OAPB* constructed above.

This leads to yet another way to construct a line parallel to a given line ℓ through a given point *P*.



- Choose any point *A* on the line ℓ.
- Draw an arc with centre *A* and radius *AP* cutting ℓ at *B*.
- Complete the rhombus *PABQ* as before.

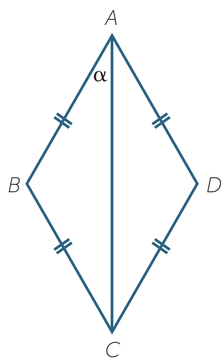
Then $PQ \parallel \ell$ because the figure *PABQ* is a rhombus.

Second property of a rhombus – Each diagonal bisects two vertex angles

Theorem

Each diagonal of a rhombus bisects the vertex angles through which it passes.

Proof



Let *ABCD* be a rhombus with the diagonal *AC* drawn.

Let $\angle BAC = \alpha$

$\angle BAC = \alpha$ (base angles of isosceles $\triangle ABC$)

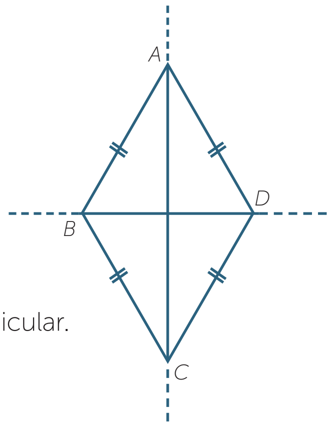
so $\angle DAC = \alpha$ (alternate angles, $BC \parallel AD$):

That is AC bisects $\angle BAD$

EXERCISE 4

Prove this result using the congruence $\triangle ABC \equiv \triangle ADC$.

The axes of symmetry of a rhombus



The exercise above showed that each diagonal of a rhombus dissects the rhombus into two congruent triangles that are reflections of each other in the diagonal,

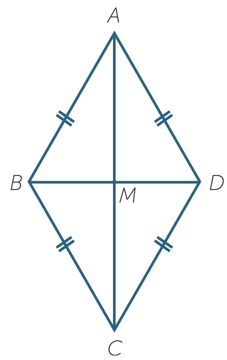
$\triangle ABC \equiv \triangle ADC$ and $\triangle BAD \equiv \triangle BCD$.

Thus the diagonals of a rhombus are axes of symmetry.
The following property shows that these two axes are perpendicular.

Third property of a rhombus – The diagonals are perpendicular

The proof given here uses the theorem about the axis of symmetry of an isosceles triangle proven at the start of this module. Two other proofs are outlined as exercises.

Theorem



The diagonals of a rhombus are perpendicular.

Proof

Let $ABCD$ be a rhombus,
with diagonals meeting at M.

To prove that $AC \perp BD$.

By the previous theorem, AM is the angle bisector of $\triangle DAB$.

Hence $AM \perp BD$, because A is the apex of the isosceles triangle ABD,

The diagonals also bisect each other because a rhombus is a parallelogram, so we usually state the property as

'The diagonals of a rhombus bisect each other at right angles.'

EXERCISE 5

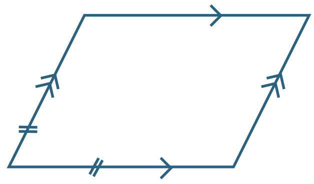
a Use congruence to prove this property.

b Use angle-chasing to prove this property.

We now turn to tests for a quadrilateral to be a rhombus. This is a matter of establishing that a property, or a combination of properties, gives us enough information for us to conclude that such a quadrilateral is a rhombus.

First test for a rhombus – A parallelogram with two adjacent sides equal

We have proved that the opposite sides of a parallelogram are equal, so if two adjacent sides are equal, then all four sides are equal and it is a rhombus.



Theorem

If two adjacent sides of a parallelogram are equal, then it is a rhombus.

This test is often taken as the definition of a rhombus.

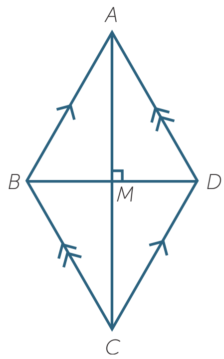
Second test for a rhombus – A quadrilateral whose diagonals bisect each other at right angles

Theorem

A quadrilateral whose diagonals bisect each other at right angles is a rhombus.

Proof

Let $ABCD$ be a quadrilateral whose diagonals bisect each other at right angles at M.



We prove that $DA = AB$. It follows similarly that

$AB = BC$ and $BC = CD$

$\triangle AMB \equiv \triangle AMD$ (SAS)

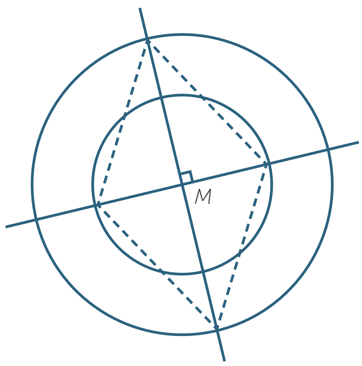
So $AB = AD$ and by the first test above $ABCD$ is a rhombus.

A quadrilateral whose diagonals bisect each other is a parallelogram, so this test is often stated as

'If the diagonals of a parallelogram are perpendicular, then it is a rhombus.'

This test gives us another construction of a rhombus.

- Construct two perpendicular lines intersecting at M.
- Draw two circles with centre M and different radii.
- Join the points where alternate circles cut the lines.



This figure is a rhombus because its diagonals bisect each other at right angles.

EXERCISE 6

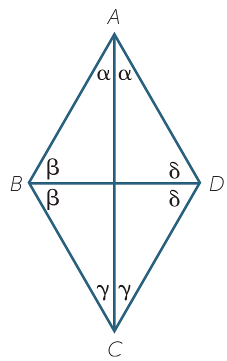
If the circles in the constructions above have radius 4cm and 6cm, what will the side length and the vertex angles of the resulting rhombus be?

Third test for a rhombus – A quadrilateral in which the diagonals bisect the vertex angles

Theorem

If each diagonal of a quadrilateral bisects the vertex angles through which it passes, then the quadrilateral is a rhombus.

Proof



Let *ABCD* be a quadrilateral, and suppose the diagonals bisect the angles, then let

$\angle DAC = \angle BAC = \alpha$ $\quad\quad$ $\angle ABD = \angle CBD = \beta$

$\angle BCA = \angle DCA = \gamma$ $\quad\quad$ $\angle CDB = \angle ADB = \delta$

To prove that *ABCD* is a rhombus.

First,	$2\alpha + 2\beta + 2\gamma + 2\delta = 360°$	(angle sum of quadrilateral *ABCD*)
	$\alpha + \beta + \gamma + \delta = 180°$	
Secondly,	$\alpha + 2\beta + \gamma = 180°$	(angle sum of $\triangle ABC$):
Combining these,	$\beta = \delta$,	
Hence	$AB \parallel DC$ and $BC \parallel AD$	(alternate angles are equal)
and	*ABCD* is a parallelogram	
and	$AD = AB$	(opposite angles are equal in $\triangle ABD$);

so *ABCD* is a rhombus because it is a parallelogram with a pair of adjacent sides equal.

Rhombuses – definition, properties, tests and symmetries

Definition of a rhombus

- A rhombus is a quadrilateral with all sides equal.

Properties of a rhombus

- The opposite sides of a rhombus are parallel.
- The opposite angles of a rhombus are equal.
- The diagonals of a rhombus bisect each vertex angle.
- The diagonals of a rhombus bisect each other at right angles.

Tests for a rhombus

A quadrilateral is a rhombus if:

- it is a parallelogram, and a pair of adjacent sides are equal,
- its diagonals bisect each other at right angles,
- its diagonals bisect each vertex angle.

Symmetries of a rhombus

- The diagonals of a rhombus are perpendicular axes of symmetry.
- The rhombus has rotation symmetry of order two in their intersection.

Extension – Quadrilaterals whose diagonals are perpendicular

The converse of a property is not necessarily a test. For example, a quadrilateral with perpendicular diagonals need not be a rhombus – just place two sticks across each other at right angles and join their endpoints. The following exercise gives an interesting characterisation of quadrilaterals with perpendicular diagonals.

Both parts of the proof are applications of Pythagoras' theorem. One half is straightforward, the other requires proof by contradiction and an ingenious construction.

EXERCISE 7

Use Pythagoras' theorem to prove that the diagonals of a convex quadrilateral are perpendicular if and only if the sum of the squares of each pair of opposite sides are equal.

SQUARES

We usually think of a square as a quadrilateral with all sides equal and all angles right angles. Now that we have dealt with the rectangle and the rhombus, we can define a square concisely as:

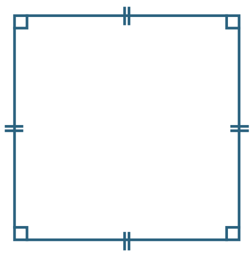
Definition of a square

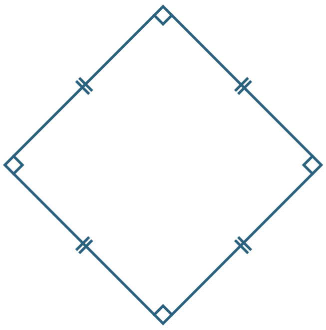
A square is a quadrilateral that is both a rectangle and a rhombus.

Properties of a square

A square thus has all the properties of a rectangle, and all the properties of a rhombus.

- Opposite sides are parallel.
- The diagonals meet each side at 45°.
- The diagonals are equal in length, and bisect each other at right angles.
- The two diagonals, and the two lines joining the midpoints of opposite sides, are axes of symmetry.





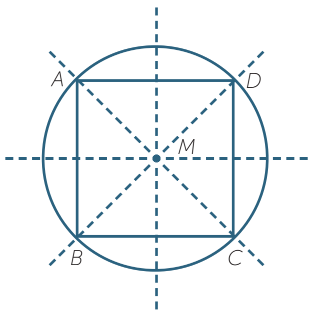
Symmetries of a square

The intersection of the two diagonals is the *circumcentre* of the *circumcircle* through all four vertices. We have already seen, in the discussion of the symmetries of a rectangle, that all four axes of symmetry meet at the circumcentre.

A square *ABCD* is congruent to itself in three other orientations,

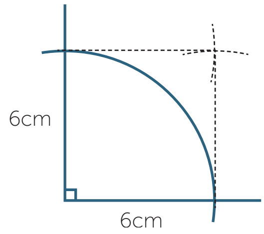
$$ABCD \equiv BCDA \equiv CDAB \equiv DABC$$

corresponding to the fact that it has *rotation symmetry of order 4*. The centre of the rotation symmetry is the circumcentre, because the vertices are equidistant from it.

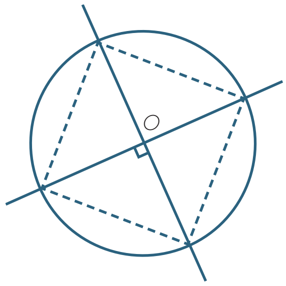


Constructing a square

The most obvious way to construct a square of side length 6cm is to construct a right angle, cut off lengths of 6cm on both arms with a single arc, and then complete the parallelogram.



Alternatively, we can combine the previous diagonal constructions of the rectangle of the rhombus. Construct two perpendicular lines intersecting at *O*, draw a circle with centre *O*, and join up the four points where the circle cuts the lines.



EXERCISE 8

What radius should the circle have for the second construction above to produce a square of side length 6cm?

Squares – definition, properties, tests and symmetries

Definition of a square

- A **square** is a quadrilateral that is both a rectangle and a rhombus.

Properties of a square

- The opposite sides of a square are parallel.
- All sides of a square are equal.
- All angles of a square are right angles.
- The diagonals of a square meet each side at 45°.
- The diagonals of a square are equal and bisect each other at right angles.

Test for a square

- The quadrilateral must be both a rectangle and a rhombus.

Symmetries of a square

- The two diagonals of a square, and the two lines joining the midpoints of opposite sides, are axes of symmetry. These four axes are all concurrent in the *circumcentre* of the *circumcircle* passing through all four vertices.
- A square has rotation symmetry of order 4 about its circumcentre.

Extension – A dissection problem

The following problem requires the construction that divides a given interval in a given ratio – see the module *Constructions*.

EXERCISE 9

a Through the vertex *A* of a square *ABCD*, construct two lines that dissect the square into three regions of equal area.

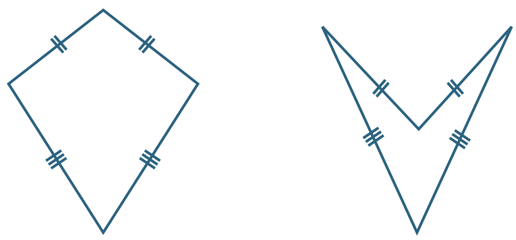
b Does this construction also work with rhombuses, rectangles and parallelograms?

KITES

Some of the distinctive properties of the diagonals of a rhombus hold also in a kite, which is a more general figure. Because of this, several important constructions are better understood in terms of kites than in terms of rhombuses.

Definition of a kite

A **kite** is a quadrilateral with two pairs of adjacent equal sides.



A kite may be convex or non-convex, as shown in the diagrams above. Only the convex cases are presented in the proofs below – the non-convex cases are similar, but are left as exercises.

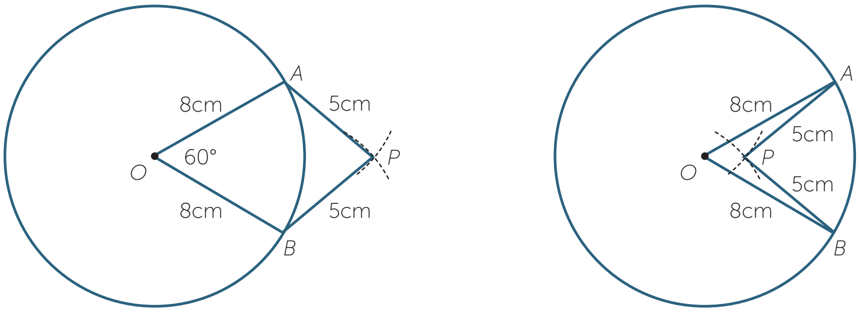
Constructing a kite using its definition

The definition allows a straightforward construction using compasses. Suppose that we want to construct a kite with side lengths 8cm and 5cm, with the two 8cm sides meeting at 60°.

- Draw a circle with centre *O* and radius 8cm.
- Draw two radii *OA* and *OB* meeting at *O* at an angle of 60°.
- Complete the kite *OAPB* by drawing circles of radius 5cm with centres at *A* and *B*.

The last two circles meet at two points *P* and P_0, one inside the large circle and one outside, giving a convex kite and a non-convex kite meeting the specifications.

Notice that the reflex angle of a non-convex kite is formed between the two shorter sides.



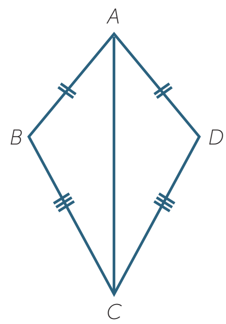
EXERCISE 10

What will the vertex angles and the lengths of the diagonals be in the kites constructed above?

First property of a kite – The axis of symmetry

Theorem

Let $ABCD$ be a kite with $AB = AD$ and $CB = CD$.



a $\triangle ABC \equiv \triangle ADC$.

b The diagonal AC is an axis of reflection symmetry of the kite.

c The axis AC bisects the vertex angles at A and C.

d $\angle B = \angle D$.

Proof

The congruence follows from the definition, and the other parts follows from the congruence.

Second property of a kite – The axis is the perpendicular bisector of the other diagonal

Theorem

The axis of a kite is the perpendicular bisector of the other diagonal.

Proof

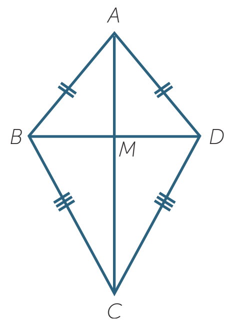
Let $ABCD$ be a kite with $AB = AD$ and $CB = CD$.

and let the diagonals of the kite meet at M.

Using the theorem about the axis of symmetry of an isosceles triangle, the bisector *AM* of the apex angle of the isosceles triangle $\triangle ABD$ is also the perpendicular bisector of its base *BD*.

Hence BM = MD and AM $\perp$ BD

So AC $\perp$ BD



EXERCISE 11

a Prove this result using congruence.

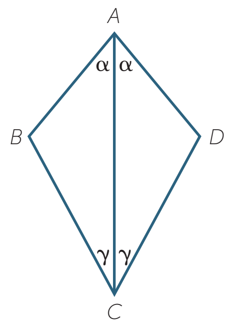
b Prove this result using angle-chasing.

Tests for a kite

The converses of some these properties of a kite are tests for a quadrilateral to be a kite.

Theorem

If one diagonal of a quadrilateral bisects the two vertex angles through which it passes, then the quadrilateral is a kite.

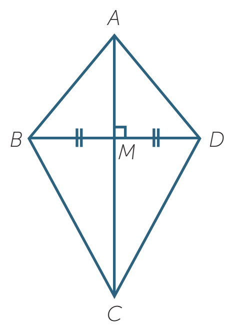


EXERCISE 12

Prove this result using the given diagram.

Theorem

If one diagonal of a quadrilateral is the perpendicular bisector of the other diagonal, then the quadrilateral is a kite.



EXERCISE 13

Prove this result using the given diagram.

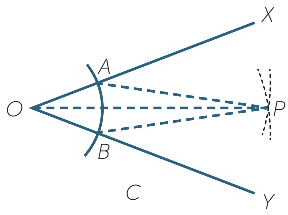
EXERCISE 14

Is it true that if a quadrilateral has a pair of opposite angles equal and a pair of adjacent sides equal, then it is a kite?

Kites and geometric constructions

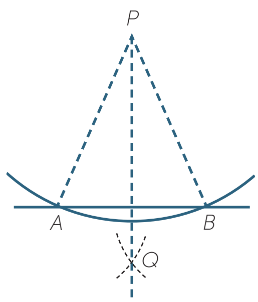
Three of the most common ruler-and-compasses constructions can be explained in terms of kites.

1 The first diagram to the right shows the construction of the **angle bisector** of $\angle XOY$. This construction works because the axis *OP* of the kite *OAPB* bisects the vertex angle at *O*.



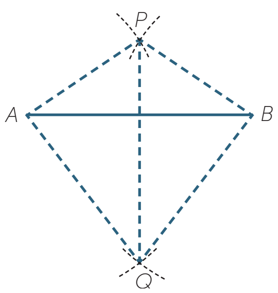
Notice that the radii of the arcs meeting at *P* need not be the same as the radius of the first arc with centre *O*.

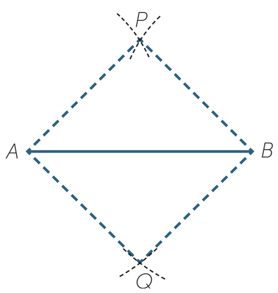
2 The second diagram to the right shows the construction of the **perpendicular to a line** *l* from a point *P*. This construction works because the diagonals of the kite *PAQB* are perpendicular.



Notice that the radii of the arcs meeting at *Q* need not be the same as the radii of the original arc with centre *P*.

3 The two diagrams below show the construction of the perpendicular bisector of *AB*. This construction works because the axis *PQ* of the kite *APBQ* bisects the other diagonal *AB* at right angles.





In the diagram to the left, the radii of the arcs meeting at *P* are not the same as the radii of the arcs meeting at *Q*. Of course it is usual in this construction, and far more convenient, to use equal radii – as in the diagram to the right – in which case the figure constructed is a rhombus.

Kites – definition, properties, tests and symmetries

Definition of a kite

- A **kite** is a quadrilateral with two pairs of adjacent sides equal.
- A kite may be convex or non-convex.

Axis of symmetry of a kite

- The line through the two vertices where equal sides meet is an **axis** of **symmetry of a kite**, called the axis of the kite.

Properties of a kite

- The angles opposite the axis of a kite are equal.
- The axis of a kite bisects the vertex angles through which it passes.
- The axis of a kite is the perpendicular bisector of the other diagonal.

Test for a kite

A quadrilateral is a kite if:

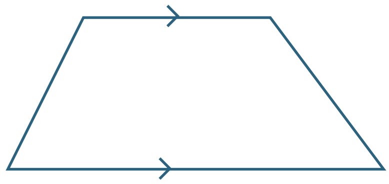
- one diagonal bisects the vertex angles through which it passes, or
- one diagonal is the perpendicular bisector of the other diagonal.

TRAPEZIA

Trapezia also have a characteristic property involving the diagonals, but the property concerns areas, not lengths or angles.

Definition of a trapezium

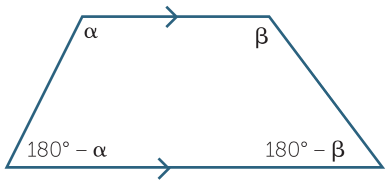
A **trapezium** is a quadrilateral with one pair of opposite sides parallel.



Although the name is Latin (the plural is 'trapezia'), it originally comes from the Greek word *trapeza*, meaning 'table'. The figure is called a 'trapezoid' in the USA.

The angles of a trapezium

Using co-interior angles, we can see that a trapezium has two pairs of adjacent supplementary angles.



Conversely, if a quadrilateral is known to have one pair of adjacent supplementary angles, then it is a trapezium.

EXERCISE 15

What sort of figure is both a kite and a trapezium?

The diagonals of a trapezium

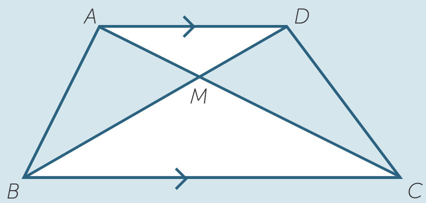
The diagonals of a convex quadrilateral dissect the quadrilateral into four triangular regions, as shown in the diagrams below. In a trapezium, two of these triangles have the same area, and the converse of this property is a test for a quadrilateral to be a trapezium.

These results are written as exercises because they are not usually regarded as standard theorems for students to know.

EXERCISE 16

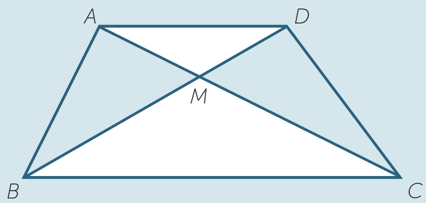
Let $ABCD$ be a trapezium with $AD \parallel BC$, and let the diagonals intersect at M.

Prove that $\triangle AMB$ and $\triangle DMC$ have the same area.



EXERCISE 17

Conversely, let $ABCD$ be a quadrilateral in which $\triangle AMB$ and $\triangle DMC$ have the same area, where M is the intersection of the diagonals. Prove that $ABCD$ is a trapezium with $AD \parallel BC$.



Extension – Isosceles trapezia

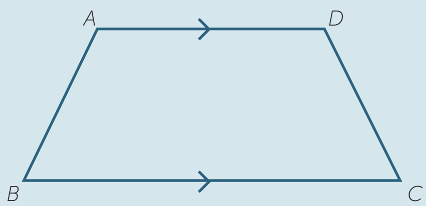
The trapezia that occur in this exercise are called **isosceles trapezia**. Further results about isosceles trapezia can be found at http://en.wikipedia.org/wiki/Isosceles_trapezoid.

EXERCISE 18

Let $ABCD$ be a trapezium with $AD \parallel BC$, and suppose that $AD < BC$, so that $ABCD$ is not a parallelogram.

a Prove that $AB = DC$ if and only if $\angle B = \angle C$.

b Prove that $AB = DC$ if and only if the line through the midpoints F of AD and G of BC is an axis of symmetry of the trapezium.

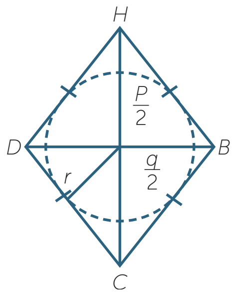


c Prove that no other line is an axis of symmetry of the isosceles trapezium.

LINKS FORWARD

This module completes the study of special quadrilaterals using congruence. Similarity is a generalisation of congruence, and when it has been developed, some further results about special quadrilaterals will become possible. The module, *Circle Geometry* will use some special quadrilaterals, and will also introduce cyclic quadrilaterals, which are quadrilaterals whose four vertices all lie on the one circle – they will be the last special quadrilaterals discussed in these modules.

All triangles have both a circumcircle and an incircle. The only quadrilaterals that have a circumcircle are those with opposite angles supplementary, the situation with incircles is interesting. For example, a rhombus always has an incircle.



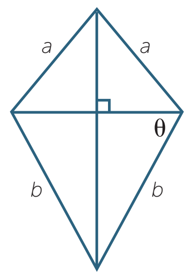
As an easy exercise show that if the lengths of the diagonals of the rhombus are p and q and the radius of the incircle is r then

$$r = \frac{pq}{2\sqrt{p^2+q^2}}$$

A kite has an incircle as well but its radius is difficult to calculate.

A kite is determined by the triangle with side lengths a, b and included angle θ. If the lengths of the diagonals are p and q show that :

$$2pq = ab \sin \theta$$



When complex numbers are graphed on Argand diagrams, many arithmetic and algebraic results are proved or illustrated using special quadrilaterals. In particular if z_1 and z_2 are two complex numbers of equal modulus then the four numbers, z_1, $z_1 + z_2$ and z_2 form a rhombus so, as a consequence, $z_1 + z_2$ and $z_1 - z_2$ are perpendicular vectors.

HISTORY AND APPLICATIONS

This illustrates very well the constant attitude in mathematics that an investigation is not complete until a theorem with a true converse has been identified. It reminds us too that logic, accompanied by the intuition of diagrams, should always be a strong motivation in geometry.

Whenever a surface is divided up, triangles and special quadrilaterals are involved, particularly when parallel lines are used in the dissection. Thus surveyors analysing suburban blocks or farming lots will try to use the simplest geometric shapes in their analysis, and architects, who often have great freedom to invent striking patterns for their building, often use special quadrilaterals other than simple squares and rectangles in their designs. Infinite tilings of the plane, for example, are possible with any other quadrilateral. Trigonometry is also an essential part of these processes, and trigonometry and geometry should be seen as a unit rather than as two disconnected topics. Several exercises in these modules have required such connections to be made.

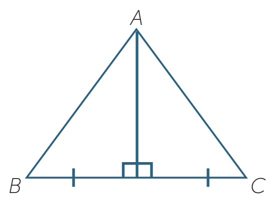
A standard problem for computer programmers is to encrypt pictures with as little storage as possible, and they typically divide up the picture into simple geometrical shapes as part of that process – a recent study uses interlocking trapezia for this purpose.

All the special quadrilaterals of this and the previous module, apart from the kite, were studied by the ancient Greeks as part of their systematic investigation of geometry. The kite was named and brought to attention in modern times, partly because that it clarifies several important geometric constructions, but also because it demonstrates that some of the properties of a rhombus hold in more general quadrilaterals.

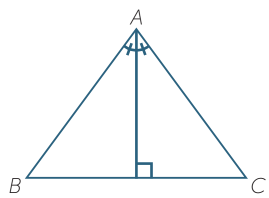
ANSWERS TO EXERCISES

EXERCISE 1

a This is a simple application of the SAS congruence test.

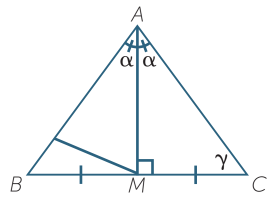


b This is a simple application of the AAS congruence test.



c Suppose AC < AB. Choose P on the internal AB so that AP = AC and join PM

	$\triangle PAM \equiv \triangle CAM$ (SAS)	
so	$\angle APM = \angle C = \gamma$	(matching angles of congruent triangles)
so	$\angle BPM = 180° - \gamma$	(straight angle)
Also	$PM = CM$	(matching sides of congruent triangles)
So	$\angle MPB = \angle MBP = 180° - \gamma$	(base angles of isosceles $\triangle MPB$)
Hence	$\alpha + \alpha + \gamma + (180° - \gamma) = 180°$	(angle sum of $\triangle ABC$)
	$\alpha = 0°$. So AC = AB	



EXERCISE 2

In the triangles APM and CQM in the given diagram:

1 $AP = QC$ (each is half the opposite side of a rectangle)

2 $\triangle MAP = \triangle MCQ$ (alternate angles, $AB \parallel DC$)

3 $\angle AMP = \angle CMQ$ (vertically opposite angles)

so $\triangle APM \equiv \triangle CQM$ (AAS).

Hence $AM = CM$ (matching sides of congruent triangles).

That is PQ bisects AC

EXERCISE 3

$AB^2 = 5^2 + 5^2 - 2 \times 5 \times 5 \times \cos 50°$
$= 50\,(1 - \cos 50°)$
$AB \approx 4.23$cm,

$PO^2 = 5^2 + 5^2 + 2 \times 5 \times 5 \times \cos 50°$
$= 50\,(1 + \cos 50°)$
$PO \approx 9.06$cm,

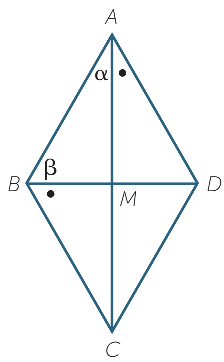
EXERCISE 4

The congruence $\triangle ABC \equiv \triangle ADC$ follows by the SSS test.

EXERCISE 5

a $\triangle ABM \equiv \triangle ADM$ (SAS or SSS).

b Let $\alpha = \angle BAM$ and $\beta = \angle ABM$.



Then	$\angle CBM = \beta$	(previous property)
and	$\angle BCM = \alpha$	(base angles of isosceles $\triangle ABC$);
so	$2\alpha + 2\beta = 180°$	(angle sum of $\triangle ABC$)
	$\alpha + \beta = 90°$	
Hence	$\angle AMB = 90°$	(angle sum of $\triangle ABM$).

or A rhombus is a parallelogram. So BM = MD. Hence $\triangle AMB \equiv \triangle ADM$ (SSS)
Thus $\angle AMB = \angle AMD$ and the diagonals are perpendicular.

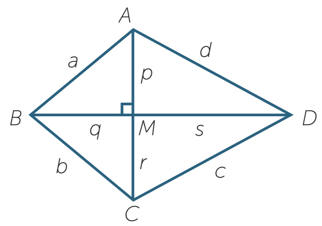
EXERCISE 6

Using Pythagoras' theorem, the side length is $2\sqrt{13}$cm. Using trigonometry, the vertex angles are about 67.38° and 112.62°.

EXERCISE 7

First, let *ABCD* be a convex quadrilateral whose diagonals meet at right angles at *M*. Let the sides and the intervals on the diagonals have lengths as on the diagram to the right. Then using Pythagoras' theorem,

$$a^2 + c^2 = (p^2 + q^2) + (r^2 + s^2)$$
$$= (q^2 + r^2) + (p^2 + s^2)$$
$$= b^2 + d^2, \text{ as required.}$$

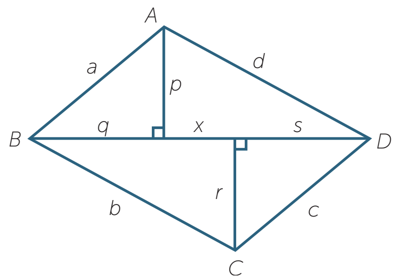


Conversely, let *ABCD* be a convex quadrilateral in which the diagonals are not perpendicular. The diagram will be as drawn on the right or its reflection, and it will be sufficient to consider only the one case. Let the lengths be as given on the figure, where $x \neq 0$ because *AC* is not perpendicular to *BD*. Using Pythagoras' theorem,

$$a^2 + c^2 = p^2 + q^2 + r^2 + s^2$$

and $b^2 + d^2 = (q + x)^2 + r^2 + (s + x)^2 + p^2$

so $(b^2 + d^2) - (a^2 + c^2) = 2x(q + s + x)$,
which is not zero because $x \neq 0$.



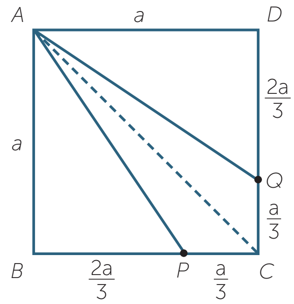
Hence if $a^2 + b^2 = c^2 + d^2$ then AC ' BD as required

EXERCISE 8

Using Pythagoras' theorem, the required radius is $3\sqrt{2}$cm.

EXERCISE 9

a Construct points P on BC and Q on DC so that



$BP : PC = DQ : QC = 2 : 1$

Then the triangles APB and APC have the same altitude AB,

and their bases are in the ratio 2 : 1, so

area $\triangle APB$: area $\triangle APC$ = 2 : 1

Similarly area $\triangle AQD$: area $\triangle AQC$ = 2 : 1. Since the diagonal AC divides the square into two congruent regions of equal area, the lines AP and AQ dissect the square into three regions of equal area.

or show that the area of each region is $\frac{a^2}{3}$

b A similar argument works

EXERCISE 10

Take the convex case first. Using equilateral triangles and Pythagoras' theorem, the axis is

$4\sqrt{3} + 3$cm and the other diagonal is 8cm. Using trigonometry, the angle at the other end of the axis is about 106.26°, and the other two angles are each about 60° + 36.87°.

In the non-convex case, the axis is $4\sqrt{3} - 3$cm and the other diagonal is still 8cm. The angle at the other end of the axis is about 253.74°, and the other two angles are each about 60° – 36.87°.

EXERCISE 11

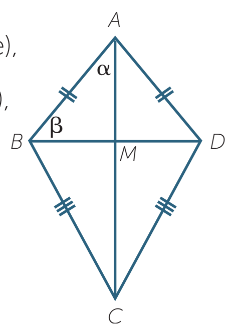
a Since the axis AC bisects the vertex angle at A,

$\triangle ABM \equiv \triangle ADM$ (SAS);

from which it follows that $\angle AMB = \angle AMD = 90°$.

b

Let	$\angle BAM = \angle DAM = \alpha$	(the axis bisects the vertex angle),
and let	$\angle ABM = \angle ADM = \beta$	(base angles of isosceles $\triangle ABD$),
Then	$2\alpha + 2\beta = 180°$	(angle sum of $\triangle ABD$)
	$\alpha + \beta = 90°$	
Hence	$\angle AMB = 90°$	(angle sum of $\triangle AMB$).



EXERCISE 12

Using the AAS congruence test, $\triangle ABC \equiv \triangle ADC$.

So $AB = AD$, $BC = DC$ and the quadrilateral is a kite.

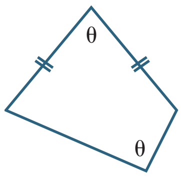
EXERCISE 13

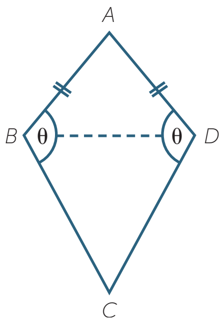
Using the SAS congruence test, $\triangle ABM \equiv \triangle ADM$ and $\triangle CBM \equiv \triangle CDM$.

So $AB = AD$ and $CB = CD$ and the quadrilateral is a kite.

EXERCISE 14

Diagram: Kite 10





If one of the equal angles is included by the given sides, then there is no reason for the figure to be a kite, as is illustrated in the diagram on the left above.

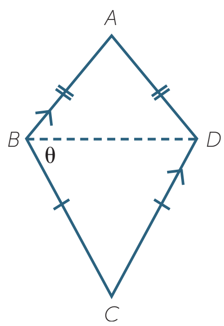
If neither equal angle is included by the equal sides, then the figure is a kite. Be careful, however, because joining the diagonal AC in the diagram on the right above would not give a congruence proof because the angle in each triangle would be non-included.

Instead, join the other diagonal BD, as in the diagram on the right above.

Then	$\angle ABD = \angle ADB$	(base angles of isosceles $\triangle ABD$),
so	$\angle CBD = \angle CDB$	(subtracting equal angles from equal angles)
so	$BC = DC$	(opposite angles are equal in $\triangle BCD$)
	$ABCD$ is a kite	

EXERCISE 15

In the diagram to the right, join BD and let $\angle DBC = \theta$.



Then	$\angle CDB = \theta$	(base angles of isosceles $\triangle CBD$)
so	$\angle ABD = \theta$	(alternate angles, $AB \parallel DC$)
so	$\angle ADB = \theta$	(base angles of isosceles $\triangle ABD$)
Hence	$AD \parallel BC$	(alternate angles are equal)

so $ABCD$ is a rhombus, because it is a parallelogram with two adjacent sides equal.

EXERCISE 16

The triangles $\triangle ABC$ and $\triangle DBC$ have the same perpendicular height and the same base BC, so area $\triangle ABC$ = area $\triangle DBC$.

Subtracting the triangle MBC from both regions,

area $\triangle AMB$ = area $\triangle DMC$.

EXERCISE 17

Adding $\triangle MBC$ to both regions,

area $\triangle ABC$ = area $\triangle DBC$

These two triangles have the same base BC, so they have the same perpendicular height. Hence $AD \parallel BC$.

EXERCISE 18

a Let $\angle B = \beta$ and construct P on BC so that $DP \parallel AB$.

Then $DP = AB$ (opposite sides of a parallelogram)

and $\angle DPC = \beta$ (corresponding angles, $AB \parallel DP$).

Suppose first that $AB = DC$.

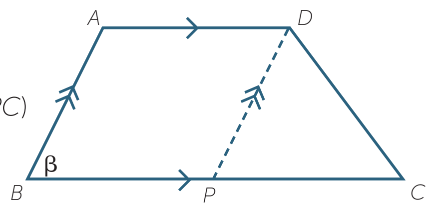
Then $PD = DC$

so $\angle C = \beta$ (base angles of isosceles $\triangle DPC$)

Conversely, suppose that $\angle C = \beta$

Then $DC = DP$ (opposite angles of $\triangle DPC$ are equal)

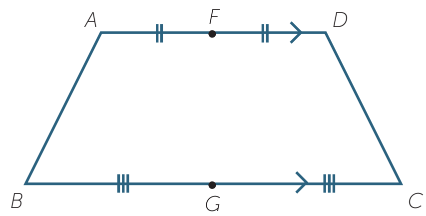
so $DC = AB$



b If FG is an axis of symmetry, then reflection in FG maps AB to DC, so $AB = DC$.

Conversely, suppose that $AB = DC$.

Then $\angle B = \angle C$ by part **a**.



Drop perpendiculars AL and DM to the line BC.

Then $AL = DM$ (opposite sides of rectangle $ALMD$)

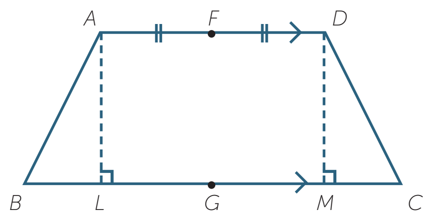
so $\triangle ABL \equiv \triangle DMC$ (AAS)

Hence $BL = CM$ (matching sides of congruent triangles) so $LG = MG$ (subtract equal lengths from equal lengths)

so $LG = AF$ (opposite sides of a rectangle)

Hence $FG \perp BC$

so reflection in FG swaps B and C, and swaps A and D, as required.



c Since $AD \neq BC$, a reflection cannot swap AD and BC.

If a reflection swapped AD and AB, then it would also swap BC and DC, so $ABCD$ would be a kite with parallel sides, so it would be a parallelogram. Similarly a reflection cannot swap AD and DC.

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