RATIONAL DOUBLE AFFINE HECKE ALGEBRAS

JOSÉ SIMENTAL RODRIGUEZ. NOTES BY BRADLEY HICKS.

Rational DAHA

In this lecture we will define the Rational Double Affine Hecke Algebra (DAHA), its presentations, and some of its subalgebras and standard modules. For us let $W \subset \mathrm{GL}(\mathfrak{h})$ be a complex reflection group, which requires $|W| < \infty$ and a subset $S = \{s \in W \mid \mathrm{codim}(\mathrm{Fix}(s)) = 1\}$ that generates W. We call elements of S *reflections* of W. For each reflection $s \in S$ we associate an element $\alpha_s \in \mathfrak{h}^*$ satisfying $\mathrm{Fix}(s) = \ker(\alpha_s)$ and $s\alpha_s = \lambda_s \alpha_s$ for some $\lambda_s \neq 1$. Note that this element is defined only up to multiplication by a nonzero scalar.

Next we need a parameter: a function $c : S \to \mathbb{C}$ that is constant on conjugacy classes, that is to say we require $c(wsw^{-1}) = c(s)$ for all $s \in S$ and $w \in W$. With these data we can introduce the **Dunkl operator** on the space of polynomial functions $\mathbb{C}[\mathfrak{h}]$.

Definition. Let $y \in \mathfrak{h}$. The **Dunkl operator** associated to y is

$$D_y = \partial_y - \sum_{s\in S} \frac{2c(s)}{1-\lambda_s} \frac{\langle \alpha_s, y\rangle}{\alpha_s}(1-s)$$

where ∂_y is the directional derivative: $\partial_y(x) = \langle y, x\rangle$ for $x \in \mathfrak{h}^*$, and $\partial_y(f)$ can be computed from any $f \in \mathbb{C}[\mathfrak{h}] = \mathrm{Sym}(\mathfrak{h}^*)$ using the Leibniz rule.

As written this is only an operator on $\mathfrak{h}^{\mathrm{reg}} := \mathfrak{h} \setminus \bigcup_{s\in S}\{\alpha_s = 0\}$ because it has pole set $\{\alpha_s = 0\}$. In fact, it is an element of the algebra $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}}) \rtimes W$. Here $\mathcal{D}(X)$ is the space of differential operators on X and the algebra has the underlying vector space $\mathcal{D}(X) \otimes \mathbb{C}W$ and the multiplication is defined to be

$$(d_1 \otimes w_1)(d_2 \otimes w_2) = d_1 w_1(d_2) \otimes w_1 w_2$$

However, even if the operator D_y has poles, it does act on $\mathbb{C}[\mathfrak{h}]$, polynomials in $\dim \mathfrak{h}$ variables, because given a polynomial $f : \mathfrak{h} \to \mathbb{C}$ the result of $(1-s)f[x] = f(x) - f(sx)$ is divisible by $\alpha_s(x)$. Hence we may think of these operators as elements of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}])$

These operators generate part of our rational DAHA:

Definition. The **Rational DAHA** $\mathcal{H}_c$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}])$ generated by

- $\mathbb{C}[\mathfrak{h}]$ acting on itself by multiplication,
- $\mathbb{C}W$,
- D_y for $y \in \mathfrak{h}$.

Note that, by definition, $\mathcal{H}_c$ acts faithfully on $\mathbb{C}[\mathfrak{h}]$.

This algebra is a deformation of $\mathcal{D}(\mathfrak{h}) \rtimes W$ in the sense that this algebra is recovered when $c \equiv 0$. We can also give presentations of $\mathcal{H}_c$. For each $s \in S$, take a nonzero element $\alpha_s^\vee \in \mathfrak{h}$ such that $s\alpha_s^\vee = \lambda_s^{-1}\alpha_s^\vee$. This element is again only defined up to a nonzero scalar, and we partially normalize so that $\langle \alpha_s, \alpha_S^\vee\rangle = 2$. Of course, this normalization is inspired by the case when W is the Weyl group of a root system, α_s is a root and $\alpha_s^\vee$ the corresponding coroot.

Theorem 1 (Etingof, Ginzburg)**.** Below we assume $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$. There is an isomorphism

$$\mathcal{H}_c \xrightarrow{\cong} \mathcal{T}(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W/\mathcal{R}$$

where $\mathcal{T}$ is tensor algebra functor and

$$\mathcal{R} = \left\langle [x, x'] = [y, y'] = 0, [y, x] = \langle y, x \rangle - \sum_{s \in S} c(s) \langle \alpha_s, y \rangle \langle \alpha_s^\vee, x \rangle s \right\rangle$$

are the relations.

$\mathcal{H}_c$ has the following notable subalgebras:

(1) $\mathbb{C}[\mathfrak{h}]$
(2) $\mathbb{C}W$
(3) $\mathcal{D}_c$, the subalgebra generated by D_y for all $y \in \mathfrak{h}$.

The following theorem of Dunkl shows the relationships between these subalgebras and other algebras.

Theorem 2 (Dunkl)**.** $\mathcal{D}_c$ is isomorphic to $\mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$, i.e. the multilinear forms on $\mathfrak{h}$. Moreover

(1) The algebra generated by $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}W$ is isomorphic to $\mathbb{C}[\mathfrak{h}] \rtimes W$.
(2) The algebra generated by $\mathbb{C}W$ and D_y is isomorphic to $\mathbb{C}[\mathfrak{h}^*] \rtimes W$.

Another important property of $\mathcal{H}_c$ is that it has a basis akin to the PBW basis for the universal enveloping algebra $U(\mathfrak{g})$ for a Lie algebra.

Theorem 3 (PBW: Etingof, Ginzburg)**.** There is an isomorphism

$$\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*] \xrightarrow{\cong} \mathcal{H}_c$$

given by $h \otimes w \otimes h^* \mapsto hwh^*$.

$\mathcal{H}_c$ has further filtered and graded properties.

(1) $\mathcal{H}_c$ has a filtering in the following way. Let $\deg W = 0$ and $\deg \mathfrak{h} = \deg \mathfrak{h}^* = 1$. This induces a filtration on $\mathcal{H}_c$ and by the PBW theorem we can identify the associated graded

$$\mathrm{gr}\, \mathcal{H}_c \xrightarrow{\cong} \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W\,.$$

(2) From the relations, we can see that $\mathcal{H}_c$ has a grading by setting $\deg W = 0$, $\deg \mathfrak{h} = -1$, and $\deg \mathfrak{h}^* = 1$. This grading is *inner*, meaning that there exists an element $h \in \mathcal{H}_c$ satisfying $[h, m] = \deg(m) m$ for a homogeneous element m. We can construct such an element (called the *Euler* element) as follows. Given a basis of y_i of $\mathfrak{h}$ and a dual basis x_i this element is

$$h = \sum_{i=1}^{\dim \mathfrak{h}} \frac{x_i y_i + y_i x_i}{2} = \sum_{i=1}^{\dim \mathfrak{h}} x_i y_i - \frac{1}{2} \dim \mathfrak{h} - \sum_{s \in S} \frac{2c(s)}{1 - \lambda_s} s$$

This element is in fact independent of the choice of basis and satisfies

$$[h, w] = 0 \qquad [h, x] = x \qquad [h, y] = -y$$

where $w \in W, y \in \mathfrak{h}, x \in \mathfrak{h}^*$.

Some Representation Theory of $\mathcal{H}_c$

Definition. Let $\mathcal{O} = \mathcal{O}_c$ be the category of finitely generated $\mathcal{H}_c$ modules which have a locally nilpotent action of $\mathfrak{h} \subset \mathcal{H}_c$.

An example object in this category is the polynomial representation $\mathbb{C}[\mathfrak{h}]$. Indeed, $\mathfrak{h}$ acts on $\mathbb{C}[\mathfrak{h}]$ by Dunkl operators, which decrease the degree of a polynomial by at least 1. Given an irreducible representation λ of the group W we can extend the action to one of algebra $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ by letting $\mathfrak{h}$ act by zero. This gives us our **standard modules**

$$\Delta_c(\lambda) := \mathrm{Ind}_{\mathbb{C}[\mathfrak{h}^*]\rtimes W}^{\mathcal{H}_c}(\lambda) = \mathcal{H}_c \underset{\mathbb{C}[\mathfrak{h}^*]\rtimes W}{\otimes} \lambda \overset{\text{PBW}}{=} \mathbb{C}[\mathfrak{h}] \otimes \lambda$$

The last equality is as a $\mathbb{C}[\mathfrak{h}]$ module and follows from the PBW theorem. As a simple example if we induce the trivial represention $\lambda = \mathbb{1}$ we have the polynomial representation $\Delta_c(\mathbb{1}) = \mathbb{C}[\mathfrak{h}]$.

Generally we have a subspace $1 \otimes \lambda \subset \Delta_c(\lambda)$, on which h acts by some scalar c_λ because h commutes with W. Hence the action of h is diagonalizable with eigenvalues of the form $c_\lambda + k$ for $k \in \mathbb{Z}_{\geq 0}$ and weight spaces

$$\Delta_c(\lambda)_{c_\lambda + k} = \mathbb{C}[\mathfrak{h}]_k \otimes \mathfrak{h}\,,$$

where $\mathbb{C}[\mathfrak{h}]_k$ are the homogeneous polynomials of degree k. This seemingly innocent fact has a couple of important consequences. First, it can be deduced that there is a unique irreducible quotient $L_c(\lambda)$ of $\Delta_c(\lambda)$. These irreducible quotients form a complete list of irreducibles in $\mathcal{O}_c$. Second, note that if $L_c(\mu)$ appears as a composition factor in $\Delta_c(\lambda)$, then $c_\mu = k + c_\lambda$ for some $k \geq 0$ (and, if $\mu \neq \lambda$, $k > 0$). It follows that if c is a parameter so that $c_\lambda - c_\mu \notin \mathbb{Z}$ for any two irreducibles $\lambda \neq \mu$, then the category $\mathcal{O}_c$ is semisimple and equivalent to the category of representations of W.

The category $\mathcal{O}_c$ is intimately related to the category of finite-dimensional representations of a certain finite Hecke algebra, as follows. First, consider the element $\delta := \prod_{s \in S} \alpha_s \in \mathbb{C}[\mathfrak{h}] \subseteq \mathcal{H}_c$. Since $\mathcal{H}_c \subseteq \mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes W$ and $\mathfrak{h}^{\text{reg}}$ is precisely the principal open set defined by δ, the non-commutative localization $\mathcal{H}_c[\delta^{-1}]$ makes sense and it follows from the formula defining the Dunkl operators that $\mathcal{H}_c[\delta^{-1}] = \mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes W$.

Now take $M \in \mathcal{O}_c$. By definition, it is finitely generated over $\mathcal{H}_c$. It is an exercise to see that, moreover, it is finitely generated over $\mathbb{C}[\mathfrak{h}]$. It follows that $M[\delta^{-1}] := \mathbb{C}[\mathfrak{h}][\delta^{-1}] \otimes_{\mathbb{C}[\mathfrak{h}]} M$ is a $\mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes W$-module that is finitely generated over $\mathbb{C}[\mathfrak{h}^{\text{reg}}]$. From the theory of D-modules it follows that $M[\delta^{-1}]$ is a W-equivariant vector bundle on $\mathfrak{h}^{\text{reg}}$ with a flat connection. Taking W-invariants, we obtain a vector bundle on $\mathfrak{h}^{\text{reg}}/W$ with a flat connection. The monodromy representation then equips a fiber of this vector bundle with an action of the fundamental group $\pi_1(\mathfrak{h}^{\text{reg}}/W)$.

This analysis can be encoded as functors

$$\mathcal{O}_c \longrightarrow \mathrm{Rep}(\mathcal{H}_c[\delta^{-1}]) \longrightarrow \mathrm{Rep}(\mathcal{D}(\mathfrak{h}^{\text{reg}}/W)) \longrightarrow \mathrm{Rep}(\pi_1(\mathfrak{h}^{\text{reg}}/W))$$

$$M \longmapsto M[\delta^{-1}] \longmapsto M[\delta^{-1}]^W \longmapsto M[\delta^{-1}]^W_v$$

where we use the fact $\mathcal{H}_c[\delta^{-1}] = \mathcal{D}(\mathfrak{h}^{\text{reg}}) \rtimes W$ to induce the second map, and the notation $M[\delta^{-1}]^W_v$ means the fiber of the bundle $M[\delta^{-1}]^W$ at a point $v \in \mathfrak{h}^{\text{reg}}/W$ (any choice of points yields isomorphic representations).

An amazing fact now is that the action of $\pi_1(\mathfrak{h}^{\text{reg}}/W)$ factors through a much smaller quotient of the group algebra $\mathbb{C}\pi_1(\mathfrak{h}^{\text{reg}}/W)$, known as the finite Hecke algebra. Here, we will only give details on the case when $W = S_n$ is the symmetric group, acting on $\mathfrak{h} = \mathbb{C}^n$ by permuting the coordinates. Note that in this case there is a single conjugacy class of reflections, and so our parameter is a single complex number $c \in \mathbb{C}$. The set $\mathfrak{h}^{\text{reg}}$ consists of points in $\mathbb{C}^n$ with pairwise distinct coordinates, and $\pi_1(\mathfrak{h}^{\text{reg}}/W)$ is the usual Artin braid group, generated by $T_1, \dots, T_{n-1}$ with relations $T_iT_j = T_jT_i$ if $|i-j| > 1$, and $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ (the element T_i represents a half loop around the hyperplane $x_i = x_{i+1}$, that descends to a loop in the quotient $\mathfrak{h}^{\text{reg}}/W$). For a module $M \in \mathcal{O}_c$, the action of $\mathbb{C}\pi_1(\mathfrak{h}^{\text{reg}}/W)$ on $M[\delta^{-1}]_v^W$ factors through the quotient

$$H_q = \mathbb{C}\pi_1(\mathfrak{h}^{\text{reg}}/W)/\langle (T_i - 1)(T_i + e^{2\pi\sqrt{-1}c})\rangle_{i=1,\dots,n-1}$$

that, up to a renormalization, coincides with the finite Hecke algebra that appeared in Monica Vazirani's lectures. To summarize, we have a functor $\text{KZ} : \mathcal{O}_c \to H_q\text{-mod}$, known as the *Knizhnik-Zamolodchikov* functor (because the connection appearing in its definition coincides with the Knizhnik-Zamolodchikov connection). This functor is exact, and it is one of the most important tools in the representation theory of the rational DAHA $\mathcal{H}_c$.

Examples

Example 1. Let $\iota = \sqrt{-1}$. Let $W = \mathbb{Z}/\ell\mathbb{Z} = \langle s \mid s^\ell = 1\rangle$ act on $\mathfrak{h} = \mathbb{C}$ via multiplication by $\eta = e^{2\pi\iota/\ell}$, i.e. $s.z = \eta z$ which is the rotation of the complex plane by the angle $2\pi/\ell$. Our reflections are $S = s^i \mid 1 \leq i \leq \ell - 1$ and our function c is determined by the numbers $c(s^i) = c_i$, or just the vector $c = (c_1, \dots, c_{\ell-1})$. Pick $x \in \mathfrak{h}^*$ and define $\alpha_s = x \in \mathfrak{h}^*$; the number λ_s is η^{-1} since the W acts via the adjoint on $\mathfrak{h}^*$. Then the Dunkl operator is

$$D_y = \partial_y - \sum_{i=1}^{\ell-1} \frac{2c_i}{1-\eta^{-i}} \frac{1-s^i}{x}$$

Then $\mathcal{H}_c$ can be presented as the algebra

$$\mathbb{C}[x, y, s]/\mathcal{R}$$

with relations

$$\mathcal{R} = \left\langle s^\ell = 1, sxs^{-1} = \eta x, sys^{-1} = \eta^{-1}y, [y, x] = 1 - \sum_{i=1}^{\ell-1} 2c_i s^i \right\rangle.$$

Example 2. Let $W = S_n$ act on $\mathfrak{h} = \mathbb{C}^n$ by permutation of the coordinates and take

$$S = \{(ij) \mid i < j\}.$$

Then $\alpha_{(ij)} = x_i - x_j$, the usual $GL_n(\mathbb{C})$ positive roots, and the numbers λ_s are all seen to be -1. Since all elements of S are conjugate in S_n we need only specify a single complex number c. Then the Dunkl operators take the form

$$D_y = \partial_y - \sum_{j \neq i} \frac{c}{x_i - x_j}(1 - (ij))$$

and $\mathcal{H}_c$ can be presented with the relations

$$\mathcal{R} = \left\langle [x_i, x_j] = [y_i, y_j] = 0, [y_j, x_i] = c(ij) \text{ if } i \neq j\ , [y_i, x_i] = 1 - c\sum_{i \neq j}(ij) \right\rangle$$

as the algebra

$$\mathcal{H}_c = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \rtimes S_n/\mathcal{R}\,.$$

Recall that the irreducible representations of S_n correspond exactly to partitions λ of n. Let $S(\lambda)$ be these modules. Then $-s\sum_{s\in S} s$ acts on $S(\lambda)$ by

$$-c\sum_{i=1}^{n} \mathrm{JM}_i$$

where $\mathrm{JM}_i = \sum_{j<i}(ji)$ are the Jucys-Murphy's elements. $S(\lambda)$ has a basis consisting of standard Young tableaux of shape λ and JM_i acts by $\mathrm{JM}_i\, t = \mathrm{ct}(\boxed{\mathrm{i}})t$, where $\mathrm{ct}(\boxed{\mathrm{i}})$ is the content of the cell $\boxed{\mathrm{i}}$ which is defined to be its column coordinate minus its row coordinate,

$$\mathrm{ct}(\boxed{\mathrm{i}}) = \mathrm{col}(\boxed{\mathrm{i}}) - \mathrm{row}(\boxed{\mathrm{i}})\,.$$

Therefore

$$c_\lambda = -\frac{n}{2} - c\sum_{\Box\in\lambda} \mathrm{ct}(\Box)$$

We note that if $L(\mu)$ appears in the Jordan-H older series for $\Delta(\lambda)$ then $c_\mu = c_\lambda + k$ for $k \in \mathbb{Z}_{\geq 0}$. For generic complex numbers c the category $\mathcal{O}_c$ is semi-simple.