

# Dynamics of spin $\frac{1}{2}$ quantum plasmas

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The fully nonlinear governing equations for spin  $\frac{1}{2}$  quantum plasmas are presented. Starting from the Pauli equation, the relevant plasma equations are derived, and it is shown that nontrivial quantum spin couplings arise, enabling studies of the combined collective and spin dynamics. The linear response of the quantum plasma in an electron-ion system is obtained and analyzed. Applications of the theory to solid state and astrophysical systems as well as dusty plasmas are pointed out.

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There is currently a great deal of interest in investigating collective plasma modes [1, 2, 3, 4, 5, 6, 7, 8] in quantum plasmas, as such plasmas could be of relevance in nano-scale electro-mechanical systems [9, 10, 11], in microplasmas and dense laser-plasmas [12], and laser interactions with atomic systems [13, 14]. For example, Refs. [1] and [3, 4, 5] used quantum transport models in order to derive modified dispersion relations for Langmuir and ion-acoustic waves, while Shukla & Stenflo [15] investigated drift modes in nonuniform quantum magnetoplasmas. Moreover, it is known that cold quantum plasmas can support new dust modes [16, 17]. In Ref. [8] it was shown that electron quantum plasmas could support highly stable dark solitons and vortices. Further examples of quantum plasmas and the range of validity of their descriptions has been discussed recently in Ref. [18]. The above studies of quantum plasmas have used models based on the Schrödinger description of the electron. It is expected that new and possible important effects could appear as further quantum effects are incorporated in models describing the quantum plasma particles. The coupling of spin to classical motion has attracted interest in the literature (see, e.g., [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]). Much work has been done concerning single particle spin effects in external field configurations, such as intense laser fields [22, 23, 24, 25, 26, 27], and the possible experimental signatures thereof. However, there have also been interest in excitations of collective modes in spin systems, such as spin waves, in a wide scientific community. For example, in Refs. [19, 20, 21] hydrodynamical models including spin was presented, and further theory concerning spin, angular momentum, and the forces related to spin was discussed in Refs. [29] and [30]. Moreover, spin waves in spinor Bose condensates has recently been discussed in, e.g., Ref. [31]. The treatment of charged particles and plasmas using quantum theory has received attention in astrophysical settings, especially in strongly mag-

netized environments [32, 33]. For example, effects of quantum field theory on the linear response of an electron gas has been analyzed [34], results concerning the spin-dependence of cyclotron decay on strong magnetic fields has been presented [35], and the propagation of quantum electro-dynamical waves in strongly magnetized plasmas has been considered [36].

In this Letter we present for the first time the fully nonlinear governing equations for spin  $\frac{1}{2}$  quantum electron plasmas. Starting from the Pauli equation describing the nonrelativistic electron, we show that the electron-ion plasma equations are subject to spin related terms. These terms give rise to a multitude of collective effects of which some are investigated in detail. Applications of the governing equations are discussed, and it is shown that under certain circumstances the collective spin effects can dominate the plasma dynamics.

We will assume that the electron wave function can be written in the product form  $\Psi = \Psi_{(1)}\Psi_{(2)}\dots\Psi_{(N)}$ , where  $N$  is the number of particle states. Thus we will here neglect the effects of entanglement and focus on the collective properties of the the quantum electron plasma. Then the non-relativistic evolution of spin  $\frac{1}{2}$  particles, as described by the two-component spinor  $\Psi_{(\alpha)}$ , is given by (see, e.g. [37])

$$i\hbar\frac{\partial\Psi_{(\alpha)}}{\partial t} = \left[ -\frac{\hbar^2}{2m_e} \left( \nabla + \frac{ie}{\hbar c} \mathbf{A} \right)^2 + \mu_B \mathbf{B} \cdot \boldsymbol{\sigma} - e\phi \right] \Psi_{(\alpha)} \quad (1)$$

where  $\alpha$  numbers the particle states,  $m_e$  is the particle mass,  $\mathbf{A}$  is the vector potential,  $e$  is the magnitude of the electron charge,  $\mu_B = -e\hbar/2m_e c$  is the electron magnetic moment,  $\phi$  is the electrostatic potential, and  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  are the Pauli spin matrices, represented by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

By introducing the decomposition of the spinors according to  $\Psi_{(\alpha)} = \sqrt{n_{(\alpha)}} \exp(iS_{(\alpha)}/\hbar)\varphi_{(\alpha)}$ , we may derive a set of  $N$  coupled fluid equations [37] for the densities  $n_{(\alpha)}$ , the velocities  $\mathbf{v}_{(\alpha)} = (1/m_e) (\nabla S_{(\alpha)} - i\hbar\varphi_{(\alpha)}^\dagger \nabla\varphi) +$

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$(e/m_e c)\mathbf{A}$ , and the spin vectors  $\mathbf{s}_{(\alpha)} = (\hbar/2)\varphi_{(\alpha)}^\dagger \boldsymbol{\sigma} \varphi_{(\alpha)}$  (where  $\varphi_{(\alpha)}$  is the 2-spinor through which the spin  $\frac{1}{2}$  properties are mediated).

Next we define the total particle density for the species with charge  $q$  according to  $n_e = \sum_{(\alpha)=1}^N p_\alpha n_{(\alpha)}$ , where  $p_\alpha$  is the probability related to the wave function  $\Psi_{(\alpha)}$ . Using the ensemble average  $\langle f \rangle = \sum_\alpha p_\alpha (n_{(\alpha)}/n) f$  for any tensorial quantity  $f$ , we define the total electron fluid velocity for charges  $\mathbf{V}_e = \langle \mathbf{v}_{(\alpha)} \rangle$  and the total electron spin density  $\mathbf{S} = \langle \mathbf{s}_{(\alpha)} \rangle$ . From these definitions we can define the microscopic velocity in the electron fluid rest frame according to  $\mathbf{w}_{(\alpha)} = \mathbf{v}_{(\alpha)} - \mathbf{V}_e$ , satisfying  $\langle \mathbf{w}_{(\alpha)} \rangle = 0$ , and the microscopic spin density  $\mathbf{S}_{(\alpha)} = \mathbf{s}_{(\alpha)} - \mathbf{S}$ , such that  $\langle \mathbf{S}_{(\alpha)} \rangle = 0$ .

We then obtain the conservation equations

$$\partial_t n_e + \nabla \cdot (n_e \mathbf{V}_e) = 0, \quad (3)$$

$$m n_e (\partial_t + \mathbf{V}_e \cdot \nabla) \mathbf{V}_e = -e n_e (\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) - \nabla \cdot \mathbf{\Pi}_e - \nabla P_e + \mathbf{C}_{ei} + \mathbf{F}_Q \quad (4)$$

and

$$n_e (\partial_t + \mathbf{V}_e \cdot \nabla) \mathbf{S} = \frac{2\mu_B n_e}{\hbar} \mathbf{B} \times \mathbf{S} - \nabla \cdot \mathbf{K}_e + \Omega_S \quad (5)$$

respectively. Here we have added the electron-ion collisions  $\mathbf{C}_{ei}$ , denoted the total quantum force density by

$$\begin{aligned} \mathbf{F}_Q &= -n_e \langle \nabla Q_{(\alpha)} \rangle - \frac{2\mu_B n_e}{\hbar} (\nabla \otimes \mathbf{B}) \cdot \mathbf{S} \\ &\quad - \frac{1}{m_e} \nabla \cdot (n_e \boldsymbol{\Sigma}) - \frac{1}{m_e} \nabla \cdot (n_e \tilde{\boldsymbol{\Sigma}}) \\ &\quad - \frac{2}{m_e} \nabla \cdot \left\{ n_e \text{Sym} \left[ (\nabla S_a) \otimes \langle (\nabla S_{(\alpha)}^a) \rangle \right] \right\}, \quad (6) \end{aligned}$$

where Sym denotes the symmetric part of the tensor, and defined the nonlinear spin fluid contribution by

$$\begin{aligned} \Omega_S &= \frac{1}{m_e} \mathbf{S} \times [\partial_a (n_e \partial^a \mathbf{S})] + \frac{1}{m_e} \mathbf{S} \times [\partial_a (n_e \langle \partial^a \mathbf{S}_{(\alpha)} \rangle)] \\ &\quad + \frac{n_e}{m_e} \left\langle \frac{\mathbf{S}_{(\alpha)}}{n_{(\alpha)}} \times [\partial_a (n_{(\alpha)})] \partial^a \mathbf{S} \right\rangle \\ &\quad + \frac{n_e}{m_e} \left\langle \frac{\mathbf{S}_{(\alpha)}}{n_{(\alpha)}} \times [\partial_a (n_{(\alpha)} \partial^a \mathbf{S}_{(\alpha)})] \right\rangle, \quad (7) \end{aligned}$$

where  $\mathbf{\Pi}_e = m_e n_e [\langle \mathbf{w}_{(\alpha)} \otimes \mathbf{w}_{(\alpha)} \rangle - \mathbf{I} \langle w_{(\alpha)}^2 \rangle / 3]$  is the trace-free anisotropic pressure tensor ( $\mathbf{I}$  is the unit tensor),  $P_e = m_e n_e \langle w_{(\alpha)}^2 \rangle$  is the isotropic scalar pressure,  $\boldsymbol{\Sigma} = (\nabla S_a) \otimes (\nabla S^a)$  is the nonlinear spin correction to the classical momentum equation,  $\tilde{\boldsymbol{\Sigma}} = \langle (\nabla S_{(\alpha)a}) \otimes (\nabla S_{(\alpha)}^a) \rangle$  is a pressure like spin term (which may be decomposed into trace-free part and trace),  $\mathbf{K} = n_e \langle \mathbf{w}_{(\alpha)} \otimes \mathbf{S}_{(\alpha)} \rangle$  is the thermal-spin coupling, and  $[(\nabla \otimes \mathbf{B}) \cdot \mathbf{S}]^a = (\partial^a B_b) S^b$ . Here the latin indices  $a, b, \dots = 1, 2, 3$  denote the vector components. We note that the momentum conservation equation (4) and the spin evolution equation (5)

still contains the explicit sum over the  $N$  states, and (as in classical fluid theory) it is necessary to impose further statistical relations in order to close the system [38]. The preceding analysis applies equally well to electrons as holes or similar condensations. We will now include the ion species, which, due to the smaller charge-to-mass ratio, are described by the classical equations of motion.

The coupling between the quantum plasma species is mediated by the electromagnetic field. By definition we let  $\mathbf{B}_{\text{tot}}$  include spin sources, i.e.  $\mathbf{B}_{\text{tot}} \equiv \mathbf{B} + \mathbf{B}_{\text{sp}}$ , such that Ampere's law in terms of  $\mathbf{B}_{\text{tot}}$  reads  $\nabla \times \mathbf{B}_{\text{tot}} = \mu_0 (\mathbf{j} + \mathbf{j}_{\text{sp}}) + c^{-2} \partial_t \mathbf{E}$ , including the magnetization spin current  $\mathbf{j}_{\text{sp}} = \nabla \times (2n\mu_B \mathbf{S}/\hbar)$  [39]. We obtain consistency with the momentum conservation equation (4) by adding a term proportional to  $\mathbf{V} \times \mathbf{B}_{\text{sp}}$  to the Lorentz force, and subtracting it from the quantum force. The above alterations are only reshuffling of terms. However, a difference do appear when closing the system using Faraday's law. By letting  $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}_{\text{tot}}$ , using  $\mathbf{B}_{\text{tot}}$  instead of  $\mathbf{B}$ , we indeed obtain a difference compared to the classical Maxwell's equations. It is the full electromagnetic fields, including spin sources, that should be used in Faraday's law. Thus, Faraday's law as presented here is therefore the correct one to use. This form also gives a Hermitian susceptibility tensor (see below), something which is not obtained otherwise.

To demonstrate the usefulness of the spin fluid equations, we investigate linear wave propagation in a magnetized plasma. For comparison we first neglect all quantum effects. Linearizing and Fourier analyzing the equations of motion, and substituting the velocities into Maxwell's equations, we obtain  $\boldsymbol{\varepsilon} \cdot \mathbf{E} = 0$ , where

$$\boldsymbol{\varepsilon} = \mathbf{I} + \begin{pmatrix} \chi_{\perp\perp} & \chi_{\perp\text{T}} & \chi_{\perp z} \\ -\chi_{\perp\text{T}} & \chi_{\text{T}\text{T}} & \chi_{\text{T}z} \\ \chi_{\perp z} & -\chi_{\text{T}z} & \chi_{zz} \end{pmatrix} + \begin{pmatrix} \frac{k_z^2 c^2}{\omega^2} & 0 & \frac{k_z k_\perp c^2}{\omega^2} \\ 0 & \frac{k^2 c^2}{\omega^2} & 0 \\ \frac{k_z k_\perp c^2}{\omega^2} & 0 & \frac{k_\perp^2 c^2}{\omega^2} \end{pmatrix} \quad (8)$$

and the standard susceptibility components are

$$\begin{aligned} \chi_{\perp\perp} &= - \sum_{\text{p.s.}} \frac{\omega_p^2 (\omega^2 - k_z^2 v_t^2)}{\omega_w^4}, \\ \chi_{\perp\text{T}} &= -i \sum_{\text{p.s.}} \frac{\omega_p^2 \omega_c (\omega^2 - k_z^2 v_t^2)}{\omega \omega_w^4}, \\ \chi_{\perp z} &= - \sum_{\text{p.s.}} \frac{\omega_p^2 k_\perp k_z v_t^2}{\omega_w^4}, \\ \chi_{\text{T}\text{T}} &= - \sum_{\text{p.s.}} \frac{\omega_p^2 (\omega^2 - k^2 v_t^2)}{\omega_w^4}, \\ \chi_{\text{T}z} &= i \sum_{\text{p.s.}} \frac{\omega_p^2 \omega_c k_\perp k_z v_t^2}{\omega \omega_w^4}, \\ \chi_{zz} &= - \sum_{\text{p.s.}} \frac{\omega_p^2 (\omega^2 - \omega_c^2 - k_\perp^2 v_t^2)}{\omega_w^4}. \end{aligned} \quad (9)$$

Here, the sums are over the particle species,  $k = (k_z^2 + k_\perp^2)^{1/2}$ ,  $\mathbf{k}_\perp$  is the perpendicular (to  $\hat{\mathbf{z}}$ ) part of the wave-vector, the  $\top$ -direction is parallel to  $\hat{\mathbf{z}} \times \mathbf{k}_\perp$ ,  $\omega_p$  is the plasma frequency ( $\omega_{pe}$  for the electrons and  $\omega_{pi}$  for the ions),  $\omega_c = qB_0/m$  is the cyclotron frequency,  $q$  and  $m$  are the particle charge and mass,  $v_t^2$  is the square of the thermal velocity times the ratio of specific heats,  $c$  is the speed of light in vacuum, and  $\omega_w^4 = \omega^2(\omega^2 - k^2v_t^2) - \omega_c^2(\omega^2 - k_z^2v_t^2)$ . For notational convenience, the subscripts denoting the various particle species have been left out.

Next we determine the equilibrium spin configuration. For many plasmas paramagnetic theory applies. Thus, in an external magnetic field  $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ , the zero order magnetization  $\mathbf{M}_{S0}$  due to the spin can be written [40]

$$\mathbf{M}_{S0} = n_0\mu_B\eta\left(\frac{\mu_B B_0}{KT}\right)\hat{\mathbf{z}} \quad (10)$$

where  $K$  is Boltzmann's constant,  $T$  is the temperature, and we have introduced the Langevin function  $\eta(x) = [\coth(x) - x^{-1}]$ . Here we have assumed that the spin contribution to the total magnetic field is small, otherwise  $B_0 \rightarrow B_0 + B_{S0}$ , where  $B_{S0} = \mu_0\mu_B n_0$ , in Eq. (10). In general, the spin-magnetization  $\mathbf{M}_S$  and the spin-vector  $\mathbf{S}$  are related by

$$\mathbf{S} = \frac{\hbar\mathbf{M}_S}{2n\mu_B} \quad (11)$$

and thus the zero order spin vector becomes  $\mathbf{S}_0 = (\hbar/2)\eta(\mu_B B_0/KT)\hat{\mathbf{z}}$ . From (11) we obtain the spin-current contribution  $\mathbf{j}_s = \nabla \times (4n_e e\mathbf{S}/m_e)$ .

Generalizing (8) to include all terms from quantum effects give extremely complicated expressions. However, for most plasmas, the parameter  $\mu_B B_0/KT$  is very small, the spins are essentially randomly orientes, and the spin quantum effects are negligible. On the other hand, for low-frequency wave motion in a highly magnetized (or low temperature) plasma, the spin effects can be appreciable. In this case the dominant contribution to the spin effects come from the component of the spin force parallel to the magnetic field,  $F_{Qz} = -(2\mu_B n_0 S_0/\hbar)\partial_z B_1$ , where  $B_1$  denotes the magnetic field perturbation, together with the part of the spin current in the  $\top$ -direction (from the part proportional to  $\nabla n \times \mathbf{S}_0$ ), and we can drop all other components as well as quantum terms that are proportional to  $\hbar^2$ , provided  $eB_0 \gg \hbar k^2$ . Keeping the above terms, including only the lowest order contributions in  $\omega/\omega_{ci}$ , the susceptibility tensor is modified to

$$\chi = \begin{pmatrix} \chi_{\perp\perp} & \chi_{\perp\top} & \chi_{\perp z} \\ -\chi_{\perp\top} & \chi_{\top\top} & \chi_{\top z} + \chi_{sp} \\ \chi_{\perp z} & -(\chi_{\top z} + \chi_{sp}) & \chi_{zz} \end{pmatrix} \quad (12)$$

where the spin-contribution is

$$\chi_{sp} = i\eta\left(\frac{\mu_B B_0}{KT}\right)\frac{\omega_{pe}^2 \hbar k_\perp k_z}{\omega(\omega^2 - k_z^2 v_{te}^2)m_e} \quad (13)$$

As an example we consider the fast and slow magnetosonic mode, which is now described by the dispersion relation  $\varepsilon_{\top\top}\varepsilon_{zz} + (\varepsilon_{\top z} + \varepsilon_{sp})^2 = 0$ . In the standard regime  $\omega_{ci}^2/\omega_{pi}^2 \ll 1$ ,  $\omega \ll k_z v_{te}$ , the dispersion relation becomes

$$(\omega^2 - k^2 c_A^2)(\omega^2 - k_z^2 c_s^2) = \omega^2 k_\perp^2 c_s^2 \left[ 1 + \eta\left(\frac{\mu_B B_0}{KT}\right) \frac{\hbar\omega_{ce}}{m_e v_{te}^2} \right] \quad (14)$$

where the ion-acoustic velocity is  $c_s = (m_e/m_i)^{1/2}v_{te}$ , the Alfvén velocity is  $c_A = (B_0^2/\mu_0 n_0 m_i)^{1/2}$  and, for simplicity, we have assumed that the ion-temperature is smaller than the electron temperature and included only electron thermal effects. Noting that  $\mu_B B_0/KT \equiv \hbar\omega_{ce}/m_e v_{te}^2$ , obviously the spin-effects are important if

$$\frac{\hbar\omega_{ce}}{m_e v_{te}^2} \gtrsim 1. \quad (15)$$

Thus, for laboratory magnetic fields, where at most  $B_0 \sim 10 - 20$  T, we need low temperature plasmas for spin effects to influence the fast and slow magnetosonic modes. However, in the vicinity of pulsars and magnetars [33], we have  $B_0 \geq 10^8$  T. For such systems, spin plasma effects can be important even in a high temperature plasma. The spin effect on the fast and slow mode is illustrated in Fig. 1. Furthermore, we point out that for modes with even lower phase velocities (which exist in for example dusty plasmas [16]), the relative importance of the spin susceptibility term is enhanced, and spin effects can be significant also under laboratory conditions.

In conclusion, we have derived the multi-fluid equations for spin  $\frac{1}{2}$  quantum plasmas, starting from the Pauli equation. In order to demonstrate the usefulness of our equations, we have analyzed the linear modes, and demonstrated that the low-frequency modes are significantly altered by the spin effects provided the condition (15) is fulfilled. In many classical plasmas spin effects can be neglected due to the random orientations of the spin vector. We stress here, however, that our results show that the spin multi-fluid equations can have important applications to such different mediums as low-temperature solid state plasmas, as well as to the accretion discs surrounding pulsars and magnetars. Furthermore, we emphasize that the spin-contributions are typically more important than the usual quantum plasma corrections [18], specifically when the inequality  $eB_0 \gg \hbar k^2$  is fulfilled.

The linearized results presented in this Letter will most likely find its experimental application in dusty plasmas, where the low phase velocity will make the relative importance of the spin contribution (13) particularly significant, enabling probing of the collective spin dynamics.

Finally, we suggest that the full nonlinear system (3)–(7) will show interesting behavior close to the electrons cyclotron frequency, when the spin vector evolution becomes resonant. Moreover, the importance of the pressure like spin terms for, e.g., astrophysical plasmas is a

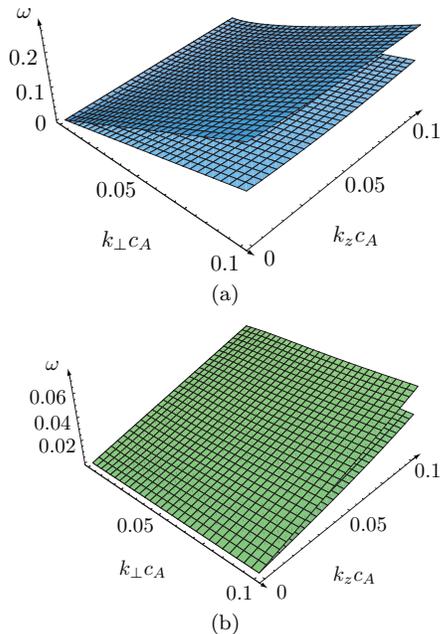


FIG. 1: The two roots of the dispersion relation (14) plotted in (a) (fast mode) and (b) (slow mode). In case (a) the lower surface is without spin and the upper surface is with spin, while in (b) the lower surface is with spin and the upper surface is without spin. We note that the contribution from the spin term can be significant, in particular for large values of the wave numbers  $k_z$  and  $k_\perp$ . Here we have used  $c_s^2/c_A^2 = 0.5$ ,  $\eta\hbar\omega_{ce}/m_e v_{te}^2 = 8$ , and normalized the frequency by the ion cyclotron frequency  $\omega_{ci}$  and the wave numbers by  $\omega_{ci}/c_A$ .

further field of investigation.

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 [38] Using  $L \gg \lambda_F$  where  $L$  is the typical fluid length scale and  $\lambda_F$  is the Fermi wavelength, we obtain [18]  $\langle \nabla Q_{(\alpha)} \rangle \approx \nabla [-(\hbar^2/2m_e n_e^{1/2}) \nabla^2 n_e^{1/2}]$ . In the Schrödinger treatment of quantum plasmas, this term is the only quantum contribution to the equations of motion (see [1, 2, 3, 4, 5, 6, 18]).  
 [39] We note that this reshuffling of terms stems from the use of a non-relativistic particle theory. Thus, it is expected that the need for such rearrangements will disappear in a fully relativistic theory.  
 [40] The full magnetization is known to include both Pauli spin paramagnetism and Landau orbit diamagnetism. However, in Eq. (11) only the Pauli contribution should be included