Fractions: the new frontier for theories of numerical development

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Recent research on fractions has broadened and deepened theories of numerical development. Learning about fractions requires children to recognize that many properties of whole numbers are not true of numbers in general and also to recognize that the one property that unites all real numbers is that they possess magnitudes that can be ordered on number lines. The difficulty of attaining this understanding makes the acquisition of knowledge about fractions an important issue educationally, as well as theoretically. This article examines the neural underpinnings of fraction understanding, developmental and individual differences in that understanding, and interventions that improve the understanding. Accurate representation of fraction magnitudes emerges as crucial both to conceptual understanding of fractions and to fraction arithmetic.

Introduction

Fractions play a central role in mathematics learning. They are theoretically important because they require a deeper understanding of numbers than that typically gained from experience with whole numbers [1]. Fractions also are educationally important because of their inherent role in more advanced mathematics*, the strong predictive relation between earlier knowledge of them and later mathematics achievement [2,3], and the difficulty many children and adults have in learning about them [4–6]. Despite the centrality of fraction knowledge, however, most current theories of numerical development are actually theories of whole number development and do not integrate fractions and whole numbers within a single framework.

Fortunately, the same types of behavioral and neural methods that have proved useful for understanding whole number representations and processes are also proving useful for understanding representations and processes involving fractions (e.g., [1,7–9]). Recent applications of these methods have greatly expanded understanding of developmental and individual differences in fraction knowledge, as well as its neural basis (Box 1). The main conclusion that has emerged from this research is that, just as understanding of magnitudes is central to understanding whole numbers [10,11], it is equally central to understanding fractions.

In the following sections, we first examine the role of fractions in current theories of numerical development and why fractions are difficult for many learners. We then review current understanding of the neural bases of fractions knowledge, developmental and individual differences in the acquisition process, and interventions for improving fractions knowledge.

The role of fractions in theories of numerical development

Most current theories of numerical development are actually theories of whole number development. When fractions are considered at all in these theories, the purpose is to contrast children's quick, effortless, and consistently successful acquisition of whole number knowledge with their slow, effortful, and incomplete acquisition of fractions knowledge [12–14].

As these theories note, there are large and important differences between acquisition of whole number and fraction knowledge. However, the failure to integrate both whole numbers and fractions within a single framework unnecessarily deprives theories of numerical development of some of their potentially most interesting content. Learning fractions requires a reorganization of numerical knowledge, one that allows a deeper understanding of numbers than is ordinarily gained through experience with whole numbers. Stated another way, fractions are an inherently important part of numerical development and theories of numerical development that exclude them are unnecessarily truncated.

The recently proposed integrated theory of numerical development [1] argues that the reason why learning fractions requires a re-organization of numerical knowledge is that children (and adults) who have not learned fractions generally assume that properties of whole numbers are properties of all numbers [4–6]. Whole numbers have unique successors, can be represented by a single symbol, are countable, never decrease with multiplication, never increase with division, and so on. None of these properties is true of fractions, however, and therefore none is true of all real numbers.

As also noted in the integrated theory of numerical development, acquiring understanding of fractions requires learning that fractions, like whole numbers, represent magnitudes that can be located on number lines [1].

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* National Mathematics Advisory Panel (2008) *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*, US Department of Education.

Box 1. The neural bases of fraction knowledge

Recent research has shown that brain regions around the intraparietal sulcus (IPS) play the same essential role in representing fractions as in representing whole numbers. The role of the IPS is evident regardless of whether the task involves automatic processing of magnitudes, magnitude comparison, or arithmetic. However, the most active areas within the IPS vary with the task.

At least in adults, processing of fraction magnitudes in the IPS is automatic: it occurs even when there is no specific task. This has been demonstrated by presenting a fraction until participants habituate to it and then presenting a novel fraction that varies in its distance from the original [54]. An adaptation effect for repeatedly viewing the original fraction is observed bilaterally in the anterior IPS. The amount of recovery of activation in this area varies with numerical distance between the habituated and novel fractions. The specific areas of activation are highly similar for fractions and whole numbers.

IPS activation was also evident on a fraction magnitude comparison task, though only on one side of the brain [7]. In particular, the right IPS was sensitive to numerical distance between the fractions, independent of the distances between numerators and between denominators of the fractions being compared.

Brain activity while solving arithmetic problems also shows strong commonalities between whole numbers and fractions [55]. An independent components analysis of brain activity during addition and subtraction of fractions, as measured by fMRI, revealed task-related components with activation in bilateral inferior parietal, left perisylvian, and ventral occipitotemporal areas, a pattern closely similar to that observed with whole number arithmetic [56]. These results suggest an underlying commonality in the neural basis of whole number and fraction knowledge.

This is actually the only property that unites real numbers. Learning to accurately represent and arithmetically combine the magnitudes of all types of real numbers – whole numbers and fractions; positives and negatives; common fractions, decimals, and percentages – is thus inherently central to numerical development [1,15].

The discussion above suggests that fraction knowledge should play a central role in mathematics learning more broadly. Recent evidence suggests that this is the case. High school students' knowledge of fractions correlates strongly (rs $>$ 0.80) with their overall mathematics achievement in both the UK and the USA [2]. Perhaps even more striking, in both the USA and the UK, fifth graders' fraction knowledge predicts their mastery of algebra and overall mathematics achievement in high school, even after controlling for IQ, reading achievement, working memory, family income and education, and whole number knowledge [2].

Why are fractions difficult?

Failure to learn fractions is a serious educational problem, one that affects a great many people. Although children in the USA receive substantial fraction instruction beginning in 3rd or 4th grade [16], a recent National Assessment of Educational Progress found that 50% of 8th graders could not correctly order the magnitudes of three fractions ($^2/_7$, $^1/_{12}$, and $^5/_9$) [17]. The problem is not limited to numbers written in common fractions notation; when asked whether .274 or .83 is larger, most 5th and 6th graders choose .274 [18]. The difficulty extends to adolescents and adults; fewer than 30% of US 11th graders translate .029 into the correct fraction [19] and community college students show similar weaknesses [9,20]. Also attesting to the problem, a representative sample of 1,000 US algebra teachers ranked lack of fraction understanding as one of the two largest problems hindering their students' algebra learning (trailing only 'word problems', many of which involve fractions)[†]. Nor is the difficulty limited to the USA; mathematics educators in other nations, including high achieving ones such as Japan, China, and Taiwan, have noted similar problems [21–23].

Students' difficulty in learning fractions has many sources. One problem was described earlier – erroneous assumptions that properties of whole numbers are properties of all numbers. Another source of difficulty is confusable relations among fraction arithmetic procedures. When fraction addition and subtraction problems have the same denominator, the denominator is maintained in the answer, but that is not the case for fraction multiplication and division. Confusion about when (and why) common denominators are maintained leads to errors such as '$^2/_5 \times {}^3/_5 = {}^6/_5$' [1]. A further source of difficulty is that children often view fractions exclusively in terms of part/whole relations, which are often emphasized in instruction [15]. The type of confusion that arises from emphasizing this single interpretation is seen in one student's explanation of why he thought $^4/_3$ had no meaning: 'You cannot have four parts of an object that is divided into three parts' [24].

Such findings have led recent commissions charged with improving mathematics education to emphasize the importance of improving students' fractions knowledge. For example, the National Mathematics Advisory Panel concluded that the most important foundational skill not presently developed appears to be proficiency with fractions...The teaching of fractions must be acknowledged as critically important and improved before an increase in student achievement in algebra can be expected'[‡].

Development of fraction knowledge

Two distinctions are crucial for understanding developmental and individual differences in fraction knowledge. One is between conceptual and procedural knowledge. Conceptual knowledge includes understanding of the properties of fractions: their magnitudes, principles, and notations. By contrast, procedural knowledge involves fluency with the four fraction arithmetic operations.

The other key distinction involves non-symbolic and symbolic knowledge. Non-symbolic knowledge involves competence with concrete stimuli (e.g., which set has a higher proportion of blue dots?); symbolic knowledge involves competence with conventional representations (which is greater, $^2/_3$ or $^4/_9$?). We organize our discussion of developmental and individual differences around these two distinctions.

Conceptual development

Non-symbolic knowledge Even infants possess a basic, non-symbolic understanding of fractions. Six-month-olds

[†] National Mathematics Advisory Panel (2008) *Report of the Subcommittee on the National Survey of Algebra I Teachers*, US Department of Education.

[‡] National Mathematics Advisory Panel (2008) *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*, US Department of Education, p. 18.

accurately discriminate between two ratios that differ by at least a factor of 2 (e.g., they dishabituate when, after repeated presentations of 2:1 ratios of yellow to blue dots, the ratio switches to 4:1) [25]. This matches their discrimination abilities with whole numbers [26]. Six-month-olds' expectations regarding continuous variables also reflect a sense of ratios; they look longer when an object that is 70% unsupported remains stationary than when one that is 15% unsupported does [27].

By age 3 years, children can draw analogies between pairs of non-symbolic fractions (e.g., $^{1}/_{2}$ of a square: $^{1}/_{2}$ of a circle:: $^{3}/_{4}$ of a square: $^{3}/_{4}$ of a circle) [28,29]. Somewhat older children use $^{1}/_{2}$ as a reference point when matching non-verbal representations of fractions. When asked which of two partially filled rectangles match a third, 6- and 7-year-olds are more accurate when the two options are on opposite sides of $^{1}/_{2}$. For example, when matching $^{3}/_{8}$, participants are more accurate when the options are sets equivalent to $^{3}/_{8}$ and $^{5}/_{8}$ than ones equivalent to $^{3}/_{8}$ and $^{1}/_{8}$ [30,31].

Although young children can accurately match fractions presented as continuous quantities, they often err when counting yields an alternative answer [31–33]. For example, kindergartners usually choose the correct (proportional) match to the target in the continuous condition shown on the left side of Figure 1. However, the same children usually choose the incorrect (numerical) match on the discrete task shown on the right side of Figure 1, because they count and choose the option that has the same number of shaded segments as the target, rather than the matching proportion [34].

Symbolic knowledge Symbolic fraction knowledge develops later than non-symbolic knowledge, but the fraction $^{1}/_{2}$ again is prominent in early understanding. Four-year-olds respond accurately when asked to give a doll half of a cookie or pizza, but do not do so for $^{1}/_{4}$ and $^{3}/_{4}$ until age seven [35,36].

Kindergartners can also give the same fraction of a set of objects to multiple people, but they do not yet understand the general inverse relation between the number of sharers and the number of objects received [37–39]. Not until age seven do children predict that sharing a set of objects with more people reduces the number of objects that each person will receive, regardless of the number of people or the size of the set being apportioned [37,39].

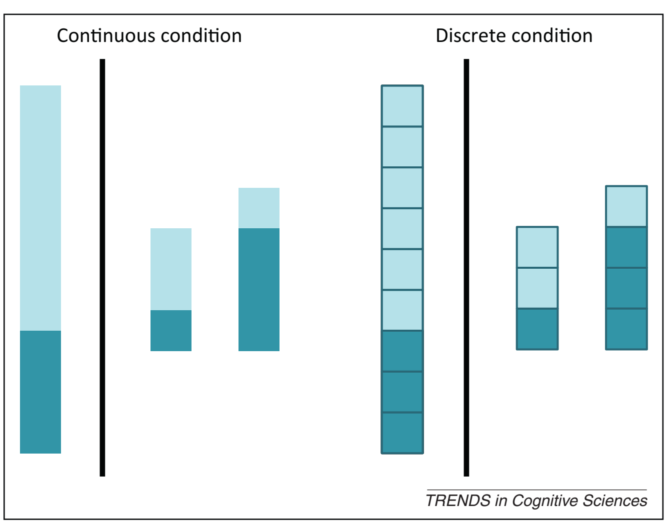

Figure 1. Examples of continuous and discrete fraction matching tasks, similar to those used in [32].

Children are usually introduced to written fraction notation as a general representational system in 3rd or 4th grade. Connecting written fractions with the magnitudes they represent poses large and enduring challenges for many children. Understanding does improve with experience, but the improvements are slow and limited [1,40–42]. When asked to order $^{1}/_{7}$, $^{5}/_{6}$, 1, and $^{4}/_{3}$, only 33% of Greek 5th graders and 58% of 9th graders answered correctly [43]. A study of US students found an increase in fraction magnitude comparison accuracy from 68% to 79% correct between 6th and 8th grade (chance was 50%) [1]; another study found that US community college students correctly answered only 70% of similar fraction comparisons problems [9].

As noted earlier, understanding fractions requires learning that many principles that apply to whole numbers do not apply to fractions. For example, whole numbers have unique successors, but fractions do not; instead, they are infinitely divisible. Children tend to come to this realization slowly, if at all. One study found that third graders did not understand infinite divisibility, but sixth graders did [42]; two other studies found that many high school students do not understand it [5,6].

Procedural development

Non-symbolic arithmetic As with non-symbolic understanding of individual fractions, non-symbolic fraction arithmetic competence begins relatively early. For example, 4-year-olds accurately predict solutions to problems involving addition and subtraction of simple non-verbal fraction representations (e.g., $^{1}/_{2}$ of a circle + $^{1}/_{4}$ of a circle) [44].

Symbolic arithmetic As with representations of individual fractions, this early competence with non-symbolic fraction arithmetic contrasts sharply with the enduring difficulty that children have with symbolic arithmetic. For example, US 4th and 6th graders correctly answered 25% and 51% of fraction addition and multiplication problems [45], 6th and 8th graders correctly answered 32% and 60% of addition, subtraction, multiplication, and division problems [1]; and another sample of 6th and 8th graders correctly answered 41% and 57% of a similar set [41].

Children make two main types of errors on symbolic fraction arithmetic problems. Independent whole number errors [1,4] involve performing the arithmetic operation independently on numerators and denominators (e.g., $^{1}/_{2} + ^{1}/_{3} = ^{2}/_{5}$). Wrong fraction operation errors involve using components that are correct for another fraction arithmetic operation on an operation where they are incorrect. A common example involves maintaining common denominators in multiplication problems, as is appropriate in addition and subtraction problems (e.g., $^{1}/_{3} \times ^{2}/_{3} = ^{2}/_{3}$). Both types of errors are common among community college students, as well as children [20].

These errors reflect a lack of understanding of the conceptual basis of fraction arithmetic procedures. Claiming that $^{1}/_{2} + ^{1}/_{3} = ^{2}/_{5}$ indicates a lack of understanding that adding positive numbers must produce answers greater than either addend (or lack of understanding that $^{2}/_{5} < ^{1}/_{2}$). Claiming that $^{1}/_{3} \times ^{2}/_{3} = ^{2}/_{3}$ indicates a lack of understanding that multiplying by a number below 1 must result in an answer smaller than the number being multiplied. This lack of conceptual understanding is also apparent in the variability of fraction arithmetic strategies. Children often generate both correct answers and errors on virtually identical fraction arithmetic problems within a single session [1,41].

These findings suggest that many children's fraction arithmetic knowledge includes a mix of correct procedures, components of procedures detached from the relevant arithmetic operation, and whole number arithmetic procedures. These children's strategy choices seem to be constrained little – if at all – by conceptual knowledge, which leads to the observed mixture of correct strategies and errors, even on highly similar problems.

Individual differences

Knowledge of fractions varies widely among children of the same age. Consistent individual differences have been found across different types of fraction knowledge, across fractions and other types of mathematical knowledge, across fraction knowledge and domain general processes, and in the same children over time.

Relations among aspects of fraction knowledge

Different aspects of conceptual knowledge of fractions are positively related. Knowing that numbers are infinitely divisible correlates positively with accurately comparing fraction magnitudes, knowing that fractions can be viewed as cases of division ($N \div M$), and knowing that there are numbers between 0 and 1 [42]. Alternative measures of fraction magnitude knowledge, in particular magnitude comparison and number line estimation, also are closely related [1,41]. Similarly, use of strategies that require conceptual knowledge, such as transforming improper fractions into mixed numbers (e.g., $^{17}/_{4} = 4\ ^{1}/_{4}$), predicts number line estimation accuracy, another measure of conceptual knowledge [1]. Different measures of procedural fraction knowledge, in particular accuracy on the four arithmetic operations, also correlate positively [40,46,47].

Conceptual and procedural knowledge of fractions also are related [40,45–48]. For example, children's accuracy in identifying pictorial equivalents of symbolic fractions and in comparing and estimating fraction magnitudes are related to their fraction arithmetic accuracy [1,47]. Note that fraction arithmetic proficiency could, in principle, have been unrelated to conceptual understanding; the algorithms could have simply been memorized, without conceptual understanding. This does not seem to be how children learn fraction arithmetic, however.

Linguistic differences in fraction names also are related to other types of fraction knowledge. In East Asian languages, fractions are named by stating the denominator first and the numerator second. For example, the phrase corresponding to 'one third' would translate as 'of three parts, one'. The East Asian phrase appears to convey the part-whole relation, and the linkage between numerator and denominator, more transparently than in English. Consistent with this view, presenting an English version of the East Asian phrasing to US second graders enables them to perform as well as Korean peers in shading squares within a matrix to match a written fraction, despite the fact that the US children perform much worse when presented the usual English phrasing [49,50].

Relations of fractions and other mathematics knowledge

Individual differences in fraction knowledge are related to individual differences in both earlier and later acquired aspects of mathematics. Both conceptual and procedural knowledge of fractions are related to previously acquired whole number arithmetic proficiency [40,46,47]. Fraction knowledge is also related to subsequently acquired mathematics, in particular algebra and the range of topics covered on high school achievement tests [1,2,51]. In two large samples of high school students, one from the USA and one from the UK, fraction competence was highly associated with both algebra knowledge (both rs > 0.60) and mathematical achievement scores (both rs > 0.80) [2]. In two studies of middle school children [1,41], fraction magnitude knowledge also was highly related to overall mathematics achievement test scores (Figure 2); the relation was present even after knowledge of fraction arithmetic was statistically controlled. Two other studies [3,51] showed that both conceptual and procedural knowledge of fractions accounted for unique variance in mathematics achievement, even after the other was statistically controlled, and that algebra proficiency is more closely related to conceptual knowledge of fractions than to conceptual knowledge of whole numbers.

Relations to domain general processes

Both conceptual and procedural fraction knowledge are also related to domain-general cognitive abilities. Fractions include two pieces of information (numerator and denominator); therefore, representing them seems likely to demand more working memory resources than representing whole numbers [52]. Fraction knowledge also requires inhibitory control, so that the numerator and denominator are not treated as independent whole numbers [4]. Not surprisingly, fraction knowledge is associated with working memory, attention, and IQ [2,40,41,46,53].

Longitudinal stability

Individual differences in fraction knowledge are quite stable over both short and long time periods; children who start ahead stay ahead, and children who start behind stay behind [3,40]. The stability is present from elementary school to high school, even after controlling for domain-general abilities (IQ, reading achievement, and working memory) and whole number arithmetic knowledge [2].

Studies that have examined individual differences over shorter periods have identified some of the factors that contribute to this stability of individual differences. Conceptual knowledge of fractions, attentive behavior, working memory, and whole number knowledge predict gains in procedural fraction knowledge from one year to the next [40]. However, when all of these variables are simultaneously controlled, only fraction conceptual knowledge and

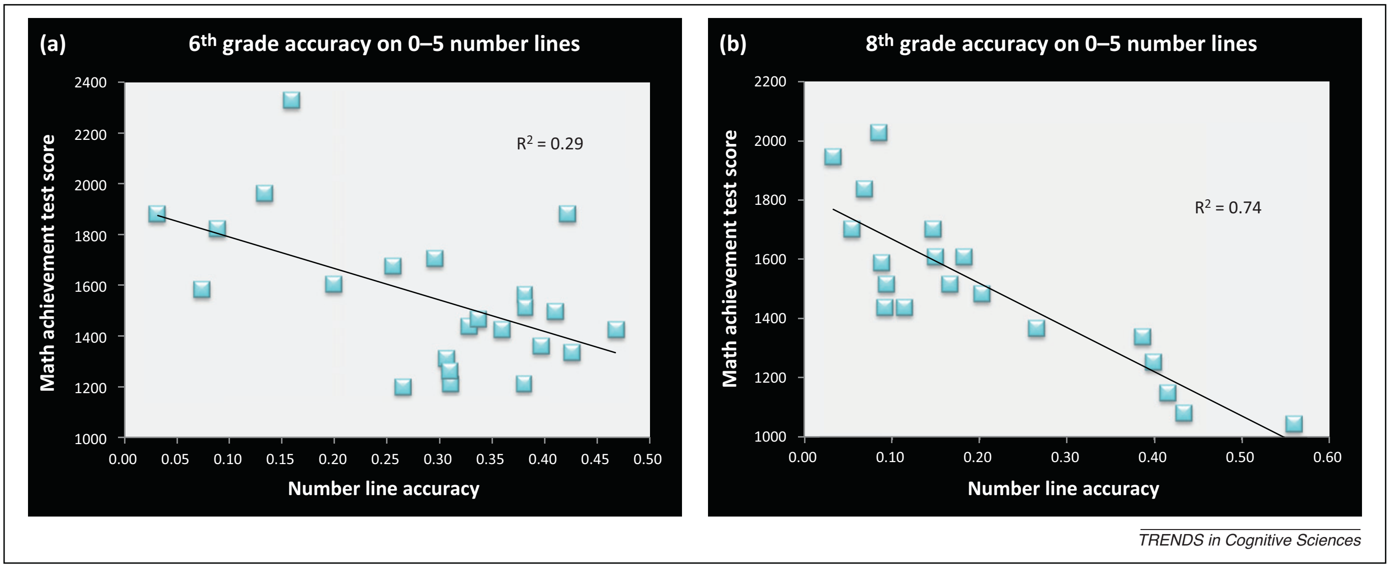

Figure 2. Relation between number line estimation accuracy and mathematics achievement test scores of **(a)** 6th and **(b)** 8th graders. Adapted from [1].

attentive behavior predict gains in procedural knowledge of fractions.

Acquisition of conceptual and procedural knowledge of fractions is mutually reinforcing. When provided with relevant instruction, children with higher initial conceptual knowledge of fractions show greater acquisition of fraction arithmetic procedures [45]; symmetrically, children with higher initial knowledge of the procedures show greater subsequent gains in conceptual knowledge [40]. This pattern helps explain consistent positive relations between conceptual and procedural knowledge of fractions [40,45–48]; each helps in acquiring the other. Box 2 provides examples of successful fractions instruction.

Concluding remarks

The exclusive focus of most theories of numerical development on whole numbers has unnecessarily excluded fascinating aspects of the growth of numerical understanding, in particular, the processes through which children come to understand that many properties of whole numbers are not properties of numbers in general, and that the one property that all real numbers share is magnitudes that can be ordered on number lines. Consistent with this analysis, recent research has shown that accurate magnitude representations play the same central role with fractions as with whole numbers, that the two have similar neural bases, and that interventions that

Box 2. How can children's fraction knowledge be improved?

Children's difficulties with fractions have motivated many interventions to improve fraction knowledge. A common feature of the successful interventions is that they help children understand how symbolic fractions map onto the magnitudes they represent [57–61].

One especially impressive demonstration of how instruction can improve knowledge of fraction magnitudes is Moss and Case's rational number curriculum [59]. This instructional approach began by teaching 4th graders broad qualitative distinctions among percentages: 100% 'means everything', 99% 'means almost everything', 50% 'means half', 1% 'means almost nothing', and 0% 'means nothing'. The children were then encouraged to estimate the percentage of a tube that was covered by material placed at various heights around its circumference, and then to compare their estimates to the results of computations involving benchmarks. For example, if a tube held 60 ml of water, they were taught that 50% full would be 30 ml, that 25% full would be 15 ml, and that they could compare estimates of other percentages to those and other benchmarks. Still later, children were taught to represent quantities as decimals and common fractions, as well as percentages; to estimate the position of fractions expressed in all three forms on number lines; and to play board games intended to reinforce what they had learned.

At the end of instruction, the fraction knowledge of 4th graders who had been taught using this curriculum was as good as that of a control group of 8th graders who had been taught with a standard curriculum. It was also as good as the knowledge of a control group of pre-service teachers.

Fuchs *et al.* [62] demonstrated that instruction focusing on fraction magnitudes can have especially large, positive effects on at-risk children (defined as those in the bottom 35% of math achievement test scores). Children in the control group were presented instruction from the widely adopted mathematics textbook that they used throughout the year. The textbook emphasized part-whole representations and a mix of conceptual and procedural instruction. Children in the intervention group received instruction that more heavily emphasized number lines and other representations designed to help children understand fraction magnitudes. Relative to the control group, the intervention group received less emphasis on part-whole understanding and on how to execute fraction arithmetic procedures, but more emphasis on other types of conceptual understanding.

Comparisons of the effects of the two types of instruction showed that the intervention led to greater improvement not only in conceptual understanding of fractions, but also in proficiency with fraction arithmetic. The intervention was most effective in raising the proficiency of children very low in initial achievement. Especially striking, improvements in understanding of fraction magnitudes mediated all of the intervention effects; children whose number line estimation improved most were the ones whose performance improved most on fraction arithmetic and other tasks.

These and other intervention studies indicate that improving understanding of fraction magnitudes should be an important goal of efforts to improve fraction knowledge more generally.

improve fraction magnitude representations also improve other mathematical capabilities.

Fractions are also of central importance for education. Many children and adults show poor quality representations of fraction magnitudes years after the subject was covered in school. Poor fraction knowledge in elementary school predicts low mathematics achievement and algebra knowledge in high school, even after controlling for general cognitive abilities, knowledge of whole number arithmetic, and family education and income. High school algebra teachers recognize this relation; they rank students' fraction knowledge as among the largest impediments to success in their course.

Recent research on developmental and individual differences demonstrates that even infants can accurately represent non-symbolically expressed fractions. The difficulty comes in mapping symbolically expressed fractions onto their magnitudes and in learning symbolic fraction arithmetic procedures. These procedures overlap in complex ways, and many students confuse fraction and whole number procedures and components of different fraction operations. Fortunately, interventions that improve fraction magnitude representations have proved effective in helping children learn fraction arithmetic procedures, as well as in helping them gain conceptual understanding (Box 2).

This research on fraction understanding raises intriguing questions for further research, several of which are listed in Box 3.

Box 3. Questions for future research

- What relations connect whole number and fractions knowledge? Are individual differences in representations of whole number and fraction magnitudes related, do earlier individual differences in whole number representations predict later differences in fraction representations, and do interventions that improve whole number magnitude representations improve fraction representations as well?
- How does knowledge of whole number division influence understanding of fractions? Given the intrinsic relation between whole number division and fractions, would improving whole number division knowledge improve learning of fractions?
- What processes lead to the consistent positive relations that have been found between conceptual and procedural knowledge of fractions? Is the relation solely due to the common role of fraction magnitudes or are other aspects of conceptual understanding, such as the infinite divisibility of fractions, also important? Do interventions that improve conceptual understanding of fraction magnitudes produce a 'learning to learn' effect, such that experiences that improve magnitude understanding enhance subsequent benefits from fraction arithmetic instruction?
- Fractions are typically taught before percentages, but Moss and Case's [59] rational number curriculum indicates that percentages can be effectively used to teach fractions. Does teaching fractions first or percentages first produce greater learning of fractions and percentages?
- How specific are the relations of neural processing of whole numbers and fractions? For example, are individual differences in brain activations on the two types of numbers related? Do interventions that improve whole number and fraction magnitude representations produce similar changes in brain activity? The integrated theory [1] predicts that such relations should emerge, but the accuracy of this prediction is unknown.

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