Math Handbook

of Formulas, Processes and Tricks

(www.mathguy.us)

Algebra and PreCalculus

Prepared by: Earl L. Whitney, FSA, MAAA

Version 3.2

July 10, 2019

Algebra Handbook
Table of Contents

Cover art by Rebecca Williams,
Twitter handle: @jolteonkitty

Algebra Handbook
Table of Contents

Algebra Handbook
Table of Contents

Algebra Handbook
Table of Contents

Algebra Handbook
Table of Contents

Algebra Handbook
Table of Contents

Useful Websites

Mathguy.us – Developed specifically for math students from Middle School to College, based on the author's extensive experience in professional mathematics in a business setting and in math tutoring. Contains free downloadable handbooks, PC Apps, sample tests, and more.

http://www.mathguy.us/

Wolfram Math World – Perhaps the premier site for mathematics on the Web. This site contains definitions, explanations and examples for elementary and advanced math topics.

http://mathworld.wolfram.com/

Purple Math – A great site for the Algebra student, it contains lessons, reviews and homework guidelines. The site also has an analysis of your study habits. Take the Math Study Skills Self-Evaluation to see where you need to improve.

http://www.purplemath.com/

Math.com – Has a lot of information about Algebra, including a good search function.

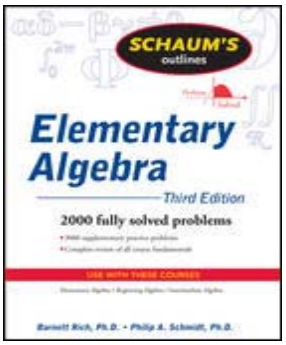
http://www.math.com/homeworkhelp/Algebra.html

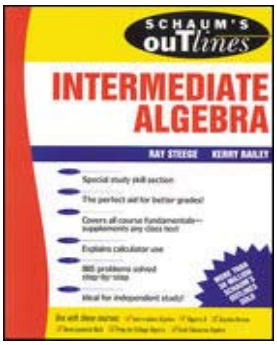
Algebra Handbook
Table of Contents

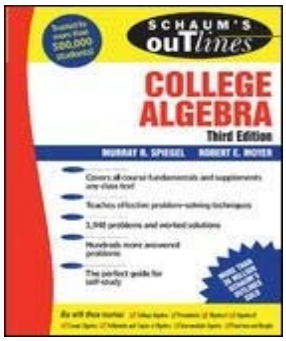
Schaum's Outlines

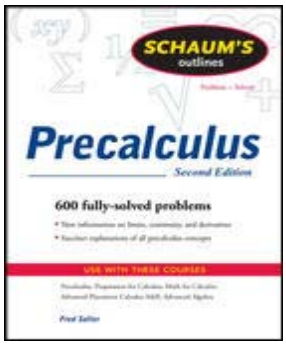
An important student resource for any high school math student is a Schaum's Outline. Each book in this series provides explanations of the various topics in the course and a substantial number of problems for the student to try. Many of the problems are worked out in the book, so the student can see examples of how they should be solved.

Schaum's Outlines are available at Amazon.com, Barnes & Noble, Borders and other booksellers.









Note: This study guide was prepared to be a companion to most books on the subject of High School Algebra. In particular, I used the following texts to determine which subjects to include in this guide.

- *Algebra 1* , by James Schultz, Paul Kennedy, Wade Ellis Jr, and Kathleen Hollowelly.
- *Algebra 2* , by James Schultz, Wade Ellis Jr, Kathleen Hollowelly, and Paul Kennedy.

Although a significant effort was made to make the material in this study guide original, some material from these texts was used in the preparation of the study guide.

Algebra
Order of Operations

To the non-mathematician, there may appear to be multiple ways to evaluate an algebraic expression. For example, how would one evaluate the following?

$$3 \cdot 4 \cdot 7 + 6 \cdot 5^2$$

You could work from left to right, or you could work from right to left, or you could do any number of other things to evaluate this expression. As you might expect, mathematicians do not like this ambiguity, so they developed a set of rules to make sure that any two people evaluating an expression would get the same answer.

PEMDAS

In order to evaluate expressions like the one above, mathematicians have defined an order of operations that must be followed to get the correct value for the expression. The acronym that can be used to remember this order is **PEMDAS**. Alternatively, you could use the mnemonic phrase "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally" or make up your own way to memorize the order of operations. The components of **PEMDAS** are:

- **P** Anything in **P**arentheses is evaluated first.
- **E** Items with **E**xponents are evaluated next.
- **M** **M**ultiplication and …
  D **D**ivision are performed next.
- **A** **A**ddition and …
  S **S**ubtraction are performed last.

Note: When there are multiple operations in the same category, for example, a division and two multiplications, the operations are performed from left to right.

Parenthetical Device. A useful device is to use apply parentheses to help you remember the order of operations when you evaluate an expression. Parentheses are placed around the items highest in the order of operations; then solving the problem becomes more natural. Using PEMDAS and this parenthetical device, we solve the expression above as follows:

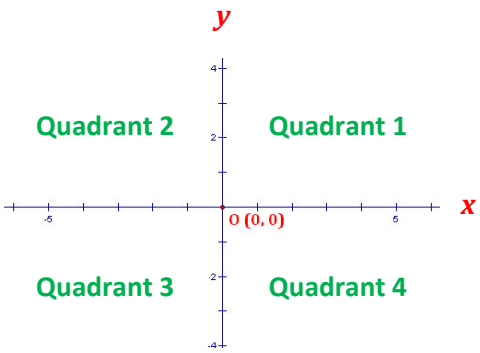
Initial Expression:	$3 \cdot 4 \cdot 7 + 6 \cdot 5^2$
Add parentheses/brackets:	$= (3 \cdot 4 \cdot 7) + [6 \cdot (5^2)]$
Solve using PEMDAS:	$= (84) + (6 \cdot 25)$
	$= (84) + (150)$
Final Answer	$= 234$

Note: Any expression which is ambiguous, like the one above, is poorly written. Students should strive to ensure that any expressions they write are easily understood by others and by themselves. Use of parentheses and brackets is a good way to make your work more understandable.

Algebra
Graphing with Coordinates

Graphs in two dimensions are very common in algebra and are one of the most common algebra applications in real life.

Coordinates



The plane of points that can be graphed in 2 dimensions is called the Rectangular Coordinate Plane or the Cartesian Coordinate Plane (named after the French mathematician and philosopher René Descartes).

- Two axes are defined (usually called the ***x***- and ***y***-axes).
- Each point on the plane has an ***x*** value and a ***y*** value, written as: ***(x-value, y-value)***
- The point (0, 0) is called the origin, and is usually denoted with the letter "O".
- The axes break the plane into 4 quadrants, as shown above. They begin with Quadrant 1 where ***x*** and ***y*** are both positive and increase numerically in a counter-clockwise fashion.

Plotting Points on the Plane

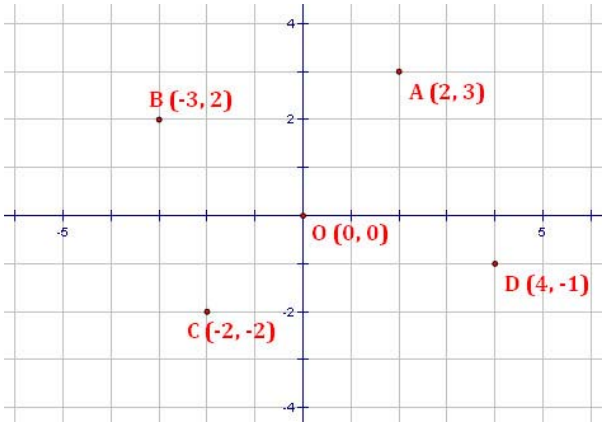
When plotting points,

- the ***x***-value determines how far right (positive) or left (negative) of the origin the point is plotted.
- The ***y***-value determines how far up (positive) or down (negative) from the origin the point is plotted.

Examples:

The following points are plotted in the figure to the right:

A = (2, 3) in Quadrant 1
B = (-3, 2) in Quadrant 2
C = (-2, -2) in Quadrant 3
D = (4, -1) in Quadrant 4
O = (0, 0) is not in any quadrant



Algebra

Linear Patterns

Recognizing Linear Patterns

The first step to recognizing a pattern is to arrange a set of numbers in a table. The table can be either horizontal or vertical. Here, we consider the pattern in a horizontal format. More advanced analysis generally uses the vertical format.

Consider this pattern:

x-value	0	1	2	3	4	5
y-value	6	9	12	15	18	21

To analyze the pattern, we calculate differences of successive values in the table. These are called first differences. If the first differences are constant, we can proceed to converting the pattern into an equation. If not, we do not have a linear pattern. In this case, we may choose to continue by calculating differences of the first differences, which are called second differences, and so on until we get a pattern we can work with.

In the example above, we get a constant set of first differences, which tells us that the pattern is indeed linear.

x-value	0	1	2	3	4	5
y-value	6	9	12	15	18	21
First Differences		3	3	3	3	3

Converting a Linear Pattern to an Equation

Creating an equation from the pattern is easy if you have constant differences and a y-value for x = 0. In this case,

- The equation takes the form $y = mx + b$, where
- *"m"* is the constant difference from the table, and
- *"b"* is the y-value when x = 0.

In the example above, this gives us the equation: $y = 3x + 6$.

Note: If the table does not have a value for x=0, you can still obtain the value of **"b"**. Simply extend the table left or right until you have an x-value of 0; then use the first differences to calculate what the corresponding y-value would be. This becomes your value of **"b"**.

Finally, it is a good idea to test your equation. For example, if $x = 4$, the above equation gives $y = (3 \cdot 4) + 6 = 18$, which is the value in the table. So we can be pretty sure our equation is correct.

Algebra
Identifying Number Patterns

When looking at patterns in numbers, is is often useful to take differences of the numbers you are provided. If the first differences are not constant, take differences again.

n	Δ
-3	
	2
-1	
	2
1	
	2
3	
	2
5	
	2
7	

When first differences are constant, the pattern represents a linear equation. In this case, the equation is: $y = 2x - 5$. The constant difference is the coefficient of x in the equation.

n	Δ	Δ^2
2		
	3	
5		2
	5	
10		2
	7	
17		2
	9	
26		2
	11	
37		

When second differences are constant, the pattern represents a quadratic equation. In this case, the equation is: $y = x^2 + 1$. The constant difference, divided by 2, gives the coefficient of x^2 in the equation.

When taking successive differences yields patterns that do not seem to level out, the pattern may be either exponential or recursive.

n	Δ	Δ^2
5		
	2	
7		2
	4	
11		4
	8	
19		8
	16	
35		16
	32	
67		

In the pattern to the left, notice that the first and second differences are the same. You might also notice that these differences are successive powers of 2. This is typical for an exponential pattern. In this case, the equation is: $y = 2^x + 3$.

n	Δ	Δ^2
2		
	1	
3		1
	2	
5		1
	3	
8		2
	5	
13		3
	8	
21		

In the pattern to the left, notice that the first and second differences appear to be repeating the original sequence. When this happens, the sequence may be recursive. This means that each new term is based on the terms before it. In this case, the equation is: $y_n = y_{n-1} + y_{n-2}$, meaning that to get each new term, you add the two terms before it.

Algebra
Completing Number Patterns

The first step in completing a number pattern is to identify it. Then, work from the right to the left, filling in the highest order differences first and working backwards (left) to complete the table. Below are two examples.

Example 1

n
-1
6
25
62
123
214

Consider in the examples the sequences of six numbers which are provided to the student. You are asked to find the ninth term of each sequence.

Example 2

n
2
3
5
8
13
21

---

n	Δ	Δ^2	Δ^3
-1			
	7		
6		12	
	19		6
25		18	
	37		6
62		24	
	61		6
123		30	
	91		
214			

Step 1: Create a table of differences. Take successive differences until you get a column of constant differences (Example 1) or a column that appears to repeat a previous column of differences (Example 2).

n	Δ	Δ^2	Δ^3
2			
	1		
3		1	
	2		0
5		1	
	3		1
8		2	
	5		1
13		3	
	8		
21			

---

n	Δ	Δ^2	Δ^3
-1			
	7		
6		12	
	19		6
25		18	
	37		6
62		24	
	61		6
123		30	
	91		**6**
214			**6**
			6

Step 2: In the last column of differences you created, continue the constant differences (Example 1) or the repeated differences (Example 2) down the table. Create as many entries as you will need to solve the problem. For example, if you are given 6 terms and asked to find the 9th term, you will need 3 (= 9 - 6) additional entries in the last column.

n	Δ	Δ^2	Δ^3
2			
	1		
3		1	
	2		0
5		1	
	3		1
8		2	
	5		1
13		3	
	8		**2**
21			**3**
			5

---

n	Δ	Δ^2	Δ^3
-1			
	7		
6		12	
	19		6
25		18	
	37		6
62		24	
	61		6
123		30	
	91		**6**
214		**36**	**6**
341	**127**	**42**	**6**
510	**169**	**48**	
727	**217**		

Step 3: Work backwards (from right to left), filling in each column by adding the differences in the column to the right.

In the example to the left, the calculations are performed in the following order:

Column Δ^2: 30 + 6 = 36; 36 + 6 = 42; 42 + 6 = 48

Column Δ: 91 + 36 = 127; 127 + 42 = 169; 169 + 48 = 217

Column n: 214 + 127 = 341; 341 + 169 = 510; 510 + 217 = **727**

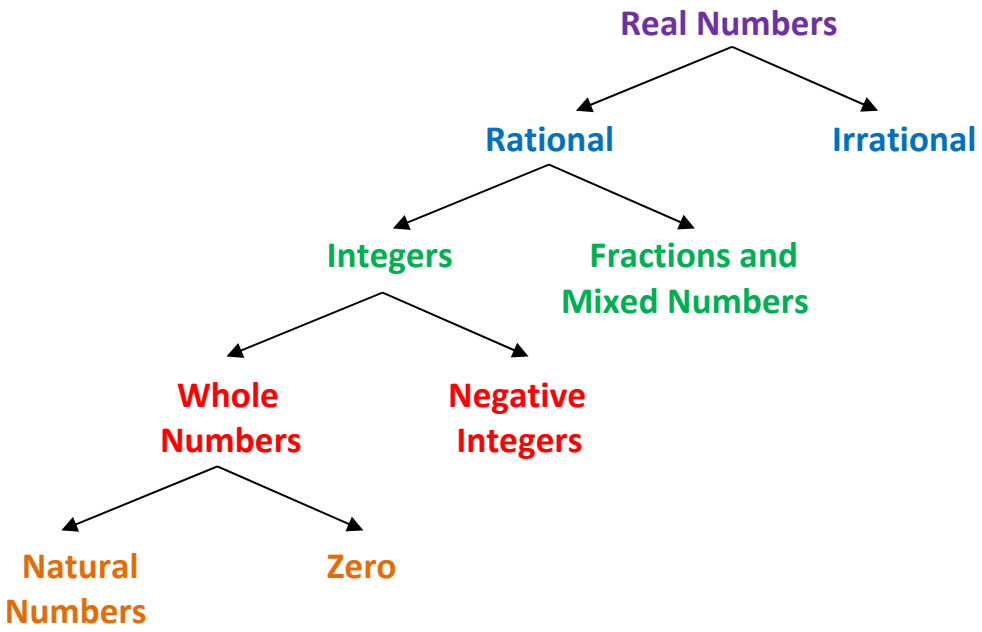
n	Δ	Δ^2	Δ^3
2			
	1		
3		1	
	2		0
5		1	
	3		1
8		2	
	5		1
13		3	
	8		**2**
21		**5**	**3**
34	**13**	**8**	**5**
55	**21**	**13**	
89	**34**		

The final answers to the examples are the ninth items in each sequence, the items in **bold red.**

Algebra
Real Number Sets

Number Set	Symbol	Definition	Examples
Natural (or, Counting) **Numbers**	**N** or **Z+**	Numbers that you would normally count with.	1, 2, 3, 4, 5, 6, …
Whole Numbers	**W**	Add the number *zero* to the set of Natural Numbers	0, 1, 2, 3, 4, 5, 6, …
Integers	**Z**	Whole numbers plus the set of negative Natural Numbers	… -3, -2, -1, 0, 1, 2, 3, …
Rational Numbers	**Q**	Any number that can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.	All integers, plus fractions and mixed numbers, such as: $-\frac{2}{3}$, $\frac{17}{6}$, $3\frac{4}{5}$
Real Numbers	**R**	Any number that can be written in decimal form, even if that form is infinite.	All rational numbers plus roots and some others, such as: $\sqrt{2}$, $\sqrt[3]{12}$, π, e

Real Number Set Tree



Algebra
Operating with Real Numbers

Absolute Value

The absolute value of something is the distance it is from zero. The easiest way to get the absolute value of a number is to eliminate its sign. Absolute values are always positive or 0.

$$|-5| = 5 \qquad |3| = 3 \qquad |0| = 0 \qquad \left|-\frac{3}{4}\right| = \frac{3}{4} \qquad |1.5| = 1.5$$

Adding and Subtracting Real Numbers

Adding Numbers with the Same Sign:

- Add the numbers without regard to sign.
- Give the answer the same sign as the original numbers.
- Examples:

$$(-6) + (-3) = -9$$
$$12 + 6 = 18$$

Adding Numbers with Different Signs:

- Ignore the signs and subtract the smaller number from the larger one.
- Give the answer the sign of the number with the greater absolute value.
- Examples:

$$(-6) + 3 = -3$$
$$(-7) + 11 = 4$$

Subtracting Numbers:

- Change the sign of the number or numbers being subtracted.
- Add the resulting numbers.
- Examples:

$$(-6) - (-3) = (-6) + 3 = -3$$
$$13 - 4 = 13 + (-4) = 9$$

Multiplying and Dividing Real Numbers

Numbers with the Same Sign:

- Multiply or divide the numbers without regard to sign.
- Give the answer a "+" sign.
- Examples:

$$(-6) \cdot (-3) = +18 = 18$$
$$12 \div 3 = +4 = 4$$

Numbers with Different Signs:

- Multiply or divide the numbers without regard to sign.
- Give the answer a "-" sign.
- Examples:

$$(-6) \cdot (3) = -18$$
$$12 \div (-3) = -4$$

Algebra
Properties of Algebra

Properties of Addition and Multiplication. For any real numbers **a**, **b**, and **c**:

Property	Definition for Addition	Definition for Multiplication
Closure Property	$a + b$ is a real number	$a \cdot b$ is a real number
Identity Property	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Inverse Property	$a + (-a) = (-a) + a = 0$	$For\ a \neq 0,\ a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$
Commutative Property	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative Property	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Distributive Property	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	

Properties of Zero. For any real number **a**:

Multiplication by 0	$a \cdot 0 = 0 \cdot a = 0$
0 Divided by Something	$For\ a \neq 0,\ \frac{0}{a} = 0$
Division by 0	$\frac{a}{0}$ is undefined (even if $a = 0$)

Algebra
Properties of Algebra

Operational Properties of Equality. For any real numbers **a**, **b**, and **c**:

Property	Definition
Addition Property	$If\ a = b, then\ a + c = b + c$
Subtraction Property	$If\ a = b, then\ a - c = b - c$
Multiplication Property	$If\ a = b, then\ a \cdot c = b \cdot c$
Division Property	$If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$

Other Properties of Equality. For any real numbers **a**, **b**, and **c**:

Property	Definition
Reflexive Property	$a = a$
Symmetric Property	$If\ a = b, then\ b = a$
Transitive Property	$If\ a = b\ and\ b = c, then\ a = c$
Substitution Property	If $a = b$, then either can be substituted for the other in any equation (or inequality).

Algebra
Solving Multi-Step Equations

Reverse PEMDAS

One systematic way to approach multi-step equations is Reverse PEMDAS. PEMDAS describes the order of operations used to evaluate an expression. Solving an equation is the opposite of evaluating it, so reversing the PEMDAS order of operations seems appropriate.

The **guiding principles** in the process are:

- Each step works toward isolating the variable for which you are trying to solve.
- Each step "un-does" an operation in **Reverse PEMDAS** order:

Subtraction	Inverses	**A**ddition
Division	Inverses	**M**ultiplication
Exponents	Inverses	Logarithms
Parentheses	Inverses	Remove Parentheses (and repeat process)

Note: Logarithms are the inverse operator to exponents. This topic is typically covered in the second year of Algebra.

The list above shows inverse operation relationships. In order to undo an operation, you perform its inverse operation. For example, to undo addition, you subtract; to undo division, you multiply. Here are a couple of examples:

Example 1

Solve:	$3x - 4 = 14$
Step 1: Add 4	$+4 \quad +4$
Result:	$3x \quad = 18$
Step 2: Divide by 3	$\div 3 \quad \div 3$
Result:	$x \quad = 6$

Notice that we add and subtract before we multiply and divide. **Reverse PEMDAS.**

Example 2

Solve:	$2 \cdot (2x + 5) - 3 = -5$
Step 1: Add 3	$+3 \quad +3$
Result:	$2 \cdot (2x + 5) \quad = -2$
Step 2: Divide by 2	$\div 2 \quad \div 2$
Result:	$(2x + 5) \quad = -1$
Step 3: Remove parentheses	
Result:	$2x + 5 \quad = -1$
Step 4: Subtract 5	$-5 \quad -5$
Result:	$2x \quad = -6$
Step 5: Divide by 2	$\div 2 \quad \div 2$
Result:	$x \quad = -3$

With this approach, you will be able to solve almost any multi-step equation. As you get better at it, you will be able to use some shortcuts to solve the problem faster. Since speed is important in mathematics, learning a few tips and tricks with regard to solving equations is likely to be worth your time.

Algebra
Tips and Tricks in Solving Multi-Step Equations

Fractional Coefficients

Fractions present a stumbling block to many students in solving multi-step equations. When stumbling blocks occur, it is a good time to develop a trick to help with the process. The trick shown below involves using the reciprocal of a fractional coefficient as a multiplier in the solution process. (Remember that a coefficient is a number that is multiplied by a variable.)

Example 1

Solve: $\frac{2}{3}x = 8$

Multiply by $\frac{3}{2}$: $\cdot\frac{3}{2} \quad \cdot\frac{3}{2}$

Result: $x = \frac{3}{2} \cdot 8 = \frac{24}{2} = 12$

Explanation: Since $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$, when we multiply them, we get 1, and $1 \cdot x = x$. Using this approach, we can avoid dividing by a fraction, which is more difficult.

Example 2

Solve: $-\frac{1}{4}x = -2$

Multiply by -4: $\cdot(-4) \quad \cdot(-4)$

Result: $x = (-2) \cdot (-4) = 8$

Explanation: -4 is the reciprocal of $-\frac{1}{4}$, so when we multiply them, we get 1. Notice the use of parentheses around the negative number to make it clear we are multiplying and not subtracting.

Another Approach to Parentheses

In the Reverse PEMDAS method, parentheses are handled after all other operations. Sometimes, it is easier to operate on the parentheses first. In this way, you may be able to re-state the problem in an easier form before solving it.

Example 3, at right, is another look at the problem in Example 2 on the previous page.

Use whichever approach you find most to your liking. They are both correct.

Example 3

Solve: $2 \cdot (2x + 5) - 3 = -5$

Step 1: Distribute the lead multiplier (2)

Result: $4x + 10 \quad - 3 = -5$

Step 2: Combine constants

Result: $4x + 7 \quad = -5$

Step 3: Subtract 7 $\quad -7 \quad -7$

Result: $4x \quad = -12$

Step 4: Divide by 4 $\quad \div 4 \quad \div 4$

Result: $x \quad = -3$

Algebra
Probability and Odds

Probability

Probability is a measure of the likelihood that an event will occur. It depends on the number of outcomes that represent the event and the total number of possible outcomes. In equation terms,

$$P(event) = \frac{number\ of\ outcomes\ representing\ the\ event}{number\ of\ total\ possible\ outcomes}$$

Example 1: The probability of a flipped coin landing as a head is 1/2. There are two equally likely events when a coin is flipped – it will show a head or it will show a tail. So, there is one chance out of two that the coin will show a head when it lands.

$$P(head) = \frac{1\ outcome\ of\ a\ head}{2\ total\ possible\ outcomes} = \frac{1}{2}$$

Example 2: In a jar, there are 15 blue marbles, 10 red marbles and 7 green marbles. What is the probability of selecting a red marble from the jar? In this example, there are 32 total marbles, 10 of which are red, so there is a 10/32 (or, when reduced, 5/16) probability of selecting a red marble.

$$P(red\ marble) = \frac{10\ red\ marbles}{32\ total\ marbles} = \frac{10}{32} = \frac{5}{16}$$

Odds

Odds are similar to probability, except that we measure the number of chances that an event will occur relative to the number of chances that the event will not occur.

$$Odds(event) = \frac{number\ of\ outcomes\ representing\ the\ event}{number\ of\ outcomes\ NOT\ representing\ the\ event}$$

In the above examples,

$$Odds(head) = \frac{1\ outcome\ of\ a\ head}{1\ outcome\ of\ a\ tail} = \frac{1}{1} \qquad Odds(red\ marble) = \frac{10\ red\ marbles}{22\ other\ marbles} = \frac{10}{22} = \frac{5}{11}$$

- Note that the numerator and the denominator in an odds calculation add to the total number of possible outcomes in the denominator of the corresponding probability calculation.
- To the beginning student, the concept of odds is not as intuitive as the concept of probabilities; however, they are used extensively in some environments.

Algebra
Probability with Dice

Single Die

Probability with a single die is based on the number of chances of an event out of 6 possible outcomes on the die. For example:

$$P(2) = \frac{1}{6} \qquad P(odd\ number) = \frac{3}{6} = \frac{1}{2} \qquad P(number < 5) = \frac{4}{6} = \frac{2}{3}$$

Two Dice

Probability with two dice is based on the number of chances of an event out of 36 possible outcomes on the dice. The following table of results when rolling 2 dice is helpful in this regard:

	1st Die					
2nd Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The probability of rolling a number with two dice is the number of times that number occurs in the table, divided by 36. Here are the probabilities for all numbers 2 to 12.

$$P(2) = \frac{1}{36} \qquad P(5) = \frac{4}{36} = \frac{1}{9} \qquad P(8) = \frac{5}{36} \qquad P(11) = \frac{2}{36} = \frac{1}{18}$$

$$P(3) = \frac{2}{36} = \frac{1}{18} \qquad P(6) = \frac{5}{36} \qquad P(9) = \frac{4}{36} = \frac{1}{9} \qquad P(12) = \frac{1}{36}$$

$$P(4) = \frac{3}{36} = \frac{1}{12} \qquad P(7) = \frac{6}{36} = \frac{1}{6} \qquad P(10) = \frac{3}{36} = \frac{1}{12}$$

$$P(odd\ number) = \frac{18}{36} = \frac{1}{2} \qquad P(number\ divisible\ by\ 3) = \frac{2+5+4+1}{36} = \frac{12}{36} = \frac{1}{3}$$

$$P(even\ number) = \frac{18}{36} = \frac{1}{2} \qquad P(number\ divisible\ by\ 4) = \frac{3+5+1}{36} = \frac{9}{36} = \frac{1}{4}$$

$$P(number\ divisible\ by\ 6) = \frac{5+1}{36} = \frac{6}{36} = \frac{1}{6}$$

Algebra
Combinations

Single Category Combinations

The number of combinations of items selected from a set, several at a time, can be calculated relatively easily using the following technique:

> ***Technique:*** Create a ratio of two products. In the numerator, start with the number of total items in the set, and count down so the total number of items being multiplied is equal to the number of items being selected. In the denominator, start with the number of items being selected and count down to 1.

Example: How many combinations of 3 items can be selected from a set of 8 items? Answer: $$\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$	**Example:** How many combinations of 4 items can be selected from a set of 13 items? Answer: $$\frac{13 \cdot 12 \cdot 11 \cdot 10}{4 \cdot 3 \cdot 2 \cdot 1} = 715$$	**Example:** How many combinations of 2 items can be selected from a set of 30 items? Answer: $$\frac{30 \cdot 29}{2 \cdot 1} = 435$$

Multiple Category Combinations

When calculating the number of combinations that can be created by selecting items from several categories, the technique is simpler:

> ***Technique:*** Multiply the numbers of items in each category to get the total number of possible combinations.

Example: How many different pizzas could be created if you have 3 kinds of dough, 4 kinds of cheese and 8 kinds of toppings? Answer: $$3 \cdot 4 \cdot 8 = 96$$	**Example:** How many different outfits can be created if you have 5 pairs of pants, 8 shirts and 4 jackets? Answer: $$5 \cdot 8 \cdot 4 = 160$$	**Example:** How many designs for a car can be created if you can choose from 12 exterior colors, 3 interior colors, 2 interior fabrics and 5 types of wheels? Answer: $$12 \cdot 3 \cdot 2 \cdot 5 = 360$$

Algebra
Statistical Measures

Statistical measures help describe a set of data. A definition of a number of these is provided in the table below:

Concept	Description	Calculation	Example 1	Example 2
Data Set	Numbers		35, 35, 37, 38, 45	15, 20, 20, 22, 25, 54
Mean	Average	Add the values and divide the total by the number of values	$\frac{35+35+37+38+45}{5} = 38$	$\frac{15+18+22+22+25+54}{6} = 26$
Median(1)	Middle	Arrange the values from low to high and take the middle value(1)	37	21(1)
Mode	Most	The value that appears most often in the data set	35	20
Range	Size	The difference between the highest and lowest values in the data set	45 – 35 = 10	54 – 15 = 39
Outliers(2)	Oddballs	Values that look very different from the other values in the data set	none	54

Notes:

(1) If there are an even number of values, the median is the average of the two middle values. In Example 2, the median is 21, which is the average of 20 and 22.

(2) The question of what constitutes an outlier is not always clear. Although statisticians seek to minimize subjectivity in the definition of outliers, different analysts may choose different criteria for the same data set.

Algebra
Introduction to Functions

Definitions

- A **Relation** is a relationship between variables, usually expressed as an equation.
- In a typical ***x*-*y*** equation, the **Domain** of a relation is the set of ***x***-values for which ***y***-values can be calculated. For example, in the relation $y = \sqrt{x}$ the domain is $x \geq 0$ because these are the values of ***x*** for which a square root can be taken.
- In a typical ***x*-*y*** equation, the **Range** of a relation is the set of ***y***-values that result for all values of the domain. For example, in the relation $y = \sqrt{x}$ the range is $y \geq 0$ because these are the values of ***y*** that result from all the values of ***x***.
- A **Function** is a relation in which each element in the domain has only one corresponding element in the range.
- A **One-to-One Function** is a function in which each element in the range is produced by only one element in the domain.

Function Tests in 2-Dimensions

Vertical Line Test – If a vertical line passes through the graph of a relation in any two locations, it is not a function. If it is not possible to construct a vertical line that passes through the graph of a relation in two locations, it is a function.

Horizontal Line Test – If a horizontal line passes through the graph of a function in any two locations, it is not a one-to-one function. If it is not possible to construct a horizontal line that passes through the graph of a function in two locations, it is a one-to-one function.

Examples:

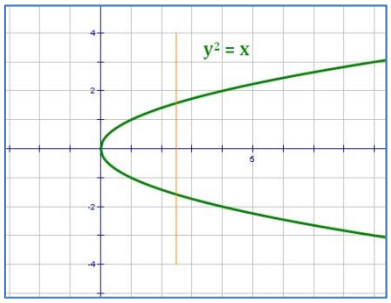


Figure 1: $y^2 = x$

Not a function.

Fails vertical line test.

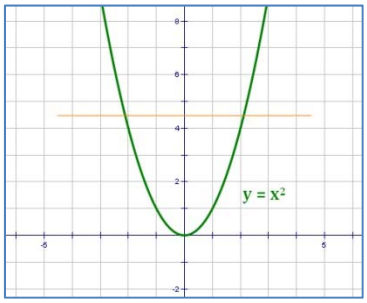


Figure 2: $y = x^2$

Is a function, but not a one-to-one function.

Passes vertical line test.

Fails horizontal line test.

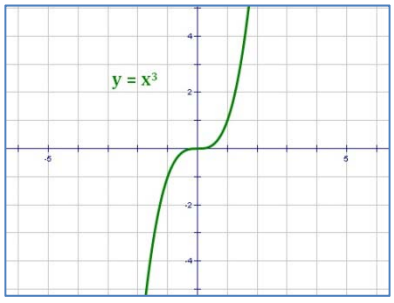


Figure 3: $y = x^3$

Is a one-to-one function.

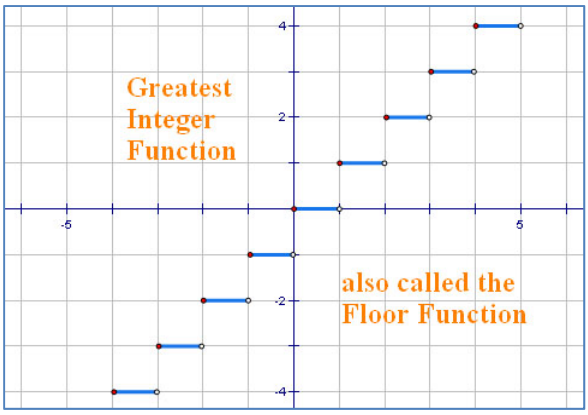
Passes vertical line test.

Passes horizontal line test.

Algebra
Special Integer Functions

Greatest Integer Function

Also called the Floor Function, this function gives the greatest integer less than or equal to a number. There are two common notations for this, as shown in the examples below.



Notation and examples:

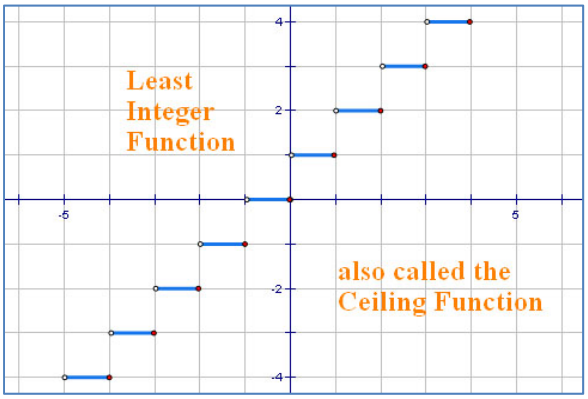
$[3.5] = 3$ $[-2.7] = -3$ $[6] = 6$

$\lfloor 2.4 \rfloor = 2$ $\lfloor -7.1 \rfloor = -8$ $\lfloor 0 \rfloor = 0$

In the graph to the right, notice the solid dots on the left of the segments (indicating the points are included) and the open lines on the right of the segments (indicating the points are not included).

Least Integer Function

Also called the Ceiling Function, this function gives the least integer greater than or equal to a number. The common notation for this is shown in the examples below.



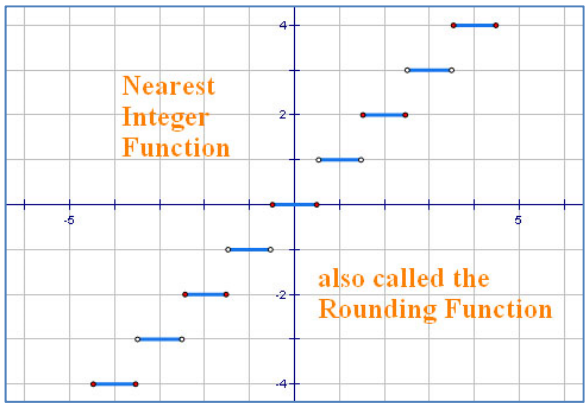
Notation and examples:

$[3.5] = 4$ $[-2.7] = -2$ $[6] = 6$

In the graph to the right, notice the open dots on the left of the segments (indicating the points are not included) and the closed dots on the right of the segments (indicating the points are included).

Nearest Integer Function

Also called the Rounding Function, this function gives the nearest integer to a number (rounding to the even number when a value ends in .5). There is no clean notation for this, as shown in the examples below.



Notation and examples:

$nint(3.5) = 4$ $nint(-2.7) = -3$ $nint(6) = 6$

In the graph to the right, notice the open dots on the left of the segments (indicating the points are not included) and the closed dots on the right of the segments (indicating the points are included).

Algebra
Operations with Functions

Function Notation

$$f(x) = x^2 + 2 \qquad g(x) = e^x + ln(x) \qquad h(x) = -3x + 4$$

Function notation replaces the variable y with a function name. The x in parentheses indicates that x is the domain variable of the function. By convention, functions tend to use the letters f, g, and h as names of the function.

Operations with Functions

Adding Functions	$(f+g)(x) = f(x) + g(x)$
Subtracting Functions	$(f-g)(x) = f(x) - g(x)$
Multiplying Functions	$(f \cdot g)(x) = f(x) \cdot g(x)$
Dividing Functions	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

The **domain** of the combination of functions is the intersection of the domains of the two individual functions. That is, the combined function has a value in its domain if and only if the value is in the domain of each individual function.

Examples:

Let: $f(x) = x^2 - 1$

$g(x) = x + 1$

Then:

$$(f+g)(x) = x^2 + x$$
$$(f-g)(x) = x^2 - x - 2$$
$$(f \cdot g)(x) = x^3 + x^2 - x - 1$$
$$\left(\frac{f}{g}\right)(x) = x - 1, \quad with\ x \neq -1$$

Note that in $\left(\frac{f}{g}\right)(x)$ there is the requirement $x \neq -1$. This is because $g(-1) = 0$ in the denominator would require dividing by 0, producing an undefined result.

Other Operations

Other operations of equality also hold for functions, for example:

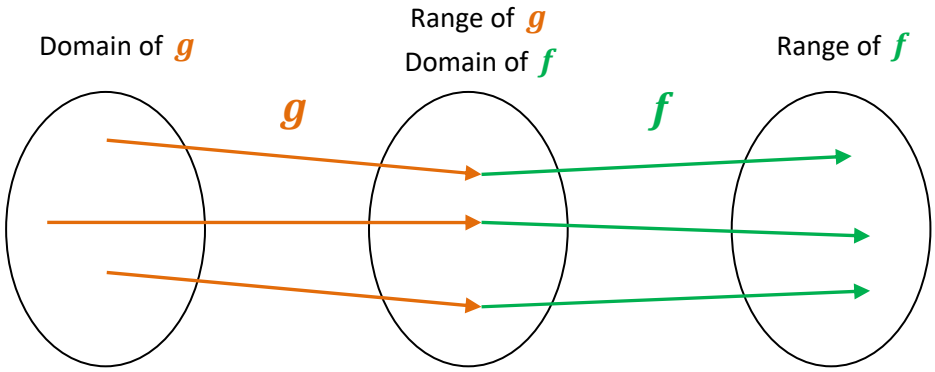
$$f^3(x) = (f(x))^3 = f(x) \cdot f(x) \cdot f(x)$$
$$f(x) \cdot \big(g(x) + h(x)\big) = f(x) \cdot g(x) + f(x) \cdot h(x)$$

Algebra
Composition of Functions

In a **Composition of Functions**, first one function is performed, and then the other. The notation for composition is, for example: $f(g(x))$ or $(f \circ g)(x)$. In both of these notations, the function g is performed first, and then the function f is performed on the result of g. Always perform the function closest to the variable first.

Double Mapping

A composition can be thought of as a double mapping. First g maps from its domain to its range. Then, f maps from the range of g to the range of f:



The Words Method

Example: Let $f(x) = x^2$

and $g(x) = x + 1$

Then: $(f \circ g)(x) = (x+1)^2$

And: $(g \circ f)(x) = x^2 + 1$

In the example,

- The function f says *square the argument*.
- The function g says *add 1 to the argument*.

Sometimes it is easier to think of the functions in words rather than in terms of an argument like x.

$(f \circ g)$ says "add 1 first, then square the result."

$(g \circ f)$ says "square first, then add 1 to the result."

Using the words method,

Calculate: $(f \circ g)(12yz)$

g: add 1 to it $(12yz) + 1$

f: square it $(12yz + 1)^2$

Calculate: $(g \circ f)(-2)$

f: square it $(-2)^2 = 4$

g: add 1 to it $4 + 1 = 5$

Algebra
Inverses of Functions

In order for a function to have an inverse, it must be a one-to-one function. The requirement for a function to be an inverse is:

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$

The notation $f^{-1}(x)$ is used for the **Inverse Function** of $f(x)$.

Another way of saying this is that **if $f(a) = b$, then $f^{-1}(b) = a$ for all a in the domain of f.**

Deriving an Inverse Function

The following steps can be used to derive an inverse function. This process assumes that the original function is expressed in terms of $f(x)$.

- Make sure the function is one-to-one. Otherwise it has no inverse. You can accomplish this by graphing the function and applying the vertical and horizontal line tests.
- Substitute the variable y for $f(x)$.
- Exchange variables. That is, change all the x's to y's and all the y's to x's.
- Solve for the new y in terms of the new x.
- (Optional) Switch the expressions on each side of the equation if you like.
- Replace the variable y with the function notation $f^{-1}(x)$.
- Check your work.

Examples:

Derive the inverse of: $f(x) = \frac{1}{3}x + 2$

Substitute y for $f(x)$:	$y = \frac{1}{3}x + 2$
Exchange variables:	$x = \frac{1}{3}y + 2$
Subtract 2:	$x - 2 = \frac{1}{3}y$
Multiply by 3:	$3x - 6 = y$
Switch sides:	$y = 3x - 6$
Change Notation:	$f^{-1}(x) = 3x - 6$

To check the result, note that:

$$(f^{-1} \circ f)(x) = 3f(x) - 6 = 3\left(\frac{1}{3}x + 2\right) - 6 = x$$

Derive the inverse of: $f(x) = 2x - 1$

Substitute y for $f(x)$:	$y = 2x - 1$
Exchange variables:	$x = 2y - 1$
Add 1:	$x + 1 = 2y$
Divide by 2:	$\frac{x+1}{2} = y$
Switch sides:	$y = \frac{x+1}{2}$
Change Notation:	$f^{-1}(x) = \frac{x+1}{2}$

To check the result, note that:

$$(f^{-1} \circ f)(x) = \frac{f(x) + 1}{2} = \frac{(2x - 1) + 1}{2} = x$$

Algebra
Transformation – Translation

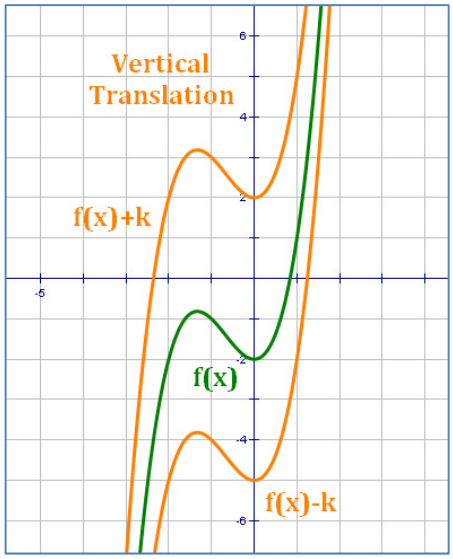
A **Translation** is a movement of the graph of a relation to a different location in the plane. It preserves the shape and orientation of the graph on the page. Alternatively, a translation can be thought of as leaving the graph where it is and moving the axes around on the plane.

In Algebra, the translations of primary interest are the vertical and horizontal translations of a graph.

Vertical Translation

Starting form: $y = f(x)$

Vertical Translation: $y = f(x) + k$



At each point, the graph of the translation is k units higher or lower depending on whether k is positive or negative. The letter k is used as a convention when moving up or down. In algebra, k usually represents a y-value of some importance.

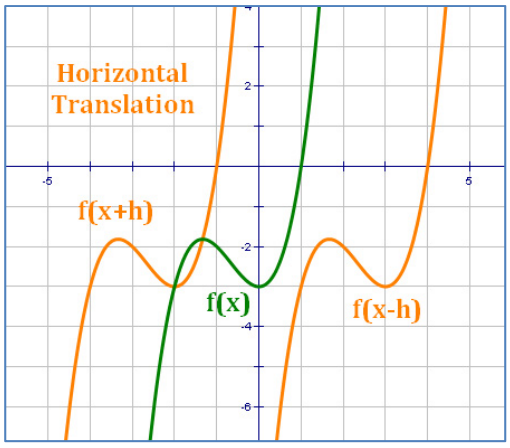
Note:

- A positive k shifts the graph up.
- A negative k shifts the graph down.

Horizontal Translation

Starting form: $y = f(x)$

Horizontal Translation: $y = f(x - h)$



At each point, the graph of the translation is h units to the left or right depending on whether h is positive or negative. The letter h is used as a convention when moving left or right. In algebra, h usually represents an x-value of some importance.

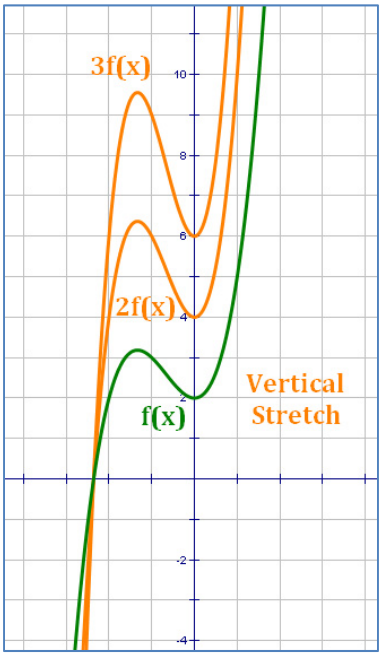
Note:

- A positive h shifts the graph to the left.
- A negative h shifts the graph to the right.

For horizontal translation, the direction of movement of the graph is counter-intuitive; be careful with these.

Algebra
Transformation – Vertical Stretch and Compression

A **Vertical Stretch or Compression** is a stretch or compression in the vertical direction, relative to the x-axis. It does not slide the graph around on the plane like a translation. An alternative view of a vertical stretch or compression would be a change in the scale of the y-axis.



Vertical Stretch

Starting form: $y = f(x)$

Vertical Stretch: $y = a \cdot f(x), \quad a > 1$

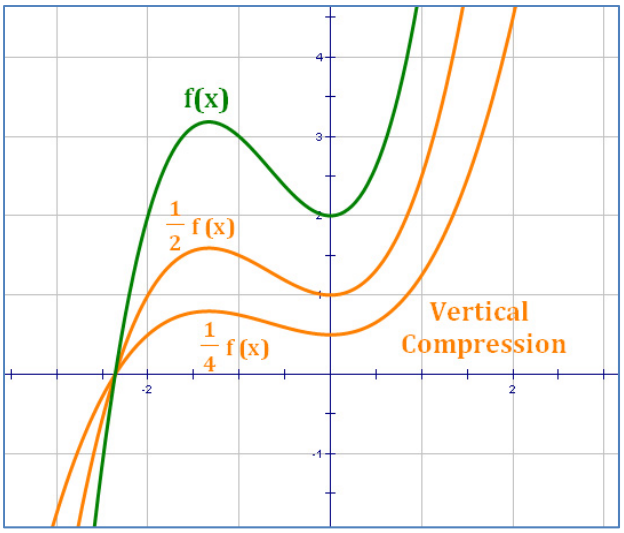
At each point, the graph is stretched vertically by a factor of a. The result is an elongated curve, one that exaggerates all of the features of the original.

Vertical Compression

Starting form: $y = f(x)$

Vertical Compression: $y = a \cdot f(x), \quad 0 < a < 1$

At each point, the graph is compressed vertically by a factor of a. The result is a flattened-out curve, one that mutes all of the features of the original.



Note: The forms of the equations for vertical stretch and vertical compression are the same. The only difference is the value of "a".

Value of "a" in $y = a \cdot f(x)$	Resulting Curve
$a < 0$	reflection
$a = 0$	x-axis
$0 < a < 1$	compression
$a = 1$	original curve
$a > 1$	stretch

Algebra
Transformation – Horizontal Stretch and Compression

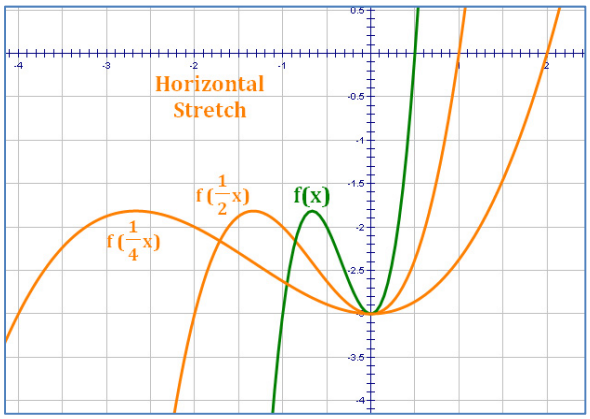
A **Horizontal Stretch or Compression** is a stretch or compression in the horizontal direction, relative to the y-axis. It does not slide the graph around on the plane like a translation. An alternative view of a horizontal stretch or compression would be a change in the scale of the x-axis.

Horizontal Stretch

Starting form: $y = f(x)$

Horizontal Stretch: $y = f(bx),\ 0 < b < 1$

At each point, the graph is stretched horizontally by a factor of $\frac{1}{b}$. The result is a widened curve, one that exaggerates all of the features of the original.

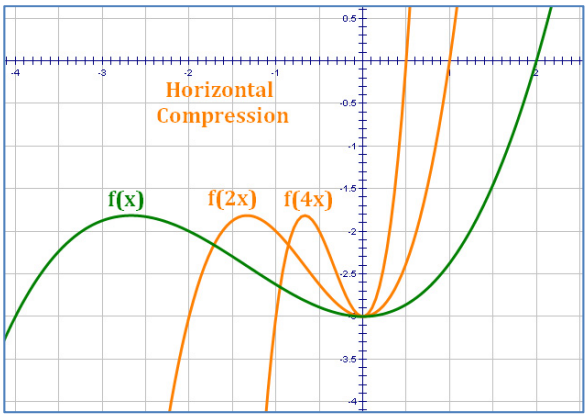


Horizontal Compression

Starting form: $y = f(x)$

Horizontal Compression: $y = f(bx),\ b > 1$

At each point, the graph is compressed horizontally by a factor of $\frac{1}{b}$. The result is a skinnier curve, one that mutes all of the features of the original.



Note: The forms of the equations for the horizontal stretch and the horizontal compression are the same. The only difference is the value of "b".

Value of "b" in $y = f(bx)$	Resulting Curve
$b < 0$	reflection
$b = 0$	horizontal line
$0 < b < 1$	stretch
$b = 1$	original curve
$b > 1$	compression

Note: For horizontal stretch and compression, the change in the graph caused by the value of "b" is counter-intuitive; be careful with these.

Algebra
Transformation – Reflection

A **Reflection** is a "flip" of the graph across a mirror in the plane. It preserves the shape the graph but can make it look "backwards."

In Algebra, the reflections of primary interest are the reflections across an axis in the plane.

X-Axis Reflection

Starting form: $y = f(x)$

x-axis Reflection: $y = -f(x)$

Note the following:

- At each point, the graph is reflected across the x-axis.
- The form of the transformation is the same as a vertical stretch or compression with $a = -1$.
- The flip of the graph over the x-axis is, in effect, a vertical transformation.

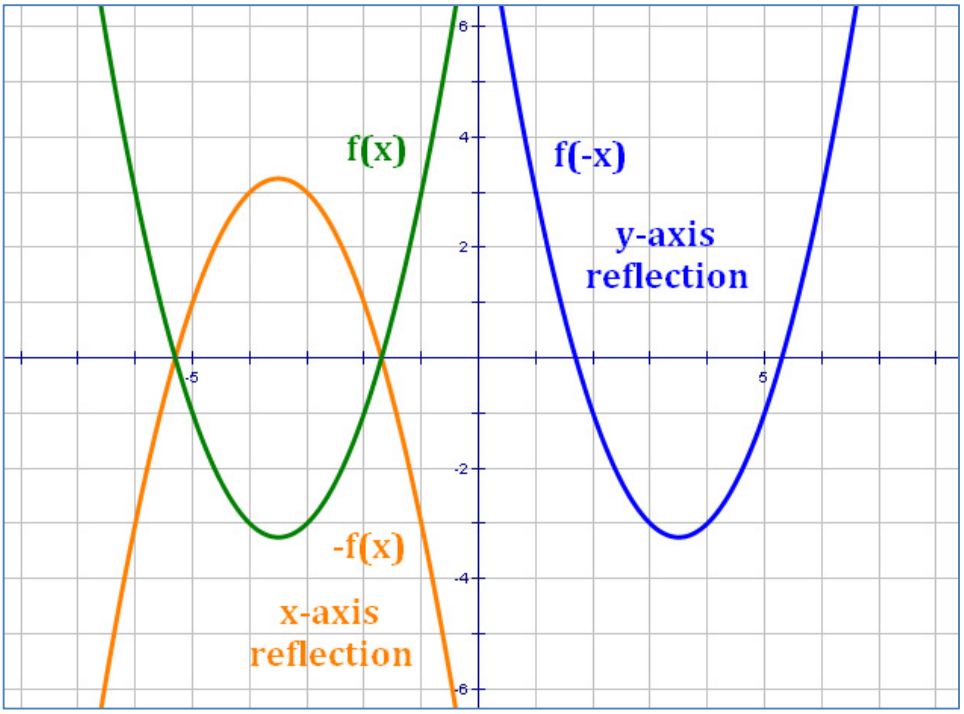
Y-Axis Reflection

Starting form: $y = f(x)$

y-axis Reflection: $y = f(-x)$

Note the following:

- At each point, the graph is reflected across the y-axis.
- The form of the transformation is the same as a horizontal stretch or compression with $b = -1$.
- The flip of the graph over the y-axis is, in effect, a horizontal transformation.

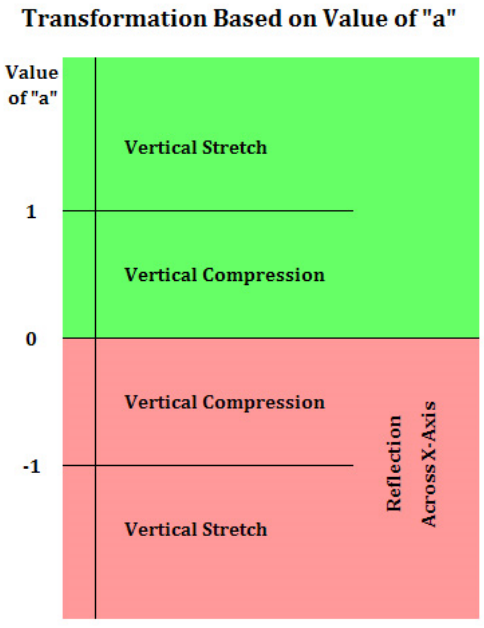


Algebra
Transformations – Summary

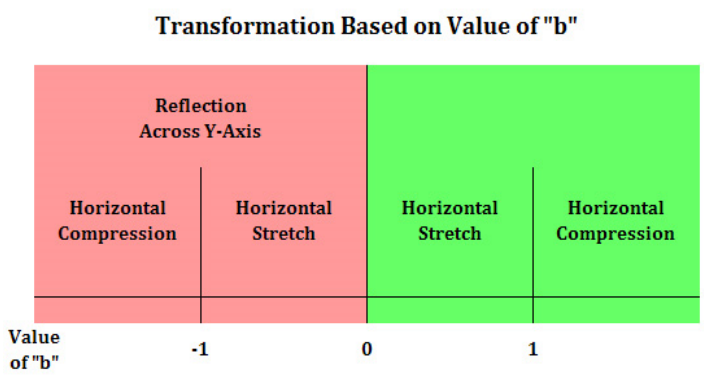
Starting form: $y = f(x)$

For purposes of the following table, the variables h and k are positive to make the forms more like what the student will encounter when solving problems involving transformations.

Transformation Summary	
Form of Transformation	**Result of Transformation**
$y = f(x) + k$ $y = f(x) - k$	Vertical translation up k units. Vertical translation down k units.
$y = f(x + h)$ $y = f(x - h)$	Horizontal translation left h units. Horizontal translation right h units.
$y = a \cdot f(x), \quad a > 1$ $y = a \cdot f(x), \quad 0 < a < 1$	Vertical stretch by a factor of a. Vertical compression by a factor of a.
$y = f(bx), \quad b > 1$ $y = f(bx), \quad 0 < b < 1$	Horizontal compression by a factor of $\frac{1}{b}$. Horizontal stretch by a factor of $\frac{1}{b}$.
$y = -f(x)$ $y = f(-x)$	Reflection across the x-axis (vertical). Reflection across the y-axis (horizontal).

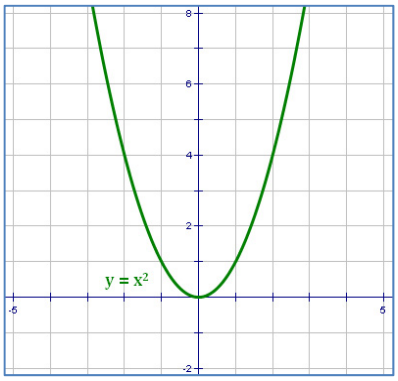


Transformations based on the values of "a" and "b" (stretches, compressions, reflections) can be represented by these graphics.

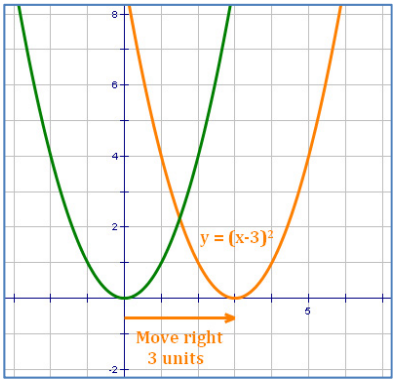


Algebra
Building a Graph with Transformations

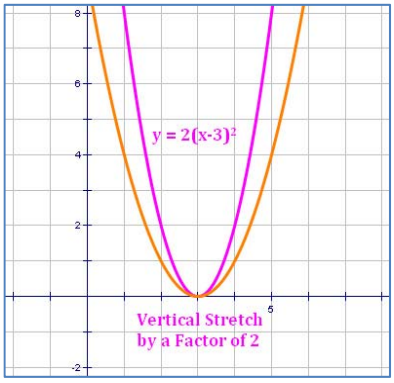
The graph of an equation can be built with blocks made up of transformations. As an **example**, we will build the graph of $y = -2(x-3)^2 + 4$.



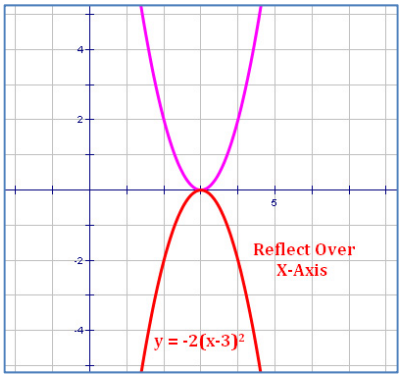
Step 1: Start with the basic quadratic equation: $y = x^2$



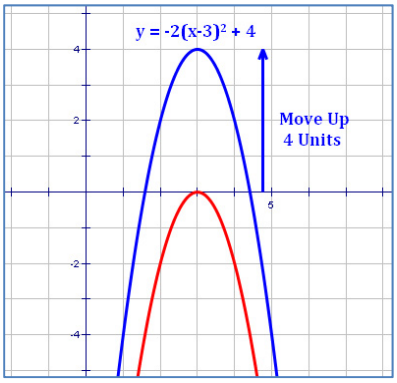
Step 2: Translate 3 units to the right to get equation: $y = (x-3)^2$



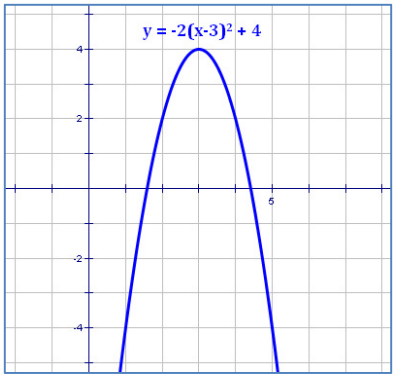
Step 3: Stretch vertically by a factor of 2 to get equation: $y = 2(x-3)^2$



Step 4: Reflect over the x-axis to get equation: $y = -2(x-3)^2$



Step 5: Translate up 4 units to get equation: $y = -2(x-3)^2 + 4$



Final Result: Show the graph of the final equation: $y = -2(x-3)^2 + 4$

Algebra
Slope of a Line

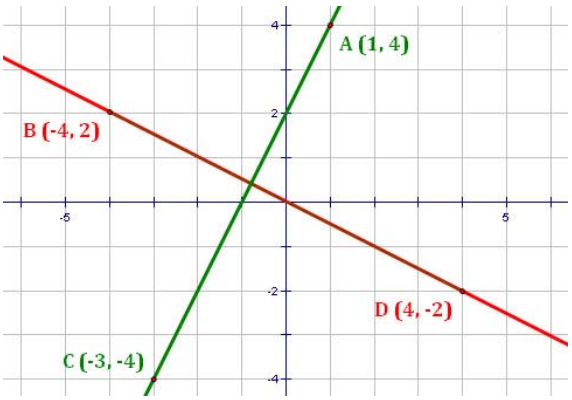
The slope of a line tells how fast it rises or falls as it moves from left to right. If the slope is rising, the slope is positive; if it is falling, the slope is negative. The letter "m" is often used as the symbol for slope.

The two most useful ways to calculate the slope of a line are discussed below.

Mathematical Definition of Slope

The definition is based on two points with coordinates (x_1, y_1) and (x_2, y_2). The definition, then, is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



Comments:

- You can select any 2 points on the line.
- A table such as the one at right can be helpful for doing your calculations.
- Note that $m = \frac{y_2 - y_1}{x_2 - x_1}$ implies that $m = \frac{y_1 - y_2}{x_1 - x_2}$. So, it does not matter which point you assign as Point 1 and which you assign as Point 2. Therefore, neither does it matter which point is first in the table.
- It is important that once you assign a point as Point 1 and another as Point 2, that you use their coordinates in the proper places in the formula.

	x-value	y-value
Point 2	x_2	y_2
Point 1	x_1	y_1
Difference	$x_2 - x_1$	$y_2 - y_1$

Examples:

For the two lines in the figure above, we get the following:

Green Line	x-value	y-value
Point A	1	4
Point C	-3	-4
Difference	4	8

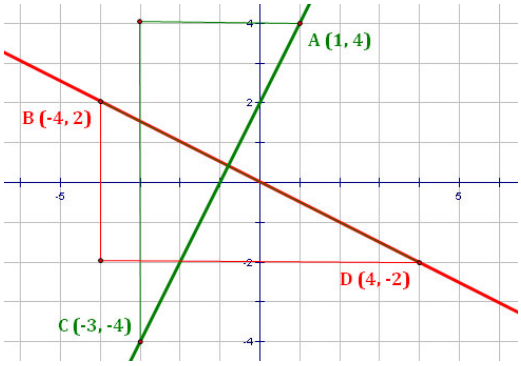
Red Line	x-value	y-value
Point D	4	-2
Point B	-4	2
Difference	8	-4

Green Line: $m = \frac{8}{4} = 2$

Red Line: $m = \frac{-4}{8} = -\frac{1}{2}$

Algebra
Slope of a Line (cont'd)

Rise over Run



An equivalent method of calculating slope that is more visual is the "Rise over Run" method. Under this method, it helps to draw vertical and horizontal lines that indicate the horizontal and vertical distances between points on the line.

The slope can then be calculated as follows:

$$m = \frac{rise}{run} = \frac{length\ of\ vertical\ line}{length\ of\ horizontal\ line}$$

The **rise** of a line is how much it **increases (positive)** or **decreases (negative)** between two points. The **run** is how far the line moves to the **right (positive)** or the **left (negative)** between the same two points.

Comments:

- You can select any 2 points on the line.
- It is important to start at the same point in measuring both the rise and the run.
- A good convention is to always start with the point on the left and work your way to the right; that way, the run (i.e., the denominator in the formula) is always positive. The only exception to this is when the run is zero, in which case the slope is undefined.
- If the two points are clearly marked as integers on a graph, the rise and run may actually be counted on the graph. This makes the process much simpler than using the formula for the definition of slope. However, when counting, make sure you get the right sign for the slope of the line, e.g., moving down as the line moves to the right is a negative slope.

Examples:

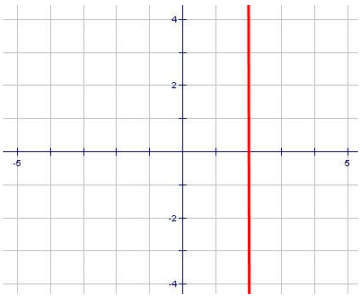
For the two lines in the figure above, we get the following:

Green Line: $m = \frac{rise\ from (-4)\ to\ 4}{run\ from\ (-3) to\ 1} = \frac{8}{4} = 2$

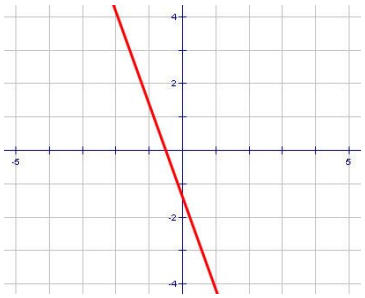
Red Line: $m = \frac{fall\ from\ 2\ to\ (-2)}{run\ from\ (-4)\ to\ 4} = \frac{-4}{8} = -\frac{1}{2}$

Notice how similar the calculations in the examples are under the two methods of calculating slopes.

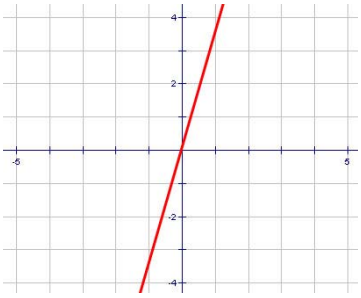
Algebra
Slopes of Various Lines



$m = undefined$
line is vertical



$m = -2\frac{4}{5}\ (big\ negative)$
line is steep and going down



$m = 3\frac{1}{2}\ (big\ positive)$
line is steep and going up

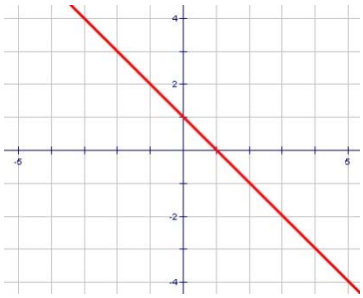
When you look at a line, you should notice the following about its slope:

- *Whether it is 0, positive, negative or undefined.*
- *If positive or negative, whether it is less than 1, about 1, or greater than 1.*

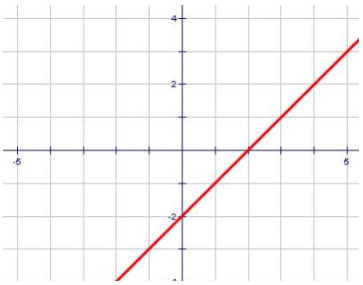
The purpose of the graphs on this page is to help you get a feel for these things.

This can help you check:

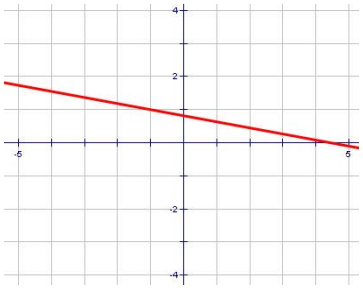
- Given a slope, whether you drew the line correctly, or
- Given a line, whether you calculated the slope correctly.



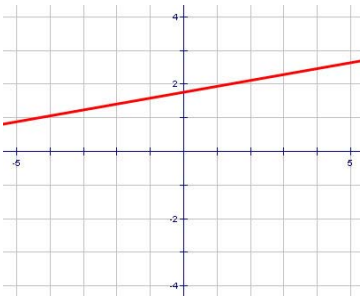
$m = -1$
line goes down at a 45° angle



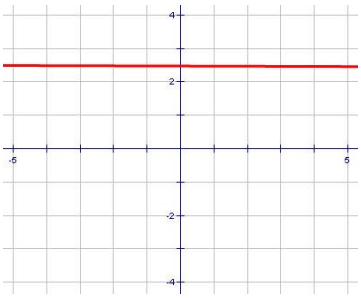
$m = 1$
line goes up at a 45° angle



$m = -\frac{3}{17}\ (small\ negative)$
line is shallow and going down



$m = \frac{2}{11}\ (small\ positive)$
line is shallow and going up



$m = 0$
line is horizontal

Algebra
Various Forms of a Line

There are three forms of a linear equation which are most useful to the Algebra student, each of which can be converted into the other two through algebraic manipulation. The ability to move between forms is a very useful skill in Algebra, and should be practiced by the student.

Standard Form

The Standard Form of a linear equation is:

$$\boldsymbol{Ax + By = C}$$

where A, B, and C are real numbers and A and B are not both zero. Usually in this form, the convention is for A to be positive.

Standard Form Examples

$3x + 2y = 6$

$2x - 7y = 14$

Why, you might ask, is this "Standard Form?" One reason is that this form is easily extended to additional variables, whereas other forms are not. For example, in four variables, the Standard Form would be: $\boldsymbol{Ax + By + Cz + Dw = E}$. Another reason is that this form easily lends itself to analysis with matrices, which can be very useful in solving systems of equations.

Slope-Intercept Form

The Slope-Intercept Form of a linear equation is the one most familiar to many students. It is:

$$\boldsymbol{y = mx + b}$$

where m is the slope and b is the y-intercept of the line (i.e., the value at which the line crosses the y-axis in a graph). m and b must also be real numbers.

Slope-Intercept Examples

$y = -3x + 6$

$y = \frac{3}{4}x - 14$

Point-Slope Form

The Point-Slope Form of a linear equation is the one used least by the student, but it can be very useful in certain circumstances. In particular, as you might expect, it is useful if the student is asked for the equation of a line and is given the line's slope and the coordinates of a point on the line. The form of the equation is:

$$\boldsymbol{(y - y_1) = m(x - x_1)}$$

where m is the slope and (x_1, y_1) is any point on the line. One strength of this form is that **equations formed using different points on the same line will be equivalent.**

Point-Slope Examples

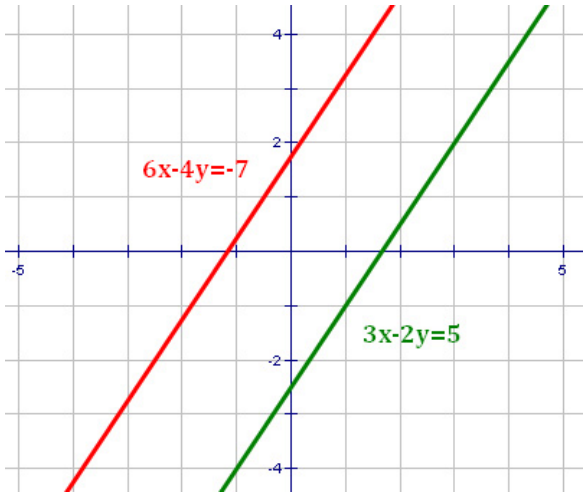
$(y - 3) = 2\,(x + 4)$

$(y + 7) = 5\left(x - \frac{2}{3}\right)$

Algebra
Slopes of Parallel and Perpendicular Lines

Parallel Lines

Two lines are parallel if their slopes are equal.



- In $y = mx + b$ form, if the values of m are the same.

  Example: $y = 2x - 3$ and $y = 2x + 1$

- In Standard Form, if the coefficients of x and y are proportional between the equations.

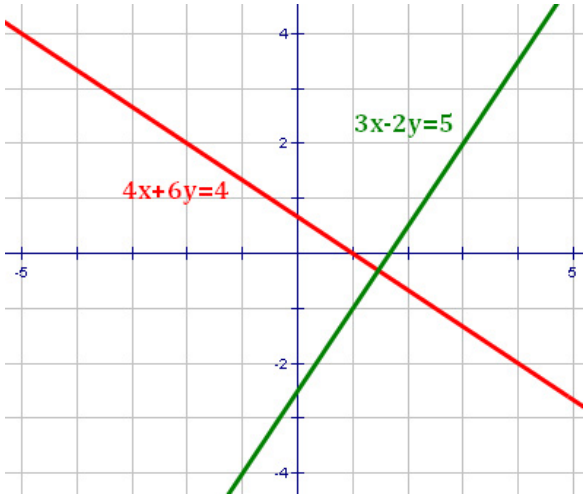
  Example: $3x - 2y = 5$ and $6x - 4y = -7$

- Also, if the lines are both vertical (i.e., their slopes are undefined).

  Example: $x = -3$ and $x = 2$

Perpendicular Lines

Two lines are perpendicular if the product of their slopes is -1. That is, if the slopes have different signs and are multiplicative inverses.



- In $y = mx + b$ form, the values of m multiply to get -1..

  Example: $y = 6x + 5$ and $y = -\frac{1}{6}x - 3$

- In Standard Form, if you add the product of the x-coefficients to the product of the y-coefficients and get zero.

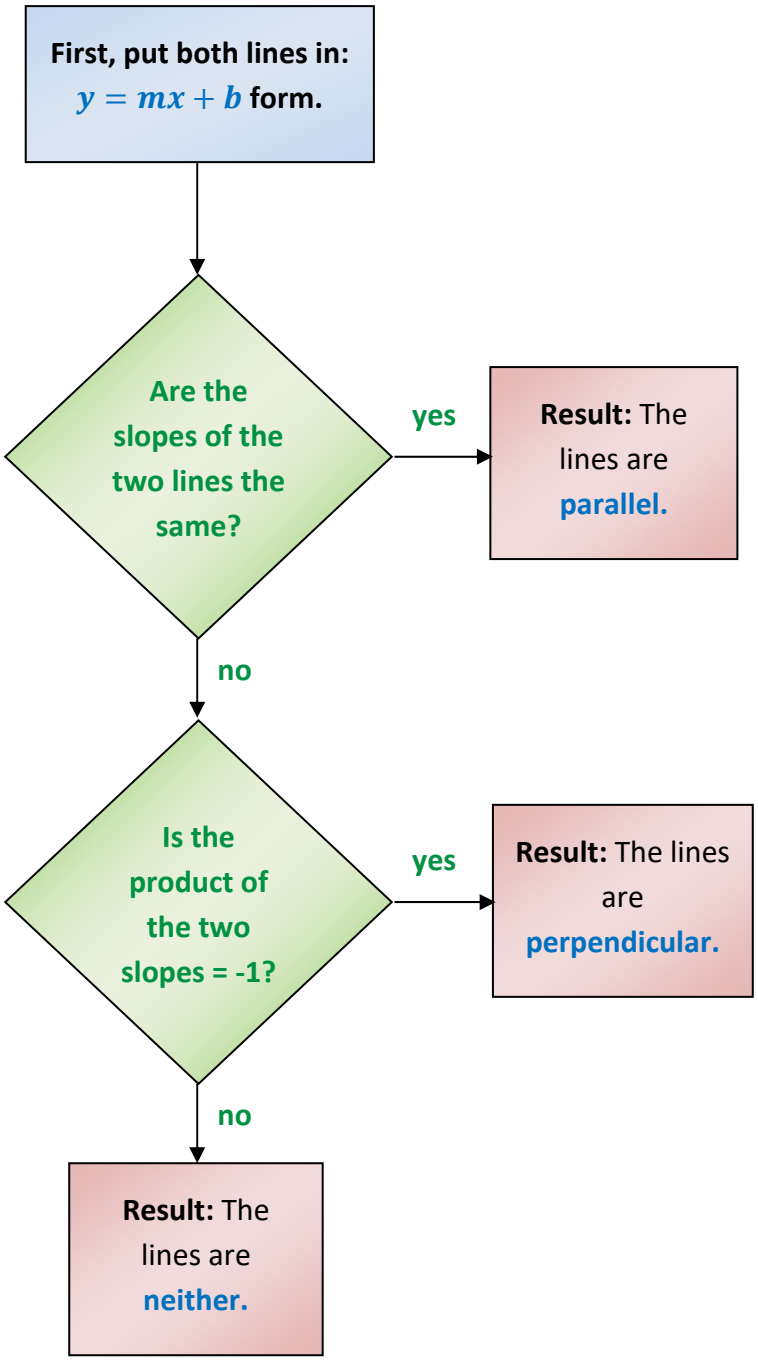
  Example: $4x + 6y = 4$ and $3x - 2y = 5$ because $(4 \cdot 3) + \big(6 \cdot (-2)\big) = 0$

- Also, if one line is vertical (i.e., m is undefined) and one line is horizontal (i.e., $m = 0$).

  Example: $x = 6$ and $y = 3$

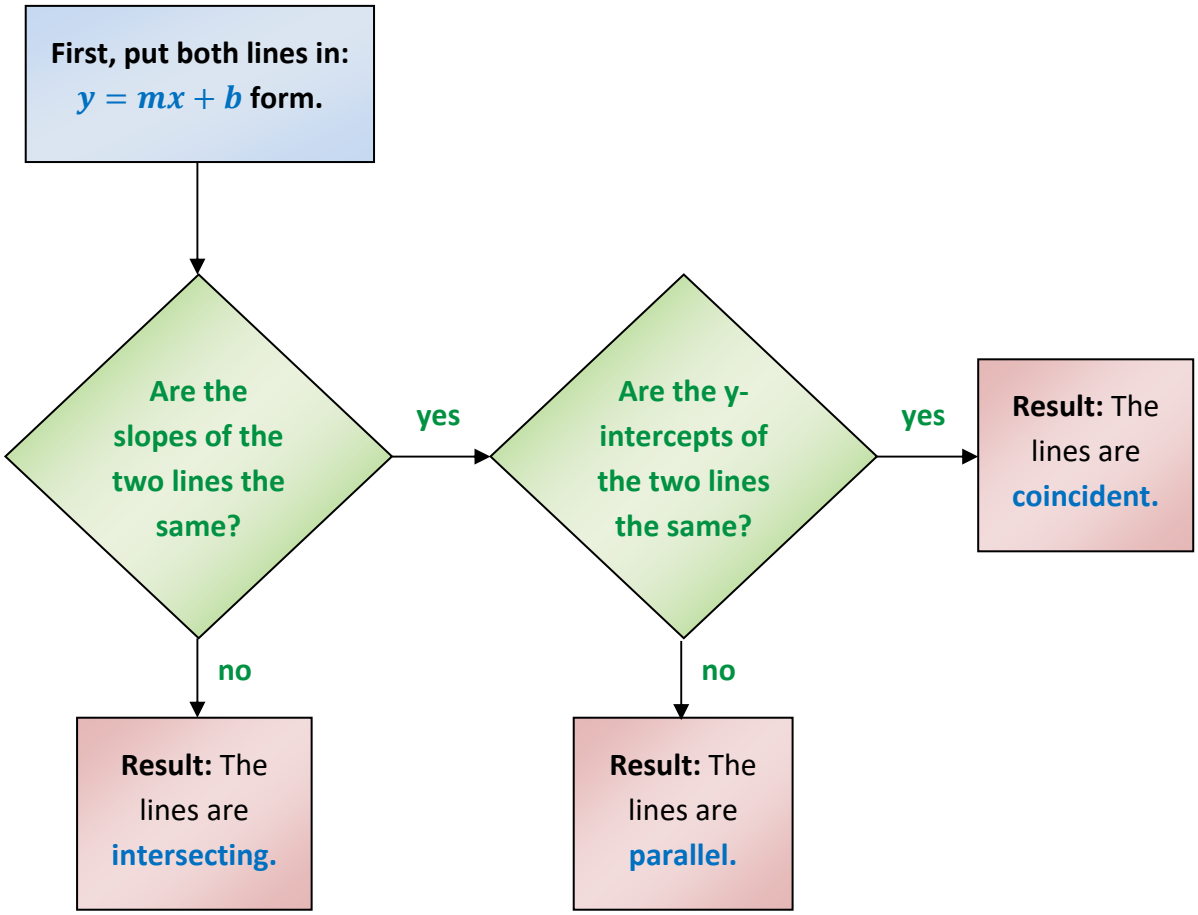
Algebra
Parallel, Perpendicular or Neither

The following flow chart can be used to determine whether a pair of lines are parallel, perpendicular, or neither.



Algebra
Parallel, Coincident or Intersecting

The following flow chart can be used to determine whether a pair of lines are parallel, coincident, or intersecting. Coincident lines are lines that are the same, even though they may be expressed differently. Technically, coincident lines are not parallel because parallel lines never intersect and coincident lines intersect at all points on the line.



The intersection of the two lines is:

- For intersecting lines, the point of intersection.
- For parallel lines, the empty set, { }.
- For coincident lines, all points on the line.

Algebra
Properties of Inequalities

For any real numbers **a**, **b**, and **c**:

Property	Definition	
Addition Property	$If\ a < b, then\ a + c < b + c$ $If\ a > b, then\ a + c > b + c$	
Subtraction Property	$If\ a < b, then\ a - c < b - c$ $If\ a > b, then\ a - c > b - c$	
Multiplication Property	**For $c > 0$,** $If\ a < b, then\ a \cdot c < b \cdot c$ $If\ a > b, then\ a \cdot c > b \cdot c$	**For $c < 0$,** $If\ a < b, then\ a \cdot c > b \cdot c$ $If\ a > b, then\ a \cdot c < b \cdot c$
Division Property	**For $c > 0$,** $If\ a < b, then\ a \div c < b \div c$ $If\ a > b, then\ a \div c > b \div c$	**For $c < 0$,** $If\ a < b, then\ a \div c > b \div c$ $If\ a > b, then\ a \div c < b \div c$
Note: all properties which hold for "<" also hold for "≤", and all properties which hold for ">" also hold for "≥".		

There is nothing too surprising in these properties. The most important thing to be obtained from them can be described as follows: **When you multiply or divide an inequality by a negative number, you must "flip" the sign.** That is, "<" becomes ">", ">" becomes "<", etc.

In addition, it is useful to note that you can flip around an entire inequality as long as you keep the "pointy" part of the sign directed at the same item. Examples:

$4 > x$ is the same as $x < 4$

$3x - 2 < y$ is the same as $y > 3x - 2$

One way to remember this is that when you flip around an inequality, you must also flip around the sign.

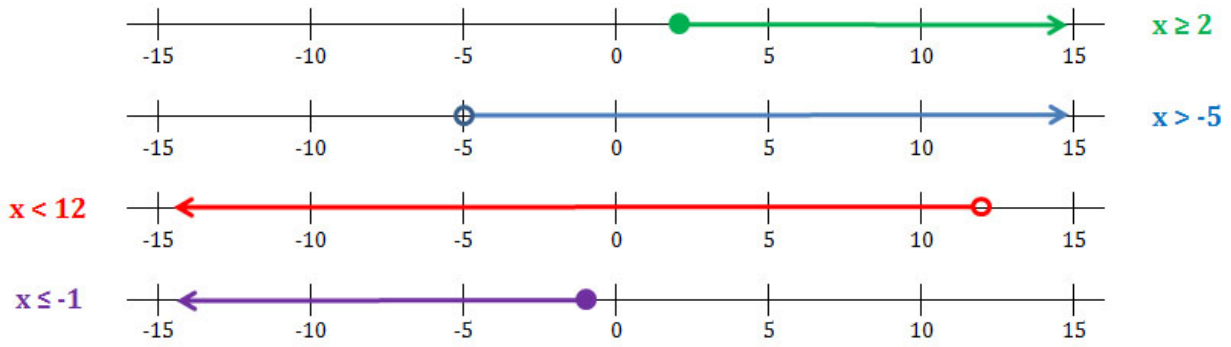
Algebra
Graphs of Inequalities in One Dimension

Inequalities in one dimension are generally graphed on the number line. Alternatively, if it is clear that the graph is one-dimensional, the graphs can be shown in relation to a number line but not specifically on it (examples of this are on the next page).

One-Dimensional Graph Components

- **The endpoint(s)** – The endpoints for the ray or segment in the graph are shown as either open or closed circles.
    - If the point is included in the solution to the inequality (i.e., if the sign is ≤ or ≥), the circle is closed.
    - If the point is not included in the solution to the inequality (i.e., if the sign is < or >), the circle is open.
- **The arrow** – If all numbers in one direction of the number line are solutions to the inequality, an arrow points in that direction.
    - For < or ≤ signs, the arrow points to the left (◄———).
    - For > or ≥ signs, the arrow points to the right (———►).
- **The line** – in a simple inequality, a line is drawn from the endpoint to the arrow. If there are two endpoints, a line is drawn from one to the other.

Examples:



Algebra
Compound Inequalities in One Dimension

Compound inequalities are a set of inequalities that must all be true at the same time. Usually, there are two inequalities, but more than two can also form a compound set. The principles described below easily extend to cases where there are more than two inequalities.

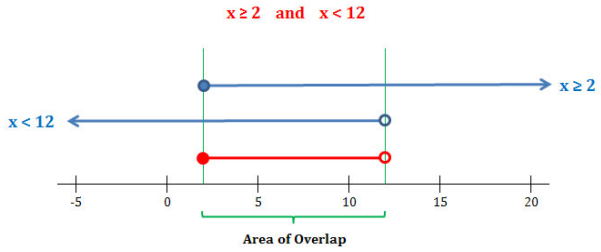
Compound Inequalities with the Word "AND"

An example of compound inequalities with the word "AND" would be:

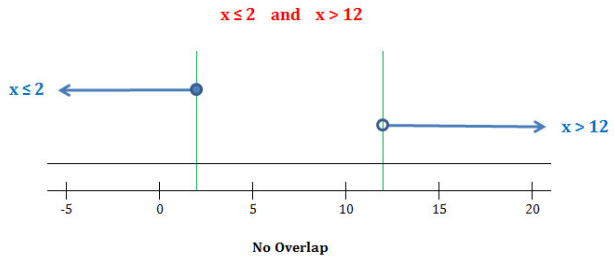
$x < 12 \quad and \quad x \geq 2$ **(Simple Form)** **or** $2 \leq x < 12$ **(Compound Form)**

These are the same conditions, expressed in two different forms.

Graphically, "AND" inequalities exist at points where the graphs of the individual inequalities overlap. This is the "intersection" of the graphs of the individual inequalities. Below are two examples of graphs of compound inequalities using the word "AND."



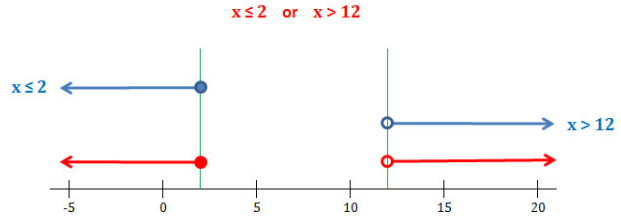
A typical "AND" example: The result is a segment that contains the points that overlap the graphs of the individual inequalities.



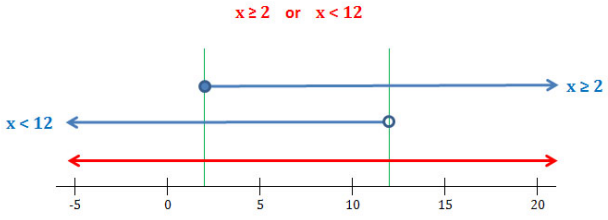
"AND" compound inequalities sometimes result in the empty set. This happens when no numbers meet both conditions at the same time.

Compound Inequalities with the Word "OR"

Graphically, "OR" inequalities exist at points where <u>any</u> of the original graphs have points. This is the "union" of the graphs of the individual inequalities. Below are two examples of graphs of compound inequalities using the word "OR."



A typical "OR" example: The result is a pair of rays extending in opposite directions, with a gap in between.



"OR" compound inequalities sometimes result in the set of all numbers. This happens when every number meets at least one of the conditions.

Algebra
Inequalities in Two Dimensions

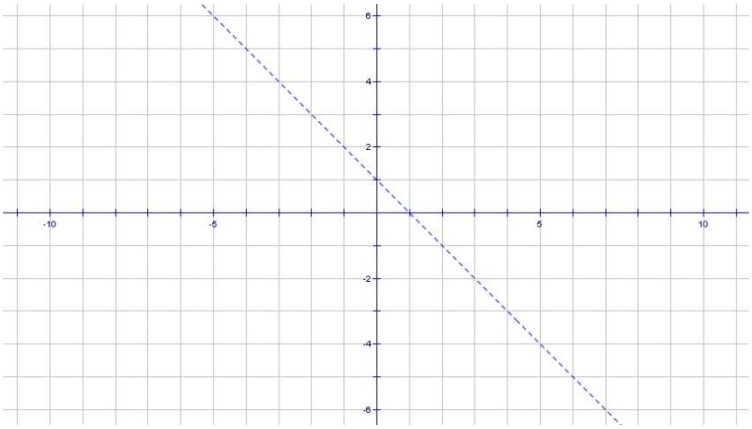
Graphing an inequality in two dimensions involves the following steps:

- Graph the underlying equation.
- Make the line solid or dotted based on whether the inequality contains an "=" sign.
  - For inequalities with **"<"** or **">"** the line is **dotted**.
  - For inequalities with **"≤"** or **"≥"** the line is **solid**.
- Determine whether the region containing the solution set is above the line or below the line.
  - For inequalities with **">"** or **"≥"** the shaded region is **above** the line.
  - For inequalities with **"<"** or **"≤"** the shaded region is **below** the line.
- Shade in the appropriate region.

Example:

Graph the solution set of the following system of inequality: $y > -x + 1$

Step 1: Graph the underlying equation.

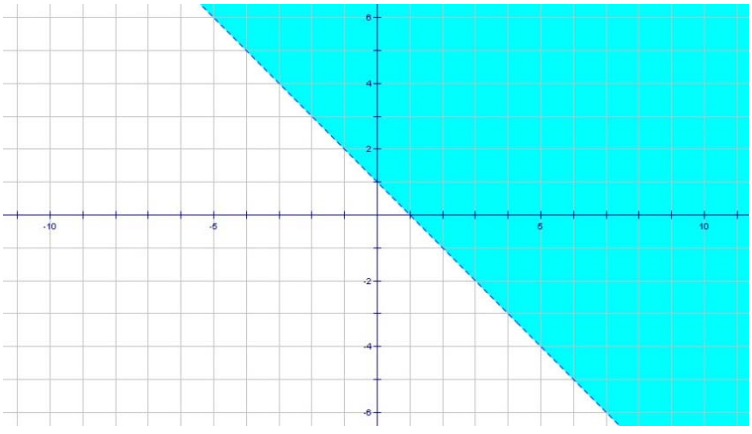


Step 2: Determine whether the line should be solid or dotted:

$y > -x + 1$ the > sign does not contain "=", so the line is **dotted**

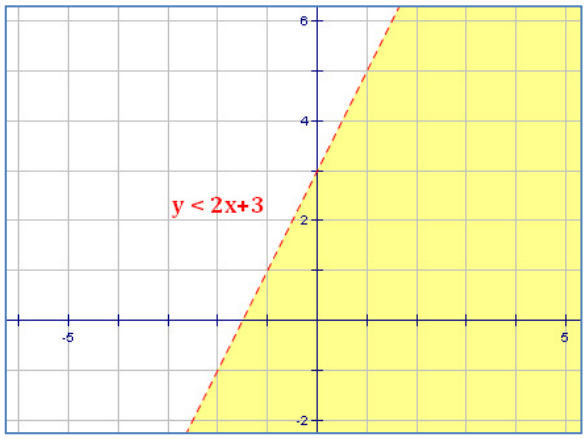
Step 3: Determine the region to be shaded based on the sign in the equation:

$y > -x + 1$ the > sign indicates shading above the line



The solution set is the shaded area.

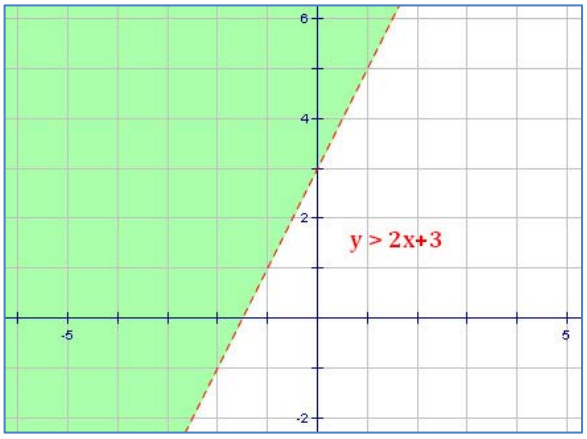
Algebra
Graphs of Inequalities in Two Dimensions



Less Than $(<)$

Dashed Line

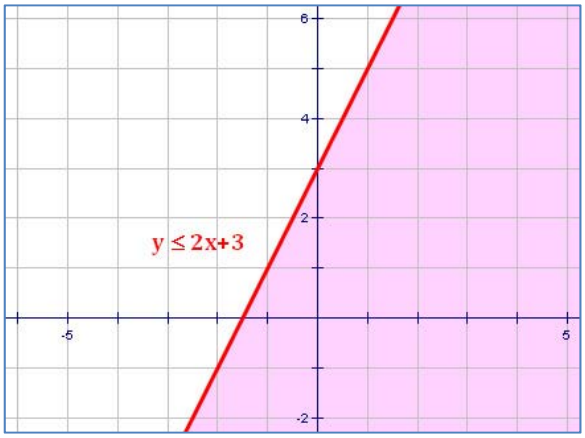
Below the Line



Greater Than $(>)$

Dashed Line

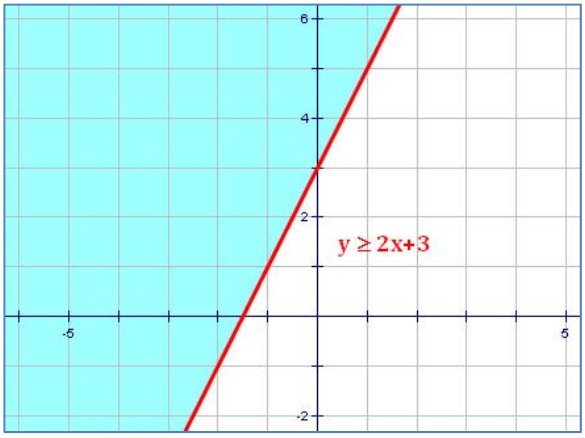
Above the Line



Less Than or Equal $(\leq)$

Solid Line

Below the Line



Greater Than or Equal $(\geq)$

Solid Line

Above the Line

Algebra
Absolute Value Functions

Equations

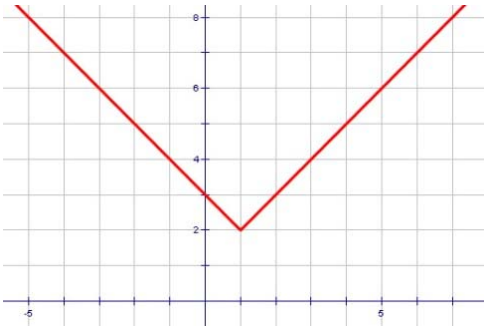
Graphs of equations involving absolute values generally have a "V" pattern. Whenever you see a "V" in a graph, think "absolute value." A general equation for an absolute value function is of the form:

$$y = |m(x - h)| + k \qquad or \qquad y = -|m(x - h)| + k$$

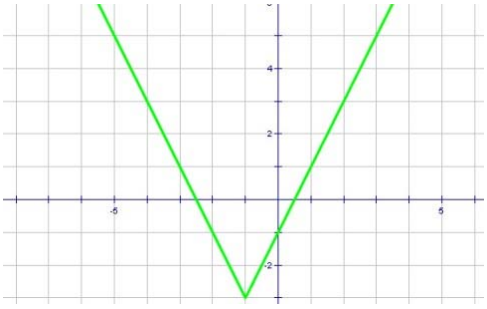
where,

- the sign indicates whether the graph opens up ("+" sign) or down ("−" sign).
- $|m|$is the absolute value of the slopes of the lines in the graph.
- (h, k) is the location of the vertex (i.e., the sharp point) in the graph.

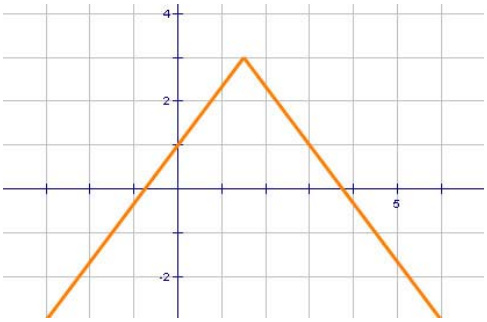
Examples:



Equation: $y = |x - 1| + 2$
Vertex = $(1, 2)$
$m = 1$; |slopes| $= 1$
Graph opens up



Equation: $y = |-2(x + 1)| - 3$
Vertex = $(-1, -3)$
$m = -2$; |slopes| $= 2$
Graph opens up

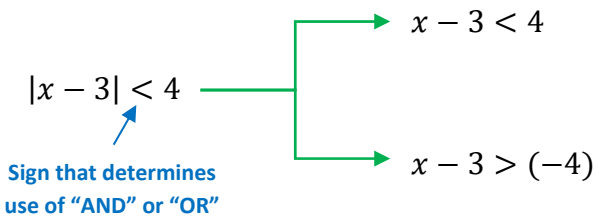


Equation: $y = -\left|\frac{4}{3}\left(x - \frac{3}{2}\right)\right| + 3$
Vertex = $(\frac{3}{2}, 3)$
$m = \frac{4}{3}$; |slopes| $= \frac{4}{3}$
Graph opens down

Algebra
Absolute Value Functions (cont'd)

Inequalities

Since a positive number and a negative number can have the same absolute value, inequalities involving absolute values must be broken into two separate equations. For example:



The first new equation is simply the original equation without the absolute value sign.

In the second new equation, two things change: (1) the sign flips, and (2) the value on the right side of the inequality changes its sign.

At this point the absolute value problem has converted into a pair of compound inequalities.

Equation 1	
Solve:	$x-3 < 4$
Step 1: Add 3	$+3 \quad +3$
Result:	$x < 7$

Equation 2	
Solve:	$x-3 > -4$
Step 1: Add 3	$+3 \quad +3$
Result:	$x > -1$

Next, we need to know whether to use "AND" or "OR" with the results. To decide which word to use, look at the sign in the inequality; then …

- Use the word **"AND"** with **"less thand"** signs.
- Use the word **"OR"** with **"greator"** signs.

Note: the English is poor, but the math is easier to remember with this trick!

The solution to the above absolute value problem, then, is the same as the solution to the following set of compound inequalities:

$x < 7 \quad and \quad x > (-1)$ The solution set is all x in the range (-1, 7)

Note: the solution set to this example is given in **"range" notation**. When using this notation,

- use parentheses ( ) whenever an endpoint is not included in the solution set, and
- use square brackets [ ] whenever an endpoint is included in the solution set.
- Always use parentheses ( ) with infinity signs ($-\infty \; or \; \infty$).

Examples:

The range: $x < 6 \; and \; x \geq 2$
Notation: $[2, 6)$

The range: $x \leq -2$
Notation: $(-\infty, -2]$

Algebra
Systems of Equations

A system of equations is a set of 2 or more equations for which we wish to determine all solutions which satisfy each equation. Generally, there will be the same number of equations as variables and a single solution to each variable will be sought. However, sometimes there is either no solution or there is an infinite number of solutions.

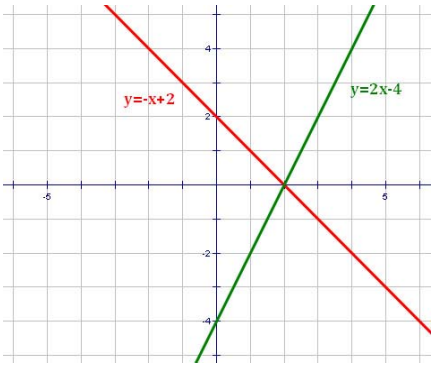
There are many methods available to solve a system of equations. We will show three of them below.

Graphing a Solution

In the simplest cases, a set of 2 equations in 2 unknowns can be solved using a graph. A single equation in two unknowns is a line, so two equations give us 2 lines. The following situations are possible with 2 lines:

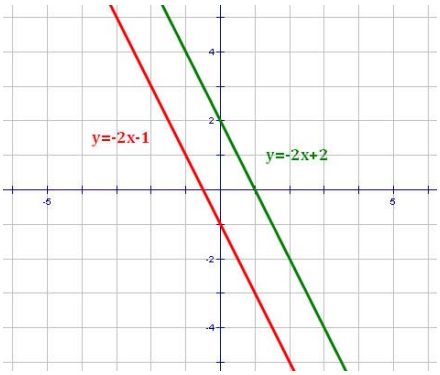
- **They will intersect.** In this case, the point of intersection is the only solution.
- **They will be the same line.** In this case, all points on the line are solutions (note: this is an infinite set).
- **They will be parallel but not the same line.** In this case, there are no solutions.

Examples



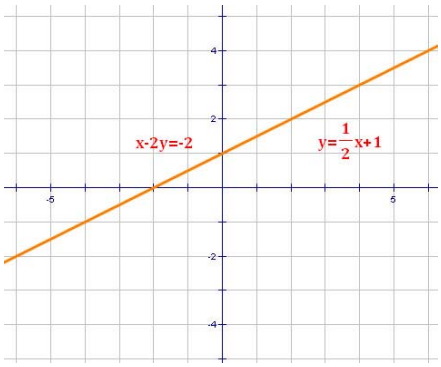
Solution Set:

The point of intersection can be read off the graph; the point (2,0).



Solution Set:

The empty set; these parallel lines will never cross.



Solution Set:

All points on the line. Although the equations look different, they actually describe the same line.

Algebra
Systems of Equations (cont'd)

Substitution Method

In the **Substitution Method**, we eliminate one of the variables by substituting into one of the equations its equivalent in terms of the other variable. Then we solve for each variable in turn and check the result. The steps in this process are illustrated in the example below.

Example: Solve for x and y if: $y = -x + 2$
and: $2x = 3y + 9.$

Step 1: Review the two equations. Look for a variable that can be substituted from one equation into the other. In this example, we see a single *"y"* in the first equation; this is a prime candidate for substitution.

We will substitute $(-x + 2)$ from the first equation for y in the second equation.

Step 2: Perform the substitution.

$2x = 3y + 9$ becomes: $2x = 3(-x + 2) + 9$

Step 3: Solve the resulting equation for the single variable that is left.

$$2x = 3(-x + 2) + 9$$
$$2x = -3x + 6 + 9$$
$$2x = -3x + 15$$

$$2x = -3x + 15$$
$$5x = 15$$
$$x = 3$$

Step 4: Substitute the known variable into one of the original equations to solve for the remaining variable.

$$y = -x + 2$$
$$y = -(3) + 2$$
$$y = -1$$

After this step, the solution is tentatively identified as: $x = 3, y = -1$, meaning the point $(3, -1)$.

Step 5: Check the result by substituting the solution into the equation not used in Step 4. If the solution is correct, the result should be a true statement. If it is not, you have made a mistake and should check your work carefully.

$$2x = 3y + 9$$
$$2(3) = 3(-1) + 9$$
$$6 = -3 + 9$$

Since this is a true mathematical statement, the solution $(3, -1)$ can be accepted as correct.

Algebra
Systems of Equations (cont'd)

Elimination Method

In the **Substitution Method**, we manipulate one or both of the equations so that we can add them and eliminate one of the variables. Then we solve for each variable in turn and check the result. This is an outstanding method for systems of equations with "ugly" coefficients. The steps in this process are illustrated in the example below. Note the flow of the solution on the page.

Example: Solve for x and y if: $y = -x + 2$
and: $2x = 3y + 9.$

Step 1: Re-write the equations in standard form.

$$x + y = 2$$
$$2x - 3y = 9$$

Step 2: Multiply each equation by a value selected so that, when the equations are added, a variable will be eliminated.

(Multiply by 2) → $2x + 2y = 4$

(Multiply by -1) → $-2x + 3y = -9$

Step 3: Add the resulting equations.

$$2x + 2y = 4$$
$$\underline{-2x + 3y = -9}$$
$$5y = -5$$

Step 4: Solve for the variable.

$$5y = -5$$
$$y = -1$$

Step 5: Substitute the result into one of the original equations and solve for the other variable.

$$y = -x + 2$$
$$-1 = -x + 2$$
$$x - 1 = 2$$
$$x = 3$$

Step 6: Check the result by substituting the solution into the equation not used in Step 5. If the solution is correct, the result should be a true statement. If it is not, you have made a mistake and should check your work.

$$2x = 3y + 9$$
$$2(3) = 3(-1) + 9$$
$$6 = -3 + 9$$

Since this is a true mathematical statement, the solution **(3, -1)** can be accepted as correct.

Algebra
Systems of Equations (cont'd)

Classification of Systems

There are two main classifications of systems of equations: Consistent vs. Inconsistent, and Dependent vs. Independent.

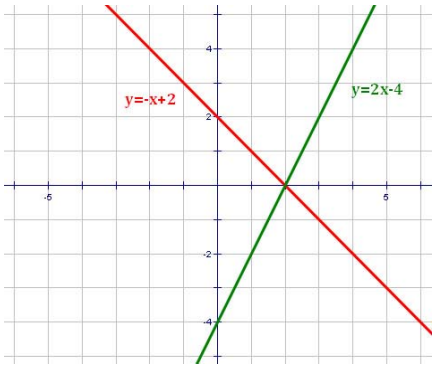
Consistent vs. Inconsistent

- **Consistent Systems** have one or more solutions.
- **Inconsistent Systems** have no solutions. When you try to solve an inconsistent set of equations, you often get to a point where you have an impossible statement, such as "$1 = 2$." This indicates that there is no solution to the system.

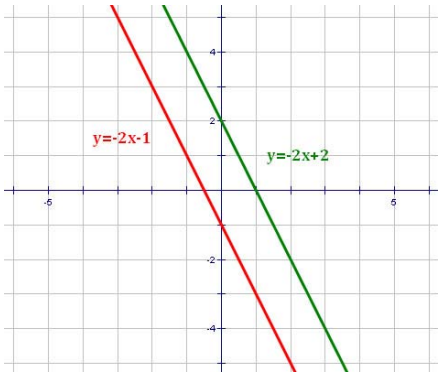
Dependent vs. Independent

- **Linearly Dependent Systems** have an infinite number of solutions. In **Linear Algebra**, a system is linearly dependent if there is a set of real numbers (not all zero) that, when they are multiplied by the equations in the system and the results are added, the final result is zero.
- **Linearly Independent Systems** have at most one solution. In **Linear Algebra**, a system is linearly independent if it is not linearly dependent. **Note:** some textbooks indicate that an independent system must have a solution. This is not correct; they can have no solutions (see the middle example below). *For more on this, see the next page.*

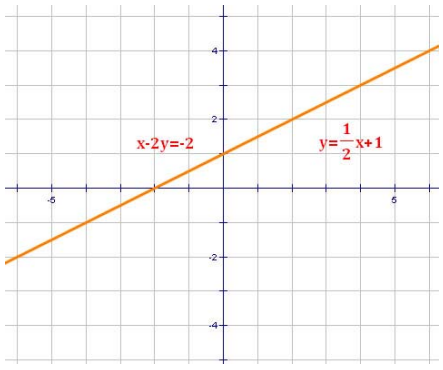
Examples



One Solution
Consistent
Independent



No Solution
Inconsistent
Independent



Infinite Solutions
Consistent
Dependent

ADVANCED

Algebra
Linear Dependence

Linear dependence is a concept from **Linear Algebra**, and is very useful in determining if solutions to complex systems of equations exist. Essentially, a system of functions $\boldsymbol{f_i}$ is defined to be **linearly dependent** if there is a set of real numbers $\boldsymbol{a_i}$ (not all zero), such that:

$a_1f_1 + a_2f_2 + a_3f_3 + \ldots + a_nf_n = 0$ or, in summation notation, $\sum_{i=1}^{n} a_if_i = 0$

If there is no set of real numbers $\boldsymbol{a_i}$, such that the above equations are true, the system is said to be **linearly independent**.

The expression $a_1f_1 + a_2f_2 + \cdots + a_nf_n$ is called a linear combination of the functions f_i. The importance of the concept of linear dependence lies in the recognition that a dependent system is redundant, i.e., the system can be defined with fewer equations. It is useful to note that a linearly dependent system of equations has a determinant of coefficients equal to 0.

Example:

Consider the following system of equations:

$$\boldsymbol{f_1 = 3x - 2y + z + 1}$$
$$\boldsymbol{f_2 = -x + y + 2z - 2}$$
$$\boldsymbol{f_3 = x + 5z - 3}$$

Notice that: $\boldsymbol{f_3 = f_1 + 2f_2}$. Therefore, the system is linearly dependent.

Checking the determinant of the coefficient matrix:

$$D = \begin{vmatrix} 3 & -2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 5 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} + 5\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} = 1(-5) + 0(7) + 5(1) = 0.$$

It should be noted that the fact that $\mathrm{D} = 0$ is sufficient to prove linear dependence only if there are no constant terms in the functions (e.g., if the problem involves vectors). If there are constant terms, it is also necessary that these terms combine "properly." There are additional techniques to test this, such as the use of augmented matrices and Gauss-Jordan Elimination.

Much of Linear Algebra concerns itself with sets of equations that are linearly independent. If the determinant of the coefficient matrix is non-zero, then the set of equations is linearly independent.

Algebra
Systems of Inequalities in Two Dimensions

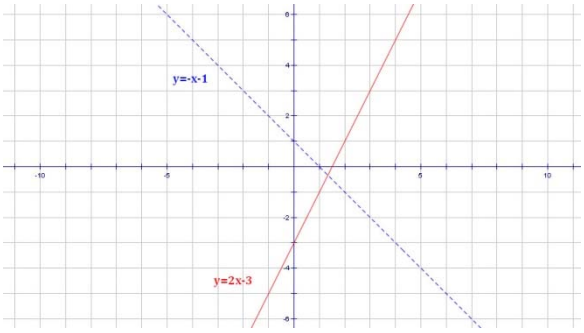
Systems of inequalities are sets of more than one inequality. To graph a system of inequalities, graph each inequality separately (including shading in the appropriate region). The solution set, then, is either the overlap of the regions of the separate inequalities ("AND" Systems) or the union of the regions of the separate inequalities ("OR" Systems).

Examples:

Graph the solution set of the following system of inequalities:

(a) $y \leq 2x - 3$ AND $y > -x + 1$ (b) $y \leq 2x - 3$ OR $y > -x + 1$

Step 1: Graph the underlying equations.



Step 2: Determine whether each line should be solid or dotted:

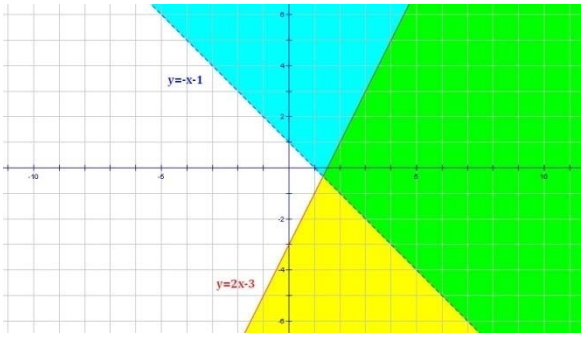
- $y \leq 2x - 3$ the ≤ sign contains "=", so the line is **solid**
- $y > -x + 1$ the > sign does not contain "=", so the line is **dotted**

Step 3: Determine the regions to be shaded based on the signs in the equations:

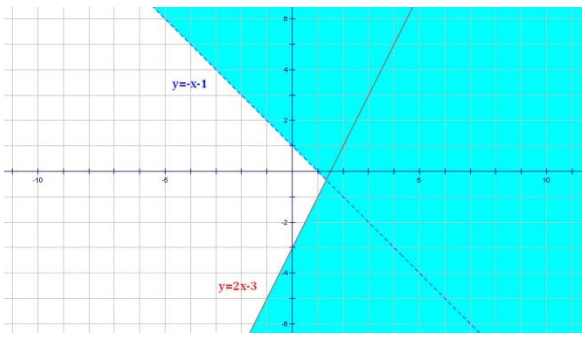
- $y \leq 2x - 3$ the ≤ sign indicates shading below the line
- $y > -x + 1$ the > sign indicates shading above the line

Step 4: Determine the final solution set.

(a) If the problem has an "AND" between the inequalities, the solution set is the overlap of the shaded areas (i.e., the green part in the graph below).



(b) If the problem has an "OR" between the inequalities, the solution set is the union of all of the shaded areas (i.e., the blue part in the graph below).



Algebra
Parametric Equations

Parametric Equations in 2 dimensions are functions that express each of the two key variables in terms of a one or more others. For example,

$$x = -t + 1$$
$$y = 2t - 3$$

Parametric equations are sometimes the most useful way to solve a problem.

Pythagorean Triples

As an example, the following parametric equations can be used to find Pythagorean Triples:

Let s, t be relatively prime integers and let $s > t$. Then, the following equations produce a set of integer values that satisfy the Pythagorean Theorem:

$$a = s^2 - t^2 \qquad b = 2st \qquad c = s^2 + t^2$$

Examples:

s	t	a	b	c	Pythagorean Relationship
3	2	5	12	13	$5^2 + 12^2 = 13^2$
4	3	7	24	25	$7^2 + 24^2 = 25^2$
5	2	21	20	29	$21^2 + 20^2 = 29^2$
5	3	16	30	34	$16^2 + 30^2 = 34^2$

Creating a Standard Equation from Parametric Equations

To create a standard equation from a set of parametric equations in two dimensions,

- Solve one parametric equation for t.
- Substitute this value of t into the other equation.
- Clean up the remaining expression as necessary.

Note: any other method of solving simultaneous equations can also be used for this purpose.

Example: Create a standard equation for the parametric equations:

$$x = -t + 1 \qquad y = 2t - 3$$

Solving for t in the first equation, we get:

$$t = -x + 1$$

Substituting into the second equation gives:

$$y = 2(-x + 1) - 3$$

Cleaning this up, we get the solution we seek:

$$y = -2x - 1$$

Algebra

Exponent Formulas

Word Description of Property	Math Description of Property	Limitations on variables	Examples
Product of Powers	$a^p \cdot a^q = a^{(p+q)}$		$x^4 \cdot x^3 = x^7$ $x^5 \cdot x^{-8} = x^{-3}$
Quotient of Powers	$\frac{a^p}{a^q} = a^{(p-q)}$	$a \neq 0$	$\frac{y^5}{y^2} = y^3$
Power of a Power	$(a^p)^q = a^{(p \cdot q)}$		$(z^4)^3 = z^{12}$ $(x^{-3})^{-5} = x^{15}$
Anything to the zero power is 1	$a^0 = 1$	$a \neq 0$	$91^0 = 1$ $(xyz^3)^0 = 1$, if $x, y, z \neq 0$
Negative powers generate the reciprocal of what a positive power generates	$a^{(-p)} = \frac{1}{a^p}$	$a \neq 0$	$x^{(-3)} = \frac{1}{x^3}$ $\left(\frac{1}{x}\right)^{-5} = x^5$
Power of a product	$(a \cdot b)^p = a^p \cdot b^p$		$(3y)^3 = 27y^3$ $[(x+1)z]^4 = (x+1)^4 z^4$
Power of a quotient	$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$	$b \neq 0$	$\left(\frac{x}{4}\right)^3 = \frac{x^3}{64}$
Converting a root to a power	$\sqrt[n]{a} = a^{(1/n)}$	$n \neq 0$	$\sqrt{x} = x^{1/2}$

Algebra
Scientific Notation

Format

A number in scientific notation has two parts:

- A number which is at least 1 and is less than 10 (i.e., it must have only one digit before the decimal point). This number is called the **coefficient**.
- A power of 10 which is multiplied by the first number.

Here are a few **examples** of regular numbers expressed in scientific notation.

$$32 = 3.2\ x\ 10^{1} \qquad 0.00034 = 3.4\ x\ 10^{-4} \qquad 1 = 1\ x\ 10^{0}$$
$$1{,}420{,}000 = 1.42\ x\ 10^{6} \qquad 1000 = 1\ x\ 10^{3} \qquad -450 = -4.5\ x\ 10^{2}$$

How many digits? How many zeros?

There are a couple of simple rules for converting from scientific notation to a regular number or for converting from a regular number to scientific notation:

- If a regular number is less than 1, the exponent of 10 in scientific notation is negative. The number of leading zeros in the regular number is equal to the absolute value of this exponent. In applying this rule, you must count the zero before the decimal point in the regular number. Examples:

Original Number	Action	Conversion
0.00034	Count 4 zeros	3.4×10^{-4}
6.234×10^{-8}	Add 8 zeros before the digits	0.000 000 062 34

- If the number is greater than 1, the number of digits after the first one in the regular number is equal to the exponent of 10 in the scientific notation.

Original Number	Action	Conversion
4,800,000	Count 6 digits after the "4"	4.8×10^{6}
9.6×10^{3}	Add 3 digits after the "9"	9,600

- As a general rule, multiplying by powers of 10 moves the decimal point one place for each power of 10.
    - Multiplying by positive powers of 10 moves the decimal to the right.
    - Multiplying by negative powers of 10 moves the decimal to the left.

Algebra
Adding and Subtracting with Scientific Notation

When adding or subtracting numbers in scientific notation:

- Adjust the numbers so they have the same power of 10. This works best if you adjust the representation of the smaller number so that it has the same power of 10 as the larger number. To do this:
  - Call the difference between the exponents of 10 in the two numbers *"n"*.
  - Raise the power of 10 of the smaller number by *"n"*, and
  - Move the decimal point of the coefficient of the smaller number *"n"* places to the left.
- Add the coefficients, keeping the power of 10 unchanged.
- If the result is not in scientific notation, adjust it so that it is.
  - If the coefficient is at least 1 and less than 10, the answer is in the correct form.
  - If the coefficient is 10 or greater, increase the exponent of 10 by 1 and move the decimal point of the coefficient one space to the left.
  - If the coefficient is less than 1, decrease the exponent of 10 by 1 and move the decimal point of the coefficient one space to the right.

Examples:

$$\begin{array}{rcr} 3.2\ x\ 10^{3} & \longrightarrow & 0.32\ x\ 10^{4} \\ +9.9\ x\ 10^{4} & \longrightarrow & +9.90\ x\ 10^{4} \\ \hline & & 10.22\ x\ 10^{4} \end{array}$$

$$= 1.022\ x\ 10^{5}$$

Explanation: A conversion of the smaller number is required prior to adding because the exponents of the two numbers are different. After adding, the result is no longer in scientific notation, so an extra step is needed to convert it into the appropriate format.

$$\begin{array}{rcr} 6.1\ x\ 10^{-2} & \longrightarrow & 6.1\ x\ 10^{-2} \\ +2.3\ x\ 10^{-2} & \longrightarrow & +2.3\ x\ 10^{-2} \\ \hline & & 8.4\ x\ 10^{-2} \end{array}$$

Explanation: No conversion is necessary because the exponents of the two numbers are the same. After adding, the result is in scientific notation, so no additional steps are required.

$$\begin{array}{rcr} 1.2\ x\ 10^{-8} & \longrightarrow & 1.20\ x\ 10^{-8} \\ -4.5\ x\ 10^{-9} & \longrightarrow & -0.45\ x\ 10^{-8} \\ \hline & & 0.75\ x\ 10^{-8} \end{array}$$

$$= 7.5\ x\ 10^{-9}$$

Explanation: A conversion of the smaller number is required prior to subtracting because the exponents of the two numbers are different. After subtracting, the result is no longer in scientific notation, so an extra step is needed to convert it into the appropriate format.

Algebra
Multiplying and Dividing with Scientific Notation

When multiplying or dividing numbers in scientific notation:

- Multiply or divide the coefficients.
- Multiply or divide the powers of 10. Remember that this means adding or subtracting the exponents while keeping the base of 10 unchanged.
    - If you are multiplying, add the exponents of 10.
    - If you are dividing, subtract the exponents of 10.
- If the result is not in scientific notation, adjust it so that it is.
    - If the coefficient is at least 1 and less than 10, the answer is in the correct form.
    - If the coefficient is 10 or greater, increase the exponent of 10 by 1 and move the decimal point of the coefficient one space to the left.
    - If the coefficient is less than 1, decrease the exponent of 10 by 1 and move the decimal point of the coefficient one space to the right.

Examples:

$$\begin{array}{r} 4\ x\ 10^{6} \\ \cdot\ 5\ x\ 10^{-4} \\ \hline 20\ x\ 10^{2} \\ = 2.0\ x\ 10^{3} \end{array}$$

Explanation: The coefficients are multiplied and the exponents are added. After multiplying, the result is no longer in scientific notation, so an extra step is needed to convert it into the appropriate format.

$$\begin{array}{r} 1.2\ x\ 10^{-2} \\ \cdot\ 2.0\ x\ 10^{-6} \\ \hline 2.4\ x\ 10^{-8} \end{array}$$

Explanation: The coefficients are multiplied and the exponents are added. After multiplying, the result is in scientific notation, so no additional steps are required.

$$\begin{array}{r} 3.3\ x\ 10^{4} \\ \div\ 5.5\ x\ 10^{-2} \\ \hline 0.6\ x\ 10^{6} \\ = 6.0\ x\ 10^{5} \end{array}$$

Explanation: The coefficients are divided and the exponents are subtracted. After dividing, the result is no longer in scientific notation, so an extra step is needed to convert it into the appropriate format.

Algebra
Introduction to Polynomials

What is a Polynomial?

A polynomial is an expression that can be written as a term or a sum of terms, each of which is the product of a scalar (the coefficient) and a series of variables. Each of the terms is also called a monomial.

Examples (all of these are polynomials):

Monomial	$3x$	$-4x^{12}y^3$
Binomial	$2x + 8$	$15xyz^7 - 12xyz$
Trinomial	$x^2 - 6x + 9$	$x^5 + 7x^4 - 3x^3$
Other	$x^4 - 4x^3 + 6x^2 - 4x + 1$	$-2xyz + 6xz^2 + 3yz - 8xz + 2z^5$

Definitions:

Scalar: A real number.

Monomial: Polynomial with one term.

Binomial: Polynomial with two terms.

Trinomial: Polynomial with three terms.

Degree of a Polynomial

The degree of a monomial is the sum of the exponents on its variables.

The degree of a polynomial is the highest degree of any of its monomial terms.

Examples:

Polynomial	Degree
6	0
$3x$	1
$3xyz$	3

Polynomial	Degree
$3x^3yz + 6x^2yz^3$	6
$15xyz^7 - 12xyz$	9
$x^5 + 7x^4 - 3x^3$	5

Algebra
Adding and Subtracting Polynomials

Problems asking the student to add or subtract polynomials are often written in linear form:

$$\text{Add: } (3x^3 + 2x - 4) + (-2x^2 + 4x + 6)$$

The problem is much more easily solved if the problem is written in column form, with each polynomial written in standard form.

Definitions

Standard Form: A polynomial in standard form has its terms written from highest degree to lowest degree from left to right.

Example: The standard form of $(x + 3x^2 + 4)$ is $(3x^2 + x + 4)$

Like Terms: Terms with the same variables raised to the same powers. Only the numerical coefficients are different.

Example: $2xz^5$, $-6xz^5$, and xz^5 are like terms.

Addition and Subtraction Steps

Step 1: Write each polynomial in standard form. Leave blank spaces for missing terms. For example, if adding $(3x^3 + 2x - 4)$, leave space for the missing x^2-term.

Step 2: If you are subtracting, change the sign of each term of the polynomial to be subtracted and add instead. Adding is much easier than subtracting.

Step 3: Place the polynomials in column form, being careful to line up like terms.

Step 4: Add the polynomials.

Examples:

Add: $(3x^3 + 2x - 4) + (-2x^2 + 4x + 6)$

Solution:

$$\begin{array}{rr} & 3x^3 \quad + 2x - 4 \\ + & -2x^2 + 4x + 6 \\ \hline & 3x^3 - 2x^2 + 6x + 2 \end{array}$$

Subtract: $(3x^3 + 2x - 4) - (-2x^2 + 4x + 6)$

Solution:

$$\begin{array}{rr} & 3x^3 \quad + 2x - 4 \\ + & 2x^2 - 4x - 6 \\ \hline & 3x^3 + 2x^2 - 2x - 10 \end{array}$$

Algebra
Multiplying Binomials

The three methods shown below are equivalent. Use whichever one you like best.

FOIL Method

FOIL stands for First, Outside, Inside, Last. To multiply using the FOIL method, you make four separate multiplications and add the results.

Example: Multiply $(2x+3)\cdot(3x-4)$

First: $2x\cdot 3x = 6x^2$

Outside: $2x\cdot(-4) = -8x$

Inside: $3\cdot(3x) = 9x$

Last: $3\cdot(-4) = -12$

The result is obtained by adding the results of the 4 separate multiplications.

F O I L

$$(2x+3)\cdot(3x-4) = 6x^2 - 8x + 9x - 12$$
$$= 6x^2 + x - 12$$

Box Method

The Box Method is pretty much the same as the FOIL method, except for the presentation. In the box method, a 2x2 array of multiplications is created, the 4 multiplications are performed, and the results are added.

Example: Multiply $(2x+3)\cdot(3x-4)$

Multiply	**3x**	**−4**
2x	$6x^2$	$-8x$
+3	$9x$	-12

The result is obtained by adding the results of the 4 separate multiplications.

$$(2x+3)\cdot(3x-4) = 6x^2 - 8x + 9x - 12$$
$$= 6x^2 + x - 12$$

Stacked Polynomial Method

A third method is to multiply the binomials like you would multiply 2-digit numbers. The name comes from how the two polynomials are placed in a "stack" in preparation for multiplication.

Example: Multiply $(2x+3)\cdot(3x-4)$

$$\begin{array}{r} (2x+3) \\ \cdot\ (3x-4) \\ \hline -8x - 12 \\ 6x^2 + 9x \\ \hline 6x^2 + x - 12 \end{array}$$

Algebra
Multiplying Polynomials

If the polynomials to be multiplied contain more than two terms (i.e., they are larger than binomials), the FOIL Method will not work. Instead, either the Box Method or the Stacked Polynomial Method should be used. Notice that each of these methods is essentially a way to apply the distributive property of multiplication over addition.

The methods shown below are equivalent. Use whichever one you like best.

Box Method

The Box Method is the same for larger polynomials as it is for binomials, except the box is bigger. An array of multiplications is created; the multiplications are performed; and like terms are added.

Example: Multiply $(x^3 - 2x^2 + 2x + 3) \cdot (2x^2 - 3x - 4)$

Multiply	$\mathbf{2x^2}$	$\mathbf{-3x}$	$\mathbf{-4}$
$\mathbf{x^3}$	$2x^5$	$-3x^4$	$-4x^3$
$\mathbf{-2x^2}$	$-4x^4$	$+6x^3$	$+8x^2$
$\mathbf{+2x}$	$+4x^3$	$-6x^2$	$-8x$
$\mathbf{+3}$	$+6x^2$	$-9x$	-12

Results:

$$(x^3 - 2x^2 + 2x + 3) \cdot (2x^2 - 3x - 4)$$

$$= \quad 2x^5$$

$$-4x^4 - 3x^4$$

$$+4x^3 + 6x^3 - 4x^3$$

$$+6x^2 - 6x^2 + 8x^2$$

$$-9x - 8x$$

Stacked Polynomial Method

In the Stacked Polynomial Method, the polynomials are multiplied using the same technique to multiply multi-digit numbers One helpful tip is to place the smaller polynomial below the larger one in the stack.

Results:

$$\begin{array}{r} x^3 - 2x^2 + 2x + 3 \\ \cdot \quad 2x^2 - 3x - 4 \\ \hline -4x^3 + 8x^2 - 8x - 12 \\ -3x^4 + 6x^3 - 6x^2 - 9x \\ 2x^5 - 4x^4 + 4x^3 + 6x^2 \\ \hline 2x^5 - 7x^4 + 6x^3 + 8x^2 - 17x - 12 \end{array}$$

Algebra
Dividing Polynomials

Dividing polynomials is performed much like dividing large numbers long-hand.

Long Division Method

This process is best described by example:

Example: $(2x^3 + 5x^2 + x - 2) \div (x + 2)$

Step 1: Set up the division like a typical long hand division problem.

$$x + 2 \overline{)\ 2x^3 + 5x^2 + x - 2}$$

Step 2: Divide the leading term of the dividend by the leading term of the divisor. Place the result above the term of like degree of the dividend.

$(2x^3) \div x = 2x^2$

$$\begin{array}{r} 2x^2 \\ x + 2 \overline{)\ 2x^3 + 5x^2 + x - 2} \end{array}$$

Step 3: Multiply the new term on top by the divisor and subtract from the dividend.

$$(2x^2)(x + 2) = 2x^3 + 4x^2$$

$$\begin{array}{r} 2x^2 \\ x + 2 \overline{)\ 2x^3 + 5x^2 + x - 2} \\ 2x^3 + 4x^2 \\ \hline x^2 + x - 2 \end{array}$$

Step 4: Repeat steps 2 and 3 on the remainder of the division until the problem is completed.

This process results in the final answer appearing above the dividend, so that:

$(2x^3 + 5x^2 + x - 2) \div (x + 2) = 2x^2 + x - 1$

$$\begin{array}{r} 2x^2 + x - 1 \\ x + 2 \overline{)\ 2x^3 + 5x^2 + x - 2} \\ 2x^3 + 4x^2 \\ \hline x^2 + x - 2 \\ x^2 + 2x \\ \hline -x - 2 \\ -x - 2 \\ \hline 0 \end{array}$$

Remainders

If there were a remainder, it would be appended to the result of the problem in the form of a fraction, just like when dividing integers. For example, in the problem above, if the remainder were 3, the fraction $\frac{3}{x+2}$ would be added to the result of the division. $(2x^3 + 5x^2 + x + 1) \div (x + 2) = 2x^2 + x - 1 + \frac{3}{x+2}$

Alternatives

This process can be tedious. Fortunately, there are better methods for dividing polynomials than long division. These include **Factoring**, which is discussed next and elsewhere in this Guide, and **Synthetic Division**, which is discussed in the chapter on Polynomials – Intermediate.

Algebra
Factoring Polynomials

Polynomials cannot be divided in the same way numbers can. In order to divide polynomials, it is often useful to factor them first. Factoring involves extracting simpler terms from the more complex polynomial.

Greatest Common Factor

The Greatest Common Factor of the terms of a polynomial is determined as follows:

Step 1: Find the Greatest Common Factor of the coefficients.

Step 2: Find the Greatest Common Factor for each variable. This is simply each variable taken to the lowest power that exists for that variable in any of the terms.

Step 3: Multiply the GCF of the coefficients by the GCF for each variable.

Example:

Find the GCF of $(18x^5y^6z + 42x^3y^7z^3 + 30x^8z^6)$

The GCF of the coefficients and each variable are shown in the box to the right. The GCF of the polynomial is the product of the four individual GCFs.

$$\text{GCF}\,(18, 42, 30) = 6$$
$$\text{GCF}\,(x^5, x^3, x^8) = x^3$$
$$\text{GCF}\,(y^6, y^7, 1) = 1$$
$$\text{GCF}\,(z,\ z^3, z^6) = z$$
$$\text{So, GCF (polynomial)} = 6x^3z$$

Factoring Steps

Step 1: Factor out of all terms the GCF of the polynomial.

Step 2: Factor out of the remaining polynomial any binomials that can be extracted.

Step 3: Factor out of the remaining polynomial any trinomials that can be extracted.

Step 4: Continue this process until no further simplification is possible.

Note: Typically only steps 1 and 2 are needed in high school algebra problems.

Examples:

Factor: $3x^4y - 18x^3y + 27x^2y$

$= 3x^2y\,(x^2 - 6x + 9)$

$= 3x^2y\,(x - 3)^2$

The factoring of the blue trinomial (2nd line) into the square of a binomial is the result of recognizing the special form it represents. Special forms are shown on the next two pages.

Factor: $6x^3y^3 - 24xy^3$

$= 6xy^3\,(x^2 - 4)$

$= 6xy^3\,(x + 2)(x - 2)$

The factoring of the blue binomial (2nd line) into binomials of lower degree is the result of recognizing the special form it represents. Special forms are shown on the next two pages.

Algebra
Special Forms of Quadratic Functions

It is helpful to be able to recognize a couple special forms of quadratic functions. In particular, if you can recognize perfect squares and differences of squares, your work will become easier and more accurate.

Perfect Squares

Perfect squares are of the form: $\boldsymbol{a^2 + 2ab + b^2 = (a + b)^2}$

$\boldsymbol{a^2 - 2ab + b^2 = (a - b)^2}$

Identification and Solution

The following steps allow the student to identify and solve a trinomial that is a perfect square:

Step 1: Notice the first term of the trinomial is a square. Take its square root.

Step 2: Notice the last term of the trinomial is a square. Take its square root.

Step 3: Multiply the results of the first 2 steps and double that product. If the result is the middle term of the trinomial, the expression is a perfect square.

Step 4: The binomial in the solution is the sum or difference of the square roots calculated in steps 1 and 2. The sign between the terms of the binomial is the sign of the middle term of the trinomial.

Example:

$\boldsymbol{4x^2 - 12xy + 9y^2}$

$\sqrt{\boldsymbol{4x^2}} = \pm\boldsymbol{2x}$ $\sqrt{\boldsymbol{9y^2}} = \pm\boldsymbol{3y}$

Notice that the middle term is double the product of the two square roots ($\boldsymbol{2x}$ and $\boldsymbol{3y}$). This is a telltale sign that the expression is a perfect square.

Identify the trinomial as a perfect square:

- Take the square roots of the first and last terms. They are $2x$ and $3y$.
- Test the middle term. Multiply the roots from the previous step, then double the result: $(2x \cdot 3y) \cdot 2 = 12xy$. The result (with a "$-$" sign in front) is the middle term of the original trinomial. Therefore, the expression is a perfect square.

To express the trinomial as the square of a binomial:

- The square roots of the first and last terms ($2x$ and $3y$) make up the binomial we seek.
- We may choose the sign of the first term, so let's choose the "$+$" sign.
- Having chosen the "$+$" sign for the first term, the second term of the binomial takes the sign of the middle term of the original trinomial ("$-$"). Therefore, the result is:

$$\boldsymbol{4x^2 - 12xy + 9y^2 = (2x - 3y)^2}$$

Algebra
Special Forms of Quadratic Functions

Differences of Squares

Differences of squares are of the form: $a^2 - b^2 = (a + b) \cdot (a - b)$

These are much easier to recognize than the perfect squares because there is no middle term to consider. Notice why there is no middle term:

$$(a + b) \cdot (a - b) = a^2 \underbrace{+ ab - ab}_{\text{these two terms cancel}} - b^2 = a^2 - b^2$$

Identification

To see if an expression is a difference of squares, you must answer "yes" to four questions:

1. Are there only two terms?
2. Is there a "$-$" sign between the two terms?
3. Is the first term a square? If so, take its square root.
4. Is the second term a square? If so, take its square root.

The solution is the product of a) the sum of the square roots in questions 3 and 4, and b) the difference of the square roots in steps 3 and 4.

Note: *A telltale sign of when an expression might be the difference of 2 squares is when the coefficients on the variables are squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, etc.*

Examples:

(1) $4x^2 - 25y^2 = (2x + 5y) \cdot (2x - 5y)$

(2) $x^2 - 49 = (x + 7) \cdot (x - 7)$

(3) $81 - 9z^2 = (9 + 3z) \cdot (9 - 3z)$

(4) $\frac{x^2}{9} - \frac{y^2}{16} = \left(\frac{x}{3} + \frac{y}{4}\right) \cdot \left(\frac{x}{3} - \frac{y}{4}\right)$

ADVANCED: Over the field of complex numbers, it is also possible to factor the sum of 2 squares:

$$a^2 + b^2 = (a + bi) \cdot (a - bi)$$

This is not possible over the field of real numbers.

Algebra
Factoring Trinomials – Simple Case Method

A common problem in Elementary Algebra is the factoring of a trinomial that is neither a perfect square nor a difference of squares.

Consider the simple case where the coefficient of x^2 is 1. The general form for this case is:

$$(x+p)\cdot(x+q) = x^2 + (p+q)x + (pq)$$

sign 1 — coefficient of x — sign 2 — constant

In order to simplify the illustration of factoring a polynomial where the coefficient of x^2 is 1, we will use the orange descriptors above for the components of the trinomial being factored.

Simple Case Method

Step 1: Set up parentheses for a pair of binomials. Put "x" in the left hand position of each binomial.

Step 2: Put **sign 1** in the middle position in the left binomial.

Step 3: Multiply **sign 1** and **sign 2** to get the sign for the right binomial. Remember:

$(+)\cdot(+) = (+)$ $\quad$ $(-)\cdot(-) = (+)$
$(+)\cdot(-) = (-)$ $\quad$ $(-)\cdot(+) = (-)$

Step 4: Find two numbers that:

(a) Multiply to get the **constant**, and

(b) Add to get the **coefficient of x**

Fill in:
___ · ___ = ___
___ + ___ = ___

Step 5: Place the numbers in the binomials so that their signs match the signs from Steps 2 and 3. **This is the final answer.**

Step 6: Check your work by multiplying the two binomials to see if you get the original trinomial.

Example: Factor $x^2 - 3x - 28$

$= (x \quad)\cdot(x \quad)$

$= (x - \quad)\cdot(x \quad)$

$= (x - \quad)\cdot(x + \quad)$

The numbers we seek are 4 and -7 because:
$4\cdot(-7) = -28$, and
$4 - 7 = -3$

$= (x-7)\cdot(x+4)$

$(x-7)\cdot(x+4)$
$= x^2 + 4x - 7x - 28$
$= x^2 - 3x - 28$

✓

Algebra
Factoring Trinomials – AC Method

There are times when the simple method of factoring a trinomial is not sufficient. Primarily this occurs **when the coefficient of x^2 is not 1**. In this case, you may use the AC method presented here, or you may use either the brute force method or the quadratic formula method (described on the next couple of pages).

AC Method

The AC Method derives its name from the first step of the process, which is to multiply the values of "a" and "c" from the general form of the quadratic equation: $y = ax^2 + bx + c$

Step 1: Multiply the values of "a" and "c".

Step 2: Find two numbers that:

(a) Multiply to get the **value of ac**, and

(b) Add to get the **coefficient of x**

Fill in:

__ · __ = ___

__ + __ = ___

Step 3: Split the middle term into two terms, with coefficients equal to the values found in Step 2. (Tip: if only one of the coefficients is negative, put that term first.)

Step 4: Group the terms into pairs.

Step 5: Factor each pair of terms.

Step 6: Use the distributive property to combine the multipliers of the common term. **This is the final answer.**

Step 7: Check your work by multiplying the two binomials to see if you get the original trinomial.

Example: Factor $6x^2 - x - 2$

$6x^2 - x - 2$

-12

$(-4) \cdot 3 = -12$

$(-4) + 3 = -1$

$6x^2 - 4x + 3x - 2$

$(6x^2 - 4x) + (3x - 2)$

$2x(3x - 2) + 1(3x - 2)$

$= (2x + 1) \cdot (3x - 2)$

$(2x + 1) \cdot (3x - 2)$

$= 6x^2 - 4x + 3x - 2$

$= 6x^2 - x - 2$

✓

Algebra
Factoring Trinomials – Brute Force Method

When the coefficient of x^2 is not 1, the factoring process becomes more difficult. There are a number of methods that can be used in this case.

If the question being asked is to find roots of the equation, and not to factor it, the student may want to use the quadratic formula whenever the coefficient of x^2 is not 1. Even if you are required to factor, and not just find roots, the quadratic formula may be a viable approach.

Brute Force Method

This method is exactly what it sounds like. Multiple equations are possible and you must try each of them until you find the one that works. Here are the steps to finding which equations are candidate solutions:

Step 1: Find all sets of whole numbers that multiply to get the coefficient of the first term in the trinomial. If the first term is positive, you need only consider positive factors.

Step 2: Find all sets of whole numbers that multiply to get the coefficient of the last term in the trinomial. You must consider both positive and negative factors.

Step 3: Create all possible products of binomials that contain the whole numbers found in the first two steps.

Step 4: Multiply the binomial pairs until you find one that results in the trinomial you are trying to factor.

Step 5: Identify the correct solution.

Example: Factor $4x^2 + 4x - 3$

Combinations that produce a product of 4 are:

1 and 4 or 2 and 2

Combinations that produce a product of -3 are:

-1 and 3 or 1 and -3

$$(x-1)(4x+3)$$
$$(x+1)(4x-3)$$
$$(x+3)(4x-1)$$
$$(x-3)(4x+1)$$
$$(2x-1)(2x+3)$$
$$(2x+1)(2x-3)$$

$$(x-1)(4x+3) = 4x^2 - x - 3$$
$$(x+1)(4x-3) = 4x^2 + x - 3$$
$$(x+3)(4x-1) = 4x^2 + 11x - 3$$
$$(x-3)(4x+1) = 4x^2 - 11x - 3$$
$$(2x-1)(2x+3) = 4x^2 + 4x - 3$$
$$(2x+1)(2x-3) = 4x^2 - 4x - 3$$

$$\mathbf{(2x-1)(2x+3)} = 4x^2 + 4x - 3$$

Notice the patterns in the candidate solutions in Step 4. Each pair of equations is identical except for the sign of the middle term in the product. Therefore, you can cut your work in half by considering only one of each pair until you see a middle term coefficient that has the right absolute value. If you have everything right but the sign of the middle term, switch the signs in the binomials to obtain the correct solution. **Remember to check your work!**

Algebra
Factoring Trinomials – Quadratic Formula Method

Quadratic Formula Method

The Quadratic Formula is designed specifically to find roots of a second degree equation. However, it can also be used as a back-door method to factor equations of second degree. The steps are:

Step 1: Apply the quadratic formula to determine the roots of the equation.

Step 2: Put each root into the form: $(x - root) = 0$.

Step 3: Show the two $(x - root)$ binomials as a product. Note that these binomials may contain fractions. We will eliminate the fractions, if possible, in the next step.

Step 4: Multiply the binomials in Step 3 by the coefficient of x^2 the following way:

(a) Break the coefficient of x^2 into its prime factors.

(b) Allocate the prime factors to the binomials in a way that eliminates the fractions.

Step 5: Check your work.

Example:

Factor: $\boldsymbol{4x^2 + 4x - 3}$

Step 1: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{4^2-4(4)(-3)}}{2(4)} = \frac{-4 \pm \sqrt{64}}{8} = \frac{-4 \pm 8}{8} = -\frac{3}{2}$ or $\frac{1}{2}$

Step 2: The two equations containing roots are: $\left(x + \frac{3}{2}\right) = 0$ and $\left(x - \frac{1}{2}\right) = 0$.

Step 3: $\left(x + \frac{3}{2}\right)\left(x - \frac{1}{2}\right)$

Step 4: The coefficient of x^2 in the original equation is 4, and $4 = 2 \cdot 2$. An inspection of the binomials in Step 3 indicates we need to multiply each binomial by 2 in order to eliminate the fractions:

$$2 \cdot \left(x + \frac{3}{2}\right) = (2x + 3) \qquad \text{and} \qquad 2 \cdot \left(x - \frac{1}{2}\right) = (2x - 1)$$

So that: $4x^2 + 4x - 3 = \boldsymbol{(2x + 3) \cdot (2x - 1)}$ in factored form

Step 5: Check (using FOIL) $(2x + 3) \cdot (2x - 1) = 4x^2 - 2x + 6x - 3 = 4x^2 + 4x - 3$, which is the equation we were trying to factor.

✓

Algebra
Solving Equations by Factoring

There are a number of reasons to factor a polynomial in algebra; one of the most common reasons is to find the zeros of the polynomial. A "zero" is a domain value (e.g., x-value) for which the polynomial generates a value of zero. Each zero is a solution of the polynomial.

In factored form, it is much easier to find a polynomial's zeros. Consider the following:

$(x-2)(x+4)(x-8)(x-\pi)(x+3)$ is the factored form of a polynomial.

If a number of items are multiplied together, the result is zero whenever any of the individual items is zero. This is true for constants and for polynomials. Therefore, if any of the factors of the polynomial has a value of zero, then the whole polynomial must be zero. We use this fact to find zeros of polynomials in factored form.

Example 1:

Find the zeros of $y = (x-2)(x+4)(x-8)(x-\pi)(x+3)$.

Step 1: Set the equation equal to zero.

$$(x-2)(x+4)(x-8)(x-\pi)(x+3) = 0$$

Step 2: The whole equation is zero whenever any of its factors is zero. For the example, this occurs when:

$(x-2) = 0$, or
$(x+4) = 0$, or
$(x-8) = 0$, or
$(x-\pi) = 0$, or
$(x+3) = 0$

The solution set, then, is:

$$x = \{2, -4, 8, \pi, -3\}$$

or, more conventionally, the x-values are put in numerical order from smallest to largest:

$$x = \{-4, -3, 2, \pi, 8\}$$

Set Notation: We may list the set of solutions to a problem by placing the solutions in braces {}, separated by commas.

Example 2:

Find the zeros of $y = x^2 - 7x + 6$

$$x^2 - 7x + 6 = 0$$
$$(x-6)(x-1) = 0$$

$(x-6) = 0$ $\quad$ $(x-1) = 0$

$x = 6$ $\quad$ $x = 1$

The solution set contains the two domain values that make the original equation zero, namely:

$$x = \{1, 6\}$$

Algebra
Introduction to Quadratic Functions

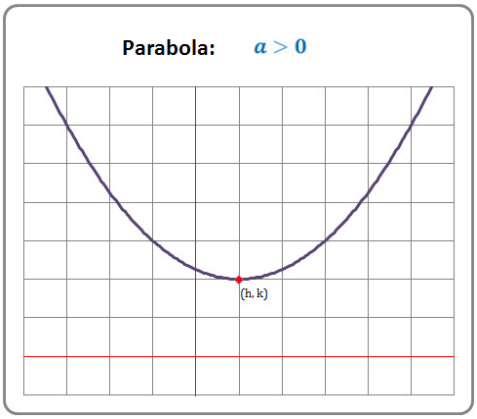
Standard Form of a Quadratic Function

The Standard Form of a second degree polynomial is:

$$\mathbf{y = ax^2 + bx + c} \qquad \text{with } a \neq 0$$

An equation of this form is called a quadratic function.

The graph of this equation is called a parabola.

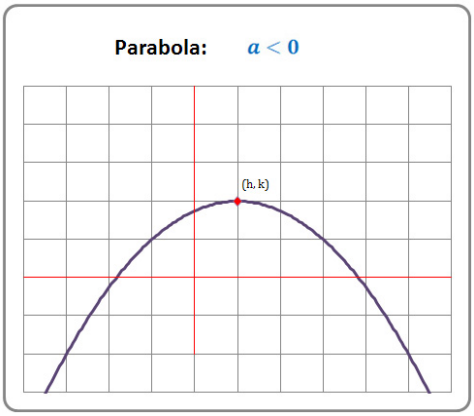


Up or down?

The direction in which the parabola opens on a graph is based on the sign ($+$ or $-$) of a in the equation.

- If $a > 0$, the parabola points down and it opens up.
- If $a < 0$, the parabola points up and it opens down.

If you forget this rule, just remember that "up or down" depends on the sign of a, and do a quick graph of $y = x^2$, where $(a = +1)$ on your paper.



Vertex and Axis of Symmetry

In Standard Form, the vertex of the parabola has coordinates: $\left(\frac{-b}{2a}, y\right)$ where "y" is calculated by substituting $\left(\frac{-b}{2a}\right)$ for "x" in the equation. The vertex is either the highest point on the graph (called a maximum) or the lowest point on the graph (called a minimum). It also lies on the axis of symmetry of the graph.

The equation $\boldsymbol{x} = \left(\frac{-b}{2a}\right)$ is called the axis of symmetry of the parabola.

Vertex Form of a Quadratic Function

A second useful form of a quadratic function is based on the vertex, and is called Vertex Form:

$$\mathbf{y = a(x - h)^2 + k} \qquad \text{where (h, k) is the vertex of the parabola}$$

It is possible to convert from Standard Form to Vertex Form and from Vertex Form to Standard Form. Both are equally correct.

Algebra
Completing the Square

A very useful method for solving quadratic equations is Completing the Square. In fact, this is the method used to derive the quadratic formula from the general quadratic equation in Standard Form. The steps involved in Completing the Square and an example are provided below:

Consider the starting equation: $\boldsymbol{ax^2 + bx + c = 0}$

Step 1: Modify the equation so that the coefficient of x^2 is 1. To do this, simply divide the whole equation by the value of "a".

Example: Consider the equation: $3x^2 + 18x - 21 = 0$

Divide by 3 to get: $x^2 + 6x - 7 = 0$

Step 2: Get rid of the pesky constant. We will generate our own.

Example: Add 7 to both sides : $x^2 + 6x = 7$

Step 3: Calculate a new constant. The required constant is the square of one half of the coefficient of x. Add it to both sides of the equation.

Example: $x^2 + 6x = 7$

Half it, then square the result: $\frac{6}{2} = 3, \quad 3^2 = 9.$

Result: $x^2 + 6x + 9 = 7 + 9$

Step 4: Recognize the left hand side of the equation as a perfect square. After all, that was the reason we selected the new constant the way we did.

Example: $(x + 3)^2 = 16$

Step 5: Take the square root of both sides. Remember the "$\pm$" sign on the constant term.

Example: $\sqrt{(x+3)^2} = \pm\sqrt{16}$

$x + 3 = \pm 4$

Step 6: Break the resulting equation into two separate equations, and solve.

Example: $x + 3 = 4$ $\qquad$ $x + 3 = -4$

$x = 1$ $\qquad$ $x = -7$

Solution: $\boldsymbol{x = \{-7, 1\}}$

Algebra
Table of Powers and Roots

Square Root	Number	Square	Cube	4th Power
$\sqrt{1} = 1.000$	1	$1^2 = 1$	$1^3 = 1$	$1^4 = 1$
$\sqrt{2} = 1.414$	2	$2^2 = 4$	$2^3 = 8$	$2^4 = 16$
$\sqrt{3} = 1.732$	3	$3^2 = 9$	$3^3 = 27$	$3^4 = 81$
$\sqrt{4} = 2.000$	4	$4^2 = 16$	$4^3 = 64$	$4^4 = 256$
$\sqrt{5} = 2.236$	5	$5^2 = 25$	$5^3 = 125$	$5^4 = 625$
$\sqrt{6} = 2.449$	6	$6^2 = 36$	$6^3 = 216$	$6^4 = 1{,}296$
$\sqrt{7} = 2.646$	7	$7^2 = 49$	$7^3 = 343$	$7^4 = 2{,}401$
$\sqrt{8} = 2.828$	8	$8^2 = 64$	$8^3 = 512$	$8^4 = 4{,}096$
$\sqrt{9} = 3.000$	9	$9^2 = 81$	$9^3 = 729$	$9^4 = 6{,}561$
$\sqrt{10} = 3.162$	10	$10^2 = 100$	$10^3 = 1{,}000$	$10^4 = 10{,}000$
$\sqrt{11} = 3.317$	11	$11^2 = 121$	$11^3 = 1{,}331$	$11^4 = 14{,}641$
$\sqrt{12} = 3.464$	12	$12^2 = 144$	$12^3 = 1{,}728$	$12^4 = 20{,}736$
$\sqrt{13} = 3.606$	13	$13^2 = 169$	$13^3 = 2{,}197$	$13^4 = 28{,}561$
$\sqrt{14} = 3.742$	14	$14^2 = 196$	$14^3 = 2{,}744$	$14^4 = 38{,}416$
$\sqrt{15} = 3.873$	15	$15^2 = 225$	$15^3 = 3{,}375$	$15^4 = 50{,}625$
$\sqrt{16} = 4.000$	16	$16^2 = 256$	$16^3 = 4{,}096$	$16^4 = 65{,}536$
$\sqrt{17} = 4.123$	17	$17^2 = 289$	$17^3 = 4{,}913$	$17^4 = 83{,}521$
$\sqrt{18} = 4.243$	18	$18^2 = 324$	$18^3 = 5{,}832$	$18^4 = 104{,}976$
$\sqrt{19} = 4.359$	19	$19^2 = 361$	$19^3 = 6{,}859$	$19^4 = 130{,}321$
$\sqrt{20} = 4.472$	20	$20^2 = 400$	$20^3 = 8{,}000$	$20^4 = 160{,}000$
$\sqrt{21} = 4.583$	21	$21^2 = 441$	$21^3 = 9{,}261$	$21^4 = 194{,}481$
$\sqrt{22} = 4.690$	22	$22^2 = 484$	$22^3 = 10{,}648$	$22^4 = 234{,}256$
$\sqrt{23} = 4.796$	23	$23^2 = 529$	$23^3 = 12{,}167$	$23^4 = 279841$
$\sqrt{24} = 4.899$	24	$24^2 = 576$	$24^3 = 13{,}824$	$24^4 = 331{,}776$
$\sqrt{25} = 5.000$	25	$25^2 = 625$	$25^3 = 15{,}625$	$25^4 = 390{,}625$

Algebra
The Quadratic Formula

The **Quadratic Formula** is one of the first difficult math formulas that students are asked to memorize. Mastering the formula, though difficult, is full of rewards. By knowing why it works and what the various parts of the formula are, a student can generate a lot of knowledge in a short period of time.

For a quadratic function of the form: $y = ax^2 + bx + c$

The formula for the roots (i.e., where $y = 0$) is: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ← Quadratic Formula

How Many Real Roots?

The discriminant is the part under the radical: $b^2 - 4ac$

- If the discriminant is negative, the quadratic function has **0** real roots. This is because a negative number under the radical results in imaginary roots instead of real roots. In this case the graph the graph will not cross the x-axis. It will be either entirely above the x-axis or entirely below the x-axis, depending on the value of "a".
- If the discriminant is zero, the quadratic function has **1** real root. The square root of zero is zero, so the radical disappears and the only root is $x = \left(\frac{-b}{2a}\right)$. In this case, the graph will appear to bounce off the x-axis; it touches the x-axis at only one spot – the value of the root.
- If the discriminant is positive, the quadratic function has **2** real roots. This is because a real square root exists, and it must be added in the formula to get one root and subtracted to get the other root. In this case, the graph will cross the x-axis in two places, the values of the roots.

Where are the Vertex and Axis of Symmetry?

The x-coordinate of the vertex is also easily calculated from the quadratic formula because the vertex is halfway between the two roots. If we average the two roots, the $\pm$ portion of the formula disappears and the resulting x-value is $x = \left(\frac{-b}{2a}\right)$. The y-value of the vertex must still be calculated, but the x-value can be read directly out of the quadratic formula.

Also, once the x-value of the vertex is known, the equation for the axis of symmetry is also known. It is the vertical line containing the vertex: $x = \left(\frac{-b}{2a}\right)$.

Algebra
Quadratic Inequalities in One Variable

Solve the inequality of the form: $ax^2 + bx + c \geq 0$

Note: The same process works for inequalities with <, ≤, or > signs.

Step 1: If "a" is negative, multiply the whole inequality by "– 1". This will make the problem easier to work with. Don't forget to change the sign of the inequality.

Example: convert $[-3x^2 + 9x - 6 \geq 0]$ to $[3x^2 - 9x + 6 \leq 0]$

Step 2: Factor out any scalars that divide into all of the terms of the inequality. This will also make the problem easier to work with.

Example: factor $[3x^2 - 9x + 6 \leq 0]$ to $[3(x^2 - 3x + 2) \leq 0]$

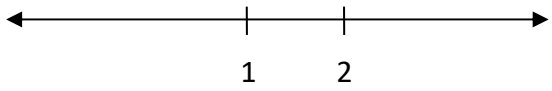
then divide by 3 to get: $[(x^2 - 3x + 2) \leq 0]$

Step 3: Solve the equation that corresponds to the inequality. The solutions of the equation are the critical values in the solution of the inequality.

Example: solve $[(x^2 - 3x + 2) = 0]$, which gives: $x = 1,\ x = 2$

The solution to the inequality, when shown on a number line, must be either outside the solutions or between the solutions. That is, either:

- $(x \leq 1$ or $x \geq 2)$
- $(x \geq 1$ and $x \leq 2)$



But, which one?

Step 4: "Or" vs. "And". Look at the inequality you are working with (at the end of Step 2). As in solving inequalities with absolute values, use the following trick to remember if the answer uses "Or" vs. "And".

- If the inequality contains a $<$ sign, use "and." Think: less thand
- If the inequality contains a $>$ sign, use "or." Think: greator
- Don't forget to use $\leq, \geq$ instead of $<, >$ if they are in the original inequality.

Example: Using this method, we start with $[x^2 - 3x + 2 \leq 0]$ and note the $<$ part of the inequality. This means the result in the example would be $\boldsymbol{(x \geq 1\ and\ x \leq 2)}$.

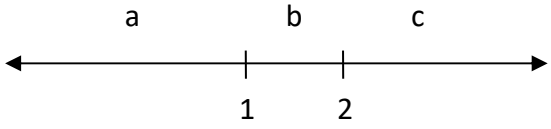
In interval notation this solution set is shown as: $\boldsymbol{x \in [1, 2]}$

In set notation, this solution is shown as: $\boldsymbol{\{x \mid 1 \leq x \leq 2\}}$

Algebra
Quadratic Inequalities in One Variable (cont'd)

Step 5: Checking your "Or" vs. "And" decision. The solutions to the equation in Step 3 break the number line into 3 distinct pieces; in the example:

- Less than 1
- Between 1 and 2
- More than 2



Test to see if the original inequality is correct for a number in each of these segments of the number line. Although you could test a derivation of the original inequality (e.g., after Steps 1 or 2), it is best to work with the original when checking your work.

So, you might test values like the following:

a. Less than 1: try the value 0

b. Between 1 and 2: try the value $\frac{3}{2}$

c. More than 2: try the value 3

In the example, you find that $\frac{3}{2}$ works, but 0 and 3 do not work. The answer must then be in the middle interval. This matches the answer obtained in Step 4.

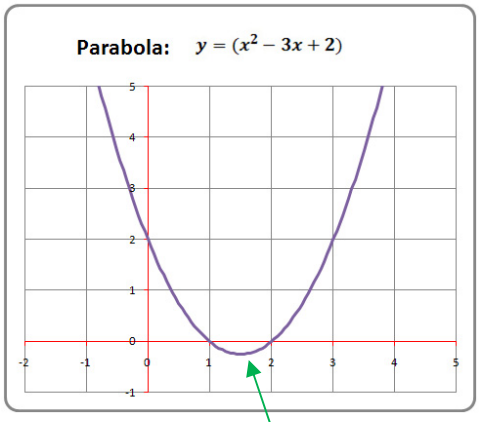
Step 5 Alternative: Another way to check your work.

An alternative way to check your work is to graph the equation corresponding to the inequality.

- Use the equation $(x^2 - 3x + 2) \leq 0$ from Step 2 only if you are sure you performed Steps 1 and 2 correctly.
- The graph of the corresponding equation is at right.
- Notice that the portion of the graph that is below zero is the portion between 1 and 2.
- Because the "≤" sign includes the equals sign, the endpoints of the intervals are included in the solution set.
- Therefore, the solution of the inequality is:

$$\{x \mid 1 \leq x \leq 2\}$$

- This matches the answer obtained in Step 4.



ADVANCED

Algebra
Fitting a Quadratic Equation through Three Points

It takes three points to define a quadratic equation, just like it takes 2 points to define a line. In general, it takes (n+1) points to define a polynomial of degree "n".

Starting with:

- the basic quadratic equation: $y = ax^2 + bx + c$, and
- three points: $(x_1, y_1), (x_2, y_2), (x_3, y_3)$,

it is possible to calculate the coefficients of the quadratic equation by substituting in the x and y values of the 3 points to create a system of 3 equations in 3 unknowns:

$$y_1 = ax_1{}^2 + bx_1 + c$$
$$y_2 = ax_2{}^2 + bx_2 + c$$
$$y_3 = ax_3{}^2 + bx_3 + c$$

Now, that's a lot of symbols, so let's look at an example.

Example:

Find the quadratic equation that passes through the three points:

$$(-1, -8), (1,\ 4), (2,\ 13)$$

Using the basic quadratic equation, and substituting in x-values and y-values, we get 3 equations in 3 unknowns:

$$-8 = a - b + c$$
$$4 = a + b + c$$
$$13 = 4a + 2b + c$$

These 3 equations can be solved by eliminating variables or by using Cramer's Rule, whichever the student finds more comfortable. Solving by either method gives:

$$\boldsymbol{a = 1,\ \ b = 6,\ \ c = -3}$$

So that: $\boldsymbol{y = x^2 + 6x - 3}$

The odd thing about this process is that in most algebra problems the student is asked to solve for $\boldsymbol{x}$ or $\boldsymbol{y}$, but that is not the case in curve fitting. Instead, the student is asked to derive a quadratic equation given 3 sets of $\boldsymbol{x}$'s and $\boldsymbol{y}$'s, which requires solving for $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ instead.

Algebra
Complex Numbers – Introduction

Definitions

$i = \sqrt{-1}$.

Imaginary Number – Any number that can be expressed in the form $a \cdot i$ where a is real.

Examples: $\sqrt{-7}$, $6i$, $\sqrt{-23}$, $\sqrt{3} \cdot i$

Complex Number – Any number that can be expressed in the form $a + bi$ where a, b are real.

Examples: $6 + \sqrt{2}\, i$, $\sqrt[3]{6} + 3i$

Note: all real numbers and all imaginary numbers are also complex numbers.

In the form $a + bi$,

- a is called the real part of the complex number, and
- bi is called the imaginary part of the complex number.

Absolute Value of a Complex Number

The absolute value of a complex number is also called its magnitude. Algebraically, it is:

$$|a + bi| = \sqrt{a^2 + b^2}$$

Conjugate of a Complex Number

The conjugate of a complex number $(a + bi)$ is denoted $\overline{(a + bi)}$; $\overline{(a + bi)} = (a - bi)$.

Examples:

Complex Number	Conjugate
$2 + 3i$	$2 - 3i$
$6i$	$-6i$
2	2

Complex Number	Conjugate
$-6 + i$	$-6 - i$
$2 + \sqrt{2}i$	$2 - \sqrt{2}i$
$\sqrt{7} + \sqrt{3}i$	$\sqrt{7} - \sqrt{3}i$

Comments about conjugates:

1. The conjugate of a conjugate is the number you stated with.

   $\overline{(a + bi)} = (a - bi)$, $\overline{(a - bi)} = (a + bi)$,

2. The product of conjugates is a real number.

   $(a + bi) \cdot (a - bi) = (a^2 + b^2)$

3. Conjugate numbers have the same absolute value.

   $|a + bi| = |a - bi| = \sqrt{a^2 + b^2}$

Algebra
Operations with Complex Numbers

Adding and Subtracting

Add or subtract both the Real and Imaginary parts:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$
$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

Example: $(1-3i)+(-3+6i)=(-2+3i)$

Multiplying

Step 1: Multiply like you multiply binomials.

Step 2: Substitute -1 for i^2 and simplify.

$$(a+bi)\cdot(c+di)=(ac-bd)+(ad+bc)i$$

Example:
$$\begin{aligned}(1-3i)\cdot(-3+6i) &= -3+6i+9i-18i^2\\ &= -3+6i+9i+18\\ &= 15+15i\end{aligned}$$

Dividing

Step 1: Multiply by a fraction whose numerator and denominator are the conjugate of the original expression's denominator.

Step 2: Substitute -1 for i^2 and simplify.

The resulting complex number will have a denominator that is free of imaginary numbers.

Remember the method, not the formula!

$$\frac{(a+bi)}{(c+di)}=\frac{(a+bi)}{(c+di)}\cdot\frac{(c-di)}{(c-di)}=\left[\frac{(ac+bd)}{(c^2+d^2)}\right]+\left[\frac{(bc-ad)}{(c^2+d^2)}\right]i$$

Example:

$$\frac{(1-3i)}{(-3+6i)}=\frac{(1-3i)}{(-3+6i)}\cdot\frac{(-3-6i)}{(-3-6i)}=\frac{-3-6i+9i+18i^2}{9-36i^2}$$

$$=\frac{-3-6i+9i-18}{9+36}=\frac{-21+3i}{45}=\mathbf{\frac{-7}{15}+\frac{1}{15}i}$$

ADVANCED

Algebra
The Square Root of i

If $i = \sqrt{-1}$, what is $\sqrt{i}$?

This is an interesting question and solving it will illustrate a very useful method in working with complex numbers. You can use this method to calculate the square root of any complex number.

- Recall that each complex number is of the form $a + bi$, where a and b are real.
- We want a complex number that, when squared, generates i.

So, we want to calculate a and b such that $(a + bi)^2 = i$

Let's solve that equation:

$$(a + bi)^2 = i$$

$$(a + bi) \cdot (a + bi) = i$$

$$a^2 + abi + abi + b^2 i^2 = i$$

$$(a^2 - b^2) + 2abi = i$$

Now, break this into 2 equations, one for the real part, and one for the imaginary part.

Here are the 2 equations: $(a^2 - b^2) = 0$ and $2ab = 1$

$a^2 = b^2$ and $b = \frac{1}{2a}$

Substituting for b, $a^2 = \left(\frac{1}{2a}\right)^2$

$$a^2 = \frac{1}{4a^2}$$

Since $a \neq 0$, and a is real, $a^4 = \frac{1}{4}$ $\xrightarrow{yields}$ $a^2 = \frac{1}{2}$

$\xrightarrow{yields}$ $a = \frac{\sqrt{2}}{2}$ or $a = -\frac{\sqrt{2}}{2}$

Since $a^2 = b^2$ $b = a$ or $b = -a$

So, our candidates for $\sqrt{i}$ are: $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$ or $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$ or $\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$ or $\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$

Let's try them: $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^2 = i$

$\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^2 = \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = -i$

So, we have found not only the two square roots of i, but also the two square roots of $-i$.

Results:

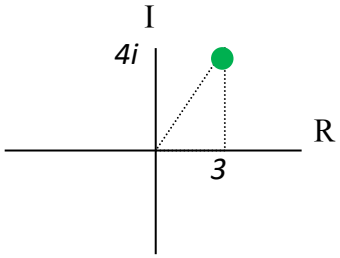
$$\sqrt{i} = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$\sqrt{-i} = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

Algebra

Complex Numbers – Graphical Representation

Complex numbers of the form $a + bi$ can be represented on a set of axes that represent Real and Imaginary numbers instead of x and y. Consider the complex number $3 + 4i$. It would look like this on a set of Real (R) and Imaginary (I) axes:



Graphical representation of $3 + 4i$

Polar Coordinates

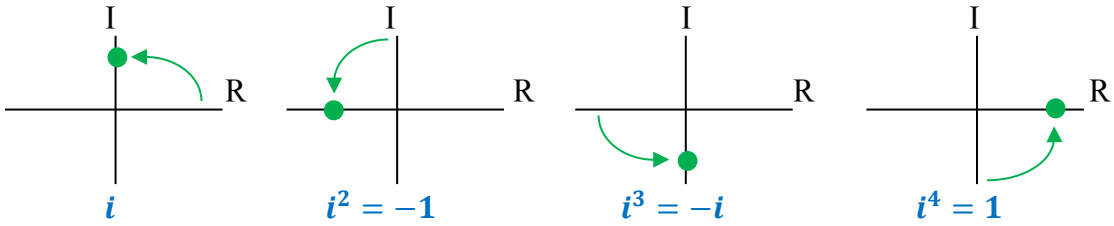
Represented in this manner, complex numbers have interesting properties (see the next page for some of these). Each complex number can be thought of as not only a pair of rectangular coordinates, e.g., (3,4), but also as a set of polar coordinates with magnitude (i.e., length) ρ and angle θ. Then, to multiply complex numbers, you multiply their magnitudes and add their angles.

Powers of i

This is a useful bit of information for seeing the value of $\boldsymbol{i^n}$ in graphical terms. Since $i = \sqrt{-1}$, algebraically, we have:

$i = \sqrt{-1}$ $\quad i^2 = -1$ $\quad i^3 = -i$ $\quad i^4 = 1$
$i^5 = \sqrt{-1}$ $\quad i^6 = -1$ $\quad i^7 = -i$ $\quad i^8 = 1$ etc.

Since $\boldsymbol{i}$ has magnitude 1, all powers $\boldsymbol{i^n}$ also have magnitude 1. Each succeeding power of $\boldsymbol{i}$, then, results only in a change of the angle θ, and can be considered a 90°rotation in the coordinate plane containing the Real and Imaginary axes, like so:



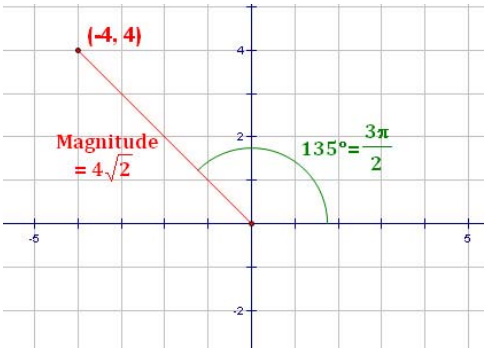
This shows the rotating pattern in the values of the powers of $\boldsymbol{i}$ every 4 increments. After 4 rotations you return to where you started.

ADVANCED

Algebra
Complex Number Operations in Polar Coordinates

Polar Coordinates

If a complex number is expressed in terms of its polar coordinates, many calculations are made much easier. First, let's investigate the relationship between a point's rectangular coordinates (x, y) and its polar coordinates (ρ, θ).



The magnitude, ρ, is the distance of the point from the origin: $\rho = \sqrt{x^2 + y^2}$

The angle, θ, is the angle the line from the point to the origin makes with the positive portion of the x-axis. Generally, this angle is expressed in radians, not degrees.

$$\tan\theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Conversion from polar coordinates to rectangular coordinates is straightforward:

$$x = \rho \cdot \cos\theta \quad \text{and} \quad y = \rho \cdot \sin\theta$$

Example: In the illustration above, the point shown has:

- Rectangular coordinates: $(-4, 4)$
- Polar Coordinates: $(4\sqrt{2}, \frac{3\pi}{2})$

Complex Number Formulas:

To see how useful this can be, consider the following formulas for complex numbers:

Multiplication: $(\rho_1, \theta_1) \cdot (\rho_2, \theta_2) = ((\rho_1 \cdot \rho_2), (\theta_1 + \theta_2))$

So, to multiply complex numbers, you multiply their magnitudes and add their angles.

Division: $(\rho_1, \theta_1) \div (\rho_2, \theta_2) = ((\rho_1 \div \rho_2), (\theta_1 - \theta_2))$

So, to divide complex numbers, you divide their magnitudes and subtract their angles.

Powers: $(\rho, \theta)^n = \left(\rho^n, (n \cdot \theta)\right)$

This results directly from the multiplication rule.

Roots: $\sqrt[n]{(\rho, \theta)} = \left(\sqrt[n]{\rho}, (\theta \div n)\right)$

This results directly from the power rule if the exponent is a fraction.

Note on $\sqrt{i}$:

Since i in polar coordinates is $\left(1, \frac{\pi}{2}\right)$,

Using the root formula, $\sqrt{i} = \left(\pm 1, \frac{\pi}{4}\right)$

In rectangular coordinates, then,

$$\sqrt{i} = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

Algebra
Complex Solutions to Quadratic Equations

Over the field of real numbers there are no roots to a quadratic function if the discriminant is less than zero. Over the field of complex numbers, however, such a quadratic function has two roots.

Quadratic Refresher

For a quadratic function of the form: $y = ax^2 + bx + c$

The formula for the roots (i.e., where $y = 0$) is: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ ← Quadratic Formula

The discriminant is the part under the radical: $b^2 - 4ac$

How Many Roots?

The following table tells us how many real or complex roots exist for a function, based on its discriminant:

Value of Discriminant	Number of Roots
$b^2 - 4ac < 0$	2 complex
$b^2 - 4ac = 0$	1 real
$b^2 - 4ac > 0$	2 real

Note: because of the "±" sign in the quadratic formula, when there are 2 complex roots they are conjugates.

Example 1: Solve $x^2 - 2x + 10 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

Example 2: Solve $2x^2 + 4x + 7 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{(4)^2 - 4(2)(7)}}{2(2)} = \frac{-4 \pm \sqrt{-40}}{4} = \frac{-4 \pm 2\sqrt{10}i}{4} = -1 \pm \frac{\sqrt{10}}{2}i$$

Algebra
Radical Rules

Simple Rules Involving Radicals

General Radical Rule	Rule for Square Roots	Example
$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$	$\sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$
$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$	$\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}$

Note also that: $\sqrt[n]{a} = a^{(1/n)}$

e.g., $\sqrt{a} = a^{(1/2)}$, so the rules for exponents also apply for roots.

Rationalizing the Denominator

Mathematicians prefer to keep radicals out of the denominator. Here are two methods to accomplish this, depending on what's in the denominator.

Case 1: *Simple radical in the denominator.* Solution: multiply the beginning expression by a fraction which is the offending radical divided by itself.

Example: $\frac{2+\sqrt{3}}{4\sqrt{5}} = \frac{2+\sqrt{3}}{4\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}+\sqrt{15}}{20}$

Case 2: *Number and radical in the denominator.* Solution: multiply by the beginning expression by a fraction which is designed to eliminate the radical from the denominator. The numerator and denominator of the fraction are created by changing the sign between the number and the radical in the denominator.

Example: $\frac{\sqrt{7}}{3-\sqrt{5}} = \frac{\sqrt{7}}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{3\sqrt{7}+\sqrt{35}}{9-5} = \frac{3\sqrt{7}+\sqrt{35}}{4}$

Algebra
Simplifying Square Roots – Two Methods

Method 1: Extracting Squares

In this method, you pull squares out from under the radical. This is the quickest method if you are comfortable with what the squares are and with dividing them out of larger numbers.

$1^2 = 1$	$11^2 = 121$
$2^2 = 4$	$12^2 = 144$
$3^2 = 9$	$13^2 = 169$
$4^2 = 16$	$14^2 = 196$
$5^2 = 25$	$15^2 = 225$
$6^2 = 36$	$16^2 = 256$
$7^2 = 49$	$17^2 = 289$
$8^2 = 64$	$18^2 = 324$
$9^2 = 81$	$19^2 = 361$
$10^2 = 100$	$20^2 = 400$

Examples: (1) $\sqrt{98} = \sqrt{49} \cdot \sqrt{2}$
$= 7\sqrt{2}$

(2) $\sqrt{9600} = \sqrt{100} \cdot \sqrt{96}$
$= \sqrt{100} \cdot \sqrt{16} \cdot \sqrt{6}$
$= 10 \cdot 4 \cdot \sqrt{6}$
$= 40\sqrt{6}$

Method 2: Extracting Prime Numbers

If you are not comfortable with Method 1, you can pull prime numbers out from under the radical and pair them up to simplify the square root.

Example: $\sqrt{54} = \sqrt{2} \cdot \sqrt{27}$
$= \sqrt{2} \cdot \sqrt{3} \cdot \sqrt{9}$
$= \sqrt{2} \cdot \sqrt{3} \cdot \sqrt{3} \cdot \sqrt{3}$
$= \sqrt{2} \cdot (\sqrt{3} \cdot \sqrt{3}) \cdot \sqrt{3}$
$= \sqrt{2} \cdot (3) \cdot \sqrt{3}$
$= 3 \cdot \sqrt{2} \cdot \sqrt{3}$
$= 3 \cdot \sqrt{6}$

Note that the last step is to re-combine roots that do <u>not</u> come in pairs.

Method 2 may take a lot longer than Method 1, but it works. A good use for Method 2 is when you try using the quicker Method 1 but get stuck – then working with primes can get you back on track toward solving the problem.

Algebra
Solving Radical Equations

When an equation involves radicals, the radicals must be eliminated in order to obtain a solution. The one special thing about these equations is that, in the process of eliminating the radical, it is possible to add another solution that is not a solution to the original problem.

Solutions that are added by the process used to solve the problem are called **Extraneous Solutions**. At the end of the problem, we must check for extraneous solutions and eliminate them.

Solving a Radical Equation

The steps to solving an equation involving radicals are:

- Isolate the radical on one side of the equation. To do this, add or subtract any variables or constants that are on the same side of the equation as the radical.
- If the radical is a square root, square both sides of the equation. If the radical is a cube root, cube both sides, etc. This should get rid of the radical.
- If there are any radicals remaining in the problem, repeat the first two steps until they are gone.
- Solve the equation that remains.
- Check all solutions to the problem using the equation in the original statement of the problem.
- Discard extraneous roots.

Example: Solve $\sqrt{2x+6}+1=x$

Starting Problem:	$\sqrt{2x+6}+1=x$
Subtract $\mathbf{1}$ from both sides:	$\sqrt{2x+6}=x-1$
Square both sides:	$2x+6=x^2-2x+1$
Subtract $2x+6$ from both sides:	$x^2-4x-5=0$
Factor:	$(x-5)(x+1)=0$
Obtain Preliminary Solutions:	$x=\{-1,5\}$
Test -1 as a solution:	$\sqrt{2(-1)+6}+1=-1\ ?$
Test 5 as a solution:	$\sqrt{4(5)+5}=5\ ?\quad\checkmark$
Identify the final Solution Set:	$x=5$

If we allowed $\sqrt{2(-1)+6}$ to be -2, the equation would work and **-1** would work as a solution. However, the square root of a number is defined to be the positive root only. **So, -1 fails as a solution to the problem.**

Algebra
Solving Radical Equations (cont'd)

Why Only Positive Square Roots?

It is generally taught in high school algebra that square roots have both positive and negative roots. So why are we now saying that the square root of a number is defined to be a positive number only?

The answer lies in the missing step, which is often not taught to high school students. What you learn is this:

$\boldsymbol{If\ x^2 = 4, then\ x = \pm 2}$. And, that is correct, but not because $\sqrt{x^2} = x$ which it does not!

The Missing Step

In the box to the right is the development with the missing step included. Notice that:

- $\sqrt{x^2} = |x|$
- When we take square roots, we have positive numbers on each side of the resulting equation. Both $|x|$ and 2 are positive.

Starting Problem:	$\boldsymbol{x^2 = 4}$
Take square roots:	$\boldsymbol{\|x\| = 2}$
Solve for x:	$\boldsymbol{x = \pm 2}$

- The two possible values for x come from solving the middle equation $\boldsymbol{|x| = 2}$.

This solves the apparent arbitrariness of when a root is only positive and when it is both positive and negative.

Inequalities

The "missing step" also provides an explanation for the method used to solve inequalities. Consider the inequality: $\boldsymbol{x^2} < 9$. The process for solving this with the missing step included is:

Starting Problem:	$\boldsymbol{x^2} < 9$	
Take square roots:	$\boldsymbol{\|x\|} < 3$	
Case i		Case ii
$\boldsymbol{x} < 3$	and	$-\boldsymbol{x} < 3$
$\boldsymbol{x} < 3$	and	$\boldsymbol{x} > -3$

Solution: $\{\, \boldsymbol{x} \mid -\boldsymbol{3} < x < 3\}$

Notice that $\boldsymbol{|x|}$ converts to $\boldsymbol{x}$ in Case I and to $-\boldsymbol{x}$ in Case ii. By the end of the problem, you see that the sign in Case ii has been flipped around from the original problem. This is not magic; it is the result of "The Missing Step" being applied in all its mathematical glory!

Algebra
Matrix Addition and Scalar Multiplication

What is a Matrix?

A matrix is an ordered set of objects (typically real or complex numbers) set up in a two-dimensional array. Matrices are very useful in algebra, statistics and other applications because they provide a concise way to carry out complex mathematical processes and methods.

Matrices have dimensions, expressed as the number of rows x the number of columns. For example, a 2x3 matrix (read "2 by 3 matrix") has 2 rows and 3 columns. Knowing the dimensions of a matrix is important because many matrix operations can only occur on matrices with certain dimensions.

Adding Matrices

Each object in a matrix is called an element. Matrices are added by adding the corresponding elements in the matrices. Matrices must have the same dimensions if they are to be added.

Example:

$$\begin{bmatrix} 2 & -3 & 1 \\ 5 & 1 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 2 & -2 \end{bmatrix}$$

1st row, 1st column: 2 + (-1) = 1
1st row, 2nd column: (-3) + 2 = -1

Scalar Multiplication

Multiplying a matrix by a scalar (i.e., a number) is accomplished by multiplying each element in the matrix by the scalar. The term scalar simply refers to "scaling" the matrix by making its values larger or smaller. Scalar multiplication can be performed on matrices of any dimensions.

Example:

$$3 \cdot \begin{bmatrix} -1 & 2 & 4 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 12 \\ -6 & 3 & 0 \end{bmatrix}$$

1st row, 1st column: $3 \cdot (-1) = -3$
1st row, 2nd column: $3 \cdot 2 = 6$

Algebra
Matrix Multiplication

Multiplying Matrices

Multiplication of matrices is a more complex process. Although the student may find it difficult at first, it is a powerful tool that is useful in many fields of mathematics and science.

Matrix multiplication can be performed only on matrices that are **conformable** (i.e., compatible in size). In order for two matrices to be multiplied together, the number of columns in the first matrix must equal the number of rows in the second matrix. If an $m \times n$ matrix is multiplied by an $n \times p$ matrix, the result is an $m \times p$ matrix. This is illustrated as follows:

must match

$$[m \times n] \cdot [n \times p] = [m \times p]$$

size of resulting matrix

To multiply matrices, multiply the elements in a row of the first matrix, element by element, by the corresponding elements in a column of the second matrix and add the results. If row *i* in the first matrix is multiplied by row *j* in the second matrix, the result is placed in row *i*, column *j* of the resulting matrix. The element in position *i, j* of a matrix A is often denoted $a_{i,j}$.

Example 1:

$$\begin{bmatrix} 2 & -3 & 1 \\ 5 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -13 \end{bmatrix}$$

Notice that multiplying a ***2 x 3*** matrix by a ***3 x 2*** matrix results in a ***2 x 2*** matrix.

1st row, 1st column: [2 · 1] + [(-3) · 2] + [1 · 3] = -1
1st row, 2nd column: [2 · (-2)] + [(-3) · (-1)] + [1 · 1] = 0
2nd row, 1st column: [5 · 1] + [1 · 2] + [(-2) · 3] = 1
2nd row, 2nd column: [5 · (-2)] + [1 · (-1)] + [(-2) · 1] = -13

Example 2:

$$\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 & 1 \\ 5 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -8 & -5 & 5 \\ -1 & -7 & 4 \\ 11 & -8 & 1 \end{bmatrix}$$

Notice that multiplying a ***3 x 2*** matrix by a ***2 x 3*** matrix results in a ***3 x 3*** matrix.

From these examples, it is clear that **matrix multiplication is not commutative**. That is, if we name two matrices A and B, **it is generally not true that** $A \cdot B = B \cdot A$. Further, if matrices are not square (i.e., having the same number of rows and columns), matrix multiplication is never commutative; that is $A \cdot B \neq B \cdot A$.

Algebra
Identity Matrices and Inverse Matrices

Identity Matrices

For real numbers, the multiplicative identity is **1** because for any real number x:

$$x \cdot \mathbf{1} = \mathbf{1} \cdot x = x$$

That is, when a number is multiplied by **1**, the result is the original number. It does not matter whether we multiply by **1** on the left or the right; in either case, the result is the original number.

The same is true for square matrices. If we multiply a square matrix by the identity matrix with the same dimensions, the result will be the original matrix. It does not matter whether we multiply by the identity matrix on the left or right; the result is the same.

Identity matrices exist for each square dimension. Identity matrices have 1's along the diagonal and 0's in every other position. For example, the following are identity matrices of rank $2, 3$ and 4. Identity matrices are generally denoted by the letter "$\boldsymbol{I}$":

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: If we define square matrix $\boldsymbol{A}$, then $\boldsymbol{A} \cdot \boldsymbol{I} = \boldsymbol{I} \cdot \boldsymbol{A} = \boldsymbol{A}$ as follows:

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

Inverse Matrices

For real numbers, multiplying a number by its inverse results in the identity, **1**. For example, $3 \cdot \frac{1}{3} = \frac{1}{3} \cdot 3 = \mathbf{1}$. Similarly, multiplying a matrix by its inverse results in the identity matrix with the same dimensions as the original matrix.

Example: Using the same square matrix $\boldsymbol{A}$ as above, $\boldsymbol{A} \cdot \boldsymbol{A^{-1}} = \boldsymbol{A^{-1}} \cdot \boldsymbol{A} = \boldsymbol{I}$ as follows:

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2.0 & -1 & 2.0 \\ 1.6 & 1 & -1.4 \\ 2.2 & 1 & -1.8 \end{bmatrix} = \begin{bmatrix} -2.0 & -1 & 2.0 \\ 1.6 & 1 & -1.4 \\ 2.2 & 1 & -1.8 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Algebra
Inverse of a 2x2 Matrix

Using matrix notation:

- Let the identity matrix of size $n \times n$ be called I
- Let an $n \times n$ matrix be called A
- Let the determinant of matrix A be denoted $|A|$
- Let the inverse of matrix A be denoted A^{-1}
- Then, $A^{-1} \cdot A = A \cdot A^{-1} = I$

Not all square matrices have inverses. In order for a matrix to have an inverse, its determinant must be non-zero. That is, matrix A has an inverse if and only if: $|A| \neq 0$.

Formula for the Inverse of a 2x2 Matrix

If a 2x2 matrix has elements a, b, c and d, such that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then:

$$A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

In words, the inverse is calculated as follows:

- From the original matrix, switch elements a and d, and change the signs of b and c.
- Divide the resulting matrix by the determinant of the original matrix. (Note: the determinant of a matrix is a scalar).

The determinant of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is calculated as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Example: 2x2 Matrix Inverse Calculation

Let: $A = \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$

Then: $|A| = \begin{vmatrix} 4 & -1 \\ -2 & 1 \end{vmatrix} = (4 \cdot 1) - ((-1) \cdot (-2)) = 2$

So: $A^{-1} = \frac{\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}}{\begin{vmatrix} 4 & -1 \\ -2 & 1 \end{vmatrix}} = \frac{\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}}{2} \Longrightarrow$ $A^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 2 \end{bmatrix}$

Finally, check to make sure: $A^{-1} \cdot A = A \cdot A^{-1} = I$

$\begin{bmatrix} 0.5 & 0.5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓ and $\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.5 & 0.5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Algebra
Matrix Division and Inverses

Multiplication and division are inverse processes with which the student is familiar when working with real numbers. Multiplication of matrices, as described above is much more complex than multiplication of real numbers. So, how do you divide matrices?

Division of real numbers can be considered to be the process of multiplying a number by the inverse of the number by which you want to divide. For example:

- $12 \div 3 = 12 \cdot \frac{1}{3} = 4$ (i.e., 12 divided by 3 is the same as 12 times $\frac{1}{3}$; the result is 4 either way).
- Dividing by 3 is the same a multiplying by the inverse of 3, which is $\frac{1}{3}$.
- 3 and $\frac{1}{3}$ are "multiplicative inverses" because when multiplied, they result in 1, which is the "multiplicative identity".

Matrix division works in a similar fashion. It is typically handled by multiplying the dividend (top) matrix by the inverse of the divisor (bottom) matrix. For matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, if we want to *divide* $\boldsymbol{B}$ by $\boldsymbol{A}$, we would instead *multiply* $\boldsymbol{B}$ by $\boldsymbol{A^{-1}}$.

Some methods for calculating inverses of a square matrices are shown in the following pages. *Only square matrices have inverses.* Identity matrices must also, by definition, be square.

Example:

Let: $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix}$ $\boldsymbol{B} = \begin{bmatrix} 1 & 3 & -3 \\ 5 & 2 & -1 \\ -2 & 3 & 4 \end{bmatrix}$

Then: $\boldsymbol{A^{-1}} = \begin{bmatrix} -2.0 & -1 & 2.0 \\ 1.6 & 1 & -1.4 \\ 2.2 & 1 & -1.8 \end{bmatrix}$ *(see the following pages for methods to calculate this inverse)*

And: $\frac{\boldsymbol{B}}{\boldsymbol{A}} = \boldsymbol{B} \cdot \boldsymbol{A^{-1}} = \begin{bmatrix} 1 & 3 & -3 \\ 5 & 2 & -1 \\ -2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2.0 & -1 & 2.0 \\ 1.6 & 1 & -1.4 \\ 2.2 & 1 & -1.8 \end{bmatrix} = \begin{bmatrix} -3.8 & -1 & 3.2 \\ -9.0 & -4 & 9.0 \\ 17.6 & 9 & -15.4 \end{bmatrix}$

Note that because matrix multiplication is not commutative, you are very likely to get a different result if you multiply $\boldsymbol{B}$ by $\boldsymbol{A^{-1}}$ on the left instead of the right. That is, in general, $\boldsymbol{B \cdot A^{-1}} \neq \boldsymbol{A^{-1} \cdot B}$.

Algebra
Determinants – The General Case

Determinants are very useful in matrix operations. The determinant of a **2 x 2 matrix** is defined to be:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

In words, the diagonals are multiplied and the product of second diagonal is subtracted from the product of the first diagonal. This process is generalized in determinants of larger matrices using what are referred to as minors. A minor is what is left of a matrix when the row and column of the element are eliminated.

The determinant of a matrix can be calculated by selecting a row or column and multiplying each element of that row or column by the determinant of its corresponding minor. The results are alternately added and subtracted to get the value of the determinant. The $\pm$ sign of each term is determined by the row and column in which it resides. The sign for the element in row m and column n is $(-1)^{(m+n)}$. The following matrices of signs show how they are applied to each row element:

2x2: $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ 3x3: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ 4x4: $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

Using minors of the first row to evaluate a **3 x 3 matrix**,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Or, using minors of the second column to evaluate the same **3 x 3 matrix**,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The results of the calculation will be the same, regardless of which row or column is selected, because of the power of matrices and determinants.

Example for a 3 x 3 matrix using minors of the first row:

$$\begin{vmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ -2 & 2 & 3 \end{vmatrix} = 3 \begin{vmatrix} -2 & -1 \\ 2 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix}$$

$$= 3(-4) - 1(1) + 1(-2) = \boxed{-15}$$

Note: this is the matrix that forms the denominator in the solution of the system of equations in the Cramer's Rule example.

The same process is followed for larger determinants. For example, a 5 x 5 determinant is first reduced to a sum of five elements each multiplied by their 4 x 4 minors. Each of the 4 x 4 minors is reduced to a sum of four elements each multiplied by their 3 x 3 minors, etc. The process is calculation intensive; today it would typically be performed using a computer.

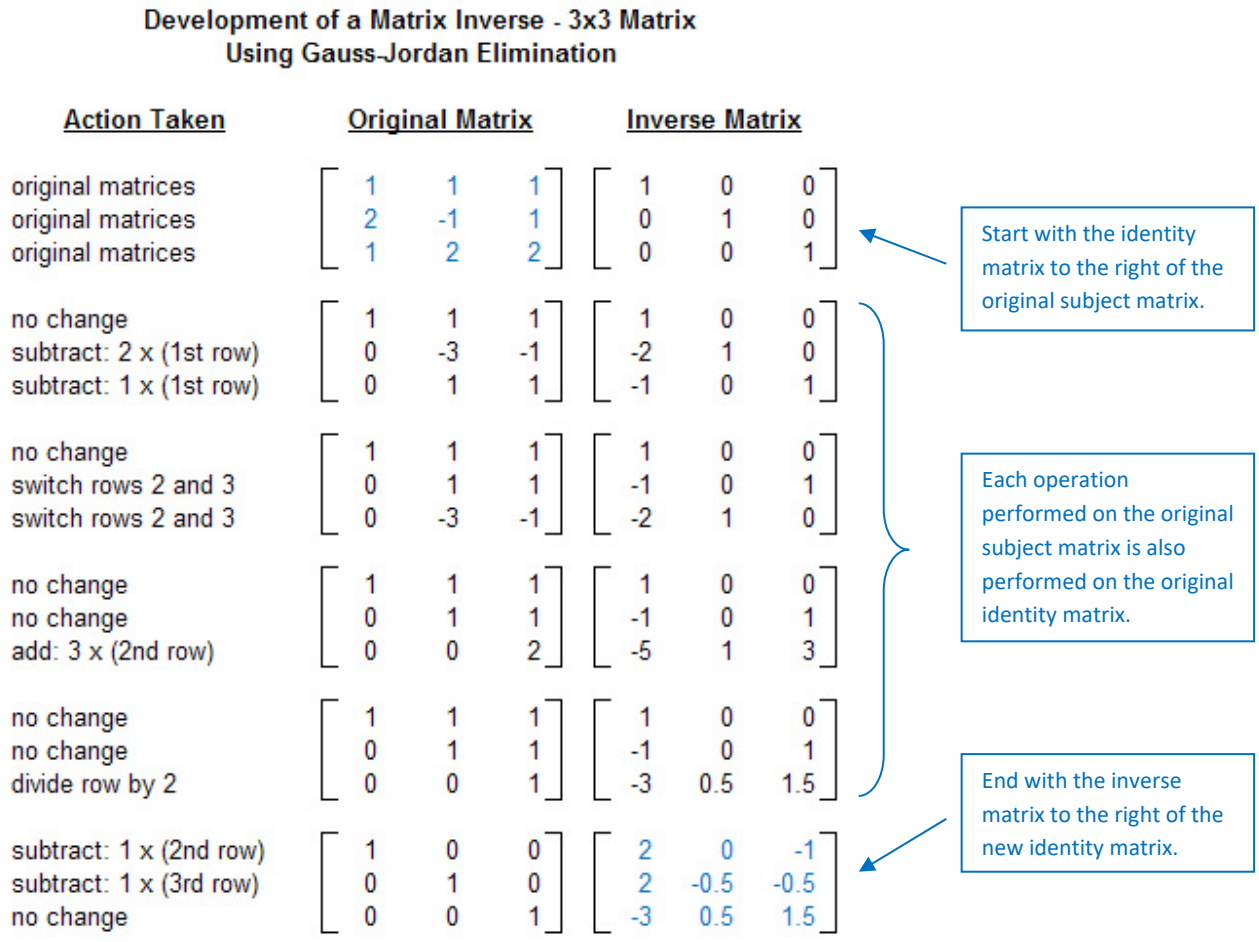
Algebra
Calculating Inverses – The General Case

The calculation of the inverse of a matrix of size greater than 2 x 2 can be performed by a process called Gauss-Jordan Elimination. The process is also called, more generically, Row Reduction. In this process, you begin with two side-by-side matrices, the one you want to invert (the subject matrix) and the identity matrix of the same size. Operations are performed on both matrices, gradually converting the original matrix to the Identity Matrix.

Allowable operations are:

- Multiplying or dividing a row by a scalar (i.e., a number).
- Switching rows.
- Adding or subtracting a multiple of one row to or from another.

When this process is complete, the original identity matrix has been converted to the inverse matrix. Below is an example of the development of an inverse of a 3 x 3 matrix using this process:



Algebra
Calculating Inverses Using Adjoint Matrices

Definitions and Notation

- Minor: A minor M_{ij} is what is left of a matrix when the i-th row and j-th column (i.e., the row and column containing element a_{ij}) of the original matrix A are eliminated.
- Cofactor: The cofactor c_{ij} of a_{ij} is $(-1)^{i+j}$ multiplied by the determinant of the minor M_{ij}. That is, $c_{ij} = (-1)^{i+j} \cdot |M_{ij}|$.
- Cofactor Matrix: The cofactor matrix is the matrix of cofactors of each element of the original matrix. It is often denoted C.
- Adjoint Matrix: The adjoint matrix is the transpose of the cofactor matrix. That is, $adj\ A = C^T$. The adjoint matrix is also called the adjugate matrix or adjunct matrix.
- Recall that the original matrix A must be square and not singular (i.e., $|A| \neq 0$) in order to have an inverse. A singular matrix has a determinant of zero.

The inverse of a matrix is equal to its corresponding adjoint matrix divided by the determinant of the original matrix. That is:

$$A^{-1} = \frac{C^T}{|A|}$$

Note: the transpose of a matrix interchanges rows and columns of the original matrix. For example:

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

Example: 2x2 Matrix Inverse Calculation

Let: $A = \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$

Then:
$c_{11} = (-1)^2 \cdot 1 = 1$
$c_{12} = (-1)^3 \cdot (-2) = 2$
$c_{21} = (-1)^3 \cdot (-1) = 1$
$c_{22} = (-1)^4 \cdot 4 = 4$

Therefore:

$$C = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}, \quad adj\ A = C^T = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

And: $|A| = \begin{vmatrix} 4 & -1 \\ -2 & 1 \end{vmatrix} = (4 \cdot 1) - ((-1) \cdot (-2)) = 2$

So: $A^{-1} = \frac{C^T}{|A|} = \frac{\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}}{|A|} = \frac{\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}}{2} = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 2 \end{bmatrix}$

Finally, check to make sure: $A^{-1} \cdot A = A \cdot A^{-1} = I$

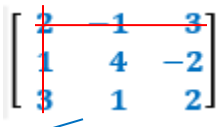
$$\begin{bmatrix} 0.5 & 0.5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark \quad \text{and} \quad \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.5 & 0.5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Algebra
Calculating Inverses Using Adjoint Matrices (cont'd)

Example: 3x3 Matrix Inverse Calculation

Let: $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix}$

Example: How to adjust A to get M_{11}:



Eliminate: Row 1 and column 1

Cofactors:

$$c_{11} = (-1)^{(1+1)} \cdot M_{11} = (-1)^2 \cdot \begin{vmatrix} 4 & -2 \\ 1 & 2 \end{vmatrix} = 10$$

$$c_{12} = (-1)^3 \cdot \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = -8 \qquad c_{23} = (-1)^5 \cdot \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = -5$$

$$c_{13} = (-1)^4 \cdot \begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} = -11 \qquad c_{31} = (-1)^4 \cdot \begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} = -10$$

$$c_{21} = (-1)^3 \cdot \begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} = 5 \qquad c_{32} = (-1)^5 \cdot \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 7$$

$$c_{22} = (-1)^4 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5 \qquad c_{33} = (-1)^6 \cdot \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} = 9$$

Adjoint:

$$C = \begin{bmatrix} 10 & -8 & -11 \\ 5 & -5 & -5 \\ -10 & 7 & 9 \end{bmatrix}, \qquad adj\ A = C^T = \begin{bmatrix} 10 & 5 & -10 \\ -8 & -5 & 7 \\ -11 & -5 & 9 \end{bmatrix}$$

Determinant of A:

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 4 & -2 \\ 1 & 2 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix}$$

$$= \quad 2 \cdot 10 \quad + \quad 1 \cdot 8 \quad + \quad 3 \cdot (-11) \quad = \quad -5$$

Inverse: $A^{-1} = \dfrac{C^T}{|A|} = \dfrac{\begin{bmatrix} 10 & 5 & -10 \\ -8 & -5 & 7 \\ -11 & -5 & 9 \end{bmatrix}}{|A|} = \dfrac{\begin{bmatrix} 10 & 5 & -10 \\ -8 & -5 & 7 \\ -11 & -5 & 9 \end{bmatrix}}{-5} = \begin{bmatrix} -2.0 & -1.0 & 2.0 \\ 1.6 & 1.0 & -1.4 \\ 2.2 & 1.0 & -1.8 \end{bmatrix}$

Finally, check to make sure: $A^{-1} \cdot A = I$

$$\begin{bmatrix} -2.0 & -1.0 & 2.0 \\ 1.6 & 1.0 & -1.4 \\ 2.2 & 1.0 & -1.8 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

*Note: If a square matrix has an inverse from either the left **or** the right, then that matrix is an inverse from both the left **and** the right, and is, therefore, the inverse of the original matrix.*

Algebra
Cramer's Rule – 2 Equations

Cramer's Rule provides a powerful and simple way to solve systems of two or three linear equations. In larger systems of equations, it is a useful way to solve for just one of the variables, without having to solve the entire system of equations. To solve an entire system of four or more equations, a better technique would be Gauss-Jordan Elimination, especially if the student is aided by a computer and spreadsheet software such as Microsoft Excel.

Cramer's Rule works as long as the determinant of variable coefficients (i.e., the determinant in the denominator) is non-zero. If this determinant is zero, then there is no unique solution to the system of equations.

General Case for 2 Equations in 2 Unknowns

The standard form of the equations is:

$$a_1x + b_1y = k_1$$
$$a_2x + b_2y = k_2$$

Using determinant notation, Cramer's Rule states that the solutions for $\boldsymbol{x}$ and $\boldsymbol{y}$ are:

$$x = \frac{\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Notice that the determinants in the denominators are the same; the columns in these determinants are the coefficients of the variables in the equations. The determinants in the numerators are almost the same as the ones in the denominators; the only difference is that the column of coefficients associated with the variable being evaluated is replaced by the equations' constant terms.

Example: Consider these equations:

$$3x - 6y = 18$$
$$x - 3y = 7$$

Then,

$$x = \frac{\begin{vmatrix} 18 & -6 \\ 7 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -6 \\ 1 & -3 \end{vmatrix}} = \frac{-12}{-3} = \mathbf{4} \qquad y = \frac{\begin{vmatrix} 3 & 18 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & -6 \\ 1 & -3 \end{vmatrix}} = \frac{3}{-3} = \mathbf{-1}$$

Algebra
Cramer's Rule – 3 Equations

General Case for 3 Equations in 3 Unknowns

The standard form of the equations is:

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$$

Using determinant notation, Cramer's Rule states that the solutions for ***x, y*** and z are:

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

As in the case with two equations, the determinants in the denominators are all the same; the columns in these determinants are the coefficients of the variables in the equations. The determinants in the numerators are almost the same as the ones in the denominators; the only difference is that the column of coefficients associated with the variable being evaluated is replaced by the equations' constant terms.

Example: Consider these equations:

$$\begin{aligned} 3x + y + z &= 7 \\ x - 2y - z &= -2 \\ -2x + 2y + 3z &= -4 \end{aligned}$$

Note that the determinant of variable coefficients must be non-zero in order to use Cramer's Rule. If this determinant is zero, there is no unique solution to the system of equations.

Using determinant notation:

$$x = \frac{\begin{vmatrix} 7 & 1 & 1 \\ -2 & -2 & -1 \\ -4 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ -2 & 2 & 3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 3 & 7 & 1 \\ 1 & -2 & -1 \\ -2 & -4 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ -2 & 2 & 3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} 3 & 1 & 7 \\ 1 & -2 & -2 \\ -2 & 2 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ 1 & -2 & -1 \\ -2 & 2 & 3 \end{vmatrix}}$$

Performing the required calculations, we obtain the unique solution:

$$x = \frac{-30}{-15} = 2 \qquad y = \frac{-45}{-15} = 3 \qquad z = \frac{30}{-15} = -2$$

Algebra
Augmented Matrices

Solving Simultaneous Equations

One relatively advanced method of solving simultaneous equations is through the use of an Augmented Matrix. A matrix is considered augmented if it consists of the matrix of the coefficients of the variables, augmented by the constant terms. In order for a system of equations to be solved in this form, they must be written in standard form.

Example:

To solve the system: $-x + 3y = 13$, $2x - y = 4$

The augmented matrix would be: $\left[\begin{array}{cc|c} -1 & 3 & 13 \\ 2 & -1 & 4 \end{array}\right]$

Gauss-Jordan Elimination

A process called Gauss-Jordan Elimination (GJE) is used to manipulate the augmented matrix to obtain a solution to the equations. GJE is also called Row Reduction because each step adjusts the values in one row of the augmented matrix. At the end of the process, the rows of the coefficient matrix are "reduced" to the Identity Matrix.

The following manipulations of the rows are allowed:

- Multiplying or dividing a row by a scalar (i.e., a number).
- Switching rows.
- Adding or subtracting a multiple of one row to or from another.

When this process is complete, the constant column of the augmented matrix has been converted to the solution of the system of equations. Why does this work? The process used is essentially the same as solving a system of equations by the elimination method. In GJE, you ignore the variable names by using matrices, but the manipulations are the same.

Inverse Matrix

This process can also be used to develop an Inverse Matrix. To do this,

- Place an identity matrix to the right of the augmented matrix at the start.
- Perform all row operations on this matrix as you progress.
- At the end, the original identity matrix will have been converted to the inverse matrix.

In the following examples, augmented matrices are manipulated to develop solutions to systems of equations and identity matrices are converted to inverse matrices.

Algebra
2x2 Augmented Matrix Examples

Problem: solve the following set of simultaneous equations using row reduction (formally called Gauss-Jordan Elimination)	2x + 8y = 36 x + 5y = 10

Action Taken	Why?	Augmented Matrix x	y	=		Inverse Matrix	
Start		x	y	=			
Starting Augmented Matrix		2	8	36		1	0
		1	5	10		0	1
Adjust Top Row							
(Row 1) ÷ (a_{11}) --> Row 1	To get a "1" in column 1	1	4	18		0.5	0
Row2 (no change)		1	5	10		0	1
Adjust 2nd Row							
Row1 (no change)		1	4	18		0.5	0
(Row 2) - (a_{21} * Row 1) --> Row 2	To get a "0" in column 1	0	1	-8		-0.5	1
Adjust 2nd Row							
Row1 (no change)		1	4	18		0.5	0
(Row 2) ÷ (a_{22}) --> Row 2	To get a "1" in column 2	0	1	-8		-0.5	1
Adjust Top Row							
(Row 1) - (a_{12} * Row 2) --> Row 1	To get a "0" in column 2	1	0	50	**= x**	2.5	-4
Row2 (no change)		0	1	-8	**= y**	-0.5	1
						Inverse Matrix	

Matrix Inverse Test	Original Matrix		Inverse Matrix		Product Matrix	
	2	8	2.5	-4	**1**	**0**
	1	5	-0.5	1	**0**	**1**

Problem: solve the following set of simultaneous equations using row reduction (formally called Gauss-Jordan Elimination)	-x + 3y = 13 2x - y = 4

Action Taken	Why?	Augmented Matrix x	y	=		Inverse Matrix	
Start		x	y	=			
Starting Augmented Matrix		-1	3	13		1	0
		2	-1	4		0	1
Adjust Top Row							
(Row 1) ÷ (a_{11}) --> Row 1	To get a "1" in column 1	1	-3	-13		-1	0
Row2 (no change)		2	-1	4		0	1
Adjust 2nd Row							
Row1 (no change)		1	-3	-13		-1	0
(Row 2) - (a_{21} * Row 1) --> Row 2	To get a "0" in column 1	0	5	30		2	1
Adjust 2nd Row							
Row1 (no change)		1	-3	-13		-1	0
(Row 2) ÷ (a_{22}) --> Row 2	To get a "1" in column 2	0	1	6		0.4	0.2
Adjust Top Row							
(Row 1) - (a_{12} * Row 2) --> Row 1	To get a "0" in column 2	1	0	5	**= x**	0.2	0.6
Row2 (no change)		0	1	6	**= y**	0.4	0.2
						Inverse Matrix	

Matrix Inverse Test	Original Matrix		Inverse Matrix		Product Matrix	
	-1	3	0.2	0.6	**1**	**0**
	2	-1	0.4	0.2	**0**	**1**

Algebra
3x3 Augmented Matrix Example

Problem: solve the following set of simultaneous equations using row reduction (formally called Gauss-Jordan Elimination)	2x - y + 3z = -7 x + 4y -2z = 17 3x +y +2z = 2

Action Taken	Why?	Augmented Matrix					Inverse Matrix		
Starting Augmented Matrix		x	y	z	=				
Starting Augmented Matrix		2	-1	3	-7		1	0	0
		1	4	-2	17		0	1	0
		3	1	2	2		0	0	1
Work Down									
Row1 * 1/2 --> Row1	To get a "1" in column 1	1	-0.5	1.5	-3.5		0.5	0	0
Row2 - (new Row1) --> Row2	To get a "0" in column 1	0	4.5	-3.5	20.5		-0.5	1	0
Row3 - (3 * new Row1) --> Row 3	To get a "0" in column 1	0	2.5	-2.5	12.5		-1.5	0	1
Work Down									
Row1 (no change)		1	-0.5	1.5	-3.5		0.5	0	0
Switch Rows 2 and 3	New Row2 looks easier to work with	0	2.5	-2.5	12.5		-1.5	0	1
Switch Rows 2 and 3	New Row2 looks easier to work with	0	4.5	-3.5	20.5		-0.5	1	0
Work Down									
Row1 (no change)		1	-0.5	1.5	-3.5		0.5	0	0
Row2 / 2.5 --> Row2	To get a "1" in column 2	0	1	-1	5		-0.6	0	0.4
Row3 * 2 --> Row3	To get rid of the fractions	0	9	-7	41		-1	2	0
Work Down									
Row1 (no change)		1	-0.5	1.5	-3.5		0.5	0	0
Row2 (no change)		0	1	-1	5		-0.6	0	0.4
Row3 - (9 * Row2) --> Row 3	To get a "0" in column 2	0	0	2	-4		4.4	2	-3.6
Work Down									
Row1 (no change)		1	-0.5	1.5	-3.5		0.5	0	0
Row2 (no change)		0	1	-1	5		-0.6	0	0.4
Row3 * .5 --> Row3	To get a "1" in column 3	0	0	1	-2		2.2	1	-1.8
Work Up									
Row1 - (Row3 * 1.5) --> Row1	To get a "0" in column 3	1	-0.5	0	-0.5		-2.8	-1.5	2.7
Row2 + Row3 --> Row2	To get a "0" in column 3	0	1	0	3		1.6	1	-1.4
Row3 (no change)		0	0	1	-2		2.2	1	-1.8
Work Up									
Row1 + (Row2 * .5) --> Row1	To get a "0" in column 2	1	0	0	1	= x	-2	-1	2
Row2 (no change)		0	1	0	3	= y	1.6	1	-1.4
Row3 (no change)		0	0	1	-2	= z	2.2	1	-1.8
							Inverse Matrix		

Matrix Inverse Test	**Original Matrix**			**Inverse Matrix**			**Product Matrix**		
	2	-1	3	-2	-1	2	**1**	**0**	**0**
	1	4	-2	1.6	1	-1.4	**0**	**1**	**0**
	3	1	2	2.2	1	-1.8	**0**	**0**	**1**

Algebra
Characteristic Equation and Eigenvalues

Characteristic Equation

The characteristic polynomial or characteristic function of a square matrix $\boldsymbol{A}$ is defined as:

$$P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}|$$

As a simple example, let's find the characteristic polynomial of matrix $\boldsymbol{A} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$.

$$P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}| = \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(1-\lambda) - (-1)(2) = \lambda^2 - 5\lambda + 4 + 2$$

$$= \lambda^2 - 5\lambda + 6$$

The characteristic equation of a square matrix is determined by setting its characteristic polynomial equal to zero. The characteristic equation of matrix $\boldsymbol{A}$, above, is:

$$\lambda^2 - 5\lambda + 6 = 0$$

Note that:

- $\boldsymbol{A}$ is an n x n square matrix.
- The characteristic polynomial is of degree n.
- There are n roots of the characteristic equation, potentially not all of which are real.

Eigenvalues and Trace

The n roots of the characteristic equation, i.e., the n values of λ, are called the eigenvalues of the matrix. The prefix "eigen" comes from the Old Dutch meaning "to own," referring to the essence of something. So, the matrix owns its eigenvalues and they are key to its existence.

For matrix $\boldsymbol{A}$ in the example above, the eigenvalues are 2 and 3 because:

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda = \{2, 3\}$$

The trace of a square matrix, $tr(A)$, is the sum of its diagonal values (from upper left to lower right). An interesting theorem relates the trace of a matrix to its eigenvalues.

Theorem: The trace of a square matrix is equal to the sum of its eigenvalues.

In the example above, the trace of $\boldsymbol{A}$ is: $tr(\boldsymbol{A}) = 4 + 1 = 5$. The sum of the eigenvalues of $\boldsymbol{A}$ is: $3 + 2 = 5$.

Algebra
Eigenvectors

Eigenvectors

Associated with each eigenvalue of a matrix $\boldsymbol{A}$ is a corresponding eigenvector of $\boldsymbol{A}$. The eigenvectors of $\boldsymbol{A}$, $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, are obtained by solving the equation: $(\boldsymbol{A} - \lambda \boldsymbol{I})\mathbf{v} = \mathbf{0}$ for each value of λ. Note that $\mathbf{0}$ is the zero vector, which is size: n x 1.

Example: For matrix $\boldsymbol{A}$, defined in the example above, let us find the eigenvectors associated with the eigenvalues $\lambda = 2$ and $\lambda = 3$.

Consider $\lambda = 2$:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\mathbf{v} = \begin{pmatrix} 4-2 & -1 \\ 2 & 1-2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

$$\begin{pmatrix} 2x - y \\ 2x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let x be any arbitrary value except zero and we can solve for y. Letting $x = 1$ gives:

$$2x - y = 2(1) - y = 2 - y = 0 \quad \Rightarrow \quad y = 2$$

So, an eigenvector associated with the eigenvalue $\lambda = 2$ of $\boldsymbol{A}$ is $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Note that any vector of the form $\begin{pmatrix} t \\ 2t \end{pmatrix}$ satisfies the above equation and so it is also an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue $\lambda = 2$.

Consider $\lambda = 3$:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\mathbf{v} = \begin{pmatrix} 4-3 & -1 \\ 2 & 1-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

$$\begin{pmatrix} x - y \\ 2x - 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Again, let $x = 1$. Then,

$$x - y = (1) - y = 1 - y = 0 \quad \Rightarrow \quad y = 1$$

So, an eigenvector associated with the eigenvalue $\lambda = 3$ of $\boldsymbol{A}$ is $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Note that any vector of the form $\begin{pmatrix} t \\ t \end{pmatrix}$ satisfies the above equation and so it is also an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue $\lambda = 3$.

Conclude: One set of eigenvectors of $\boldsymbol{A}$ is: $\mathbf{v} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

Algebra
2x2 Eigenvalues and Eigenvectors – General Case

Characteristic Equation

Let: $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Eigenvalues

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4ad + 4bc}}{2} = \frac{(a+d) \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4bc}}{2}$$

$$= \frac{(a+d) \pm \sqrt{a^2 - 2ad + d^2 + 4bc}}{2} = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

Let: $\Delta = \sqrt{(a-d)^2 + 4bc}$

$$\lambda = \frac{a + d \pm \Delta}{2}$$

Eigenvectors

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\mathbf{v} = 0, \qquad \mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Working with the top row: $(a-\lambda)x + by = 0$

$$(\lambda - a)x = by$$

For each λ, then, $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ \lambda - a \end{pmatrix}$

Finally, you may wish to simplify $\mathbf{v}$ by reducing its elements to lowest terms, if possible.

Algebra
Calculating Inverses Using Characteristic Equations

Hamilton-Cayley Theorem: Every square matrix satisfies its own characteristic equation.

That is, if a matrix $\boldsymbol{A}$ is substituted for λ in matrix $\boldsymbol{A}$'s characteristic equation, the result is a true statement, i.e., $P(\boldsymbol{A}) = \boldsymbol{0}$, where $P(\boldsymbol{A})$ is the characteristic polynomial of $\boldsymbol{A}$.

Example: In the case of matrix $\boldsymbol{A} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$, defined above, with characteristic equation $\lambda^2 - 5\lambda + 6 = 0$, it must be true that $\boldsymbol{A}^2 - 5\boldsymbol{A} + 6\boldsymbol{I} = \boldsymbol{0}$, where $\boldsymbol{0}$ is the appropriate zero matrix. Let's demonstrate that this is true.

$$\boldsymbol{A}^2 - 5\boldsymbol{A} + 6\boldsymbol{I} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}^2 - 5\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & -5 \\ 10 & -1 \end{pmatrix} - 5\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 14 - 5(4) + 6 & -5 - 5(-1) + 0 \\ 10 - 5(2) + 0 & -1 - 5(\ 1) + 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

Using Hamilton-Cayley to Calculate Inverses

By setting $P(\boldsymbol{A}) = \boldsymbol{0}$, we can develop another method for calculating inverse matrices. Consider the case of a 3 x 3 matrix $\boldsymbol{A}$. The characteristic equation of $\boldsymbol{A}$ must be of the form:

$a\lambda^3 + b\lambda^2 + c\lambda + d = 0,$ for some values a, b, c, d, with $a \neq 0$.

Then, by the Hamilton-Cayley Theorem, it must be true that:

$$a\boldsymbol{A}^3 + b\boldsymbol{A}^2 + c\boldsymbol{A} + d\boldsymbol{I} = \boldsymbol{0}$$

Multiply both sides by $\boldsymbol{A}^{-1}$ on the right (or the left) to get:

$$a\boldsymbol{A}^3\boldsymbol{A}^{-1} + b\boldsymbol{A}^2\boldsymbol{A}^{-1} + c\boldsymbol{A}\boldsymbol{A}^{-1} + d\boldsymbol{I}\boldsymbol{A}^{-1} = \boldsymbol{0}$$

$$a\boldsymbol{A}^2 + b\boldsymbol{A} + c\boldsymbol{I} + d\boldsymbol{A}^{-1} = \boldsymbol{0}$$

$$d\boldsymbol{A}^{-1} = -(a\boldsymbol{A}^2 + b\boldsymbol{A} + c\boldsymbol{I})$$

$$\boldsymbol{A}^{-1} = -\frac{a\boldsymbol{A}^2 + b\boldsymbol{A} + c\boldsymbol{I}}{d}$$

Algebra
Calculating Inverses Using Characteristic Equations (cont'd)

Example: Find the inverse of the 3 x 3 matrix $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{pmatrix}$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = \begin{vmatrix} 2-\lambda & -1 & 3 \\ 1 & 4-\lambda & -2 \\ 3 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 4-\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & -2 \\ 3 & 2-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & 4-\lambda \\ 3 & 1 \end{vmatrix} = 0$$

$$(2-\lambda)(\lambda^2 - 6\lambda + 10) + (-\lambda + 8) + 3(3\lambda - 11) = 0$$

$\lambda^3 - 8\lambda^2 + 14\lambda + 5 = 0$ is the characteristic equation of $\boldsymbol{A}$.

By the Hamilton-Cayley Theorem, then,

$$\boldsymbol{A}^3 - 8\boldsymbol{A}^2 + 14\boldsymbol{A} + 5\boldsymbol{I} = \boldsymbol{0}$$

Multiply both sides by $\boldsymbol{A}^{-1}$ on the right (or the left) to get:

$$\boldsymbol{A}^3\boldsymbol{A}^{-1} - 8\boldsymbol{A}^2\boldsymbol{A}^{-1} + 14\boldsymbol{A}\boldsymbol{A}^{-1} + 5\boldsymbol{I}\boldsymbol{A}^{-1} = \boldsymbol{0}$$

$$\boldsymbol{A}^2 - 8\boldsymbol{A} + 14\boldsymbol{I} + 5\boldsymbol{A}^{-1} = \boldsymbol{0} \qquad \text{(next, solve for } \boldsymbol{A}^{-1}\text{)}$$

$$\boldsymbol{A}^{-1} = -\frac{\boldsymbol{A}^2 - 8\boldsymbol{A} + 14\boldsymbol{I}}{5} = -\frac{1}{5}(\boldsymbol{A}^2 - 8\boldsymbol{A} + 14\boldsymbol{I})$$

$$= -\frac{1}{5}\left[\begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{pmatrix} - 8\begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{pmatrix} + 14\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= -\frac{1}{5}\left[\begin{pmatrix} 12 & -3 & 14 \\ 0 & 13 & -9 \\ 13 & 3 & 11 \end{pmatrix} - 8\begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 2 \end{pmatrix} + 14\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= -\frac{1}{5}\begin{pmatrix} 12 - 8(2) + 14 & -3 - 8(-1) & 14 - 8(3) \\ 0 - 8(1) & 13 - 8(4) + 14 & -9 - 8(-2) \\ 13 - 8(3) & 3 - 8(1) & 11 - 8(2) + 14 \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} 10 & 5 & -10 \\ -8 & -5 & 7 \\ -11 & -5 & 9 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 2 \\ 1.6 & 1 & -1.4 \\ 2.2 & 1 & -1.8 \end{pmatrix}$$

This result matches the inverse developed using the augmented matrix method that was illustrated earlier in this chapter.

Algebra

Exponent Formulas

Word Description of Property	Math Description of Property	Limitations on variables	Examples
Product of Powers	$a^p \cdot a^q = a^{(p+q)}$		$x^4 \cdot x^3 = x^7$ $x^5 \cdot x^{-8} = x^{-3}$
Quotient of Powers	$\frac{a^p}{a^q} = a^{(p-q)}$	$a \neq 0$	$\frac{y^5}{y^2} = y^3$
Power of a Power	$(a^p)^q = a^{(p \cdot q)}$		$(z^4)^3 = z^{12}$ $(x^{-3})^{-5} = x^{15}$
Anything to the zero power is 1	$a^0 = 1$	$a \neq 0$	$91^0 = 1$ $(xyz^3)^0 = 1$, if $x, y, z \neq 0$
Negative powers generate the reciprocal of what a positive power generates	$a^{(-p)} = \frac{1}{a^p}$	$a \neq 0$	$x^{(-3)} = \frac{1}{x^3}$ $\left(\frac{1}{x}\right)^{-5} = x^5$
Power of a product	$(a \cdot b)^p = a^p \cdot b^p$		$(3y)^3 = 27y^3$ $[(x+1)z]^4 = (x+1)^4 z^4$
Power of a quotient	$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$	$b \neq 0$	$\left(\frac{x}{4}\right)^3 = \frac{x^3}{64}$
Converting a root to a power	$\sqrt[n]{a} = a^{(1/n)}$	$n \neq 0$	$\sqrt{x} = x^{1/2}$

Algebra

Logarithm Formulas

Word Description of Property	Math Description of Property	Limitations on variables	Examples
Definition of logarithm	$(\log_b x = a)$ implies $(b^a = x)$	$b, x > 0$ $b \neq 1$	$\log_3 x = 4$ implies $3^4 = x$ $\log_7(-49)$ is undefined
Log (base anything) of 1 is zero	$\log_b 1 = 0$	$b > 0$ $b \neq 1$	$\log_{32} 1 = 0$ $\ln 1 = 0$
Exponents and logs are inverse operators, leaving what you started with	$b^{(\log_b x)} = x$	$b, x > 0$ $b \neq 1$	$3^{(\log_3 92)} = 92$ $e^{(\ln x)} = x$
Logs and exponents are inverse operators, leaving what you started with	$\log_b(b^x) = x$	$b, x > 0$ $b \neq 1$	$\log_6(6^{xyz}) = xyz$ $\ln\ (e^{4y}) = 4y$
The log of a product is the sum of the logs	$\log_b(m \cdot n) = \log_b m + \log_b n$	$m, n, b > 0$ $b \neq 1$	$\log_2(32x) = 5 + \log_2 x$ $\ln(8e) = \ln(8) + 1$
The log of a quotient is the difference of the logs	$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$	$m, n, b > 0$ $b \neq 1$	$\log_3\left(\frac{3}{x}\right) = 1 - \log_3 x$ $\ln\left(\frac{12}{e}\right) = \ln(12) - 1$
The log of something to a power is the power times the log	$\log_b(m^p) = p \cdot \log_b m$	$m, b > 0$ $b \neq 1$	$\log_4(x^{23}) = 23 \cdot \log_4 x$ $\ln(3^z) = z \cdot \ln(3)$
Change the base to whatever you want by dividing by the log of the old base	$\log_b m = \frac{\log_a m}{\log_a b}$	$m, a, b > 0$ $a, b \neq 1$	$\log_{100} x = \frac{\log_{10} x}{2}$

ADVANCED

Algebra

e

What is "e"?

- e is a transcendental number, meaning that it is not the root of any polynomial with integer coefficients.
- e is the base of the natural logarithms.

What Makes "e" so Special?

e shows up over and over in mathematics, especially in regard to limits, derivatives, and integrals. In particular, it is noteworthy that:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \qquad e = \lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{n!}}\right) \qquad \frac{d}{dx}(e^x) = e^x \qquad \int_1^e \frac{dx}{x} = 1$$

Perhaps, most interestingly, the following equation, called **Euler's Equation**, relates five seemingly unrelated mathematical constants to each other.

$$\boldsymbol{e^{i\pi}+1=0}$$

Some Series Representations of e

$$e = \sum_{k=0}^{\infty}\frac{1}{k!} = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\cdots$$

There are many more series involving e. A sampling of these is provided at:

http://mathworld.wolfram.com/e.html.

$$e = \left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\right]^{-1} = \frac{1}{1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\cdots}$$

Decimal Expansion

e =2.7 1828 1828 4590 4523 5360 2874 7135 2662 4977 5724 7093 6999 5957 4966 …

The web site http://antwrp.gsfc.nasa.gov/htmltest/gifcity/e.2mil shows the decimal expansion of e to over 2 million digits.

Algebra
Table of Exponents and Logarithms

Definition: $b^a = c$ **if and only if** $\log_b c = a$

$2^0 = 1$	$\log_2 1 = 0$
$2^1 = 2$	$\log_2 2 = 1$
$2^2 = 4$	$\log_2 4 = 2$
$2^3 = 8$	$\log_2 8 = 3$
$2^4 = 16$	$\log_2 16 = 4$
$2^5 = 32$	$\log_2 32 = 5$
$2^6 = 64$	$\log_2 64 = 6$
$2^7 = 128$	$\log_2 128 = 7$
$2^8 = 256$	$\log_2 256 = 8$
$2^9 = 512$	$\log_2 51\,2 = 9$
$2^{10} = 1024$	$\log_2 1024 = 10$

$3^0 = 1$	$\log_3 1 = 0$
$3^1 = 3$	$\log_3 3 = 1$
$3^2 = 9$	$\log_3 9 = 2$
$3^3 = 27$	$\log_3 27 = 3$
$3^4 = 81$	$\log_3 81 = 4$
$3^5 = 243$	$\log_3 243 = 5$

$4^0 = 1$	$\log_4 1 = 0$
$4^1 = 4$	$\log_4 4 = 1$
$4^2 = 16$	$\log_4 16 = 2$
$4^3 = 64$	$\log_4 64 = 3$
$4^4 = 256$	$\log_4 256 = 4$

$5^0 = 1$	$\log_5 1 = 0$
$5^1 = 5$	$\log_5 5 = 1$
$5^2 = 25$	$\log_5 25 = 2$
$5^3 = 125$	$\log_5 125 = 3$
$5^4 = 625$	$\log_5 625 = 4$

$6^0 = 1$	$\log_6 1 = 0$
$6^1 = 6$	$\log_6 6 = 1$
$6^2 = 36$	$\log_6 36 = 2$
$6^3 = 216$	$\log_6 216 = 3$

$7^0 = 1$	$\log_7 1 = 0$
$7^1 = 7$	$\log_7 7 = 1$
$7^2 = 49$	$\log_7 49 = 2$
$7^3 = 343$	$\log_7 343 = 3$

$8^0 = 1$	$\log_8 1 = 0$
$8^1 = 8$	$\log_8 8 = 1$
$8^2 = 64$	$\log_8 64 = 2$
$8^3 = 512$	$\log_8 512 = 3$

$9^0 = 1$	$\log_9 1 = 0$
$9^1 = 9$	$\log_9 9 = 1$
$9^2 = 81$	$\log_9 81 = 2$
$9^3 = 729$	$\log_9 729 = 3$

$10^0 = 1$	$\log_{10} 1 = 0$
$10^1 = 10$	$\log_{10} 10 = 1$
$10^2 = 100$	$\log_{10} 100 = 2$
$10^3 = 1000$	$\log_{10} 1000 = 3$

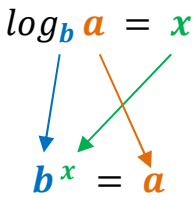
$11^0 = 1$	$\log_{11} 1 = 0$
$11^1 = 11$	$\log_{11} 11 = 1$
$11^2 = 121$	$\log_{11} 121 = 2$
$11^3 = 1331$	$\log_{11} 1331 = 3$

Algebra
Converting Between Exponential and Logarithmic Forms

To convert between an exponential expression and a logarithmic expression, it is often helpful to use the "first-last-middle" rule to perform the conversion. If necessasy, set the expression equal to x before applying the rule.

Note: the "first-last-middle" rule requires that the logarithmic or exponential portion of the expression be on the left-hand side of the equation.

Converting from Logarithmic Form to Exponential Form



using "**first-last-middle**"

Examples:

1) Solve for x: $log_4 64 = x$.

   First is "4", **last** is "x" and **middle** is "64." So, $4^x = 64$.

   Then, $4^1 = 4;\ 4^2 = 16;\ 4^3 = 64$ ✓

   So, we have: $\boldsymbol{x = 3}$

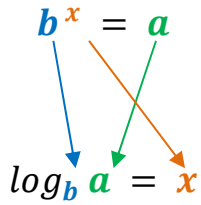
2) Solve for x: $\ln e = x$
   (remember ln is shorthand for log_e)

   Using first-last-middle,

   $log_e\, e = x$ converts to: $e^x = e$

   So, we have: $\boldsymbol{x = 1}$

Converting from Exponential Form to Logarithmic Form



using "**first-last-middle**"

Examples:

1) Convert the expression, $2^5 = 32$ to logarithmic form.

   First is "2", **last** is "32" and **middle** is "5".

   So, we have: $\boldsymbol{log_2 32 = 5}$

2) Convert the expression, $7^3 = 343$ to logarithmic form.

   Using first-last-middle,

   $7^3 = 343$ converts to: $log_7 343 = 3$

   So, we have: $\boldsymbol{log_7 343 = 3}$

Algebra
Expanding Logarithmic Expressions

Expanding a logarithmic expression is a process best described by example. Each step of the process is described and illustrated in the example below.

Expand: $log_2\left(\frac{6a^3b^2c^3}{9c^7}\right)$

When expanded:

- Each item in the numerator will become a term preceded by a "+" sign
- Each item in the denominator will become a term preceded by a "–" sign.
- All exponents become term coefficients in expanded form.

Step 1: Simplify terms in the original expression, if possible:

$$log_2\left(\frac{6a^3b^2c^3}{9c^7}\right) = log_2\left(\frac{2a^3b^2}{3c^4}\right)$$

Step 2: Write the *log* of all of the items in parentheses in the simplified expression:

$$log_2 2 \quad log_2 a \quad log_2 b \quad log_2 3 \quad log_2 c$$

Step 3: Write the exponents from the simplified expression as coefficients of each log:

$$log_2 2 \quad \mathbf{3}log_2 a \quad \mathbf{2}log_2 b \quad log_2 3 \quad \mathbf{4}log_2 c$$

Step 4: Write the signs ("+" for items in the numerator; "−" for items in the denominator):

$$log_2 2 + 3log_2 a + 2log_2 b - log_2 3 - 4log_2 c$$

Step 5: Simplify terms in the solution, if possible:

$$1 + 3log_2 a + 2log_2 b - log_2 3 - 4log_2 c$$

Result: $$log_2\left(\frac{6a^3b^2c^3}{9c^7}\right) = 1 + 3log_2 a + 2log_2 b - log_2 3 - 4log_2 c$$

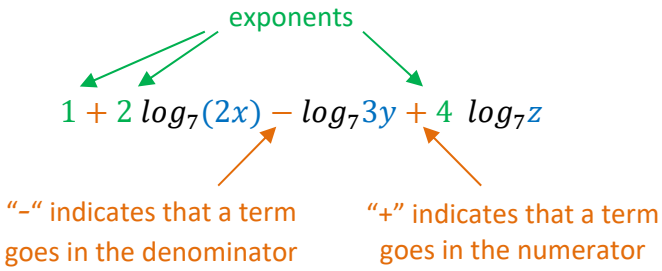
Algebra
Condensing Logarithmic Expressions

Condensing a logarithmic expression is a process best described by example. Each step of the process is described and illustrated in the example below.

Condense: $1 + 2\ log_7(2x) - log_7 3y + 4\ log_7 z$

Step 1: Review the expression and identify each element.

- The argument of each log will become the base of an exponential term.
- The coefficient of each log will become an exponent on that term
- The sign of each term determines whether the exponential term goes in the numerator (+) or denominator of the condensed expression.



When condensed, each term will become exponential in form. All terms in the numerator will be multiplied together. All terms in the denominator will be multiplied together.

Step 2: Set up the log expression with the proper base and parentheses to contain the various terms. If there is at least one negative sign, set up a fraction inside the parentheses:

$$log_7 \left(\frac{\quad}{\quad} \right)$$

Step 3: Convert any constants to powers of the base of the log:

$$log_7 \left(\frac{7^1 \ldots}{\quad} \right)$$

Step 4: Bring in each term containing a variable as an exponential expression with the proper exponent and base:

$$log_7 \left(\frac{7\ (2x)^2\ z^4}{3y} \right)$$

Step 5: Simplify to the extent possible:

$$log_7 \left(\frac{7\ (2x)^2\ z^4}{3y} \right) = log_7 \left(\frac{7 \cdot 4x^2\ z^4}{3y} \right) = log_7 \left(\frac{28\ x^2\ z^4}{3y} \right)$$

Algebra

Condensing Logarithmic Expressions – More Examples

Suggestions for doing problems:

- Use parentheses liberally to help yourself see what is going on in the problem.
- Do the problems one step at a time, working carefully down the page.
- Leave yourself a lot of room to do the work; there may be a lot of steps.

Some additional examples to help see how the various rules fit together:

$$(8\log_b x) + \left(\frac{1}{4}\log_b y\right) - (2\log_b z) = \log_b\left(\frac{x^8 \cdot y^{\left(\frac{1}{4}\right)}}{z^2}\right)$$

$$6 + \ln(3e^5) = 6 + (\ln 3) + \ln(e^5) = 11 + \ln 3$$

$$2 + 3 \cdot \log_4(x^2) + \log_{12} 1 = \log_4(16) + \log_4(x^6) + 0 = \log_4(16x^6)$$

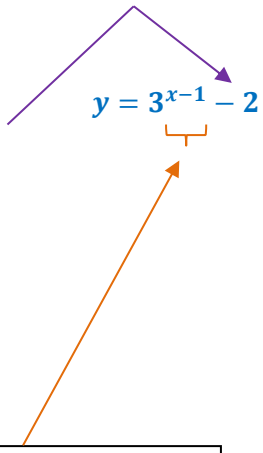
$$\log_{100} x + \log_{10} x = \frac{\log_{10}(x)}{\log_{10}(100)} + \log_{10} x = \frac{3}{2}\log_{10} x$$

$$\log_4\left(\frac{1}{16}\right) - 6^{(\log_6 4)} = -2 - 4 = -6$$

Algebra
Graphing an Exponential Function

Graphing an exponential or logarithmic function is a process best described by example. Each step of the process is described and illustrated in the examples over the next few pages.

Graph the function: $y = 3^{x-1} - 2$



Step 1: The horizontal asymptote occurs at the value of the constant term. This is because the exponential term approaches zero as the exponent becomes more and more negative.

$y = -2$ is the asymptote.

Step 2: Select points for the graph:

In selecting points for an exponential curve, good choices often relate to the value of the exponent. Choose values that make the exponent -1, 0 and 1; or -1, 0 and 2.

In this example, select x so that:

$x - 1 = -1$ so, $x = 0$

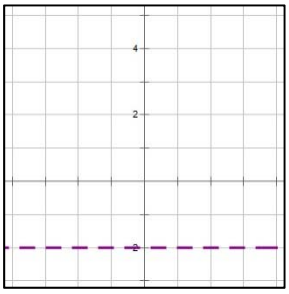
$x - 1 = 0$ so, $x = 1$

$x - 1 = 1$ so, $x = 2$

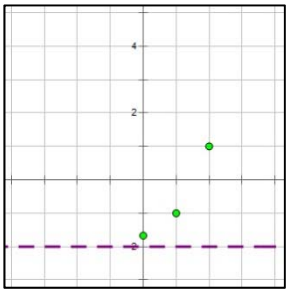
x	y
0	$y = 3^{0-1} - 2 = -1.67$
1	$y = 3^{1-1} - 2 = -1$
2	$y = 3^{2-1} - 2 = 1$

Step 3: Graph the exponential function:

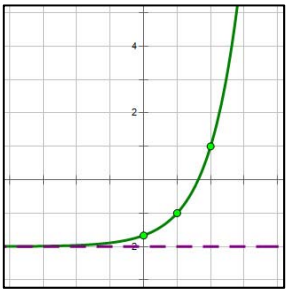
3A: Graph the asymptote $y = -2$



3B: Graph the points



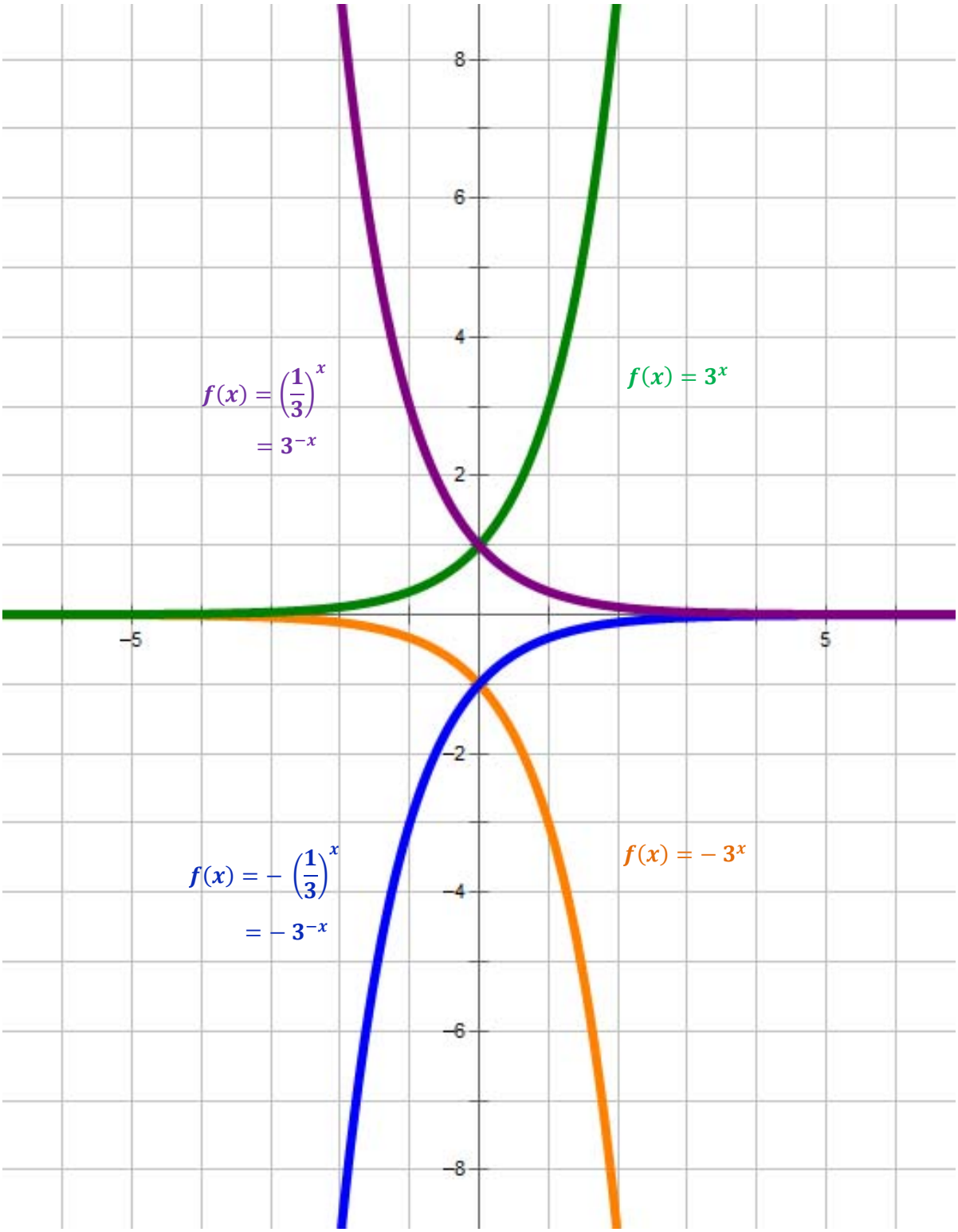
3C: Sketch in the curve



Four Exponential Function Graphs

Impact of varying base (above 1 and below 1)

Note: exponential and logarithmic functions that are inverses of each other (on this page and the one following) are the same color.



Algebra
Graphing a Logarithmic Function

Method 1: Direct Method

Graph the sample Function: $y = log_4(x - 1) + 3$

Step 1: Find the vertical asymptote at the value of x that makes the argument of the log zero.

$x - 1 = 0$ so, $\boldsymbol{x = 1}$ is the asymptote.

$$\boldsymbol{y = log_4\,(x - 1) + 3}$$

Step 2: Select points for the graph:

In selecting points for a logarithmic curve, good choices often relate to the value of the argument. Choose values that make the argument 1 and the base of the logarithm (1 and 4 in this example).

In this example, select x so that:

$x - 1 = 1$ so, $x = 2$

$x - 1 = 4$ so, $x = 5$

See what happens when we do this in the table to the right:

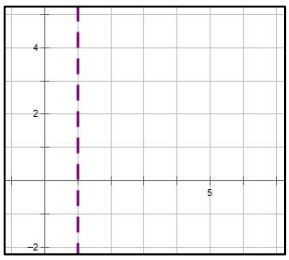
x	y
2	$y = log_4\,(2 - 1) + 3 = log_4\,(1) + 3 = \mathbf{3}$
5	$y = log_4\,(5 - 1) + 3 = log_4\,(4) + 3 = \mathbf{4}$

By selecting x's so that the values in parentheses are 1 and the base of the log, the calculation of y's becomes easy.

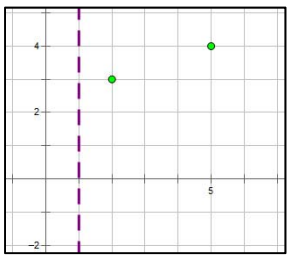
Note that two points may be sufficient to graph the curve if we have also drawn the asymptote.

Step 3: Graph the logarithmic function:

3A: Graph the asymptote $x = 1$



3B: Graph the points



3C: Sketch in the curve

Algebra
Graphing a Logarithmic Function (cont'd)

Natural Logarithm Notation: $\ln(\ldots)$

"ln" stands for "logarithmus naturalis" (Latin for "natural logarithm") and is the logarithm with base "e". The "ln" notation was invented in the late 1800's, when a lot of work was done by hand (there were no electronic calculators or computers). Those who worked with logarithms often used natural logarithms extensively, so this shorthand notation became quite popular, and has survived to this day.

So, when you see: $\ln(\ldots)$ think $\log_e(\ldots)$.

Method 1: Direct Method when the base of the logarithm is "e"

Step 1: Find the vertical asymptote at the value of x that makes the argument of the log zero.

$y = \ln(x-1) + 3$

$x - 1 = 0$ so, $x = 1$ is the asymptote.

Step 2: Select points for the graph:

In selecting points for a logarithmic curve, good choices often relate to the value of the argument. Choose values that make the argument 1 and the base of the logarithm (1 and e in this example).

In this example, select x so that:

$x - 1 = 1$ so, $x = 2$

$x - 1 = e$ so, $x = e + 1$

You need to know that $e \sim 2.7$.
So, $e + 1 \sim 3.7$.

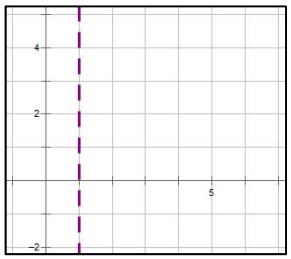
x	y
2	$y = \ln(2-1) + 3 = \ln(1) + 3 = 3$
3.7	$y = \ln(e+1-1) + 3 = \ln(e) + 3 = 4$

By selecting x's so that the values in parentheses are 1 and the base of the log, the calculation of y's becomes easy.

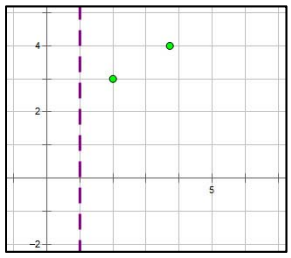
Note that two points may be sufficient to graph the curve if we have also drawn the asymptote.

Step 3: Graph the logarithmic function:

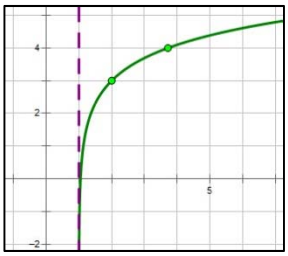
3A: Graph the asymptote $x = 1$



3B: Graph the points



3C: Sketch in the curve



Algebra
Graphing a Logarithmic Function (cont'd)

Method 2: Double-Inverse Method

Step 1: Find the inverse function: $y = log_4(x-1) + 3$

Switch variables: $x = log_4(y-1) + 3$

Subtract the constant: $-3 \qquad -3$

Result: $x - 3 = log_4(y-1)$

Take powers of the log base: $4^{x-3} = 4^{log_4(y-1)}$

Simplify: $4^{x-3} = y - 1$

Subtract the constant: $+1 \quad +1$

Resulting INVERSE Function: $4^{x-3} + 1 = y$ or $y = 4^{x-3} + 1$

Step 2: Find the vertical asymptote at the value of x that makes the argument of the log zero.

$x - 1 = 0$

so, $x = 1$ is the asymptote.

Step 3: Select points for the graph:

In selecting points for an exponential curve, good choices often relate to the value of the exponent. Choose values that make the exponent -1, 0 and 1; or -1, 0 and 2.

In this example, select x so that:

$x - 3 = -1$ so, $x = 2$

$x - 3 = 0$ so, $x = 3$

$x - 3 = 1$ so, $x = 4$

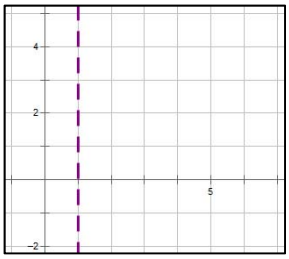
x	y
2	$y = 4^{2-3} + 1 = 1.25$
3	$y = 4^{3-3} + 1 = 2$
4	$y = 4^{4-3} + 1 = 5$

Step 4: Switch the x and y values to get points for the logarithmic function:

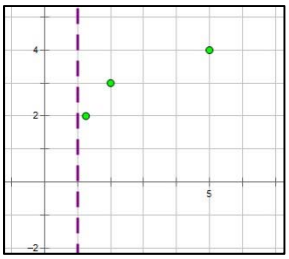
x	y
1.25	2
2	3
5	4

Step 5: Graph the logarithmic function:

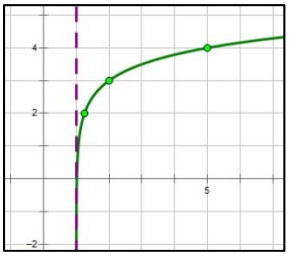
5A: Graph the asymptote $x = 1$



5B: Graph the points



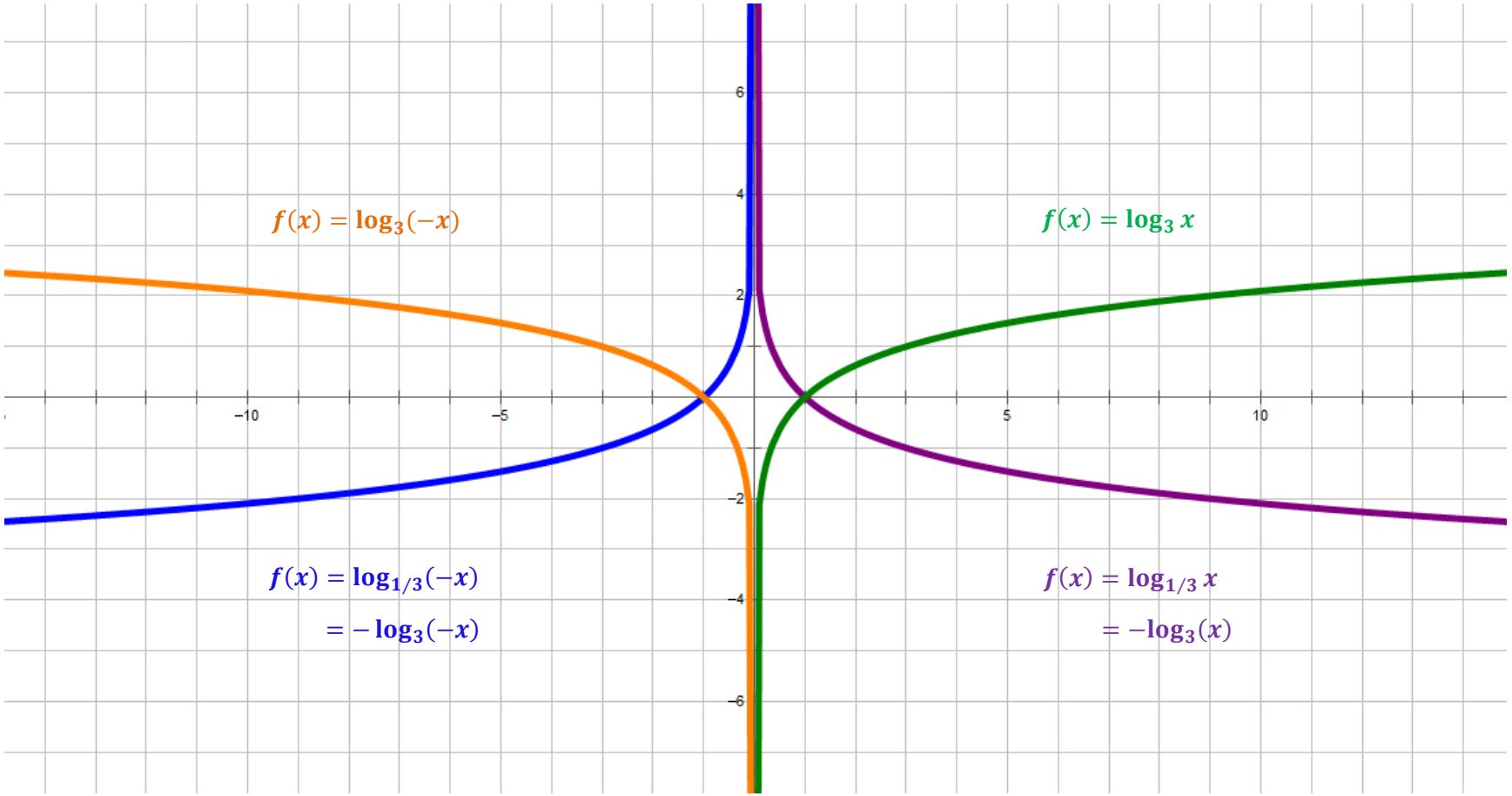
5C: Sketch in the curve



Four Logarithmic Graphs

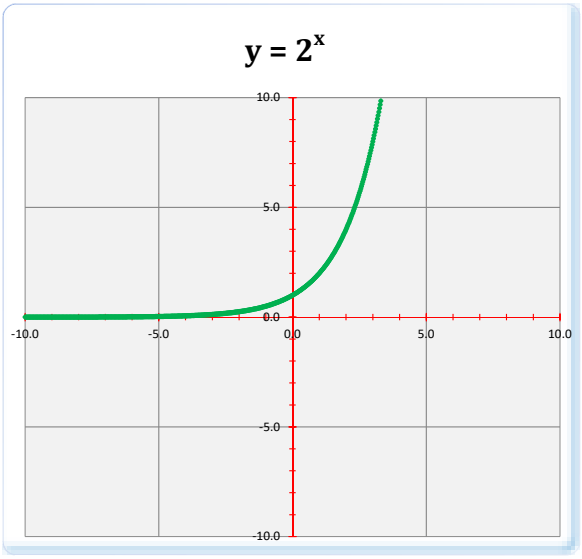
Impact of varying base (above 1 and below 1)

Note: exponential and logarithmic functions that are inverses of each other (on the earlier page and on this one) are the same color.

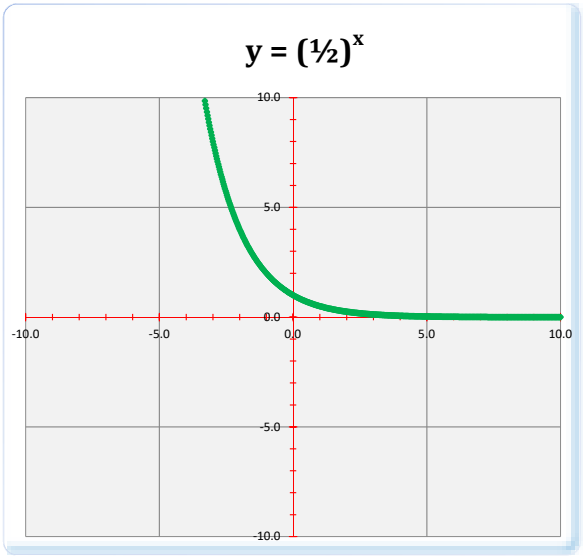


Algebra
Graphs of Various Functions

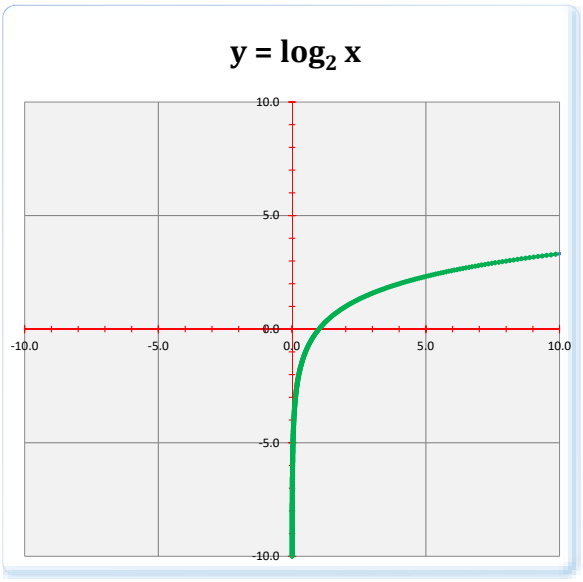
$y = 2^x$



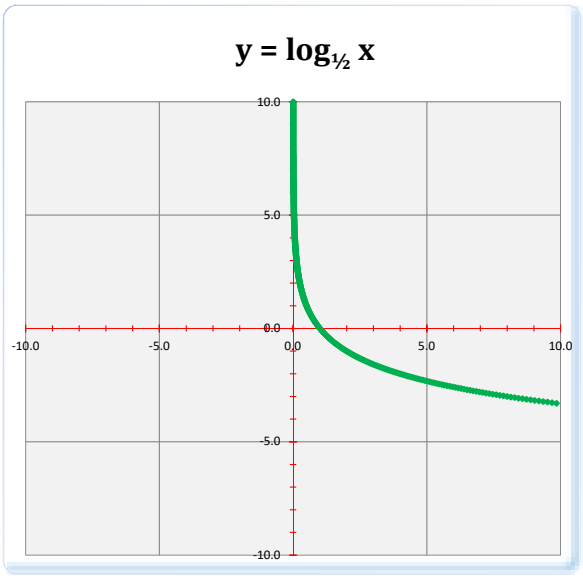
$y = (½)^x$



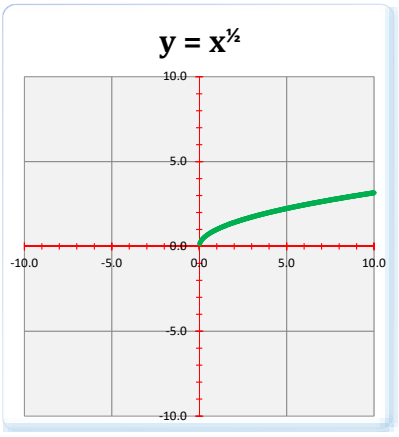
$y = \log_2 x$



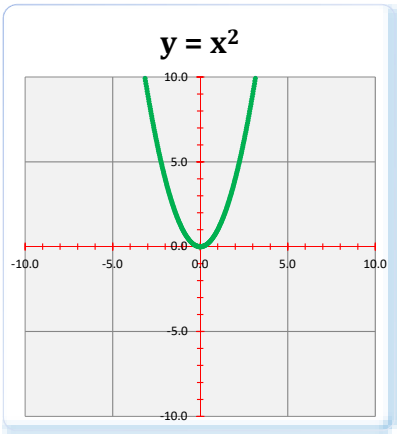
$y = \log_{½} x$



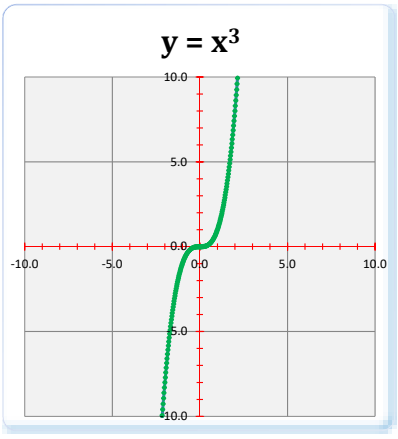
$y = x^{½}$



$y = x^2$



$y = x^3$



Algebra
Applications of Exponential Functions

Exponential Growth and Decay

$$\begin{pmatrix} end \\ value \end{pmatrix} = \begin{pmatrix} start \\ value \end{pmatrix} \cdot (1 + growth)^n$$

$$\begin{pmatrix} end \\ value \end{pmatrix} = \begin{pmatrix} start \\ value \end{pmatrix} \cdot (1 - decay)^n$$

Note: if you let the end value be represented by $\boldsymbol{A(t)}$ and the start value by $\boldsymbol{P}$, you get similar formulas to the interest formulas below.

Interest Formulas

Let: $A(t)$ = Amount of money at time (t)
P = Principal (starting amount of money); note that $P = A(0)$
r = the annual rate of interest (e.g., 4% or .04)
n = the number of times per year that interest is credited

Compound interest pays interest a number of times during the year; that is, in periods after the first, interest is paid on the original amount invested plus interest earned in prior periods.

Compound interest paid n times a year: $\boldsymbol{A(t) = P \cdot \left(1 + \frac{r}{n}\right)^{nt}}$

Simple case. If interest is compounded on an annual basis, we get the simplest formula:

Annual interest paid once a year: $\boldsymbol{A(t) = P \cdot (1 + r)^t}$

Continuous compounding. The more frequent the compounding of interest, the more money you get. The best return on your money occurs if interest is compounded continuously. Because of the definition of the mathematical constant "e", we get the following formula (known as the **"Pert"** formula).

Interest compounded continuously: $\boldsymbol{A(t) = P \cdot (e)^{rt}}$

Algebra
Solving Exponential and Logarithmic Equations

Logarithms and exponents are inverse operations in the same way addition and subtraction are, and in the same way multiplication and division are. So, to undo an exponent, you take a logarithm, and to undo a logarithm, you take an exponent.

Solving an Exponential Equation

Sometimes an equation has a variable in an exponent. To solve this kind of equation, follow these steps:

- Isolate the term with the exponent on one side of the equation. That is, undo any additions, subtractions, multiplications, and divisions in the equation.
- Take a logarithm of both sides of the equation. Use the base that exists in the exponential term.
- Solve the equation that remains.

Example:

Start:	$2 + 4 \cdot 3^{(x-2)} = 110$
Subtract 2:	$4 \cdot 3^{(x-2)} = 108$
Divide by 4:	$3^{(x-2)} = 27$
Take logs:	$\log_3\left(3^{(x-2)}\right) = \log_3 27$
Simplify:	$(x-2) = 3$
Add 2:	$\boldsymbol{x = 5}$

Note: in the example at right, the base of 3 is selected for the logarithms because it is the base of the exponent in the problem.

Solving a Logarithmic Equation

To solve an equation with a logarithm in it, follow these steps:

- Isolate the logarithm on one side of the equation. That is, undo any additions, subtractions, multiplications, and divisions in the equation.
- Take the base of the logarithm to the power of both sides of the equation. Use the same base that exists in the logarithmic term.
- Solve the equation that remains.

Example:

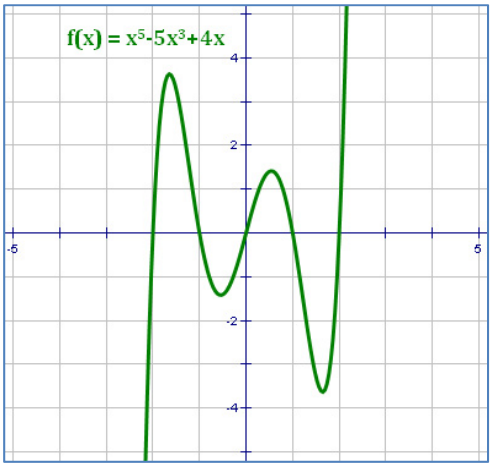
Start:	$-1 + \frac{2}{3}\log_2(x+1) = 1$
Add 1:	$\frac{2}{3}\log_2(x+1) = 2$
Multiply by $\frac{3}{2}$:	$\log_2(x+1) = 3$
Exponentiate:	$2^{\log_2(x+1)} = 2^3$
Simplify:	$(x+1) = 8$
Subtract 1:	$\boldsymbol{x = 7}$

Note: in the example at right, the base of 2 is selected for use in exponentiation because it is the base of the logarithm in the problem.

Algebra
Polynomial Function Graphs

Definitions

- **Local Maximum** – The largest value that a function takes in a neighborhood around the point. There must be smaller values on both sides of the local maximum.
- **Local Minimum** – The smallest value that a function takes in a neighborhood around the point. There must be larger values on both sides of the local minimum.
- A function is **Increasing** over an interval if it generates larger values as $\boldsymbol{x}$ increases over the same interval.
- A function is **Decreasing** over an interval if it generates smaller values as $\boldsymbol{x}$ increases over the same interval.



The graph at the right has two local maxima and two local minima. It also has three intervals where it is increasing and two intervals where it is decreasing.

Characteristics of the Graph of a Polynomial

If $P(x)$ is a polynomial of degree n, then …

- $P(x)$ is continuous over all values of x.
- Roots (i.e., zeros) of $P(x)$ exist wherever the graph intersects the x-axis. There are at most n of them.
- The y-intercept of the graph occurs at $P(0)$, which is the constant term of the polynomial.
- $P(x)$ will have at most $n-1$ local extrema (either maxima or minima). For example, a 5th degree polynomial will have at most $\mathbf{4}$ extrema; the example above has $\mathbf{4}$ extrema.
- At each extreme, the slope of a line tangent to the curve will be zero. However, if the slope of the tangent line to the curve at a point is zero, the point is not necessarily an extreme.
- At a local maximum, the polynomial must be increasing on the left and decreasing on the right.
- At a local minimum, the polynomial must be decreasing on the left and increasing on the right.

ADVANCED

Algebra
Finding Extrema with Derivatives

Derivatives

The derivative of a monomial is: $\frac{d}{dx}(x^n) = nx^{n-1}$

The notation $\frac{d}{dx}$ means "take a derivative with respect to the variable x." We need to know two other things about derivatives in order to find extrema with them.

- The derivative of a sum is the sum of the derivatives. That is:

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big)$$

- Derivatives provide information about the slopes of lines tangent to the curve at each point. Since the slope of a tangent line at a minimum or maximum is zero, we can calculate the derivative of a polynomial and set it equal to zero to find the x-values of its extrema.

Finding Extrema with Derivatives

If $P(x)$ is a polynomial, any extrema lie at points where $\frac{d}{dx}\big(P(x)\big) = 0.$

Example 1: Take the general quadratic equation: $y = ax^2 + bx + c$

We know that the graph of this equation has a single maximum or minimum which is the vertex of the parabola. Taking a derivative of this formula and setting it equal to zero, we get:

$$\frac{d}{dx}(ax^2 + bx + c) = 2ax + b = 0$$

Solving for x, we get : $x = -\frac{b}{2a}$, which we already know is the x-value of the vertex.

Example 2:

Find local maxima and minima for the cubic equation: $y = x^3 + 3x^2 - 9x + 5$

$$\frac{d}{dx}(x^3 + 3x^2 - 9x + 5) = 3x^2 + 6x - 9 = 0$$

Solving this for x, we find local maxima or minima may exist at $x = \{-3, 1\}$.

One caution: When a derivative is equal to zero, it only provides the possibility of an extreme; it does not guarantee an extreme. It is possible for the slope of a curve to be zero and not have either a local maximum or minimum. For an example of this, look at $y = x^3$ where $x = 0$.

Algebra
Factoring Higher Degree Polynomials

Sum and Difference of Cubes

The sum of cubes is of the form: $a^3 + b^3 = (a + b) \cdot (a^2 - ab + b^2)$

The difference of cubes is of the form: $a^3 - b^3 = (a - b) \cdot (a^2 + ab + b^2)$

Notice the following about these two formulas:

- Each formula factors into a binomial and a trinomial.
- Each term in the binomial is of degree 1.
- Each term in the trinomial is of degree 2.
- Each formula has one minus sign in its factorization.
- In each formula, the second term of the binomial factor takes the sign of the second term of the original expression.

Identification

To see if an expression is a sum or difference of cubes, you must answer "yes" to three questions:

1. Are there only two terms?
2. Is the first term a cube? If so, take its cube root to get the value of a.
3. Is the second term a cube? If so, take its cube root to get the value of b.

The solution is based on the appropriate formula above, substituting the cube roots of the two terms for a and b. Be careful!

Note: *A telltale sign of when an expression might be the sum or difference of 2 cubes is when the coefficients on the variables are cubes and the exponents on the variables are multiples of 3.*

Examples:

(1) $8x^3 + 125y^6 = (2x + 5y^2) \cdot (4x^2 - 10xy^2 + 25y^4)$

(2) $x^3 - 8 = (x - 2) \cdot (x^2 + 2x + 4)$

(3) $27 - 64z^3 = (3 - 4z) \cdot (9 + 12z + 16z^2)$

(4) $\frac{x^3}{216} + \frac{y^3}{27} = \left(\frac{x}{6} + \frac{y}{3}\right) \cdot \left(\frac{x^2}{36} - \frac{xy}{18} + \frac{y^2}{9}\right)$

Table of Cubes

$1^3 = 1$
$2^3 = 8$
$3^3 = 27$
$4^3 = 64$
$5^3 = 125$
$6^3 = 216$
$7^3 = 343$
$8^3 = 512$
$9^3 = 729$
$10^3 = 1{,}000$
$11^3 = 1{,}331$
$12^3 = 1{,}728$

Algebra
Factoring Higher Degree Polynomials

Variable Substitution

On occasion you will encounter a polynomial that looks familiar except that the exponents on the variables are higher than usual. In this case, a technique called **Variable Substitution** may be useful.

The steps for variable substitution are:

- Identify which kind of equation the problem resembles.
- Identify what terms are likely to require substitution. Often there are only one or two terms that need to be substituted.
- Create new variables for purposes of substitution.
- Re-write the problem in terms of the new variables.
- Solve the problem in terms of the new variables.
- Substitute the original variables into the solution.
- Perform any additional work that is needed based on the original variables.
- Check your work.

Example 1:

Factor: $x^4 + 3x^2y^2 - 4y^4$

This looks like a typical trinomial factoring problem except for the large exponents.

Create two new variables: $u = x^2$ and $v = y^2$

Re-write the expression: $u^2 + 3uv - 4v^2$

Factor the expression: $= (u + 4v)(u - v)$

Substitute original variables: $= (x^2 + 4y^2)(x^2 - y^2)$

Perform additional work: $= (x^2 + 4y^2)(x + y)(x - y)$

Check your work by multiplying the factored form to see if you get the original polynomial.

Example 2:

Factor: $8x^3 + 125y^6$

This looks like a sum of cubes.

Create two new variables: $u = 2x$ and $v = 5y^2$

Use the sum of cubes formula: $u^3 + v^3 = (u + v) \cdot (u^2 - uv + v^2)$

Substitute original variables: $= (2x + 5y^2)(4x^2 - 10xy^2 + 25y^4)$

Check your work by multiplying the factored form to see if you get the original polynomial.

Algebra
Factoring Higher Degree Polynomials

Synthetic Division

Synthetic Division is a short-cut to dividing polynomials by a linear factor. Here's how it works. We will use an example to illustrate the process.

Example 1: $(2x^3 + 5x^2 + x - 2) \div (x + 2)$

Step 1: In the linear term $(\boldsymbol{x - r})$ take the value $\boldsymbol{r}$ as the divisor. In the example, the divisor will be $\boldsymbol{-2}$. We use the letter r to indicate that the value is actually a root of the equation. So, in synthetic division, the root is used as the divisor.

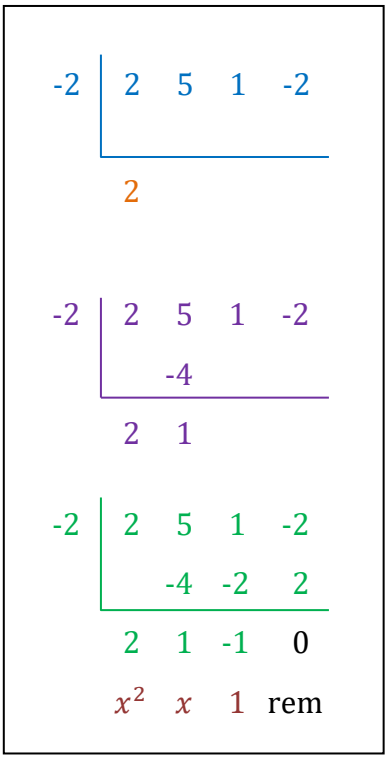
Step 2: Line up the coefficients of the terms from highest degree to lowest degree in a row to the right of the divisor. If a term is missing, use a zero for the coefficient of that term. We will call this array of coefficients the dividend.

Step 3: Bring the leading coefficient down below the line.

Step 4: Multiply the divisor by the number just placed below the line and put the result above the line and one column to the right. Add the two numbers in that column to get a number below the line for that column.

Step 5: Repeat Step 4 until all of the columns have been completed.

The final result is a set of coefficients of the polynomial that results from the division. The exponents of the terms of the resulting polynomial begin one lower than the degree of the original polynomial.



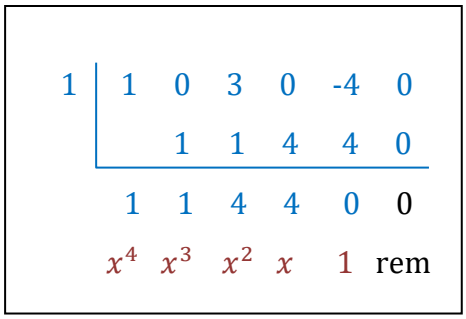
In the example, the result is $\boldsymbol{2x^2 + x - 1}$, with a remainder of **0**. The remainder of **0** is a good indication that the division was performed properly.

Example 2: $(x^5 + 3x^3 - 4x) \div (x - 1)$

From the synthetic division to the right, we get:

$$\boldsymbol{(x^5 + 3x^3 - 4x) \div (x - 1) = x^4 + x^3 + 4x^2 + 4x}$$

There is no constant term and no remainder in the solution to this example.



Algebra
Comparing Synthetic Division to Long Division

Advantages of Synthetic Division

Synthetic division has the following advantages over long division:

- The divisor is a possible root of the polynomial; it is a root if the remainder is zero.
- It is shorter.
- It is much quicker.
- It works by addition and multiplication instead of by subtraction and division. Because of this, it is much less prone to error.

Comparison of Methods

It is instructive to compare synthetic division and long division to get a better idea of why synthetic division works. Consider the division: $(2x^3 + 5x^2 + x - 2) \div (x + 2)$

The two methods of performing this division are laid out below. Notice the following correspondences between the examples:

- **Root vs. Factor.** Synthetic division uses the root of the polynomial as the divisor. Long division uses the whole factor. The signs on the root are opposite in the two methods.
- **Dividend.** The dividends in the two methods are the same (except that synthetic division leaves out the variables).
- **Second Row.** The second row in synthetic division corresponds to the "secondary" coefficients of each division in long division (but with opposite signs).
- **Answer Row.** In synthetic division the answer row (of coefficients) is calculated directly by adding the values in the rows above it. In long division, it is necessary to subtract expressions to determine another expression that must be divided by the divisor to get the next term of the answer.
- **Adding Variables.** In synthetic division, it is necessary to add the variables after the answer is determined. In long division, the answer is provided directly.

Synthetic Division

-2	2	5	1	-2
		-4	-2	2
	2	1	-1	0

Long Division

$$
\begin{array}{r}
 & 2x^2 + 1x - 1 \\
\hline
x + 2 \;\big|\; 2x^3 + 5x^2 + 1x - 2 \\
 \quad 2x^3 + 4x^2 \\
\hline
 \qquad x^2 + 1x - 2 \\
 \quad x^2 + 2x \\
\hline
 \qquad -1x - 2 \\
 \qquad -1x - 2 \\
\hline
 \qquad 0
\end{array}
$$

Algebra
Zeros of Polynomials

Developing Possible Roots

If a polynomial has the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Then,

- $P(x)$ will have exactly n complex roots. For example, a 5th degree polynomial will have exactly 5 complex roots. Note: some of these roots may be the same, and some of them may be real.
- $P(x)$ will have exactly $(n - 2k)$ real roots, where k is a whole number. For example, a 5th degree polynomial will have either 5 real roots, 3 real roots, or 1 real root.
- **Descartes' Rule of Signs.** *(Note how this ties into the bullet above.)*
  - The number of positive real roots of a polynomial $P(x)$ is equal to the number of sign changes in $P(x)$, or is less than this by a multiple of 2.
  - The number of negative real roots of a polynomial $P(x)$ is equal to the number of sign changes in $P(-x)$, or is less than this by a multiple of 2. **Note**: to generate $P(-x)$ quickly, just change the signs of the terms with odd exponents.
- $P(x)$ will have an even number of non-real roots. For example, a 5th degree polynomial will have either 0 non-real roots, 2 non-real roots, or 4 non-real roots. Further, the non-real roots exist in conjugate pairs; so if $(a + bi)$ is a root of $P(x)$, then so is $(a - bi)$.
- **Rational Root Theorem.** Any rational roots have the characteristic $r = \frac{\text{factor of } a_0}{\text{factor of } a_n}$. This fact is especially useful if the lead coefficient of the polynomial is 1; in this case, any real roots are factors of the constant term. This fact, in combination with the ease of synthetic division, makes finding integer roots a quick process.

Example:

What can we say about the roots of $P(x) = x^4 - 2x^3 + x^2 - 4x + 4$? (note: 4 sign changes)

- First, note that $P(-x) = x^4 + 2x^3 + x^2 + 4x + 4$ (note: zero sign changes)
- So, $P(x)$ has 4 complex roots. 0, 2, or 4 of them are real; all real roots are positive.
- The real roots must be 1, 2, or 4 (the positive factors of the constant term 4).
- To find out more, we have to test the possible real root values.

Algebra
Zeros of Polynomials

Testing Possible Roots

The following two theorems are very useful in testing possible **roots** (**zeros**) of Polynomials.

Factor Theorem: $(x - r)$ is a factor of a polynomial $P(x)$ if and only if $P(r) = 0$.

Remainder Theorem: If $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$.

Methods of Testing Possible Roots

If a polynomial can be factored, then **first, factor the polynomial**; the problem will be easier to solve after factoring. In addition, if you are able to produce linear or quadratic factors, the roots of those factors will be roots of the polynomial.

After factoring, the following methods can be used to test possible roots of a polynomial.

- **Use synthetic division to test possible roots.** Because synthetic division is quick, several potential roots can be tested in a short period of time.
- **Substitute possible roots into the polynomial** to see if the remainder is zero. If $\boldsymbol{P(a) = 0}$, then $\boldsymbol{a}$ is a root of $\boldsymbol{P(x)}$.
- **Graph the polynomial.** Real roots exist wherever the graph crosses the x-axis. Although this method may help find the approximate location of roots, it is not a reliable method for determining exact values of roots.

Example: Factor and find the roots of $\boldsymbol{P(x) = x^4 - 2x^3 + x^2 - 4x + 4}$

Using synthetic division:

1	1	-2	1	-4	4
		1	-1	0	-4
	1	-1	0	-4	0

Trying first the possible root $x = 1$, then the possible root $x = 2$, we find that they both work. So,

$$\boldsymbol{P(x) = x^4 - 2x^3 + x^2 - 4x + 4}$$
$$\boldsymbol{= (x - 1)(x - 2)(x^2 + x + 2)}$$

2	1	-1	0	-4
		2	2	4
	1	1	2	0

Using the quadratic formula on the quadratic factor in this expression we find two non-real roots. So the four roots are:

$$\boldsymbol{x = \left\{1, 2, \frac{-1 + i\sqrt{7}}{2}, \frac{-1 - i\sqrt{7}}{2}\right\}}$$

Algebra
Intersections of Curves

General Case (Bezout's Theorem)

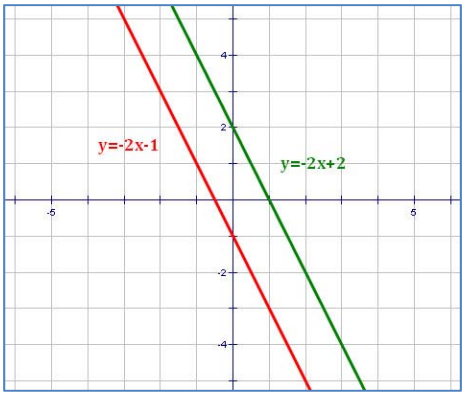
Bezout's Theorem states that the maximum number of intersections of two distinct curves in the complex field is the product of the degrees of the curves. (Note: for these purposes, a line is considered a curve of degree 1.) For graphs in two dimensions, if the degrees of two distinct curves are r and s, then:

- There are at most $(r \cdot s)$ intersections of the two curves.
- There are also cases where fewer than $(r \cdot s)$ intersections exist.
- To solve for the points of intersection, either set the two equations equal to each other or use variable substitution; then solve.

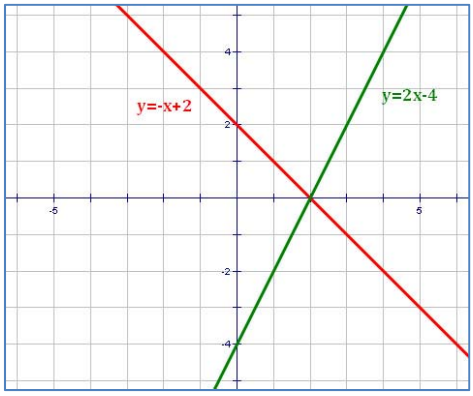
To apply this theorem, it is useful to set up each curve as an equation equal to zero. Examples are provided below and on the pages that follow.

Two Lines

Two distinct lines may have either zero or one point of intersection, as shown in the following illustrations:



Parallel Lines:
0 points of intersection



Intersecting Lines:
1 point of intersection

Lines have the form: $y = mx + b$, so the equations of any two lines can be written as:

$$y - m_1 x - b_1 = 0 \qquad \text{and} \qquad y - m_2 x - b_2 = 0$$

Notice that both lines are of degree 1; i.e., $r = 1$ and $s = 1$. Using Bezout's Theorem:

- The maximum number of intersections is: $\mathbf{1 \cdot 1 = 1}$.
- There may be less than one intersection.

Algebra
Intersections of Curves

A Line and a Parabola

The general forms for a line and a parabola are:

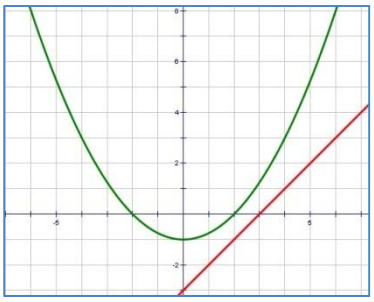
- **Line:** $y = mx + t$
- **Parabola:** $y = ax^2 + bx + c$

Note: we use the letter "t" instead of "b" in the equation of a line in order to avoid confusion with the coefficient "b" in the equation of a parabola.

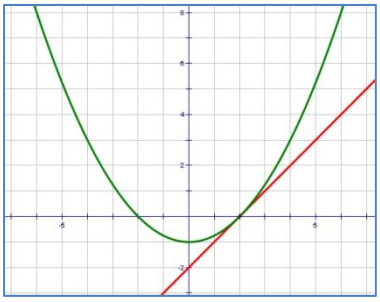
For purposes of Bezout's Theorem, these convert to:

- **Line:** $y - mx - t = 0$ $\quad Degree = 1$
- **Parabola:** $y - ax^2 - bx - c = 0$ $\quad Degree = 2$

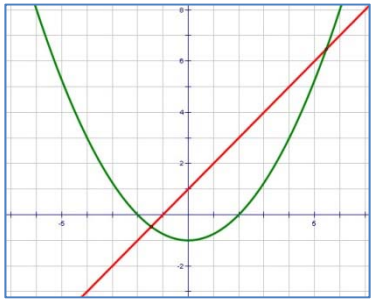
Using Bezout's Theorem, the maximum number of intersections is: $1 \cdot 2 = 2$. Cases for 0, 1, and 2 intersections are provided below:



0 points of intersection



1 point of intersection



2 points of intersection

Finding the Point(s) of Intersection

In order to find any points of intersection, set the two original equations equal to each other and solve:

$$ax^2 + bx + c = mx + t$$

$$ax^2 + (b - m)x + (c - t) = 0$$

This equation can be solved for x by any of the methods used to find the roots of a quadratic equation. The value of y can be calculated for each value of x by substituting x into either of the original equations.

Algebra
Intersections of Curves

A Circle and an Ellipse

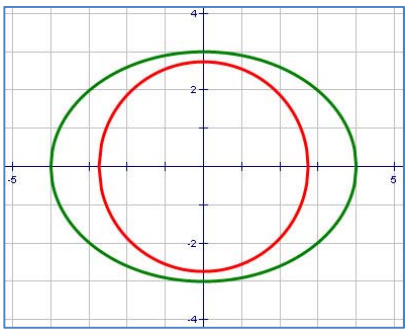
The general forms for a circle and an ellipse are:

- **Circle:** $x^2 + y^2 = r^2$
- **Ellipse:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

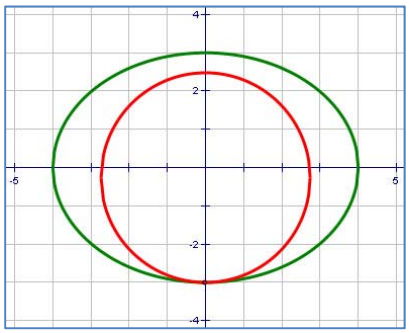
For purposes of Bezout's Theorem, these convert to:

- **Circle:** $x^2 + y^2 - r^2 = 0$ $\quad Degree = 2$
- **Ellipse:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ $\quad Degree = 2$

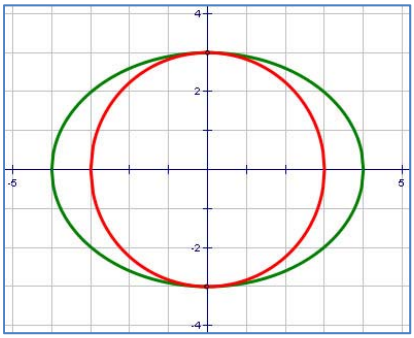
Using Bezout's Theorem, the maximum number of intersections is: $2 \cdot 2 = 4$. Cases for 0, 1, 2, 3 and 4 intersections are provided below:



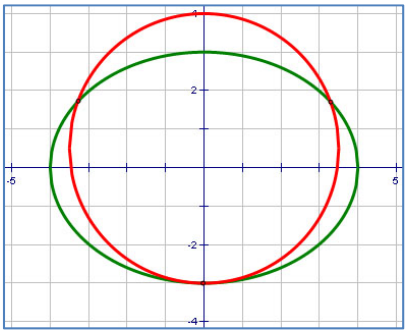
0 points of intersection



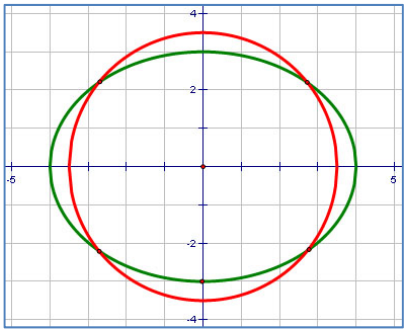
1 point of intersection



2 points of intersection



3 points of intersection



4 points of intersection

Algebra
Domains of Rational Functions

Rational Expressions are fractions with polynomials in both the numerator and denominator. If the rational expression is a function, it is a **Rational Function**.

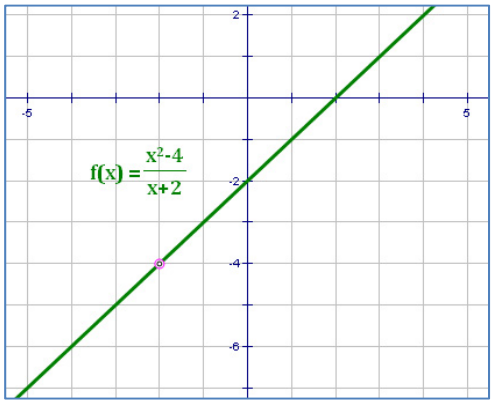
Finding the Domain of a Rational Function

The domain (e.g., x-values) of a rational function is the set of all values that result in valid range values (e.g., y-values). Generally, there are two situations where a value is not included in the domain of a rational function:

- Any x that generates a zero in the denominator.
- Any x that generates a square root of a negative number.

Example 1:

Consider the rational function: $\boldsymbol{f(x) = \frac{x^2-4}{x+2}}$.



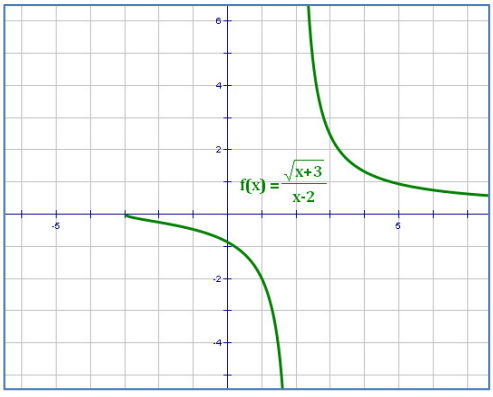
Since there are no square roots, the only value for which we cannot calculate $\boldsymbol{f(x)}$ is where $\boldsymbol{x + 2 = 0}$ or, where $\boldsymbol{x = -2.}$ So the domain is all real $\boldsymbol{x}$ except $\boldsymbol{x = -2,}$ or:

$$\{x \mid x \neq -2\}$$

Notice the hole in the graph of the function at the point $(-2, -4)$. This indicates that the function does not have a value for $\boldsymbol{x = -2}$.

Example 2:

Consider the function: $\boldsymbol{f(x) = \frac{\sqrt{x+3}}{x-2}}$



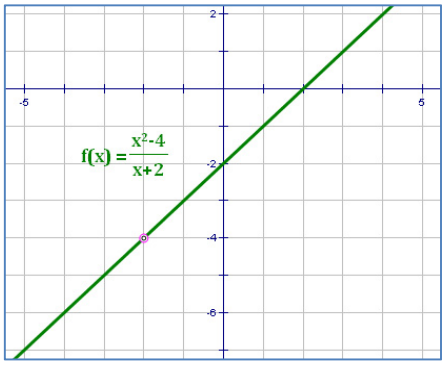
This function has no valid x-values for $x < -3$ because they would generate the square root of a negative number in the numerator. In addition, the denominator would be zero if $\boldsymbol{x = 2}$. So the domain is all real $\boldsymbol{x}$ greater than **-3** except $\boldsymbol{x = 2,}$ or:

$$\{x \mid x > -3 \ \ and \ \ x \neq 2\}$$

Algebra
Holes and Asymptotes

Holes

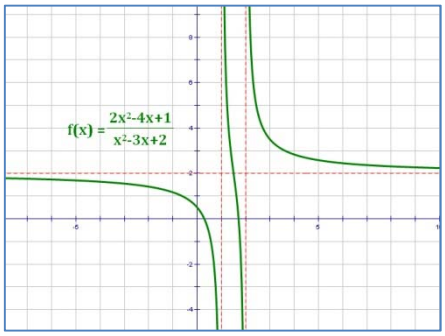
A **hole in a graph** exists whenever a factor $(x - a)$ occurs more times in the numerator than in the denominator of a rational function.



Example: In the function $f(x) = \frac{x^2-4}{x+2}$ the factor $(x + 2)$ is in both the numerator and the denominator. In fact, the function can be reduced to $f(x) = x - 2$ except at the point $x = -2$ where the function is undefined.

Vertical Asymptotes

A **vertical asymptote** exists whenever a factor $(x - a)$ occurs more times in the denominator than in the numerator of a rational function.



Example: In $f(x) = \frac{2x^2-4x+1}{x^2-3x+2} = \frac{2x^2-4x+1}{(x-1)(x-2)}$ the factors $(x - 1)$ and $(x - 2)$ occur in the denominator but not in the numerator of the function, so they generate vertical asymptotes. The vertical asymptotes are shown as red dotted lines at $x = 1$ and $x = 2$ in the graph at right.

Horizontal Asymptotes

There are three separate cases for **horizontal asymptotes** of a rational function $\frac{P(x)}{Q(x)}$:

1. If **the degree of $P(x)$ > the degree of $Q(x)$**, there is no horizontal asymptote.
2. If **the degree of $P(x)$ = the degree of $Q(x)$,** a horizontal asymptote exists at the line:

$$y = \frac{lead\ coefficient\ of\ P(x)}{lead\ coefficient\ of\ Q(x)}.$$

3. If **the degree of $P(x)$ < the degree of $Q(x)$**, a horizontal asymptote exists at the line $y = 0$.

Example: In the function $f(x) = \frac{2x^2-4x+1}{x^2-3x+2}$ the degrees of the polynomials in the numerator and denominator are the same, and the ratio of their lead coefficients is $\frac{2}{1} = 2$. The location of the horizontal asymptote is shown as the red dotted line $y = 2$ in the graph above.

Algebra
Graphing Rational Functions

Rational functions are of two types:

- **Simple rational functions** are of the form $y = \frac{a}{x-h} + k$ or an equivalent form that does not contain a polynomial of degree higher than 1 (i.e., no x^2, x^3, etc. – just x's and constants).

- **General rational functions** are of the form $y = \frac{P(x)}{Q(x)}$ where either $P(x)$ or $Q(x)$ is a polynomial of degree 2 or higher (i.e., contains an $x^2, x^3, etc.$).

In general, it is a good idea to find the asymptotes for a function first, and then find points that help graph the curve. The domain and any holes can typically be easily identified during this process. The range and the end behavior become identifiable once the function is graphed.

Simple Rational Functions

If you can put a rational function in the form $y = \frac{a}{x-h} + k$, here's what you get:

Vertical Asymptote: Occurs at $x = h$. The vertical asymptote is easy to find because it occurs at $x = h$. At this value of x, the denominator is $h - h = 0$, and you cannot divide by zero. Hence, as x approaches h, the denominator of $\frac{a}{x-h}$ becomes very small, and the graph shoots off either up or down.

Horizontal Asymptote: Occurs at $y = k$. The function cannot have a value of $y = k$ because that would require the lead term, $\frac{a}{x-h}$ to be zero, which can never happen since $a \neq 0$. Hence, the function will approach $y = k$, but will never reach it.

Domain: All Real $x \neq h$. No value of x exists at any vertical asymptote.

Range: All Real $y \neq k$. No value of y exists at a horizontal asymptote in simple rational functions.

Holes: None.

End Behavior: Both ends of the function tend toward the horizontal asymptote, so:

$As\ x \to -\infty, y \to k$ and $As\ x \to \infty, y \to k$

Algebra
Simple Rational Functions - Example

Example: $y = \frac{5}{x-1} - 2$

Recall that the simple rational form is: $y = \frac{a}{x-h} + k$

First, note that $h = 1$ ***and*** $k = -2$

Vertical Asymptote: Occurs at $x = 1$ because if $x = 1$, the denominator, $x - 1$, would be zero.

Horizontal Asymptote: Occurs at $y = -2$ because the lead term, $\frac{5}{x-1}$, can never be zero. Hence, the function can approach $y = -2$, but will never reach it.

Domain: All Real $x \neq 1$. No value of x exists at any vertical asymptote.

Range: All Real $y \neq -2$. No value of y exists at a horizontal asymptote in a simple rational function.

Holes: None.

End Behavior: Both ends of the function tend toward the horizontal asymptote, so:

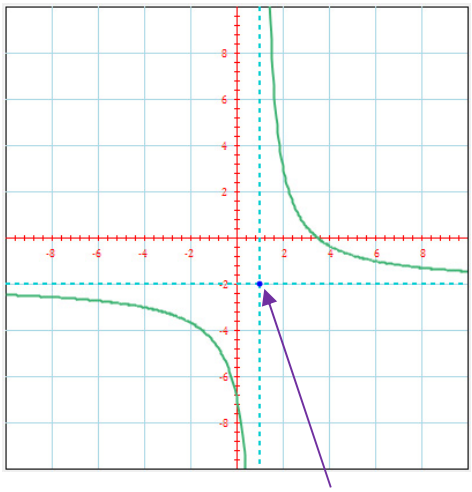
$As\ x \to -\infty, y \to -2$ and $As\ x \to \infty, y \to -2$

Graphing:

Step 1. Graph the vertical and horizontal asymptotes (the dashed horizontal and vertical lines shown).

Step 2. Pick some x-values and calculate the corresponding y-values. I like to pick a couple of x-values to the left of the vertical asymptote ($x = 1$) and a couple of x-values to its right. So, let's try some.

x	$y = \frac{5}{x-1} - 2$
-2	-3.67
-1	-4.5
0	-7
2	3
3	0.5
4	-0.33



Note that the intersection of the asymptotes has coordinates (h, k).

Step 3. Draw a curve on each side of the vertical asymptote: through the points on that side and approaching both the horizontal and vertical asymptotes.

Algebra
General Rational Functions

General rational functions are of the form: $\boldsymbol{y = \frac{P(x)}{Q(x)}}$

The easiest way to graph a general rational function is to factor both the numerator and denominator and simplifying the resulting fraction.

Example: in $y = \frac{(x+3)(x+2)}{(x-1)(x+2)}$ the $(x+2)$ in the numerator and denominator can be eliminated to obtain the function to be graphed: $y = \frac{(x+3)}{(x-1)}$.

Vertical Asymptotes and Holes: Any root (also called a "zero") of the denominator of a rational function (prior to simplification) will produce either a vertical asymptote or a hole.

Vertical Asymptote: If r is a root of the denominator is also a root of the simplified denominator, then $x = r$ is a vertical asymptote of the function.

Hole: If r is a root of the denominator and is not a root of the simplified denominator, then $x = r$ defines the location of a hole in the function.

Horizontal Asymptote: One way to find the horizontal asymptotes of a general rational function (also, see the section on *"Holes and Asymptotes"*, above) is to eliminate all terms of the polynomials in both the numerator and denominator except the ones with the single greatest exponent of all the terms. Then,

- If all terms are eliminated from the numerator, the horizontal asymptote occurs at $y = 0$.

  Example: $\boldsymbol{y = \frac{x+3}{x^2-5x+6} \rightarrow y = \frac{nothing}{x^2}}$ has a horizontal asymptote at $y = 0$.
  Note that all terms in the numerator were eliminated because none of them had the greatest exponent in the rational function, which in this example is 2.

- If a term remains in both the numerator and denominator, the horizontal asymptote occurs at the reduced form of the remaining terms.

  Example: $\boldsymbol{y = \frac{2x^2+3}{3x^2-5x+6} \rightarrow y = \frac{2x^2}{3x^2} = \frac{2}{3}}$ has a horizontal asymptote at $y = \frac{2}{3}$.

- If all terms are eliminated from the denominator, the function does not have a horizontal asymptote.

  Example: $\boldsymbol{y = \frac{x^2-5x+6}{x-3} \rightarrow y = \frac{x^2}{nothing}}$ does not have a horizontal asymptote.
  Note that all terms in the denominator were eliminated because none of them had the greatest exponent in the rational function, which in this example is 2.

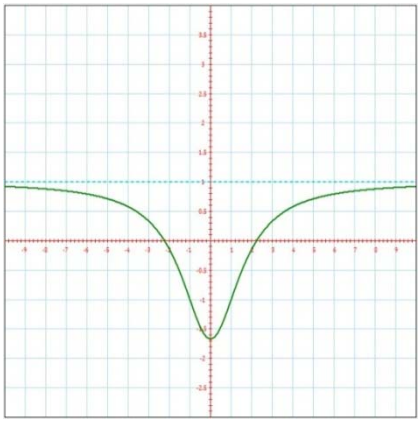
Algebra
General Rational Functions (cont'd)

Domain: The domain is always "all Real x" except where there is a vertical asymptote or a hole. No function value is associated with x at either a vertical asymptote or a hole (or when an even root of a negative number is required).

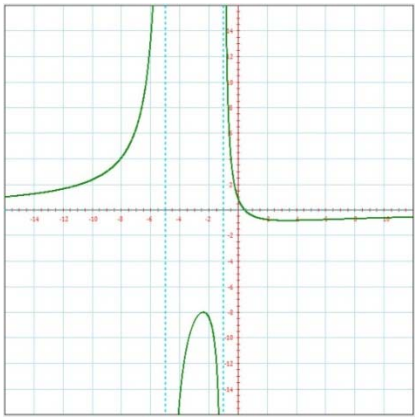
Range: The range is a bit trickier. You will need to look at the graph to determine the range. You might think that no y-value would exist at a horizontal asymptote, like in simple rational functions. However, it is possible for a function to cross over its horizontal asymptote and then work its way back to the asymptote as $x \to -\infty$ or as $x \to \infty$. Odd but true (see below, right).

For oddities in the range of a function, check these out these two rational functions:

$$y = \frac{x^2-5}{x^2+3}$$

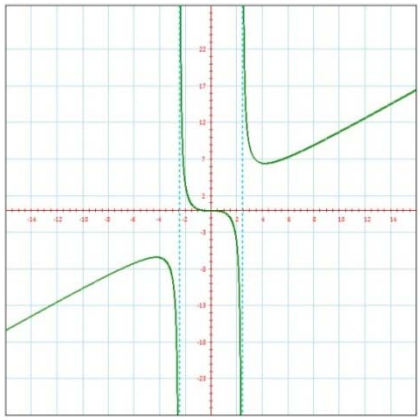


$$y = \frac{-10x+5}{x^2+6x+5}$$



End Behavior: Both ends of the function tend toward the horizontal asymptote if there is one. However, if there is not one, you can look at the graph to determine end behavior. Note that the function below does not have a horizontal asymptote:

$$y = \frac{x^3}{x^2-6}$$



In this function,

$As\ x \to -\infty, y \to -\infty,$

$As\ x \to \infty, y \to \infty$

Although this function does not have a horizontal asymptote, it does have a "slant asymptote": the line $y = x$.

Algebra
General Rational Functions - Example

Example: $y = \frac{2x^2+x-3}{x^2-1}$

Factor both the numerator and the denominator: $y = \frac{(x-1)(2x+3)}{(x+1)(x-1)}$

Get the Roots: $\frac{Numerator:\ x=-1.5,\ \ x=1}{Denominator:\ x=-1,\ \ x=1}$

Simplify: Since 1 is a root of both the numerator and the denominator, the function may be simplified as follows:

$$y = \frac{(x-1)(2x+3)}{(x+1)(x-1)} = \frac{\cancel{(x-1)}(2x+3)}{(x+1)\cancel{(x-1)}} = \frac{2x+3}{x+1}$$

Vertical Asymptotes and Holes: "-1" and "1" are roots of the original denominator, so they must generate either vertical asymptotes or holes.

Vertical Asymptote: After simplification, this function still contains "-1" as a root in the denominator. Therefore, **$x = -1$ is a vertical asymptote** of the function.

Hole: "1" is a root of the denominator of the original function but is not a root of the denominator of the simplified function. Therefore, this function has **a hole at $x = 1$.**

Horizontal Asymptote: Eliminate all terms of both polynomials except any with the single greatest exponent of all the terms. In this case:

$\boldsymbol{y = \frac{2x^2}{x^2}}$ $\rightarrow$ **$y = 2$ is a horizontal asymptote.** Since a term remains in both the numerator and denominator, the horizontal asymptote occurs at the reduced form of the remaining terms.

Domain: All Real x except where there is a vertical asymptote or a hole.
So, the domain is **all Real $x \neq -1$ *or* 1.**

We must graph the function in order to get a good look at its **range and end behavior**. We must plot points on both sides of the vertical asymptote.

(graph on next page)

Algebra
General Rational Functions – Example (cont'd)

Graphing:

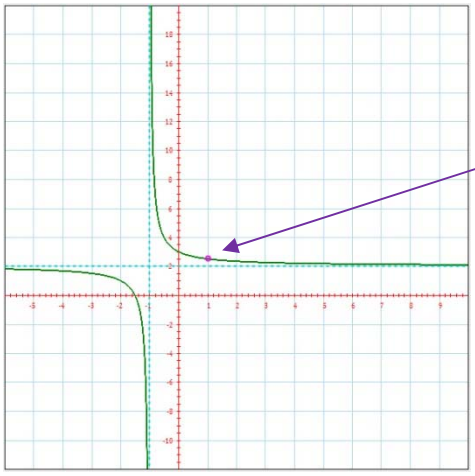
Step 1. Graph the vertical and horizontal asymptotes.

Step 2. Pick some x-values on each side of the vertical asymptote and calculate the corresponding y-values.

x	$y = \frac{2x+3}{x+1}$
-4	1.67
-3	1.5
-2	1
0	3
1	2.5 (a hole)
2	2.33

Step 3. Draw a curve on each side of the vertical asymptote: through the points on that side and approaching both the horizontal and vertical asymptotes.

Step 4: Draw an open circle at the point of any holes.



Range: The range can be determined from the graph.
It appears that the range excludes only the horizontal asymptote and the hole.
So the range is: **all Real $y \neq 1, 2.5$.**

End Behavior: In this function,

$As\ x \to -\infty, y \to 1,$ $As\ x \to \infty, y \to 1$

Algebra
Operating with Rational Expressions

Performing operations with rational expressions is very similar to performing operations with fractions. After all, that's really what rational expressions are – fractions involving polynomials.

Addition and Subtraction

To add or subtract rational expressions:

- Factor both the numerator and denominator as much as possible.
- Create a common denominator.
- Add or subtract expressions.
- Simplify.

Example: $\frac{x}{x+1}+\frac{1}{x+2}$

$$= \frac{x+2}{x+2}\cdot\frac{x}{x+1}+\frac{1}{x+2}\cdot\frac{x+1}{x+1}$$

$$= \frac{(x^2+2x)+(x+1)}{(x+2)(x+1)} = \frac{(x^2+3x+1)}{(x+2)(x+1)}$$

After the numerators are added, you must check to see if the new numerator can be factored. If so, further simplification may be possible. **Note:** no further simplification is possible in this example.

Multiplication and Division

To multiply or divide rational expressions:

- Factor both the numerator and denominator as much as possible.
- Multiply or divide expressions. (Remember, to divide, "flip that guy and multiply.") Cancel all factors that appear in both the numerator and denominator.
- Simplify.

Example: $\frac{x^2+2x-8}{x^2+2x-3}\cdot\frac{x^2+4x+3}{x^2+6x+8}$

$$= \frac{(x+4)(x-2)}{(x+3)(x-1)}\cdot\frac{(x+1)(x+3)}{(x+4)(x+2)}$$

$$= \frac{\cancel{(x+4)}(x-2)}{\cancel{(x+3)}(x-1)}\cdot\frac{(x+1)\cancel{(x+3)}}{\cancel{(x+4)}(x+2)} = \frac{(x-2)(x+1)}{(x-1)(x+2)}$$

Note that you can cross out common factors in the numerator and denominator across the expressions that are being multiplied.

Algebra
Solving Rational Equations

Solving Rational Equations

Solving rational equations involves one main rule: **Get rid of the denominators!**

A fuller listing of the steps involved is:

- Multiply by whatever expressions are necessary to eliminate the denominators.
- Pay careful attention to which values make the expressions you use equal to zero (because you are not allowed to multiply both sides of an equation by zero).
- Solve the remaining problem.
- Check each answer to see if it is a solution to the original problem. Note: as long as you do not multiply by zero, your solutions are likely to be valid.

Example 1: Solve $\frac{x+1}{x-3} = \frac{x-3}{x+1}$

First note that x cannot be 3 or -1 since each of these creates an undefined fraction in the original problem.

The easiest way to start this problem is to cross multiply to get:

$$(x+1)^2 = (x-3)^2$$

Then, $x^2 + 2x + 1 = x^2 - 6x + 9$

So, $8x = 8$

And finally, $x = 1$

Check work:

$$\frac{x+1}{x-3} \stackrel{?}{=} \frac{x-3}{x+1}$$

$$\frac{1+1}{1-3} \stackrel{?}{=} \frac{1-3}{1+1}$$

$$\frac{2}{-2} = \frac{-2}{2} \quad \checkmark$$

Example 2: Solve $\frac{x}{x+8} = \frac{1}{x+3}$

First note that x cannot be -8 or -3 since each of these creates an undefined fraction in the original problem.

The easiest way to start this problem is to cross multiply to get:

$$x^2 + 3x = x + 8$$

Then, $x^2 + 2x - 8 = 0$

So, $(x+4)(x-2) = 0$

And finally, $x = -4 \text{ or } 2$

Check -4:

$$\frac{x}{x+8} \stackrel{?}{=} \frac{1}{x+3}$$

$$\frac{-4}{-4+8} \stackrel{?}{=} \frac{1}{-4+3}$$

$$\frac{-4}{4} = \frac{1}{-1} \quad \checkmark$$

Check 2:

$$\frac{2}{2+8} \stackrel{?}{=} \frac{1}{2+3}$$

$$\frac{2}{10} = \frac{1}{5} \quad \checkmark$$

Algebra
Solving Rational Inequalities

Solving Rational Inequalities

Solving Rational Inequalities is a bit more involved than solving Rational Equations. The key issue in the Inequalities is whether in the step where you eliminate the denominator, you are multiplying by a negative number. Remember, when you multiply by a negative number, you must flip an inequality sign.

The steps involved are similar to those for solving rational equations:

- Multiply by whatever expressions are necessary to eliminate the denominators.
- Identify when the denominators are positive and when they are negative; set up cases for each situation. Within each case, you will need to meet multiple conditions (i.e., using the word "**and**" between conditions within a case).
- Pay careful attention to which values make the expression you multiply equal to zero (because you are not allowed to multiply both sides of an inequality by zero).
- Solve the remaining problems. Any of the cases produces valid results, so you must combine the solutions for the various cases with "**or's**."
- Check sample answers in each range you develop to see if they are solutions to the original problem. Alternatively, graph the solution to see if the results are correct.

Example: Solve $\frac{(3x-5)}{(x-1)} > 2$

We want to eliminate the denominator $(x-1)$ but we need to create 2 cases:

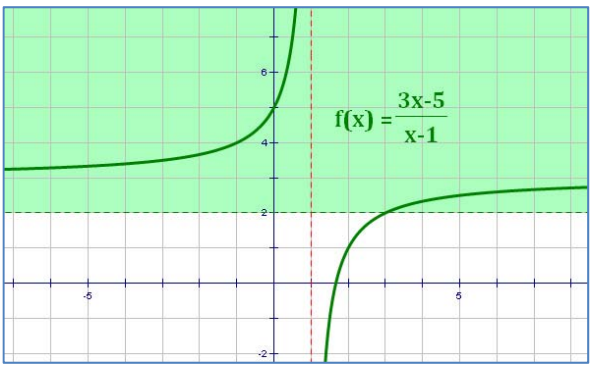
Case 1: $x-1>0 \rightarrow x>1$
Then: $3x-5>2x-2$
So, $x>3$
The solution here requires:
$x>1 \quad and \quad x>3$
Which simplifies to: $x>3$

Case 2: $x-1<0 \rightarrow x<1$
Then: $3x-5<2x-2$
So, $x<3$
The solution here requires:
$x<1 \quad and \quad x<3$
Which simplifies to: $x<1$

The combined result of the two cases is:

$$x<1 \quad or \quad x>3$$

To check the result, we graph the function and see where it produces a y-value above 2; this is the set of x-values where the dark green curve intersects the light green region in the graph below.



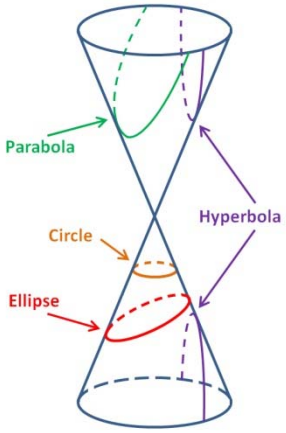
Algebra

Introduction to Conic Sections

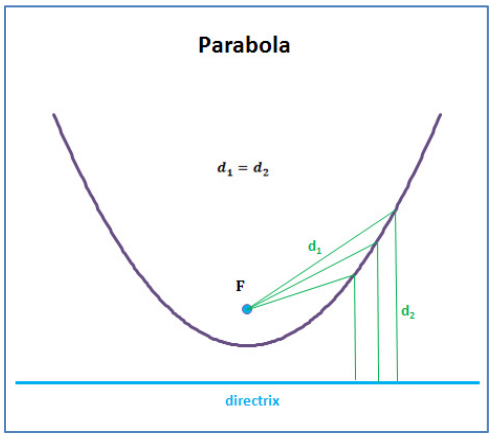
The intersection of a cone and a plane is called a **conic section**. There are four types of curves that result from these intersections that are of particular interest:

- Parabola
- Circle
- Ellipse
- Hyperbola

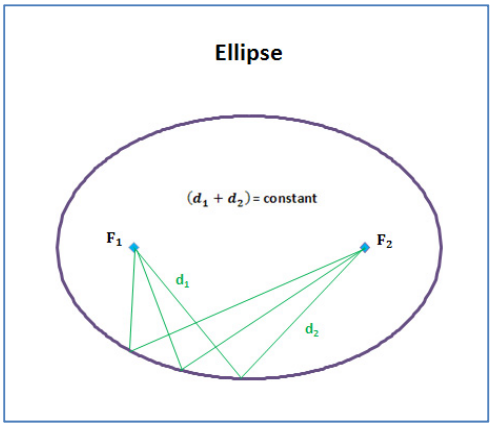
Each of these has a geometric definition, from which the algebraic form is derived.



Geometric Definitions

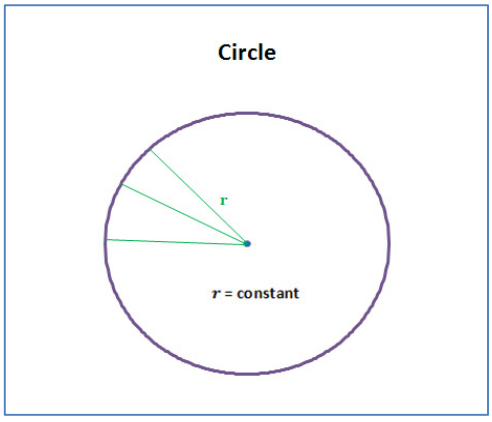


Parabola – The set of all points that are the same distance from a point (called the focus) and a line (called the Directrix).

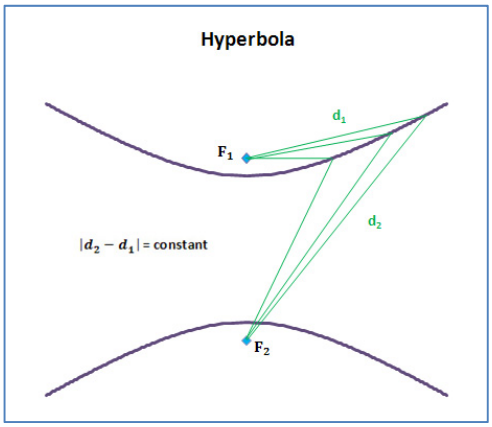


Ellipse – The set of all points for which the sum of the distances to two points (called foci) is constant.

Circle – The set of all points that are the same distance from a point (called the center). The distance is called the radius.



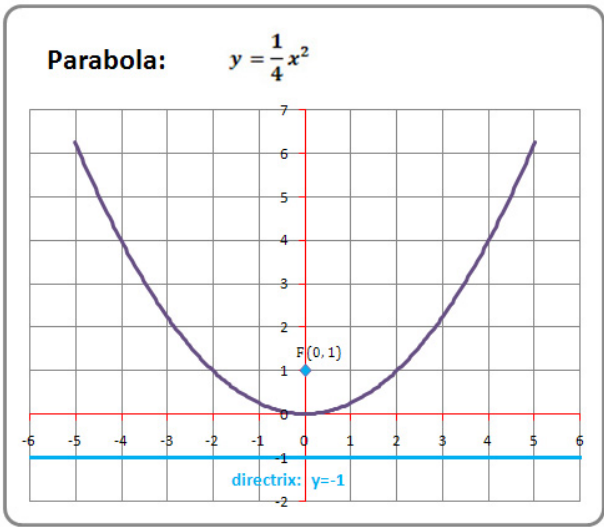
Hyperbola – The set of all points for which the difference of the distances to two points (called foci) is constant.



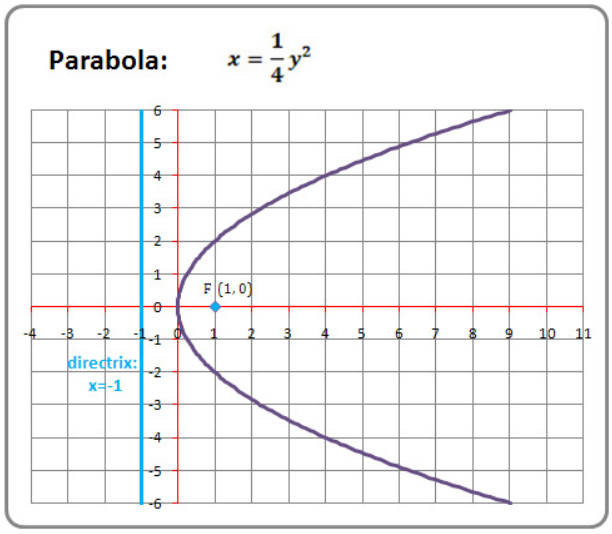
Algebra

Parabola with Vertex at the Origin (Standard Position)

Horizontal Directrix



Vertical Directrix

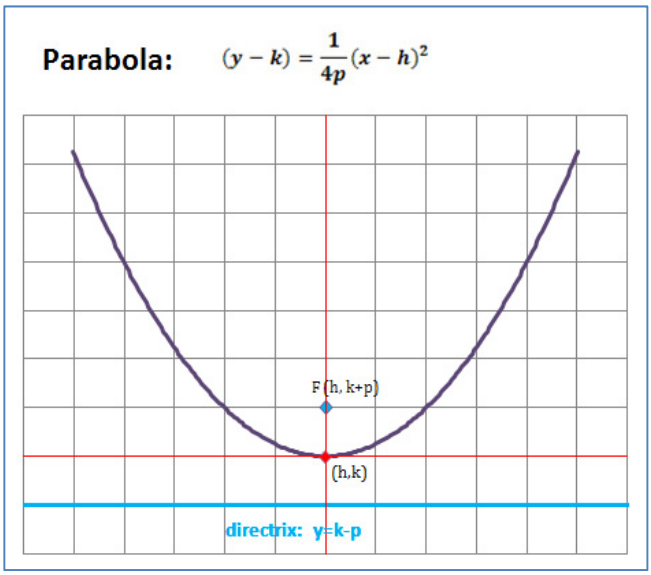


Characteristics of a Parabola in Standard Position		
	Horizontal Directrix	**Vertical Directrix**
Equation	$y = \frac{1}{4p}x^2$	$x = \frac{1}{4p}y^2$
If $p > 0$	opens up	opens right
If $p < 0$	opens down	opens left
Eccentricity (“e”)	$e = 1$	$e = 1$
Value of p (in illustration)	$p = 1$	$p = 1$
Vertex	$(0, 0)$ - the origin	$(0, 0)$ - the origin
Focus	$(0, p)$	$(p, 0)$
Directrix	$y = -p$	$x = -p$
Axis of symmetry	$x = 0$ (y-axis)	$y = 0$ (x-axis)

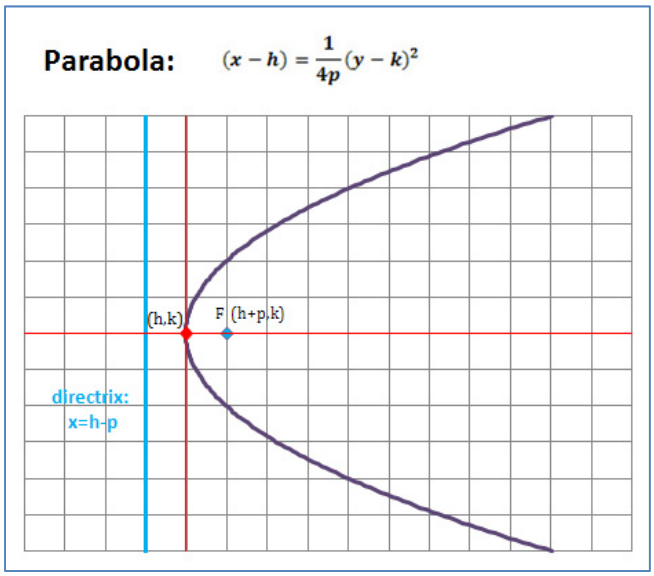
Algebra

Parabola with Vertex at (h, k)

Horizontal Directrix



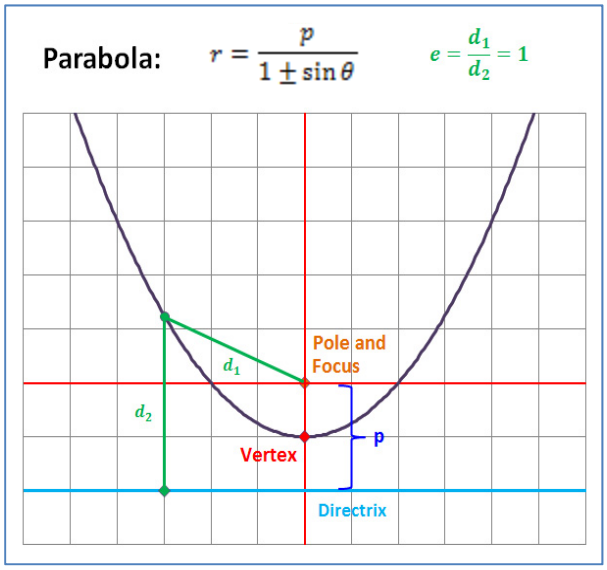
Vertical Directrix



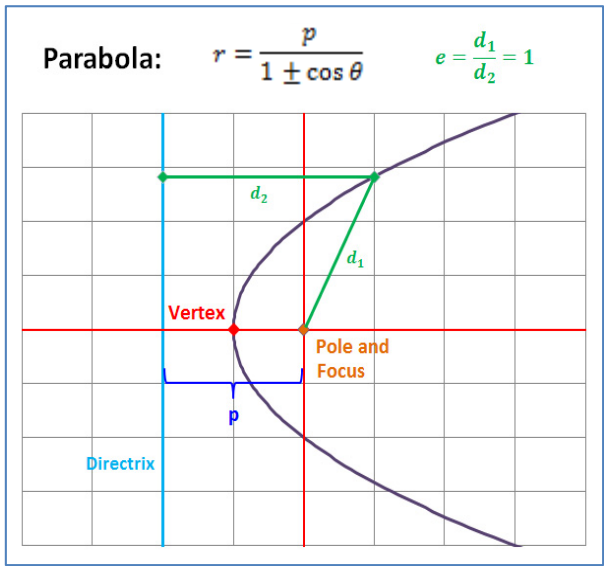
Characteristics of a Parabola with Vertex at Point (h, k)		
	Horizontal Directrix	**Vertical Directrix**
Equation	$(y-k)=\frac{1}{4p}(x-h)^2$	$(x-h)=\frac{1}{4p}(y-k)^2$
If $p>0$	opens up	opens right
If $p<0$	opens down	opens left
Eccentricity (“e”)	$e=1$	$e=1$
Vertex	(h,k)	(h,k)
Focus	$(h,k+p)$	$(h+p,k)$
Directrix	$y=k-p$	$x=h-p$
Axis of symmetry	$x=h$	$y=k$

Algebra
Parabola in Polar Form

Horizontal Directrix

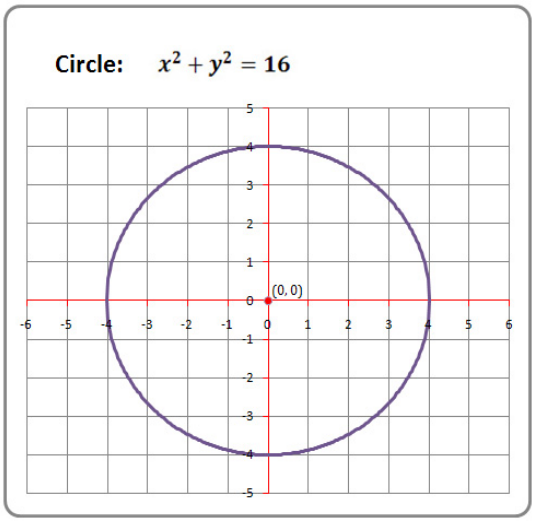


Vertical Directrix

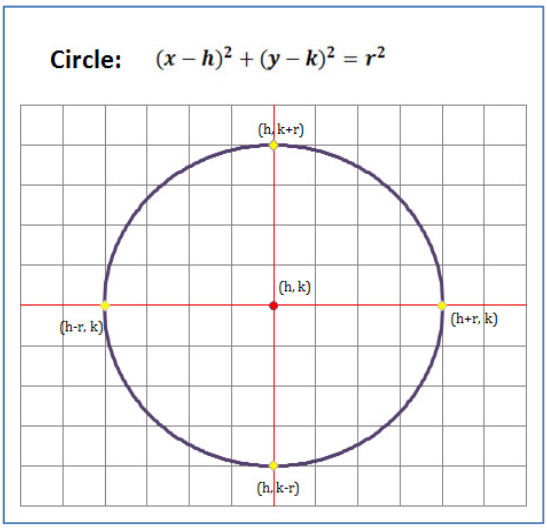


Characteristics of a Parabolas in Polar Form		
	Horizontal Directrix	**Vertical Directrix**
Equation (simplified)	$r = \frac{p}{1 \pm \sin\theta}$	$r = \frac{p}{1 \pm \cos\theta}$
If " $-$ " in denominator	opens up Directrix below Pole	opens right Directrix left of Pole
If " $+$ " in denominator	opens down Directrix above Pole	opens left Directrix right of Pole
Eccentricity ("e")	$e = 1$	$e = 1$
Focal Parameter ("p")	p = distance between the Directrix and the Focus *Note: "p" in Polar Form is different from "p" in Cartesian Form*	
Coordinates of Key Points: *(change all instances of "–p" below to "p" if "+" is in the denominator)*		
Vertex	$(0, -p/2)$	$(-p/2, 0)$
Focus	$(0,0)$	$(0,0)$
Directrix	$y = -p$	$x = -p$

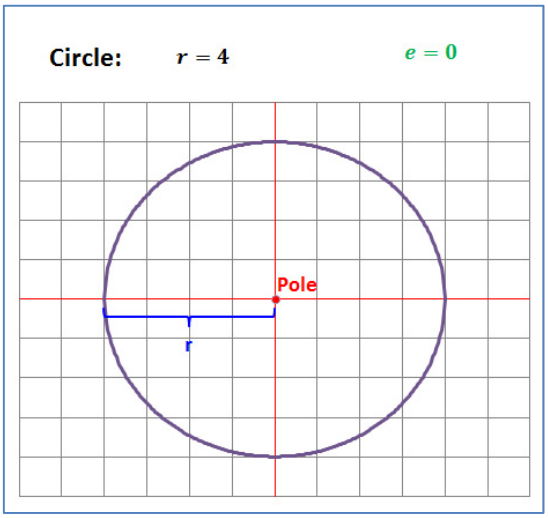
Algebra
Circles



Characteristics of a Circle in Standard Position	
Equation	$x^2 + y^2 = r^2$
Center	$(0,0)$ - the origin
Radius	r
In the example	$r = 4$



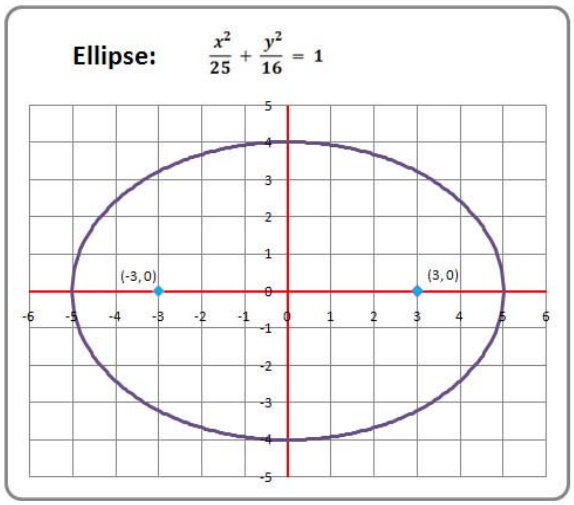
Characteristics of a Circle Centered at Point (h, k)	
Equation	$(x-h)^2 + (y-k)^2 = r^2$
Center	(h, k)
Radius	r



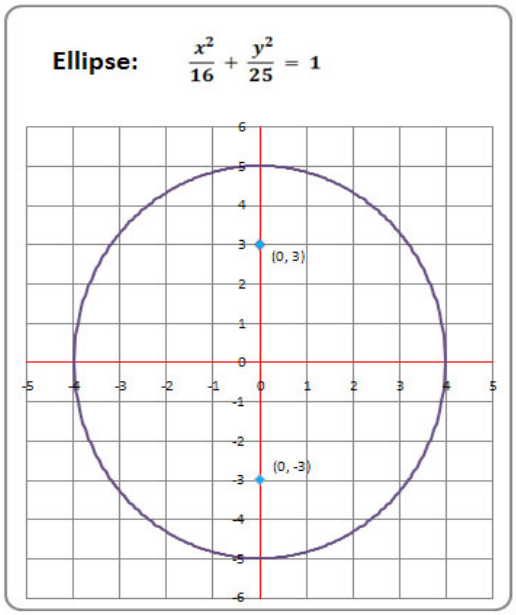
Characteristics of a Circle in Polar Form	
Equation	$r = constant$
Pole	$(0, 0)$
Radius	r

Algebra
Ellipse Centered on the Origin (Standard Position)

Horizontal Major Axis



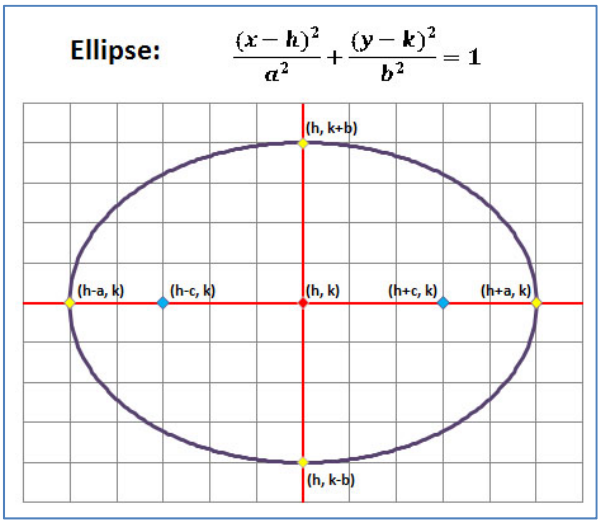
Vertical Major Axis



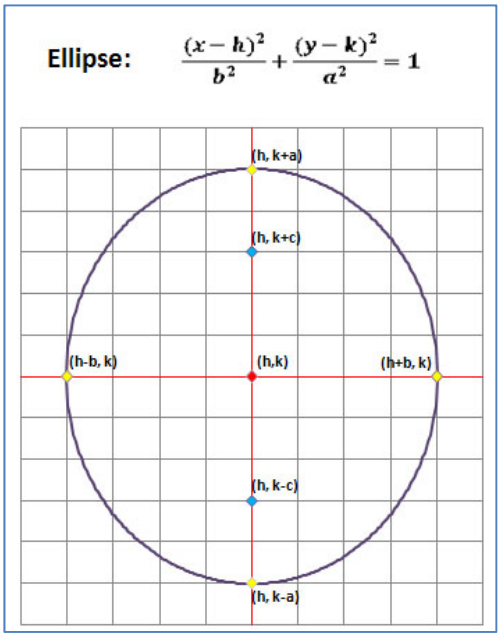
Characteristics of an Ellipse in Standard Position		
	Horizontal Major Axis	**Vertical Major Axis**
In the above example	$a = 5,\ b = 4,\ c = 3$	$a = 5,\ b = 4,\ c = 3$
Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$
Values of "a" and "b"	$a > b$	
Value of "c"	$c^2 = a^2 - b^2$	
Eccentricity ("e")	$e = c/a \qquad 0 < e < 1$	
Center	$(0,0)$ - the origin	
Major Axis Vertices	$(\pm a, 0)$	$(0, \pm a)$
Minor Axis Vertices	$(0, \pm b)$	$(\pm b, 0)$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Directrixes (not shown)	$x = \pm a/e$	$y = \pm a/e$

Algebra
Ellipse Centered at (h, k)

Horizontal Major Axis



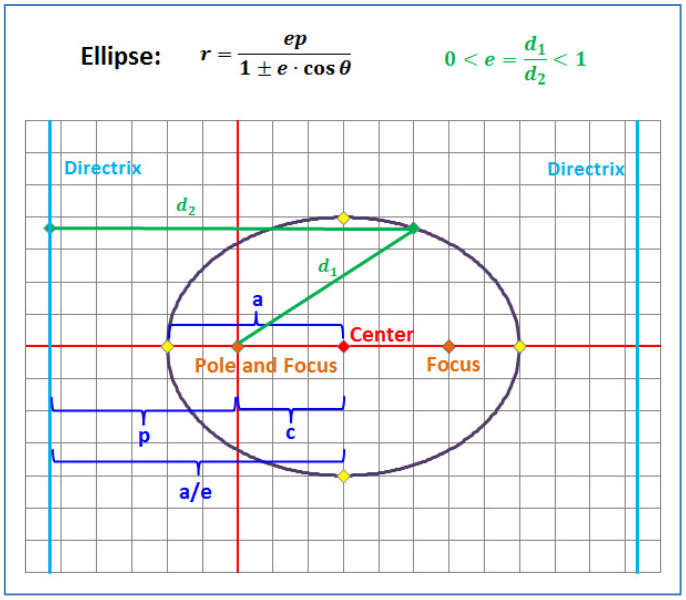
Vertical Major Axis



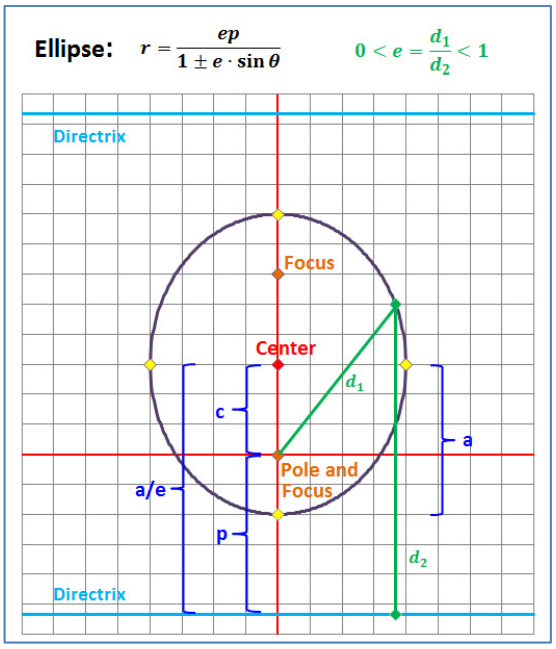
Characteristics of an Ellipse Centered at Point (h, k)		
	Horizontal Major Axis	**Vertical Major Axis**
Equation	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$
Values of "a" and "b"	$a > b$	
Value of "c"	$c^2 = a^2 - b^2$	
Eccentricity ("e")	$e = c/a \qquad 0 < e < 1$	
Center	(h, k)	
Major Axis Vertices	$(h \pm a,\ k)$	$(h,\ k \pm a)$
Minor Axis Vertices	$(h,\ k \pm b)$	$(h \pm b,\ k)$
Foci	$(h \pm c,\ k)$	$(h,\ k \pm c)$
Directrixes (not shown)	$x = h \pm a/e$	$y = k \pm a/e$

Algebra
Ellipse in Polar Form (Pole = One Focus)

Horizontal Major Axis



Vertical Major Axis

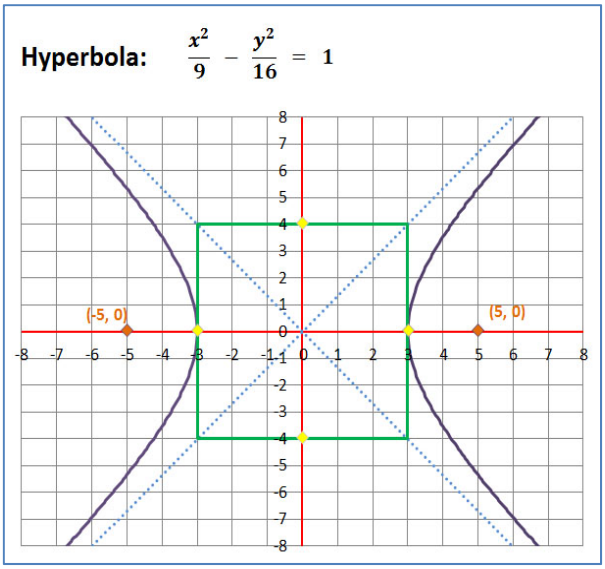


Characteristics of an Ellipse in Polar Form		
	Horizontal Major Axis	**Vertical Major Axis**
Equation	$r = \frac{ep}{1 \pm e \cdot \cos\theta}$	$r = \frac{ep}{1 \pm e \cdot \sin\theta}$
Value of "a"	a = distance from the Center to each major axis Vertex	
Value of "c"	c = distance from the Center to each Focus	
Eccentricity ("e")	$e = c/a \qquad 0 < e < 1$	
Focal Parameter ("p")	p = distance from each Focus to its Directrix $= a/e - c$	
Coordinates of Key Points:		
If " $-$ " in denominator	all coordinate values are shown below	
If " $+$ " in denominator	change all instances of "c", below, to "$-c$"	
Center	$(c, 0)$	$(0, c)$
Major Axis Vertices	$(c \pm a, 0)$	$(0, c \pm a)$
Foci	$(c \pm c, 0)$	$(0, c \pm c)$
Directrixes	$x = c \pm a/e$	$y = c \pm a/e$

Algebra
Hyperbola Centered on the Origin (Standard Position)

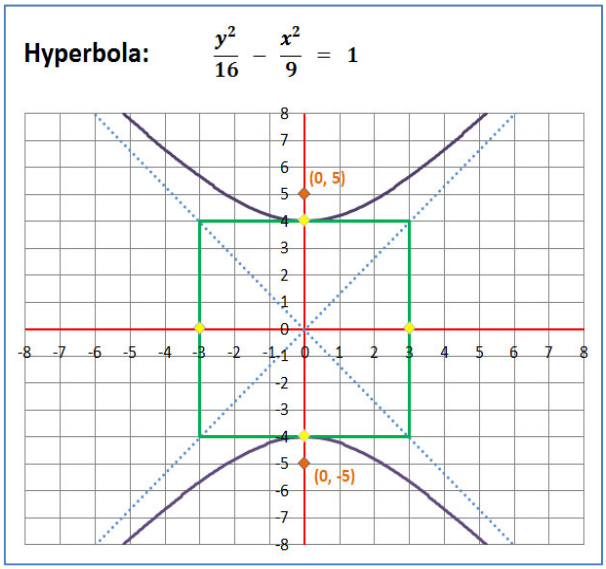
Horizontal Transverse Axis

Hyperbola: $\frac{x^2}{9} - \frac{y^2}{16} = 1$



Vertical Transverse Axis

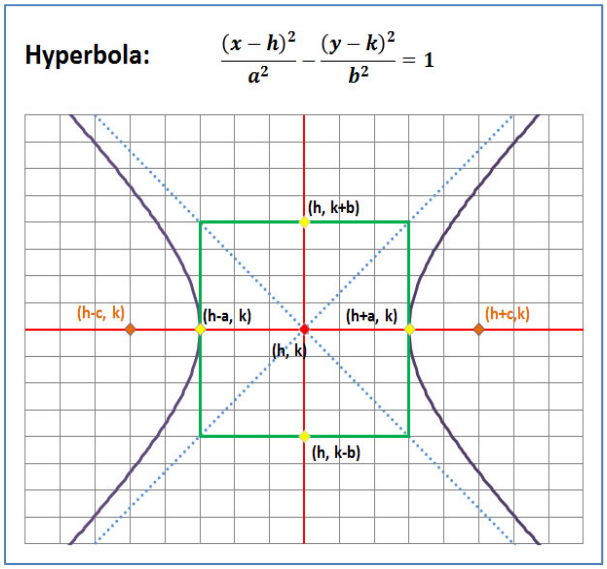
Hyperbola: $\frac{y^2}{16} - \frac{x^2}{9} = 1$



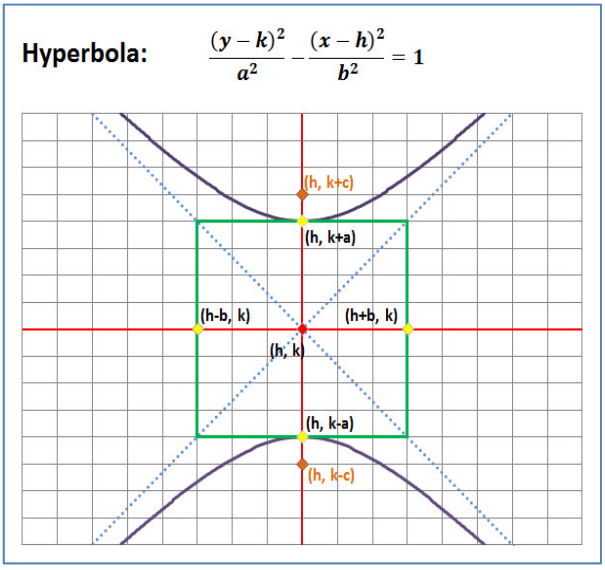
Characteristics of a Hyperbola in Standard Position		
	Horizontal Transverse Axis	**Vertical Transverse Axis**
In the above example	$a = 3,\ b = 4,\ c = 5$	$a = 4,\ b = 3,\ c = 5$
Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Value of "c"	$c^2 = a^2 + b^2$	
Eccentricity ("e")	$e = c/a \qquad e > 1$	
Center	$(0,0)$ - the origin	
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Foci	$(\pm c, 0)$	$(0, \pm c)$
Asymptotes	$y = \pm \frac{b}{a} x$	$y = \pm \frac{a}{b} x$
Directrixes (not shown)	$x = \pm a/e$	$y = \pm a/e$

Algebra
Hyperbola Centered at (h, k)

Horizontal Transverse Axis



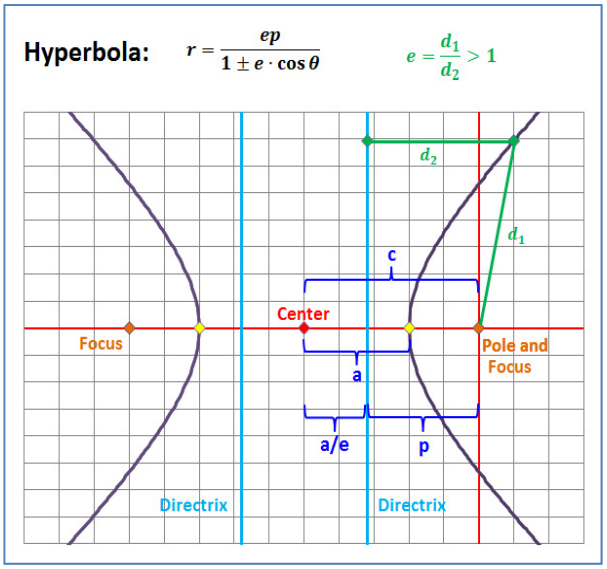
Vertical Transverse Axis



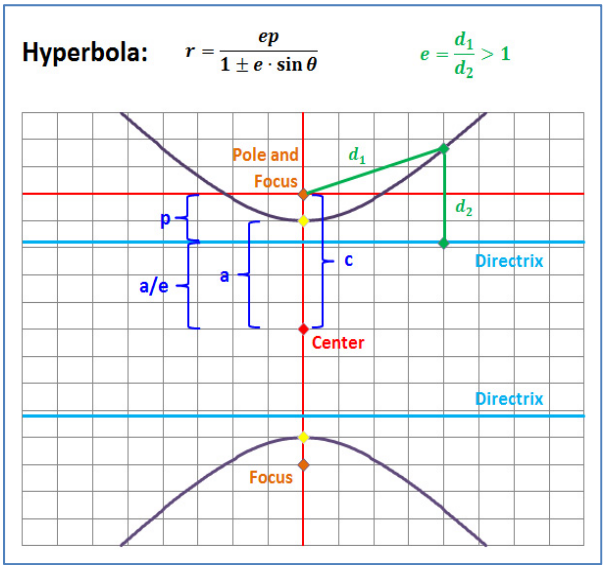
Characteristics of a Hyperbola Centered at Point (h, k)		
	Horizontal Transverse Axis	**Vertical Transverse Axis**
Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Value of "c"	$c^2 = a^2 + b^2$	
Eccentricity ("e")	$e = c/a \qquad e > 1$	
Center	(h, k)	
Vertices	$(h \pm a,\ k)$	$(h,\ k \pm a)$
Foci	$(h \pm c,\ k)$	$(h,\ k \pm c)$
Asymptotes	$(y-k) = \pm\frac{b}{a}(x-h)$	$(y-k) = \pm\frac{a}{b}(x-h)$
Directrixes (not shown)	$x = h \pm a/e$	$y = k \pm a/e$

Algebra
Hyperbola in Polar Form (Pole = One Focus)

Horizontal Transverse Axis



Vertical Transverse Axis



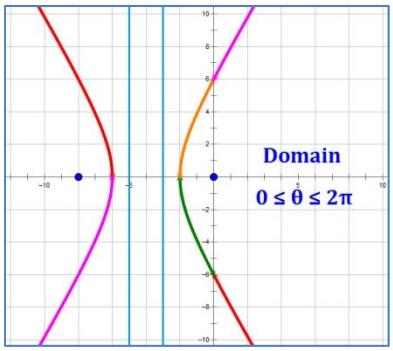
Characteristics of a Hyperbola in Polar Form		
	Horizontal Transverse Axis	**Vertical Transverse Axis**
Equation	$r = \frac{ep}{1 \pm e \cdot \cos\theta}$	$r = \frac{ep}{1 \pm e \cdot \sin\theta}$
Value of "a"	a = distance from the Center to each Vertex	
Value of "c"	c = distance from the Center to each Focus	
Eccentricity ("e")	$e = c/a \qquad e > 1$	
Focal Parameter ("p")	p = distance from each Focus to its Directrix $= c - a/e$	
Coordinates of Key Points:		
If " $-$ " in denominator	all coordinate values are shown below	
If " $+$ " in denominator	change all instances of "$-c$", below, to "c"	
Center	$(-c, 0)$	$(0, -c)$
Vertices	$(-c \pm a, 0)$	$(0, -c \pm a)$
Foci	$(-c \pm c, 0)$	$(0, -c \pm c)$
Directrixes	$x = -c \pm a/e$	$y = -c \pm a/e$

Algebra
Hyperbola in Polar Form (Pole = One Focus)
Partial Construction Over the Domain: 0 to 2π

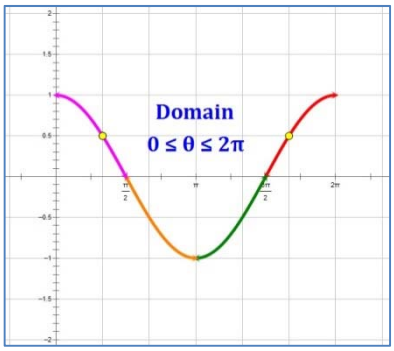
It is instructive to look at partial constructions of a hyperbola in polar form. Let's take a look at a curve constructed by varying θ from 0 to 2π, quadrant by quadrant:

$$\textbf{curve: } r = \frac{6}{1 - 2\cos\theta} \qquad \textbf{parameters: } a = 2, \quad c = 4, \quad e = 2, \quad p = 3$$

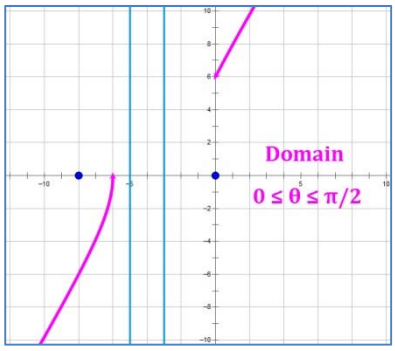
In the plots below, each quadrant in the domain is represented by a separate color. The portion of the curve added in each illustration is presented as a thicker line than the rest of the curve. The Foci of the curve are dark blue points and the Directrixes are light blue vertical lines.



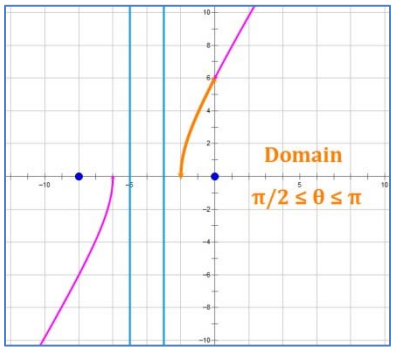
The **final curve** looks like this. The curve is plotted over the domain $0 \le \theta \le 2\pi$ but could also be plotted over the domain $-\pi \le \theta \le \pi$.



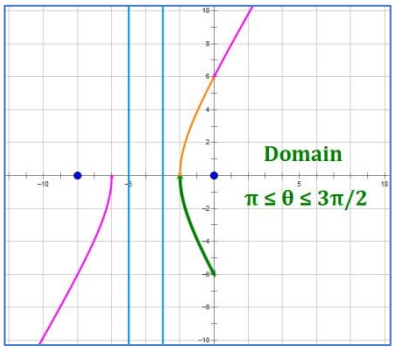
The **cosine function** has a major impact on how the curve graphs. Note the two yellow points where $\cos\theta = 0.5$. At these points, the curve is undefined.



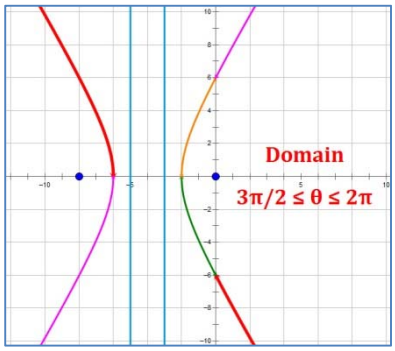
Q I: Domain $0 \le \theta \le \pi/2$. Note that the curve starts out on the left and switches to the right at $\theta = \pi/3$, where the curve is undefined.



Q II: Domain $\pi/2 \le \theta \le \pi$. The curve continues on the right side of the graph and gently curves down to the x-axis.



Q III: Domain $\pi \le \theta \le 3\pi/2$. The curve continues its gentle swing below the x-axis. Q III is essentially a reflection of the curve in Q II over the x-axis.



Q IV: Domain $3\pi/2 \le \theta \le 2\pi$. The curve continues on the right and switches to the left at $\theta = 5\pi/3$, where the curve is undefined.

Algebra
General Conic Equation – Classification

The General Case of the Conic Equation is:

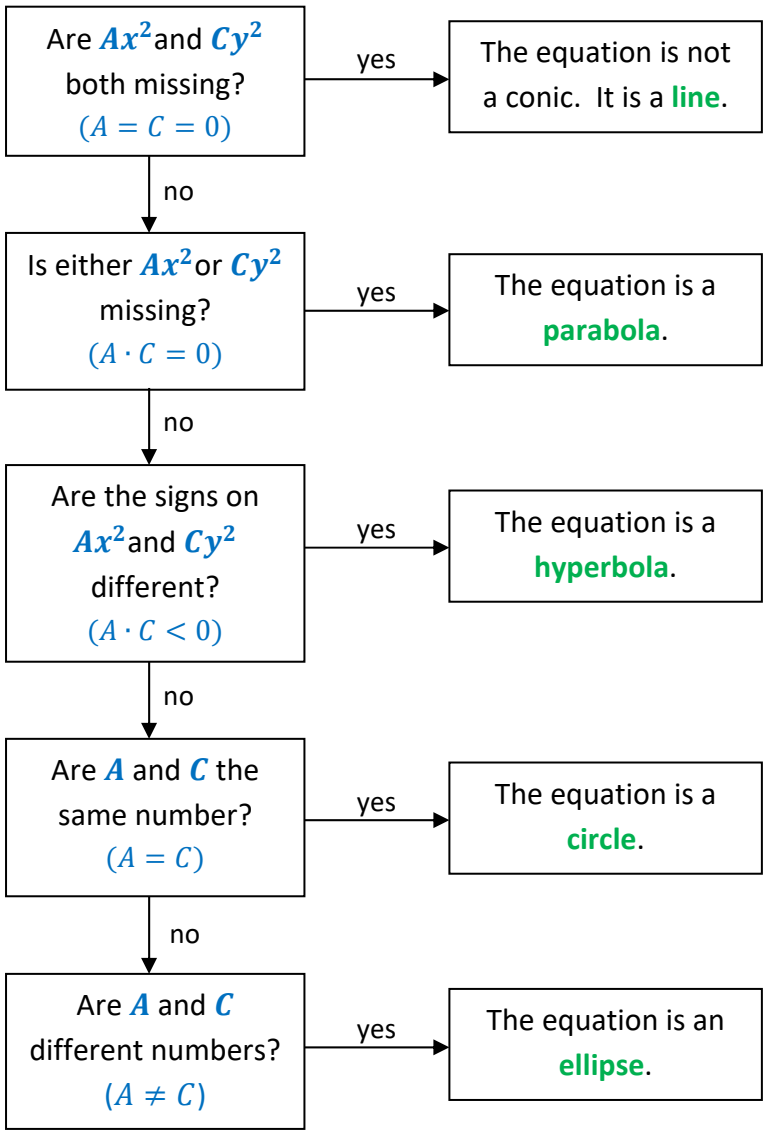
$$\boldsymbol{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0}$$

The second term may be omitted if the curve is not rotated relative to the axes in the Cartesian Plane, giving the simpler form:

$$\boldsymbol{Ax^2 + Cy^2 + Dx + Ey + F = 0}$$

Conic Classification Tree

In this form, it is relatively easy to identify which type of curve the equation represents, using the following decision tree:



Examples:

$3x - 2y - 4 = 0$
$-2x + 7y + 1 = 0$

$3y^2 - 2x + 7y - 3 = 0$
$x^2 + 4x + 2y - 1 = 0$

$4x^2 - 9y^2 - 8x + 27y + 2 = 0$
$-3x^2 + y^2 + 6x - 12y - 15 = 0$

$x^2 + y^2 + 4x - 10y = 0$
$-4x^2 - 4y^2 + 16x - 8y + 5 = 0$

$5x^2 + 3y^2 + 20x - 12y - 3 = 0$
$-x^2 - 2y^2 + 8x + 12y + 13 = 0$

Algebra
General Conic Equation – Manipulation

After a conic equation is classified, it must be algebraically manipulated into the proper form. The steps involved are:

1. If there are negative coefficients in front of the square terms (Ax^2 and/or Cy^2), you may choose to eliminate them by multiplying the entire equation by -1.
2. Group the **x**-terms on the left, the **y**-terms on the right, and move the constant to the right side of the **=** sign. Set up parentheses around the **x**-terms and the **y**-terms.
3. Factor out the coefficients of the x^2 and y^2 terms.
4. Complete the squares for both the **x**-terms and the **y**-terms. Be careful to add the same numbers to both the right and left sides of the equations.
5. Reduce the completed squares to squared-binomial form.
6. If necessary, divide both sides by the required scalar and rearrange terms to obtain the proper form.

Example 1:

Solve: Equation	$-3x^2 + y^2 + 6x - 12y - 15 = 0$
Step 1: Change signs	$+3x^2 - y^2 - 6x + 12y + 15 = 0$
Step 2: Group variables	$(3x^2 - 6x + \_\_) - (y^2 - 12y + \_\_) = -15$
Step 3: Factor coefficients	$3(x^2 - 2x + \_\_) - (y^2 - 12y + \_\_) = -15$
Step 4: Complete Squares	$3(x^2 - 2x + 1) - (y^2 - 12y + 36) = -15 + 3 - 36$
Step 5: Reduce Square Terms	$3(x-1)^2 - (y-6)^2 = -48$
Step 6: Divide by (-48)	$-\frac{(x-1)^2}{16} + \frac{(y-6)^2}{48} = 1$
Rearrange Terms	$\frac{(y-6)^2}{48} - \frac{(x-1)^2}{16} = 1$

The final result is a hyperbola with center (1, 6) and a vertical transverse axis.

Example 2:

Solve: Equation	$-4x^2 - 4y^2 + 16x - 8y + 5 = 0$
Step 1: Change signs	$+4x^2 + 4y^2 - 16x + 8y - 5 = 0$
Step 2: Group variables	$(4x^2 - 16x + \_\_) + (4y^2 + 8y + \_\_) = 5$
Step 3: Factor Coefficients	$4(x^2 - 4x + \_\_) + 4(y^2 + 2y + \_\_) = 5$
Step 4: Complete Squares	$4(x^2 - 4x + 4) + 4(y^2 + 2y + 1) = 5 + 16 + 4$
Step 5: Reduce Square Terms	$4(x-2)^2 + 4(y+1)^2 = 25$
Step 6: Divide by 4	$(x-2)^2 + (y+1)^2 = \frac{25}{4}$

The final result is a circle with center (2, -1) and radius $\frac{5}{2}$.

ADVANCED

Algebra
Parametric Equations of Conic Sections

Parabola (note: $4ap = 1$)

Parametric Equations Centered at the Origin	Parametric Equations Centered at (h, k)
$x = 2pt$ $y = pt^2$	$x = 2pt + h$ $y = pt^2 + k$

Circle

Parametric Equations Centered at the Origin	Parametric Equations Centered at (h, k)
$x = r \cdot cos(t)$ $y = r \cdot sin(t)$	$x = r \cdot cos(t) + h$ $y = r \cdot sin(t) + k$

Ellipse

Parametric Equations Centered at the Origin	Parametric Equations Centered at (h, k)
$x = a \cdot cos(t)$ $y = b \cdot sin(t)$	$x = a \cdot cos(t) + h$ $y = b \cdot sin(t) + k$

Hyperbola

Parametric Equations Centered at the Origin	Parametric Equations Centered at (h, k)
$x = a \cdot sec(t)$ $y = b \cdot tan(t)$	$x = a \cdot sec(t) + h$ $y = b \cdot tan(t) + k$

Algebra
Introduction to Sequences and Series

Definitions

- A **Sequence** is an ordered set of numbers.
- A **Term** is an element in the set of ordered numbers.
- An **Infinite Sequence** has no end. A **Finite Sequence** has a final term.
- An **Explicit Formula** is one that specifically defines the terms of the sequence based on the number of the term. By convention, the number of the term is usually expressed in terms of the variables n or k. We talk of the n^{th} *term* or the k^{th} *term* of the sequence or series.
- A **Recursive Formula** defines a term based on one or more previous terms.
- A **Series** is an ordered summation of a sequence.

Example (Sequence and Series):

Consider the sequence defined by the explicit formula: $t_n = 3n + 1.$

The notation t_n refers to the n^{th} term of the sequence. So, we can construct both a sequence and a series from this. Here are the first seven terms of the sequence and the series:

n	1	2	3	4	5	6	7
t_n	4	7	10	13	16	19	22
Sum of t_n	4	11	21	34	50	69	91

Example (Recursive Formula):

One of the simplest and most famous recursive formulas is the **Fibonacci Sequence**, defined as:

$$t_n = t_{n-1} + t_{n-2}$$

This simply means that each term is the sum of the two terms before it. The Fibonacci Sequence begins with a pair of ones, and uses the recursive formula to obtain all other terms:

n	1	2	3	4	5	6	7
t_n	1	1	2	3	5	8	13
Sum of t_n	1	2	4	7	12	20	33

This sequence has some very interesting properties, which will be discussed on another page.

ADVANCED

Algebra
The Fibonacci Sequence

The **Fibonacci Sequence** was first published in 1202 by Leonardo Fibonacci (of Pisa). It starts with a pair of ones and continues with the recursive formula: $t_n = t_{n-1} + t_{n-2}$. The beginning of the sequence looks like this:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

Properties of the Fibonacci Sequence

It is possible to spend a long time exploring the properties of this simple sequence. Here are a few of the more interesting properties:

Illustration 1: Add the sequence to create a series.

n	1	2	3	4	5	6	7	8	9	10
t_n	1	1	2	3	5	8	13	21	34	55
s_n	1	2	4	7	12	20	33	54	88	143

Notice that : $\boldsymbol{s_n = t_{n+2} - 1}$.

That is, the n^{th} sum is one less than the term of the original sequence two positions further to the right!

Illustration 2: Calculate the squares and add the resulting sequence to create a series.

n	1	2	3	4	5	6	7	8	9	10
t_n	1	1	2	3	5	8	13	21	34	55
${t_n}^2$	1	1	4	9	25	64	169	441	1156	3025
S_n	1	2	6	15	40	104	273	714	1870	4895

Notice that : $\boldsymbol{S_n = t_n \cdot t_{n+1}}$.

That is, the n^{th} sum of the squares is the product of the two terms from the original sequence, one of which is in the same position and one of which is one position to the right!

The Golden Ratio ($\mathbf{f}$)

Ratios of successive values of the Fibonacci Sequence approach the Golden Ratio: $\mathbf{f} = \frac{1+\sqrt{5}}{2}$.

$\frac{8}{5} = 1.6$ $\frac{13}{8} = 1.625$ $\frac{21}{13} = 1.615$ $\frac{34}{21} = 1.619$ ...

The approximate value of $\mathbf{f} = \frac{1+\sqrt{5}}{2} = 1.618034$

One way to express this result is:

$$\lim_{n\to\infty} \left(\frac{t_{n+1}}{t_n}\right)^n = \mathbf{f}$$

Algebra
Summation Notation and Properties

Summation Notation

Mathematicians are fond of finding shorthand ways of expressing things, so they invented notation for the summation of numbers. If we consider the series for $t_n = 3n + 1$, the notation for the series would be:

$$s_n = \sum_{k=1}^{n} (3k - 1)$$

This simply means that the nth term of the series is defined by adding the first n term of the sequence for $t_n = 3n + 1$.

Example:

n	1	2	3	4	5	6	7
t_n	4	7	10	13	16	19	22
s_n	4	11	21	34	50	69	91

Note that $s_3 = t_1 + t_2 + t_3 = 4 + 7 + 10 = 21$.

Although it looks complicated at first, after you write a few series longhand, you will begin to appreciate the shorthand notation.

Summation Properties

Here are a couple of useful properties of summations, all of which are based on the algebraic properties of addition, multiplication and equality. a_k and b_k are two series. c and d are real.

$$\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$$

You can factor a constant out of a summation if is a factor of all the terms.

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

The sum of two series can be broken out into the summations for each series.

$$\sum_{k=1}^{n} (c \cdot a_k + d \cdot b_k) = c \cdot \sum_{k=1}^{n} a_k + d \cdot \sum_{k=1}^{n} b_k$$

This is basically the distributive property of multiplication over addition.

Algebra

Some Interesting Summation Formulas

The following are a few interesting summation series. The development of some series like this may be possible with algebra, but others require either calculus or the calculus of finite differences. Note: parentheses are used in the formulas to aid reading them; the parentheses are not required.

$\sum_{k=1}^{n}(c) = nc$	$c + c + \cdots + c = nc$
$\sum_{k=1}^{n}(k) = \frac{n(n+1)}{2}$	$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
$\sum_{k=1}^{n}(k^2) = \frac{n(n+1)(2n+1)}{6}$	$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n}(k^3) = \left(\frac{n(n+1)}{2}\right)^2$	$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$
$\sum_{k=0}^{\infty}(x^k) = \frac{1}{1-x} \quad for\ \{-1 < x < 1\}$	$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$
$\sum_{k=0}^{\infty}\left(\frac{x^k}{k!}\right) = e^x$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$
$\sum_{k=1}^{\infty}\left[\frac{1}{k}\left(\frac{x-1}{x}\right)^k\right] = \ln x \quad for\ x \geq \frac{1}{2}$	$\left(\frac{x-1}{x}\right) + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots = \ln x$
$\sum_{k=1}^{\infty}\left[(-1)^{(k-1)}\left(\frac{x^k}{k}\right)\right] = \ln(1+x)$ $for\ \{-1 < x \leq 1\}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \ln(1+x)$
$\sum_{k=0}^{\infty}\left[(-1)^{(k)}\left(\frac{x^{(2k)}}{(2k)!}\right)\right] = \cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x$
$\sum_{k=1}^{\infty}\left[(-1)^{(k-1)}\left(\frac{x^{(2k-1)}}{(2k-1)!}\right)\right] = \sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots = \sin x$

Algebra
Arithmetic Sequences

An **Arithmetic Sequence** is one in which the terms are separated by a constant. The constant, often indicated by the letter d, is called the Common Difference. Arithmetic sequences, then meet the condition:

$$t_{n+1} = t_n + d, \quad \text{where } d \text{ is the common difference.}$$

Example:

n	1	2	3	4	5	6
t_n	9	12	15	18	21	24
First Differences		3	3	3	3	3

In this sequence, the common difference is 3. If there is not a common difference, the sequence is not arithmetic.

n^{th} Term of an Arithmetic Sequence

The formula for the n^{th} term of an arithmetic sequence is:

$$t_n = t_1 + (n-1) \cdot d$$

The problem with this formula is the $(n-1)$ that gets multiplied by d. Sometimes this is hard to remember. An alternative method would be to first calculate a term zero,

$$t_0 = t_1 - d$$

Then: $t_n = t_0 + n \cdot d$, which seems a nicer formula

The value of this alternative is that it also allows the student to establish a formula in $y = mx + b$ form for the sequence, where t_0 is the y-intercept, and d, the common difference, is the slope.

Example: In the above example, the n^{th} term of the sequence can be written:

$$t_n = 9 + (n-1) \cdot 3$$

Or, first calculate: $t_0 = 9 - 3 = 6$

Then: $t_n = 6 + 3n$ or $t_n = 3n + 6$

These two equations are equivalent.

Either method works; the student should use whichever one they find more comfortable.

Algebra
Arithmetic Series

An **Arithmetic Series** is the sum of the elements of an arithmetic sequence. The sum of the first n terms of an arithmetic sequence is:

$$s_n = n \cdot \left(\frac{t_1 + t_n}{2}\right)$$

In words, the sum is the product of n and the average term.

Since $t_n = t_1 + (n-1) \cdot d$, we can derive the following formula:

$$s_n = n \cdot \left(t_1 + \frac{(n-1) \cdot d}{2}\right)$$

Or, perhaps better:

$$s_n = n \cdot \left(t_0 + \frac{(n+1) \cdot d}{2}\right)$$

$$= (n \cdot t_0) + d\left(\frac{n(n+1)}{2}\right)$$

Notice that the last term is the sum of the first n integers. That is,

$$\sum_{k=1}^{n}(k) = \frac{n(n+1)}{2}$$

This last equation provides a method for solving many arithmetic series problems.

Example: Find the 8th sum of the sequence $t_n = 6 + 3n$.

$$\sum_{k=1}^{8}(3k+6) = \left(\sum_{t=1}^{8} 6\right) + 3 \cdot \left(\sum_{k=1}^{8} k\right)$$

$$= (8 \cdot 6) + 3 \cdot \left(\frac{8 \cdot 9}{2}\right) = 48 + 108 = 156$$

(n and $(n+1)$ point to $\frac{8 \cdot 9}{2}$; n, t_0, and d point to $(8 \cdot 6)$)

To check this, let's build a table:

n	1	2	3	4	5	6	7	8
t_n	9	12	15	18	21	24	27	30
s_n	9	21	36	54	75	99	126	156

✓

Algebra
Pythagorean Means

Consider a set of n values. We can take a mean of these n values in several ways. The three classical methods of calculating a mean are called Pythagorean Means.

Arithmetic Mean

The arithmetic mean is the one that students are most familiar with. It is also called the average. It is simply the sum of the n items, divided by n.

$$\textbf{arithmetic mean} \;=\; \mathbf{A} \;=\; \frac{\mathbf{1}}{\boldsymbol{n}}\sum \boldsymbol{x_i} \;=\; \frac{\boldsymbol{x_1 + x_2 + \ldots + x_n}}{\boldsymbol{n}}$$

Example: Calculate the arithmetic mean of 5 test scores: 92, 94, 85, 72, 99

$$\boldsymbol{A} \;=\; \frac{92 + 94 + 86 + 74 + 99}{5} \;=\; 89$$

Trick: A shortcut to calculating an arithmetic mean:

- Estimate a value for the average by "eyeballing" the values. For the example above, it looks like 90 would be a good estimate.
- Subtract the estimate from each value to get a set of n differences.
- Add the n differences and divide by n.
- Add the result to the original estimate. The result is the arithmetic mean of the original set of values. In the above example, the result is: $90 + (-1) = 89$.

Trick Example:	
Score	**Value vs. 90**
92	+2
94	+4
86	-4
74	-16
99	+9
Total	-5
Average	-1

Geometric Mean

The geometric mean is the n-th root of the product of the n values.

$$\textbf{geometic mean} \;=\; \boldsymbol{G} \;=\; \left(\prod x_1\right)^{\left(\frac{1}{n}\right)} \;=\; \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$$

Example: Calculate the geometric mean of 2, 9, and 12:

$$\boldsymbol{G} \;=\; \sqrt[3]{2 \cdot 9 \cdot 12} \;=\; \sqrt[3]{216} \;=\; 6$$

Algebra
Pythagorean Means (cont'd)

Harmonic Mean

The harmonic mean is reciprocal of the arithmetic mean of the reciprocals of the n values. It has applications in science.

$$\textbf{harmonic mean} = \mathbf{H} = \left(\frac{1}{n}\cdot\sum\frac{1}{x_i}\right)^{(-1)} = \frac{1}{\frac{1}{n}\cdot\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)}$$

$$= \frac{n}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)}$$

Example: Since the harmonic mean looks so odd, it is useful to look at a real-life example. Consider a car that travels 15 miles at 30 miles per hour, then another 15 miles at 15 miles per hour. *The average speed of the car over this distance is generated by the harmonic mean.*

First, calculate the average speed from basic principles:

- 15 miles at 30 miles per hour takes 30 minutes.
- 15 miles at 15 miles per hour takes 60 minutes.
- Total trip is 30 miles in 90 minutes, for an average speed of 20 miles per hour.

Now, calculate the harmonic mean of the two speeds:

$$\boldsymbol{H} = \frac{2}{\left(\frac{1}{30}+\frac{1}{15}\right)} = \frac{2}{\left(\frac{3}{30}\right)} = \frac{60}{3} = 20\,mph$$

Comparing Means

Compare the values of the three Pythagorean Means of 3, 6, and 12:

$$\mathbf{A}\text{: arithmetic mean} = \frac{3+6+12}{3} = \mathbf{7}$$

$$\mathbf{G}\text{: geometic mean} = \sqrt[3]{3\cdot 6\cdot 12} = \mathbf{6}$$

$$\mathbf{H}\text{: harmonic mean} = \frac{3}{\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{12}\right)} = \mathbf{5\frac{1}{7}}$$

In general, it is true that:

$$\boldsymbol{H < G < A}$$

However, if the values being averaged are all the same,

$$\boldsymbol{H = G = A}$$

Algebra
Geometric Sequences

An **Geometric Sequence** is one in which the ratio of successive terms is the same. The ratio, often indicated by the letter r, is called the Common Ratio. Geometric sequences, then meet the condition:

$$t_{n+1} = r \cdot t_n, \quad \text{where } r \text{ is the common ratio.}$$

Example:

n	1	2	3	4	5	6
t_n	6	12	24	48	96	192
First Ratios		2	2	2	2	2

In this sequence, the common ratio is 2. If there is not a common ratio, the sequence is not geometric.

n^{th} Term of an Geometric Sequence

The formula for the n^{th} term of an arithmetic sequence is:

$$t_n = t_1 \cdot r^{(n-1)}$$

The problem with this formula is the $(n-1)$ that is the exponent of r. Sometimes this is hard to remember. An alternative method would be to first calculate a term zero,

$$t_0 = \frac{t_1}{r}$$

Then: $t_n = t_0 \cdot r^n$, which seems a nicer formula

Example: In the above example, the n^{th} term of the sequence can be written:

$$t_n = 6 \cdot 2^{(n-1)}$$

Or, first calculate: $t_0 = \frac{6}{2} = 3$

Then: $t_n = 3 \cdot 2^n$

These two equations are equivalent.

Either method works; the student should use whichever one they find more comfortable.

Algebra
Geometric Series

An **Geometric Series** is the sum of the elements of an geometric sequence. The sum of the first n terms of an geometric sequence is:

$$s_n = t_1 \cdot \left(\frac{1-r^n}{1-r}\right)$$

Infinite Geometric Series

Of particular interest are **Infinite Geometric Series**. These series never end; they go on forever. An infinite geometric series may have a sum as the series goes to infinity. The sums along the way are called **Partial Sums**. The formula above works for the partial sums of an infinite geometric series.

Starting with the above formula, for a series that does not end, consider the case where $|r| < 1$:

$$S = \lim_{n\to\infty}\left[t_1 \cdot \left(\frac{1-r^n}{1-r}\right)\right]$$

So,
$$S = t_1 \cdot \left(\frac{1}{1-r}\right)$$

The r^n term shrinks as n gets larger, and in the infinite case, it disappears altogether.

Convergence

An infinite series **converges** if it approaches a single value as more terms are added. Otherwise the series **diverges**.

Example: Show examples of where the series: $\left(\frac{1}{1-x}\right) = 1 + x + x^2 + x^3 + x^4 + \cdots$

For $x = \frac{1}{2}$ the series gives: $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ This looks good!

For $x = 2$ the series gives: $-1 = 1 + 2 + 4 + 8 + 16 + \cdots$ Uh oh! This looks very wrong!

Notice that in this series, the common ratio $r = x$. The reason why the first value of x works and the second does not is because this series converges only when $|x| < 1$. This is very common for infinite series with increasing exponents.

The set of values for which a series converges is called the **Interval of Convergence**. For the series in the example, the interval of convergence is $|x| < 1$ or $\{\, x \mid -1 < x < 1 \,\}$.

Algebra

A Few Special Series

Series Involving π

$\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)=\frac{\pi^2}{6}$	$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$
$\sum_{k=1}^{\infty}\left(\frac{1}{k^4}\right)=\frac{\pi^4}{90}$	$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots=\frac{\pi^4}{90}$

Series Involving e

$\sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)=e$	$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots=e$
$\sum_{k=0}^{\infty}\left(\frac{1-2k}{(2k)!}\right)=\frac{1}{e}$	$1-\frac{1}{2!}-\frac{3}{4!}-\frac{5}{6!}-\frac{7}{8!}-\frac{9}{10!}-\cdots=\frac{1}{e}$

Cubes of Natural Numbers

Consider the Series of cubes of the natural numbers:

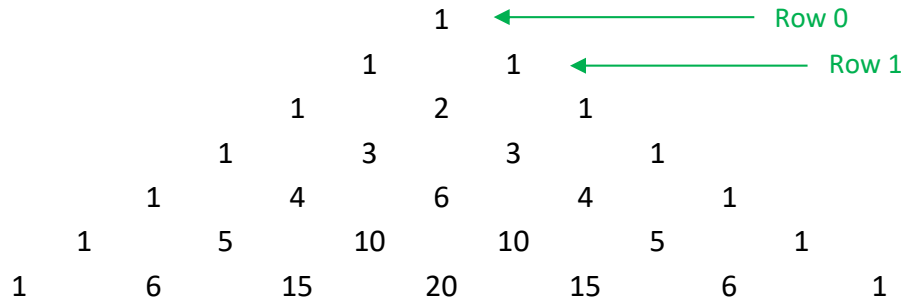
n	1	2	3	4	5	6	7
$t_n = n^3$	1	8	27	54	125	216	343
s_n	1	9	36	100	225	441	784
s_n	1^2	3^2	6^2	10^2	15^2	21^2	28^2

So, the sums of cubes are squares. In fact successive sums are the squares of the **Triangle Numbers**. The Triangle Numbers are the sums of the sequence of natural numbers:

n	1	2	3	4	5	6	7
$t_n = n$	1	2	3	4	5	6	7
T_n	1	3	6	10	15	21	28

Algebra
Pascal's Triangle

Pascal's Triangle starts with the number one at the top and creates entries in successive rows by adding the two numbers above it. For example, in the 4th row, the number $\boldsymbol{6}$ is the sum of the $\boldsymbol{3}$ to its upper left and the $\boldsymbol{3}$ to its upper right. The triangle continues forever and has some very interesting properties.



Properties of Pascal's Triangle

The Triangle starts with Row 0. The number of a row is the value of the second number in the row. Then, the $\boldsymbol{r^{th}}$ number in row $\boldsymbol{n}$ is given by:

$$a_{n,r} = \frac{n!}{r!\,(n-r)!} = {}_nC_r$$

The symbol ${}_nC_r$ comes from Probability Theory, and represents the number of ways $\boldsymbol{r}$ items can be selected from a set of $\boldsymbol{n}$ items. This value is also a Binomial Coefficient; the binomial coefficients are the coefficients of the variables in the expansion of $(a+b)^n$.

Here are a few other interesting properties of Pascal's Triangle:

- It is symmetric along a vertical line containing the top entry.
- The nth row contains $(n+1)$ elements.
- The sum of the entries in row n is 2^n.
- It contains the natural numbers in the second diagonal.
- It contains the triangle numbers in the third diagonal.
- ${}_{n+1}C_r = {}_nC_{r-1} + {}_nC_r$, which is how the triangle is formed in the first place.

Two-Outcome Experiments

In a two-outcome experiment, like flipping a coin, the probability of an event occurring exactly r times in an experiment of n trials is given by the expression at right. This is because there are ${}_nC_r$ outcomes for the event out of a total of 2^n total possible outcomes.

$$P = \frac{{}_nC_r}{2^n}$$

Algebra
Binomial Expansion

4 Steps to a Constructing a Binomial Expansion $(a+b)^n$

1. Start with the binomial coefficients
2. Add in the powers of the first term of the binomial
3. Add in the powers of the second term of the binomial
4. Simplify

Consider the following example: Expand and simplify $(2x-3)^4$

Step 1: Start with the binomial coefficients

$$\binom{4}{0} \quad +\binom{4}{1} \quad +\binom{4}{2} \quad +\binom{4}{3} \quad +\binom{4}{4}$$

Step 2: Add in the powers of the first term of the binomial

$$\binom{4}{0}(2x)^4 \quad +\binom{4}{1}(2x)^3 \quad +\binom{4}{2}(2x)^2 \quad +\binom{4}{3}(2x)^1 \quad +\binom{4}{4}(2x)^0$$

Step 3: Add in the powers of the second term of the binomial

$$\binom{4}{0}(2x)^4(-3)^0 + \binom{4}{1}(2x)^3(-3)^1 + \binom{4}{2}(2x)^2(-3)^2 + \binom{4}{3}(2x)^1(-3)^3 + \binom{4}{4}(2x)^0(-3)^4$$

Step 4: Simplify:

$$= (1)(16x^4)(1) + (4)(8x^3)(-3) + (6)(4x^2)(9) + (4)(2x)(-27) + (1)(1)(81)$$

$$\mathbf{= 16x^4 - 96x^3 + 144x^2 - 216x + 81}$$

Notice the following about the expansion:

1. There are $(\boldsymbol{n}+\mathbf{1})$ terms, where $\boldsymbol{n}$ is the exponent of the binomial being expanded.
2. $\boldsymbol{n}$ is the top number in every binomial coefficient.
3. The bottom numbers in the binomial coefficients count up from **0** to $\boldsymbol{n}$.
4. When a term of the original binomial is negative, the signs in the solution alternate.
5. The exponent of the first term in the original binomial counts down from $\boldsymbol{n}$ to **0**.
6. The exponent of the second term in the original binomial counts up from **0** to $\boldsymbol{n}$.
7. The exponents of the two terms in the original binomial add to $\boldsymbol{n}$ in every term of the expansion.

ADVANCED

Algebra
Gamma Function and $n!$

Definition

The Gamma Function is defined by the following definite integral for $x > 0$:

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1} dt$$

Initially, this integral appears daunting, but it simplifies nicely under certain conditions, and has some very interesting properties.

Properties and Values

The following properties and values of the Gamma Function are of particular interest:

- $\Gamma(x+1) = x!$ for integer values of x
- $\Gamma(x+1) = x \cdot \Gamma(x)$ for values of x where $\Gamma(x)$ exists
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(x) \cdot \Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$ for $\{0 < x < 1\}$
- $\Gamma(x) \neq 0$ for any value of x
- Some other functions relate to the Gamma Function. Examples:

Digamma Function: $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ **Beta Function:** $B(x,y) = \dfrac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}$

Factorials

$\Gamma(1) = 0! = 1$
$\Gamma(2) = 1! = 1$
$\Gamma(3) = 2! = 2$
$\Gamma(4) = 3! = 6$
$\Gamma(5) = 4! = 24$
$\Gamma(6) = 5! = 120$
. . .

Stirling's Formula

For large values of n, **Stirling's Formula** provides the approximation: $n! \sim \sqrt{2n\pi} \cdot \left(\frac{n}{e}\right)^n$

Example: Direct Calculation: $100! \sim 9.3326 \text{ x } 10^{157}$

Using Stirling's Formula: $100! \sim \sqrt{200\,\pi} \cdot \left(\frac{100}{e}\right)^{100} \sim 9.3248 \text{ x } 10^{157}$

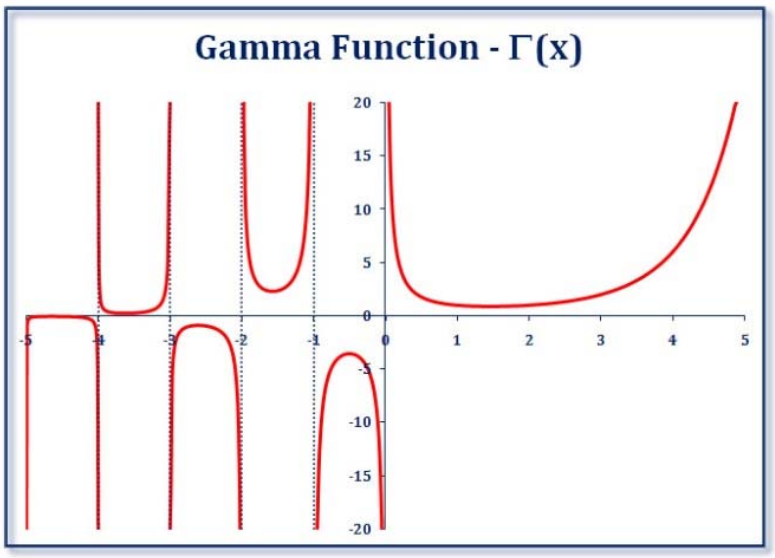
This represents an error of less than 0.1%.

ADVANCED

Algebra
Graphing the Gamma Function

Gamma Function Graph

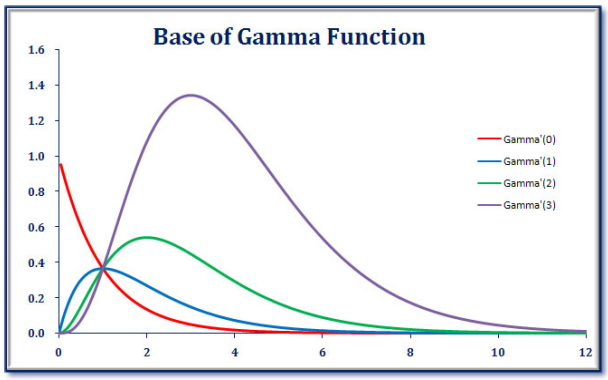
Here is a graph of the Gamma Function. For $x > 0$, the function is continuous and passes through all the factorials.

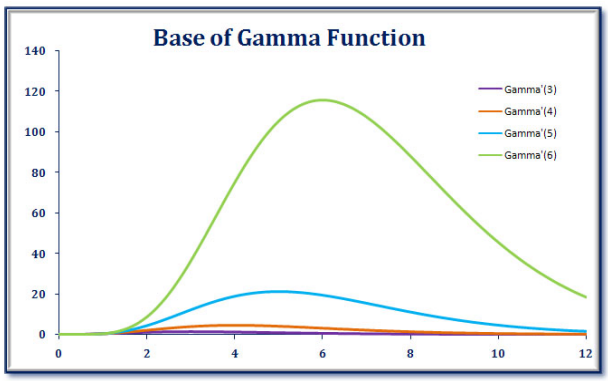


- For $0 < x < 1$, the graph moves asymptotically vertical as $x \Rightarrow 0$.
- For $x < 0$, the function has vertical asymptotes at each integer value and forms a "U" between the integers, with alternating positive and negative values by interval.
- $\Gamma(x)$ approaches $y = 0$ as x becomes increasingly negative.

Calculating $\Gamma(x)$

Each value of $\Gamma(x)$ can be calculated using the definition of the Gamma Function on the previous page. Recall that a definite integral is a measure of the area under the curve of the function being integrated. Based on this, we have the following examples of $\Gamma(x)$ values and graphs that illustrate the curves which determine those values.





$$\Gamma(0) = \int_0^\infty \left(\frac{e^{-t}}{t}\right) dt$$

$$\Gamma(1) = \int_0^\infty (e^{-t})\, dt$$

$$\Gamma(2) = \int_0^\infty (e^{-t}\, t)\, dt$$

$$\Gamma(3) = \int_0^\infty (e^{-t}\, t^2)\, dt$$

$$\Gamma(4) = \int_0^\infty (e^{-t}\, t^3)\, dt$$

$$\Gamma(5) = \int_0^\infty (e^{-t}\, t^4)\, dt$$

Algebra Handbook
Index

Algebra Handbook
Index

Algebra Handbook
Index

Algebra Handbook
Index

Algebra Handbook
Index

Algebra Handbook
Index

Algebra Handbook
Index

Algebra Handbook
Index

Algebra Handbook
Index

Algebra Handbook
Index