Complex Analysis Exam 1

October 29, 2018

Name: ______________________

1. **Let $f(z)$ be a branch of $log\frac{1+z}{1-z}$. Find the power series of this function centered at the point $z = 0$ and find its radius of convergence.**

   Proof: $f(z) = log(1+z) - log(1-z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1} - \sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n+1}{n+1} z^{n+1} = 2\sum_{n=1}^{\infty} \frac{1}{2n+1} z^{2n+1}$. The radius of convergence is

   $$R = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}} = \frac{1}{\lim_{n\to\infty} \sqrt[2n+1]{|a_{2n+1}|}} = \lim_{n\to\infty} \sqrt[2n+1]{2n+1} = \lim_{n\to\infty} \sqrt[n]{n} = 1. \quad \square$$

2. **Find a linear transformation $f(z)$ that maps the points $1, 2$ and i to the points $i, 1, 2$, respectively.**

   Solution Consider the cross ratio $(z, 1, 2, i) = (w, i, 1, 2)$ where $w = f(z)$, i.e.,

   $$\frac{z-1}{z-i} \cdot \frac{2-i}{2-1} = \frac{w-i}{w-2} \cdot \frac{1-2}{1-i},$$

   i.e., $(z-1)(2-i)(1-i)(w-2) = -(w-i)(z-i)$, i.e., $[(z-1)(2-i)(1-i)+(z-i)]w = i(z=i) + 2(2-i)(1-i)(z-1)$, i.e.,

   $$w = f(z) = \frac{(-5i+2)z - 1 + 6i}{(2-3i)z - 1 + 2i}.$$

3. **Evaluate the integral $\oint_C \frac{\bar{z}}{|z|} dz$ where $C = \partial\Delta(4)$ is the circle with counter-clockwise orientation.**

   Solution: The curve C is defined by $\phi : [0, 2\pi] \to \mathbb{C}, t \mapsto 4cos\ t + 4i\ sin\ t = 4e^{it}$. Then

   $$\int_C \frac{\bar{z}}{|z|} dz = \int_0^{2\pi} \frac{4e^{-it}}{4} 4ie^{it} dt = \int_0^{2\pi} 4idt = 8\pi i.$$

4. **Compute**

$$\oint_C \frac{dz}{z^2+1},$$

where $C = \partial\Delta(-i,1)$ **with counterclockwise.**

Solution: By Cauchy integral formula, by setting $f(\zeta) = \frac{1}{\zeta - i}$,

$$\oint_C \frac{dz}{z^2+1} = \oint_C \frac{f(\zeta)}{\zeta-(-i)} = 2\pi i f(-i) = -\pi.,$$

5. **Let** $f : \mathbb{C} \to \mathbb{C}$ **be a linear transformation such that** $f(x+ix) \in \mathbb{R}$ **for any** $x \in \mathbb{R}$**. If** $f(2) = i$**, find** $f(2i) =?$

Solution: Since $2i$ and 2 are mutual reflection points with respect to the line $L = \{x + ix \mid x \in \mathbb{R}\}$, and since $f(L) = \mathbb{R}$, we know that $f(2i)$ and $f(2)$ are mutual reflection points with respect to the real axis. Since $f(2) = i$, it implies $f(2i) = -i$. □

6. **If** $f(z) = \sum_{n=0}^{\infty} a_n z^n$ **has the radius of convergence** R**,** $0 < R < \infty$**. Show that there exists at least one point** $z_0 \in \partial\Delta(R)$ **such that** f **cannot extend holomorphically on a neighborhood of** z_0**.** [1]

Proof: Suppose there is no such point z_0. We seek a contradiction. In fact, for any point $z \in \partial\Delta(R)$, there exists a neighborhood $\Delta(z, \epsilon_z)$ on which f can extend holomorphically. Then we obtain an open covering $\mathcal{U} = \{\Delta(z, \epsilon_z)\}$ of $\partial\Delta(R)$. Since $\partial\Delta(R)$ is compact, there exists a finite subcover of $\mathcal{U}$: $\Delta(z_1, \epsilon_{z_1}), \Delta(z_2, \epsilon_{z_2}), \ldots\ldots, \Delta(z_m, \epsilon_{z_m})$. This implies that f extend holomorphically on $\Delta(R')$ where $R' > R$, which is a contradiction.

[1] By f extends holomorphically at $z_0 \in \partial\Delta(R)$, it means that there is another holomorphic function $g(z)$ defined on a disk $\Delta(z_0, r)$ such that $g(z) = f(z)$ for all $z \in \Delta(R) \cap \Delta(z_0, r)$.