Design of Multi-loop control systems

Consider a single loop system as shown in Fig.1.

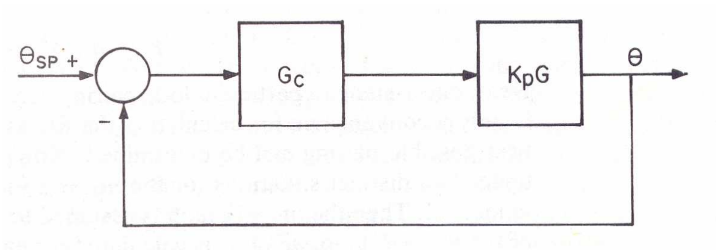


Fig.1

Suppose controller is fixed, substantial changes in K_p invariably lead to deteriorate the control system response (see Figure 2).

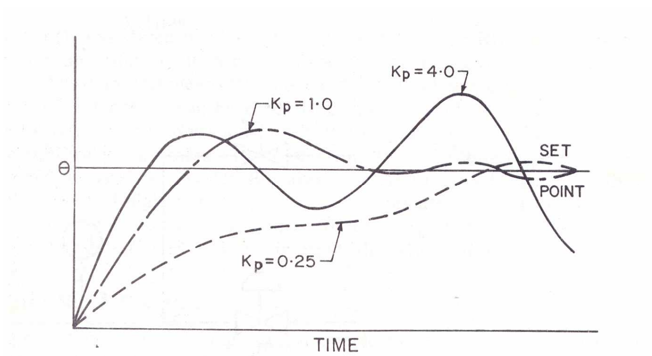


Fig.2

Now consider a 2×2 control system in Fig. 3

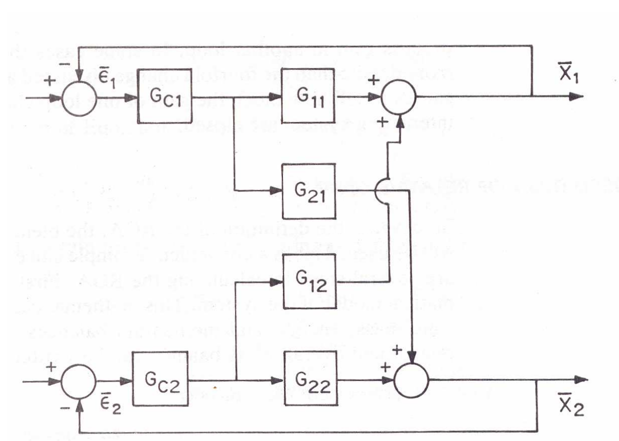


Fig.3

Let us consider open loop 1 (Fig.4)

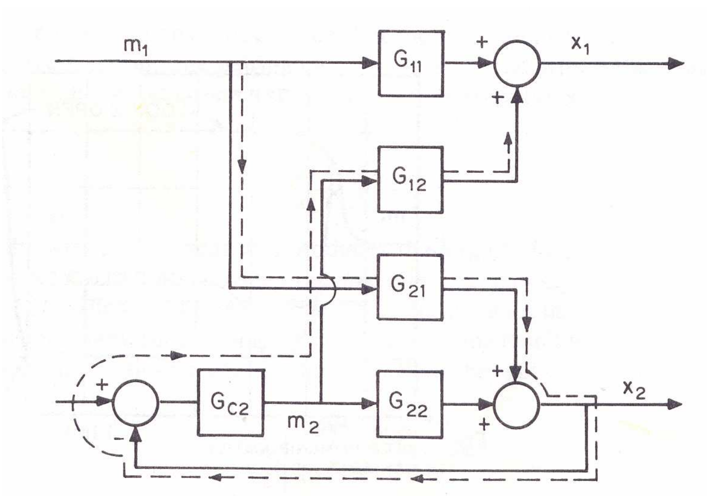


Fig.4

One may notice that there are two transmission paths from m_1 to x_1.let us define a relative gain from m_1 to x_1.as:

$$\lambda = \frac{Gain\ m_1 - x_1, loop2\ open}{Gain\ m_1 - x_1, loop2\ close}$$

When $\lambda = 0.5$, the responses of x_1 to a unit step input at m_1 is shown in Fig. 5

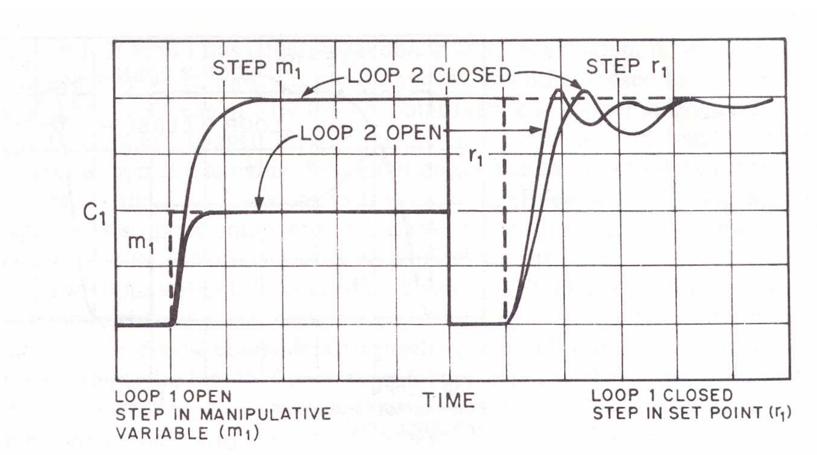
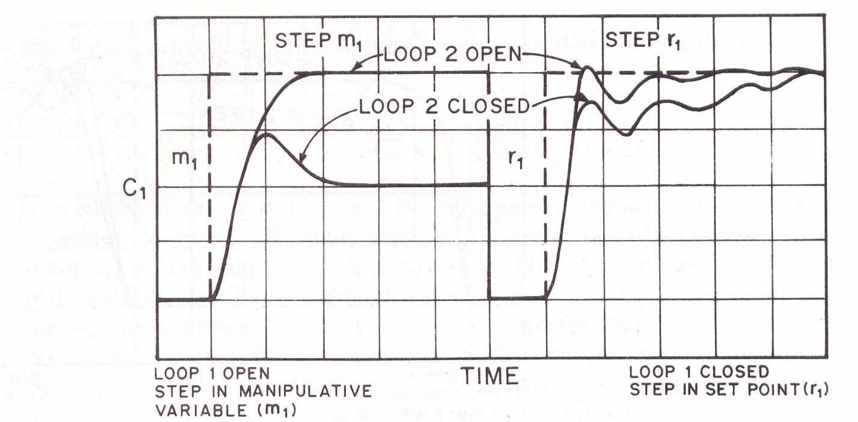


Fig.5

In this case, when loop 2 is closed, the open loop gain of $m_1 - x_1$ becomes doubled. The increase in the loop gain results in more oscillation in the closed loop response as shown.

On the other hand, when $\lambda = 2$, the open loop and closed loop responses are also given in Fig. 6.



In this latter case, the open-loop gain decrease when loop 2 is switched from open to close. As a result, the close of loop 1 leads the system to a more sluggish response to the r_1 input.

The increase or decrease of the loop gain is a result of closing loop2, and , hence, is considered loop interaction. From the above example, λ is a measure of such interaction and is named as relative gain of loop 1. You may also find the other relative gain for loop2. But, in this case, the two relative gains will be equal.

Algebraic Properties of the RGA

1. $\sum_j g_{ij}\bar{g}_{ji} = \sum_j \lambda_{ij} = 1, \quad \forall i$
2. $P_1\Lambda\{G\}\,P_2 = \Lambda\{P_1G\,P_2\}$, P_1 and P_2 are two permutation matrices.
3. $\Lambda\{G\} = \Lambda\{S_1G\,S_2\}$, S_1 and S_2 are two diagonal matrices.
4. If transfer matrix, G, is diagonal or triangular, then: $\Lambda\{G\}=I$.

   [Proof]:

   Let, $G = \begin{bmatrix} x & 0 & 0...0 \\ x & x & 0...0 \\ .... & & \\ x & x & x...x \end{bmatrix}$

   Then,

$$G^{-1} = \overline{G} = \begin{bmatrix} x & 0 & 0...0 \\ x & x & 0...0 \\ .... & & \\ x & x & x...x \end{bmatrix}$$

Thus, $g_{ij}\overline{g}_{ji} = 0 = \lambda_{\mathrm{ij}}, \quad \forall \mathrm{j} \neq \mathrm{i}$

and, $g_{ii}\overline{g}_{ii} = 1 = \lambda_{\mathrm{ii}}, \quad \forall \mathrm{j} = \mathrm{i}$

So, $\Lambda = I$

5. $$\frac{d\overline{g}_{ji}}{\overline{g}_{ji}} = -\lambda_{\mathrm{ij}}\frac{dg_{ij}}{g_{ij}}$$

$$\overline{g}_{ij} = \frac{adj[A]}{\det[G]} = \frac{(-1)^{i+j}\det[G^{ij}]}{\det[G]}$$

$$\frac{d\overline{g}_{ji}}{dg_{ij}} = -\frac{\frac{d\det[G]}{dg_{ij}}(-1)^{i+j}\det[G^{ij}]}{\det[G(g_{ij})]^2} = \frac{\det[G^{ij}]^2}{\det[G(g_{ij})]^2} = -\overline{g}_{ji}^2$$

$$\frac{d\overline{g}_{ji}}{\overline{g}_{ji}} = -\overline{g}_{ji}dg_{ij} = -\overline{g}_{ji}g_{ij}\frac{dg_{ij}}{g_{ij}} = -\lambda_{ij}\frac{dg_{ij}}{g_{ij}}$$

5. $$\frac{d\lambda_{\mathrm{ij}}}{\lambda_{\mathrm{ij}}} = (1-\lambda_{\mathrm{ij}})\frac{dg_{ij}}{g_{ij}}, \quad \text{and} \quad \frac{d\lambda_{\mathrm{ij}}}{\lambda_{\mathrm{ij}}} = \frac{\lambda_{\mathrm{ij}}-1}{\lambda_{\mathrm{ij}}}\frac{d\overline{g}_{ij}}{\overline{g}_{ij}}$$

[Proof];

$$\lambda_{\mathrm{ij}} = g_{ij}\overline{g}_{ji} \Rightarrow d\lambda_{\mathrm{ij}} = dg_{ij}\overline{g}_{ji} + g_{ij}d\overline{g}_{ji}$$

$$\Rightarrow \frac{d\lambda_{\mathrm{ij}}}{\lambda_{\mathrm{ij}}} = \frac{dg_{ij}\overline{g}_{ji} + g_{ij}d\overline{g}_{ji}}{g_{ij}\overline{g}_{ji}} = \frac{dg_{ij}}{g_{ij}} + (\frac{d\overline{g}_{ji}}{\overline{g}_{ji}}) = (1-\lambda_{ij})\frac{dg_{ij}}{g_{ij}}$$

or,

$$\Rightarrow \frac{d\lambda_{\mathrm{ij}}}{\lambda_{\mathrm{ij}}} = \left(\frac{dg_{ij}}{g_{ij}}\right) + \frac{d\overline{g}_{ji}}{\overline{g}_{ji}} = (1-\frac{1}{\lambda_{ij}})\frac{d\overline{g}_{ij}}{\overline{g}_{ij}} = \left[\frac{\lambda_{ij}-1}{\lambda_{ij}}\right]\frac{d\overline{g}_{ij}}{\overline{g}_{ij}}$$

RGA-implications:

1. Pairing loops on λ_{ij} values that are positive and close to 1.

2. Reasonable Pairings: $0.5 < \lambda_{ij} < 4.0$

3. Pairing on negative λ_{ij} values results in at least one of the following;
   a. Closed loop system is unstable,
   b. Loop with negative λ_{ij} is unstable,
   c. Closed loop system becomes unstable if loop with negative is λ_{ij} turned off.

4. Plants with large RGA-elements are always ill-conditioned. (i.e., a plant with a large $\gamma(G)$ may have small RGA-elements)

5. Plants with large RGA-elements around the crossover frequency are fundamentally difficult to control because of sensitivity to input uncertainties.
   -----→decouplers or other inverse-based controllers should not be used for plants with large RGA-elements.

6. Large RGA-element implies sensitivity to element-by-element uncertainty.

7. If the sign of RGA-element changes from $s=0$ to $s=\infty$, then there is a RHP-zero in G or in some subsystem of G.

8. The RGA-number can be used to measure diagonal dominance:
   RGA-number = $\| \Lambda(G)-I \|_{min}$.
   For decentralized control,, pairings with RGA-number at crossover frequency close to one is preferred.

9. For integrity of whole plant, we should avoid input-output pairing on negative RGA-element.

10. For stability, pairing on an RGA-number close to zero at crossover frequency is preferred.

The Relative Disturbance Gain (RDG)

Ref: **Galen Stanley, Maria Marino-Galarraga, and T. J. McAvoy**, Shortcut Operability Analysis. 1. The relative disturbance gain, I&EC, Process Des. Dev. 1985,24, 1181-1188

The use of RDG:

1. To decide if interaction resulting from a disturbance is favorable or unfavorable.
2. To decide whether or not decoupling should be used and what type of decoupling structure is best.

$$y_1 = k_{11}m_1 + k_{12}m_2 + k_{F1}d$$
$$y_2 = k_{21}m_1 + k_{22}m_2 + k_{F2}d$$

$$\left(\frac{\partial m_1}{\partial d}\right)_{y_1, m_2} = -\frac{k_{F1}}{k_{11}}$$

$\left(\frac{\partial m_1}{\partial d}\right)_{y_1, y_2}$ is derived when both y_1 and y_2 are held still:

$$y_1 = k_{11}m_1 + k_{12}m_2 + k_{F1}d = 0 \qquad (2)$$
$$y_2 = k_{21}m_1 + k_{22}m_2 + k_{F2}d = 0$$

so that:

$$m_2 = \frac{1}{k_{22}}\left[-k_{21}m_1 - k_{F2}d\right] \qquad (3)$$

Substitute Eq.(3) into E.(2), we have:

$$\left[k_{11} - \frac{k_{12}k_{21}}{k_{22}}\right]m_1 + \left[k_{F1} - \frac{k_{12}k_{F2}}{k_{22}}\right]d = 0$$

Thus,

$$\left(\frac{\partial m_1}{\partial d}\right)_{y_1, y_2} = \frac{-k_{F1} + \dfrac{k_{12}k_{F2}}{k_{22}}}{k_{11} - \dfrac{k_{12}k_{21}}{k_{22}}} = \frac{k_{12}k_{F2} - k_{F1}k_{22}}{k_{11}k_{22} - k_{12}k_{21}} \qquad (4)$$

So,

$$\beta_1 = \frac{\left(\frac{\partial m_1}{\partial d}\right)_{y_1,y_2}}{\left(\frac{\partial m_1}{\partial d}\right)_{y_1,m_2}} = -\frac{k_{11}}{k_{F1}} \times \frac{k_{12}k_{F2} - k_{F1}k_{22}}{k_{11}k_{22} - k_{12}k_{21}} = \frac{k_{11}k_{22}}{k_{F1}k_{22}} \times \frac{k_{12}k_{F2} - k_{F1}k_{22}}{k_{11}k_{22} - k_{12}k_{21}} \qquad (5)$$

$$= -\frac{k_{12}k_{F2} - k_{F1}k_{22}}{k_{F1}k_{22}} \times \frac{k_{11}k_{22}}{k_{11}k_{22} - k_{12}k_{21}} = \left[1 - \frac{k_{12}k_{F2}}{k_{F1}k_{22}}\right]\lambda$$

Similarly, we have:

$$\beta_1 = \frac{\left(\frac{\partial m_1}{\partial d}\right)_{y_1,y_2}}{\left(\frac{\partial m_1}{\partial d}\right)_{y_1,m_2}} = \left[1 - \frac{k_{21}k_{F1}}{k_{F2}k_{11}}\right]\lambda$$

$$\frac{k_{F2}k_{12}}{k_{F1}k_{22}} = 1 - \frac{\beta_1}{\lambda} = \frac{\lambda - \beta_1}{\lambda} \qquad \Rightarrow \qquad \frac{k_{F2}}{k_{F1}} = \frac{\lambda - \beta_1}{\lambda} \times \frac{k_{22}}{k_{12}}$$

Similarly, $\Rightarrow \quad \frac{k_{F1}}{k_{F2}} = \frac{\lambda - \beta_2}{\lambda} \times \frac{k_{11}}{k_{21}}$

So, $\frac{k_{F2}}{k_{F1}} \times \frac{k_{F1}}{k_{F2}} = 1 = \left(\frac{\lambda - \beta_2}{\lambda} \times \frac{k_{11}}{k_{21}}\right) \times \left(\frac{\lambda - \beta_1}{\lambda} \times \frac{k_{22}}{k_{12}}\right)$

$$\Rightarrow \quad \left(\frac{\lambda - \beta_2}{\lambda}\right)\left[\frac{\lambda - \beta_1}{\lambda}\right] = \frac{k_{11}}{k_{21}}\frac{k_{22}}{k_{12}} = 1 - \frac{1}{\lambda}$$

or,

$$(\beta_1 - \lambda)(\beta_2 - \lambda) = \lambda(\lambda - 1)$$

$$(\beta_2 - \lambda) = \frac{\lambda(\lambda - 1)}{(\beta_1 - \lambda)} \qquad \Rightarrow \qquad \beta_2 = \frac{\lambda(\lambda - 1)}{(\beta_1 - \lambda)} + \lambda = \frac{(\beta_1 - 1)\lambda}{(\beta_1 - \lambda)} = \frac{1 - \beta_1}{\lambda - \beta_1}\lambda$$

It can be shown that:

$$\frac{\text{Multi-loop } e_1 \text{ area}}{\text{SISO idealy decoupled } e_1 \text{ area}} \propto \beta_1 \quad \text{and}$$
$$\frac{\text{Multi-loop } e_2 \text{ area}}{\text{SISO idealy decoupled } e_2 \text{ area}} \propto \beta_2$$

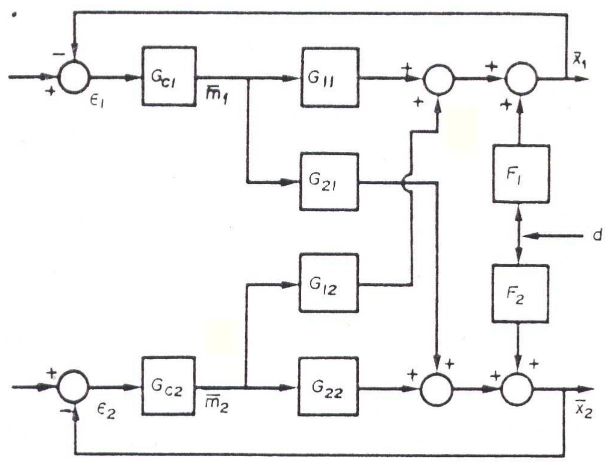


Figure 4. General 2 × 2 system.

$$\frac{e_1}{d} = \frac{F_2 G_{12} G_{c2} - F_1(1+G_{22}G_{c2})}{(1+G_{11}G_{c1})(1+G_{22}G_{c2}) - G_{c2}G_{c1}G_{12}G_{21}} \qquad (6)$$

If d is a unit step, then the area under e_1 curve is given as:

$$\int_0^\infty e_1 dt = \lim_{s\to 0} e_1(s) = \frac{k_{F2}k_{12}\frac{k_{c2}}{\tau_{R2}} - \frac{k_{F1}k_{22}k_{c2}}{\tau_{R2}}}{\frac{k_{c1}k_{11}}{\tau_{R1}}\frac{k_{c2}k_{22}}{\tau_{R2}} - \frac{k_{c2}}{\tau_{R2}}\frac{k_{c1}}{\tau_{R1}}k_{12}k_{21}} = \frac{k_{F2}k_{12} - k_{F1}k_{22}}{\frac{k_{c1}k_{11}k_{22}}{\tau_{R1}}\left(1 - \frac{k_{12}k_{21}}{k_{11}k_{22}}\right)}$$

$$= \lambda \frac{\tau_{R1}}{k_{c1}k_{11}}\left(k_{F2}\frac{k_{12}}{k_{22}} - k_{F1}\right)$$

On the other hand, when loop 2 is opened, the area under $e_1$becomes:

$$\int_0^\infty e_1^o dt = -\frac{\tau'_{R1}}{k'_{c1}}\frac{k_{F1}}{k_{11}}$$

Thus,

$$\frac{\int_0^\infty e_1 dt}{\int_0^\infty e_1^o dt} = -\frac{\tau_{R1}}{\tau_{R1}^{'}} \frac{k_{c1}^{'}}{k_{c1}} \times \left(\frac{k_{F2} k_{12}}{k_{F1} k_{22}} - 1 \right) \lambda = \frac{\tau_{R1}}{\tau_{R1}^{'}} \frac{k_{c1}^{'}}{k_{c1}} \times \beta_1 = f_1 \beta_1$$

Similarly, we have:

$$\frac{\int_0^\infty e_2 dt}{\int_0^\infty e_2^o dt} = \frac{\tau_{R2}}{\tau_{R2}^{'}} \frac{k_{c2}^{'}}{k_{c2}} \times \beta_2 = f_2 \beta_2$$

Notice that the PI parameters in the interacting loops are used to be more conservative than those in single loops. In another words,

$$f_1 \geq 1; \quad f_2 \geq 1$$

The multi-loop control should be beneficial when the sum of absolute values of the

Remarks:

1. If λ is assumed not vary with frequency, and the process under study is FOPDT, λ>1, f_1 lies in the range 1< f_1 <2, while 0.5<l<1, f_1 lies in the range 1< f_1 <3.
2. When f_1 =1, β is equal to the ratio of response areas.
3. If β is small and f_1 =is close to one, then the interacting control is favored for that particular disturbance.
4. If β is large, the interacting control is un favorable for that particular disturbance.

The Relative Gain for Non-square Multivariable Systems

(J.C. Chang and C.C. Yu, CES Vol.45, pp. 1309-1323 1990)

Consider a non-square MV system.

$$y_{m\times 1}(s) = G_{m\times n}(s) u_{n\times 1}(s)$$

Define Moore-Penrose pseudo-inverse of the matrix $G(s)$ as:

$$G^+(s) = \left(G^T G\right)^{-1} G^T(s)$$

Then, under close-loop control, the steady-state control input will be:

$u = G^+(0) y^d$ and $\left(\frac{\partial u_i}{\partial y_j}\right)_{CL} = g_{ij}^+(0)$.

Thus, the non-square relative gain is defined similarly to the square RGA, that is:

$$\tilde{\Lambda} = \left(\frac{\partial y_i}{\partial u_j}\right)_{OL} \left\{\left(\frac{\partial y_i}{\partial u_j}\right)_{CL}\right\}^{-1} = G(0) \otimes \left[G^+(0)\right]^T$$

Properties of the non-square RGA

1. Row sum of $\tilde{\Lambda}$:

$$RS = [rs(1), \quad rs(2), \quad \cdots, \quad rs(m)] = \left[\sum_{j=1}^{n} \tilde{\lambda}_{1j}\ , \quad \sum_{j=1}^{n} \tilde{\lambda}_{2j}\ , \quad \cdots \ , \quad \sum_{j=1}^{n} \tilde{\lambda}_{mj}\ ,\right]^T;$$

Where, $rs(i) = \left[G(0)G^+(0)\right]_{ii}$

2. $$CS = [cs(1), \quad cs(2), \quad \cdots, \quad cs(n)] = \left[\sum_{j=1}^{n} \tilde{\lambda}_{j1}\ , \quad \sum_{j=1}^{n} \tilde{\lambda}_{j2}\ , \quad \cdots \ , \quad \sum_{j=1}^{n} \tilde{\lambda}_{jn}\ ,\right]^T = [1, \quad 1, \quad \cdots, \quad 1]^T$$

Where, $cs(i) = \left[G^+G(0)\right]_{ii}$; (Note: $G^+G = (G^TG)^{-1}G^TG = I$)

3. $0 \le rs(i) \le 1, \quad \forall i = 1, 2, \cdots, m$

4. $\sum_{i=1}^{m} rs(i) = \sum_{j=1}^{n} cs(i) = n$

Note: $\sum_{i=1}^{m} rs(i) = \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{\lambda}_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} \tilde{\lambda}_{ij} = \sum_{j=1}^{n} cs(j) = n$

5. Non-square RGA is invariant under input scaling, but is variant under output scaling:

$$(GS) \otimes \left[(GS)^+\right]^T = \left(G \otimes G^+\right)^T \qquad (SG) \otimes \left[(SG)^+\right]^T \ne \left(G \otimes G^+\right)^T$$

6. Let P_1 and P_2 are permutation matrices. Then, $\tilde{\Lambda}(P_1GP_2) = P_1\tilde{\Lambda}(G)P_2$

A. Multi-loop BLT-Tuning:

I. BLT-1 method:

a. Calculate the Ziegler-Nichol settings for each PI controller by using the diagonal element of G, i.e. $g_{i,i}$.
b. Assume a detuning factor "F", and calculate controller settings for loops.

$$k_{c,i} = k_{ZN,i} / F; \quad \tau_{R,i} = \left(\tau_{R,i}\right)_{ZN} F$$

c. Define: $W_{(i\omega)} = -1 + \det\left[I + G_{(i\omega)}G_{c(i\omega)}\right]$

d. Calculate the closed-loop function $L_{c(i\omega)}$):

$$L_{c(i\omega)} = 20\log\left|\frac{W_{(i\omega)}}{1+W_{(i\omega)}}\right|$$

e. Calculate the detuning factor F until the peak in the Lc log modulus curve is equal to 2N, that is:

$$L_{cm} = \underset{\omega}{Max}\left\{20\log\left|\frac{W_{(i\omega)}}{1+W_{(i\omega)}}\right|\right\} = 2N$$

II. BLT-2

a. Find BLT-1 PI controllers.
b. Choose a second detuning factor F_D. F_D should be greater than one.
c. Compute $\tau_{D,j}$ as:

$$\tau_{D,j} = \frac{\left(\tau_{D,j}\right)_{ZN}}{F_D}$$

d. Calculate $W_{(i\omega)}$ and $L_c(i\omega)$.

e. Change F_D until $L_C^{\max}$ is minimized, maintaining $F_D \geq 1$. The trivial case may result where $L_C^{\max}$ is minimized for $F_D = \infty$, i.e., no derivative action.

f. Reduce F in the P and I modes, until $L_C^{\max} = 2N$.

III. BLT-3

The objective is to estimate the level of imbalance in detuning the BLT-1 controller and compensate for it.

Consider the PI controller:

$$u_j = u_j(0) + k_{C,j}\left(e_j + \frac{1}{\tau_{R,j}}\int_0^t e_j dt\right); \qquad u_j(0) = 0$$

At steady state,

$$\lim_{t\to\infty}\left[u_j(t)\right]=\frac{k_{C,j}}{\tau_{R,j}}\int_0^\infty e_j(t)dt$$

So,

$$\int_0^\infty e_j(t)dt=\frac{\tau_{R,j}u_j(\infty)}{k_{C,j}}$$

Notice that:

$$u(\infty)=G^{-1}(0)R-G^{-1}G_L(0)d(\infty)$$

For unit step set-point input:

$$\begin{aligned}u_j(\infty)&=G^{-1}(0)[0,\dots,0,1,0,\dots,0]^T\\&=[\overline{g}_{i,j}(0);\, i,\, j=1,\dots,N][0,\dots,0,1,0,\dots,0]^T\end{aligned}$$

For unit step load disturbance:

$$u_i(\infty)=i\,th\ \ row\, of\ \ G^{-1}(0)G_L(0)=\sum_{j=1}^{N}[\overline{g}_{i,j}(0)g_{L,j}(0)]$$

Then, ITE_j becomes:

$$ITE_j=\frac{u_j(\infty)\tau_{R,j}}{k_{C,j}}$$

Let,

$$S_j=\sum_{i=1}^{N}\left|\frac{ITE_j}{N}\right|+\left|ITE_{j-load}\right|$$

$$S_j=\left|\frac{\tau_{R,j}}{k_{C,j}}\right|\times\left\{\sum_{i=1}^{N}\left|\frac{\overline{g}_{j,i}(0)}{N}\right|+\left|\overline{g}_{j,i}(0)g_{L,i}(0)\right|\right\}$$

Let $S_{\max}=\underset{j}{Max}\,S_j$

$$F_j=F\sqrt{\frac{S_{\max}}{S_j}}$$

The PI controller parameters becpme:

$$k_{c,i}=k_{ZN,i}/F_j;\quad \tau_{R,i}=\left(\tau_{R,i}\right)_{ZN}F_j$$

IV. BLT-4

a. BLT-3 is used to get individual PI controllers as described above.
b. BLT-2 procedure is used with individual F_D factors for each loop:

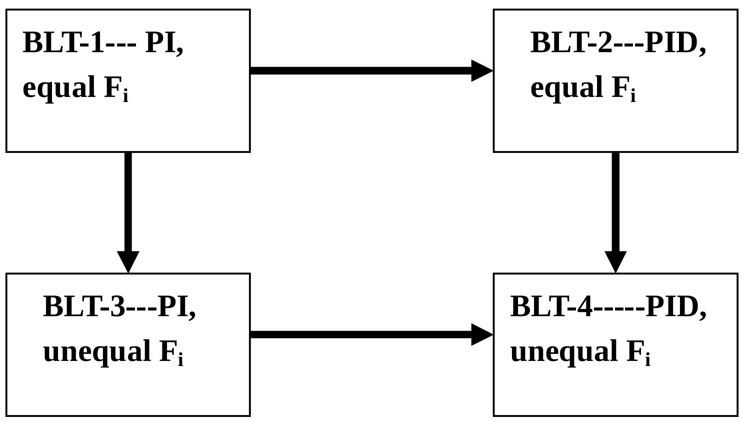
$$F_{D,j} = F_D \sqrt{\frac{S_{\max}}{S_j}}$$

V. Tyreus Load-Rejection Criterion (TLC)

The best variable pairing is the one that gives the smallest magnitudes for each element of X,(i.e. X_i) of the following:

$$X_{(i\omega)} = \left([I + GG_C]^{-1} G_L L\right)_{(i\omega)}$$

VI. Summary



B. Parallel-design method---Modified Z-N methods for TITO Processes

This method is based on A modified Z-N method for SISO control system. To derive this modified Z-N method, ageneral formulation is to start with a given point of the Nyquist curve of the process:

$$G_p(j\omega) = r_p e^{j(-\pi+\varphi_p)} \quad (1)$$

And to find a regulator GR

$$G_R(j\omega) = k\left(1 + j\left(\omega\tau_D - \frac{1}{j\tau_R\omega}\right)\right) \qquad (2)$$

To move this point to $B = r_s e^{j(-\pi+\varphi_s)}$ (3)

An amplitude margin (i.e. gain margin) design corresponding to $\varphi_s = 0$ and

$r_s = \frac{1}{A_m}$.

A phase margin design corresponds to $r_s = 1$ and $\varphi_s = \varphi_m$

From Eqs.(1)~Equ.(3), we have: $r_s e^{j(-\pi+\varphi_s)} = r_p r_R e^{j(-\pi+\varphi_p+\varphi_R)}$, so that

$r_R = \frac{r_s}{r_p}$ and $\varphi_R = \varphi_s - \varphi_p$

In other words,

$$G_R(j\omega) = k\left(1 + j\left(\omega\tau_D - \frac{1}{j\tau_R\omega}\right)\right) = r_R e^{j(\varphi_R)} = r_R\cos\varphi_R + jr_R\sin\varphi_R$$

Or,

$$k = r_R\cos\varphi_R = \frac{r_s}{r_p}\cos\left(\varphi_s - \varphi_p\right) \text{ and } \left(\omega\tau_D - \frac{1}{\tau_R\omega}\right) = \tan\left(\varphi_s - \varphi_p\right)$$

The gain is uniquely determined. Only one equation determines τ_R and τ_D.

Let $\tau_D = \alpha\tau_R$, where α is often chosen as $\alpha \approx 0.25$. Another method to specify α is as follows:

$$\alpha = \frac{0.413}{3.302\kappa + 1}, \text{ where } \kappa = \left|\frac{g(0)}{g(j\omega_c)}\right|$$

From $\left(\omega\tau_D - \frac{1}{\tau_R\omega}\right) = \tan^{-1}\left(\varphi_s - \varphi_p\right)$, τ_D can be solved to obtain:

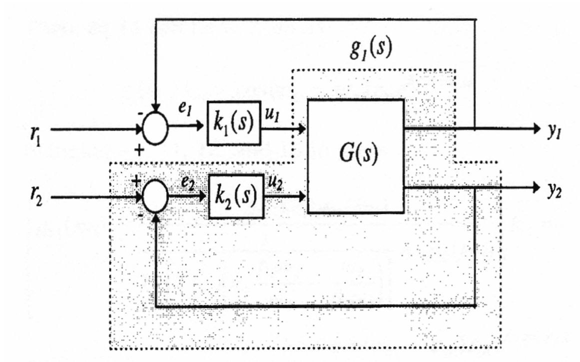
$$\tau_D = \frac{1}{2\omega}\left[-\tan(\varphi_s - \varphi_p) + \sqrt{4\alpha + \tan^2(\varphi_s - \varphi_p)}\right] \text{ and}$$

$$\tau_R = \frac{1}{\alpha}\tau_D$$

Consider a stable 2×2 process :

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

$$\begin{bmatrix} c_1(s) \\ c_2(s) \end{bmatrix} = \begin{bmatrix} c_1(s) & 0 \\ 0 & c_2(s) \end{bmatrix}$$



$$g_1 = g_{11} - \frac{c_2 g_{12} g_{21}}{1 + c_2 g_{22}} = g_{11} - \frac{g_{12} g_{21}}{c_2^{-1} + g_{22}}$$

$$g_2 = g_{22} - \frac{g_{12} g_{21}}{c_1^{-1} + g_{11}}$$

Let

$$A_i = r_{ai} e^{j(-\pi + \varphi_{ai})} = g_i(j\omega_i)$$

$$B_i = r_{bi} e^{j(-\pi + \varphi_{bi})} = g_i(j\omega_i) c_i(j\omega_i)$$

$$c_i(j\omega) = k\left(1 + j\left(\omega \tau_{Di} + \frac{1}{j\tau_{Ri}\omega}\right)\right) \quad ; \quad i = 1, 2$$

Take PI controller as example.

$$c_i(j\omega) = k_{ci}\left(1 - j\tan(\varphi_{bi} - \varphi_{ai})\right) \quad ; \quad i = 1, 2$$

And, $g_i(j\omega_i) k_{ci} = \cos(\varphi_{bi} - \varphi_{ai}) r_{bi} e^{j(-\pi + \varphi_{ai})}$

$$r_{ai}e^{j(-\pi+\varphi_{ia})}\cdot k_{ci}(1-j\tan(\varphi_{bi}-\varphi_{ai}))=r_{bi}e^{j(-\pi+\varphi_{ia})}$$

$$\Downarrow$$

$$\frac{r_{bi}}{r_{ai}}e^{j(\varphi_{ai}-\varphi_{bi})}=\frac{r_{bi}}{r_{ai}}\cos(\varphi_{ai}-\varphi_{bi})+j\frac{r_{bi}}{r_{ai}}\sin(\varphi_{ai}-\varphi_{bi})=k_{ci}(1-j\tan(\varphi_{bi}-\varphi_{ai}))$$

$$\Downarrow$$

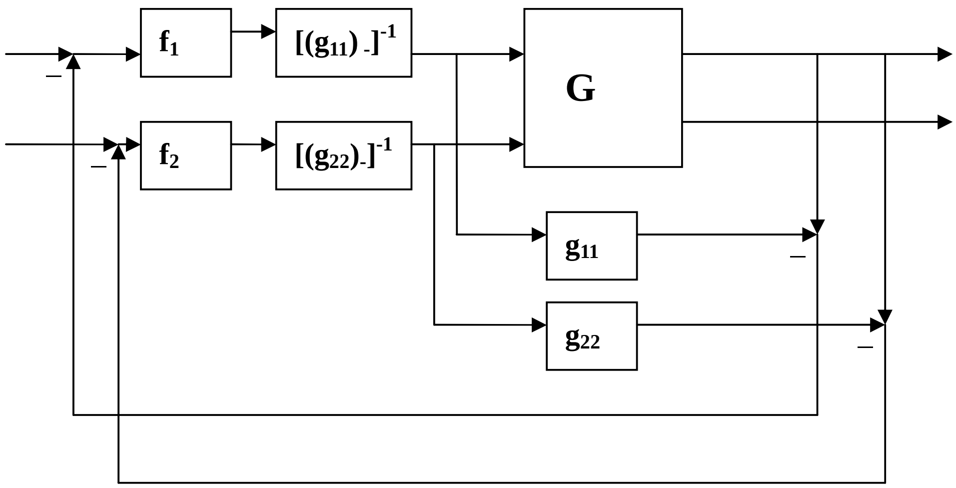
$$k_{ci}=\frac{r_{bi}}{r_{ai}}\cos(\varphi_{ai}-\varphi_{bi})$$

$$k_{ci}\cdot g_i(j\omega)=\frac{r_{bi}}{r_{ai}}\cos(\varphi_{ai}-\varphi_{bi})\cdot r_{ai}e^{j(-\pi+\varphi_{ia})}=r_{bi}\cos(\varphi_{ai}-\varphi_{bi})\cdot e^{j(-\pi+\varphi_{ia})}$$

By setting i equal one and two, one will obtain two equations with k_{c1} and k_{c2} as unknowns, and, thus, can be solved. But, there are very tedious procedures to find the controller gains (such as:such k_{c1} and k_{c2}) and frequency ω_{11} and ω_{22} that satisfy the phase criteria. (see the reference: I&EC Res. 1998, 37, 4725-4733, Q-G Wang, T-H Lee, and Y. Zhang)

C. Independent design method

---IMC Multi-loop PID Controller



$$G_{C,i} = \left(G_{i,i}\right)_{-}^{-1} f_i \ ; \ i = 1,\dots,n$$

The stability is guaranteed for any stable IMC filter that satisfies either of the following:

$$\left| f_i(i\omega) \right| < f_{R,i}^{*}(i\omega) = \frac{\left| g_{i,i}(i\omega) \right|}{\sum_{j,j\neq i} \left| g_{i,j}(i\omega) \right|} \ ; \ i = 1,2,\dots,n$$

$$\left| f_i(i\omega) \right| < f_{C,i}^{*}(i\omega) = \frac{\left| g_{i,i}(i\omega) \right|}{\sum_{j,j\neq i} \left| g_{j,i}(i\omega) \right|} \ ; \ i = 1,2,\dots,n$$

Imc Row interaction measure [Economou and Morari]

$$R_i(i\omega) = \frac{1}{1 + f_{R,i}^{*}(i\omega)} = \frac{\sum_{j,j\neq i} \left| g_{i,j}(i\omega) \right|}{\sum_{j} \left| g_{i,j}(i\omega) \right|} \ ; \ 0 \le \omega \le \infty$$

$$C_i(i\omega) = \frac{1}{1 + f_{C,i}^{*}(i\omega)} = \frac{\sum_{j,j\neq i} \left| g_{j,i}(i\omega) \right|}{\sum_{j} \left| g_{j,i}(i\omega) \right|} \ ; \ 0 \le \omega \le \infty$$

For significant interaction: $0.5 \le \ R_i, C_i \ \le 1 \quad \Rightarrow \quad f^{*} < 1$

For small interaction: $0.0 \le \; R_i, C_i \; \le 0.5 \quad \Rightarrow \quad f^* > 1$

D. Chien-Huang-Yang's multi-loop PID---with no proportional and derivative kicks

1. Controllers for SISO loop:

Controller: $u(s) = k_C \left\{ -y(s) + \frac{1}{\tau_R s}[r(s) - y(s)] - \tau_D s y(s) \right\}$

$$\frac{y}{r} = \frac{k_C/(\tau_R s)G_p}{1 + k_C/(\tau_R s)G_p}$$

a. Time constant dominant processes:

$$G_P = \frac{\mathrm{Re}^{-Ls}}{s}\;;\; R = \text{slope of the initial unit step response}$$

$$G_P = \frac{\mathrm{Re}^{-Ls}}{s} \approx \frac{R(1 - Ls)}{s}$$

$$\frac{y}{r} = \frac{1 - Ls}{\left(\frac{\tau_R}{Rk_C} - \tau_R L\right)s^2 + (\tau_R - L)s + 1} \approx \frac{1 - Ls}{\tau_C^2 s^2 + 1.414\tau_C s + 1}$$

$$\Rightarrow \quad k_C = \frac{(1.414\tau_C + L)}{R(\tau_C^2 + 1.414\tau_C L + L^2)} \;;\; \tau_R = 1.414\tau_C + L$$

b. Deadtime dominant processes:

$$G_P = \frac{k_P\,\mathrm{e}^{-Ls}}{\tau s + 1} \approx \frac{k_P(1 - Ls)}{\tau s + 1}$$

$$\frac{y}{r} = \frac{1 - Ls}{\left(\frac{\tau_R \tau}{k_C k_P} - \tau_R L\right)s^2 + \left(\frac{\tau_R}{k_C k_P} + \tau_R - L\right)s + 1}$$

$$\approx \quad \frac{1 - Ls}{\tau_C^2 s^2 + 1.414\tau_C s + 1}$$

$$\Rightarrow \quad k_C = \frac{1}{k_P}\frac{-\tau_C^2 + 1.414\tau_C \tau + L\tau}{\tau_C^2 + 1.414\tau_C \tau + L^2};$$

$$\Rightarrow \quad \tau_R = \frac{-\tau_C^2 + 1.414\tau_C \tau + L\tau}{\tau + L}$$

Derivation of the PID controller parameters is similar to the above PI derivations except that the deadtime approximation:

$$e^{-Ls} \approx \frac{1 - 0.5Ls}{1 + 0.5Ls}$$

Appendix: Derivation of PID Tuning Rules

The closed-loop transfer function between controlled variable (y) and setpoint (r) is

$$\frac{y}{r} = \frac{(K_c/\tau_i s)G_p}{1 + \dfrac{K_c(\tau_i\tau_d s^2 + \tau_i s + 1)}{\tau_i s}G_p} \quad \text{(A.1)}$$

For time constant dominant processes, the process model, G_p, can be approximated using Padé approximation as

$$G_p = \frac{Re^{-Ls}}{s} \approx \frac{R}{s}\frac{1 - (L/2)s}{1 + (L/2)s} \quad \text{(A.2)}$$

Substituting eq A.2 into A.1 and simplifying, we get

$$\frac{y}{r} = (1 - (L/2)s) \Bigg/ \left[\left(\frac{L\tau_i}{2K_cR} - \frac{L\tau_i\tau_d}{2}\right)s^3 + \left(\frac{\tau_i}{K_cR} + \tau_i\tau_d - \frac{L\tau_i}{2}\right)s^2 + \left(\tau_i - \frac{L}{2}\right)s + 1\right] \quad \text{(A.3)}$$

Let us assume our desired closed-loop servo response to be a underdamped system with damping coefficient of 0.707. This corresponds to a closed-loop system with about 5% overshoot. The desired closed-loop servo response is

$$\left(\frac{y}{r}\right)_{\text{desired}} = \frac{e^{-Ls}}{\tau_{cl}^2 s^2 + 1.414\tau_{cl}s + 1} \approx \frac{1}{(\tau_{cl}^2 s^2 + 1.414\tau_{cl}s + 1)}\frac{1 - (L/2)s}{1 + (L/2)s} \quad \text{(A.4)}$$

where τ_{cl} is an user-specified closed-loop effective time constant. Equating eqs A.3 and A.4 and doing some algebraic manipulation, we can solve for the PID tuning parameters as

$$K_c = \frac{1.414\tau_{cl} + L}{R\left(\tau_{cl}^2 + 0.707\tau_{cl}L + \frac{L^2}{4}\right)} \qquad (A.5)$$

$$\tau_i = 1.414\tau_{cl} + L \qquad (A.6)$$

$$\tau_d = \frac{(L^2/4) + 0.707\tau_{cl}L}{1.414\tau_{cl} + L} \qquad (A.7)$$

For processes with deadtime greater than $^1/_5$ of the process time constant, it is better for controller tuning purposes to model the processes as a first-order-plus deadtime model. With the same Padé approximation as

$$G_p = \frac{K_p e^{-Ls}}{\tau s + 1} \approx \frac{K_p}{\tau s + 1}\frac{1 - (L/2)s}{1 + (L/2)s} \qquad (A.8)$$

Substituting eq A.8 into A.1 and simplifying, we obtain

$$\frac{y}{r} = (1 - (L/2)s) \Bigg/ \Bigg[\left(\frac{L\tau_i\tau}{2K_cK_p} - \frac{L\tau_i\tau_d}{2}\right)s^3 + \left(\frac{\tau_i L}{2K_cK_p} + \frac{\tau_i\tau}{K_cK_p} + \tau_i\tau_d - \frac{L\tau_i}{2}\right)s^2 + \left(\frac{\tau_i}{K_cK_p} + \tau_i - L/2\right)s + 1\Bigg] \qquad (A.9)$$

Again, equating eqs A.9 and A.4 and doing some algebraic manipulation, we can solve for the PID tuning parameters as

$$K_c = \frac{\tau L + (L^2/4) + 1.414\tau_{cl}\tau - \tau_{cl}^2}{K_p(\tau_{cl}^2 + 0.707\tau_{cl}L + L^2/4)} \qquad (A.10)$$

$$\tau_i = \frac{\tau L + (L^2/4) + 1.414\tau_{cl}\tau - \tau_{cl}^2}{\tau + (L/2)} \qquad (A.11)$$

$$\tau_d = \frac{0.707\tau\tau_{cl}L + (L^2/4)\tau - \tau_{cl}^2(L/2)}{\tau L + (L^2/4) + 1.414\tau_{cl}\tau - \tau_{cl}^2} \qquad (A.12)$$

By selecting τ_{cl} as in Figure 2, the negative terms in eqs A.10–A.12 will not cause any problem in changing the signs of the PID tuning parameters. With the τ_{cl} selection as in Figure 2, combining with eqs A.5–A.7 and A.10–A.12, the final PID tuning rules in Table 2 can be obtained.

2. Controllers for multi-loop system

$$At\ \omega \to 0\,;\ \left(\frac{y}{u_1}\right)_{\text{loop 2 closed}} = g_{1,1}\left(1-\frac{k_{1,2}k_{2,1}}{k_{1,1}k_{2,2}}\right) = \frac{g_{1,1}}{RGA(\lambda)}$$

$$At\ \omega \to \infty\,;\ \left(\frac{y}{u_1}\right)_{\text{loop 2 closed}} = g_{1,1}$$

a. For RGA>1, multi-loop controller tuning based on the process model in the main loop should provide satisfactory closed loop results. This is because:

b. For RGA < 1,

$$k_{C,i} = \left(k_C\right)_{\text{based on main loop}} RGA(\lambda_{i,i})$$

$$\tau_{R,i} = \frac{\left(\tau_{R,i}\right)_{\text{based on main loop}}}{RGA(\lambda_{i,i})}$$

$$\tau_{D,i} = \left(\tau_{D,i}\right)_{\text{based on main loop}} RGA(\lambda_{i,i})$$

The closed-loop time constant is chosen according to the value of L/τ in three different ranges, that is: $L/\tau < 0.2$, $0.2 < L/\tau < 0.5$, and $L/\tau > 0.5$.

For details, see the original paper.

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IX. Robustness of Closed-loop System.

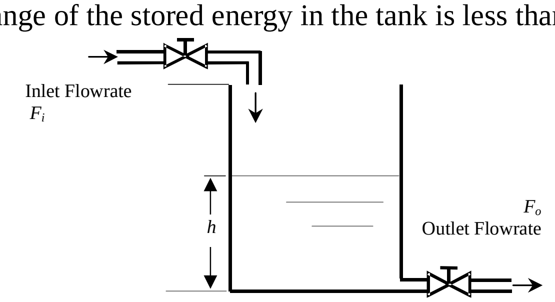
The final pairing and the controller tuning is checked for robustness by plotting DSO and DSI as functions of frequency, [Doyle and Stein]. The singular values below 0.3-0.2 indicate a lack of stability robustness.

$$DSO_{(i\omega)} = \underline{\sigma}[I + \left(GG_C\right)^{-1}]_{(i\omega)}$$

$$DSI_{(i\omega)} = \underline{\sigma}[I + \left(G_C G\right)^{-1}]_{(i\omega)}$$

E. Design Method based on Passivity

1. Hardware simplicity and relative effortlessness to achieve failure tolerant design, multi-loop control is the most widely used strategy in the industrial process control.
2. Current multi-loop control design approaches can be classified into three categories: detuning methods (Luyben, 1986), independent design methods (Skogestard and Morari, 1989), and sequential design methods (Mayne, Chiu and Arkun, 1992).
3. Loop interactions have to be taken into considerations, as they may have deteriorating effects on both control performance and closed-loop stability.
4. It is desirable if the multi-lop control system is decentralized unconditionally stable (i.e., any subset of the control loops can be independently to an arbitrary degree or even turned off without endangering close-loop stability.
5. Independent design is based on the basis of the paired transfer function while satisfying some stability constraints due to process interactions.
6. Perhaps the mostwidely used decentralized stability conditions are those μ-interaction measure.
7. Passivity Concept:
   The rate of change of the stored energy in the tank is less than the power supplied to it.



Potential energy stored in the tank: $S(h) = \frac{1}{2} Ah\rho \; gh = \frac{1}{2} A\rho gh^2$

Increment of potential energy per unit time: $w(t) = \rho F_i(t) gh(t)$

The rate of change of the storage function:

$$\frac{dS}{dt} = -C_v \rho gh\sqrt{h} + \rho g F_i h = -C_v \rho gh\sqrt{h} + w < w \qquad \forall h > 0$$

The rate of change of the stored energy in the tank is less than the power supplied to it. Therefore this process is said to be strictly passive.

Passive(Willems 1972)*:* if a non-negative ***storage*** function $S(x)$ can be found s.t.:

$S(0)=0$ and $S(x)-S(x^0) \le \int_{t0}^{t} y^T(\tau)u(\tau)d\tau$ for all $t>t0\ge 0$, $x0$, $x\in X$, $u\in U$.

Strictly passive: if $S(x)-S(x^0) < \int_{t0}^{t} y^T(\tau)u(\tau)d\tau$

Where, y is the output of a system, u is the input to the system.

- KYP Lemma
  - Nonlinear control affine systems (Hill & Moylan 1976)

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

where $x \in X \subset R^n$, $u \in U \subset R^m$, $y \in Y \subset R^m$

The process is passive if

$$L_f S(x) = \frac{\partial S^T(x)}{\partial x} f(x) \le 0,$$
$$L_g S(x) = \frac{\partial S^T(x)}{\partial x} g(x) = h^T(x)$$

- KYP Lemma

A linear system (Willems 1972) $G(s):=(A,B,C,D)$ is passive if there exists a positive definite matrix P such that:

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D - D^T \end{bmatrix} \le 0$$

The system is strictly passive if

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D - D^T \end{bmatrix} < 0$$

Definition:

An LTI system S: G(s) is *passive* if :

(1) G(s) is analytic in Re(s)>0;

(2) G(jw)+G*(jw)≥0 for all that jw is not a pole of G(s);

(3) If there are poles of G(s) on the imaginary axis, they are non-repeated and the residue matrices at the poles are Hermitian and positive semi-definite.

$G(s)$ is *strictly passive* if:

(1) G(s) is analytic in Re(s) ≥ 0;

(2) $G(jw)+G^*(jw)>0 \quad \forall \omega \in (-\infty, \infty)$.

Theorem 1: For a given stable non-passive process with a transfer function matrix **G**(s), there exists a diagonal, stable, and passive transfer function matrix **W**(s)=w(s)**I** such that **H**(s)=**G**(s)+**W**(s) is passive.

[Proof]:

$$\lambda_{\min}(H(j\omega)+H^*(j\omega)) = \lambda_{\min}(G(j\omega)+G^*(j\omega)+(W(j\omega)+W^*(j\omega))$$

Since both (G+G*) and (W+W*) are Hermitian, from the Weyl inequality, we have:

$$\begin{aligned}\lambda_{\min}(H(j\omega)+H^*(j\omega)) &\geq \lambda_{\min}(G(j\omega)+G^*(j\omega))+\lambda_{\min}(W(j\omega)+W^*(j\omega)) \\ &= \lambda_{\min}(G(j\omega)+G^*(j\omega))+2\,\mathrm{Re}(W(j\omega))\end{aligned}$$

Thus, if:

$$\mathrm{Re}(W(j\omega)) \geq \frac{1}{2}\lambda_{\min}(G(j\omega)+G^*(j\omega))$$

H(s) can be render passive. On the other hand, if

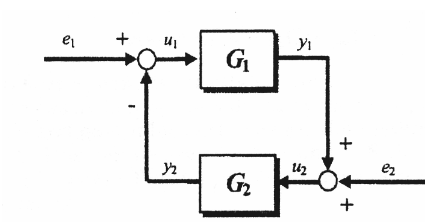
$$\mathrm{Re}(W(j\omega)) > \frac{1}{2}\lambda_{\min}(G(j\omega)+G^*(j\omega))$$

H(s) will be strictly passive.

Properties of Passive Systems:

- A passive system is minimum phase. The phase of a linear process is within [-90º, 90º]
- Passive systems are Lyapunov stable
- A passive system is of relative degree < 2
- Passive systems can have infinite gain (e.g., 1/s)

Passivity Theorem :



If *G1* is strictly passive and *G2* is passive, then the closed-loop system is *L*2 stable.

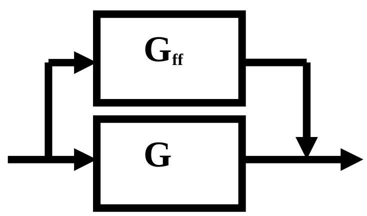
- A strictly passive process can be stabilized by any passive controller (including multi-loop PID controllers) even if it is highly nonlinear and/or highly coupled

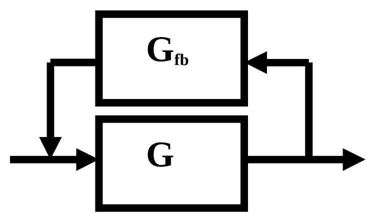
⇨ Control design based on passivity

- Excess or shortage of passivity of a process can be used to analyse whether this process can be easily controlled

  ⇨ Passivity based controllability study

A non-passive process can be made passive using feedforward and/or feedback passification:



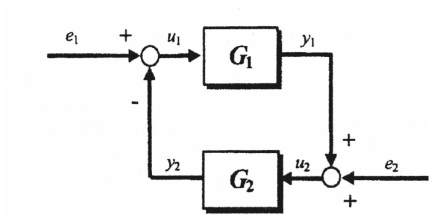


The excess or shortage of passivity can be quantified using:

- Input Feedforward Passivity (IFP) (Sepulchre et al 1997) - If a system G with a *negative* feedforward of νI is passive, then G has excessive input feedforward passivity, i.e., G is IFP(ν).

- Output Feedback Passivity (OFP) (Sepulchre et al 1997) - If a system G with a *positive* feedback of ρI is passive, then G has excessive output feedback passivity, i.e., G is OFP (ρ).

Agin, use the following figure:



If *G*1 is IFP(ν) and *G*2 is OFP(ρ), then the closed-loop system is stable if ρ+ν>0. In other words, a processs shortage of passivity can be compensated by another process's excess of passivity.

- **Passivity Index**

The excessive IFP of a system G(s) can be quantified by a frequency dependent passivity index

$$\nu_F[G(s),\omega] \stackrel{\Delta}{=} \lambda_{\min}\left(\frac{1}{2}[G(j\omega)+G^*(j\omega)]\right)$$

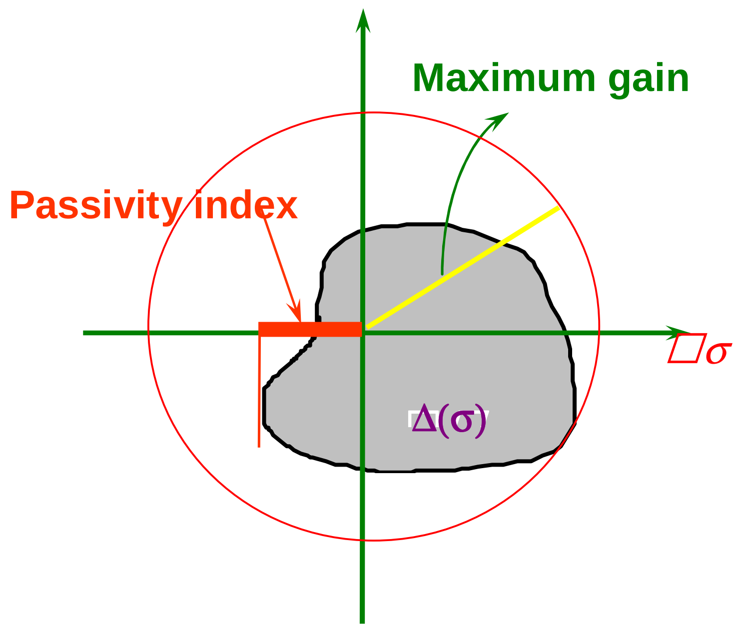
Assume the true process is $G_T(s) = G(s) + \Delta(s)$

The passivity index of the true process can be estimated as

$$\begin{aligned}\nu(G_T(j\omega)) &= -\lambda_{\min}\left\{\frac{1}{2}\left[\Delta(j\omega)+\Delta^*(j\omega)\right]+\frac{1}{2}\left[G(j\omega)+G^*(j\omega)\right]\right\} \\ &\le -\lambda_{\min}\left\{\frac{1}{2}\left[\Delta(j\omega)+\Delta^*(j\omega)\right]\right\} - -\lambda_{\min}\left\{\frac{1}{2}\left[G(j\omega)+G^*(j\omega)\right]\right\} \\ &= \nu(G(j\omega)) + \nu(\Delta(j\omega))\end{aligned}$$

Properties of the Passivity Index

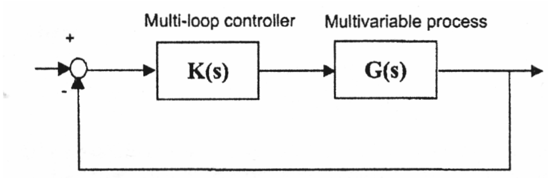
1. Comprises gain & phase information of the uncertainty



2. *Always* no greater than the maximum singular value.

$$|\nu_F[\Delta(s),\omega]| \le \sigma_{\max}[\Delta(j\omega)] \text{ for any } \omega \in R$$

Passivity Theorem 2: If the multivariable process is strictly passive, then the closed-loop system is stable if the multi-loop controller is passive.



Theorem 1: A closed-loop system comprising a stable subsystem G(s) and a decentralized controller K(s)=diag(k_i(s)), w(s) is a stable and minmum phase, and

$$\nu(W(j\omega)) < -\nu(G^{+}(j\omega))$$

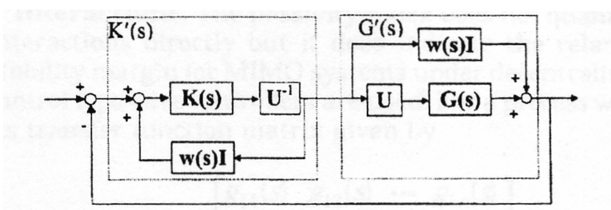
Then the closed-loop system will be decentralized unconditional stable, if

K(s)=diag{ $k_i^{'}(s)$ } is passive, where,

$$k_i^{'}(s) = k_i^{+}[1 - w(s)k_i^{+}(s)]^{-1} \quad \text{and} \quad k_i^{+} = U_{ii}k_i$$

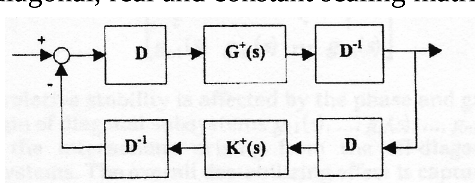
$$K^{'}(s) = U^{-1}K(s)\left[I - w(s)U^{-1}K(s)\right]^{-1}$$

Notice that the above figure is equivalent to the one in the following:



Rescaling of the system

Let D(s) be a diagonal, real and constant scaling matrix.



The scaling matrix D(s) is to make

$$\nu(D^{-1}G^{+}D(j\omega)) < \nu(G^{+}(j\omega))$$

and

$$D^{-1}G^{+}(0)D + D^{-1}\left[G^{+}(0)\right]^{+} D > 0$$

Design procedures:

1. Find matrix U and calculate $G^{+}(s)$.
2. Check the pairing. Examine the proposed pairing using DIC condition:

   $$G^{+}(0)M + M\left[G^{+}(0)\right]^{T} > 0$$

3. Use matrix M obtained in the step 2 to derive D, $D = M^{1/2}$
4. Calculate $\nu(D^{-1}G^{+}(j\omega)D)$ for different frequency points. These frequency

   points form a set $\Omega \in \left[0, \omega_E\right]$ where ω_E is the frequency which is high enough

   sych tant $\nu(D^{-1}G^{+}(j\omega)D) \rightarrow 0$ for $\omega > \omega_E$.
5. For each loop of the controller, solve problem:

   $$\min_{k_{ci}, \tau_{Ri}} (-\gamma_i)$$

   such that

   $$\left| \frac{1}{1 + G_{ii}^{+}(j\omega)k_{c,i}^{+}\left[1 + \dfrac{1}{j\tau_{R,i}\omega}\right]} \frac{\gamma_i}{j\omega} \right| < 1$$

   and

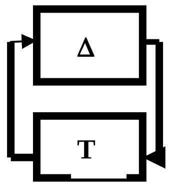
   $$\tau_{R,i}^2 \geq \frac{k_{c,i}^{+}\nu_s(\omega)}{\left[1 - k_{c,i}^{+}\nu_s(\omega)\right]\omega^2}, \quad \forall \omega \in R, \quad i = 1, \cdots, n$$

6. Obtain the final controller settings: $k_{c,i} = U_{ii}k_{c,i}^{+}$

This method is limited to open-loop stable processes.

$$\nu_F(\Delta(s),\omega) \geq -\nu_F(W(s),\omega), \;\; \forall \omega \in \mathbf{R}$$

Robust Stability Condition



If the uncertainty is passive, then the controller is only required to render system T strictly passive to achieve robust stability even if Δ is very large.

If the uncertainty's passivity index is bounded by

$$\nu_F(\Delta(s),\omega) \geq -\nu_F(W(s),\omega), \quad \forall \omega \in \mathbf{R}$$

where $W(s)$ is minimum phase, the closed-loop system will be robust stable if system

$$T(s)[I - W(s)T(s)]^{-1}$$

is strictly passive.

The basic idea:

1. Characterise the uncertainty in terms of passivity using IFP or OFP.
2. Derive the robust stability condition for systems with uncertainties bounded by their passivity indices.
3. Develop a systematic procedure to design the robust controller which satisfies the above stability condition.

Passivity Based Robust Control Design

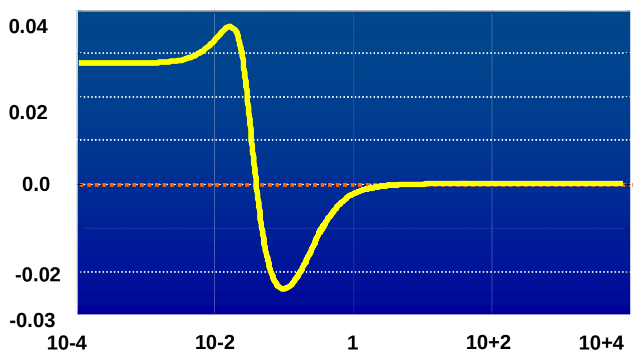
- Blended approach
  - Design a controller that satisfies the small gain condition at high frequencies and satisfies the passivity condition at low frequencies (Bao, Lee et al 1998)
    - Based on the bilinear transformation

- Multi-objective control design
  - Design a controller that satisfies the passivity condition for robust stability and achieves H∞ control performance (Bao, Lee et al 2000, 2003)
    - Based on KYP lemma and Semi-Definite Programming

Example:

$$G(s) = \begin{bmatrix} \dfrac{0.126e^{-6s}}{60s+1} & \dfrac{-0.101e^{-12s}}{(48s+1)(45s+1)} \\ \dfrac{0.094e^{-8s}}{38s+1} & \dfrac{-0.12e^{-8s}}{35s+1} \end{bmatrix}$$

Passivity index



F. Design by Sequential Loop Closing

Advantages of sequential design:

1. Each step in the design procedure involves designing only one SISO controller.
2. Limited degree of failure tolerance is guaranteed: If stability has been achieved

after the design of each loop, the system will remain stable if loop fail or are taken out of service in the reverse order of they were designed.

3. During startup, the system will be stable if the loops are brought into service in the same order as they have been designed.
4.

Problems with sequential design:

1. The final controller design, and thus the control quality achieved, may depend on the order in which the controllers in the individual loops are designed.
2. Only one output is usually considered at a time, and the closing of subsequent loops may alter the response of previously designed loops, and thus make iteration necessary.
3. The transfer function between input u_k and output y_k may contain RHP zeros that do not corresponding to the RHP zeros of G(s).

Notations:

1. G(s): the $n \times n$ matrix of the plant, $G(s) = \{g_{ij}(s); \ i, j = 1, \cdots, n\}$
2. $C(s) = diag\{c_i(s); \ i = 1, \cdots, n\}$
3. $S = (I + GC)^{-1}$; $\qquad H = I - S = GC(I + GC)^{-1}$
4. $\tilde{G} = diag\{g_{ii}(s); \quad i = 1, \cdots, n\}$
5. $\tilde{S} = diag\{s_i(s); \quad i = 1, \cdots, n\} = diag\{\frac{1}{1 + g_{ii}c_i}; \quad i = 1, \cdots, n\}$
6. $\tilde{H} = diag\{h_i(s); \quad i = 1, \cdots, n\} = diag\{\frac{g_{ii}c_i}{1 + g_{ii}c_i}; \quad i = 1, \cdots, n\}$
7. $\Gamma = \tilde{G}G^{-1} = \{\gamma_{ij}; \ i, j = 1, \cdots, n\}$
8. $CLDG = \tilde{G}G^{-1}G_d$
9. $E = (G - \tilde{G})\tilde{G}^{-1}$
10. $G = \begin{bmatrix} G_k & \vdots \\ \cdots & \ddots \end{bmatrix}$; $\mathrm{C} = \begin{bmatrix} C_k & \vdots \\ \cdots & \ddots \end{bmatrix}$;
11. $S_k = \left(I + G_k C_k\right)^{-1}$; $H_k = G_k C_k \left(I + G_k C_k\right)^{-1}$
12. $\hat{H}_k = \begin{bmatrix} H_k & 0 \\ 0 & \tilde{h}_i \end{bmatrix}$; $\hat{S}_k = \begin{bmatrix} S_k & 0 \\ 0 & \tilde{s}_i \end{bmatrix}$; $\quad i = k+1, \ K+2, \ \cdots N$

$$S=(I+GC)^{-1}=[I+\tilde{G}C+(G-\tilde{G})C]^{-1}$$
$$=\left\{\left[I+(G-\tilde{G})C\left(I+\tilde{G}C\right)^{-1}\right]\left(I+\tilde{G}C\right)\right\}^{-1}$$
$$=\left\{\left[I+(G-\tilde{G})\tilde{G}^{-1}\tilde{G}C\left(I+\tilde{G}C\right)^{-1}\right]\left(I+\tilde{G}C\right)\right\}^{-1}$$
$$=\left(I+\tilde{G}C\right)^{-1}\left(I+E\tilde{H}\right)^{-1}=\tilde{S}\left(I+E\tilde{H}\right)^{-1}$$

Design procedures:

In each of the following step, $S=\widehat{S}_k(I+E_k\widehat{H}_k)^{-1}; E_k=(G-\hat{G}_k)\left(\widehat{G}_k\right)^{-1}$

Determine c_i such that $\left\|W_pSW_D\right\|_{\infty}$ is minimized.

Step 0. Initialization. Determine the order of loop closing by estimating the required bandwidth in each loop. Also estimate the individual loop designs in terms of $\tilde{H}$.

Step 1. Design of controller c_1 by considering output 1 only. In this case, we have

$\hat{G}_k=\tilde{G}_k$ and $\hat{H}_k=\tilde{H}$

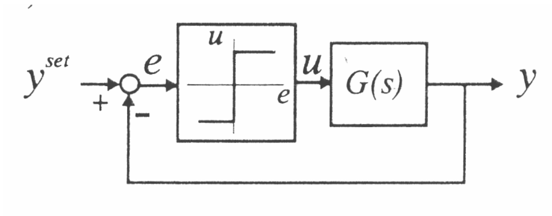
Step k. Design of controller ck by consider outputs 1 to k. Here,

$\hat{G}_k=diag\{\tilde{G}_k, g_{ii}\};\quad i=k+1,k+2,\cdots,n$ and

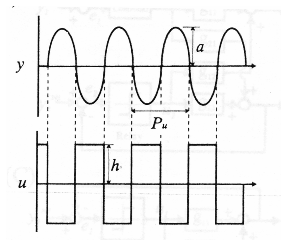
$\hat{H}_k=daig\{H_{k-1},\tilde{h}_i\}\ ;\quad i=k,k+1,\cdots,n$

Sequential Design Using Relay feedback Tests of Shen and Yu

The relay feedback system for SISO auto-tuning is as shown in the follwing figure:



When constant cycles appear after the system has been activated, the ultimate gain and ultimate frequency of the open-loop system can be approximated by measuring the magnitude and period (see the following figure) and by the following equations:



$$K_u = \frac{4h}{\pi a}; \quad \omega_u = \frac{2\pi}{P_u}$$

The Z-N tuning method can be used to determine the controller parameters:

PI Controller: $K_c = 0.45K_u, \quad \tau_R = P_u / 1.2,$

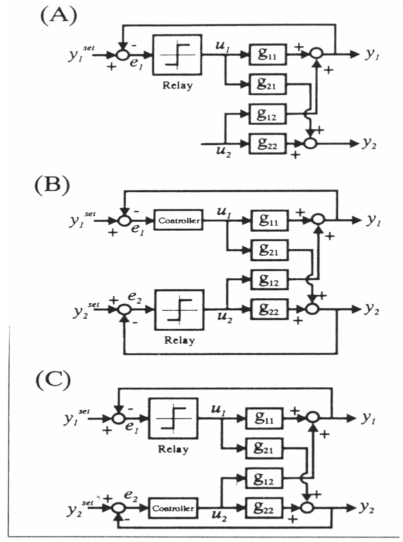
PID Controller: $K_c = 0.60K_u, \quad \tau_R = P_u / 1.2, \quad \tau_D = 1.25P_u$

Or, use the Tyreus-Luyben's formula to give more conservative response:

PI Controller: $K_c = K_u / 3.2, \quad \tau_R = 2.2P_u,$

PID Controller: $K_c = K_u / 2.2, \quad \tau_R = 2.2P_u, \quad \tau_D = P_u / 6.3$

To avoid the difficult mathematics envolved in the formulation of sequential design, Shen and Yu suggested to use the relay-feedback test as shown in the following figure:



The controller for a 2×2 system is suggested:

PI Controller: $K_c = K_{c,ZN}/3, \quad \tau_R = 2\ P_u$

Analysis:

The sequential design is derived by considering the multi-loop control system as coupled SISO loops. For a 2×2 system as example, the equivalent SISO loops are:

$$g_1(s) = g_{1,1}(s)\ \{1-(1-\frac{1}{\lambda(s)})\ h_2(s)\}$$

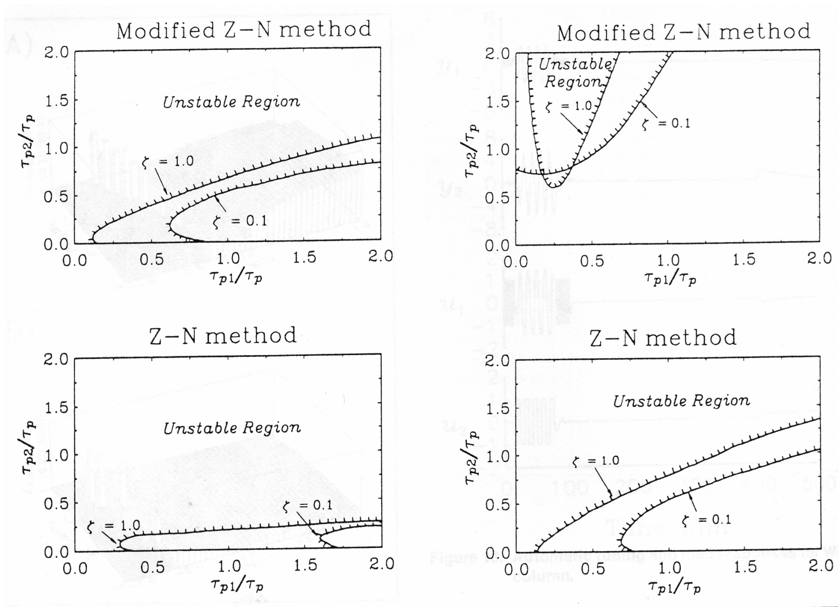
$$g_2(s) = g_{2,2}(s)\ \{1-(1-\frac{1}{\lambda(s)})\ h_1(s)\}$$

Where, $h_i(s) = \frac{g_{C,i}g_{i,i}}{1+g_{C,i}g_{i,i}}; \quad i=1,2$

Notice that, if there is damping in g_1 or g_2, this damping should come from either h_1 or h_2. According to tis study, a closed system having an FOPDT process and a modified ZN tuned PI controller will result in a closed-loop system (i.e. h_1 and h_2) having damping factor greater than 0.6. It is thus postulate that the open-loop transfer functions $g_1(s)$ and $g_2(s)$ can be approximated by:

$$G(s) = \frac{k_p}{\tau^2 s^2 + 2\tau\zeta s + 1}\cdot\frac{\tau_{p2}s+1}{\tau_{p1}s+1}\cdot e^{-\theta s}$$

Then, the stability region of the equivalent SISO loops are explored with the parameters: $\tau_{p1}, \tau_p = 0 \sim 10,\ k_p = 1,\ \tau = 5,\ \zeta = 0.1 \sim 1,\ \theta/\tau = 0.02 \sim 0.2$. The results are given in the following figure. It can be seen that the modified ZN tuning formula proposed greatly improve the stability.



On the other hand, the convergence of the sequential design for the multi-loop controller is formulated as the problem of finding the roots of simultaneous algebraic equation using sequential iterations.
The simultaneous equations are obtained from the conditions of phase crossover for the two loops, that is:

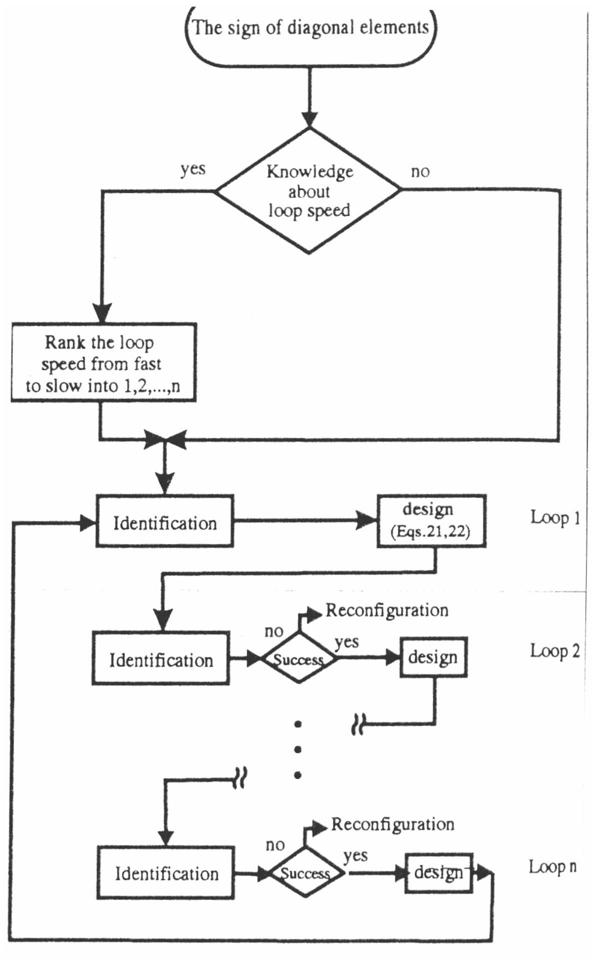
$$F_1(j_{\omega u,1}, j\omega_{u,2}) = \tan^{-1}\frac{\mathrm{Im}\left[g_1(j_{\omega u,1}, j\omega_{u,2})\right]}{\mathrm{Re}\left[g_1(j\omega_{u,1}, j\omega_{u,2})\right]} = -\pi$$

$$F_2(j_{\omega u,1}, j\omega_{u,2}) = \tan^{-1}\frac{\mathrm{Im}\left[g_2(j_{\omega u,1}, j\omega_{u,2})\right]}{\mathrm{Re}\left[g_2(j\omega_{u,1}, j\omega_{u,2})\right]} = -\pi$$

The convergence of the iteration is guaranteed by a sufficient condition of the following:

$$\frac{\left(\dfrac{\partial F_1}{\partial \omega_{u,2}}\right)_{\omega_{u,1}}\left(\dfrac{\partial F_2}{\partial \omega_{u,1}}\right)_{\omega_{u,2}}}{\left(\dfrac{\partial F_1}{\partial \omega_{u,1}}\right)_{\omega_{u,2}}\left(\dfrac{\partial F_2}{\partial \omega_{u,2}}\right)_{\omega_{u,1}}} < 1$$

The procedures of this proposed sequential design are summarized with the flow chart as shown.



Design of Multi-loop control systems

Consider a single loop system as shown in Fig.1.

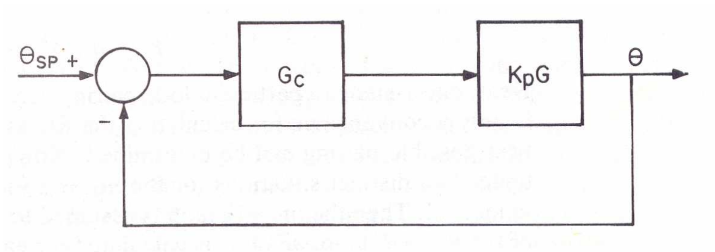


Fig.1

Suppose controller is fixed, substantial changes in K_p invariably lead to deteriorate the control system response (see Figure 2).

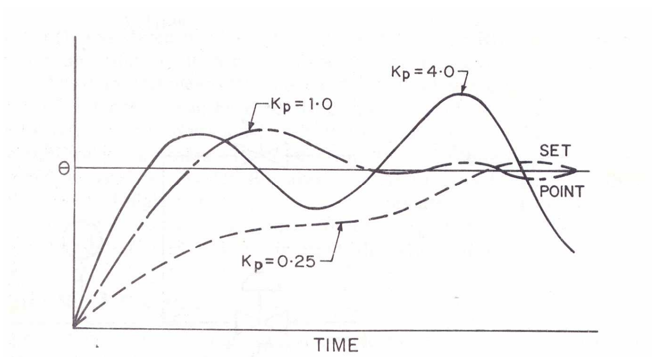


Fig.2

Now consider a 2×2 control system in Fig. 3

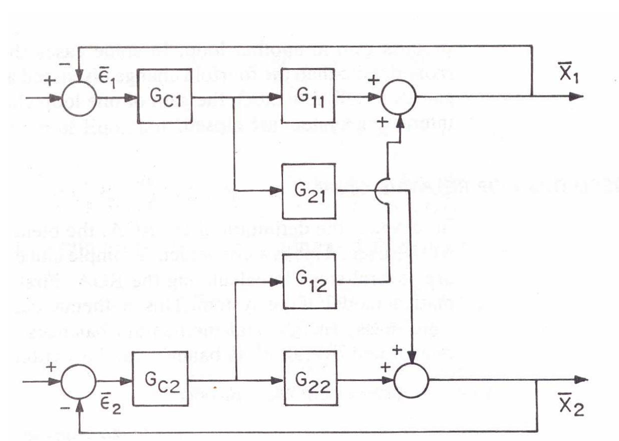


Fig.3

Let us consider open loop 1 (Fig.4)

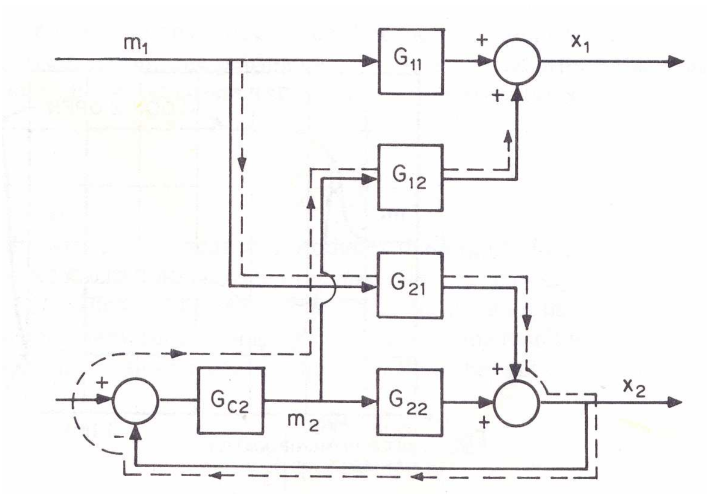


Fig.4

One may notice that there are two transmission paths from m_1 to x_1.let us define a relative gain from m_1 to x_1.as:

$$\lambda = \frac{Gain\ m_1 - x_1, loop2\ open}{Gain\ m_1 - x_1, loop2\ close}$$

When $\lambda = 0.5$, the responses of x_1 to a unit step input at m_1 is shown in Fig. 5

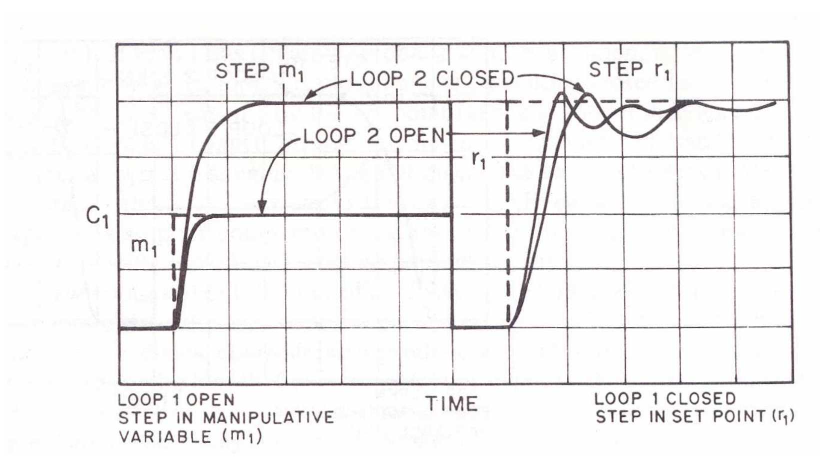
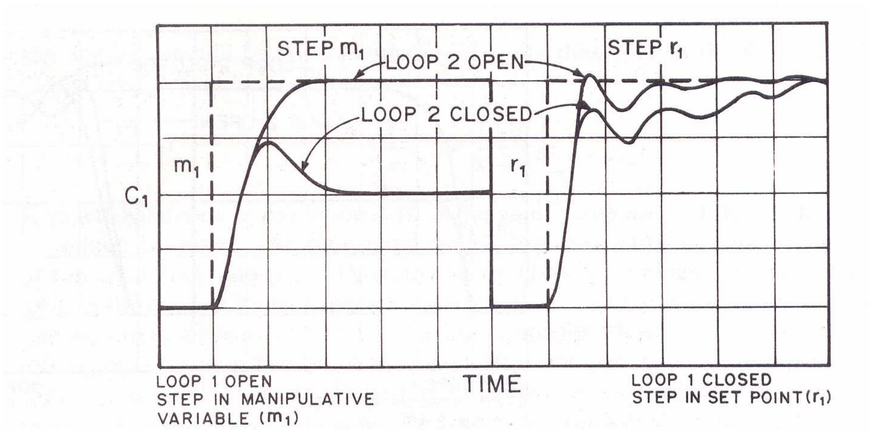


Fig.5

In this case, when loop 2 is closed, the open loop gain of $m_1 - x_1$ becomes doubled. The increase in the loop gain results in more oscillation in the closed loop response as shown.

On the other hand, when $\lambda = 2$, the open loop and closed loop responses are also given in Fig. 6.



In this latter case, the open-loop gain decrease when loop 2 is switched from open to close. As a result, the close of loop 1 leads the system to a more sluggish response to the r_1 input.

The increase or decrease of the loop gain is a result of closing loop2, and , hence, is considered loop interaction. From the above example, λ is a measure of such interaction and is named as relative gain of loop 1. You may also find the other relative gain for loop2. But, in this case, the two relative gains will be equal.

Algebraic Properties of the RGA

1. $\sum_j g_{ij}\bar{g}_{ji} = \sum_j \lambda_{ij} = 1, \quad \forall i$
2. $P_1\Lambda\{G\} P_2 = \Lambda\{P_1 G P_2\}$, P_1 and P_2 are two permutation matrices.
3. $\Lambda\{G\} = \Lambda\{S_1 G S_2\}$, S_1 and S_2 are two diagonal matrices.
4. If transfer matrix, G, is diagonal or triangular, then: $\Lambda\{G\}=I$.

   [Proof]:

   Let, $G = \begin{bmatrix} x & 0 & 0 \ldots 0 \\ x & x & 0 \ldots 0 \\ \ldots. & & \\ x & x & x \ldots x \end{bmatrix}$

   Then,

$$G^{-1} = \overline{G} = \begin{bmatrix} x & 0 & 0...0 \\ x & x & 0...0 \\ .... & & \\ x & x & x...x \end{bmatrix}$$

Thus, $g_{ij}\overline{g}_{ji} = 0 = \lambda_{ij}, \quad \forall j \neq i$

and, $g_{ii}\overline{g}_{ii} = 1 = \lambda_{ii}, \quad \forall j = i$

So, $\Lambda = I$

5. $$\frac{d\overline{g}_{ji}}{\overline{g}_{ji}} = -\lambda_{ij}\frac{dg_{ij}}{g_{ij}}$$

$$\overline{g}_{ij} = \frac{adj[A]}{\det[G]} = \frac{(-1)^{i+j}\det[G^{ij}]}{\det[G]}$$

$$\frac{d\overline{g}_{ji}}{dg_{ij}} = -\frac{\frac{d\det[G]}{dg_{ij}}(-1)^{i+j}\det[G^{ij}]}{\det[G(g_{ij})]^2} = \frac{\det[G^{ij}]^2}{\det[G(g_{ij})]^2} = -\overline{g}_{ji}^2$$

$$\frac{d\overline{g}_{ji}}{\overline{g}_{ji}} = -\overline{g}_{ji}dg_{ij} = -\overline{g}_{ji}g_{ij}\frac{dg_{ij}}{g_{ij}} = -\lambda_{ij}\frac{dg_{ij}}{g_{ij}}$$

5. $$\frac{d\lambda_{ij}}{\lambda_{ij}} = (1-\lambda_{ij})\frac{dg_{ij}}{g_{ij}}, \quad \text{and} \quad \frac{d\lambda_{ij}}{\lambda_{ij}} = \frac{\lambda_{ij}-1}{\lambda_{ij}}\frac{d\overline{g}_{ij}}{\overline{g}_{ij}}$$

[Proof];

$$\lambda_{ij} = g_{ij}\overline{g}_{ji} \Rightarrow d\lambda_{ij} = dg_{ij}\overline{g}_{ji} + g_{ij}d\overline{g}_{ji}$$

$$\Rightarrow \frac{d\lambda_{ij}}{\lambda_{ij}} = \frac{dg_{ij}\overline{g}_{ji} + g_{ij}d\overline{g}_{ji}}{g_{ij}\overline{g}_{ji}} = \frac{dg_{ij}}{g_{ij}} + (\frac{d\overline{g}_{ji}}{\overline{g}_{ji}}) = (1-\lambda_{ij})\frac{dg_{ij}}{g_{ij}}$$

or,

$$\Rightarrow \frac{d\lambda_{ij}}{\lambda_{ij}} = \left(\frac{dg_{ij}}{g_{ij}}\right) + \frac{d\overline{g}_{ji}}{\overline{g}_{ji}} = (1-\frac{1}{\lambda_{ij}})\frac{d\overline{g}_{ij}}{\overline{g}_{ij}} = \left[\frac{\lambda_{ij}-1}{\lambda_{ij}}\right]\frac{d\overline{g}_{ij}}{\overline{g}_{ij}}$$

RGA-implications:

1. Pairing loops on λ_{ij} values that are positive and close to 1.

2. Reasonable Pairings: $0.5 < \lambda_{ij} < 4.0$

3. Pairing on negative λ_{ij} values results in at least one of the following;
   a. Closed loop system is unstable,
   b. Loop with negative λ_{ij} is unstable,
   c. Closed loop system becomes unstable if loop with negative is λ_{ij} turned off.

4. Plants with large RGA-elements are always ill-conditioned. (i.e., a plant with a large $\gamma(G)$ may have small RGA-elements)

5. Plants with large RGA-elements around the crossover frequency are fundamentally difficult to control because of sensitivity to input uncertainties.
   -----→decouplers or other inverse-based controllers should not be used for plants with large RGA-elements.

6. Large RGA-element implies sensitivity to element-by-element uncertainty.

7. If the sign of RGA-element changes from s=0 to s=∞, then there is a RHP-zero in G or in some subsystem of G.

8. The RGA-number can be used to measure diagonal dominance:
   RGA-number = $\| \Lambda(G)-I \|_{min}$.
   For decentralized control,, pairings with RGA-number at crossover frequency close to one is preferred.

9. For integrity of whole plant, we should avoid input-output pairing on negative RGA-element.

10. For stability, pairing on an RGA-number close to zero at crossover frequency is preferred.

The Relative Disturbance Gain (RDG)

Ref: **Galen Stanley, Maria Marino-Galarraga, and T. J. McAvoy**, Shortcut Operability Analysis. 1. The relative disturbance gain, I&EC, Process Des. Dev. 1985,24, 1181-1188

The use of RDG:

1. To decide if interaction resulting from a disturbance is favorable or unfavorable.
2. To decide whether or not decoupling should be used and what type of decoupling structure is best.

$$y_1 = k_{11}m_1 + k_{12}m_2 + k_{F1}d$$
$$y_2 = k_{21}m_1 + k_{22}m_2 + k_{F2}d$$

$$\left(\frac{\partial m_1}{\partial d}\right)_{y_1, m_2} = -\frac{k_{F1}}{k_{11}}$$

$\left(\frac{\partial m_1}{\partial d}\right)_{y_1, y_2}$ is derived when both y_1 and y_2 are held still:

$$y_1 = k_{11}m_1 + k_{12}m_2 + k_{F1}d = 0$$
$$y_2 = k_{21}m_1 + k_{22}m_2 + k_{F2}d = 0 \qquad (2)$$

so that:

$$m_2 = \frac{1}{k_{22}}\left[-k_{21}m_1 - k_{F2}d\right] \qquad (3)$$

Substitute Eq.(3) into E.(2), we have:

$$\left[k_{11} - \frac{k_{12}k_{21}}{k_{22}}\right]m_1 + \left[k_{F1} - \frac{k_{12}k_{F2}}{k_{22}}\right]d = 0$$

Thus,

$$\left(\frac{\partial m_1}{\partial d}\right)_{y_1, y_2} = \frac{-k_{F1} + \frac{k_{12}k_{F2}}{k_{22}}}{k_{11} - \frac{k_{12}k_{21}}{k_{22}}} = \frac{k_{12}k_{F2} - k_{F1}k_{22}}{k_{11}k_{22} - k_{12}k_{21}} \qquad (4)$$

So,

$$\beta_1 = \frac{\left(\frac{\partial m_1}{\partial d}\right)_{y_1,y_2}}{\left(\frac{\partial m_1}{\partial d}\right)_{y_1,m_2}} = -\frac{k_{11}}{k_{F1}} \times \frac{k_{12}k_{F2} - k_{F1}k_{22}}{k_{11}k_{22} - k_{12}k_{21}} = \frac{k_{11}k_{22}}{k_{F1}k_{22}} \times \frac{k_{12}k_{F2} - k_{F1}k_{22}}{k_{11}k_{22} - k_{12}k_{21}} \qquad (5)$$

$$= -\frac{k_{12}k_{F2} - k_{F1}k_{22}}{k_{F1}k_{22}} \times \frac{k_{11}k_{22}}{k_{11}k_{22} - k_{12}k_{21}} = \left[1 - \frac{k_{12}k_{F2}}{k_{F1}k_{22}}\right]\lambda$$

Similarly, we have:

$$\beta_1 = \frac{\left(\frac{\partial m_1}{\partial d}\right)_{y_1,y_2}}{\left(\frac{\partial m_1}{\partial d}\right)_{y_1,m_2}} = \left[1 - \frac{k_{21}k_{F1}}{k_{F2}k_{11}}\right]\lambda$$

$$\frac{k_{F2}k_{12}}{k_{F1}k_{22}} = 1 - \frac{\beta_1}{\lambda} = \frac{\lambda - \beta_1}{\lambda} \qquad \Rightarrow \qquad \frac{k_{F2}}{k_{F1}} = \frac{\lambda - \beta_1}{\lambda} \times \frac{k_{22}}{k_{12}}$$

Similarly, $\Rightarrow \quad \frac{k_{F1}}{k_{F2}} = \frac{\lambda - \beta_2}{\lambda} \times \frac{k_{11}}{k_{21}}$

So, $\frac{k_{F2}}{k_{F1}} \times \frac{k_{F1}}{k_{F2}} = 1 = \left(\frac{\lambda - \beta_2}{\lambda} \times \frac{k_{11}}{k_{21}}\right) \times \left(\frac{\lambda - \beta_1}{\lambda} \times \frac{k_{22}}{k_{12}}\right)$

$$\Rightarrow \quad \left(\frac{\lambda - \beta_2}{\lambda}\right)\left[\frac{\lambda - \beta_1}{\lambda}\right] = \frac{k_{11}}{k_{21}}\frac{k_{22}}{k_{12}} = 1 - \frac{1}{\lambda}$$

or,

$$(\beta_1 - \lambda)(\beta_2 - \lambda) = \lambda(\lambda - 1)$$

$$(\beta_2 - \lambda) = \frac{\lambda(\lambda - 1)}{(\beta_1 - \lambda)} \qquad \Rightarrow \qquad \beta_2 = \frac{\lambda(\lambda - 1)}{(\beta_1 - \lambda)} + \lambda = \frac{(\beta_1 - 1)\lambda}{(\beta_1 - \lambda)} = \frac{1 - \beta_1}{\lambda - \beta_1}\lambda$$

It can be shown that:

$$\frac{\text{Multi-loop } e_1 \text{ area}}{\text{SISO idealy decoupled } e_1 \text{ area}} \propto \beta_1 \quad \text{and}$$

$$\frac{\text{Multi-loop } e_2 \text{ area}}{\text{SISO idealy decoupled } e_2 \text{ area}} \propto \beta_2$$

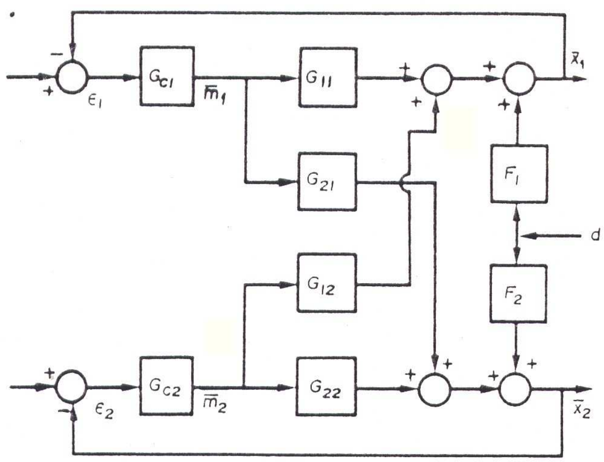


Figure 4. General 2 × 2 system.

$$\frac{e_1}{d} = \frac{F_2 G_{12} G_{c2} - F_1(1 + G_{22} G_{c2})}{(1 + G_{11} G_{c1})(1 + G_{22} G_{c2}) - G_{c2} G_{c1} G_{12} G_{21}} \quad (6)$$

If d is a unit step, then the area under e_1 curve is given as:

$$\int_0^\infty e_1 dt = \lim_{s \to 0} e_1(s) = \frac{k_{F2} k_{12} \frac{k_{c2}}{\tau_{R2}} - \frac{k_{F1} k_{22} k_{c2}}{\tau_{R2}}}{\frac{k_{c1} k_{11}}{\tau_{R1}} \frac{k_{c2} k_{22}}{\tau_{R2}} - \frac{k_{c2}}{\tau_{R2}} \frac{k_{c1}}{\tau_{R1}} k_{12} k_{21}} = \frac{k_{F2} k_{12} - k_{F1} k_{22}}{\frac{k_{c1} k_{11} k_{22}}{\tau_{R1}} \left(1 - \frac{k_{12} k_{21}}{k_{11} k_{22}}\right)}$$

$$= \lambda \frac{\tau_{R1}}{k_{c1} k_{11}} \left(k_{F2} \frac{k_{12}}{k_{22}} - k_{F1}\right)$$

On the other hand, when loop 2 is opened, the area under $e_1$becomes:

$$\int_0^\infty e_1^o dt = -\frac{\tau_{R1}^{'}}{k_{c1}^{'}} \frac{k_{F1}}{k_{11}}$$

Thus,

$$\frac{\int_0^\infty e_1 dt}{\int_0^\infty e_1^o dt} = -\frac{\tau_{R1}}{\tau_{R1}^{'}}\frac{k_{c1}^{'}}{k_{c1}}\times\left(\frac{k_{F2}k_{12}}{k_{F1}k_{22}} - 1\right)\lambda = \frac{\tau_{R1}}{\tau_{R1}^{'}}\frac{k_{c1}^{'}}{k_{c1}}\times\beta_1 = f_1\beta_1$$

Similarly, we have:

$$\frac{\int_0^\infty e_2 dt}{\int_0^\infty e_2^o dt} = \frac{\tau_{R2}}{\tau_{R2}^{'}}\frac{k_{c2}^{'}}{k_{c2}}\times\beta_2 = f_2\beta_2$$

Notice that the PI parameters in the interacting loops are used to be more conservative than those in single loops. In another words,

$$f_1 \geq 1; \quad f_2 \geq 1$$

The multi-loop control should be beneficial when the sum of absolute values of the

Remarks:

1. If λ is assumed not vary with frequency, and the process under study is FOPDT, $\lambda>1$, f_1 lies in the range $1< f_1 <2$, while $0.5<l<1$, f_1 lies in the range $1< f_1 <3$.
2. When $f_1 =1$, β is equal to the ratio of response areas.
3. If β is small and f_1 =is close to one, then the interacting control is favored for that particular disturbance.
4. If β is large, the interacting control is un favorable for that particular disturbance.

The Relative Gain for Non-square Multivariable Systems

(J.C. Chang and C.C. Yu, CES Vol.45, pp. 1309-1323 1990)

Consider a non-square MV system.

$y_{m\times 1}(s) = G_{m\times n}(s)u_{n\times 1}(s)$

Define Moore-Penrose pseudo-inverse of the matrix $G(s)$ as:

$G^{+}(s) = \left(G^T G\right)^{-1} G^T(s)$

Then, under close-loop control, the steady-state control input will be:

$u = G^{+}(0)y^d$ and $\left(\frac{\partial u_i}{\partial y_j}\right)_{CL} = g_{ij}^{+}(0)$.

Thus, the non-square relative gain is defined similarly to the square RGA, that is:

$$\tilde{\Lambda} = \left(\frac{\partial y_i}{\partial u_j}\right)_{OL} \left\{\left(\frac{\partial y_i}{\partial u_j}\right)_{CL}\right\}^{-1} = G(0) \otimes \left[G^{+}(0)\right]^{T}$$

Properties of the non-square RGA

1. Row sum of $\tilde{\Lambda}$:

$$RS = [rs(1), \quad rs(2), \quad \cdots, \quad rs(m)] = \left[\sum_{j=1}^{n} \tilde{\lambda}_{1j}\ , \quad \sum_{j=1}^{n} \tilde{\lambda}_{2j}\ , \quad \cdots \quad , \quad \sum_{j=1}^{n} \tilde{\lambda}_{mj}\ , \right]^{T};$$

Where, $rs(i) = \left[G(0)G^{+}(0)\right]_{ii}$

2. $$CS = [cs(1), \quad cs(2), \quad \cdots, \quad cs(n)] = \left[\sum_{j=1}^{n} \tilde{\lambda}_{j1}\ , \quad \sum_{j=1}^{n} \tilde{\lambda}_{j2}\ , \quad \cdots \quad , \quad \sum_{j=1}^{n} \tilde{\lambda}_{jn}\ , \right]^{T} = [1, \quad 1, \quad \cdots, \quad 1]^{T}$$

Where, $cs(i) = \left[G^{+}G(0)\right]_{ii}$; (Note: $G^{+}G = (G^{T}G)^{-1}G^{T}G = I$)

3. $0 \le rs(i) \le 1, \quad \forall i = 1, 2, \cdots, m$

4. $\sum_{i=1}^{m} rs(i) = \sum_{j=1}^{n} cs(i) = n$

Note: $\sum_{i=1}^{m} rs(i) = \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{\lambda}_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} \tilde{\lambda}_{ij} = \sum_{j=1}^{n} cs(j) = n$

5. Non-square RGA is invariant under input scaling, but is variant under output scaling:

$$(GS) \otimes \left[(GS)^{+}\right]^{T} = \left(G \otimes G^{+}\right)^{T} \qquad (SG) \otimes \left[(SG)^{+}\right]^{T} \neq \left(G \otimes G^{+}\right)^{T}$$

6. Let P_1 and P_2 are permutation matrices. Then, $\tilde{\Lambda}(P_1 G P_2) = P_1 \tilde{\Lambda}(G) P_2$

A. Multi-loop BLT-Tuning:

I. BLT-1 method:

a. Calculate the Ziegler-Nichol settings for each PI controller by using the diagonal element of G, i.e. $g_{i,i}$.
b. Assume a detuning factor “F”, and calculate controller settings for loops.

$$k_{c,i} = k_{ZN,i} / F; \quad \tau_{R,i} = \left(\tau_{R,i}\right)_{ZN} F$$

c. Define: $W_{(i\omega)} = -1 + \det\left[I + G_{(i\omega)} G_{c(i\omega)}\right]$

d. Calculate the closed-loop function $L_{c(i\omega)}$):

$$L_{c(i\omega)} = 20\log\left|\frac{W_{(i\omega)}}{1+W_{(i\omega)}}\right|$$

e. Calculate the detuning factor F until the peak in the Lc log modulus curve is equal to 2N, that is:

$$L_{cm} = \underset{\omega}{Max}\left\{20\log\left|\frac{W_{(i\omega)}}{1+W_{(i\omega)}}\right|\right\} = 2N$$

II. BLT-2

a. Find BLT-1 PI controllers.
b. Choose a second detuning factor F_D. F_D should be greater than one.
c. Compute $\tau_{D,j}$ as:

$$\tau_{D,j} = \frac{\left(\tau_{D,j}\right)_{ZN}}{F_D}$$

d. Calculate $W_{(i\omega)}$ and $L_c(i\omega)$.

e. Change F_D until $L_C^{\max}$ is minimized, maintaining $F_D \geq 1$. The trivial case may result where $L_C^{\max}$ is minimized for $F_D = \infty$, i.e., no derivative action.

f. Reduce F in the P and I modes, until $L_C^{\max} = 2N$.

III. BLT-3

The objective is to estimate the level of imbalance in detuning the BLT-1 controller and compensate for it.

Consider the PI controller:

$$u_j = u_j(0) + k_{C,j}\left(e_j + \frac{1}{\tau_{R,j}}\int_0^t e_j dt\right); \qquad u_j(0) = 0$$

At steady state,

$$\lim_{t\to\infty}\left[u_j(t)\right]=\frac{k_{C,j}}{\tau_{R,j}}\int_0^\infty e_j(t)dt$$

So,

$$\int_0^\infty e_j(t)dt=\frac{\tau_{R,j}u_j(\infty)}{k_{C,j}}$$

Notice that:

$$u(\infty)=G^{-1}(0)R-G^{-1}G_L(0)d(\infty)$$

For unit step set-point input:

$$u_j(\infty)=G^{-1}(0)[0,...,0,1,0,...,0]^T$$
$$=[\overline{g}_{i,j}(0);i,\ j=1,...,N][0,...,0,1,0,...,0]^T$$

For unit step load disturbance:

$$u_i(\infty)=i\,th\ \ row\,of\ \ G^{-1}(0)G_L(0)=\sum_{j=1}^{N}[\overline{g}_{i,j}(0)g_{L,j}(0)]$$

Then, ITE_j becomes:

$$ITE_j=\frac{u_j(\infty)\tau_{R,j}}{k_{C,j}}$$

Let,

$$S_j=\sum_{i=1}^{N}\left|\frac{ITE_j}{N}\right|+\left|ITE_{j-load}\right|$$

$$S_j=\left|\frac{\tau_{R,j}}{k_{C,j}}\right|\times\left\{\sum_{i=1}^{N}\left|\frac{\overline{g}_{j,i}(0)}{N}\right|+\left|\overline{g}_{j,i}(0)g_{L,i}(0)\right|\right\}$$

Let $S_{\max}=\underset{j}{Max}\,S_j$

$$F_j=F\sqrt{\frac{S_{\max}}{S_j}}$$

The PI controller parameters becpme:

$$k_{c,i}=k_{ZN,i}/F_j;\quad \tau_{R,i}=\left(\tau_{R,i}\right)_{ZN}F_j$$

IV. BLT-4

a. BLT-3 is used to get individual PI controllers as described above.
b. BLT-2 procedure is used with individual F_D factors for each loop:

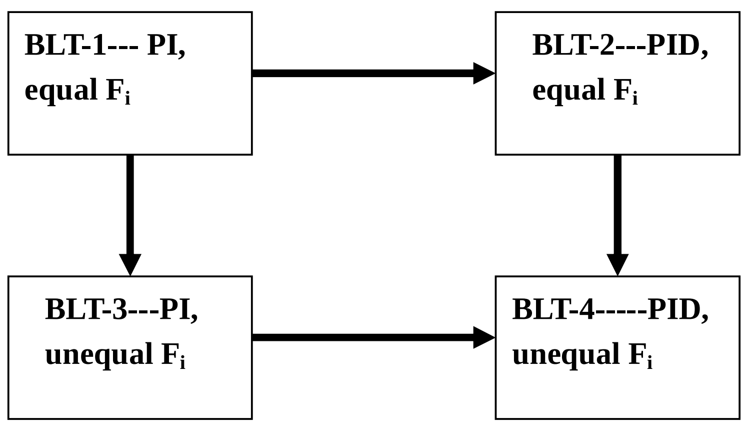
$$F_{D,j} = F_D \sqrt{\frac{S_{\max}}{S_j}}$$

V. Tyreus Load-Rejection Criterion (TLC)

The best variable pairing is the one that gives the smallest magnitudes for each element of X,(i.e. X_i) of the following:

$$X_{(i\omega)} = \left([I + GG_C]^{-1} G_L L\right)_{(i\omega)}$$

VI. Summary



B. Parallel-design method---Modified Z-N methods for TITO Processes

This method is based on A modified Z-N method for SISO control system. To derive this modified Z-N method, ageneral formulation is to start with a given point of the Nyquist curve of the process:

$$G_p(j\omega) = r_p e^{j(-\pi+\varphi_p)} \tag{1}$$

And to find a regulator GR

$$G_R(j\omega) = k\left(1 + j\left(\omega\tau_D - \frac{1}{j\tau_R\omega}\right)\right) \quad (2)$$

To move this point to $B = r_s e^{j(-\pi+\varphi_s)}$ (3)

An amplitude margin (i.e. gain margin) design corresponding to $\varphi_s = 0$ and

$$r_s = \frac{1}{A_m}.$$

A phase margin design corresponds to $r_s = 1$ and $\varphi_s = \varphi_m$

From Eqs.(1)~Equ.(3), we have: $r_s e^{j(-\pi+\varphi_s)} = r_p r_R e^{j(-\pi+\varphi_p+\varphi_R)}$, so that

$r_R = \frac{r_s}{r_p}$ and $\varphi_R = \varphi_s - \varphi_p$

In other words,

$$G_R(j\omega) = k\left(1 + j\left(\omega\tau_D - \frac{1}{j\tau_R\omega}\right)\right) = r_R e^{j(\varphi_R)} = r_R \cos\varphi_R + jr_R \sin\varphi_R$$

Or,

$k = r_R \cos\varphi_R = \frac{r_s}{r_p}\cos\left(\varphi_s - \varphi_p\right)$ and $\left(\omega\tau_D - \frac{1}{\tau_R\omega}\right) = \tan\left(\varphi_s - \varphi_p\right)$

The gain is uniquely determined. Only one equation determines τ_R and τ_D.

Let $\tau_D = \alpha\tau_R$, where α is often chosen as $\alpha \approx 0.25$. Another method to specify α is as follows:

$\alpha = \frac{0.413}{3.302\kappa + 1}$, where $\kappa = \left|\frac{g(0)}{g(j\omega_c)}\right|$

From $\left(\omega\tau_D - \frac{1}{\tau_R\omega}\right) = \tan^{-1}\left(\varphi_s - \varphi_p\right)$, τ_D can be solved to obtain:

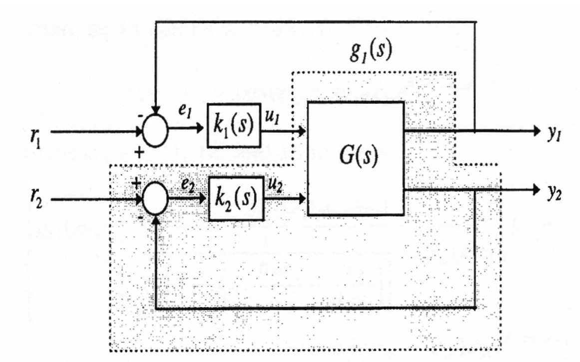
$\tau_D = \frac{1}{2\omega}\left[-\tan(\varphi_s - \varphi_p) + \sqrt{4\alpha + \tan^2(\varphi_s - \varphi_p)}\right]$ and

$$\tau_R = \frac{1}{\alpha}\tau_D$$

Consider a stable 2×2 process :

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

$$\begin{bmatrix} c_1(s) \\ c_2(s) \end{bmatrix} = \begin{bmatrix} c_1(s) & 0 \\ 0 & c_2(s) \end{bmatrix}$$



$$g_1 = g_{11} - \frac{c_2 g_{12} g_{21}}{1 + c_2 g_{22}} = g_{11} - \frac{g_{12} g_{21}}{c_2^{-1} + g_{22}}$$

$$g_2 = g_{22} - \frac{g_{12} g_{21}}{c_1^{-1} + g_{11}}$$

Let

$$A_i = r_{ai} e^{j(-\pi + \varphi_{ai})} = g_i(j\omega_i)$$

$$B_i = r_{bi} e^{j(-\pi + \varphi_{bi})} = g_i(j\omega_i) c_i(j\omega_i)$$

$$c_i(j\omega) = k\left(1 + j\left(\omega \tau_{Di} + \frac{1}{j \tau_{Ri} \omega}\right)\right) \quad ; \quad i = 1, 2$$

Take PI controller as example.

$$c_i(j\omega) = k_{ci}\left(1 - j \tan(\varphi_{bi} - \varphi_{ai})\right) \quad ; \quad i = 1, 2$$

And, $g_i(j\omega_i) k_{ci} = \cos(\varphi_{bi} - \varphi_{ai}) r_{bi} e^{j(-\pi + \varphi_{ai})}$

$$r_{ai}e^{j(-\pi+\varphi_{ia})}\cdot k_{ci}(1-j\tan(\varphi_{bi}-\varphi_{ai}))=r_{bi}e^{j(-\pi+\varphi_{ia})}$$

$$\Downarrow$$

$$\frac{r_{bi}}{r_{ai}}e^{j(\varphi_{ai}-\varphi_{bi})}=\frac{r_{bi}}{r_{ai}}\cos(\varphi_{ai}-\varphi_{bi})+j\frac{r_{bi}}{r_{ai}}\sin(\varphi_{ai}-\varphi_{bi})=k_{ci}(1-j\tan(\varphi_{bi}-\varphi_{ai}))$$

$$\Downarrow$$

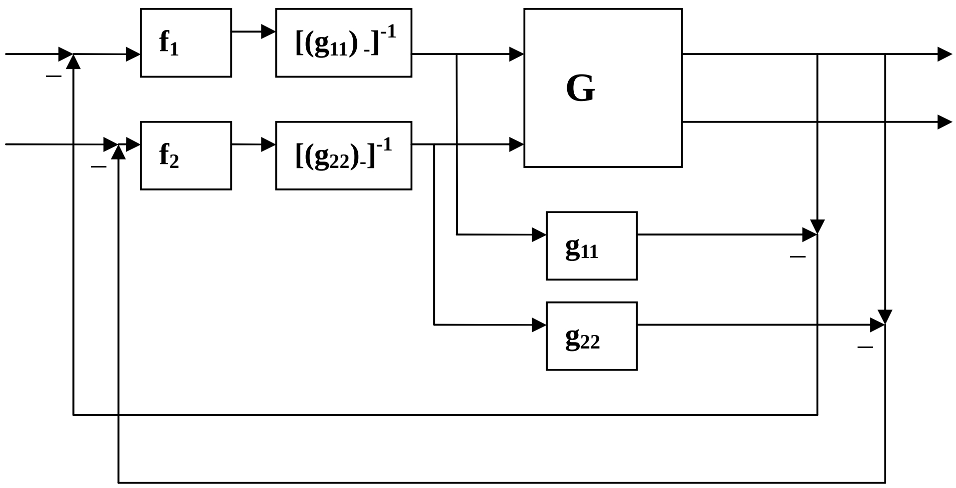
$$k_{ci}=\frac{r_{bi}}{r_{ai}}\cos(\varphi_{ai}-\varphi_{bi})$$

$$k_{ci}\cdot g_i(j\omega)=\frac{r_{bi}}{r_{ai}}\cos(\varphi_{ai}-\varphi_{bi})\cdot r_{ai}e^{j(-\pi+\varphi_{ia})}=r_{bi}\cos(\varphi_{ai}-\varphi_{bi})\cdot e^{j(-\pi+\varphi_{ia})}$$

By setting i equal one and two, one will obtain two equations with k_{c1} and k_{c2} as unknowns, and, thus, can be solved. But, there are very tedious procedures to find the controller gains (such as:such k_{c1} and k_{c2}) and frequency ω_{11} and ω_{22} that satisfy the phase criteria. (see the reference: I&EC Res. 1998, 37, 4725-4733, Q-G Wang, T-H Lee, and Y. Zhang)

C. Independent design method

---IMC Multi-loop PID Controller



$$G_{C,i} = \left(G_{i,i}\right)_{-}^{-1} f_i \; ; i = 1,\ldots,n$$

The stability is guaranteed for any stable IMC filter that satisfies either of the following:

$$\left|f_i(i\omega)\right| < f_{R,i}^{*}(i\omega) = \frac{\left|g_{i,i}(i\omega)\right|}{\sum_{j,j\neq i}\left|g_{i,j}(i\omega)\right|} \;\; ; \;\; i = 1,2,\ldots,n$$

$$\left|f_i(i\omega)\right| < f_{C,i}^{*}(i\omega) = \frac{\left|g_{i,i}(i\omega)\right|}{\sum_{j,j\neq i}\left|g_{j,i}(i\omega)\right|} \;\; ; \;\; i = 1,2,\ldots,n$$

Imc Row interaction measure [Economou and Morari]

$$R_i(i\omega) = \frac{1}{1+f_{R,i}^{*}(i\omega)} = \frac{\sum_{j,j\neq i}\left|g_{i,j}(i\omega)\right|}{\sum_{j}\left|g_{i,j}(i\omega)\right|} \;\; ; \;\; 0 \le \omega \le \infty$$

$$C_i(i\omega) = \frac{1}{1+f_{C,i}^{*}(i\omega)} = \frac{\sum_{j,j\neq i}\left|g_{j,i}(i\omega)\right|}{\sum_{j}\left|g_{j,i}(i\omega)\right|} \;\; ; \;\; 0 \le \omega \le \infty$$

For significant interaction: $0.5 \le \; R_i, C_i \; \le 1 \quad \Rightarrow \quad f^{*} < 1$

For small interaction: $0.0 \le \; R_i, C_i \; \le 0.5 \quad \Rightarrow \quad f^* > 1$

D. Chien-Huang-Yang's multi-loop PID---with no proportional and derivative kicks

1. Controllers for SISO loop:

Controller: $u(s) = k_C \left\{ -y(s) + \frac{1}{\tau_R s}[r(s) - y(s)] - \tau_D s y(s) \right\}$

$$\frac{y}{r} = \frac{k_C/(\tau_R s)G_p}{1 + k_C/(\tau_R s)G_p}$$

a. Time constant dominant processes:

$$G_P = \frac{\mathrm{R}e^{-Ls}}{s}\ ;\ R = \text{slope of the initial unit step response}$$

$$G_P = \frac{\mathrm{R}e^{-Ls}}{s} \approx \frac{R(1-Ls)}{s}$$

$$\frac{y}{r} = \frac{1-Ls}{\left(\frac{\tau_R}{Rk_C} - \tau_R L\right)s^2 + (\tau_R - L)s + 1} \approx \frac{1-Ls}{\tau_C^2 s^2 + 1.414\tau_C s + 1}$$

$$\Rightarrow \quad k_C = \frac{(1.414\tau_C + L)}{R(\tau_C^2 + 1.414\tau_C L + L^2)}\ ;\ \tau_R = 1.414\tau_C + L$$

b. Deadtime dominant processes:

$$G_P = \frac{k_P\, e^{-Ls}}{\tau s + 1} \approx \frac{k_P(1-Ls)}{\tau s + 1}$$

$$\frac{y}{r} = \frac{1 - Ls}{\left(\frac{\tau_R \tau}{k_C k_P} - \tau_R L\right)s^2 + \left(\frac{\tau_R}{k_C k_P} + \tau_R - L\right)s + 1}$$

$$\approx \quad \frac{1 - Ls}{\tau_C^2 s^2 + 1.414\tau_C s + 1}$$

$$\Rightarrow \quad k_C = \frac{1}{k_P}\frac{-\tau_C^2 + 1.414\tau_C\tau + L\tau}{\tau_C^2 + 1.414\tau_C\tau + L^2};$$

$$\Rightarrow \quad \tau_R = \frac{-\tau_C^2 + 1.414\tau_C\tau + L\tau}{\tau + L}$$

Derivation of the PID controller parameters is similar to the above PI derivations except that the deadtime approximation:

$$e^{-Ls} \approx \frac{1 - 0.5Ls}{1 + 0.5Ls}$$

Appendix: Derivation of PID Tuning Rules

The closed-loop transfer function between controlled variable (y) and setpoint (r) is

$$\frac{y}{r} = \frac{(K_c/\tau_i s)\,G_p}{1 + \dfrac{K_c(\tau_i\tau_d s^2 + \tau_i s + 1)}{\tau_i s}G_p} \tag{A.1}$$

For time constant dominant processes, the process model, G_p, can be approximated using Padé approximation as

$$G_p = \frac{Re^{-Ls}}{s} \approx \frac{R}{s}\frac{1 - (L/2)s}{1 + (L/2)s} \tag{A.2}$$

Substituting eq A.2 into A.1 and simplifying, we get

$$\frac{y}{r} = (1 - (L/2)s)\Bigg/\left[\left(\frac{L\tau_i}{2K_cR} - \frac{L\tau_i\tau_d}{2}\right)s^3 + \left(\frac{\tau_i}{K_cR} + \tau_i\tau_d - \frac{L\tau_i}{2}\right)s^2 + \left(\tau_i - \frac{L}{2}\right)s + 1\right] \tag{A.3}$$

Let us assume our desired closed-loop servo response to be a underdamped system with damping coefficient of 0.707. This corresponds to a closed-loop system with about 5% overshoot. The desired closed-loop servo response is

$$\left(\frac{y}{r}\right)_{\text{desired}} = \frac{e^{-Ls}}{\tau_{cl}^2 s^2 + 1.414\tau_{cl}s + 1} \approx \frac{1}{(\tau_{cl}^2 s^2 + 1.414\tau_{cl}s + 1)}\frac{1 - (L/2)s}{1 + (L/2)s} \tag{A.4}$$

where τ_{cl} is an user-specified closed-loop effective time constant. Equating eqs A.3 and A.4 and doing some algebraic manipulation, we can solve for the PID tuning parameters as

$$K_c = \frac{1.414\tau_{cl} + L}{R\left(\tau_{cl}^2 + 0.707\tau_{cl}L + \frac{L^2}{4}\right)} \quad \text{(A.5)}$$

$$\tau_i = 1.414\tau_{cl} + L \quad \text{(A.6)}$$

$$\tau_d = \frac{(L^2/4) + 0.707\tau_{cl}L}{1.414\tau_{cl} + L} \quad \text{(A.7)}$$

For processes with deadtime greater than $^1/_5$ of the process time constant, it is better for controller tuning purposes to model the processes as a first-order-plus deadtime model. With the same Padé approximation as

$$G_p = \frac{K_p e^{-Ls}}{\tau s + 1} \approx \frac{K_p}{\tau s + 1} \frac{1 - (L/2)s}{1 + (L/2)s} \quad \text{(A.8)}$$

Substituting eq A.8 into A.1 and simplifying, we obtain

$$\frac{y}{r} = (1 - (L/2)s) \Bigg/ \left[\left(\frac{L\tau_i\tau}{2K_cK_p} - \frac{L\tau_i\tau_d}{2}\right)s^3 + \left(\frac{\tau_i L}{2K_cK_p} + \frac{\tau_i\tau}{K_cK_p} + \tau_i\tau_d - \frac{L\tau_i}{2}\right)s^2 + \left(\frac{\tau_i}{K_cK_p} + \tau_i - L/2\right)s + 1\right] \quad \text{(A.9)}$$

Again, equating eqs A.9 and A.4 and doing some algebraic manipulation, we can solve for the PID tuning parameters as

$$K_c = \frac{\tau L + (L^2/4) + 1.414\tau_{cl}\tau - \tau_{cl}^2}{K_p(\tau_{cl}^2 + 0.707\tau_{cl}L + L^2/4)} \quad \text{(A.10)}$$

$$\tau_i = \frac{\tau L + (L^2/4) + 1.414\tau_{cl}\tau - \tau_{cl}^2}{\tau + (L/2)} \quad \text{(A.11)}$$

$$\tau_d = \frac{0.707\tau\tau_{cl}L + (L^2/4)\tau - \tau_{cl}^2(L/2)}{\tau L + (L^2/4) + 1.414\tau_{cl}\tau - \tau_{cl}^2} \quad \text{(A.12)}$$

By selecting τ_{cl} as in Figure 2, the negative terms in eqs A.10–A.12 will not cause any problem in changing the signs of the PID tuning parameters. With the τ_{cl} selection as in Figure 2, combining with eqs A.5–A.7 and A.10–A.12, the final PID tuning rules in Table 2 can be obtained.

2. Controllers for multi-loop system

$$At\ \omega \rightarrow 0\,;\ \left(\frac{y}{u_1}\right)_{\text{loop 2 closed}} = g_{1,1}\left(1-\frac{k_{1,2}k_{2,1}}{k_{1,1}k_{2,2}}\right) = \frac{g_{1,1}}{RGA(\lambda)}$$

$$At\ \omega \rightarrow \infty\,;\ \left(\frac{y}{u_1}\right)_{\text{loop 2 closed}} = g_{1,1}$$

a. For RGA>1, multi-loop controller tuning based on the process model in the main loop should provide satisfactory closed loop results. This is because:

b. For RGA < 1,

$$k_{C,i} = \left(k_C\right)_{\text{based on main loop}} RGA(\lambda_{i,i})$$

$$\tau_{R,i} = \frac{\left(\tau_{R,i}\right)_{\text{based on main loop}}}{RGA(\lambda_{i,i})}$$

$$\tau_{D,i} = \left(\tau_{D,i}\right)_{\text{based on main loop}} RGA(\lambda_{i,i})$$

The closed-loop time constant is chosen according to the value of L/τ in three different ranges, that is: L/τ < 0.2, 0.2 < L/τ < 0.5, and L/τ > 0.5.

For details, see the original paper.

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IX. Robustness of Closed-loop System.

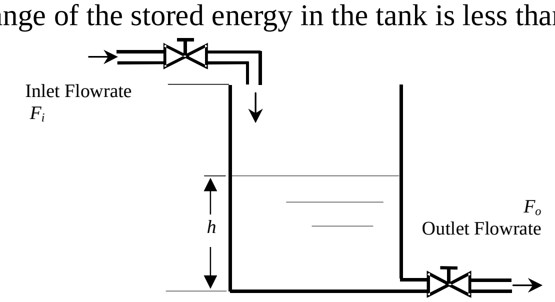
The final pairing and the controller tuning is checked for robustness by plotting DSO and DSI as functions of frequency, [Doyle and Stein]. The singular values below 0.3-0.2 indicate a lack of stability robustness.

$$DSO_{(i\omega)} = \underline{\sigma}[I + \left(GG_C\right)^{-1}]_{(i\omega)}$$

$$DSI_{(i\omega)} = \underline{\sigma}[I + \left(G_C G\right)^{-1}]_{(i\omega)}$$

E. Design Method based on Passivity

1. Hardware simplicity and relative effortlessness to achieve failure tolerant design, multi-loop control is the most widely used strategy in the industrial process control.
2. Current multi-loop control design approaches can be classified into three categories: detuning methods (Luyben, 1986), independent design methods (Skogestard and Morari, 1989), and sequential design methods (Mayne, Chiu and Arkun, 1992).
3. Loop interactions have to be taken into considerations, as they may have deteriorating effects on both control performance and closed-loop stability.
4. It is desirable if the multi-lop control system is decentralized unconditionally stable (i.e., any subset of the control loops can be independently to an arbitrary degree or even turned off without endangering close-loop stability.
5. Independent design is based on the basis of the paired transfer function while satisfying some stability constraints due to process interactions.
6. Perhaps the mostwidely used decentralized stability conditions are those μ-interaction measure.
7. Passivity Concept:
   The rate of change of the stored energy in the tank is less than the power supplied to it.



Potential energy stored in the tank: $S(h) = \frac{1}{2} Ah\rho \ \ gh = \frac{1}{2} A\rho gh^2$

Increment of potential energy per unit time: $w(t) = \rho F_i(t) gh(t)$

The rate of change of the storage function:

$$\frac{dS}{dt} = -C_v \rho gh\sqrt{h} + \rho g F_i h = -C_v \rho gh\sqrt{h} + w < w \qquad \forall h > 0$$

The rate of change of the stored energy in the tank is less than the power supplied to it. Therefore this process is said to be strictly passive.

Passive(Willems 1972)*:* if a non-negative ***storage*** function $S(x)$ can be found s.t.:

$S(0)=0$ and $S(x)-S(x^0)\le\int_{t0}^{t} y^T(\tau)u(\tau)d\tau$ for all $t>t0\ge 0$, $x0$, $x\in X$, $u\in U$.

Strictly passive: if $S(x)-S(x^0)<\int_{t0}^{t} y^T(\tau)u(\tau)d\tau$

Where, y is the output of a system, u is the input to the system.

- KYP Lemma
  - Nonlinear control affine systems (Hill & Moylan 1976)

    $\dot{x}=f(x)+g(x)u$

    $y=h(x)$

    where $x\in X\subset R^n$, $u\in U\subset R^m$, $y\in Y\subset R^m$

    The process is passive if

    $$L_f S(x)=\frac{\partial S^T(x)}{\partial x}f(x)\le 0,$$

    $$L_g S(x)=\frac{\partial S^T(x)}{\partial x}g(x)=h^T(x)$$

- KYP Lemma

  A linear system (Willems 1972) $G(s):=(A,B,C,D)$ is passive if there exists a positive definite matrix P such that:

  $$\begin{bmatrix} A^T P+PA & PB-C^T \\ B^T P-C & -D-D^T \end{bmatrix}\le 0$$

  The system is strictly passive if

  $$\begin{bmatrix} A^T P+PA & PB-C^T \\ B^T P-C & -D-D^T \end{bmatrix}< 0$$

Definition:

An LTI system S: G(s) is *passive* if :

(1) G(s) is analytic in Re(s)>0;

(2) G(jw)+G*(jw)≥0 for all that jw is not a pole of G(s);

(3) If there are poles of G(s) on the imaginary axis, they are non-repeated and the residue matrices at the poles are Hermitian and positive semi-definite.

$G(s)$ is *strictly passive* if:

(1) G(s) is analytic in Re(s) ≥ 0;

(2) $G(jw)+G^*(jw)>0 \quad \forall\omega\in(-\infty,\ \infty)$.

Theorem 1: For a given stable non-passive process with a transfer function matrix **G**(s), there exists a diagonal, stable, and passive transfer function matrix **W**(s)=w(s)**I** such that **H**(s)=**G**(s)+**W**(s) is passive.

[Proof]:

$$\lambda_{\min}(H(j\omega)+H^*(j\omega)) = \lambda_{\min}(G(j\omega)+G^*(j\omega)+(W(j\omega)+W^*(j\omega))$$

Since both (G+G*) and (W+W*) are Hermitian, from the Weyl inequality, we have:

$$\begin{aligned}\lambda_{\min}(H(j\omega)+H^*(j\omega)) &\geq \lambda_{\min}(G(j\omega)+G^*(j\omega))+\lambda_{\min}(W(j\omega)+W^*(j\omega)) \\ &= \lambda_{\min}(G(j\omega)+G^*(j\omega))+2\,\mathrm{Re}(W(j\omega))\end{aligned}$$

Thus, if:

$$\mathrm{Re}(W(j\omega)) \geq \frac{1}{2}\lambda_{\min}(G(j\omega)+G^*(j\omega))$$

H(s) can be render passive. On the other hand, if

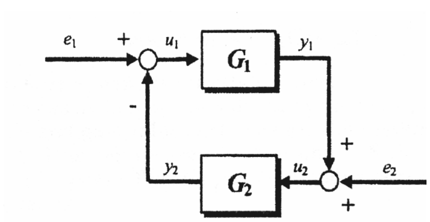
$$\mathrm{Re}(W(j\omega)) > \frac{1}{2}\lambda_{\min}(G(j\omega)+G^*(j\omega))$$

H(s) will be strictly passive.

Properties of Passive Systems:

- A passive system is minimum phase. The phase of a linear process is within [-90º, 90º]
- Passive systems are Lyapunov stable
- A passive system is of relative degree < 2
- Passive systems can have infinite gain (e.g., 1/s)

Passivity Theorem :



If *G1* is strictly passive and *G2* is passive, then the closed-loop system is *L*2 stable.

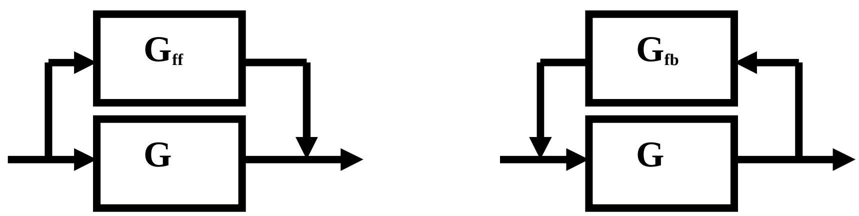
- A strictly passive process can be stabilized by any passive controller (including multi-loop PID controllers) even if it is highly nonlinear and/or highly coupled

⇨ Control design based on passivity

- Excess or shortage of passivity of a process can be used to analyse whether this process can be easily controlled

  ⇨ Passivity based controllability study

A non-passive process can be made passive using feedforward and/or feedback passification:

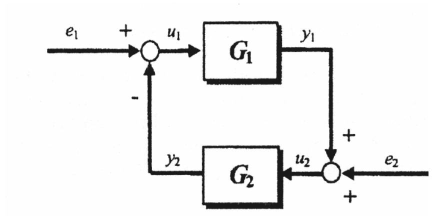


The excess or shortage of passivity can be quantified using:

- Input Feedforward Passivity (IFP) (Sepulchre et al 1997) - If a system G with a *negative* feedforward of νI is passive, then G has excessive input feedforward passivity, i.e., G is IFP(ν).

- Output Feedback Passivity (OFP) (Sepulchre et al 1997) - If a system G with a *positive* feedback of ρI is passive, then G has excessive output feedback passivity, i.e., G is OFP (ρ).

Agin, use the following figure:



If *G*1 is IFP(ν) and *G*2 is OFP(ρ), then the closed-loop system is stable if ρ+ν>0. In other words, a processs shortage of passivity can be compensated by another process's excess of passivity.

- **Passivity Index**

The excessive IFP of a system G(s) can be quantified by a frequency dependent passivity index

$$\nu_F[G(s),\omega] \stackrel{\Delta}{=} \lambda_{\min}\left(\frac{1}{2}[G(j\omega)+G^*(j\omega)]\right)$$

Assume the true process is $G_T(s) = G(s) + \Delta(s)$

The passivity index of the true process can be estimated as

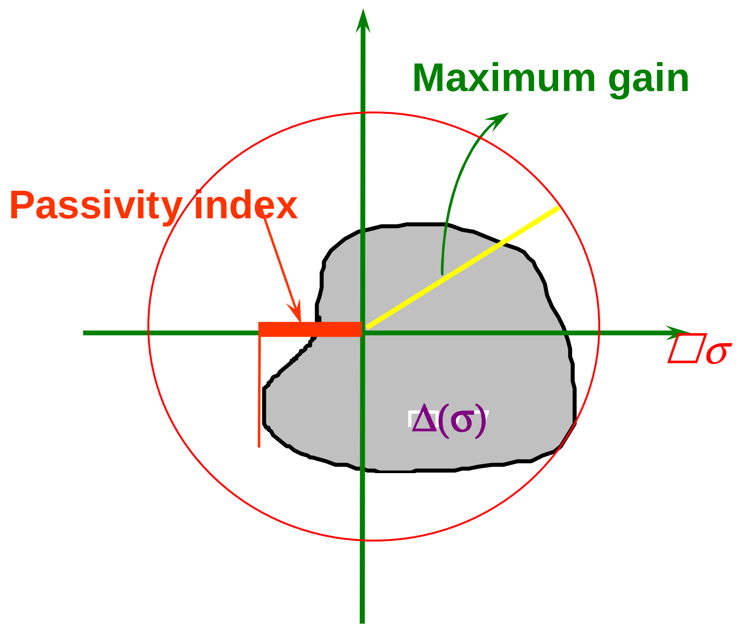
$$\nu(G_T(j\omega)) = -\lambda_{\min}\left\{\frac{1}{2}\left[\Delta(j\omega)+\Delta^*(j\omega)\right]+\frac{1}{2}\left[G(j\omega)+G^*(j\omega)\right]\right\}$$

$$\leq -\lambda_{\min}\left\{\frac{1}{2}\left[\Delta(j\omega)+\Delta^*(j\omega)\right]\right\} - -\lambda_{\min}\left\{\frac{1}{2}\left[G(j\omega)+G^*(j\omega)\right]\right\}$$

$$= \nu(G(j\omega)) + \nu(\Delta(j\omega))$$

Properties of the Passivity Index

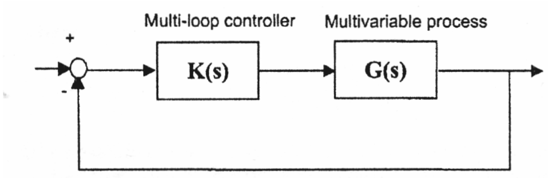
1. Comprises gain & phase information of the uncertainty



2. *Always* no greater than the maximum singular value.

$$|\nu_F[\Delta(s),\omega]| \leq \sigma_{\max}[\Delta(j\omega)] \text{ for any } \omega \in R$$

Passivity Theorem 2: If the multivariable process is strictly passive, then the closed-loop system is stable if the multi-loop controller is passive.



Theorem 1: A closed-loop system comprising a stable subsystem G(s) and a decentralized controller $K(s)=\mathrm{diag}(k_i(s))$, w(s) is a stable and minmum phase, and

$$\nu(W(j\omega)) < -\nu(G^+(j\omega))$$

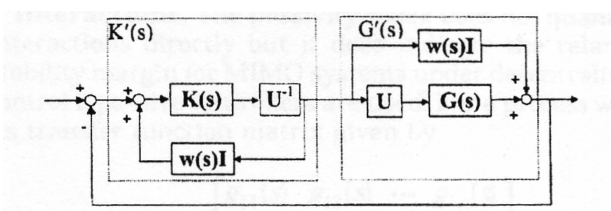
Then the closed-loop system will be decentralized unconditional stable, if

$K(s)=\mathrm{diag}\{\, k_i^{'}(s) \,\}$ is passive, where,

$$k_i^{'}(s) = k_i^+[1 - w(s)k_i^+(s)]^{-1} \quad \text{and} \quad k_i^+ = U_{ii}k_i$$

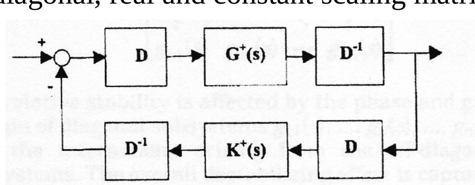
$$K^{'}(s) = U^{-1}K(s)\left[I - w(s)U^{-1}K(s)\right]^{-1}$$

Notice that the above figure is equivalent to the one in the following:



Rescaling of the system

Let D(s) be a diagonal, real and constant scaling matrix.



The scaling matrix D(s) is to make

$$\nu(D^{-1}G^+D(j\omega)) < \nu(G^+(j\omega))$$

and

$$D^{-1}G^{+}(0)D + D^{-1}\left[G^{+}(0)\right]^{+} D > 0$$

Design procedures:

1. Find matrix U and calculate $G^{+}(s)$.
2. Check the pairing. Examine the proposed pairing using DIC condition:

   $$G^{+}(0)M + M\left[G^{+}(0)\right]^{T} > 0$$

3. Use matrix M obtained in the step 2 to derive D, $D = M^{1/2}$
4. Calculate $\nu(D^{-1}G^{+}(j\omega)D)$ for different frequency points. These frequency points form a set $\Omega \in \left[0, \omega_E\right]$ where ω_E is the frequency which is high enough sych tant $\nu(D^{-1}G^{+}(j\omega)D) \to 0$ for $\omega > \omega_E$.
5. For each loop of the controller, solve problem:

   $$\min_{k_{ci},\tau_{Ri}}(-\gamma_i)$$

   such that

   $$\left| \frac{1}{1 + G_{ii}^{+}(j\omega)k_{c,i}^{+}\left[1 + \frac{1}{j\tau_{R,i}\omega}\right]} \frac{\gamma_i}{j\omega} \right| < 1$$

   and

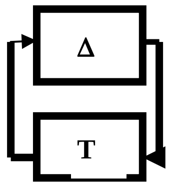
   $$\tau_{R,i}^{2} \geq \frac{k_{c,i}^{+}\nu_s(\omega)}{\left[1 - k_{c,i}^{+}\nu_s(\omega)\right]\omega^{2}}, \quad \forall \omega \in R, \quad i = 1, \cdots, n$$

6. Obtain the final controller settings: $k_{c,i} = U_{ii}k_{c,i}^{+}$

This method is limited to open-loop stable processes.

$$\nu_F(\Delta(s),\omega) \geq -\nu_F(W(s),\omega), \ \forall \omega \in \mathbf{R}$$

Robust Stability Condition



If the uncertainty is passive, then the controller is only required to render system T strictly passive to achieve robust stability even if Δ is very large.

If the uncertainty's passivity index is bounded by

$$\nu_F(\Delta(s),\omega) \geq -\nu_F(W(s),\omega), \quad \forall \omega \in \mathbf{R}$$

where $W(s)$ is minimum phase, the closed-loop system will be robust stable if system

$$T(s)[I - W(s)T(s)]^{-1}$$

is strictly passive.

The basic idea:
1. Characterise the uncertainty in terms of passivity using IFP or OFP.
2. Derive the robust stability condition for systems with uncertainties bounded by their passivity indices.
3. Develop a systematic procedure to design the robust controller which satisfies the above stability condition.

Passivity Based Robust Control Design

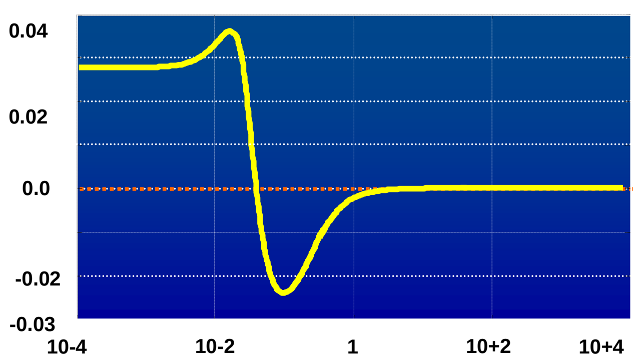
- Blended approach
  - Design a controller that satisfies the small gain condition at high frequencies and satisfies the passivity condition at low frequencies (Bao, Lee et al 1998)
    - Based on the bilinear transformation

- Multi-objective control design
  - Design a controller that satisfies the passivity condition for robust stability and achieves H∞ control performance (Bao, Lee et al 2000, 2003)
    - Based on KYP lemma and Semi-Definite Programming

Example:

$$G(s)=\begin{bmatrix} \dfrac{0.126e^{-6s}}{60s+1} & \dfrac{-0.101e^{-12s}}{(48s+1)(45s+1)} \\ \dfrac{0.094e^{-8s}}{38s+1} & \dfrac{-0.12e^{-8s}}{35s+1} \end{bmatrix}$$

Passivity index



F. Design by Sequential Loop Closing

Advantages of sequential design:

1. Each step in the design procedure involves designing only one SISO controller.
2. Limited degree of failure tolerance is guaranteed: If stability has been achieved

after the design of each loop, the system will remain stable if loop fail or are taken out of service in the reverse order of they were designed.

3. During startup, the system will be stable if the loops are brought into service in the same order as they have been designed.
4.

Problems with sequential design:

1. The final controller design, and thus the control quality achieved, may depend on the order in which the controllers in the individual loops are designed.
2. Only one output is usually considered at a time, and the closing of subsequent loops may alter the response of previously designed loops, and thus make iteration necessary.
3. The transfer function between input u_k and output y_k may contain RHP zeros that do not corresponding to the RHP zeros of G(s).

Notations:

1. G(s): the $n \times n$ matrix of the plant, $G(s) = \{g_{ij}(s);\ \ i, j = 1, \cdots, n\}$
2. $C(s) = diag\{c_i(s);\ \ i = 1, \cdots, n\}$
3. $S = (I + GC)^{-1}$; $\quad H = I - S = GC(I + GC)^{-1}$
4. $\tilde{G} = diag\{g_{ii}(s);\ \ i = 1, \cdots, n\}$
5. $\tilde{S} = diag\{s_i(s);\ \ i = 1, \cdots, n\} = diag\{\frac{1}{1 + g_{ii}c_i};\ \ i = 1, \cdots, n\}$
6. $\tilde{H} = diag\{h_i(s);\ \ i = 1, \cdots, n\} = diag\{\frac{g_{ii}c_i}{1 + g_{ii}c_i};\ \ i = 1, \cdots, n\}$
7. $\Gamma = \tilde{G}G^{-1} = \{\gamma_{ij};\ \ i, j = 1, \cdots, n\}$
8. $CLDG = \tilde{G}G^{-1}G_d$
9. $E = (G - \tilde{G})\tilde{G}^{-1}$
10. $G = \begin{bmatrix} G_k & \vdots \\ \cdots & \ddots \end{bmatrix}$; $\mathrm{C} = \begin{bmatrix} C_k & \vdots \\ \cdots & \ddots \end{bmatrix}$;
11. $S_k = \left(I + G_kC_k\right)^{-1}$; $H_k = G_kC_k\left(I + G_kC_k\right)^{-1}$
12. $\hat{H}_k = \begin{bmatrix} H_k & 0 \\ 0 & \tilde{h}_i \end{bmatrix}$; $\hat{S}_k = \begin{bmatrix} S_k & 0 \\ 0 & \tilde{s}_i \end{bmatrix}$; $\quad i = k + 1,\ K + 2,\ \cdots N$

$$S = (I + GC)^{-1} = [I + \tilde{G}C + (G - \tilde{G})C]^{-1}$$
$$= \left\{ \left[I + (G - \tilde{G})C\left(I + \tilde{G}C\right)^{-1} \right] \left(I + \tilde{G}C\right) \right\}^{-1}$$
$$= \left\{ \left[I + (G - \tilde{G})\tilde{G}^{-1}\tilde{G}C\left(I + \tilde{G}C\right)^{-1} \right] \left(I + \tilde{G}C\right) \right\}^{-1}$$
$$= \left(I + \tilde{G}C\right)^{-1}\left(I + E\tilde{H}\right)^{-1} = \tilde{S}\left(I + E\tilde{H}\right)^{-1}$$

Design procedures:

In each of the following step, $S = \widehat{S}_k(I + E_k\widehat{H}_k)^{-1}; E_k = (G - \hat{G}_k)\ \left(\widehat{G}_k\right)^{-1}$

Determine c_i such that $\left\| W_p S W_D \right\|_{\square}$ is minimized.

Step 0. Initialization. Determine the order of loop closing by estimating the required bandwidth in each loop. Also estimate the individual loop designs in terms of $\tilde{H}$.

Step 1. Design of controller c_1 by considering output 1 only. In this case, we have

$\hat{G}_k = \tilde{G}_k$ and $\hat{H}_k = \tilde{H}$

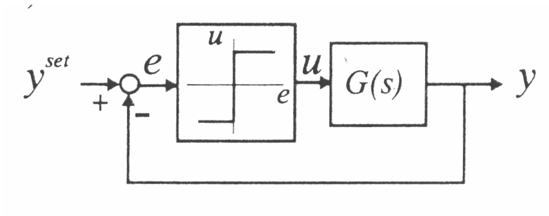
Step k. Design of controller ck by consider outputs 1 to k. Here,

$\hat{G}_k = diag\{\tilde{G}_k, g_{ii}\};\ \ i = k+1, k+2, \cdots, n$ and

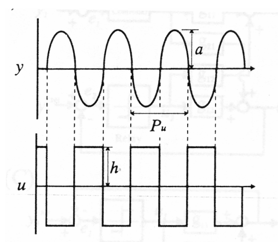
$\hat{H}_k = daig\{H_{k-1,}\tilde{h}_i\}\ ;\ \ i = k, k+1, \cdots, n$

Sequential Design Using Relay feedback Tests of Shen and Yu

The relay feedback system for SISO auto-tuning is as shown in the follwing figure:



When constant cycles appear after the system has been activated, the ultimate gain and ultimate frequency of the open-loop system can be approximated by measuring the magnitude and period (see the following figure) and by the following equations:



$$K_u = \frac{4h}{\pi a}; \quad \omega_u = \frac{2\pi}{P_u}$$

The Z-N tuning method can be used to determine the controller parameters:

PI Controller: $K_c = 0.45K_u, \quad \tau_R = P_u / 1.2,$

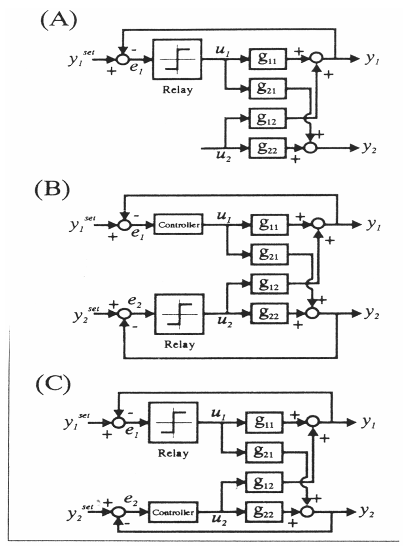
PID Controller: $K_c = 0.60K_u, \quad \tau_R = P_u / 1.2, \quad \tau_D = 1.25P_u$

Or, use the Tyreus-Luyben's formula to give more conservative response:

PI Controller: $K_c = K_u / 3.2, \quad \tau_R = 2.2P_u,$

PID Controller: $K_c = K_u / 2.2, \quad \tau_R = 2.2P_u, \quad \tau_D = P_u / 6.3$

To avoid the difficult mathematics envolved in the formulation of sequential design, Shen and Yu suggested to use the relay-feedback test as shown in the following figure:



The controller for a 2×2 system is suggested:

PI Controller: $K_c = K_{c,ZN}/3, \quad \tau_R = 2\ P_u$

Analysis:

The sequential design is derived by considering the multi-loop control system as coupled SISO loops. For a 2×2 system as example, the equivalent SISO loops are:

$$g_1(s) = g_{1,1}(s)\ \{1-(1-\frac{1}{\lambda(s)})\ h_2(s)\}$$

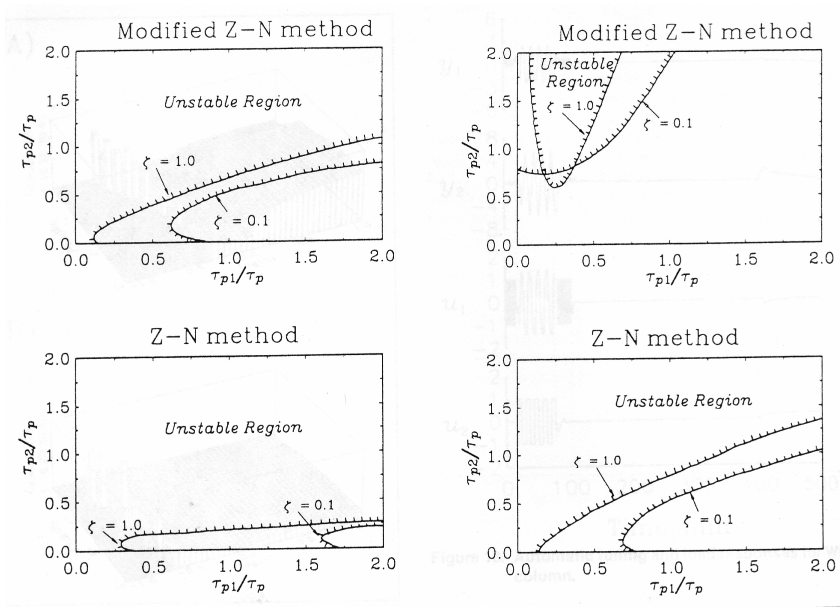
$$g_2(s) = g_{2,2}(s)\ \{1-(1-\frac{1}{\lambda(s)})\ h_1(s)\}$$

Where, $h_i(s) = \dfrac{g_{C,i} g_{i,i}}{1+g_{C,i} g_{i,i}}; \quad i=1,2$

Notice that, if there is damping in g_1 or g_2, this damping should come from either h_1 or h_2. According to tis study, a closed system having an FOPDT process and a modified ZN tuned PI controller will result in a closed-loop system (i.e. h_1 and h_2) having damping factor greater than 0.6. It is thus postulate that the open-loop transfer functions $g_1(s)$ and $g_2(s)$ can be approximated by:

$$G(s) = \frac{k_p}{\tau^2 s^2 + 2\tau\zeta s + 1} \cdot \frac{\tau_{p2} s + 1}{\tau_{p1} s + 1} \cdot e^{-\theta s}$$

Then, the stability region of the equivalent SISO loops are explored with the parameters: $\tau_{p1}, \tau_p = 0 \sim 10,\ k_p = 1,\ \tau = 5,\ \zeta = 0.1 \sim 1,\ \theta/\tau = 0.02 \sim 0.2$. The results are given in the following figure. It can be seen that the modified ZN tuning formula proposed greatly improve the stability.



On the other hand, the convergence of the sequential design for the multi-loop controller is formulated as the problem of finding the roots of simultaneous algebraic equation using sequential iterations.

The simultaneous equations are obtained from the conditions of phase crossover for the two loops, that is:

$$F_1(j_{\omega u,1}, j\omega_{u,2}) = \tan^{-1}\frac{\mathrm{Im}\left[g_1(j_{\omega u,1}, j\omega_{u,2})\right]}{\mathrm{Re}\left[g_1(j\omega_{u,1}, j\omega_{u,2})\right]} = -\pi$$

$$F_2(j_{\omega u,1}, j\omega_{u,2}) = \tan^{-1}\frac{\mathrm{Im}\left[g_2(j_{\omega u,1}, j\omega_{u,2})\right]}{\mathrm{Re}\left[g_2(j\omega_{u,1}, j\omega_{u,2})\right]} = -\pi$$

The convergence of the iteration is guaranteed by a sufficient condition of the following:

$$\frac{\left(\dfrac{\partial F_1}{\partial \omega_{u,2}}\right)_{\omega_{u,1}}\left(\dfrac{\partial F_2}{\partial \omega_{u,1}}\right)_{\omega_{u,2}}}{\left(\dfrac{\partial F_1}{\partial \omega_{u,1}}\right)_{\omega_{u,2}}\left(\dfrac{\partial F_2}{\partial \omega_{u,2}}\right)_{\omega_{u,1}}} < 1$$

The procedures of this proposed sequential design are summarized with the flow chart as shown.