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Riemann zeta function

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I declare that I carried out this bachelor thesis independently, and only with the cited sources, literature and other professional sources.

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Abstrakt: Riemannova zeta funkce je v současné matematice důležitým nástrojem analytické teorie čísel s aplikacemi zejména v kvantové mechanice, teorii pravděpodobnosti a statistice. Zavedena Bernhardem Riemannem v roce 1859, zeta funkce je ústředním objektem mnoha doposud nevyřešených problémů a z dosavadních výsledků je zřejmý její význam pro další vývoj na poli teorie čísel. Tato práce se soustředí na základní vlastnosti Riemannovy zeta funkce, zejména problematiku kořenů zahrnující dokázaná tvrzení o rozložení kořenů vně i uvnitř kritického pásu, formulaci Riemannovy hypotézy a problematiku iracionality vybraných hodnot zeta funkce včetně důkazu iracionality $\zeta(3)$.

Klíčová slova: Riemann, zeta, iracionalita, kořen

Title: Riemann zeta function

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Abstract: Riemann zeta function represents an important tool in analytical number theory with various applications in quantum mechanics, probability theory and statistics. First introduced by Bernhard Riemann in 1859, zeta function is a central object of many outstanding problems. From previous results follows the importance of zeta function for further development in the field of number theory. This thesis provides basic properties of the Riemann zeta function. In particular, we prove theorems concerning the distribution of its roots outside and inside the critical strip which leads to the formulation of the Riemann hypothesis and theorems concerning the irrationality of selected values of the Riemann zeta function including the proof of the irrationality of $\zeta(3)$.

Keywords: Riemann, zeta, irrationality, root

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List of Symbols

ln	...	natural logarithm
log	...	principal value of complex logarithm
$\mathbf{R}$	...	set of real numbers
$\mathbf{C}$	...	set of complex numbers
$\mathbf{Z}$	...	set of integers
$\mathbf{N}$	...	set of natural numbers $1, 2, 3, \ldots$
$\mathbf{Q}$	...	set of rational numbers
$\mathbf{N}_0$	...	set of natural numbers including zero
$\{c_n\}_{n=0}^{\infty}$	...	sequence $\{c_0, c_1, c_2, \ldots\} \subset \mathbf{C}$
Γ	...	gamma function
$n!$	...	n factorial
$\binom{n}{k}$	...	binomial coefficient
$\Re(s)$	...	real part of s
$\Im(s)$	...	imaginary part of s
ζ	...	Riemann zeta function
ζ_e	...	Euler zeta function
$[1, ..., n]$	...	least common multiple of numbers $1, 2, ..., n$
$\deg_p n$	...	number of times that given prime p divides n
$b_n = O(a_n), n \to \infty$	...	there exists sufficiently large n that b_n is at most $C \in \mathbf{R}$ multiplied by a_n in absolute value
$a \mapsto b$	...	transformation of a into b
$a := b$	...	a is defined as b
$[a]$	...	integer part of a
$\equiv$	...	is equivalent

Introduction

The Riemann zeta function is a complex function of one complex variable with great importance in pure mathematics. Its properties deeply bind the Riemann zeta function with many results and conjectures surrounding the prime numbers. This thesis provides basic properties of the Riemann zeta function and focuses on the proof of Apéry's theorem.

The thesis is divided into chapters, each containing proved theorems, comments, and references to further reading. The aim is to present the subject matter in an illustrative manner.

In the first chapter the construction of the Riemann zeta function is introduced. We start from the Euler zeta function and by the means of analytic continuation we find an expression which holds in $\mathbf{C} \setminus \{1\}$.

In the second chapter the number-theoretic properties of the Riemann zeta function are discussed. In particular, the Euler product formula which represents a connection to prime numbers is proved and an expression for the zeta values at even positive integers is introduced which is further used to show that these numbers are irrational. Finally, Apéry's theorem that states the irrationality of $\zeta(3)$ is proved.

The third chapter deals with basic results in the analysis of the distribution of zeta zeros. Theorems concerning the trivial roots of the Riemann zeta function are proved and the famous Riemann hypothesis along with other results in this area are presented.

The fourth chapter contains additional supporting computations related to the proof of Apéry's theorem. These computations are presented in a separate chapter for their complexity and in order to make the proof of Apéry's theorem more transparent.

1. Definition

1.1 Euler zeta function

As a starting point let us consider the Euler zeta function. It is a generalization of the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ appearing in the famous Basel problem[1] which was solved by Euler in 1735.

Definition 1.1.1. For $s \in \mathbf{C}$ let

$$\zeta_e(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1}$$

whenever the sum converges. (1.1) is called *the Euler zeta function.*

Lemma 1.1.2. *The series (1.1) defines holomorphic function $\zeta_e(s)$ for $s \in \mathbf{C}$, $\Re(s) > 1$.*

Proof. Let $s = \sigma + it, \sigma, t \in \mathbf{R}$. First we have

$$|n^s| = \left|e^{s \log n}\right| = \left|e^{\sigma \log n} \cdot e^{it \log n}\right| = e^{\sigma \log n} = n^\sigma.$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is absolutely convergent if and only if $\sigma > 1$. Let $\epsilon > 0$. Then there exists $n_0 \in \mathbf{N}$ such that for all $N, M \in \mathbf{N}, N > M \geq n_0$:

$$\left|\sum_{n=1}^{N} \frac{1}{n^s} - \sum_{n=1}^{M} \frac{1}{n^s}\right| = \left|\sum_{n=M+1}^{N} \frac{1}{n^s}\right| \leq \sum_{n=M+1}^{N} \left|\frac{1}{n^s}\right| = \sum_{n=M+1}^{N} \frac{1}{n^\sigma} < \epsilon,$$

where $\sigma \geq 1 + \xi$, for $\xi > 0$ since

$$\sum_{n=M+1}^{N} \frac{1}{n^\sigma} \leq \sum_{n=M+1}^{N} \frac{1}{n^{1+\xi}} \leq \int_M^N \frac{1}{x^{1+\xi}} dx =$$

$$= -\frac{1}{\xi}\left[\frac{1}{x^\xi}\right]_M^N = \frac{1}{\xi} \cdot \frac{N^\xi - M^\xi}{M^\xi N^\xi} \leq \frac{1}{\xi} \cdot \frac{N^\xi}{M^\xi N^\xi} = \frac{1}{\xi} \cdot \frac{1}{M^\xi} < \epsilon,$$

for $M > n_0$. Hence $\zeta_e(s)$ is uniformly convergent in any region in which $\sigma \geq 1+\xi$, $\xi > 0$. The sum $\sum_{n=1}^{\infty} \frac{1}{n^s}$ therefore defines a holomorphic function $\zeta_e(s)$ for $\sigma > 1$. □

1.2 Analytic continuation

By definition 1.1.1, the function $\zeta_e(s)$ is defined only in the half-plane $\Re(s) > 1$. It is only natural to ask whether the holomorphic function $\zeta_e(s)$ can be continued beyond this region. To answer this question it will be useful to prove the following.

[1]The Basel problem asks for finding the closed form and precise estimation of the value of the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$. For further references see [7], [8].

Lemma 1.2.1. *For $s \in \mathbf{C}$, $\Re(s) > 1$ the following equation, where $\Gamma(s)$ denotes the gamma function, holds:*

$$\zeta_e(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^{s-1}}{e^u - 1} du. \tag{1.2}$$

Proof. We shall start from the definition of the gamma function Γ:

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \quad s \in \mathbf{C}, \Re(s) > 0.$$

By substitution $t = nv$ we get

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt = n^s \int_0^\infty v^{s-1} e^{-nv} dv$$

and therefore

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty v^{s-1} e^{-nv} dv.$$

Further let $N \in \mathbf{N}$ and by summing both sides for $1 \leq n \leq N$ (finite sum) we obtain

$$\sum_{n=1}^{N} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty v^{s-1} \left(\sum_{n=1}^{N} e^{-nv} \right) dv = \frac{1}{\Gamma(s)} \int_0^\infty v^{s-1} \left(\frac{1 - e^{-Nv}}{e^v - 1} \right) dv.$$

Since $e^v - 1$ has a simple root at the point 0 this integral converges for $\Re(s) > 1$. Now we need to show that

$$\zeta_e(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} := \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^s} = \lim_{N\to\infty} \left(\frac{1}{\Gamma(s)} \int_0^\infty v^{s-1} \left(\frac{1 - e^{-Nv}}{e^v - 1} \right) dv \right) =$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty v^{s-1} \lim_{N\to\infty} \left(\frac{1 - e^{-Nv}}{e^v - 1} \right) dv = \frac{1}{\Gamma(s)} \int_0^\infty \frac{v^{s-1}}{e^v - 1} dv.$$

The switching of limit and integral holds by the existence of integrable majorant

$$\left| \frac{1 - e^{-Nv}}{e^v - 1} \right| \leq \left| \frac{1}{e^v - 1} \right|.$$

□

According to [26], let us define a notation which will simplify the following lemma.

Definition 1.2.2. For a function f defined in $\mathbf{C} \setminus \mathbf{R}$ and for every $x \in \mathbf{R}$ for which the limit

$$\lim_{y\to 0+} f(x + iy), \quad \lim_{y\to 0-} f(x + iy)$$

exists let us denote this limit by

$$(f(x))_+ , \quad (f(x))_- .$$

Furthermore if $c, d \in \mathbf{R} \cup \{\pm\infty\}$ then the integral

$$\int_c^d (f(x))_+, \quad \int_c^d (f(x))_-$$

will be called *the integral of the function $f(z) \equiv f(x+iy)$ along the segment $[c, d]$ on the upper (lower resp.) side of the real axis.*

Lemma 1.2.3. *Let $0 < R < 2\pi$ and let $L(R)$ denote the curve which consists of the segment $[-\infty, -R]$ on the lower side of the real axis, the circumference $C(0, R)$ and the segment $[-R, -\infty]$ on the upper side of the real axis. Then for $s \in \boldsymbol{C} \setminus \boldsymbol{Z}, \Re(s) > 0$ the following equation holds:*

$$\int_0^\infty \frac{u^s}{e^u - 1} du = \frac{1}{2i \sin \pi s} \int_{L(R)} \frac{z^s}{1 - e^{-z}} dz. \tag{1.3}$$

Proof. [2] Let us consider the function $\psi(z) := \frac{z}{1-e^{-z}} \cdot z^{s-1}$. Let $G := \mathbf{C} \setminus (-\infty, 0]$. Then $\psi(z)$ is a holomorphic function in G. Now, for every $z \in \mathbf{R}, z < 0$ we have $(\psi(z))_+ = \psi(z)$ and $(\psi(z))_- = e^{-2\pi(s-1)i}\psi(z)$. Therefore for $s \in \mathbf{C}, \Re(s) > 0$ and every real $N > 0$ we get

$$\int_0^{-N} \left(\frac{z}{1 - e^{-z}} \cdot z^{s-1}\right)_+ dz + \int_{-N}^0 \left(\frac{z}{1 - e^{-z}} \cdot z^{s-1}\right)_- dz =$$

$$= \left(1 - e^{-2\pi(s-1)i}\right) \int_0^{-N} \left(\frac{z}{1 - e^{-z}} \cdot z^{s-1}\right) dz.$$

The integrals are convergent and represent functions of s holomorphic in the half-plane $\Re(s) > 0$. By Cauchy's theorem it follows that the left side is equal to

$$\int_{-N}^{-R} \left(\frac{z}{1 - e^{-z}} \cdot z^{s-1}\right)_- dz + \int_{-R}^{-N} \left(\frac{z}{1 - e^{-z}} \cdot z^{s-1}\right)_+ dz + \int_{C_R} \left(\frac{z}{1 - e^{-z}} \cdot z^{s-1}\right) dz,$$

where C_R denotes the circumference $C(0, R)$ with radius $R < 2\pi$. Since these integrals converge for all $s \in \mathbf{C}$, this expression represents an entire function of s. After passing to the limit $N \to \infty$, the integrals above will represent an entire function of s since the first two integrals tend to finite limits. However,

$$\left(1 - e^{-2\pi(s-1)i}\right) \int_0^{-\infty} \left(\frac{z}{1 - e^{-z}} \cdot z^{s-1}\right) dz =$$

$$= 2i \sin \pi(s-1) \int_0^{-\infty} \left(\frac{z}{1 - e^{-z}} \cdot (-z)^{s-1}\right) dz =$$

$$= 2i \sin \pi s \int_0^\infty \frac{z^s}{e^z - 1} dz$$

and therefore

$$2i \sin \pi s \int_0^\infty \frac{z^s}{e^z - 1} dz =$$

$$= \int_{-\infty}^{-R} \left(\frac{z^s}{1 - e^{-z}}\right)_- dz + \int_{-R}^{-\infty} \left(\frac{z^s}{1 - e^{-z}}\right)_+ dz + \int_{C_R} \frac{z^s}{1 - e^{-z}} dz =$$

$$= \int_{L(R)} \frac{z^s}{1 - e^{-z}} dz$$

holds for every $s \in \mathbf{C}, \Re(s) > 0$. □

Being sufficiently prepared we shall start from lemma 1.2.3 and derive a formula for the analytic continuation of $\zeta_e(s)$ to $\mathbf{C} \setminus \{1\}$.

[2] For a general proof for integrals of the form $\int_0^\infty u^{s-1}\varphi(u)du$ see [26, p.418–420]. Here we shall follow this reasoning with the function $\varphi(z) = \frac{z}{e^z - 1}$.

Theorem 1.2.4. *Let $0 < R < 2\pi$ and $L(R)$ the curve from lemma 1.2.3. Then*

$$\zeta(s) := \frac{\Gamma(1-s)}{2\pi i} \int_{L(R)} \frac{z^{s-1}e^z}{1-e^z} dz$$

is the analytic continuation of $\zeta_e(s)$ which holds for $s \in \boldsymbol{C} \setminus \{1\}$, with a simple pole at 1 with residue 1.

Proof. From lemma 1.2.3 we have

$$\int_0^\infty \frac{z^s}{e^z-1} dz = \frac{1}{2i \sin \pi s} \int_{L(R)} \frac{z^s}{1-e^{-z}} dz$$

for $s \in \mathbf{C} \setminus \mathbf{Z}$. Replacing s by $s-1$ and using lemma 1.2.1 we get

$$\zeta_e(s)\Gamma(s) = \frac{1}{2i \sin \pi(s-1)} \int_{L(R)} \frac{z^{s-1}}{1-e^{-z}} dz = \frac{1}{2i \sin \pi s} \int_{L(R)} \frac{z^{s-1}e^z}{1-e^z} dz$$

which holds for $s \in \mathbf{C} \setminus \mathbf{Z}, \Re(s) > 0$. However, from the reflection formula [14, p.58–59]

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad s \in \mathbf{C} \setminus \mathbf{Z},$$

we obtain

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{L(R)} \frac{z^{s-1}e^z}{1-e^z} dz$$

for all $s \in \mathbf{C} \setminus \mathbf{Z}$. The integral is convergent for all $s \in \mathbf{C}$ and thus the only possible singularities are at the poles of $\Gamma(1-s)$: $s = 1, 2, ...$ From lemma 1.1.2 we already know that $\zeta_e(s)$ is holomorphic in $2, 3, ...$ and from the uniqueness of analytic continuation follows that $\zeta(s)$ is holomorphic at these points. Hence $s = 1$ is a simple pole and

$$\int_{L(R)} \frac{e^z}{1-e^z} dz = \int_{L(R)} \frac{1}{e^{-z}-1} dz = \int_{\widetilde{L}(R)} \frac{1}{e^z-1} dz = 2\pi i,$$

where $\widetilde{L}(R)$ is a curve which consists of the segment $[+\infty, R]$ on the upper side of the real axis then $C(0, R)$ and the segment $[R, +\infty]$ on the lower side of the real axis. Further,

$$\Gamma(s-1) = -\frac{1}{s-1} + ...$$

and therefore the residue at the pole $s = 1$ is 1. □

1.3 Definition of the Riemann zeta function

Definition 1.3.1. The analytic continuation ζ of the Euler zeta function ζ_e to $\mathbf{C} \setminus \{1\}$ from theorem 1.2.4 is called *the Riemann zeta function.*

The method we used to analytically continue ζ_e is one of Riemann's original methods [24]. Other methods can be found in [28, p.13–27].

2. Number-theoretic analysis

2.1 Prime theory relations

In this section we shall provide basic results relating the Riemann zeta function to the prime numbers. We shall start with the reminding of the Fundamental theorem of arithmetic. For detailed proof and further references see [11].

Theorem 2.1.1 (Fundamental theorem of arithmetic)**.** *Every $n \in \boldsymbol{N} \setminus \{1\}$ can be represented in exactly one way apart from rearrangement as a product of one or more primes.*

Now we shall prove the Euler product formula [9].

Theorem 2.1.2 (Euler product formula)**.** *For any given $s \in \boldsymbol{C}, \Re(s) > 1$ the following formula*

$$\zeta(s) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-s}} \tag{2.1}$$

where p_n denotes the n-th prime number, holds.

Proof. Let us consider the series

$$\sum_{n=1}^{\infty} p_n^{-s}, \quad s \in \mathbf{C}, \Re(s) > 1.$$

This, being merely a selection of the series

$$\sum_{n=1}^{\infty} n^{-s}, \quad s \in \mathbf{C}, \Re(s) > 1,$$

is for any given $\epsilon \in \mathbf{R}, \epsilon > 0$ absolutely and uniformly convergent in every half-plane $G_\epsilon := \{s \in \mathbf{C}; \Re(s) \geq 1 + \epsilon\}$ by lemma 1.1.2. We would like to show that the product

$$\prod_{n=1}^{\infty} (1 - p_n^{-s}), \quad s \in \mathbf{C}, \Re(s) > 1 \tag{2.2}$$

is absolutely and uniformely convergent in every $G_\epsilon, \epsilon \in \mathbf{R}, \epsilon > 0$. The absolute convergence follows from the uniform convergence of

$$\left|p_1^{-s}\right| + \left|p_2^{-s}\right| + \ldots + \left|p_n^{-s}\right| + \ldots \tag{2.3}$$

in every G_ϵ. Let us further denote by M the upper bound of the sum of series (2.3) for $s \in \mathbf{C}, \Re(s) \geq 1 + \epsilon$. We take the partial sum of (2.3) to obtain

$$\left|\sum_{n=1}^{N} \left(1 - p_n^{-s}\right)\right| \leq \prod_{n=1}^{N} \left(1 + \left|p_n^{-s}\right|\right) \leq e^{\sum_{n=1}^{N} \left|p_n^{-s}\right|} \leq e^M.$$

Consequently

$$\begin{aligned}
\left|\prod_{n=1}^{N} \left(1 - p_n^{-s}\right)\right| &\leq \left|1 - p_1^{-s}\right| + \sum_{k=2}^{N} \left|\prod_{j=1}^{k} \left(1 - p_j^{-s}\right) - \prod_{j=1}^{k-1} \left(1 - p_j^{-s}\right)\right| \\
&= \left|1 - p_1^{-s}\right| + \sum_{k=2}^{N} \left|\prod_{j=1}^{k-1} \left(1 - p_j^{-s}\right)\right| \left|p_k^{-s}\right| \\
&\leq \left|1 - p_1^{-s}\right| + e^M \sum_{k=2}^{N} \left|p_k^{-s}\right|.
\end{aligned}$$

Thus, the product (2.2) is uniformely convergent in every $G_\epsilon, \epsilon \in \mathbf{R}, \epsilon > 0$ and therefore represents a function holomorphic there. Let us take a finite number $N \in \mathbf{N}$ of factors $|p_n^{-s}|$. Then after multiplying a finite number of absolutely convergent series, we obtain

$$\prod_{n\leq N}\left(1+\frac{1}{p_n^s}+\frac{1}{p_n^{2s}}+\dots\right)=1+\frac{1}{n_1^s}+\frac{1}{n_2^s}+\dots,$$

where $n_1, n_2, \dots$ are integers none of whose prime factors exceed N. Since both sides of 2.1 are holomorphic in $s \in \mathbf{C}, \Re(s) > 1$ it is sufficient to prove this formula for $s \in \mathbf{R}, s > 1$. It therefore follows

$$0\leq\left|\zeta(s)-\prod_{p_n\leq N}\frac{1}{1-p_n^{-s}}\right|=\left|\zeta(s)-\left(1+\frac{1}{n_1^s}+\frac{1}{n_2^s}+\dots\right)\right|\leq\sum_{j=N+1}^{\infty}\frac{1}{j^s}$$

by theorem 2.1.1. The last term tends to 0 as $N \to \infty$. □

Euler's original proof employs the sieve of Eratosthenes and can be found in [5, p.99–105]. This formula plays a major role in the prime number theory since it relates all natural numbers and all primes. For further information and references see [5] or [6].

2.2 Irrationality of the zeta values at even positive integers

We shall start with the definition of Bernoulli numbers.

Definition 2.2.1. Consider a series $\{B_k\}_{k=0}^{\infty} \subset \mathbf{R}$ defined by recursion:

$$\begin{aligned} B_0 &:= 1 \\ B_k &:= -\frac{1}{k+1}\sum_{j=0}^{k-1}\binom{k+1}{j}B_j, \quad k=1,2,\dots. \end{aligned}$$

We call B_k *the Bernoulli numbers.*

We shall now express the values of ζ at even positive integers in terms of the Bernoulli numbers.

Theorem 2.2.2. *For every $k \in \mathbf{N}$ the following Euler's formula*

$$\zeta(2k)=\frac{(-1)^{k-1}\cdot 2^{2k-1}B_{2k}}{(2k)!}\cdot\pi^{2k},$$

where B_k are the Bernoulli numbers from the definition 2.2.1, holds.

Proof. Let us start from the formula, which can be obtained by the use of Fourier series,[1]

$$\frac{t}{e^t-1}+\frac{t}{2}-1=\sum_{n=1}^{\infty}\frac{2t^2}{t^2+4\pi^2n^2},\quad t\in\mathbf{R}\setminus\{0\}. \tag{2.4}$$

[1]For $x \in [-\pi, \pi], \alpha \neq 0$ we have after using standard Fourier series

$$\cosh\alpha x=\frac{\sinh\alpha\pi}{\alpha\pi}\left(1+\sum_{n=1}^{\infty}(-1)^n\frac{2\alpha^2}{\alpha^2+n^2}\cos nx\right)\quad\text{now}$$

For $t = 0$ the formula (2.4) is understood as the limit $t \to 0$. Let us define the function $f : \mathbf{C} \setminus \{2\pi k i, k \in \mathbf{Z}\} \to \mathbf{C}$ by $f(z) := \frac{z}{e^z - 1}$. Then there exists the Taylor series of f around the point $z = 0$, which is a removable singularity, with a radius of convergence 2π. Thus there exist $a_k \in \mathbf{R}$ such that

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k, \quad |t| < 2\pi, t \in \mathbf{R}.$$

Apparently for every $t \in \mathbf{R}$ (instead of $t = 0$ we consider the limit $t \to 0$)

$$\frac{e^t - 1}{t} = \frac{1}{t} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} - 1 \right) = \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!}$$

and therefore

$$1 = \frac{e^t - 1}{t} \cdot \frac{t}{e^t - 1} = \left(\sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \right) \cdot \left(\sum_{k=0}^{\infty} \frac{t^k}{(k+1)!} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_k}{k!(n-k+1)!} t^n$$

holds for every $|t| < 2\pi, t \in \mathbf{R}$. Thus, after comparing both sides, we obtain

$$\begin{aligned} B_0 &:= 1 \\ B_k &:= -\frac{1}{k+1} \sum_{j=0}^{k-1} \binom{k+1}{j} B_j, \quad k = 1, 2, ... \end{aligned}$$

Since (2.4) defines an even function for every $t \in \mathbf{R}$ and therefore its odd derivatives at $t = 0$ are zero, we get $B_{2k+1} = 0$ for every $k \in \mathbf{N}$. Hence

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k}, \quad |t| < 2\pi, t \in \mathbf{R}$$

and

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k} = \sum_{n=1}^{\infty} \frac{2t^2}{t^2 + 4\pi^2 n^2}, \quad |t| < 2\pi, t \in \mathbf{R}.$$

For fixed $n \in \mathbf{N}$ and $|t| < 2\pi, t \in \mathbf{R}$ we further have

$$\frac{2t^2}{t^2 + 4\pi^2 n^2} = 2 \frac{\left(\frac{t}{2\pi n}\right)^2}{1 + \left(\frac{t}{2\pi n}\right)^2} = 2 \left(\frac{t}{2\pi n}\right)^2 \sum_{k=0}^{\infty} (-1)^k \left(\frac{t}{2\pi n}\right)^{2k} = 2 \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{t}{2\pi n}\right)^{2k}$$

---

put $x = \pi$. Then we get

$$\alpha\pi \coth \alpha\pi - 1 = \sum_{n=1}^{\infty} \frac{2\alpha^2}{\alpha^2 + n^2}, \quad \alpha \neq 0.$$

The left side goes to 0 as $\alpha \to 0$ and therefore in this sence the formula is valid for all $\alpha \in \mathbf{R}$. Here we put $\alpha\pi = \frac{t}{2}$ to obtain

$$\coth \frac{t}{2} = \frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{e^t + 1}{e^t - 1} = \frac{2}{e^t - 1} + 1, \quad t \in \mathbf{R}$$

and

$$\frac{t}{e^t - 1} + \frac{t}{2} - 1 = \sum_{n=1}^{\infty} \frac{2t^2}{t^2 + 4\pi^2 n^2}, \quad t \in \mathbf{R}$$

for $t = 0$ in the sence of limit.
See `http://www.karlin.mff.cuni.cz/~rokyta/vyuka/general/tahaky/zeta_2n.pdf`.

and thus
$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k} = 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2\pi n)^{2k}} t^{2k}, \quad |t| < 2\pi, t \in \mathbf{R}. \tag{2.5}$$
The infinite series on the right side of (2.5) converges absolutely for $|t| < 2\pi, t \in \mathbf{R}$ since
$$2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\frac{|t|}{2\pi n} \right)^{2k} = \sum_{n=1}^{\infty} \frac{2t^2}{4\pi^2 n^2 - t^2} < \infty, \quad |t| < 2\pi, t \in \mathbf{R}.$$
Therefore we can change the order of summing and get
$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k} = \sum_{k=1}^{\infty} \left(\frac{(-1)^{k-1}}{2^{2k-1}\pi^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \right) t^{2k}, \quad |t| < 2\pi, t \in \mathbf{R}$$
which yields the formula we were seeking[2]. □

Corollary 2.2.3. *The values of the Riemann zeta function at even positive integers are transcendental.*

Proof. Clearly for $n \in \mathbf{N}$ is $B_n \in \mathbf{Q}$. Taking into account the expression of $\zeta(2k)$ for $k \in \mathbf{N}$ in the theorem 2.2.2 and the fact that π is transcendental [20] we can see that for $k \in \mathbf{N}$ $\zeta(2k)$ is transcendental. □

2.3 Apéry's theorem

In this section we shall prove Apéry's theorem [1]. We shall follow the reasoning in [23]. Let us start with an easy observation.

Remark 2.3.1. Let $x \in \mathbf{R} \setminus \{0\}$ and for $k \in \mathbf{N}$ let $a_1, ..., a_k \in \mathbf{R}$. Denoting
$$\begin{aligned} A_0 &:= \frac{1}{x} \\ A_k &:= \frac{a_1 \dots a_k}{x(x+a_1)\dots(x+a_k)} \end{aligned}$$
we obtain
$$\frac{1}{x} - \frac{a_1 \dots a_K}{x(x+a_1)\dots(x+a_K)} = A_0 - A_K = \sum_{k=1}^{K} (A_{k-1} - A_k) = \sum_{k=1}^{K} \frac{a_1 \dots a_{k-1}}{(x+a_1)\dots(x+a_k)}$$
for $K \in \mathbf{N}$.

Applying this observation, let us introduce a useful expression for $\zeta(3)$.

[2] There is also a Ramanujan's formula which presents an analogue to the Euler's formula in theorem 2.2.2 for $\zeta(2k+1)$. For $\alpha\beta = \pi^2$ and any positive integer n:
$$\alpha^{-n} \left[\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \left(\frac{1}{e^{2k\alpha} - 1} \right) + \frac{1}{2}\zeta(2n+1) \right] - (-\beta)^{-n} \left[\sum_{k=1}^{\infty} \frac{1}{k^{2n-1}} \left(\frac{1}{e^{2k\beta} - 1} \right) + \frac{1}{2}\zeta(2n-1) \right] =$$
$$= 2^{2n} \sum_{j=0}^{n+1} (-1)^{j+1} \frac{B_{2j}}{(2j)!} \frac{B_{2n+2-2j}}{(2n+2-2j)!} \alpha^j \beta^{n-1-j}.$$
For detailed proof and further reference see [21] or [2].

Lemma 2.3.2. [3]

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}.$$

Proof. Putting $x := n^2$, $a_k := -k^2$ in remark 2.3.1 and taking $K = n-1$ we obtain

$$\sum_{k=1}^{n-1} \frac{(-1)^{k-1}(k-1)!^2}{(n^2-1^2)\dots(n^2-k^2)} =$$

$$= \frac{1}{n^2} - \frac{(-1)^{n-1}(n-1)!^2}{n^2(n^2-1^2)\dots(n^2-(n-1)^2)}$$

$$= \frac{1}{n^2} - \frac{(-1)^{n-1}(n-1)!^2}{n^2(n-1)(n+1)\dots(n-(n-1))(n+(n-1))}$$

$$= \frac{1}{n^2} - \frac{2(-1)^{n-1}}{n^2 \frac{(2n)!}{(2n-n)!n!}} = \frac{1}{n^2} - \frac{2(-1)^{n-1}}{n^2\binom{2n}{n}}.$$

Let us consider the coefficient

$$e_{n,k} := \frac{1}{2k^3} \cdot \frac{1}{\binom{n+k}{k}\binom{n}{k}} = \frac{k!^2(n-k)!}{2k^3(n+k)!}$$

and since

$$\begin{aligned}(-1)^k n(e_{n,k} - e_{n-1,k}) &= \frac{(-1)^k n k!^2}{2k^3} \cdot \frac{(n-k-1)!}{(n+k)!}(-2k) \\ &= \frac{(-1)^{k-1}k!^2 n(n-k-1)!}{k^2(n+k)!} \\ &= \frac{(-1)^{k-1}n(k-1)!^2}{n(n^2-1^2)\dots(n^2-k^2)} \\ &= \frac{(-1)^{k-1}(k-1)!^2}{(n^2-1^2)\dots(n^2-k^2)}\end{aligned}$$

we obtain

$$\sum_{k=1}^{n-1}(-1)^k n(e_{n,k} - e_{n-1,k}) = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}(k-1)!^2}{(n^2-1^2)\dots(n^2-k^2)} = \frac{1}{n^2} - \frac{2(-1)^{n-1}}{n^2\binom{2n}{n}}$$

and by dividing both sides by n we get

$$\sum_{k=1}^{n-1}(-1)^k(e_{n,k} - e_{n-1,k}) = \frac{1}{n^3} - \frac{2(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

[3]Similarly it can be shown that $\zeta(2) = 3\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}}$. One may indeed assert that

$$\zeta(k) = C \cdot \sum_{n=1}^{\infty} \frac{(-1)^{k(n-1)}}{n^k\binom{2n}{n}}$$

for $k = 2, 3, \dots$ where $C \in \mathbf{Q}$, however, whether it is so remains yet unknown [23].

By summing both sides over $1 \leq n \leq N, N \in \mathbf{N}$ we obtain

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n^3} - 2\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}} &= \sum_{n=1}^{N}\sum_{k=1}^{n-1}(-1)^k(e_{n,k} - e_{n-1,k}) \\
&= \sum_{k=1}^{N-1}(-1)^k(e_{N,k} - e_{k,k}) \\
&= \sum_{k=1}^{N-1} \frac{(-1)^k}{2k^3\binom{N+k}{k}\binom{N}{k}} + \frac{1}{2}\sum_{k=1}^{N-1} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}}.
\end{aligned}$$

It thus follows that

$$\sum_{n=1}^{N} \frac{1}{n^3} = \frac{5}{2}\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}} - \frac{1}{2}\frac{(-1)^{N-1}}{N^3\binom{2N}{N}} + \sum_{k=1}^{N-1} \frac{(-1)^k}{2k^3\binom{N+k}{k}\binom{N}{k}}. \tag{2.6}$$

As we can see the limit $N \to \infty$ of (2.6) yields the assertion, however, it only remains to show that

$$\lim_{N\to\infty} \sum_{k=1}^{N-1} \frac{(-1)^k}{2k^3\binom{N+k}{k}\binom{N}{k}} = 0 \tag{2.7}$$

and

$$\lim_{N\to\infty} \frac{(-1)^{N-1}}{2N^3\binom{2N}{N}} = 0. \tag{2.8}$$

First we shall prove (2.7). For fixed $N \in \mathbf{N}$ and $k = 1, \ldots, N-1$ let us denote

$$A_{N,k} := \frac{k!^2(N-k)!}{2(N+k)!} = \frac{1}{2\binom{N+k}{k}\binom{N}{k}}.$$

It follows that

$$|A_{N,k}| = \frac{1}{2\binom{N+k}{k}\binom{N}{k}} \leq \frac{1}{2N}$$

since

$$\binom{N+k}{k} \geq 1, \quad \binom{N}{k} \geq N, \quad N \in \mathbf{N}, k = 1, \ldots, N-1.$$

Thus we obtain

$$\left|\sum_{k=1}^{N-1} A_{N,k}\frac{(-1)^k}{k^3}\right| \leq \sum_{k=1}^{N-1} \frac{1}{k^3}|A_{N,k}| \leq \sum_{k=1}^{N-1} \frac{1}{k^3} \cdot \frac{1}{2N} < \frac{1}{2N} \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^3}}_{\zeta(3)}$$

which goes to 0 as $N \to \infty$. (2.8) follows quite easily by

$$\left|\frac{(-1)^{N-1}}{2N^3\binom{2N}{N}}\right| = \frac{1}{2N^3\binom{2N}{N}} \leq \frac{1}{2N^3},$$

since $\binom{2N}{N} \geq 1$ for $N \in \mathbf{N}$. This goes to 0 as $N \to \infty$. $\square$

At this point let us introduce Dirichlet's criterion for irrationality which we would like to use to prove the irrationality of $\zeta(3)$.

Theorem 2.3.3 (Criterion for irrationality). *Let $\beta \in \boldsymbol{R}$. Then β is irrational if there exist $\delta \in \boldsymbol{R}, \delta > 0$ and a sequence $\left\{\frac{p_n}{q_n}\right\}_{n=1}^{\infty}$ of rational numbers such that for all $n \in \boldsymbol{N}$: $\frac{p_n}{q_n} \neq \beta$ for which*

$$\left|\beta - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}, \quad n \in \boldsymbol{N}.$$

The expression in lemma 2.3.2, however, does not imply the irrationality of $\zeta(3)$. To see this we shall prove the following.

Lemma 2.3.4. *For $n \in \boldsymbol{N}$ and $1 \leq k \leq n$ denote*

$$c_{n,k} := \sum_{j=1}^{n} \frac{1}{j^3} + \sum_{j=1}^{k} \frac{(-1)^{j-1}}{2j^3\binom{n}{j}\binom{n+j}{j}}.$$

Then

$$2c_{n,k}[1,\dots,n]^3\binom{n+k}{k} \in \boldsymbol{Z}, \quad 1 \leq k \leq n,$$

where $[1,\dots,n]$ denotes the least common multiple of numbers $1,2,\dots,n$.

Proof. Let p be a prime. Then for $n \in \mathbf{N}$ let us introduce the symbol $\deg_p n$ which we shall use for the number of times that p divides n. Then we have

$$\deg_p[1,\dots,n] = \left[\frac{\ln n}{\ln p}\right]$$

since $\deg_p[1,\dots,n] =: c \in \mathbf{N}_0$ is the greatest number such that

$$p^c \leq n.$$

By the properties of the deg function (see [23, p.6]) we have for $m \in \mathbf{N}_0$, $m \leq n$

$$\deg_p \binom{n}{m} \leq \left[\frac{\ln n}{\ln p}\right] - \deg_p m.$$

We want to show that

$$\deg_p\left(m^3\frac{\binom{m+n}{m}\binom{n}{m}}{\binom{n+k}{k}}\right) \leq \deg_p[1,\dots,n]^3$$

since

$$2[1,\dots,n]^3\binom{n+k}{k}c_{n,k} = 2[1,\dots,n]^3\frac{\sum_{m=1}^{n} a_m}{[1,\dots,n]^3} + [1,\dots,n]^3\sum_{m=1}^{k}\frac{(-1)^{m-1}}{\frac{m^3\binom{n}{m}\binom{m+n}{m}}{\binom{n+k}{k}}},$$

where $a_m \in \mathbf{Z}$ for $m = 1,\dots,n$. Since

$$\binom{n+k}{k}\binom{k}{m} = \binom{n+k}{k-m}\binom{n+m}{m}$$

we obtain

$$
\begin{aligned}
\deg_p\left(\frac{m^3\binom{n}{m}\binom{m+n}{m}}{\binom{n+k}{k}}\right) &= \deg_p\left(\frac{m^3\binom{n}{m}\binom{k}{m}}{\binom{n+k}{k-m}}\right)\\
&\le 3\deg_p m + \left[\frac{\ln n}{\ln p}\right] + \left[\frac{\ln k}{\ln p}\right] - 2\deg_p m\\
&= \deg_p m + \deg_p[1,\dots,n] + \deg_p[1,\dots,k]\\
&\le 3\deg_p[1,\dots,n]
\end{aligned}
$$

since $m \le k \le n$. □

More consise proof of lemma 2.3.4 can be found in [23, p.6]. Our aim is to apply theorem 2.3.3. However, according to lemma 2.3.4, the convergence of the quantities $c_{n,k}$ for $k = n$ needs to be accelerated.

Definition 2.3.5. For $k, n \in \mathbf{N}_0, k \le n$ let us consider two sequences

$$
\begin{aligned}
a_{n,k}^{(0)} &:= \binom{n+k}{k} c_{n,k},\\
b_{n,k}^{(0)} &:= \binom{n+k}{k}.
\end{aligned}
$$

Apparently,

$$
\frac{a_{n,k}^{(0)}}{b_{n,k}^{(0)}} = c_{n,k} \to \zeta(3), \quad n \to \infty
$$

uniformely in k. In order to accelerate the convergence of $c_{n,k}$ for $k = n$ we shall transform these sequences as follows:

$$
\begin{aligned}
a_{n,k}^{(0)} &\mapsto \binom{2n-k}{n} c_{n,n-k}\\
&\mapsto \binom{n}{k}\binom{2n-k}{n} c_{n,n-k}\\
&\mapsto \sum_{k_1=0}^{k}\binom{k}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n} c_{n,n-k_1}\\
&\mapsto \binom{n}{k}\sum_{k_1=0}^{k}\binom{k}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n} c_{n,n-k_1}\\
&\mapsto \sum_{k_2=0}^{k}\binom{k}{k_2}\binom{n}{k_2}\sum_{k_1=0}^{k_2}\binom{k_2}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n} c_{n,n-k_1},
\end{aligned}
$$

denoting the last term $a_{n,k}^{(1)}$ and in the very same manner we shall transform $b_{n,k}^{(0)}$.

$$
\begin{aligned}
b_{n,k}^{(0)} &\mapsto \binom{2n-k}{n}\\
&\mapsto \binom{n}{k}\binom{2n-k}{n}
\end{aligned}
$$

$$\mapsto \sum_{k_1=0}^{k} \binom{k}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n}$$

$$\mapsto \binom{n}{k}\sum_{k_1=0}^{k} \binom{k}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n}$$

$$\mapsto \sum_{k_2=0}^{k} \binom{k}{k_2}\binom{n}{k_2}\sum_{k_1=0}^{k_2} \binom{k_2}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n}$$

Here we denote the last term $b_{n,k}^{(1)}$. Since we applied the transformations to both sequences the ratio of corresponding elements retains the property that

$$\frac{a_{n,k}^{(1)}}{b_{n,k}^{(1)}} \longrightarrow \zeta(3), \quad n \to \infty$$

uniformly in k. For $k = n$ we shall denote the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$.

Lemma 2.3.6. *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be the sequences from definition 2.3.5. Consider the recursion*

$$n^3u_n + (n-1)^3u_{n-2} = (34n^3 - 51n^3 + 27n - 5)u_{n-1}, \quad n \geq 2. \tag{2.9}$$

Then $a_0 = 0, a_1 = 6, a_n = u_n, n \geq 2$ and $b_0 = 1, b_1 = 5, b_n = u_n, n \geq 2$. Moreover, for all $n \in \boldsymbol{N}_0 : b_n \in \boldsymbol{Z}$ and for all $n \in \boldsymbol{N}_0$ are a_n rational numbers with denominator dividing $2[1,\ldots,n]^3$.

Proof. Notice that for $n \in \mathbf{N}$ we have

$$\begin{aligned}
\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}^2\binom{n}{j}\binom{k}{j}\binom{2n-j}{n} &= \\
&= \sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{k}^2\binom{n}{j}\binom{k}{j}\binom{2n-j}{n} \\
&= \sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n}{j}\binom{k}{j}\binom{2n-j}{n} \\
&= \sum_{j=0}^{n}\binom{n}{j}\binom{2n-j}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{k}{j} \\
&= \sum_{j=0}^{n}\binom{n}{j}\binom{2n-j}{n}\left(\sum_{k=0}^{j-1}\binom{n}{k}^2\binom{k}{l} + \sum_{k=j}^{n}\binom{n}{k}^2\binom{k}{j}\right)
\end{aligned}$$

since $j \leq n$ we have

$$\sum_{k=j}^{n}\binom{n}{k}^2\binom{k}{j} = \sum_{k=j}^{n}\binom{n}{j}\binom{n-j}{k-j}\binom{n}{k} = \binom{n}{j}\sum_{k-j}^{n}\binom{n-j}{n-k}\binom{n}{k}$$

and

$$\sum_{k=0}^{j-1}\binom{n}{k}^2\binom{k}{j} = 0 = \binom{n}{j}\sum_{k=0}^{j-1}\binom{n-j}{n-k}\binom{n}{k}. \tag{2.10}$$

The left side of (2.10) is 0 since $k < j$ and by the convetion $\binom{k}{j} = 0$ and the right side is 0 since $k < j$ and thus $\binom{n-j}{n-k} = 0$. Therefore we get

$$\sum_{j=0}^{n} \binom{n}{j} \binom{2n-j}{n} \left(\sum_{k=0}^{j-1} \binom{n}{k}^2 \binom{k}{l} + \sum_{k=j}^{n} \binom{n}{k}^2 \binom{k}{j} \right) =$$

$$\begin{aligned}
&= \sum_{j=0}^{n} \binom{n}{j} \binom{2n-j}{n} \left(\binom{n}{j} \sum_{k=0}^{j-1} \binom{n-j}{n-k} \binom{n}{k} + \binom{n}{l} \sum_{k=j}^{n} \binom{n-j}{n-k} \binom{n}{k} \right) \\
&= \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-j}{n} \sum_{k=0}^{n} \binom{n-j}{n-k} \binom{n}{k} \\
&= \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-j}{n}^2
\end{aligned}$$

due to the identity

$$\binom{n-j}{k-j} \binom{n}{j} = \binom{n}{k} \binom{k}{j}$$

and Vandermonde's convolution

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n-j}{k-j} = \binom{2n-j}{n}.$$

Let us consider $n-k$ instead of k. This relates $a_{n,k}^{(1)}$ and $b_{n,k}^{(1)}$ to the following. Let us further denote

$$\begin{aligned}
b_{n,k}^{(2)} &:= \binom{n}{k}^2 \binom{n+k}{k}^2, \\
a_{n,k}^{(2)} &:= b_{n,k}^{(2)} c_{n,k}.
\end{aligned}$$

For $k > n$ or $k < 0$ are $b_{n,k}^{(2)}$ and $a_{n,k}^{(2)}$ defined by 0. Noting that

$$a_n = \sum_{k=0}^{n} b_{n,k}^{(2)} c_{n,k}, \qquad b_n = \sum_{k=0}^{n} b_{n,k}^{(2)}$$

we want to show

$$\sum_{k=0}^{n} \left[(n+1)^3 b_{n+1,k}^{(2)} - (34n^3 + 51n^2 + 27n + 5) b_{n,k}^{(2)} + n^3 b_{n-1,k}^{(2)} \right] = 0, \quad n \geq 1$$

which is equation (2.9) written for $(n+1)$ instead of n. We shall further denote

$$B_{n,k} := \begin{cases} 4(2n+1)\left(k(2k+1) - (2n+1)^2\right) \binom{n}{k}^2 \binom{n+k}{k}^2, & 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

For such $B_{n,k}$ we have[4]

$$B_{n,k} - B_{n,k-1} = (n+1)^3 \binom{n+1}{k}^2 \binom{n+1+k}{k}^2$$

[4]For detailed proof see lemma 4.0.8.

$$
\begin{aligned}
&- \quad (34n^3+51n^2+27n+5)\binom{n}{k}^2\binom{n+k}{k}^2 \\
&+ \quad n^3\binom{n-1}{k}^2\binom{n-1+k}{k}^2
\end{aligned}
$$

which can be written as

$$
B_{n,k} - B_{n,k-1} = (n+1)^3 b^{(2)}_{n+1,k} - (34n^3+51n^2+27n+5)b^{(2)}_{n,k} + n^3 b^{(2)}_{n-1,k}. \quad (2.11)
$$

Denoting

$$
P(n) = 34n^3+51n^2+27n+5
$$

and by summing both sides of (2.11) for $0 \le k \le n$ we obtain

$$
B_{n,n} = (n+1)^3 b_{n+1} - P(n)b_n + n^3 b_{n-1} - (n+1)^3 b^{(2)}_{n+1,n+1} + n^3 b^{(2)}_{n-1,n}. \quad (2.12)
$$

From the definition of $B_{n,k}$ we have

$$
B_{n,n} = -4(2n+1)^2(n+1)\binom{2n}{n}^2
$$

and from the definition of $b^{(2)}_{n,k}$ we have

$$
-(n+1)^3 b^{(2)}_{n+1,n+1} + n^3 b^{(2)}_{n-1,n} = -4(2n+1)^2(n+1)\binom{2n}{n}^2.
$$

Thus from (2.12) it follows that

$$
(n+1)^3 b_{n+1} - P(n)b_n + n^3 b_{n-1} = 0, \quad n \ge 1.
$$

Therefore the sequence $\{b_n\}_{n=0}^{\infty}$ satisfies recurrence (2.9). To prove that the sequence $\{a_n\}_{n=0}^{\infty}$ satisfies (2.9) we notice that

$$
\begin{aligned}
&(n+1)^3 b^{(2)}_{n+1,k} c_{n+1,k} - P(n) b^{(2)}_{n,k} c_{n,k} + n^3 b^{(2)}_{n-1,k} c_{n-1,k} \\
&\qquad = (B_{n,k} - B_{n,k-1})c_{n,k} + (n+1)^3 b^{(2)}_{n+1,k}(c_{n+1,k} - c_{n,k}) \\
&\qquad\qquad - n^3 b^{(2)}_{n-1,k}(c_{n,k} - c_{n-1,k}). \quad (2.13)
\end{aligned}
$$

Denoting

$$
A_{n,k} := \begin{cases} B_{n,k}c_{n,k} + \frac{5(2n+1)(-1)^{k-1}k}{n(n+1)}\binom{n}{k}\binom{n+k}{k}, & 0 \le k \le n, \\ 0, & \text{otherwise} \end{cases}
$$

we deduce that (2.13) becomes $A_{n,k} - A_{n,k-1}$[5], precisely

$$
(n+1)^3 b^{(2)}_{n+1,k} c_{n+1,k} - P(n) b^{(2)}_{n,k} c_{n,k} + n^3 b^{(2)}_{n-1,k} c_{n-1,k} = A_{n,k} - A_{n,k-1}. \quad (2.14)
$$

By summing both sides of (2.14) for $0 \le k \le n$ we obtain

$$
(n+1)^3\left(a_{n+1} - b^{(2)}_{n+1,n+1}c_{n+1,n+1}\right) - P(n)a_n + n^3\left(a_{n-1} + b^{(2)}_{n-1,n}c_{n-1,n}\right) = A_{n,n}.
$$

[5] For detailed proof see lemma 4.0.10.

Since[6]

$$A_{n,n} = B_{n,n}c_{n,n} + \frac{5(2n+1)(-1)^{n-1}}{n+1}\binom{2n}{n} = -(n+1)^3 b^{(2)}_{n+1,n+1}c_{n+1,n+1} \quad (2.15)$$

we obtain

$$(n+1)^3 a_{n+1} - P(n)a_n + n^3 a_{n-1} = 0, \quad n \geq 1$$

and thus both the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ satisfy recurrence (2.9). Moreover, considering the construction of $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ and taking into account lemma 2.3.4 we obtain that for all $n \in \mathbf{N} : b_n \in \mathbf{Z}$ and for all $n \in \mathbf{N}$ are a_n rational numbers with denominator dividing $2[1, \ldots, n]^3$. □

Theorem 2.3.7 (Apéry's theorem). *$\zeta(3)$ is irrational.*

Proof. First we have

$$P(n-1) = 34n^3 - 51n^2 + 27n - 5, \quad n \in \mathbf{N}.$$

From lemma 2.3.6 it follows that

$$n^3 a_n - P(n-1)a_{n-1} + (n-1)^3 a_{n-2} = 0, \quad n \geq 2,$$

$$n^3 b_n - P(n-1)b_{n-1} + (n-1)^3 b_{n-2} = 0, \quad n \geq 2.$$

After multiplying the first equation by b_{n-1} and the second by a_{n-1} and subtracting them, we obtain

$$n^3(a_n b_{n-1} - a_{n-1}b_n) = (n-1)^3(a_{n-1}b_{n-2} - a_{n-2}b_{n-1}), \quad n \geq 2. \quad (2.16)$$

Since $a_0 = 0, a_1 = 6, b_0 = 1, b_1 = 5$ we get

$$a_1 b_0 - a_0 b_1 = 6$$

and from equation (2.16) it follows by induction that

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3}, \quad n \in \mathbf{N}.$$

Let us write

$$\zeta(3) - \frac{a_n}{b_n} =: x_n, \quad n \in \mathbf{N}.$$

Then

$$x_n - x_{n-1} = \frac{a_{n-1}}{b_{n-1}} - \frac{a_n}{b_n} = \frac{a_{n-1}b_n - a_n b_{n-1}}{b_n b_{n-1}} = \frac{-6}{n^3 b_n b_{n-1}}$$

with $x_\infty := \lim_{n\to\infty} x_n = 0$. Therefore we get

$$\left|\zeta(3) - \frac{a_n}{b_n}\right| = |x_n| = \left|\sum_{j=n+1}^{\infty}(x_j - x_{j-1})\right| = \sum_{j=n+1}^{\infty}\frac{6}{j^3 b_j b_{j-1}}$$

to obtain

$$\zeta(3) - \frac{a_n}{b_n} = O\left(b_n^{-2}\right), \quad n \to \infty. \quad (2.17)$$

[6] For detailed proof see lemma 4.0.11.

Using formula (2.9) for $\{u_n\}_{n=0}^{\infty}$ we can estimate the asymptotic behaviour of $\{b_n\}_{n=0}^{\infty}$. Apparently, we have

$$b_n - \left(34 - \frac{51}{n} + \frac{27}{n^2} - \frac{5}{n^3}\right) b_{n-1} + \left(1 - \frac{3}{n} + \frac{3}{n^2} - \frac{1}{n^3}\right) b_{n-2} = 0, \quad n \geq 2.$$

Thus it is sufficient to consider the recursion

$$b_n - 34b_{n-1} + b_{n-2} = 0, \quad n \geq 2.$$

Since the characteristic polynom $\lambda^2 - 34\lambda + 1$ has zeros

$$\lambda_{1,2} = 17 \pm 12\sqrt{2} = (1 \pm \sqrt{2})^4$$

we obtain

$$b_n = c_1(1+\sqrt{2})^{4n} + c_2(1-\sqrt{2})^{4n}, \quad c_1, c_2 \in \mathbf{R}$$

and thus

$$b_n = O(\alpha^n), \quad n \to \infty,$$

where $\alpha = (1+\sqrt{2})^4$. From lemma 2.3.4 it follows that a_n are not integers but if we take

$$\begin{aligned} p_n &:= 2[1,\ldots,n]^3 a_n, \\ q_n &:= 2[1,\ldots,n]^3 b_n, \quad n \in \mathbf{N}_0 \end{aligned}$$

we obtain $p_n, q_n \in \mathbf{Z}$. Since, by the Prime number theorem [4], [10],

$$[1,\ldots,n] = \prod_{p \leq n} p^{\frac{\ln n}{\ln p}} \leq \prod_{p \leq n} n \approx n^{\frac{n}{\ln n}} = e^n, \quad n \in \mathbf{N}$$

where p is a prime, we have $q_n = O\left(\alpha^n e^{3n}\right), n \to \infty$. Thus

$$\zeta(3) - \frac{p_n}{q_n} = O\left(\alpha^{-2n}\right) = O\left(q_n^{-(1+\delta)}\right), \quad n \to \infty$$

where

$$\delta = \frac{\ln\alpha - 3}{\ln\alpha + 3} > 0,$$

follows from

$$\frac{\alpha^{-2n}}{q_n^{-(1+\delta)}} = \frac{\alpha^{-2n}}{\left(\alpha^n e^{3n}\right)^{-(1+\delta)}} = \left(\alpha^{-1+\delta} e^{3(1+\delta)}\right)^n, \quad n \in \mathbf{N}$$

and

$$\alpha^{-1+\delta} e^{3(1+\delta)} = 1.$$

Therefore we have found $\delta \in \mathbf{R}, \delta > 0$ and a sequence of rational numbers $\left\{\frac{p_n}{q_n}\right\}_{n=0}^{\infty}$: $\frac{p_n}{q_n} \neq \zeta(3)$ for all $n \in \mathbf{N}_0$ such that

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}.$$

This yields the assertion by theorem 2.3.3. □

Remark 2.3.8. Alternative proofs of Apéry's theorem are not as transparent as Apéry's original proof. In 1979 Beukers proved 2.3.7 introducing integrals involving the shifted Legendre polynomials. This proof can be found in [19] or [3]. Other proofs can be found in [30] and [22].

2.4 Recent results

Although the problem of the irrationality of values of the Riemann zeta function at positive integers is yet to be fully solved, there are significant results in the area. In this section we shall briefly summarize these results.[7] First we shall present the Rivoal's theorem [25], [31].

Theorem 2.4.1 (Rivoal's theorem)**.** *The sequence* $\zeta(3), \zeta(5), \zeta(7), \ldots$ *contains infinitely many irrational numbers. More precisely, the following estimate holds for the dimension* $\delta(a)$ *of the spaces generated over* $\boldsymbol{Q}$ *by* $1, \zeta(3), \zeta(5), \ldots, \zeta(a-2), \zeta(a)$ *with an odd integer* a*:*

$$\delta(a) \geq \frac{\ln a}{1 + \ln 2}\left(1 + o(1)\right), \quad n \to \infty.$$

In [31] the following results are proved.

Theorem 2.4.2. *For every odd integer* $b \geq 1$ *at least one of the numbers*

$$\zeta(b+2), \zeta(b+4), \ldots, \zeta(8b-3), \zeta(8b-1)$$

is irrational.

Theorem 2.4.3. *There are odd integers* $a_1 \leq 145$ *and* $a_2 \leq 1971$ *such that* $1, \zeta(3), \zeta(a_1), \zeta(a_2)$ *are lineary independent over* $\boldsymbol{Q}$*.*

In [29] W. Zudilin strenghtens the results from [31] and proves the following theorem.

Theorem 2.4.4. *At least one of the numbers* $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational.*

[7]For further reading see `http://wain.mi.ras.ru/zw/` where is a list of references and links to be found.

3. Analysis of roots

In this section we shall summarize the results and analyze the roots of the Riemann zeta function in $\mathbf{C}\setminus\{1\}$. This will lead us to the formulation of the famous Riemann hypothesis.

Theorem 3.0.5. *$\zeta(s) \neq 0$ for all $s \in \boldsymbol{C}, \Re(s) > 1$.*

Proof. For $s \in \mathbf{C}, \Re(s) =: \sigma > 1$ it follows from theorem 2.1.2 that

$$\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\dots\left(1-\frac{1}{N^s}\right)\zeta(s) = 1 + \frac{1}{m_1} + \frac{1}{m_2} + \dots$$

where $m_1, m_2, \dots$ are the integers of whose prime factors exceed $N, N \in \mathbf{N}$. Hence

$$\left|\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\dots\left(1-\frac{1}{N^s}\right)\zeta(s)\right| \geq 1 - \frac{1}{(N+1)^\sigma} - \frac{1}{(N+2)^\sigma} - \dots > 0$$

for sufficiently large N. Thus

$$|\zeta(s)| > 0, \quad s \in \mathbf{C}, \Re(s) > 1.$$

□

3.1 Trivial roots

At this point we shall investigate the roots outside[1] the so-called critical strip which is the area $\{s \in \mathbf{C}; 0 \leq \Re(s) \leq 1\}$. For this purpose let us introduce an important tool in the theory of the Riemann zeta function which was first introduced by Riemann [24].

Theorem 3.1.1 (Functional equation)**.** *The function $\zeta(s), s \in \boldsymbol{C}\setminus\{1\}$ satisfies the functional equation*

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s). \tag{3.1}$$

Theorem 3.1.1 can be proved in many various ways. These proofs can be found in [28, p.13–44]. The importance of the functional equation (3.1) lies in the connection of the half-plane $\Re(s) > \frac{1}{2}$ and the half-plane $\Re(s) < \frac{1}{2}$ and hence it allows one to deduce the behaviour of the function $\zeta(s)$ in the half-plane $\Re(s) > \frac{1}{2}$ from its behaviour in the half-plane $\Re(s) < \frac{1}{2}$ and vice versa. We shall use 3.1 for finding the roots in the half-plane $\Re(s) < 0$.

Theorem 3.1.2. *In the half-plane $\Re(s) < 0$ the only roots of the function $\zeta(s)$ are the points $-2, -4, -6, \dots$. These are simple roots.*

Proof. Taking into account the functional equation (3.1) it follows that the function $\zeta(s)$ has only those roots which are poles of the product

$$\Gamma(s)\cos\frac{\pi s}{2}.$$

[1]We already know that $\zeta(s) \neq 0$ for all $s \in \mathbf{C}, \Re(s) > 1$ from theorem 3.0.5

Similary, in (3.1) by substituing s to $1-s$ we obtain

$$\zeta(s) = 2^s \pi^{s-1} \cos \frac{\pi(1-s)}{2} \Gamma(1-s)\zeta(1-s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s)$$

and then the roots are those of

$$\sin \frac{\pi s}{2} \Gamma(1-s).$$

Since $\Gamma(1-s)$ has no zeros in the half-plane $\Re(s) < 0$ and $\sin \frac{\pi s}{2}$ has simple roots at $s = -2, -4, -6, \dots$ the theorem holds. $\square$

The proof of theorem 3.1.2 is given in [26] or in [28]. The points $-2, -4, -6, \dots$ are called trivial roots.

3.2 Non-trivial roots

From theorems 3.1.2 and 3.0.5 it follows clearly that if there are other roots, all have to lie in the critical strip. We shall prove that there are infinitely many roots in the strip $\{s \in \mathbf{C}; 0 \le \Re(s) \le 1\}$.

Theorem 3.2.1. *There are infinitely many zeros of the function $\zeta(s)$ in the critical strip $\{s \in \boldsymbol{C}; 0 \le \Re(s) \le 1\}$.*

Proof. First we have

$$\overline{\zeta(s)} = \zeta(\bar{s}), \quad s \in \mathbf{C} \setminus \{1\}$$

where $\bar{s}$ denotes the complex conjugate of s. This follows from theorem 1.2.4 and from the properties of complex conjugation. Hence it is sufficient to study the roots in $\{s \in \mathbf{C}; \Im(s) \ge 0, 0 \le \Re(s) \le 1\}$. For $T \in \mathbf{R}, T > 0$ let us denote by $N(T)$ the number of zeros of $\zeta(s)$ of the form $\sigma + it$ where $0 < t \le T$. In [24] Riemann states that

$$N(T) = \frac{T}{2\pi}\left(\ln \frac{T}{2\pi} - 1\right) + O(\ln T), \quad T \to \infty. \tag{3.2}$$

(3.2) was proved by von Mangoldt in 1905 [28, p.214]. The assertion follows from (3.2). $\square$

Remark 3.2.2. Another approach to the proof of theorem 3.2.1 can be found in [26, p.430–431]. It involves the so-called ξ-function

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

The zeros of the ξ-function are precisely the non-trivial zeros of the ζ function since the trivial ones are removed by poles $-2, -4, -6, \dots$ of $\Gamma\left(\frac{s}{2}\right)$. It is interesting to notice that

$$\xi(s) = \xi(1-s).$$

Remark 3.2.3. In 1896 de la Vallée Poussin and Hadamard indepently proved the Prime number theorem which is equivalent to the statement that there are no zeros of $\zeta(s)$ with real part equal to 1 [10], [4] .

Previous results lead us to the formulation of the famous Riemann hypothesis. The consequences of the Riemann hypothesis are in detail discussed in [28, p.336–387].

Conjecture 3.2.4 (Riemann hypothesis)**.** *All the non-trivial zeros of the Riemann zeta function* ζ *lie on the critical line* $\{s \in \boldsymbol{C}; \Re(s) = \frac{1}{2}\}$.

Stated by Riemann in [24], the Riemann hypothesis has been neither proved nor disproved so far. There are, however, significant results in the area. First we shall introduce Hardy's theorem.

Theorem 3.2.5 (Hardy's theorem)**.** *The function* $\zeta(s)$ *has infinitely many zeros on the critical line* $\{s \in \boldsymbol{C}; \Re(s) = \frac{1}{2}\}$.

Hardy's theorem was first proved by Hardy in 1914 [12]. Several different proofs can be found in [28, p.256–264].

Theorem 3.2.6. *For* $T \in \boldsymbol{R}, T > 0$ *let us denote by* $N_0(T)$ *the number of zeta zeros of the form* $\frac{1}{2} + it, 0 < t \leq T$. *Then there are* $T_0 \in \boldsymbol{R}, T_0 > 0$ *and* $A \in \boldsymbol{R}$, $A > 0$ *such that for all* $T \geq T_0$*:*

$$N_0(T) > AT.$$

This theorem was proved by Hardy and Littlewood in 1921 [13]. Theorem 3.2.5 simply states that $N_0(T) \to \infty$ as $T \to \infty$. This result is strenghtened by theorem 3.2.6. However, this is further strenghtened by Selberg [27].

Theorem 3.2.7 (Selberg's theorem)**.** *There are* $T_0 \in \boldsymbol{R}$, $T_0 > 0$ *and* $A \in \boldsymbol{R}$, $A > 0$ *such that for all* $T \geq T_0$*:*

$$N_0(T) > AT \ln T.$$

For more recent results and further reference see [16], [15], [17], [18].

4. Additional computations

In this section we present additional computations necessary for the proof of Apéry's theorem in the section Number-theoretic analysis. We shall also adapt the notation from this section.

Lemma 4.0.8. *For $0 \leq k \leq n, n \in \boldsymbol{N}$ we have*

$$\begin{aligned}\Delta B := B_{n,k} - B_{n,k-1} \;=\; & (n+1)^3 \binom{n+1}{k}^2 \binom{n+1+k}{k}^2 \\ & -(34n^3+51n^2+27n+5)\binom{n}{k}^2\binom{n+k}{k}^2 \\ & +n^3\binom{n-1}{k}^2\binom{n-1+k}{k}^2.\end{aligned}$$

Proof. For $0 \leq k \leq n$ we have

$$\begin{aligned}\Delta B \;=\; & 4(2n+1)\left(k(2k+1)-(2n+1)^2\right)\binom{n}{k}^2\binom{n+k}{k}^2 \\ & -4(2n+1)\left((k-1)(2k-1)-(2n+1)^2\right)\binom{n}{k-1}^2\binom{n+k-1}{k-1}^2 \\ =\; & \binom{n}{k}^2\binom{n+k}{k}^2\left(-4+4k+8k^2-24n+8kn+16k^2n-48n^2-32n^3\right) \\ & -\binom{n}{k-1}^2\binom{n+k-1}{k-1}^2(-12k+8k^2-16n-24kn+16k^2n \\ & -48n^2-32n^3)\end{aligned}$$

Denoting

$$\begin{aligned}p_{1a} &:= -4+4k+8k^2-24n+8kn+16k^2n-48n^2-32n^3 \\ p_{1b} &:= -12k+8k^2-16n-24kn+16k^2n-48n^2-32n^3, \quad n \in \mathbf{N}\end{aligned}$$

we have

$$\begin{aligned}\Delta B \;=\; & \binom{n}{k}^2\binom{n+k}{k}^2 p_{1a} - \left(\frac{k}{n-k+1}\right)^2\left(\frac{k}{n+k}\right)^2\binom{n}{k}^2\binom{n+k}{k}^2 p_{1b} \\ =\; & \binom{n}{k}^2\binom{n+k}{k}^2\Big[-34n^3+2n^3-51n^2+3n^2-27n+3n-5+1+16k^2n \\ & +8kn+8k^2+4k-\frac{4k^4(-3k+2k^2-4n-6kn+4k^2n-12n^2-8n^3)}{(k+n)^2(n-k+1)^2}\Big] \\ =\; & (-34n^3-51n^2-27n-5)\binom{n}{k}^2\binom{n+k}{k}^2\end{aligned}$$

$$+\binom{n}{k}^2\binom{n+k}{k}^2\Big[2n^3+3n^2+3n+1+16k^2n+8kn+8k^2+4k$$
$$-\frac{4k^4(-3k+2k^2-4n-6kn+4k^2n-12n^2-8n^3)}{(k+n)^2(n-k+1)^2}\Big].$$

Let us denote

$$\begin{aligned}p_2 \quad := \quad & \Big[2n^3+3n^2+3n+1+16k^2n+8kn+8k^2+4k \\ & -\frac{4k^4(-3k+2k^2-4n-6kn+4k^2n-12n^2-8n^3)}{(k+n)^2(n-k+1)^2}\Big]\end{aligned}$$

and

$$R_1 := (-34n^3-51n^2-27n-5)\binom{n}{k}^2\binom{n+k}{k}^2.$$

Hence we obtain

$$\begin{aligned}\Delta B &= R_1+\left(\frac{n-k+1}{n+1}\right)^2\left(\frac{n+1}{n+k+1}\right)^2\binom{n+1}{k}^2\binom{n+1+k}{k}^2 p_2 \\ &= R_1+\left(\frac{n-k+1}{n+k+1}\right)^2\binom{n+1}{k}^2\binom{n+k+1}{k}^2 p_2 \\ &= R_1+(n+1)^3\binom{n+1}{k}^2\binom{n+k+1}{k}^2\left(p_2-(n+1)^3\right)\end{aligned}$$

and, after denoting

$$R_2 := (n+1)^3\binom{n+1}{k}^2\binom{n+k+1}{k}^2,$$

we get

$$\Delta B \quad = \quad R_1+R_2+\binom{n+1}{k}^2\binom{n+k+1}{k}^2\cdot\frac{n^3(-k+k^2+n-2kn+n^2)^2}{(k+n)^2(1+k+n)^2}.$$

Since

$$\binom{n+1}{k}^2\binom{n+k+1}{k}^2 =$$
$$= \left(\frac{(n+1+k)(n+k)}{(n+1)n}\right)^2\left(\frac{n(n+1)}{(n+1-k)(n-k)}\right)^2\binom{n-1}{k}^2\binom{n-1-k}{k}^2$$

we have

$$\Delta B = R_1+R_2+n^3\cdot\underbrace{\frac{(-k+k^2+n-2kn+n^2)^2}{(n+1-k)^2(n-k)^2}}_{=1}\binom{n-1}{k}^2\binom{n-1-k}{k}^2$$

which yields the assertion. □

Lemma 4.0.9. *For $1 \le k \le n$, $n \in \boldsymbol{N}$ we have*

$$c_{n,k} - c_{n-1,k} = \frac{(-1)^k k!^2 (n-k-1)!}{n^2 (n+k)!}.$$

Proof. First we have

$$\begin{aligned}
c_{n,k} - c_{n-1,k} &= \sum_{j=1}^{n} \frac{1}{j^3} + \sum_{j=1}^{k} \frac{(-1)^{j-1}}{2j^3 \binom{n}{j}\binom{n+j}{j}} - \sum_{j=1}^{n-1} \frac{1}{j^3} - \sum_{j=1}^{k} \frac{(-1)^{j-1}}{2j^3 \binom{n-1}{j}\binom{n+j-1}{j}} \\
&= \frac{1}{n^3} + \sum_{j=1}^{k} \frac{(-1)^{j-1}}{2j^3} \left(\frac{1}{\binom{n}{j}\binom{n+j}{j}} - \frac{1}{\binom{n-1}{j}\binom{n+j-1}{j}} \right) \\
&= \frac{1}{n^3} + \sum_{j=1}^{k} \frac{(-1)^{j-1}}{2j^3} \cdot \frac{1 - \frac{n+j}{n-1}}{\binom{n}{j}\binom{n+j}{j}} \\
&= \frac{1}{n^3} + \sum_{j=1}^{k} \frac{(-1)^j (n-j-1)!(j-1)!^2}{(n+j)!}.
\end{aligned}$$

Further

$$\begin{aligned}
&\frac{(-1)^j (n-j-1)!(j-1)!^2}{(n+j)!} = \\
&= \frac{(-1)^j (j-1)!^2 (n-j-1)! \left(j^2 + (n+j)(n-j)\right)}{n^2 (n+j)!} \\
&= \frac{(-1)^j j!^2 (n-j-1)! - (-1)^{j-1} (j-1)!^2 (n-j)! (n+j)}{n^2 (n+j)!} \\
&= \frac{(-1)^j j!^2 (n-j-1)!}{n^2 (n+j)!} - \frac{(-1)^{j-1} (j-1)!^2 (n-j)!}{n^2 (n+j-1)!}
\end{aligned}$$

and therefore

$$\begin{aligned}
&\frac{1}{n^3} + \sum_{j=1}^{k} \frac{(-1)^j (n-j-1)!(j-1)!^2}{(n+j)!} = \\
&= \frac{1}{n^3} + \sum_{j=1}^{k} \left(\frac{(-1)^j j!^2 (n-j-1)!}{n^2 (n+j)!} - \frac{(-1)^{j-1} (j-1)!^2 (n-j)!}{n^2 (n+j-1)!} \right) \\
&= \frac{1}{n^3} - \frac{(n-1)!}{n^2 n!} + \frac{(-1)^k k!^2 (n-k-1)!}{n^2 (n+k)!} \\
&= \frac{(-1)^k k!^2 (n-k-1)!}{n^2 (n+k)!}.
\end{aligned}$$

□

Lemma 4.0.10. *For $1 \le k \le n$, $n \in \boldsymbol{N}$ we have*

$$(n+1)^3 b_{n+1,k}^{(2)} c_{n+1,k} - P(n) b_{n,k}^{(2)} c_{n,k} + n^3 b_{n-1,k}^{(2)} c_{n-1,k} = A_{n,k} - A_{n,k-1}.$$

Proof. We shall start with

$$\begin{aligned} A_{n,k}-A_{n,k-1} &= B_{n,k}c_{n,k}+\frac{5(2n+1)(-1)^{k-1}k}{n(n+1)}\binom{n}{k}\binom{n+k}{k} \\ &- B_{n,k-1}c_{n,k-1}-\frac{5(2n+1)(-1)^{k}(k-1)}{n(n+1)}\binom{n}{k-1}\binom{n+k-1}{k-1}. \end{aligned}$$

Clearly

$$\begin{aligned} R &:= (n+1)^3 b^{(2)}_{n+1,k}c_{n+1,k}-P(n)b^{(2)}_{n,k}c_{n,k}+n^3 b^{(2)}_{n-1,k}c_{n-1,k} \\ &= (B_{n,k}-B_{n,k-1})c_{n,k}+(n+1)^3 b^{(2)}_{n+1,k}(c_{n+1,k}-c_{n,k}) \\ &\qquad -n^3 b^{(2)}_{n-1,k}(c_{n,k}-c_{n-1,k}). \end{aligned}$$

From

$$c_{n,k}=c_{n,k-1}+\frac{(-1)^{k-1}}{2k^3\binom{n}{k}\binom{n+k}{k}}$$

and lemma 4.0.9 it follows, after denoting

$$R_1 :=_{n,k} c_{n,k}-B_{n,k-1}c_{n,k-1},$$

that

$$\begin{aligned} R &= R_1-B_{n,k-1}\frac{(-1)^{k-1}}{2k^3\binom{n}{k}\binom{n+k}{k}} \\ &\quad +(n+1)^3 b^{(2)}_{n+1,k}\frac{(-1)^k k!^2(n-k)!}{(n+1)^2(n+k+1)!}-n^3 b^{(2)}_{n-1,k}\frac{(-1)^k k!^2(n-k-1)!}{n^2(n+k)!} \\ &= R_1-4(2n+1)\left(2k^2-3k-4n^2-4n\right)\binom{n}{k-1}^2\binom{n+k-1}{k-1}^2\frac{(-1)^{k-1}}{2k^3\binom{n}{k}\binom{n+k}{k}} \\ &\quad +(n+1)\binom{n+1}{k}^2\binom{n+k+1}{k}^2\frac{(-1)^k k!^2(n-k)!}{(n+k+1)!} \\ &\quad -n\binom{n-1}{k}^2\binom{n+k-1}{k}^2\frac{(-1)^k k!^2(n-k-1)!}{(n+k)!} \\ &= R_1+\frac{2(2n+1)(-1)^k k(2k^2-3k-4n^2-4n)}{(n-k+1)^2(n+k)^2}\binom{n}{k}\binom{n+k}{k} \\ &\quad +(n+1)\frac{(n+k+1)^2}{(n-k+1)^2}\frac{(-1)^k k!^2(n-k)!}{(n+k+1)!}\binom{n}{k}^2\binom{n+k}{k}^2 \\ &\quad -n\frac{(n-k)^2}{(n+k)^2}\frac{(-1)^k k!^2(n-k-1)!}{(n+k)!}\binom{n}{k}^2\binom{n+k}{k}^2 \\ &= R_1+\frac{2(2n+1)(-1)^k(2k^2-3k-4n^2-4n)}{(n-k+1)^2(n+k)^2}\binom{n}{k}\binom{n+k}{k} \\ &\quad +\frac{(n+1)(n+k+1)(-1)^k}{(n-k+1)^2}\binom{n}{k}\binom{n+k}{k}-\frac{n(n-k)(-1)^k}{(n+k)^2}\binom{n}{k}\binom{n+k}{k}. \end{aligned}$$

Hence we obtain

$$R=R_1+\binom{n}{k}\binom{n+k}{k}\frac{5(2n+1)(-1)^{k-1}k}{n(n+1)}$$

$$+\binom{n}{k}\binom{n+k}{k}\Bigg[\frac{2(2n+1)(-1)^k k(2k^2-3k-4n^2-4n)}{(n-k+1)^2(n+k)^2}$$
$$+\frac{(n+1)(n+k+1)(-1)^k}{(n-k+1)^2}-\frac{n(n-k)(-1)^k}{(n+k)^2}-\frac{5(2n+1)(-1)^{k-1}k}{n(n+1)}\Bigg]$$

and thus, after denoting

$$R_2 := \binom{n}{k}\binom{n+k}{k}\frac{5(2n+1)(-1)^{k-1}k}{n(n+1)},$$

$$\begin{aligned}
R &= R_1+R_2+\binom{n}{k}\binom{n+k}{k}\left(\frac{5(-1)^k(2n+1)}{k^2-k-n(n+1)}-\frac{5(2n+1)(-1)^{k-1}k}{n(n+1)}\right)\\
&= R_1+R_2+\binom{n}{k}\binom{n+k}{k}\frac{5(-1)^{k-1}(k-1)k^2(2n+1)}{n(n+1)(k-k^2+n+n^2)}\\
&= R_1+R_2\\
&\quad +\binom{n}{k-1}\binom{n+k-1}{k-1}\frac{(n-k+1)(n+k)}{k^2}\cdot\frac{5(-1)^{k-1}(k-1)k^2(2n+1)}{n(n+1)(k-k^2+n+n^2)}\\
&= R_1+R_2+\binom{n}{k-1}\binom{n+k-1}{k-1}\frac{5(-1)^k(k-1)(2n+1)}{n(n+1)}.
\end{aligned}$$

□

Lemma 4.0.11. *For $n \in \boldsymbol{N}$ we have*

$$A_{n,n} = B_{n,n}c_{n,n} + \frac{5(2n+1)(-1)^{n-1}}{n+1}\binom{2n}{n} = -(n+1)^3 b^{(2)}_{n+1,n+1}c_{n+1,n+1}.$$

Proof. From the definition of $A_{n,n}$ we have

$$A_{n,n} = B_{n,n}c_{n,n} + \frac{5(2n+1)(-1)^{n-1}}{n+1}\binom{2n}{n}.$$

Further

$$-(n+1)^3 b^{(2)}_{n+1,n+1}c_{n+1,n+1} =$$

$$\begin{aligned}
&= B_{n,n}\left(c_{n,n}+c_{n+1,n}-c_{n,n}+\frac{(-1)^n}{2(n+1)^3\binom{2n+2}{n+1}}\right)\\
&= B_{n,n}\left(c_{n,n}+\frac{(-1)^n n!^2}{(n+1)^2(2n+1)!}+\frac{(-1)^n}{2(n+1)^3\binom{2n+2}{n+1}}\right)
\end{aligned}$$

from lemma 4.0.9 and therefore

$$\begin{aligned}
-(n+1)^3 b^{(2)}_{n+1,n+1}c_{n+1,n+1} &= B_{n,n}\left(c_{n,n}+\frac{5(-1)^n n!^2}{4(n+1)^2(2n+1)!}\right)\\
&= B_{n,n}c_{n,n}+\frac{5(-1)^{n-1}(n+1)n!^2}{4(2n+1)!}\binom{2n+2}{n+1}^2\\
&= B_{n,n}c_{n,n}+\frac{5(-1)^{n-1}(2n+1)}{n+1}\binom{2n}{n}.
\end{aligned}$$

□

Conclusion

The properties of the Riemann zeta function have been widely investigated since 1859 when Bernhard Riemann published his famous article *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*. Riemann proposed a conjecture which is considered to be one of the most important problems in contemporary mathematics - the Riemann hypothesis. This hypothesis could have far-reaching impact once proven true or false.

Generally, the problem of irrationality, let alone transcendence of special constants is indeed demanding and requires a deep theoretical insight. The approach to the proof of Apéry's theorem we chose is a complex but illustrative one and does not require advanced techniques as does the proof based on shifted Legendre polynomials. Thus, the reasoning we followed could be considered elementary yet not easy. Proving the irrationality of the zeta values at odd positive integers remains an outstanding problem of the theory since there is no obvious way how to generalize these proofs.

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