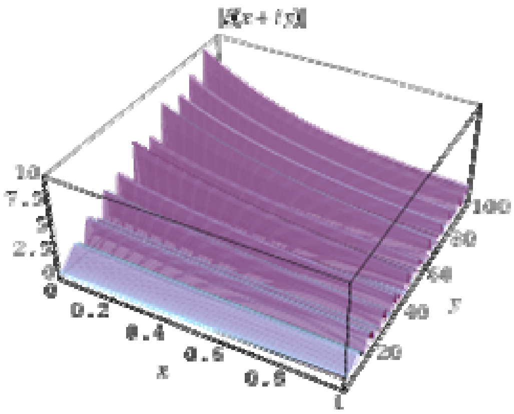
The Riemann zeta function is an extremely important special function of mathematics and physics that arises in definite integration and is intimately related with very deep results surrounding the prime number theorem. While many of the properties of this function have been investigated, there remain important fundamental conjectures (most notably the Riemann hypothesis) that remain unproved to this day. The Riemann zeta function $\zeta(s)$ is defined over the complex plane for one complex variable, which is conventionally denoted s (instead of the usual z) in deference to the notation used by Riemann in his 1859 paper that founded the study of this function (Riemann 1859). It is implemented in *Mathematica* as `Zeta`[*s*].



The plot above shows the "ridges" of $|\zeta(x+iy)|$ for $0 < x < 1$ and $1 < y < 100$. The fact that the ridges appear to decrease monotonically for $0 \leq x \leq 1/2$ is not a coincidence since it turns out that monotonic decrease implies the Riemann hypothesis (Zvengrowski and Saidak 2003; Borwein and Borwein 2003, pp. 95-96).

On the real line with $x > 1$, the Riemann zeta function can be defined by the integral

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1}\, du, \qquad (1)$$

where $\Gamma(x)$ is the gamma function. If x is an integer n, then we have the identity

$$\frac{u^{n-1}}{e^u - 1} = \frac{e^{-u} u^{n-1}}{1 - e^{-u}} = e^{-u} u^{n-1} \sum_{k=0}^\infty e^{-ku} = \sum_{k=1}^\infty e^{-ku} u^{n-1}, \qquad (2)$$

so

$$\int_0^\infty \frac{u^{n-1}}{e^u - 1}\, du = \sum_{k=1}^\infty \int_0^\infty e^{-ku} u^{n-1}\, du. \qquad (3)$$

To evaluate $\zeta(n)$, let $y \equiv k u$ so that $d y = k d u$ and plug in the above identity to obtain

$$\zeta(n) = \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} \int_0^{\infty} e^{-k u} u^{n-1} d u \tag{4}$$

$$= \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} \int_0^{\infty} e^{-y} \left(\frac{y}{k}\right)^{n-1} \frac{d y}{k} \tag{5}$$

$$= \frac{1}{\Gamma(n)} \sum_{k=1}^{\infty} \frac{1}{k^n} \int_0^{\infty} e^{-y} y^{n-1} d y. \tag{6}$$

Integrating the final expression in (6) gives $\Gamma(n)$, which cancels the factor $1/\Gamma(n)$ and gives the most common form of the Riemann zeta function,

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \tag{7}$$

which is sometimes known as a *p*-series.

The Riemann zeta function can also be defined in terms of multiple integrals by

$$\zeta(n) = \underbrace{\int_0^1 \cdots \int_0^1}_{n} \frac{\prod_{i=1}^{n} d x_i}{1 - \prod_{i=1}^{n} x_i}, \tag{8}$$

and as a Mellin transform by

$$\int_0^{\infty} \operatorname{frac}\left(\frac{1}{t}\right) t^{s-1} d t = -\frac{\zeta(s)}{s} \tag{9}$$

for $0 < \mathbb{R}[s] < 1$, where $\operatorname{frac}(x)$ is the fractional part (Balazard and Saias 2000).

It appears in the unit square integral

$$\int_0^1 \int_0^1 \frac{[-\ln(x y)]^s}{1 - x y} d x d y = \Gamma(s+2) \zeta(s+2), \tag{10}$$

valid for $\mathbb{R}[s] > 1$ (Guillera and Sondow 2005). For s a nonnegative integer, this formula is due to Hadjicostas (2002), and the special cases $s = 0$ and $s = 1$ are due to Beukers (1979).

Note that the zeta function $\zeta(s)$ has a singularity at $s = 1$, where it reduces to the divergent harmonic series.

The Riemann zeta function satisfies the reflection functional equation

$$\zeta(1-s) = 2(2\pi)^{-s}\cos\left(\tfrac{1}{2}s\pi\right)\Gamma(s)\zeta(s) \tag{11}$$

(Hardy 1999, p. 14; Krantz 1999, p. 160), a similar form of which was conjectured by Euler for real s (Euler, read in 1749, published in 1768; Ayoub 1974; Havil 2003, p. 193). A symmetrical form of this functional equation is given by

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s) \tag{12}$$

(Ayoub 1974), which was proved by Riemann for all complex s (Riemann 1859).

As defined above, the zeta function $\zeta(s)$ with $s \equiv \sigma + it$ a complex number is defined for $\mathbb{R}[s] > 1$. However, $\zeta(s)$ has a unique analytic continuation to the entire complex plane, excluding the point $s = 1$, which corresponds to a simple pole with complex residue 1 (Krantz 1999, p. 160). In particular, as $s \to 1$, $\zeta(s)$ obeys

$$\lim_{s\to 1}\left[\zeta(s) - \frac{1}{s-1}\right] = \gamma, \tag{13}$$

where γ is the Euler-Mascheroni constant (Whittaker and Watson 1990, p. 271).

To perform the analytic continuation for $\mathbb{R}[s] > 0$, write

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s} + \sum_{n=1}^{\infty}\frac{1}{n^s} = 2\sum_{n=2,4,\ldots}^{\infty}\frac{1}{n^s} \tag{14}$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{(2k)^s} \tag{15}$$

$$= 2^{1-s}\sum_{k=1}^{\infty}\frac{1}{k^s}, \tag{16}$$

so rewriting in terms of $\zeta(s)$ immediately gives

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s} + \zeta(s) = 2^{1-s}\zeta(s). \tag{17}$$

Therefore,

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \tag{18}$$

Here, the sum on the right-hand side is exactly the Dirichlet eta function $\eta(s)$ (sometimes also called the alternating zeta function). While this formula defines $\zeta(s)$ for only the right half-plane $\mathbb{R}[s] > 0$, equation (◇) can be used to analytically continue it to the rest of the complex plane. Analytic continuation can also be performed using Hankel functions. A globally convergent series for the Riemann zeta function (which provides the analytic continuation of $\zeta(s)$ to the entire complex plane except $s=1$) is given by

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s} \tag{19}$$

(Havil 2003, p. 206), where $\binom{n}{k}$ is a binomial coefficient, which was conjectured by Knopp around 1930, proved by Hasse (1930), and rediscovered by Sondow (1994). This equation is related to renormalization and random variates (Biane *et al.* 2001) and can be derived by applying Euler's series transformation with $n=0$ to equation (18).

Hasse (1930) also proved the related globally (but more slowly) convergent series

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{1-s} \tag{20}$$

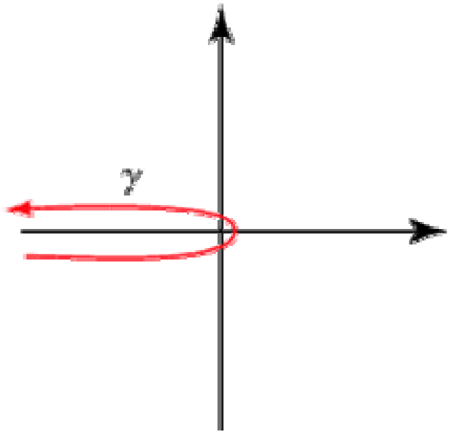
that, unlike (19), can also be extended to a generalization of the Riemann zeta function known as the Hurwitz zeta function $\zeta(s,a)$. $\zeta(s,a)$ is defined such that

$$\zeta(s) \equiv \zeta(s,1). \tag{21}$$

(If the singular term is excluded from the sum definition of $\zeta(s,a)$, then $\zeta(s) = \zeta(s,0)$ as well.) Expanding $\zeta(s)$ about $s=1$ gives

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n, \tag{22}$$

where γ_n are the so-called Stieltjes constants.



The Riemann zeta function can also be defined in the complex plane by the contour integral

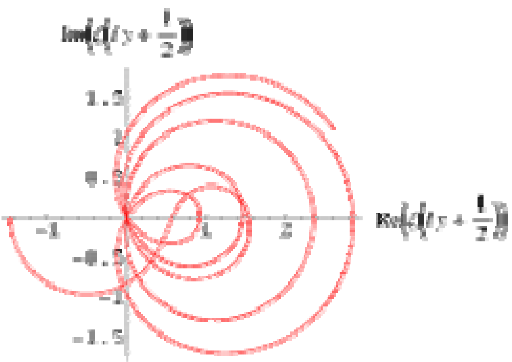
$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} \oint_\gamma \frac{u^{z-1}}{e^{-u}-1}\, du \tag{23}$$

for all $z \neq 1$, where the contour is illustrated above (Havil 2003, pp. 193 and 249-252).

Zeros of $\zeta(s)$ come in (at least) two different types. So-called "trivial zeros" occur at *all* negative even integers $s = -2$, -4, -6, ..., and "nontrivial zeros" at certain

$$s = \sigma + it \tag{24}$$

for s in the "critical strip" $0 < \sigma < 1$. The Riemann hypothesis asserts that the nontrivial Riemann zeta function zeros of $\zeta(s)$ all have real part $\sigma = \mathbb{R}[s] = 1/2$, a line called the "critical line." This is now known to be true for the first 250×10^9 roots.



The plot above shows the real and imaginary parts of $\zeta(1/2 + yi)$ (i.e., values of $\zeta(z)$ along the critical strip) as y is varied from 0 to 35 (Derbyshire 2004, p. 221).

The Riemann zeta function can be split up into

$$\zeta\left(\tfrac{1}{2}+i t\right)=Z(t)\, e^{-i \vartheta(t)}, \tag{25}$$

where $Z(t)$ and $\vartheta(t)$ are the Riemann-Siegel functions.

The Riemann zeta function is related to the Dirichlet lambda function $\lambda(\nu)$ and Dirichlet eta function $\eta(\nu)$ by

$$\frac{\zeta(\nu)}{2^{\nu}}=\frac{\lambda(\nu)}{2^{\nu}-1}=\frac{\eta(\nu)}{2^{\nu}-2} \tag{26}$$

and

$$\zeta(\nu)+\eta(\nu)=2\lambda(\nu) \tag{27}$$

(Spanier and Oldham 1987).

It is related to the Liouville function $\lambda(n)$ by

$$\frac{\zeta(2s)}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^{s}} \tag{28}$$

(Lehman 1960, Hardy and Wright 1979). Furthermore,

$$\frac{\zeta^{2}(s)}{\zeta(2s)}=\sum_{n=1}^{\infty}\frac{2^{\omega(n)}}{n^{s}}, \tag{29}$$

where $\omega(n)$ is the number of distinct prime factors of n (Hardy and Wright 1979, p. 254).

For $-2n$ a positive even integer -2, -4, ...,

$$\zeta'(-2n)=\frac{(-1)^{n}\zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}}, \tag{30}$$

giving the first few as

$$\zeta'(-2) = -\frac{\zeta(3)}{4\pi^{2}} \tag{31}$$

$$\zeta'(-4) = \frac{3\zeta(5)}{4\pi^{4}} \tag{32}$$

$$\zeta'(-6) = -\frac{45\,\zeta(7)}{8\,\pi^6} \tag{33}$$

$$\zeta'(-8) = \frac{315\,\zeta(9)}{4\,\pi^8} \tag{34}$$

(Sloane's A117972 and A117973). For $n = -1$,

$$\zeta'(-1) = \tfrac{1}{12} - \ln A, \tag{35}$$

where A is the Glaisher-Kinkelin constant. Using equation (◇) gives the derivative

$$\zeta'(0) = -\tfrac{1}{2}\ln(2\pi), \tag{36}$$

which can be derived directly from the Wallis formula (Sondow 1994). $\zeta'(0)/\zeta(0) = \ln(2\pi)$ can also be derived directly from the Euler-Maclaurin summation formula (Edwards 2001, pp. 134-135). In general, $\zeta^{(n)}(0)$ can be expressed analytically in terms of π, $\zeta(n)$, the Euler-Mascheroni constant γ, and the Stieltjes constants γ_i, with the first few examples being

$$\zeta''(0) = \gamma_1 + \tfrac{1}{2}\gamma^2 - \tfrac{1}{24}\pi^2 - \tfrac{1}{2}[\ln(2\pi)]^2 \tag{37}$$

$$\zeta'''(0) = 3\ln(2\pi)\,\gamma_1 + 3\,\gamma\gamma_1 + \tfrac{3}{2}\gamma_2 - \zeta(3) - \tfrac{1}{2}[\ln(2\pi)]^3 - \tfrac{1}{8}\pi^2\ln(2\pi) + \tfrac{3}{2}\gamma^2\ln(2\pi) + \gamma^3. \tag{38}$$

Derivatives $\zeta^{(n)}(1/2)$ can also be given in closed form, for example,

$$\zeta'\left(\tfrac{1}{2}\right) = \tfrac{1}{4}\left[(\pi + 2\gamma + 6\ln 2 + 2\ln\pi)\,\zeta\left(\tfrac{1}{2}\right)\right] \tag{39}$$

$$= -3.92264613\ldots \tag{40}$$

(Sloane's A114875).

The derivative of the Riemann zeta function for $\Re[s] > 1$ is defined by

$$\zeta'(s) = -\sum_{k=1}^{\infty}\frac{\ln k}{k^s} = -\sum_{k=2}^{\infty}\frac{\ln k}{k^s}. \tag{41}$$

$\zeta'(2)$ can be given in closed form as

$$\zeta'(2) = \tfrac{1}{6}\pi^2\,[\gamma + \ln(2\pi) - 12\ln A] \tag{42}$$

$$= -0.93754825431\ldots \tag{43}$$

(Sloane's A073002), where A is the Glaisher-Kinkelin constant (given in series form by Glaisher 1894).

The series for $\zeta(s)$ about $s=1$ is

$$\zeta(s) = \frac{1}{(s-1)^2} - \gamma_1 + \gamma_2 (s-1) - \frac{1}{2} \gamma_3 (s-1)^2 + \ldots, \quad (44)$$

where γ_i are Stieltjes constants.

In 1739, Euler found the rational coefficients C in $\zeta(2n) = C\pi^{2n}$ in terms of the Bernoulli numbers. Which, when combined with the 1882 proof by Lindemann that π is transcendental, effectively proves that $\zeta(2n)$ is transcendental. The study of $\zeta(2n+1)$ is significantly more difficult. Apéry (1979) finally proved $\zeta(3)$ to be irrational, but no similar results are known for other odd n. As a result of Apéry's important discovery, $\zeta(3)$ is sometimes called Apéry's constant. Rivoal (2000) and Ball and Rivoal (2001) proved that there are infinitely many integers n such that $\zeta(2n+1)$ is irrational, and subsequently that at least one of $\zeta(5)$, $\zeta(7)$, ..., $\zeta(21)$ is irrational (Rivoal 2001). This result was subsequently tightened by Zudilin (2001), who showed that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, or $\zeta(11)$ is irrational.

A number of interesting sums for $\zeta(n)$, with n a positive integer, can be written in terms of binomial coefficients as the binomial sums

$$\zeta(2) = 3\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} \quad (45)$$

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \quad (46)$$

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} \quad (47)$$

(Guy 1994, p. 257; Bailey *et al.* 2006, p. 70). Apéry arrived at his result with the aid of the k^{-3} sum formula above. A relation of the form

$$\zeta(5) = Z_5 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^5 \binom{2k}{k}} \quad (48)$$

has been searched for with Z_5 a rational or algebraic number, but if Z_5 is a root of a polynomial of degree 25 or less, then the Euclidean norm of the coefficients must be larger than 1.24×10^{383}, and if $\zeta(5)$ if algebraic of degree 25 or less, then the norm of coefficients must exceed 1.98×10^{380} (Bailey *et al.* 2006, pp. 70-71, updating Bailey and Plouffe). Therefore, no such sums for $\zeta(n)$ are known for $n \geq 5$.

The identity

$$\sum_{k=1}^{\infty}\frac{1}{k^2-x^2} = \sum_{n=0}^{\infty}\zeta(2n+2)x^{2n} \quad (49)$$

$$= \frac{1-\pi x\cot(\pi x)}{2x^2} \quad (50)$$

$$= 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}\left(1-\frac{x^2}{k^2}\right)}\prod_{m=1}^{k-1}\frac{1-\frac{4x^2}{m^2}}{1-\frac{x^2}{m^2}} \quad (51)$$

$$= \frac{3\,{}_4F_3\left(1,2,1-2x,1+2x;\frac{3}{2},2-x,2+x;\frac{1}{4}\right)}{2\left(1-x^2\right)} \quad (52)$$

for x is complex number not equal to a nonzero integer gives an Apéry-like formula for even positive n (Bailey *et al.* 2006, pp. 72-77).

The Riemann zeta function $\zeta(2n)$ may be computed analytically for even n using either contour integration or Parseval's theorem with the appropriate Fourier series. An unexpected and important formula involving a product over the primes was first discovered by Euler in 1737,

$$\zeta(s)(1-2^{-s}) = \left(1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots\right)\left(1-\frac{1}{2^s}\right) \quad (53)$$

$$= \left(1+\frac{1}{2^s}+\frac{1}{3^s}+\ldots\right)-\left(\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\ldots\right) \quad (54)$$

$$\zeta(s)(1-2^{-s})(1-3^{-s}) = \left(1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\ldots\right)-\left(\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\ldots\right) \quad (55)$$

$$\zeta(s)(1-2^{-s})(1-3^{-s})\cdots(1-p_n^{-s})\cdots = \zeta(s)\prod_{n=1}^{\infty}(1-p_n^{-s}) \quad (56)$$

$$= 1. \quad (57)$$

Here, each subsequent multiplication by the nth prime p_n leaves only terms that are powers of p^{-s}. Therefore,

$$\zeta(s) = \left[\prod_{n=1}^{\infty}(1-p_n^{-s})\right]^{-1}, \quad (58)$$

which is known as the Euler product formula (Hardy 1999, p. 18; Krantz 1999, p. 159), and called "the golden key" by Derbyshire (2004, pp. 104-106). The formula can also be written

$$\zeta(s) = (1 - 2^{-s})^{-1} \prod_{\substack{q \equiv 1 \\ (\mathrm{mod}\ 4)}} (1 - q^{-s})^{-1} \prod_{\substack{r \equiv 3 \\ (\mathrm{mod}\ 4)}} (1 - r^{-s})^{-1}, \quad (59)$$

where q and r are the primes congruent to 1 and 3 modulo 4, respectively.

For even $n \geq 2$,

$$\zeta(n) = \frac{2^{n-1} |B_n| \pi^n}{n!}, \quad (60)$$

where B_n is a Bernoulli number (Mathews and Walker 1964, pp. 50-53; Havil 2003, p. 194). Another intimate connection with the Bernoulli numbers is provided by

$$B_n = (-1)^{n+1} n \zeta(1-n) \quad (61)$$

for $n \geq 1$, which can be written

$$B_n = -n \zeta(1-n) \quad (62)$$

for $n \geq 2$. (In both cases, only the even cases are of interest since $B_n = 0$ trivially for odd n.) Rewriting (62),

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \quad (63)$$

for $n = 1$, 3, ... (Havil 2003, p. 194), where B_n is a Bernoulli number, the first few values of which are $-1/12$, 1/120, $-1/252$, 1/240, ... (Sloane's A001067 and A006953).

Although no analytic form for $\zeta(n)$ is known for odd n,

$$\zeta(3) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k}{k^2}, \quad (64)$$

where H_k is a harmonic number (Stark 1974). In addition, $\zeta(n)$ can be expressed as the sum limit

$$\zeta(n) = \lim_{x \to \infty} \frac{1}{(2x+1)^n} \sum_{k=1}^{x} \left[\cot\left(\frac{k}{2x+1}\right)\right]^n \quad (65)$$

for $n = 3$, 5, ... (Apostol 1973, given incorrectly in Stark 1974).

For $\mu(n)$ the Möbius function,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad (66)$$

(Havil 2003, p. 209).

The values of $\zeta(n)$ for small positive integer values of n are

$$\zeta(1) = \infty \qquad (67)$$
$$\zeta(2) = \frac{\pi^2}{6} \qquad (68)$$
$$\zeta(3) = 1.2020569032\ldots \qquad (69)$$
$$\zeta(4) = \frac{\pi^4}{90} \qquad (70)$$
$$\zeta(5) = 1.0369277551\ldots \qquad (71)$$
$$\zeta(6) = \frac{\pi^6}{945} \qquad (72)$$
$$\zeta(7) = 1.0083492774\ldots \qquad (73)$$
$$\zeta(8) = \frac{\pi^8}{9450} \qquad (74)$$
$$\zeta(9) = 1.0020083928\ldots \qquad (75)$$
$$\zeta(10) = \frac{\pi^{10}}{93555}. \qquad (76)$$

Euler gave $\zeta(2)$ to $\zeta(26)$ for even n (Wells 1986, p. 54), and Stieltjes (1993) determined the values of $\zeta(2)$, ..., $\zeta(70)$ to 30 digits of accuracy in 1887. The denominators of $\zeta(2n)$ for $n = 1$, 2, ... are 6, 90, 945, 9450, 93555, 638512875, ... (Sloane's A002432). The numbers of decimal digits in the denominators of $\zeta(10^n)$ for $n = 0$, 1, ... are 1, 5, 133, 2277, 32660, 426486, 5264705, ... (Sloane's A114474).

An integral for positive even integers is given by

$$\zeta(2n) = \frac{(-1)^{n+1}\, 2^{2n-3}\, \pi^{2n}}{\left(2^{2n}-1\right)(2n-2)!} \int_0^1 E_{2(n-1)}(x)\, dx, \qquad (77)$$

and integrals for positive odd integers are given by

$$\zeta(2n+1) = \frac{(-1)^n\, 2^{2n-1}\, \pi^{2n+1}}{\left(2^{2n+1}-1\right)(2n)!} \int_0^1 E_{2n}(x) \tan\left(\tfrac{1}{2}\pi x\right) dx \qquad (78)$$

$$= \frac{(-1)^n 2^{2n-1} \pi^{2n+1}}{(2^{2n+1}-1)(2n)!} \int_0^1 E_{2n}(x) \cot\left(\tfrac{1}{2}\pi x\right) dx \quad (79)$$

$$= \frac{(-1)^n 2^{2n} \pi^{2n+1}}{(2n+1)!} \int_0^1 B_{2n+1}(x) \tan\left(\tfrac{1}{2}\pi x\right) dx \quad (80)$$

$$= \frac{(-1)^{n+1} 2^{2n} \pi^{2n+1}}{(2n+1)!} \int_0^1 B_{2n+1}(x) \cot\left(\tfrac{1}{2}\pi x\right) dx, \quad (81)$$

where $E_n(x)$ is an Euler polynomial and $B_n(x)$ is a Bernoulli polynomial (Cvijović and Klinowski 2002; J. Crepps, pers. comm., Apr. 2002).

The value of $\zeta(0)$ can be computed by performing the inner sum in equation (◇) with $s = 0$,

$$\zeta(0) = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k}, \quad (82)$$

to obtain

$$\zeta(0) = -\sum_{n=0}^{\infty} \frac{\delta_{0,n}}{2^{n+1}} = -\frac{1}{2^{0+1}} = -\frac{1}{2}, \quad (83)$$

where $\delta_{0,n}$ is the Kronecker delta.

Similarly, the value of $\zeta(-1)$ can be computed by performing the inner sum in equation (◇) with $s = -1$,

$$\zeta(-1) = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1), \quad (84)$$

which gives

$$\zeta(-1) = -\frac{1}{3} \sum_{n=0}^{\infty} \frac{\delta_{0,n} - n\delta_{1,n}}{2^{n+1}} \quad (85)$$

$$= -\frac{1}{3} \left(\frac{1}{2^{0+1}} - \frac{1}{2^{1+1}} \right) = -\frac{1}{12}. \quad (86)$$

This value is related to a deep result in renormalization theory (Elizalde *et al.* 1994, Elizalde 1995, Bloch 1996, Lepowski 1999).

It is apparently not known if the value

$$\zeta\left(\tfrac{1}{2}\right) = -1.46035450880\ldots \tag{87}$$

(Sloane's A059750) can be expressed in terms of known mathematical constants. This constant appears, for example, in Knuth's series.

Rapidly converging series for $\zeta(n)$ for n odd were first discovered by Ramanujan (Zucker 1979, Zucker 1984, Berndt 1988, Bailey *et al.* 1997, Cohen 2000). For $n > 1$ and $n \equiv 3 \pmod 4$,

$$\zeta(n) = \frac{2^{n-1}\pi^n}{(n+1)!}\sum_{k=0}^{(n+1)/2}(-1)^{k-1}\binom{n+1}{2k}B_{n+1-2k}B_{2k} - 2\sum_{k=1}^{\infty}\frac{1}{k^n\left(e^{2\pi k}-1\right)}, \tag{88}$$

where B_k is again a Bernoulli number and $\binom{n}{k}$ is a binomial coefficient. The values of the left-hand sums (divided by π^n) in (88) for $n = 3$, 7, 11, ... are 7/180, 19/56700, 1453/425675250, 13687/390769879500, 7708537/21438612514068750, ... (Sloane's A057866 and A057867). For $n \geq 5$ and $n \equiv 1 \pmod 4$, the corresponding formula is slightly messier,

$$\zeta(n) = \frac{(2\pi)^n}{(n+1)!\,(n-1)}\sum_{k=0}^{(n+1)/4}(-1)^k(n+1-4k)\binom{n+1}{2k}B_{n+1-2k}B_{2k} - 2\sum_{k=1}^{\infty}\frac{e^{2\pi k}\left(1+\frac{4\pi k}{n-1}\right)-1}{k^n\left(e^{2\pi k}-1\right)^2} \tag{89}$$

(Cohen 2000).

Defining

$$S_{\pm}(n) \equiv \sum_{k=1}^{\infty}\frac{1}{k^n\left(e^{2\pi k}\pm 1\right)}, \tag{90}$$

the first few values can then be written

$$\zeta(3) = \tfrac{7}{180}\pi^3 - 2S_-(3) \tag{91}$$

$$\zeta(5) = \tfrac{1}{294}\pi^5 - \tfrac{72}{35}S_-(5) - \tfrac{2}{35}S_+(5) \tag{92}$$

$$\zeta(7) = \tfrac{19}{56700}\pi^7 - 2S_-(7) \tag{93}$$

$$\zeta(9) = \tfrac{125}{3704778}\pi^9 - \tfrac{992}{495}S_-(9) - \tfrac{2}{495}S_+(9) \tag{94}$$

$$\zeta(11) = \tfrac{1453}{425675250}\pi^{11} - 2S_-(11) \tag{95}$$

$$\zeta(13) = \frac{89}{257\,432\,175}\pi^{13} - \frac{16\,512}{8255} S_-(13) - \frac{2}{8255} S_+(13) \quad (96)$$

$$\zeta(15) = \frac{13\,687}{390\,769\,879\,500}\pi^{15} - 2\, S_-(15) \quad (97)$$

$$\zeta(17) = \frac{397\,549}{112\,024\,529\,867\,250}\pi^{17} - \frac{261\,632}{130\,815} S_-(17) - \frac{2}{130\,815} S_+(17) \quad (98)$$

$$\zeta(19) = \frac{7\,708\,537}{21\,438\,612\,514\,068\,750}\pi^{19} - 2\, S_-(19) \quad (99)$$

$$\zeta(21) = \frac{68\,529\,640\,373}{1\,881\,063\,815\,762\,259\,253\,125}\pi^{21} - \frac{4\,196\,352}{2\,098\,175} S_-(21) - \frac{2}{2\,098\,175} S_+(21) \quad (100)$$

(Plouffe 1998).

Another set of related formulas are

$$\zeta(3) = \frac{\pi^3}{28} + \frac{16}{7}\sum_{n=1}^{\infty}\frac{1}{n^3\left(e^{\pi n}+1\right)} - \frac{2}{7}\sum_{n=1}^{\infty}\frac{1}{n^3\left(e^{2\pi n}+1\right)} \quad (101)$$

$$\zeta(5) = 24\sum_{n=1}^{\infty}\frac{1}{n^5\left(e^{\pi n}-1\right)} - \frac{259}{10}\sum_{n=1}^{\infty}\frac{1}{n^5\left(e^{2\pi n}-1\right)} - \frac{1}{10}\sum_{n=1}^{\infty}\frac{1}{n^5\left(e^{4\pi n}-1\right)} \quad (102)$$

$$\zeta(5) = -\frac{7\pi^5}{1840} + \frac{328}{115}\sum_{n=1}^{\infty}\frac{1}{n^5\left(e^{\pi n}-1\right)} - \frac{419}{460}\sum_{n=1}^{\infty}\frac{1}{n^5\left(e^{2\pi n}-1\right)} - \frac{9}{115}\sum_{n=1}^{\infty}\frac{1}{n^5\left(e^{3\pi n}-1\right)} + \frac{261}{1840}\sum_{n=1}^{\infty}\frac{1}{n^5\left(e^{6\pi n}-1\right)} - \frac{9}{1840}\sum_{n=1}^{\infty}\frac{1}{n^5\left(e^{12\pi n}-1\right)} \quad (103)$$

$$\zeta(7) = \frac{304}{13}\sum_{n=1}^{\infty}\frac{1}{n^7\left(e^{\pi n}-1\right)} - \frac{103}{4}\sum_{n=1}^{\infty}\frac{1}{n^7\left(e^{2\pi n}-1\right)} - \frac{19}{52}\sum_{n=1}^{\infty}\frac{1}{n^7\left(e^{4\pi n}-1\right)} \quad (104)$$

$$\zeta(9) = \frac{64}{3}\sum_{n=1}^{\infty}\frac{1}{n^9\left(e^{\pi n}-1\right)} + \frac{441}{20}\sum_{n=1}^{\infty}\frac{1}{n^9\left(e^{2\pi n}-1\right)} - 32\sum_{n=1}^{\infty}\frac{1}{n^9\left(e^{3\pi n}-1\right)} - \frac{4763}{60}\sum_{n=1}^{\infty}\frac{1}{n^9\left(e^{4\pi n}-1\right)} + \frac{529}{8}\sum_{n=1}^{\infty}\frac{1}{n^9\left(e^{6\pi n}-1\right)} - \frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{n^9\left(e^{12\pi n}-1\right)} \quad (105)$$

(Plouffe 2006).

Multiterm sums for odd $\zeta(n)$ include

$$\zeta(5) = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}} - \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_{k-1}^{(2)}}{k^3\binom{2k}{k}} \quad (106)$$

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_{k-1}^{(4)}}{k^3\binom{2k}{k}} \tag{107}$$

$$\zeta(9) = \frac{9}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^9\binom{2k}{k}} - \frac{5}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_{k-1}^{(2)}}{k^7\binom{2k}{k}} + 5\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_{k-1}^{(4)}}{k^5\binom{2k}{k}} + \frac{45}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_{k-1}^{(6)}}{k^3\binom{2k}{k}} - \frac{25}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_{k-1}^{(2)}H_{k-1}^{(4)}}{k^5\binom{2k}{k}} \tag{108}$$

$$\zeta(11) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{11}\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_{k-1}^{(4)}}{k^7\binom{2k}{k}} - \frac{75}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}H_{k-1}^{(8)}}{k^3\binom{2k}{k}} + \frac{125}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\left[H_{k-1}^{(4)}\right]^2}{k^3\binom{2k}{k}} \tag{109}$$

(Borwein and Bradley 1997, 1997; Bailey *et al.* 2006, p. 71), where $H_n^{(r)}$ is a generalized harmonic number.

G. Huvent (2002) found the beautiful formula

$$\zeta(5) = -\frac{16}{11}\sum_{n=1}^{\infty}\frac{[2(-1)^n+1]\,h_n}{n^4}. \tag{110}$$

A number of sum identities involving $\zeta(n)$ include

$$\sum_{n=2}^{\infty}[\zeta(n)-1] = 1 \tag{111}$$

$$\sum_{n=2,4,\ldots}^{\infty}[\zeta(n)-1] = \frac{3}{4} \tag{112}$$

$$\sum_{n=3,5,\ldots}^{\infty}[\zeta(n)-1] = \frac{1}{4} \tag{113}$$

$$\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1] = \frac{1}{2}. \tag{114}$$

Sums involving integers multiples of the argument include

$$\sum_{n=1}^{\infty}[\zeta(2n)-1] = \tfrac{3}{4} \qquad (115)$$

$$\sum_{n=1}^{\infty}[\zeta(3n)-1] = \tfrac{1}{3}\left[-(-1)^{2/3} H_{\left(3-i\sqrt{3}\right)/2} + (-1)^{1/3} H_{\left(3+i\sqrt{3}\right)/2}\right] \qquad (116)$$

$$\sum_{n=1}^{\infty}[\zeta(4n)-1] = \tfrac{1}{8}(7-2\coth\pi), \qquad (117)$$

where H_n is a harmonic number.

Two surprising sums involving $\zeta(x)$ are given by

$$\sum_{k=2}^{\infty}\frac{(-1)^k \zeta(k)}{k} = \gamma \qquad (118)$$

$$\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k} = 1-\gamma, \qquad (119)$$

where γ is the Euler-Mascheroni constant (Havil 2003, pp. 109 and 111-112). Equation (118) can be generalized to

$$\sum_{k=2}^{\infty}\frac{(-x)^k \zeta(k)}{k} = x\gamma + \ln(x!) \qquad (120)$$

(T. Drane, pers. comm., Jul. 7, 2006) for $-1 < x \leq 1$.

Other unexpected sums are

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)2^{2n}} = \ln\pi - 1 \qquad (121)$$

(Tyler and Chernhoff 1985; Boros and Moll 2004, p. 248) and

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)} = \ln(2\pi) - 1. \qquad (122)$$

(121) is a special case of

$$\sum_{k=1}^{\infty} \frac{\zeta(2k, z)}{k(2k+1)2^{2k}} = (2z-1)\ln\left(z-\tfrac{1}{2}\right) - 2z + 1 + \ln(2\pi) - 2\ln\Gamma(z), \tag{123}$$

where $\zeta(s, a)$ is a Hurwitz zeta function (Danese 1967; Boros and Moll 2004, p. 248).

Considering the sum

$$S_n = \sum_{k=2}^{n-2} \frac{\zeta(k)\zeta(n-k)}{2^k}, \tag{124}$$

then

$$\lim_{n\to\infty} S_n = \ln 2, \tag{125}$$

where $\ln 2$ is the natural logarithm of 2, which is a particular case of

$$\lim_{n\to\infty} \sum_{k=2}^{n-2} \zeta(k)\zeta(n-k)x^{k-1} = x^{-1} - \psi_0(-x) - \gamma, \tag{126}$$

where $\psi_0(z)$ is the digamma function and γ is the Euler-Mascheroni constant, which can be derived from

$$\sum_{k=2}^{\infty} \zeta(k)x^{k-1} = -\psi_0(1-x) - \gamma \tag{127}$$

(B. Cloitre, pers. comm., Dec. 11, 2005; cf. Borwein *et al.* 2000, eqn. 27).

A generalization of a result of Ramanujan (who gave the $m=1$ case) is given by

$$\sum_{k=1}^{\infty} \frac{1}{[k(k+1)]^{2m+1}} = -2\sum_{k=0}^{m} \zeta(2k) \frac{2m+1-2k}{2m+i-2k} \tag{128}$$

(B. Cloitre, pers. comm., Sep. 20, 2005).

An additional set of sums over $\zeta(n)$ is given by

$$C_1 = \sum_{n=2}^{\infty} \frac{\zeta(n)}{n!} \tag{129}$$

$$= \int_0^\infty \frac{I_1(2\sqrt{u}) - \sqrt{u}}{(e^u - 1)\sqrt{u}}\, du \quad (130)$$

$$= \int_0^\infty \frac{{}_0\tilde{F}_1(;2;u) - 1}{e^u - 1}\, du \quad (131)$$

$$\approx 1.078189 \quad (132)$$

$$C_2 = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n!} \quad (133)$$

$$= \sum_{n=1}^{\infty} e^{1/n^2} - 1 \quad (134)$$

$$= \int_0^1 \frac{u\, {}_0F_2(;\tfrac{3}{2},2;\tfrac{1}{4}u^4)}{e^u - 1}\, du \quad (135)$$

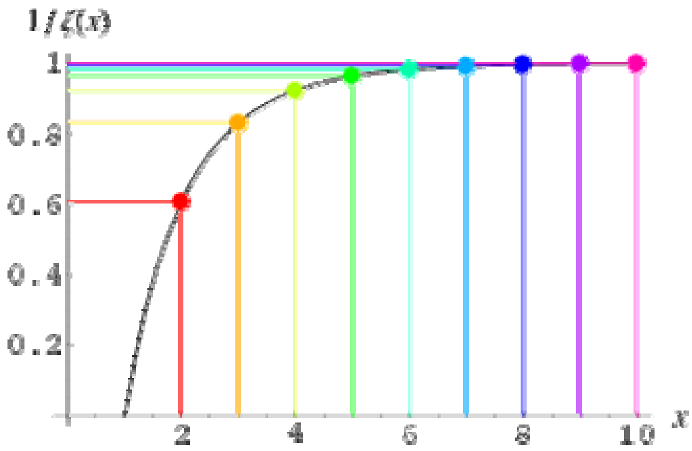
$$\approx 2.407447 \quad (136)$$

$$C_3 = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n)!} \quad (137)$$

$$= \int_0^\infty \frac{u + {}_0\tilde{F}_1(;2;-u) - {}_0\tilde{F}_1(;2;u)}{2(1 - e^u)}\, du \quad (138)$$

$$\approx 0.869002. \quad (139)$$

(Sloane's A093720, A076813, and A093721), where $I_n(z)$ is a modified Bessel function of the first kind, ${}_p\tilde{F}_q$ is a regularized hypergeometric function. These sums have no known closed-form expression.



The inverse of the Riemann zeta function $1/\zeta(p)$, plotted above, is the asymptotic density of pth-powerfree numbers (i.e., squarefree numbers, cubefree numbers, etc.). The following table gives the number $Q_p(n)$ of pth-powerfree numbers $\leq n$ for several values of n.

p	$1/\zeta(p)$	$Q_p(10)$	$Q_p(100)$	$Q_p(10^3)$	$Q_p(10^4)$	$Q_p(10^5)$	$Q_p(10^6)$
2	0.607927	7	61	608	6083	60794	607926
3	0.831907	9	85	833	8319	83190	831910
4	0.923938	10	93	925	9240	92395	923939
5	0.964387	10	97	965	9645	96440	964388
6	0.982953	10	99	984	9831	98297	982954

SEE ALSO: Abel's Functional Equation, Berry Conjecture, Critical Line, Critical Strip, Debye Functions, Dirichlet Beta Function, Dirichlet Eta Function, Dirichlet Lambda Function, Euler Product, Harmonic Series, Hurwitz Zeta Function, Khinchin's Constant, Lehmer's Phenomenon, Montgomery's Pair Correlation Conjecture, *p*-Series, Periodic Zeta Function, Prime Number Theorem, Psi Function, Riemann Hypothesis, Riemann P-Series, Riemann-Siegel Functions, Riemann-von Mangoldt Formula, Riemann Zeta Function *zeta*(2), Riemann Zeta Function Zeros, Stieltjes Constants, Voronin Universality Theorem, Xi-Function