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INFINITE CONFORMAL SYMMETRY IN TWO-DIMENSIONAL QUANTUM FIELD THEORY

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We present an investigation of the massless, two-dimensional, interacting field theories Their basic property is their invariance under an infinite-dimensional group of conformal (analytic) transformations It is shown that the local fields forming the operator algebra can be classified according to the irreducible representations of Virasoro algebra, and that the correlation functions are built up of the "conformal blocks" which are completely determined by the conformal invariance Exactly solvable conformal theories associated with the degenerate representations are analyzed In these theories the anomalous dimensions are known exactly and the correlation functions satisfy the systems of linear differential equations

1. Introduction

Conformal symmetry was introduced into quantum field theory about twelve years ago due to the scaling ideas in the second-order phase transition theory (see [1] and references therein). According to the scaling hypothesis, the interaction of the fields of the order parameters in the critical point is invariant with respect to the scale transformations

$$\xi^a \to \lambda \xi^a, \tag{1.1}$$

where ξ^a are the coordinates, $a = 1, 2, \dots, D$. In the quantum field theory the scale symmetry (1.1) takes place provided the stress-energy tensor is traceless

$$T_a^a(\xi) = 0 \tag{1.2}$$

Under the condition (1.2) the theory possesses not only the scale symmetry but is also invariant with respect to the coordinate transformations

$$\xi^a \to \eta^a(\xi) \tag{1.3}$$

having the property that the metric tensor transforms as

$$g_{ab} \to \frac{\partial \xi^{a'}}{\partial \eta^a} \frac{\partial \xi^{b'}}{\partial \eta^b} g_{a'b'} = \rho(\xi) g_{ab}, \tag{1.4}$$

where $\rho(\xi)$ is a certain function. Coordinate transformations of this type constitute the *conformal group*. These transformations can be easily described, properties of the conformal group being different for the cases $D>2$ and $D=2$. If $D>2$, the conformal group is finite-dimensional and consists of translations, rotations, dilatations and special conformal transformations (see [2, 3]) Kinematic manifestation of this symmetry and its dynamical realization in the quantum field theory has been investigated in many papers (see for example, [2–4]). In particular, it has been shown that the local fields $A_j(\xi)$, involved in the conformal theory, should possess anomalous scale dimensions d_j, i.e they transform as follows under the transformation (1)

$$A_j \to \lambda^{-d_j} A_j, \tag{1.5}$$

where the parameters d_j are non-negative. Computation of the spectrum $\{d_j\}$ of the anomalous dimensions is the most important problem of the theory since these quantities determine the critical exponents.

To solve this problem, in [4] the bootstrap approach based on the operator algebra hypothesis has been proposed. Let us describe it in some detail since it is most suitable for our purposes. The operator algebra is a strong version of the Wilson operator product expansion [5], namely, if the existence of an infinite set of local fields $A_j(\xi)$ is assumed, then the set of operators $\{A_j(0)\}$ is assumed to be complete in the sense specified below. The set $\{A_j\}$ contains the identity operator I as well as all coordinate derivatives of each field involved. The completeness of the set $\{A_j(0)\}$ means that any state can be generated by the linear action of these operators This condition is equivalent to the operator algebra

$$A_i(\xi)A_j(0) = \sum_k C_{ij}^k(\xi) A_k(0), \tag{1.6}$$

where the structure constants $C_{ij}^k(\xi)$ are the c-number functions which should be single-valued so that local properties be taken into account. The relation (1.6) is understood as an exact expansion of the correlation functions

$$\langle A_i(\xi)A_j(0)A_{l_1}(\xi_1)\ldots A_{l_N}(\xi_N)\rangle = \sum_k C_{ij}^k(\xi)\langle A_k(0)A_{l_1}(\xi_1)\ldots A_{l_N}(\xi_N)\rangle,$$

which is convergent in some finite domain of ξ, the domain being certainly dependent on the location of $\xi_1, \ldots, \xi_N$. The most rigid requirement, considered as the main dynamical principle of this approach, is associativity of the operator algebra (1.6). This requirement leads to an infinite system of equations for the structure constants $C_{ij}^k(\xi)$. Since the conformal symmetry fixes the form of the functions $C_{ij}^k(\xi)$ up to some numerical parameters (which are the anomalous dimensions and numerical factors), this system of equations has to determine these parameters However in the multidimensional theory ($D > 2$) this system proves to be too complicated to be solved exactly, the main difficulty being the classification of the fields A_j entering the algebra.

The situation is somewhat better in two dimensions. The main reason is that the conformal group is infinite-dimensional in this case, it consists of the conformal analytical transformations. To describe this group, it is convenient to introduce the complex coordinates

$$z = \xi^1 + i\xi^2, \qquad \bar{z} = \xi^1 - i\xi^2, \tag{1 7}$$

the metric having the form

$$ds^2 = dz\,d\bar{z} \tag{1 8}$$

The conformal group of the two-dimensional space which will be denoted by $\mathcal{G}$, consists of all substitutions of the form

$$z \to \zeta(z), \qquad \bar{z} \to \bar{\zeta}(\bar{z}), \tag{1 9}$$

where ζ and $\bar{\zeta}$ are arbitrary analytical functions

For our purposes it will be convenient to consider the space coordinates ξ^1, ξ^2 as complex variables, i.e to deal with the complex space $\mathbb{C}^2$ Therefore in general we shall treat the coordinates (1.7) not as complex conjugated but as two independent complex variables; the same is supposed for the functions (1 9) This space $\mathbb{C}^2$ has the complex metric (1.8) The euclidean plane and Minkowski space-time can be obtained as appropriate real sections of this complex space

In the complex case it is clear from (1 9) that the conformal group $\mathcal{G}$ is a direct product

$$\mathcal{G} = \Gamma \otimes \bar{\Gamma}, \tag{1 10}$$

where Γ ($\bar{\Gamma}$) is a group of the analytical substitutions of the variable z ($\bar{z}$) In what follows we shall often concentrate on properties of the group Γ, keeping in mind that the same properties hold for $\bar{\Gamma}$

Infinitesimal transformations of the group Γ are

$$z \to z + \varepsilon(z), \tag{1.11}$$

where $\varepsilon(z)$ is an infinitesimal analytical function It can be represented as an infinite Lourant series

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} \varepsilon_n z^{n+1} \tag{1.12}$$

Therefore the Lie algebra of the group Γ coincides with the algebra of differential operators

$$l_n = z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z}, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{1.13}$$

the commutation relations having the form

$$[l_n, l_m] = (n-m)l_{n+m}. \tag{1.14}$$

The generators $\bar{l}_n$ of the group $\bar{\Gamma}$ satisfy the same commutation relations, the operators l_n and $\bar{l}_m$ being commutative. We shall denote the algebra (1.14) as $\mathfrak{L}_0$.

The generators l_{-1}, l_0, l_{+1} form the subalgebra $\mathrm{sl}(2,\mathbb{C}) \subset \mathfrak{L}_0$. The corresponding subgroup $\mathrm{SL}(2,\mathbb{C}) \subset \Gamma$ consists of the projective transformations

$$z \to \zeta = \frac{az+b}{cz+d}, \qquad ad - bc = 1 \tag{1.15}$$

Note that the projective transformations are uniquely invertible mappings of the whole z-plane on itself and these are the only conformal transformations with this property

This is the first paper of the series we intend to devote to the general properties of the two-dimensional quantum field theory, invariant with respect to the conformal group $\mathcal{G}$* In this paper we give the general classification of the fields $A_j(\xi)$ entering the operator algebra (1.6) according to the representations of the conformal group and investigate special "exactly solvable" cases of the conformal quantum field theory associated with degenerate representations. In more detail we shall show the following

(i) The components of the stress-energy tensor $T_{ab}(\xi)$ (satisfying (1.2)) represent the generators of the conformal group $\mathcal{G}$ in the quantum field theory. The algebra of these generators is the central extension of the algebra $\mathfrak{L}_0$ (1.14) and coincides with the *Virasoro algebra* $\mathfrak{L}_c$. The value of the central charge c is the parameter of the theory

(ii) Among the fields $A_j(\xi)$ forming the operator algebra, there are some *primary fields* $\phi_n(\xi)$ which transform in the simplest way

$$\phi_n(z,\bar{z}) \to \left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^{\Delta_n}\left(\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}\bar{z}}\right)^{\bar{\Delta}_n}\phi_n(\zeta,\bar{\zeta}) \tag{1.16}$$

* Although the projective group (1 15) and the complete conformal group $\mathcal{G}$ are both consequences of (1 2) and therefore appear in the quantum field theory together, we found it instructive to consider first the general consequences of the projective symmetry The corresponding formulae, which are certainly no other than the particular case $D = 2$ of the results of refs [2–4], are presented in appendix A

under the substitutions (1 9) Here Δ_n and $\bar{\Delta}_n$ are real non-negative parameters. In fact, the combinations $d_n = \Delta_n + \bar{\Delta}_n$ and $s_n = \Delta_n - \bar{\Delta}_n$ are the anomalous scale dimension and the spin of the field ϕ_n, respectively* We shall often refer to the quantities Δ_n and $\bar{\Delta}_n$ as to the *dimensions* of the field. The simplest example of the primary field is the identity operator I A nontrivial theory involves more than one primary field and the index n is introduced to distinguish between them

(iii) A complete set of the fields $A_j(\xi)$ consists of *conformal families* $[\phi_n]$, each corresponding to a certain primary field ϕ_n. The primary field ϕ_n belongs to the *conformal family* $[\phi_n]$ and, in some sense, serves as the ancestor of the family. Each conformal family also contains infinitely many other secondary fields (descendants). Dimensions of these secondary fields form integer spaced series

$$\Delta_n^{(k)} = \Delta_n + k, \qquad \bar{\Delta}_n^{(\bar{k})} = \bar{\Delta}_n + \bar{k}, \tag{1 17}$$

where $k, \bar{k} = 0, 1, 2, \ldots$ Variations of any secondary field $A \in [\phi_n]$ under the infinitesimal conformal transformations (1.11) are expressed linearly in terms of representations of the same conformal family $[\phi_n]$. So, each conformal family corresponds to some representation of the conformal group $\mathcal{G}$. In accordance with (1 10), this representation is a direct product $[\phi_n] = V_n \otimes \bar{V}_n$, where V_n and $\bar{V}_n$ are representations of the Virasoro algebra $\mathfrak{L}_c$**, in general, these representations are irreducible.

(iv) Correlation functions of any secondary fields can be expressed in terms of the correlators of the corresponding primary fields by means of special linear differential operators Therefore, all information about the conformal quantum field theory is accumulated in the correlators of the primary field ϕ_n.

(v) The structure constants $C_{ij}^k(\xi)$ of the operator algebra (1 6) can, in principle, be computed in terms of the coefficients C_{nm}^l of the primary field ϕ_l in the operator product expansion of $\phi_n \phi_m$. Therefore, the bootstrap equations (i.e the associativity condition for the operator algebra) can be reduced to equations imposing constraints upon these coefficients and the dimensions Δ_n of the primary field.

(vi) At a given value of the charge c there are infinitely many special values of the dimension Δ such that the representation $[\phi_\Delta]$ proves to be degenerate The most important property of the corresponding "degenerate" primary field ϕ_Δ is that the correlation functions involving this field, satisfy special linear differential equations, the simplest example of which is the hypergeometry equation

(vii) If the parameter c satisfies the equation

$$\frac{\sqrt{25-c} - \sqrt{1-c}}{\sqrt{25-c} + \sqrt{1-c}} = \frac{p}{q}, \tag{1 18}$$

* The spin s_n of a local field can take an integer or half-integer value only

** The representation V_n is known as the Verma modulus over the Virasoro algebra (see, for example, [6]) This representation is evidently characterized by the parameter Δ_n only where p and q are positive integers, the "minimal" conformal quantum field theory can be constructed so that it be exactly solvable in the following sense (i) A finite number of conformal families $[\phi_n]$ is involved in the operator algebra, each of them being degenerate, (ii) all anomalous dimensions Δ_n are known exactly, (iii) all correlation functions of the theory can be computed as solutions of special systems of linear partial differential equations. There are infinitely many conformal quantum field theories of this type, each associated with a certain solution of (1.18), the simplest nontrivial example ($c = \frac{1}{2}$) describing the critical theory of the two-dimensional Ising model. We suppose that other "minimal" conformal theories describe second-order phase transitions in some two-dimensional spin systems with discrete symmetry groups.

Apart from second-order phase transitions in two dimensions, there is another application of the conformal quantum field theory. This is the dual theory. From the mathematical point of view dual models are no other than special kinds of the two-dimensional conformal quantum field theory. This is natural in view of their association with the string theory. Quantum fields describe the degrees of freedom associated with the string, the conformal symmetry being a manifestation of the reparametrization invariance of the world surface swept out by the string. In fact, the dual amplitudes are expressed in terms of correlation functions of some local fields (vertex operators). In standard models (like the Veneziano model) vertex operators are related in a simple way to free massless fields. We suppose that if considerably interacting fields are incorporated into the theory, it can produce new types of dual models with more suitable physical properties.

2. Stress-energy tensor in the conformal quantum field theory

Consider an arbitrary correlation function of the form

$$\langle X \rangle = \langle A_{j_1}(\xi_1) \ldots A_{j_N}(\xi_N) \rangle, \tag{2.1}$$

where $A_{j_k}(\xi)$ are local fields, and perform an infinitesimal coordinate transformation

$$\xi^a \to \xi^a + \varepsilon^a(\xi). \tag{2.2}$$

As is well known in quantum field theory, the following relation is valid

$$\sum_{k=1}^{N} \langle A_{j_1}(\xi_1) \ldots A_{j_{k-1}}(\xi_{k-1}) \delta_\varepsilon A_{j_k}(\xi_k) A_{j_{k+1}}(\xi_{k+1}) \ldots A_{j_N}(\xi_N) \rangle$$

$$+ \int d^2\xi \, \partial^a \varepsilon^b(\xi) \langle T_{ab}(\xi) X \rangle = 0, \tag{2.3}$$

where the field $T_{ab}(\xi)$ is the stress-energy tensor and $\delta_\varepsilon A_j$ denotes variations of the fields A_j under the transformation (2.2). Due to their local properties, these variations are linear combinations of a finite number of derivatives of the function $\varepsilon(\xi)$ taken at the point $\xi = \xi_k$, the coefficients being certain local fields It follows from (2.3) that

$$\partial_a \langle T^{ab}(\xi) X \rangle = 0 \tag{2.4}$$

everywhere but at the points $\xi_1, \xi_2, \ldots, \xi_N$ In the conformal quantum field theory the trace of the stress-energy tensor vanishes, $T_a^a = 0$. Therefore in two dimensions this tensor has only two independent components which can be chosen in the form

$$T(\xi) = T_{11} - T_{22} + 2iT_{12},$$
$$\bar{T}(\xi) = T_{11} - T_{22} - 2iT_{12} \tag{2.5}$$

Combining relations (1 2) and (2.4), it is easy to find that these components satisfy the Cauchy-Riemann equations

$$\partial_{\bar{z}} \langle T(\xi) X \rangle = 0,$$
$$\partial_z \langle \bar{T}(\xi) X \rangle = 0, \tag{2 6}$$

where z and $\bar{z}$ are defined by (1 7) So, each of the fields T and $\bar{T}$ is an analytic function of the single variable (z and $\bar{z}$, respectively) and we shall write

$$T = T(z), \qquad \bar{T} = \bar{T}(\bar{z}) \tag{2 7}$$

Take now the correlation function*

$$\langle T(z) X \rangle \tag{2 8}$$

It is the analytic function of z that is single-valued (due to its local properties) and regular everywhere but at the points $z = z_k$, $z_k = \xi_k^1 + i\xi_k^2$, where it has poles, the orders and residues of these poles being determined by the conformal properties of the fields $A_{j_k}(\xi)$. Actually, for the conformal coordinate transformations (1 11) the relation (2 3) can be reduced to the form

$$\langle \delta_\varepsilon X \rangle = \oint_C \mathrm{d}\zeta\, \varepsilon(\zeta) \langle T(\zeta) X \rangle, \tag{2 9}$$

* Here and below we generally consider correlation functions in the complex space $\mathbb{C}^2$, see the introduction

where $\delta_\varepsilon X$ is a variation of the product $X = A_{j_1}(\xi_1) \dots A_{j_N}(\xi_N)$ under the transformation (1.11) and the contour C encloses all singular points z_k, $k=1, \dots, N$. Equivalently, the following relation is valid

$$\delta_\varepsilon A_j(z, \bar z) = \oint_{C_z} d\zeta\, \varepsilon(\zeta) T(\zeta) A_j(z, \bar z), \tag{2.10}$$

where the contour C_z surrounds the point z. The same formula (with the substitution $T \to \bar T$) holds for the variation $\delta_{\bar\varepsilon} A_j$ of the field A_j under the infinitesimal transformation

$$\bar z \to \bar z + \bar\varepsilon(z) \tag{2.11}$$

of the group $\bar\Gamma$. Therefore the fields $T(z)$ and $\bar T(\bar z)$ represent the generators of the conformal group $\Gamma \otimes \bar\Gamma$ in the quantum field theory.

The conformal transformation laws for general fields A_j will be considered in the next section. Now we are interested in the conformal properties of the fields $T(z)$ and $\bar T(\bar z)$ themselves which are obviously related to the algebra of the conformal group generators. The variations $\delta_\varepsilon T$ and $\delta_\varepsilon \bar T$ should be expressed linearly in terms of the same fields T and $\bar T$ and their derivatives and may also include the c-number Schwinger terms. Taking into account tensorial properties of the field $T(z)$ and the locality condition, write down the following most general expression for the variation $\delta_\varepsilon T$:

$$\delta_\varepsilon T(z) = \varepsilon(z) T' + 2\varepsilon'(z) T(z) + \tfrac{1}{12} c \varepsilon'''(z), \tag{2.12}$$

where the prime denotes the z-derivative*. For the variation $\delta_{\bar\varepsilon} T$ it is possible to get

$$\delta_{\bar\varepsilon} T(z) = 0 \tag{2.13}$$

* Formula (2.12) corresponds to the following transformation of $T(z)$ under the finite conformal substitution (1.9)

$$T(z) \to \left(\frac{d\zeta}{dz}\right)^2 T(\zeta) + \tfrac{1}{12} c \{\zeta, z\},$$

where $\{\zeta, z\}$ is the Schwartz derivative [12]

$$\{\zeta, z\} = \left(\frac{d^3\zeta}{dz^3} \Big/ \frac{d\zeta}{dz}\right) - \frac{3}{2}\left(\frac{d^2\zeta}{dz^2} \Big/ \frac{d\zeta}{dz}\right)^2$$

Note, that the Schwartz derivative satisfies the following composition law

$$\{w, z\} = \left(\frac{d\zeta}{dz}\right)^2 \{w, \zeta\} + \{\zeta, z\}$$

The numerical constant c in the relation (2 12) is not determined by the general principles, it should be treated as the parameter of the theory The variation $\delta_{\bar{\varepsilon}}\bar{T}$ satisfies the same relation (2.1), the respective constant $\bar{c}$ being equal to c. The constant c can take real positive values. These statements result from the reality condition for the stress-energy tensor in euclidean space and Minkowski space-time

If none of the points z_k, $k = 1, 2, \ldots, N$ in (2.1) is equal to infinity, the correlation function $\langle T(z)X\rangle$ should be regular at $z = \infty$. This means that, as can be easily verified by means of the transformation law (2.12), that the function $\langle T(z)X\rangle$ decreases as

$$T(z) \sim \frac{1}{z^4} \qquad \text{at} \qquad z \to \infty . \tag{2.14}$$

In the quantum field theory the correlation functions (2 1) are represented as vacuum expectation values of the time-ordered products of the local field operators $A_j(\xi)$. In our case it is convenient to introduce the coordinates σ and τ according to the formulae

$$z = \exp(\tau + i\sigma), \qquad \bar{z} = \exp(\tau - i\sigma). \tag{2.15}$$

Choosing σ and τ as real, σ being an angular variable, $0 < \sigma \leqslant \pi$, one gets the euclidean real section. Correlation functions in this euclidean space can be represented as

$$\langle X\rangle = \langle 0|T\left[A_{j_1}(\sigma_1, \tau_1) \,..\, A_{j_N}(\sigma_N, \tau_N)\right]|0\rangle, \tag{2.16}$$

where the chronological ordering should be performed with respect to the "euclidean time" τ. In the operator formalism the variations $\delta_\varepsilon A_j$ can be expressed in terms of equal time commutators

$$\delta_\varepsilon A_j(\sigma, \tau) = \left[T_\varepsilon, A_j(\sigma, \tau)\right], \tag{2 17}$$

where the generators T_ε are defined by the formula

$$T_\varepsilon = \oint_{\log|z| = \tau} \varepsilon(z) T(z)\, \mathrm{d}z \tag{2.18}$$

Note that due to eqs (2.7) these operators are in fact τ-independent.

The relation (2 12) becomes

$$\left[T_\varepsilon, T(z)\right] = \varepsilon(z)T'(z) + 2\varepsilon'(z)T(z) + \tfrac{1}{12}c\varepsilon'''(z) \tag{2 19}$$

It is useful to introduce the operators $L_n, \bar{L}_n$, $n = 0, \pm 1, \pm 2, \ldots$ as coefficients of the Lourant expansions

$$T(z)=\sum_{n=-\infty}^{\infty}\frac{L_n}{z^{n+2}},\qquad \bar{T}(\bar{z})=\sum_{n=-\infty}^{\infty}\frac{\bar{L}_n}{\bar{z}^{n+2}} \tag{2 20}$$

It follows from (2 19) that the operators L_n satisfy the commutation relations.

$$[L_n, L_m]=(n-m)L_{n+m}+\tfrac{1}{12}c(n^3-n)\delta_{n+m,0} \tag{2.21}$$

Clearly, the same relations are satisfied by the $\bar{L}_n$'s, the operators L_n and $\bar{L}_m$ being commutative The algebra (2.21) of the conformal generators L_n is the central extension of the algebra (1 14)★ This is well known in the dual theory and the algebra (2 21) is called the Virasoro algebra [11]; we shall denote it as $\mathfrak{L}_c$

Like the algebra $\mathfrak{L}_0$, the Virasoro algebra $\mathfrak{L}_c$ contains a subalgebra sl(2,C), generated by the operators L_{-1}, L_0, L_{+1} (note that the c-number term in (2 21) vanishes for $n=0, \pm 1$) In particular, the operators L_{-1} and $\bar{L}_{-1}$ generate translations whereas L_0 and $\bar{L}_0$ generate infinitesimal dilatations of the coordinates z and $\bar{z}$. In the coordinate system σ, τ defined by (2 15) the operator

$$H=L_0+\bar{L}_0, \tag{2 22}$$

is a generator of "time" shifts. It plays the role of the hamiltonian. Note, that the "infinite past" $\tau\to-\infty$ and the "infinite future" $\tau\to\infty$ correspond to the points $z=0$ and $z=\infty$, respectively.

The vacuum $|0\rangle$ in (2.16) is the ground state of the hamiltonian (2 22) The vacuum must satisfy the equations

$$L_n|0\rangle=0, \qquad \text{if} \qquad n\geqslant -1, \tag{2 23}$$

since otherwise the stress-energy tensor would have been singular at $z=0$. Note that the operators L_n with $n\geqslant-1$ generate the conformal transformations which are regular at $z=0$ Therefore eqs (2 23) are manifestations of the conformal invariance of the vacuum The transformations generated by the operators L_n with $n\leqslant-2$ are singular at $z=0$; these operators distort the vacuum

$$L_n|0\rangle=\text{new states} \qquad \text{if} \qquad n\leqslant-2 \tag{2 24}$$

The field $T(z)$ should also be regular at $z=\infty$ Similarly to (2.23), it implies that

$$\langle 0|L_n=0 \qquad \text{if} \qquad n\leqslant 1 \tag{2.25}$$

Since in the Minkowski space-time (which can be obtained if imaginary values of τ

★ This central extension has been discovered by Gelfand and Fuks [10] are dealt with), the field $T(z)$ must be real, the operators L_n satisfy the conjugation relation

$$L_n^+ = L_{-n} \tag{2 26}$$

Note that the generators L_{-1}, L_0, L_1 annihilate both the "in" and "out" vacuua

$$\langle 0|L_s = L_s|0\rangle = 0, \qquad s = 0, \pm 1 \tag{2 27}$$

These equations are manifestation of the regularity of projective transformations mentioned in the introduction. Eqs. (2 27) are self-consistent because the c-number term in (2 21) vanishes for $n = 0, \pm 1$.

Eqs (2.23), (2.25) and the commutation relations (2 21) enable one to compute any correlation function of the form*

$$\langle T(\zeta_1) . \ T(\zeta_N)\bar{T}(\eta_1) \quad \bar{T}(\eta_M)\rangle = \langle T(\zeta_1) .. \ T(\zeta_N)\rangle\langle\bar{T}(\eta_1) . \ \bar{T}(\eta_M)\rangle . \tag{2 28}$$

In particular, a two-point function is given by the formula

$$\langle T(\zeta_1)T(\zeta_2)\rangle = c(\zeta_1 - \zeta_2)^{-4}, \tag{2 29}$$

which shows that $c > 0$.

3. Ward identities and conformal families

Consider the variation $\delta_\varepsilon A_j(\xi)$ of a certain local field A_j under the infinitesimal conformal transformation (1 11) Due to its local properties, this variation is a linear combination of the function $\varepsilon(z)$ and a finite number of its derivatives taken at the point $z = \xi^1 + i\xi^2$

$$\delta_\varepsilon A_j(z) = \sum_{k=0}^{\nu_j} B_j^{(k-1)}(z)\frac{\mathrm{d}^k}{\mathrm{d}z^k}\varepsilon(z), \tag{3 1}$$

where $B_j^{(k-1)}$ are local fields belonging to the set $\{A_j\}$ and ν_j is a certain integer In

* It can be shown that these correlators coincide with those of the fields

$$T^{(0)} = \varphi_z\varphi_z + 2\alpha_0\varphi_{zz}$$

where φ is a free massless boson field and the parameter α_0 is defined by the formula

$$c = 1 + 24\alpha_0^2$$

(3 1) we have omitted the argument $\bar{z}$ which is not important here. The study of infinitesimal translations and dilatations of the variable shows that the first and second coefficients in (3.1) are

$$B_j^{(-1)}(z)=\frac{\partial}{\partial z}A_j(z), \qquad B_j^0(z)=\Delta_j A_j(z), \tag{3.2}$$

where Δ_j is the dimension of the field A_j. It is evident that the dimensions of the fields $B_j^{(k-1)}$ in (3 1) are equal to

$$\Delta_{j,(k-1)}=\Delta_j+1-k, \qquad k=0,1,\ldots,\nu_j. \tag{3.3}$$

Let us take again the correlation function (2 8). As has already been mentioned in the previous section, this correlator is a single-valued analytic function of z, possessing the poles at $z=z_k$, $k=1,2,\ldots N$. In virtue of (2 10) and (3.1) it is possible to write down the relation

$$\langle T(z)A_{j_1}(z_1)\ldots A_{j_N}(z_N)\rangle=\sum_{l=1}^{N}\sum_{k=0}^{\nu_l}k!(z-z_l)^{-k-1}\langle A_{j_1}(z_1)\ldots$$
$$A_{j_{l-1}}(z_{l-1})B_{j_l}^{(k-1)}(z_l)A_{j_{l+1}}(z_{l+1})\ldots A_{j_N}(z_N)\rangle. \tag{3 4}$$

This formula is a general form of the conformal Ward identities.

In a physically suitable theory the dimensions Δ_j of all the fields A_j should satisfy the inequality

$$\Delta_j\geqslant 0, \tag{3 5}$$

since otherwise the theory will possess correlations increasing with distance In what follows we shall suppose that the only field with zero dimensions $\Delta=\bar{\Delta}=0$ is the identity operator I. Comparing (3.3) with condition (3.5) we see that the sum in (3.1) contains a finite number of terms $\nu_j\leqslant\Delta_j+1$ Another important conclusion following from (3 3) is that the spectrum of dimensions $\{\Delta_j\}$ in any two-dimensional conformal quantum field theory consists of the infinite integer spaced series

$$\Delta_n^{(k)}=\Delta_n+k, \qquad k=0,1,2,\ldots \tag{3.6}$$

Here Δ_n denotes the minimal dimension of each series, whereas the index n labels the series. The same is obviously valid for the dimensions $\bar{\Delta}_j$, i e. the spectrum $\{\bar{\Delta}_j\}$ also consists of the series

$$\bar{\Delta}_n^{(k)}=\bar{\Delta}_n+k, \qquad k=0,1,2,\ldots \tag{3 7}$$

Let ϕ_n be the field with the dimensions Δ_n and $\bar{\Delta}_n$. The variation (3.1) of this field has the simplest possible form

$$\delta_\varepsilon \phi_n(z) = \varepsilon(z)\frac{\partial}{\partial z}\phi_n(z) + \Delta_n \varepsilon'(z)\phi_n(z), \tag{3.8}$$

since the corresponding fields $B^{(k-1)}$ with $k>0$ would have dimensions smaller than Δ_n. A similar formula holds for the variation $\delta_{\bar{\varepsilon}}\phi_n$. The finite form of this conformal transformation law is given by (1.16). We shall call the operators ϕ_n having the transformation laws (1.16) the *primary fields*. Note that formula (3.8) is equivalent to the commutation relation:

$$[L_m, \phi_n(z)] = z^{m+1}\frac{\partial}{\partial z}\phi_n(z) + \Delta_n(m+1)z^m\phi_n(z), \tag{3.9}$$

which are satisfied by the vertex operators of the dual theory [8, 9].

If all the fields $A_j(\xi)$ entering the correlation function (2.8) are primary, the general relation (3.4) is reduced to the form

$$\langle T(z)\phi_1(z_1)\dots\phi_N(z_N)\rangle = \sum_{i=1}^{N}\left\{\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right\}\langle\phi_1(z_1)\dots\phi_N(z_N)\rangle, \tag{3.10}$$

where $\Delta_1, \Delta_2, \dots, \Delta_N$ are dimensions of the primary fields $\phi_1, \phi_2, \dots, \phi_N$, respectively. Note that this Ward identity explicitly relates the correlation functions $\langle T(z)\phi_1\dots\phi_N\rangle$ to the correlators $\langle\phi_1\dots\phi_N\rangle$. It is also noteworthy that the projective conformal Ward identities (A.6) can be directly derived from (3.10) if one takes into account the asymptotic condition (2.14).

The primary fields themselves cannot form the closed operator algebra. In fact, there are infinitely many other fields associated with each of the primary fields ϕ_n. We shall refer to these fields as to the secondary fields with respect to the primary fields ϕ_n. The dimensions of the secondary fields form the integer spaced series, mentioned above. These fields together with the primary field ϕ_n constitute a *conformal family* $[\phi_n]$. It is essential that under the transformations every member of each conformal family transforms in terms of the representatives of the same conformal family. So, each conformal family forms some irreducible representation of the conformal algebra. The complete set of the fields $\{A_j\}$ consists of some number (which can be infinite) of the conformal families

$$\{A_j\} = \bigoplus_n [\phi_n]. \tag{3.11}$$

To understand the nature of these secondary fields, consider the product

$T(\zeta)\phi_n(z, \bar{z})$ This product can be expanded according to (1.6), the coefficients C_{ij}^k being single-valued analytic functions of $(\zeta - z)$ in virtue of relation (2.7) and the local properties of the fields $T(\zeta)$ and $\phi_n(z, \bar{z})$. Therefore this product can be represented as

$$T(\zeta)\phi_n(z) = \sum_{k=0}^{\infty} (\zeta - z)^{-2+k} \phi_n^{(-k)}(z), \tag{3 12}$$

where we have again omitted the dependences of the fields on the variable $\bar{z}$ The dimensions of the fields $\phi_n^{(-k)}$ are given by (3 7). The singular terms in (3 12) are completely determined by the transformation law (3.8) (remember (2.10)). Thus the first two coefficients in (3.12) are

$$\phi_n^{(-1)}(z) = \frac{\partial}{\partial z}\phi_n(z), \qquad \phi_n^{(0)}(z) = \Delta_n \phi_n(z). \tag{3 13}$$

The coefficients $\phi_n^{(-k)}$, $k = 2, 3, \ldots,$ of the regular terms in (3.12) are new local fields To make sure of the existence of these fields, it is possible to expand the Ward identity (3.10) in power series, say, in $z - z_1$. These new fields are representatives of the conformal family $[\phi_n]$, $\phi_n^{(-k)} \in [\phi_n]$ The conformal properties of these secondary fields $\phi_n^{(-k)}$ are more complicated than those of the primary field ϕ_n. The infinitesimal conformal transformation and comparison of both sides of (3.12) yield

$$\begin{aligned}\delta_\varepsilon \phi_n^{(-k)}(z) &= \varepsilon(z)\frac{\partial}{\partial z}\phi_n^{(-k)}(z) + (\Delta_n + k)\varepsilon'(z)\phi_n^{(-k)}(z) \\ &+ \sum_{l=1}^{k} \frac{k+l}{(l+1)!}\left[\frac{\mathrm{d}^{l+1}}{\mathrm{d}z^{l+1}}\varepsilon(z)\right]\phi_n^{(l-k)}(z) \\ &+ \tfrac{1}{12}c\frac{1}{(k-2)!}\left[\frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}\varepsilon(z)\right]\phi_n(z)\end{aligned} \tag{3 14}$$

The fields $\phi_n^{(-k)}$ are not the only ones belonging to the conformal family $[\phi_n]$ Consider, for instance, the operator product expansion

$$\begin{aligned}T(\zeta)\phi_n^{(-k_2)}(z) &= \tfrac{1}{12}c(\zeta - z)^{-k_2-2}(k_2^3 - k_2)\phi_n(z) \\ &+ \sum_{l=1}^{k_2} (\zeta - z)^{-l-2}(l + k_2)\phi_n^{(l-k_2)}(z) \\ &+ \sum_{k_1=0}^{\infty} (\zeta - z)^{-2+k_1}\phi_n^{(-k_1-k_2)}(z).\end{aligned} \tag{3 15}$$

The operators accompanying the singular terms in (3 15) are unambiguously determined by formula (3.14). In particular

$$\phi_n^{(-1,-k)}(z) = \frac{\partial}{\partial z}\phi_n^{(-k)}(z), \qquad \phi_n^{(0,-k)}(z) = (\Delta_n + k)\phi_n^{(-k)}(z) \tag{3 16}$$

The new local fields $\phi_n^{(-k_1,-k_2)}$ with $k_1 > 1$ also belong to the conformal family $[\phi_n]$ The variations $\delta_\varepsilon\phi^{(-k_1,-k_2)}$ are expressed in terms of the fields $\phi_n^{(-l_1,-l_2)}$, $\phi_n^{(-l)}$ and ϕ_n

Considering the operator products $T(\zeta)\phi_n^{(-k_1-k_2)}(z)$,. etc., one can discover an infinite set of the secondary fields

$$\phi_n^{(-k_1,-k_2,\quad -k_N)}(z), \tag{3 17}$$

where $k_i \geqslant 1$ and $N = 1,2,..$. The fields (3 17) can be defined by the explicit formula

$$\phi_n^{(-k_1,\quad,-k_N)}(z) = L_{-k_1}(z). \quad L_{-k_N}(z)\phi_n(z), \tag{3 18}$$

where the operators $L_{-k}(z)$ are given by the contour integrals

$$L_{-k}(z) = \oint \frac{d\zeta T(\zeta)}{(\zeta - z)^{k+1}} \tag{3 19}$$

The integration contours associated with each of the operators $L_{-k_i}(z)$ in (3.18) enclose the point z as well as the points $\zeta_{i+1}, \zeta_{i+2}, \quad, \zeta_N$, which are the integration variables, corresponding to the operators L to the right of L_{-k_i}★ The dimensions of the fields (3 17) are

$$\Delta_n^{(k_1,\quad,k_N)} = \Delta_n + k_1 + \quad \cdot + k_N \tag{3 20}$$

An infinite set of the fields (3 17) constitutes the conformal family $[\phi_n]$ These fields are not linearly independent (see below) In fact, in general the fields (3 17) with $k_1 \leqslant k_2 \leqslant \quad \leqslant k_N$ form the basis★★ Note that

$$\phi_n^{(-1,-k_1-k_2\quad -k_N)} = \frac{\partial}{\partial z}\phi_n^{(-k_1,-k_2,\quad,-k_N)} \tag{3 21}$$

Therefore the conformal family $[\phi_n]$ naturally includes all the derivatives of each field involved It can be derived from (3.18) that the variations $\delta_\varepsilon\phi_n^{\{k\}}$, $\{k\} = (-k_1, \quad, -k_N)$ are expressed in terms of the fields, belonging to the same conformal family $[\phi_n]$, and therefore each conformal family corresponds to some representation of the conformal algebra.

★ One can easily verify that the operators (3 19), where $= 0, \pm 1, \pm 2,$, satisfy the Virasoro algebra (2 21) Obviously, the operators L_n introduced in sect 2 are no other than $L_n(0)$

★★ This statement does not hold for some special values of Δ_n, see sect 5

To describe the structure of the representation it is convenient to turn again to the operator formalism. Let us introduce the vectors (primary states)

$$|n\rangle = \phi_n(0)|0\rangle . \tag{3.22}$$

Using the properties (2.23) of the vacuum and the commutation relations (3.9) one can get

$$L_m|n\rangle = 0 \qquad \text{if} \qquad m>0 ,$$

$$L_0|n\rangle = \Delta_n|n\rangle \tag{3.23}$$

It follows from (3.18) that

$$\phi_n^{(-k_1, \ldots ,-k_N)}(0)|0\rangle = L_{-k_1} \ldots L_{-k_N}|n\rangle . \tag{3.24}$$

So, the conformal family $[\phi_n]$ is isomorphic to the space of states, generated from the primary state $|n\rangle$ by the negative components L_m, $m<0$* In the representation theory this space is known as the Verma modulus V_n (see, for example, [6]). Due to the relations (2.21), there are linear dependences between the vectors (3 24). As has been mentioned above, in all cases, excluding certain special values of Δ_n (see sect 5), the states (3 24) with $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_N$ form the basis in V_n Note that the vectors (3 24) are the eigenstates of the operator L_0, the eigenvalues being given by (3 20)

So far we have dealt only with the subgroup Γ of the conformal group $\mathcal{G}$. Actually, more precise definitions are required. Since the complete conformal group is the direct product (1.10), the representations $[\phi_n]$ are, in fact, the direct products of the representations of Γ and $\bar{\Gamma}$

$$[\phi_n] = V_n \otimes \bar{V}_n . \tag{3 25}$$

This means that it contains not only the vectors (3.24) but also all the states of the form

$$\phi_n^{\{k\}\{\bar{k}\}}(0)|0\rangle = L_{-k_1} \ldots L_{-k_N}\bar{L}_{-\bar{k}_1} \ldots \bar{L}_{-\bar{k}_M}|n\rangle , \tag{3.26}$$

where

$$\{k\} = (-k_1, -k_2, \ldots , -k_N) , \qquad \{\bar{k}\} = (-\bar{k}_1, -\bar{k}_2, \ldots , -\bar{k}_M)$$

k_i and $\bar{k}_j$ are independent positive integers. Remember that the operators L and $\bar{L}$

* This statement is not precise because we neglected the $\bar{z}$ dependence of the fields, the correct definition is given below

are commutative. According to (1.16), the primary state $|n\rangle$ satisfies, besides (3.23), the equations

$$\bar{L}_m|n\rangle = 0, \quad \text{if} \quad m > 0,$$
$$\bar{L}_0|n\rangle = \bar{\Delta}_n|n\rangle. \tag{3.27}$$

Therefore each conformal family $[\phi_n]$ is characterized by two parameters Δ_n and $\bar{\Delta}_n$.

Because of the conformal invariance, the two-point functions $\langle\phi_n(\xi_1)\phi_m(\xi_l)\rangle$ vanish unless the fields ϕ_n and ϕ_m have the same dimensions (see appendix A). Moreover, the system of the primary fields can always be chosen to be orthonormal

$$\langle\phi_n(z_1, \bar{z}_1)\phi_m(z_2, \bar{z}_2)\rangle = \delta_{nm}(z_1 - z_2)^{-2\Delta_n}(\bar{z}_1 - \bar{z}_2)^{-2\bar{\Delta}_n}. \tag{3.28}$$

Let us define the "out" primary states by the formula

$$\langle n| = \lim_{z,\bar{z}\to\infty} \langle 0|\phi_n(z, \bar{z}) z^{2L_0}\bar{z}^{2\bar{L}_0}. \tag{3.29}$$

These vectors satisfy the equations

$$\langle n|L_m = 0, \quad \text{if} \quad m < 0,$$
$$\langle n|L_0 = \Delta_n < n|, \tag{3.30}$$

and the same equation with the substitution $L \to \bar{L}$, $\Delta_n \to \bar{\Delta}_n$. Like in (3.26), we have

$$\lim_{z,\bar{z}\to\infty} \langle 0|\phi_n^{\{k\}\{\bar{k}\}}(z, \bar{z}) z^{2L_0}\bar{z}^{2\bar{L}_0} = \langle n|L_{k_N}L_{k_{N-1}}\cdots L_{k_1}\bar{L}_{\bar{k}_M}\cdots\bar{L}_{\bar{k}_1} \tag{3.31}$$

The orthonormality condition (3.28) can be rewritten as

$$\langle n|m\rangle = \delta_{nm} \tag{3.32}$$

The conformal Ward identities make it possible to express explicitly any correlation function as

$$\langle T(\zeta_1)\ldots T(\zeta_M)\phi_1(z_1)\ldots\phi_N(z_N)\rangle, \tag{3.33}$$

in terms of the correlator

$$\langle\phi_1(z_1)\ldots\phi_N(z_N)\rangle. \tag{3.34}$$

Here $\phi_1, \ldots, \phi_N$ are certain primary fields This can be done by successively applying the relation

$$\langle T(\zeta)T(\zeta_1)\ldots T(\zeta_M)\phi_1(z_1)\ldots\phi_N(z_N)\rangle$$
$$=\left\{\sum_{i=1}^{N}\left[\frac{\Delta_i}{(\zeta-z_i)^2}+\frac{1}{\zeta-z_i}\frac{\partial}{\partial z_i}\right]+\sum_{j=1}^{M}\left[\frac{2}{(\zeta-\zeta_j)^2}+\frac{1}{\zeta-\zeta_j}\frac{\partial}{\partial\zeta_j}\right]\right\}$$
$$\times\langle T(\zeta_1)\ldots T(\zeta_M)\phi_1(z_1)\ldots\phi_N(z_N)\rangle$$
$$+\sum_{j=1}^{M}\frac{c}{(\zeta-\zeta_j)^4}\langle T(\zeta_1)\ldots T(\zeta_{j-1})T(\zeta_{j+1})\ldots T(\zeta_M)\phi_1(z_1)\ldots\phi_N(z_N)\rangle \tag{3.35}$$

The first term in (3 35) is of the same origin as (3 10), whereas the second term is due to the c-number term in the transformation law (2.12)*

Using the correlation functions (3 33) one can also compute any correlators of the form

$$\langle\phi_1^{\{k_1\}}(z_1)\ldots\phi_N^{\{k_N\}}(z_N)\rangle, \tag{3.36}$$

where $\phi_i^{\{k_i\}}$ are some secondaries of the field ϕ_i, since these secondary fields are no other than the coefficients in the operator product expansions like (3 12), (3 15), etc Actually in this way the correlators (3 36) are expressed in terms of the correlation functions (3 34) by means of linear differential operators. The general expression is rather cumbersome and we present the simplest example only**

$$\langle\phi_n^{(-k_1,-k_2,\ldots-k_M)}(z)\phi_1(z_1)\ldots\phi_N(z_N)\rangle$$
$$=\hat{\mathcal{L}}_{-k_M}(z,z_i)\hat{\mathcal{L}}_{-k_{M-1}}(z,z_i)\ldots\hat{\mathcal{L}}_{-k_1}(z,z_k)\langle\phi_n(z)\phi_1(z_1)\ldots\phi_N(z_N)\rangle, \tag{3.37}$$

where the differential operators $\hat{\mathcal{L}}_{-k}$ are given by the formula

$$\hat{\mathcal{L}}_{-k}(z,z_i)=\sum_{i=1}^{N}\left[\frac{(1-k)\Delta_i}{(z-z_k)^k}-\frac{1}{(z-z_i)^{k-1}}\frac{\partial}{\partial z_i}\right] \tag{3.38}$$

* Obviously, the fields $T(z)$ and $\bar{T}(\bar{z})$ are not primary fields they belong to the conformal family $[I]$ of the identity operator

** To obtain (4 5) in the simplest way one can substitute the explicit formula (3 18) and deform the integration contours so as to enclose them around the singularities $z_1, z_2, \ldots, z_N$

Thus the conformal Ward identities enable one to express any correlation functions in terms of the correlators of the primary fields (3 34) Hence, all the information about the conformal quantum field theory is contained in these correlators

4. Conformal properties of the operator algebra

In the quantum field theory the correlation functions (2 1) should obey the operator algebra (1.6). The conformal symmetry imposes hard restrictions on the coefficients $C_{ij}^{k}(\xi)$ Consider the product of two primary fields $\phi_n(\xi)\phi_m(0)$ The operator product expansion can be represented as

$$\phi_n(z,\bar{z})\phi_m(0,0)=\sum_{p}\sum_{\{k\}}\sum_{\{\bar{k}\}} C_{nm}^{p,\{k\},\{\bar{k}\}} \times z^{\Delta_p-\Delta_n-\Delta_m+\Sigma_i k_i}\bar{z}^{\bar{\Delta}_p-\bar{\Delta}_n-\bar{\Delta}_m+\Sigma\bar{k}_i}\phi_p^{\{k\}\{\bar{k}\}}(0,0), \qquad (4\ 1)$$

where $\phi_p^{\{k\}\{\bar{k}\}}$ are the secondary fields, belonging to the conformal family $[\phi_p]$ Both sides of (4 1) should exhibit the same conformal properties The transformation law of the left-hand side is determined by (3 8), the conformal properties of each term in the right-hand side can be derived, in principle, from (3 18) The requirement of the conformal invariance of (4.1) leads to the relations for the numerical constants $C_{nm}^{p\{k\}\{\bar{k}\}}$ with different $\{k\}$'s but with the same index (see appendix B) In principle, these relations can be solved recurrently, the solution being represented as

$$C_{nm}^{p\{k\}\{\bar{k}\}}=C_{nm}^{p}\beta_{nm}^{p,\{k\}}\bar{\beta}_{nm}^{p\{\bar{k}\}}, \qquad (4\ 2)$$

where C_{nm}^{p} are the constants of the primary fields ϕ_p themselves and the factors β ($\bar{\beta}$) are expressed unambiguously in terms of the dimensions $\Delta_n, \Delta_m, \Delta_p$ ($\bar{\Delta}_n, \bar{\Delta}_m, \bar{\Delta}_p$) only, the condition $\beta_{nm}^{p\{0\}}=\bar{\beta}_{nm}^{p\{0\}}=1$ is implied The factorized (in terms of β) form of (4 2) is a consequence of (3 25) The expansion (4 1) can be rewritten as

$$\phi_n(z,\bar{z})\phi_m(0,0)=\sum_{p} C_{nm}^{p} z^{\Delta_p-\Delta_n-\Delta_m}\bar{z}^{\bar{\Delta}_p-\bar{\Delta}_n-\bar{\Delta}_m}\Psi_p(z,\bar{z}|0,0), \qquad (4\ 3)$$

where

$$\Psi_p(z,\bar{z}|0,0)=\sum_{\{k\}\{\bar{k}\}}\beta_{nm}^{p\{k\}}\bar{\beta}_{nm}^{p\{\bar{k}\}}z^{\Sigma k_i}\bar{z}^{\Sigma\bar{k}_i}\phi_p^{\{k\}\{\bar{k}\}}(0,0) \qquad (4\ 4)$$

is the contribution of the conformal family $[\phi_p]$ Let us stress that the conformal properties of the "bilocal" operators (4.4) coincide with those of the product $\phi_n(z,\bar{z})\phi_m(0,0)$, all the coefficients in the power series (4 4) being unambiguously determined by this requirement. Unfortunately, equations, determining these coefficients are too complicated to be solved exactly. The first few coefficients β are presented in appendix B for the particular case $\Delta_n = \Delta_m$.

The constants C_{nm}^p in (4.3) and the values of the dimensions $\Delta_n, \bar{\Delta}_n$ are not determined by the conformal symmetry itself. These numerical parameters are the most important dynamical characteristics of the conformal quantum field theory Note that under the orthonormality condition (3.28) the coefficients $C_{nm}^l = C_{nml}$ are symmetric functions of the indices n, m, l and coincide with the numerical factors in the three-point functions.

$$\langle n|\phi_m(z,\bar{z})|l\rangle = C_{nml} z^{\Delta_n - \Delta_m - \Delta_l} \bar{z}^{\bar{\Delta}_n - \bar{\Delta}_m - \bar{\Delta}_l}, \tag{4.5}$$

where for simplicity we put two points equal to 0 and ∞. To determine the parameters C_{nm}^l and Δ_n it is necessary to apply some dynamical principle. In the bootstrap approach described in the introduction, the associativity of the operator algebra (1.6) is taken as the main dynamical principle. As is shown in appendix C, the associativity condition is equivalent to the crossing symmetry of the four-point correlation functions

$$\langle A_{j_1}(\xi_1) A_{j_2}(\xi_2) A_{j_3}(\xi_3) A_{j_4}(\xi_4)\rangle \tag{4.6}$$

Thanks to the relations discussed at the end of the previous section, it is sufficient to consider the four-point functions of the primary fields

$$\langle \phi_k(\xi_1)\phi_l(\xi_2)\phi_n(\xi_3)\phi_m(\xi_4)\rangle. \tag{4.7}$$

Due to the projective invariance (see appendix A), the four-point functions essentially depend only on two anharmonic quotients

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \qquad \bar{x} = \frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_3 - \bar{z}_4)}{(\bar{z}_1 - \bar{z}_3)(\bar{z}_2 - \bar{z}_4)}. \tag{4.8}$$

Therefore it is convenient to set $z_1 = \bar{z}_1 = \infty$, $z_2 = \bar{z}_2 = 1$, $z_3 = x$, $\bar{z}_3 = \bar{x}$, $z_4 = \bar{z}_4 = 0$ and to define the functions

$$G_{nm}^{lk}(x,\bar{x}) = \langle k|\phi_l(1,1)\phi_n(x,\bar{x})|m\rangle. \tag{4.9}$$

In terms of these functions the crossing symmetry condition is

$$G_{nm}^{lk}(x,\bar{x}) = G_{nl}^{mk}(1-x, 1-\bar{x}) = x^{-2\Delta_n}\bar{x}^{-2\bar{\Delta}_n} G_{nk}^{lm}\left(\frac{1}{x}, \frac{1}{\bar{x}}\right) \tag{4.10}$$

Substituting the expansion (4.3) for the product $\phi_n(x,\bar{x})\phi_m(0,0)$ one can rewrite (4.9) as

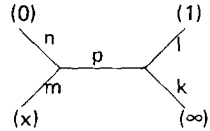
$$G_{nm}^{lk}(x,\bar{x}) = \sum_p C_{nm}^p C_{klp} A_{nm}^{lk}(p|x,\bar{x}), \tag{4.11}$$

where each of the "partial waves"

$$A^{lk}_{nm}(p|x,\bar{x}) = (C^{p}_{kl})^{-1} x^{\Delta_p-\Delta_n-\Delta_m} \bar{x}^{\bar{\Delta}_p-\bar{\Delta}_m-\bar{\Delta}_n}$$

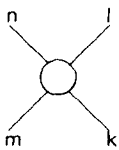
$$\times \langle k|\phi_l(1,1)\Psi_p(x,\bar{x}|0,0)|0\rangle \tag{4 12}$$

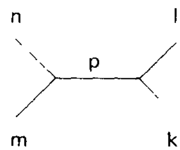
represents the "s-channel" contribution of the conformal family $[\phi_p]$ to the four-point function (4.9) It is convenient to introduce the diagrams associated with these amplitudes

$A^{lk}_{nm}(p|x,\bar{x}) =$ 

(4 13)

Then the "partial wave" decomposition (4.11) can be represented as

$G^{lk}_{nm}(x,\bar{x}) =$ 

$= \sum_p C^{p}_{nm} C_{lkp}$ 

(4.14)

It is clear from (4.4) that the amplitudes (4.12) have the following factorized form

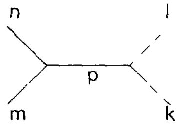
$$A^{lk}_{nm}(p|x,\bar{x}) = \mathcal{F}^{lk}_{nm}(p|x)\bar{\mathcal{F}}^{lk}_{nm}(p|\bar{x}), \tag{4.15}$$

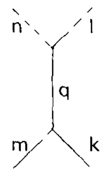
where, for instance, the function $\mathcal{F}$ is given by the power series

$$\mathcal{F}^{lk}_{nm}(p|x) = x^{\Delta_p-\Delta_n-\Delta_m} \sum_{\{k\}} \beta^{p\{k\}}_{nm} x^{\Sigma k_i} \frac{\langle k|\phi_l(1,1)L_{-k_1}\cdots L_{-k_N}|p\rangle}{\langle k|\phi_l(1,1)|p\rangle} \tag{4.16}$$

The matrix elements in the right-hand side of (4.16) can be computed exactly with the use of the commutation relations (3.9) and eqs. (3.30). Therefore, the functions (4 16) are completely determined by the conformal symmetry. These functions depend on six parameters. five dimensions Δ_n, Δ_m, Δ_k, Δ_l, Δ_p and the central charge c. We shall call (4.16) the *conformal blocks*, because any correlation function (4.7) is built up of these functions $\mathcal{F}$

The crossing symmetry conditions for the four-point functions (4.9) can be represented as the following diagrammic equations

$\sum_p C^p_{nm} C_{lkp}$ 

$= \sum_q C^q_{nl} C_{mkq}$ 

(4 17)

The analytic form of these equations is

$$\sum_p C^p_{nm} C_{lkp} \mathcal{F}^{lk}_{nm}(p|x) \bar{\mathcal{F}}^{lk}_{nm}(p|\bar{x}) = \sum_q C^q_{nl} C_{mkq} \mathcal{F}^{mk}_{nl}(q|1-x) \bar{\mathcal{F}}^{mk}_{nl}(q|1-\bar{x}) \tag{4.18}$$

If the conformal blocks $\mathcal{F}$ are known, (4 18) yields a system of equations, determining the constants C^l_{nm} and the dimensions $\Delta_n, \bar{\Delta}_n$ Therefore, the computation of the conformal blocks (4.16) for general values of Δ_n's is the problem of principle importance for the conformal quantum field theory. The first few terms of the power expansion for these functions are given in appendix B, where the case $\Delta_n = \Delta_m = \Delta_k = \Delta_l = \Delta$ is considered for the sake of simplicity. Although the conformal blocks are not yet known for the general case, there are the special values of the dimensions Δ (associated with the degenerate representation of the Virasoro algebra, see sect 5) such that the corresponding conformal blocks can be computed exactly, being the solutions of certain linear differential equations The simplest example is the hypergeometric function. In these special cases the bootstrap eq (4 18) can be solved completely.

5. Degenerate conformal families

The representation V_Δ of the Virasoro algebra is irreducible unless the dimension Δ takes some special values [6, 7]. For these values the vector space V_Δ proves to contain a special vector (*the null vector*) $|\chi\rangle \in V_\Delta$ satisfying the equations

$$L_n|\chi\rangle = 0, \qquad \text{if} \qquad n > 0,$$
$$L_0|\chi\rangle = (\Delta + K)|\chi\rangle, \tag{5 1}$$

characteristic of the primary fields Here K is some positive integer. For example, one can easily verify that the vector

$$|\chi\rangle = \left[L_{-2} + \frac{3}{2(2\Delta+1)} L^2_{-1}\right]|\Delta\rangle, \tag{5 2}$$

(where $|\Delta\rangle$ denotes the primary state of the dimension Δ) satisfies (5 1) with $K=2$, provided Δ takes any of the two values

$$\Delta = \tfrac{1}{16}\left[5 - c \pm \sqrt{(c-1)(c-25)}\,\right]. \tag{5 3}$$

In general, the jessenull vector $|\chi\rangle$ can be considered as the primary state of its own Verma modulus $V_{\Delta+K}$ Therefore the representation V_Δ proves to be reducible One obtains the irreducible representation $V_\Delta^{(ir)}$ if the null vector $|\chi\rangle$ (together with all the states belonging to $V_{\Delta+K}$) is formally put equal to zero

$$|\chi\rangle = 0 \tag{5 4}$$

Note that eq (5.4) does not lead to contradictions since due to (5 1) the null vector is orthogonal to any state of V_Δ and, in particular, has the zero norm

$$\langle\psi|\chi\rangle = 0\,, \qquad |\Psi\rangle \in V_\Delta\,,$$

$$\langle\chi|\chi\rangle = 0 \tag{5 5}$$

In the conformal quantum field theory the meaning of this phenomenon is the following. If the dimension Δ of some primary field ϕ_Δ happens to take one of the special values mentioned above, then the conformal family $[\phi_\Delta]$, formally computed according to (3 18) proves to contain the special secondary field $\chi_{\Delta+K} \in [\phi_\Delta]$, which possesses the conformal properties of a primary field, i e. satisfies the commutation relations of the type (3 9). This field corresponds to the null vector $|\chi\rangle \in V_\Delta$ and we call it the *null field* For example, if Δ is given by (5.3) the operator

$$\chi_{\Delta+2} = \phi_\Delta^{(-2)} + \frac{3}{2(2\Delta+1)}\frac{\partial^2}{\partial z^2}\phi_\Delta \tag{5 6}$$

is the null field.

Formally, the extra primary field $\chi_{\Delta+K}$ originates from the conformal family $[\chi_{\Delta+K}]$ which is imbedded into $[\phi_\Delta]$ Note, however, that any correlation functions of the form

$$\langle\chi_{\Delta+K}(z)\phi_1(z_1) \quad \phi_N(z_N)\rangle$$

vanishes So, the null field $\chi_{\Delta+K}$ can be self-consistently regarded as zero

$$\chi_{\Delta+K} = 0\,. \tag{5.7}$$

This condition obviously kills all the secondary fields of the null field

$$[\chi_{\Delta+K}] = 0 \tag{5.8}$$

If eq. (5.7) is applied, one gets the true irreducible conformal family $[\phi_\Delta]$ of the original primary field ϕ_Δ. In this case the conformal family contains "less" fields than usual and we call it a *degenerate conformal family.* We shall also call degenerate the corresponding primary field ϕ_Δ

All the special values of Δ, corresponding to the reducible representations V_Δ, have been listed by Kac [7] (see also [6]) These values, which can be labelled by two positive integers n and m, are given by the formula

$$\Delta_{(n,m)} = \Delta_0 + \left(\tfrac{1}{2}\alpha_+ n + \tfrac{1}{2}\alpha_- m\right)^2, \tag{5 9}$$

where

$$\Delta_0 = \tfrac{1}{24}(c-1), \tag{5 10}$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \tag{5 11}$$

If $\Delta = \Delta_{(n,m)}$, then the corresponding null vector has the dimension

$$\Delta_{(n,m)} + nm. \tag{5 12}$$

Let us denote the degenerate primary field $\phi_{\Delta_{(n\ m)}}$ having the dimension $\Delta_{(n,m)}$ as $\psi_{(n,m)}$* Note that

$$\Delta_{(1,1)} = 0 \tag{5 13}$$

It can be shown that the field $\psi_{(1,1)}$ is z-independent, i.e.**

$$\frac{\partial}{\partial z}\psi_{(1,1)} = 0. \tag{5 14}$$

The dimensions $\Delta_{(1,2)}$ and $\Delta_{(2,1)}$ are just the two values given by (5.3).

Consider the correlation functions of the form

$$\langle \psi_{(n,m)}(z)\phi_1(\xi_1).\ .\phi_N(\xi_N)\rangle \tag{5.15}$$

* This notation is not complete because it says nothing about the second dimension Δ of the primary field This fact, which should be always kept in mind, does not violate the conclusions we make below

** If both dimensions Δ and $\bar{\Delta}$ of the field ψ are zero this field does not depend on the coordinates at all and coincides with the identity operator I

An important property of these correlation functions is that they satisfy the linear partial differential equations, the maximal order of derivatives being nm* To make this evident let us recall that the correlation functions of any secondary fields

$$\langle \psi_{(n,m)}^{(-k_1, \dots, -k_L)}(z)\phi_1(\xi_1)\dots \phi_N(\xi_N)\rangle \tag{5.16}$$

can be expressed in terms of the correlation function (5 15) by means of the linear differential operators (see (3 37)). The null field $\chi_{\Delta+nm}$ is a certain linear combination of the secondary fields $\Psi_{(n,m)}^{(-k_1, \dots, -k_L)}$ Therefore, the differential equation for (5.15) follows directly from eq. (5 7). For example, taking into account (5.6) and (3 37), for the degenerate field $\psi_{(1,2)}(z)$ one gets

$$\left\{\frac{3}{2(2\delta+1)}\frac{\partial^2}{\partial z^2}-\sum_{i=1}^{N}\frac{\Delta_i}{(z-z_i)^2}-\sum_{i=1}^{N}\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right\}$$
$$\times\langle \Psi_{(1,2)}(z)\phi_1(z_1)\dots\phi_N(z_N)\rangle=0, \tag{5.17}$$

where $\delta=\Delta_{(1,2)}$ and $\Delta_1,\dots,\Delta_N$ are the dimensions of the primary fields $\phi_1,\dots,\phi_N$, respectively. The correlation function, involving the field $\psi_{(2,1)}$, satisfies the same differential equation, the only difference being $\delta=\Delta_{(2\,1)}$** The differential equation, satisfied by the degenerate fields $\psi_{(1,3)}$ and $\psi_{(3,1)}$, is presented in appendix D as another example

In the case of the four-point functions

$$\Psi_{(n,m)}(z|z_1,z_2,z_3)=\langle\psi_{(n,m)}(z)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle, \tag{5 18}$$

the partial differential equations can be reduced to ordinary ones. Actually in this

* The simplest example of these equations is (5 14)

** The following interpretation of eq (5 17) is worth noting Let $\psi(z)$ stand for one of the fields $\psi_{(1\,2)}$ or $\psi_{(2\,1)}$, δ being the corresponding dimension $\Delta_{(1\,2)}$ or $\Delta_{(2\,1)}$ Then the field $\psi(z)$ satisfies the operator equation

$$\frac{\partial^2}{\partial z^2}\psi(z)=\gamma\, T(z)\psi(z)\,, \tag{$*$}$$

where $\gamma=\frac{2}{3}(2\delta+1)$, whereas the singular operator product is regularized by means of the subtractions

$$T(z)\psi(z)=\lim_{\zeta\to z}\left\{T(\zeta)\psi(z)-\frac{\delta}{(\zeta-z)^2}\psi(z)-\frac{1}{\zeta-z}\frac{\partial}{\partial z}\psi(z)\right\}$$

The classical limit of eq (*) (which corresponds to the choice $\psi=\psi_{(1\,2)}$ and $c\to\infty$) is an essential part of classical theory of the Liouville equation (see, for example, [13]) We suppose that eq (*) plays the analogous role in the quantum theory of this equation, which is apparently associated with the string theory [14] We intend to discuss this point in another paper case the relations (A.) can be solved for the derivatives $\partial/\partial z_i$, $i=1,2,3$ For example substituting these derivatives into (5 17) one gets the Riemann ordinary differential equation

$$\left\{\frac{3}{2(2\delta+1)}\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\sum_{i=1}^{3}\left[\frac{1}{z-z_i}\frac{\mathrm{d}}{\mathrm{d}z}-\frac{\Delta_i}{(z-z_i)^2}\right]\right.$$

$$\left.+\sum_{j<i}\frac{\delta+\Delta_{ij}}{(z-z_i)(z-z_j)}\right\}\Psi(z|z_1,z_2,z_3)=0, \qquad (5\ 19)$$

where $\Delta_{12}=\Delta_1+\Delta_2-\Delta_3$, etc., $\delta=\Delta_{(1,2)}$, $\Psi=\Psi_{(1,2)}$ or $\delta=\Delta_{(2,1)}$, $\Psi=\Psi_{(2\ 1)}$ So, for the cases $(n,m)=(1,2)$ or $(2,1)$ the four-point function (5 18) can be expressed in terms of the hypergeometric function

Consider the operator algebra containing the degenerate fields. Some important information about this operator algebra can be obtained from the differential equations discussed above. For example, consider the product $\psi(z)\phi_\Delta(z_1)$ where ϕ_Δ is some primary field of the dimension Δ whereas $\psi(z)$ temporarily stands for one of the degenerate fields $\psi_{(1,2)}(z)$ or $\psi_{(2,1)}(z)$ Let us substitute the expansion

$$\psi(z)\phi_\Delta(z_1)=\mathrm{const}(z-z_1)^{\kappa}\left[\phi_{\Delta'}(z_1)+\beta^{(-1)}(z-z_1)\phi_{\Delta'}^{(-1)}(z_1)+\quad\right], \qquad (5\ 20)$$

into the differential eq. (5.17) In (5.20) $\phi_{\Delta'}$ denotes some primary field of the dimension Δ', $\kappa=\Delta'-\Delta-\delta$ where δ is the dimension of the field ψ, i.e one of the values given by (5 3). Considering the most singular term at $z\to z_1$, one immediately obtains the characteristic equation, determining the exponent

$$\frac{3\kappa(\kappa-1)}{2(2\delta+1)}-\Delta+\kappa=0 \qquad (5\ 21)$$

To describe the solutions of this equation it is convenient to introduce the following parametrization of the dimensions

$$\delta(\alpha)=\Delta_0+\tfrac{1}{4}\alpha^2, \qquad (5.22)$$

where Δ_0 is defined by (5 10). If $\Delta=\Delta(\alpha)$, the two solutions of (5 21) are given by the formulae

$$\Delta'_{(1)}=\Delta_0+\tfrac{1}{4}(\alpha+\alpha_\pm)^2,$$

$$\Delta'_{(2)}=\Delta_0+\tfrac{1}{4}(\alpha-\alpha_\pm)^2, \qquad (5\ 23)$$

where $\alpha_\pm$ are given by (5.11) and α_+ (α_-) is chosen if $\psi = \psi_{(1,2)}$ ($\psi = \psi_{(2,1)}$) Let $\phi_\alpha(z)$ be the primary field with the dimension (5.22) The result of the above calculation can be represented by the following symbolic formulae

$$\psi_{(1,2)}\phi_{(\alpha)} = \left[\phi_{(\alpha-\alpha_+)}\right] + \left[\phi_{(\alpha+\alpha_+)}\right],$$

$$\psi_{(2,1)}\phi_{(\alpha)} = \left[\phi_{(\alpha-\alpha_-)}\right] + \left[\phi_{(\alpha+\alpha_-)}\right] \qquad (5\ 24)$$

Here the square brackets denote the contributions of the corresponding conformal families to the operator product expansion of $\psi(z)\phi_{(\alpha)}(z_1)$. In (5 24) overall factors, standing in front of these contributions are omitted These factors cannot certainly be determined by simple calculations like the one performed above* As we shall see in the next section, some of these coefficients could vanish

It can be shown that the "fusion rule" (5.24) is generalized to the cases of arbitrary degenerate fields $\psi_{(n\ m)}$ as follows.

$$\psi_{(n,m)}\phi_\alpha = \sum_{l=1-m}^{1+m} \sum_{k=1-n}^{1+n} \left[\phi_{(\alpha+l\alpha_- +k\alpha_+)}\right], \qquad (5\ 25)$$

where the variable k runs through the even (odd) values provided the index n is odd (even), the same is valid for the variable l and the index m So in the general case the sum in (5 25) contains nm terms in agreement with the fact that the degenerate field $\psi_{(n\ m)}$ satisfies the nm-order differential equation.

We see that the differential equations satisfied by the degenerate fields impose hard constraints on the operator algebra Certainly, in the general case these differential equations do not provide enough information to determine the correlation functions (5.15) completely Even in the cases of the four-point functions (5 18) one has to take into account the $\bar{z}$-dependence of the fields and local properties In the next section we shall study the "minimal models" of the conformal quantum field theory in which all primary fields involved are degenerate

6. Minimal theories

Consider the "fusion rule" (5.24) The substitution $\phi_{(\alpha)} = \psi_{(1,2)}$ yields

$$\psi_{(1,2)}\psi_{(1,2)} = \left[\psi_{(1,1)}\right] + \left[\psi_{(1\ 3)}\right] \qquad (6\ 1)$$

Here (5 9) is taken into account. Similarly, one gets for $m > 1$

$$\psi_{(1\ 2)}\psi_{(1,m)} = \left[\psi_{(1,m-1)}\right] + \left[\psi_{(1,m+1)}\right] \qquad (6\ 2)$$

* To determine these factors in the quantum field theory one should take into account the associativity condition for the operator algebra and local properties of the fields

So, if the degenerate field $\psi_{(1,2)}$ is involved in the operator algebra, in the general case this algebra includes also all the degenerate fields $\psi_{(1,m)}$. Moreover, assuming that the operator algebra also includes the degenerate field $\psi_{(2,1)}$ and using (5.24), one can obtain all the degenerate fields $\psi_{(n,m)}$. In the "fusion rule" (5.24) the fields $\psi_{(1,2)}$ and $\psi_{(2,1)}$ act as the "shift operators"

$$\psi_{(1,2)}\psi_{(n,m)} = \left[\psi_{(n,m-1)}\right] + \left[\psi_{(n,m+1)}\right], \tag{6.3a}$$

$$\psi_{(2,1)}\psi_{(n,m)} = \left[\psi_{(n-1,m)}\right] + \left[\psi_{(n+1,m)}\right]. \tag{6.3b}$$

The following remark is necessary Using the rules (8.3) formally, one would get as a result all the fields of dimension $\Delta_{(n,m)}$ given by (5.9) where the integers n, m take the zero and negative values as well as positive values In fact, the fields of dimension $\Delta_{(n,m)}$ with the zero and negative n, m drop out from the algebra, i.e the operator algebra developed by "fusing" the fields $\psi_{(1,2)}$ and $\psi_{(2,1)}$. $\psi_{(2,1)}$ proves to contain the degenerate fields $\psi_{(n,m)}$ $(n, m > 0)$ only To understand the nature of this phenomenon, consider, for instance, the product $\psi_{(1,2)}\psi_{(2,1)}$ Analyzing the differential equation for the degenerate field $\psi_{(1,2)}$, one gets, according to (6 3a),

$$\psi_{(1,2)}\psi_{(2,1)} = C_1\left[\phi_{(2,0)}\right] + C_2\left[\Psi_{(2,2)}\right], \tag{6.4}$$

where $\phi_{(2,0)}$ denotes the primary field of the dimension $\Delta_{(2,0)} = \Delta_0 + (\alpha_+)^2$ In (6 4) we have explicitly written out the numerical coefficients C_1 and C_2 of the corresponding primary fields in the operator product expansion. In the above symbolic formulae like (6.1)–(6.3) such coefficients are omitted On the other hand, the field $\psi_{(2,1)}$, also being degenerate, satisfies the differential eq (5 17) which leads to the expansion

$$\psi_{(1,2)}\psi_{(2,1)} = C_1'\left[\phi_{(0,2)}\right] + C_2'\left[\Psi_{(2,2)}\right], \tag{6.5}$$

where the field $\phi_{(0,2)}$ has the dimension $\Delta_{(0,2)} = \Delta_0 + (\alpha_-)^2$ and C_1', C_2' are some numerical coefficients The comparison of this formula with (6 4) yields that $C_1 = C_1' = 0$ and $C_2 = C_2'$. Hence, the expansion of the product $\psi_{(1,2)}\psi_{(2,1)}$ contains the contribution of only one conformal family

$$\psi_{(1,2)}\psi_{(2,1)} = \left[\psi_{(2,2)}\right] \tag{6.6}$$

We shall call the phenomenon described above the *truncation* of the operator algebra* It can be shown that for the degenerate fields $\psi_{(n,m)}$ this is the general

* It is interesting to understand the connection of the truncation phenomenon with the monodromy properties of the differential equations satisfied by the correlation functions This problem can be most easily investigated for the four-point differential equations If all the fields involved are degenerate, the space of solutions of the differential equations proves to contain the subspace invariant under the monodromy transformations The solutions, belonging to this subspace, correspond to the degenerate fields $\psi_{(k,l)}$ $(k, l > 0)$ in (6 7) and these very solutions contribute to the correlation function

situation. the degenerate conformal families $[\psi_{(n,m)}]$ with $n, m > 0$ actually appear only in the "fusion rules" like (6.3) The general "fusion rules" for the degenerate fields have the form*

$$\psi_{(n_1,m_1)}\psi_{(n_2,m_2)} = \sum_{k=|n_1-n_2|+1}^{n_1+n_2-1} \sum_{l=|m_1-m_2|+1}^{m_1+m_2-1} [\psi_{(k,l)}], \tag{6 7}$$

where the variable k (l) runs over the even integers, provided $n_1 + n_2$ $(m_1 + m_2)$ is odd and vice versa

So, the degenerate fields (more precisely, the degenerate conformal families) form the closed operator algebra. This observation gives rise to the idea of conformal quantum field theory in which all the primary fields are degenerate To examine this possibility let us concentrate once again on the Kac formula (5 9) It is clear that there are three distinct domains of the parameter c. If $c \geqslant 25$ the second term in (5 9) is negative and the dimensions $\Delta_{(n,m)}$ become negative for sufficiently large n and m. If $25 > c > 1$, the dimensions $\Delta_{(n,m)}$ are, in general, complex Neither possibility is acceptable in the quantum field theory** Therefore in what follows we shall consider the domain

$$0 < c \leqslant 1 \tag{6 8}$$

To understand the properties of the spectrum (5 9) clearly, let us consider the "diagram of dimensions" shown in fig 1. The vertical and horizontal axes in this figure correspond to the values of the parameters n and m in (5 9). The "physical" (i e. the positive integer) values of these parameters are shown by dots. The dotted line has the slope.

$$\operatorname{tg}\theta = -\frac{\alpha_+}{\alpha_-} = \frac{\sqrt{25-c} - \sqrt{1-c}}{\sqrt{25-c} + \sqrt{1-c}} \tag{6 9}$$

The value (5.22) of the dimension is associated with each point of the plane in fig. 1, the parameter α being proportional to the distance between the point and the dotted line

* The "fusion rule" (6 7) can be obtained from the following formula

$$\psi_{(n\ m)} = (\Psi_{(1\ 2)})^{m-1}(\Psi_{(2\ 1)})^{n-1},$$

for the degenerate field $\psi_{(n\ m)}$ Although this formula scarcely has a precise mathematical meaning, one can use it to derive (6 7) assuming the associativity and taking into account the truncation phenomenon

** To avoid misunderstanding let us stress that these statements by no means exclude the possibility of quantum field theory existing at $c > 1$, but rather prevent from including the degenerate fields in the operator algebra

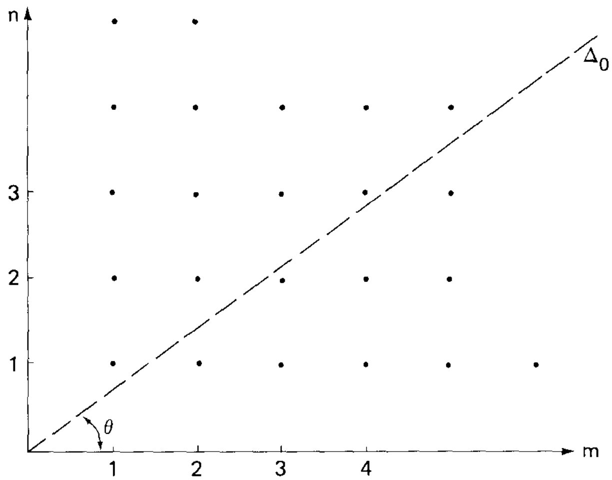


Fig 1 "Diagram of dimensions" The dimension $\Delta = \Delta_0 + \frac{1}{4}\alpha^2$ is associated with each point of the plane, α being proportional to the distance between the point and the dotted line The dots with coordinates (n, m) corresponds to the dimensions $\Delta_{(n\ m)}$ described by Kac formula (5 9)

If the slope (6.9) takes an arbitrary irrational value, the dotted line in fig 1 passes arbitrarily close to some of the dots. Since at $c < 1$, Δ_0 is negative, we meet again with the problem of negative dimensions. Let us consider, however, the cases of the rational slope

$$\operatorname{tg} \theta = -\alpha_- / \alpha_+ = p/q, \tag{6.10}$$

where p and q are positive integers. The characteristic feature of the corresponding values of c is that each degenerate representation $V_{\Delta_{(n\ m)}}$ contains not only one but infinitely many null vectors of different dimension This is evident from (5 9) and (6.10) In these cases the irreducible conformal families $[\psi_{(n,m)}]$ obtained by nullification of all the null fields, contain considerably fewer fields than the usual families and we call the conformal quantum field theories, corresponding to (6 10) and involving these degenerate fields $\psi_{(n,m)}$, minimal theories. It is important that in the minimal theories the correlation functions satisfy infinitely many differential equations, obtained by nullification of all the corresponding null fields* This fact enables one to prove that the operator algebra of degenerate fields in the minimal theories possesses not only "truncation from below", described in the beginning of the section, but also the "truncation from above". Namely, if one starts with the fields $\psi_{(n,m)}$ with $0 < n < p$, $0 < m < q$, the degenerate fields with $n \geqslant p$ or $m \geqslant q$ drop out from the "fusion rules" (6 7) (like the fields $\phi_{(2\ 0)}$ and $\phi_{(0,2)}$ in (6 4), (6.5)) In other words, the conformal families $[\psi_{(n,m)}]$ with $0 < n < p$, $0 < m < q$ form the

* In fact, these differential equations are not all independent they follow from two "basic" equations

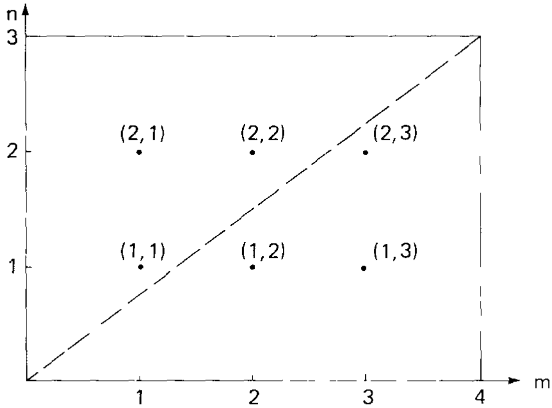


Fig 2 Diagram of dimensions corresponding to the case $\mathrm{tg}\,\theta = \frac{3}{4}$ ($c = \frac{1}{2}$) The degenerate conformal families associated with the dots inside the rectangle form the closed operator algebra

closed algebra which can be treated as the operator algebra of the quantum field theory Note that (under the condition (6 10)) $n = p$, $m = q$ are the coordinates of the nearest dot in fig. 1 which the dotted line passes through The degenerate fields with the dimensions associated with the dots inside the rectangle $0 < n < p$, $0 < m < q$, shown in figs 2 and 3, form the closed operator algebra Due to the diagonal symmetry of this rectangle there are $\frac{1}{2}(p-1)(q-1)$ different dimensions

Consider in more detail the simplest nontrivial example of the minimal theory corresponding to the case

$$p/q = \tfrac{3}{4}, \tag{6 11}$$

which occurs if

$$c = \tfrac{1}{2} \tag{6 12}$$

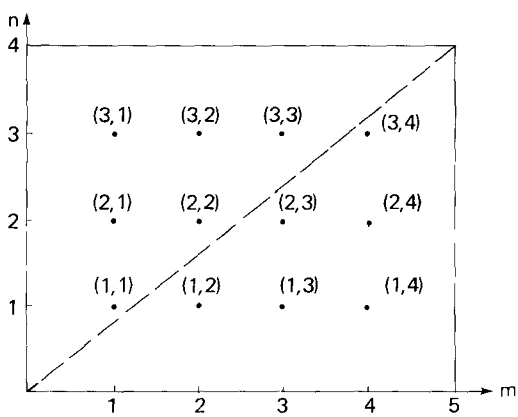


Fig 3 Diagram of dimensions for the case $\mathrm{tg}\,\theta = \frac{4}{5}$ ($c = \frac{7}{10}$)

The "diagram of dimensions" for this case is shown in fig 2 Let us demonstrate the "truncation from above", using this example. The dimensions corresponding to the dots in fig 2 are

$$\Delta_{(1,1)} = \Delta_{(2,3)} = 0,$$

$$\Delta_{(2,1)} = \Delta_{(1,3)} = \tfrac{1}{2},$$

$$\Delta_{(1,2)} = \Delta_{2,2)} = \tfrac{1}{16} \tag{6.13}$$

Respectively, there are three degenerate fields* which we shall denote by

$$I = \psi_{(1,1)} = \psi_{(2,3)},$$

$$\varepsilon = \Psi_{(2,1)} = \psi_{(1,3)},$$

$$\sigma = \psi_{(1,2)} = \psi_{(2,2)}. \tag{6.14}$$

Consider, for instance, the product ε ε. The field ε, being equal to $\psi_{(2,1)}$, satisfies the second order differential eq. (5 17). Therefore, according to (6.36), one gets

$$\varepsilon \cdot \varepsilon = \psi_{(2,1)}\psi_{(2,1)} = c_1[I] + c_2\left[\Psi_{(3,1)}\right], \tag{6 15}$$

where the field $\psi_{(3,1)}$ has the dimension $\Delta_{(3,1)} = \frac{5}{3}$. On the other hand, since $\varepsilon = \psi_{(1,3)}$, this field satisfies the third order differential equation (D.8) and hence

$$\varepsilon \;\; \varepsilon = \Psi_{(1,3)}\psi_{(1,3)} = c_1'[I] + c_2'\left[\psi_{(1,3)}\right] + c_3'\left[\psi_{F(1,5)}\right], \tag{6 16}$$

where the field $\psi_{(1,5)}$ has the dimension $\Delta_{(1,5)} = \frac{5}{2}$. Comparing (6 16) and (6.15), one concludes that in fact

$$\varepsilon \cdot \varepsilon = [I]. \tag{6.17}$$

By similar considerations the following "fusion rules" for the fields (6.14) can be obtained.

$$I \;\; \varepsilon = [\varepsilon], \qquad \varepsilon \cdot \varepsilon = [I],$$

$$I \;\; \sigma = [\sigma], \qquad \varepsilon \cdot \sigma = [\sigma],$$

$$I \;\; I = [I], \qquad \sigma \;\; \sigma = [I] + [\varepsilon]. \tag{6.18}$$

* Certainly, the analysis of the dimensions (6 13) does not prove that the operator algebra contains only three primary fields To elucidate the structure of the fields constituting the operator algebra one should take into account the $\bar{z}$-dependence and the local properties of the fields For the model under consideration this is done in appendix E

It is shown in appendix E that this minimal theory describes the critical point of the two-dimensional Ising model, the primary fields σ, ε and I being identified with the local spin, energy density and identity operators, respectively

In fig 3 the "diagram of dimensions" for the minimal theory characterized by the values

$$p/q=\tfrac{4}{5}, \qquad c=\tfrac{7}{10}, \tag{6 19}$$

is presented as another example The corresponding numerical values of the dimensions are

$$\begin{aligned}
&\Delta_{(1,1)}=\Delta_{(3,4)}=0, \\
&\Delta_{(1,2)}=\Delta_{(3,3)}=\tfrac{1}{10}, \\
&\Delta_{(1,3)}=\Delta_{(3,2)}=\tfrac{3}{5}, \\
&\Delta_{(1,4)}=\Delta_{(3,1)}=\tfrac{3}{2}, \\
&\Delta_{(2,2)}=\Delta_{(2,3)}=\tfrac{3}{80}, \\
&\Delta_{(2,4)}=\Delta_{(2,1)}=\tfrac{7}{16}
\end{aligned} \tag{6 20}$$

Note that due to the inequalities (6 8) the integers p and q in (6 10) are restricted as follows

$$\tfrac{2}{3}<p/q<1 \tag{6 21}$$

Nevertheless, there are infinitely many rational numbers, satisfying (6 21) and each of them corresponds to some minimal model of the conformal quantum field theory. We suppose that the minimal theories describe second order phase transitions in two-dimensional systems with discrete symmetry groups* In any case each of the minimal models seems to deserve a most detailed investigation. Note that the anomalous dimensions associated with each of the minimal model are known exactly (they are given by the Kac formula (5.9)), whereas the correlation functions can be computed in the following way. At first one has to derive the corresponding conformal blocks as solutions of the respective differential equations with the

* V Dotzenko has noticed that the spectrum of dimensions associated with the minimal model

$$p/q=\tfrac{5}{6}, \qquad c=\tfrac{4}{5}$$

contains some dimensions characteristic of the three-state Potts model appropriate initial conditions. Then, substituting these conformal blocks into the bootstrap eq (4 18) and taking into account the local properties of the fields, one should calculate the structure constants C_{nm}^{l} of the operator algebra, which provide enough information to construct the correlation functions. For the minimal theory (6.11) this computation is presented in appendix E. In the general case it has not yet been performed.

We are obliged to B Feigin for numerous consultations about the representations of the Virasoro algebra and to A.A Migdal for useful discussions The two of us (AB and AZ) are very grateful to D Makagonenko and A A. Anselm for the kind hospitality in the Scientific Center in Komarovo during January 1983 where this work was completed.

Appendix A

Let L_{-1}, L_0, L_{+1} and $\bar{L}_{-1}, \bar{L}_0, \bar{L}_{+1}$ be generators of the infinitesimal projective transformations

$$z \to z + \varepsilon_{-1} + \varepsilon_0 z + \varepsilon_1 z^2,$$

$$\bar{z} \to \bar{z} + \bar{\varepsilon}_{-1} + \bar{\varepsilon}_0 \bar{z} + \bar{\varepsilon}_1 \bar{z}^2, \tag{A 1}$$

where ε and $\bar{\varepsilon}$ are infinitesimal parameters. The operators L_s, $s = 0, \pm 1$ satisfy the commutation relations

$$[L_0, L_{\pm 1}] = \pm L_{\pm 1},$$

$$[L_1, L_{-1}] = 2L_0. \tag{A 2}$$

The same relations are satisfied by the $\bar{L}$'s, the L's and $\bar{L}$'s being commutative The operators $P^0 = L_{-1} + \bar{L}_{-1}$ and $P^1 = -i(L_{-1} - \bar{L}_{-1})$ are components of the total momentum, whereas $M = i(L_0 - \bar{L}_0)$ and $D = L_0 + \bar{L}_0$ are generators of the rotations (Lorentz boosts in the Minkowski space-time) and dilatations, respectively The operators L_1 and $\bar{L}_1$ correspond to the special conformal transformations The vacuum of the conformal quantum field theory satisfies the relations

$$\langle 0 | L_s = L_s | 0 \rangle = 0, \qquad s = 0, \pm 1, \tag{A 3}$$

which are equivalent to the asymptotic condition (2 14).

We shall call the local field $O_l(z, \bar{z})$ *quasiprimary*, provided it satisfies the commutation relations.

$$[L_s, O_l(z, \bar{z})] = \left[z^{s+1} \frac{\partial}{\partial z} + (s+1)\Delta_l z^s \right] O_l(z, \bar{z}),$$

$$[\bar{L}_s, O_l(z, \bar{z})] = \left[\bar{z}^{s+1} \frac{\partial}{\partial \bar{z}} + (s+1)\bar{\Delta}_l \bar{z}^s \right] O_l(z, \bar{z}), \tag{A 4}$$

where $s = 0, \pm 1$. The constants Δ_l and $\bar{\Delta}_l$ are dimensions of the field O_l. These relations mean that the fields $O_l(z, \bar{z})$ transform according to formula (1 16) under the projective transformations (1 15) This distinguishes them from the primary fields ϕ_n which transform according to (1.16) with respect to all conformal transformations (1 9)*. In the conformal quantum field theory the complete set of local fields A_j, forming the algebra (1 6), can be constituted by an infinite number of quasiprimary fields and their coordinate derivatives of all orders

$$\{A_j\} = \left\{O_l, \frac{\partial}{\partial z}O_l, \frac{\partial}{\partial \bar{z}}O_l, \frac{\partial^2}{\partial z^2}O_l, \ldots\right\}. \tag{A 5}$$

Consider an N-point correlation function of the quasiprimary fields It follows from (A.3) and (A.4) that this correlation function satisfies the equations

$$\hat{\Lambda}_s\langle O_{l_1}(z_1, \bar{z}_1) \ldots O_{l_N}(z_N, \bar{z}_N)\rangle = 0, \tag{A 6}$$

where $s = 0, \pm 1$ and $\hat{\Lambda}_s$ are the differential operators

$$\hat{\Lambda}_{-1} = \sum_{i=1}^{N} \frac{\partial}{\partial z_i},$$

$$\hat{\Lambda}_0 = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} + \Delta_i\right),$$

$$\hat{\Lambda}_1 = \sum_{i=1}^{N} \left(z_i^2 \frac{\partial}{\partial z_i} + 2z_i\Delta_i\right), \tag{A 7}$$

where $\Delta_1, \Delta_2, \ldots, \Delta_N$ are dimensions of the fields $O_{l_1}, \ldots O_{l_N}$, respectively Eqs (A 6) are the projective Ward identities. Note that these Ward identities follow directly from the general relation (2 9) For the infinitesimal projective transformations the function $\varepsilon(z)$ is regular in the finite part of the z-plane and due to the asymptotic condition (2.14) the contour integral in (2.9) vanishes. Let us stress that for the general conformal transformations the analytic function $\varepsilon(z)$ has singularities Therefore the corresponding Ward identities cannot be reduced to the closed equations for the correlation functions like (A.6) The general solution of eqs (A.6) (and the analogous equations obtained by the substitution $z_i \to \bar{z}_i$, $\Delta_i \to \bar{\Delta}_i$) is

$$\langle O_{l_1}(z_1, \bar{z}_1) \ldots O_{l_N}(z_N, \bar{z}_N)\rangle = \prod_{i<j}(z_i - z_j)^{\gamma_{ij}}(\bar{z}_i - \bar{z}_j)^{\bar{\gamma}_{ij}} Y\left(x_{ij}^{kl}, \bar{x}_{ij}^{kl}\right), \tag{A 8}$$

* Obviously, any primary field is quasiprimary whereas there are infinitely many quasiprimary fields which are secondaries

where γ_{ij} and $\bar{\gamma}_{ij}$ are arbitrary solutions of the equations

$$\sum_{j \neq i} \gamma_{ij} = 2\Delta_i, \qquad \sum_{j \neq i} \bar{\gamma}_{ij} = 2\Delta_i, \tag{A.9}$$

whereas Y is an arbitrary function of $2(N-3)$ anharmonic quotients

$$x_{ij}^{kl} = \frac{(z_i - z_j)(z_k - z_l)}{(z_i - z_l)(z_k - z_j)}, \qquad \bar{x}_{ij}^{kl} = \frac{(\bar{z}_i - \bar{z}_j)(\bar{z}_k - \bar{z}_l)}{(\bar{z}_i - \bar{z}_l)(\bar{z}_k - \bar{z}_j)} \tag{A.10}$$

In the particular cases $N = 2$ and $N = 3$ the correlation functions are determined by formulae (A 8)–(A 10) completely up to the numerical factor Namely,

$$\langle O_{l_1}(z_1, \bar{z}_1) O_{l_2}(z_2, \bar{z}_2) \rangle = \begin{cases} 0 & \text{if} \quad \Delta_{l_1} \neq \Delta_{l_2} \quad \text{or} \quad \bar{\Delta}_{l_1} \neq \bar{\Delta}_{l_2} \\ (z_1 - z_2)^{-2\Delta_{l_1}} (\bar{z}_1 - \bar{z}_2)^{-2\bar{\Delta}_{l_1}} D_{l_1} & \text{if} \\ \Delta_{l_1} = \Delta_{l_2} \quad \text{and} \quad \bar{\Delta}_{l_1} = \bar{\Delta}_{l_2}, \end{cases} \tag{A 11}$$

for $N = 2$ and

$$\langle O_{l_1}(z_1, \bar{z}_1) O_{l_2}(z_2, \bar{z}_2) O_{l_3}(z_3, \bar{z}_3) \rangle = Y_{l_1 l_2 l_3} \prod_{i<j} (z_i - z_j)^{-\Delta_{ij}} (\bar{z}_i - \bar{z}_j)^{-\bar{\Delta}_{ij}}, \tag{A 12}$$

for $N = 3$ where D_l and $Y_{l_1 l_2 l_3}$ are constants and

$$\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3 \quad \text{etc.},$$

$$\bar{\Delta}_{12} = \bar{\Delta}_1 + \bar{\Delta}_2 - \bar{\Delta}_3 \quad \text{etc} \tag{A 13}$$

Note that the functions (A 11) and (A 12) are single-valued in the euclidean space (obtained by the substitution $\bar{z}_i = z_i^*$), provided the spins $S_l = \Delta_l - \bar{\Delta}_l$ of all the fields involved take integer or half-integer values

In the conformal quantum field theory the expansion (1.6) can be represented in the form

$$O_{l_1}(z, \bar{z}) O_{l_2}(0,0) = \sum_{l_3} \sum_{k\ \bar{k}=0}^{\infty} Y_{l_1 l_2}^{l_3, k, \bar{k}} z^{\Delta_3 + k - \Delta_1 - \Delta_2} \bar{z}^{\bar{\Delta}_3 + \bar{k} - \bar{\Delta}_1 - \bar{\Delta}_2}$$

$$\times \left[\frac{\partial^{k+\bar{k}}}{\partial \zeta^k \partial \bar{\zeta}^{\bar{k}}} O_{l_3}(\zeta, \bar{\zeta}) \right]_{\zeta, \bar{\zeta}=0}, \tag{A 14}$$

where $Y_{l_1 l_2}^{l_3, k, \bar{k}}$ are constants, k and $\bar{k}$ being integers The transformation properties of the both sides of this equation with respect to the projective transformations (A 1) must coincide Commuting both sides of (A 14) with the projective generators L_s, $s = 0, \pm 1$ and using (A 4), one gets equations relating the coefficients $Y_{l_1 l_2}^{l, k \bar{k}}$ with different values of k. Solving these equations, one can rewrite (A 14) as

$$O_l(z, \bar{z}) O_l(0,0) = \sum_{l'} G_{ll}^{l'} z^{\Delta' - 2\Delta} \bar{z}^{\bar{\Delta}' - 2\bar{\Delta}}$$

$$\times F\left(\Delta', 2\Delta', z\frac{\partial}{\partial \zeta}\right) F\left(\bar{\Delta}', 2\bar{\Delta}', \bar{z}\frac{\partial}{\partial \bar{\zeta}}\right) O_{l'}(\zeta, \bar{\zeta})|_{\zeta, \bar{\zeta}=0} \quad \text{(A 15)}$$

where the case $l_1 = l_2$ is considered for the sake of simplicity, $\Delta_{l_1} = \Delta_{l_2} = \Delta$, $\Delta_{l'} = \Delta'$ In (A 15) $G_{ll}^{l'}$ are the constants, coinciding with $Y_{ll}^{l',0\,0}$ in (A 14) and $F(a, c, x)$ denotes the degenerate hypergeometric function

Obviously, each conformal family $[\phi_n] = V_n \times \bar{V}_n$ (see sect 3) contains infinitely many quasiprimary fields These fields correspond to the states satisfying the equations

$$L_1|l\rangle = \bar{L}_1|l\rangle = 0,$$

$$L_0|l\rangle = \Delta_l|l\rangle, \qquad \bar{L}_0|l\rangle = \bar{\Delta}_l|l\rangle \quad \text{(A 16)}$$

It can be shown that the basis in $[\phi_n]$ can be constituted by the states

$$(L_{-1})^n (\bar{L}_{-1})^{\bar{n}} |l\rangle, \quad \text{(A 17)}$$

where $n, \bar{n} = 0, 1, 2, \ldots$ and $|l\rangle$ are the quasiprimary states, belonging to $[\phi_n]$. This statement is equivalent to (A 5) because the operators L_{-1} and $\bar{L}_{-1}$ are associated with the derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$

Appendix B

Here we shall demonstrate that the coefficients $\beta_{nm}^{l\{k\}}$ in (4.2) are determined completely by the requirement of the conformal symmetry of the expansion (4 1), considering the particular case $\Delta_n = \Delta_m = \Delta$ for the sake of simplicity. Applying both sides of (4 1) to the vacuum state, one gets the equation

$$\phi_\Delta(z, \bar{z})|\Delta\rangle = \sum_l C_{\Delta\Delta}^{\Delta_l} z^{\Delta_l - 2\Delta} \bar{z}^{\bar{\Delta}_l - 2\bar{\Delta}} \varphi_\Delta(z) \bar{\varphi}_{\bar{\Delta}}(\bar{z}) |\Delta_l\rangle, \quad \text{(B 1)}$$

where $|\Delta\rangle$ is the primary state of the dimensions $\Delta, \bar{\Delta}$ and the operator $\varphi_\Delta(z)$ is given by the series

$$\varphi_\Delta(z) = \sum_{\{k\}} z^{\Sigma k_i} \beta_{\Delta\Delta}^{\Delta_l, \{k\}} L_{-k_1} \ldots L_{-k_N} \quad \text{(B 2)}$$

The same formula with the substitution $z \to \bar{z}$, $\beta \to \bar{\beta}$, $L \to \bar{L}$ holds for $\bar{\varphi}_{\bar{\Delta}}(\bar{z})$ Let us consider the state

$$|z, \Delta'\rangle = \varphi_\Delta(z)|\Delta'\rangle . \tag{B 3}$$

It can be represented as the power series

$$|z, \Delta'\rangle = \sum_{N=0}^{\infty} z^N |N, \Delta'\rangle , \tag{B 4}$$

where the vectors $|N, \Delta'\rangle$ satisfy the equations

$$L_0|N, \Delta'\rangle = (\Delta_l + N)|N, \Delta'\rangle . \tag{B 5}$$

To compute these vectors let us apply the operators L_n to both sides of (B 1) This leads to the equations

$$\left[z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z} + \Delta(n+1)z^n\right]|z, \Delta'\rangle = L_n|z, \Delta'\rangle \tag{B 6}$$

Substituting the power series (B 4) one gets

$$L_n|N+n, \Delta'\rangle = [N + (n-1)\Delta + \Delta']|N, \Delta'\rangle \tag{B 7}$$

Actually, one can consider eqs. (B 7) with $n = 1, 2$ only because in virtue of (2 21) the remaining equations follow from these two Solving these equations one can compute the power series (B.4) order by order In the first three orders the result is

$$|z, \Delta'\rangle = \left[1 + \tfrac{1}{2}zL_{-1} + \tfrac{1}{4}z^2\frac{\Delta'+1}{2\Delta'+1}L_{-1}^2 + z^2\frac{\Delta'(\Delta'-1) + 2\Delta(2\Delta'+1)}{c(2\Delta'+1) + 2\Delta'(8\Delta'-5)}\right.$$
$$\left.\times\left(L_{-2} + \frac{3}{2(2\Delta'+1)}L_{-1}^2\right) + \cdots\right]|\Delta'\rangle \tag{B 8}$$

This formula gives the first three coefficients β in (B 2)

Obviously the conformal block $\mathcal{F}(\Delta, \Delta', x) \equiv \mathcal{F}_{\Delta\Delta}^{\Delta\Delta}(\Delta'|x)$ is given by the scalar product

$$\mathcal{F}(\Delta, \Delta', x) = x^{\Delta' - 2\Delta}\langle 1, \Delta'|x, \Delta'\rangle . \tag{B 9}$$

The first few terms of the power expansion of this function can be directly obtained from (B 8)

$$\mathcal{F}(\Delta, \Delta', x) = x^{\Delta'-2\Delta}\left\{1 + \tfrac{1}{2}\Delta' x + \frac{\Delta'(\Delta'+1)^2}{4(2\Delta'+1)}x^2\right.$$
$$\left. + \frac{[\Delta'(1-\Delta') - 2\Delta(2\Delta'+1)]^2}{2(2\Delta'+1)[c(2\Delta'+1) + 2\Delta'(8\Delta'-5)]}x^2 + \quad\right\} . \tag{B.10}$$

Appendix C

Consider the associative algebra determined by the relations

$$A_I A_J = \sum_K C_{IJ}^K A_K \tag{C.1}$$

Eq (1 6) is just (C 1) where each of the indices, say I, combines the space coordinate ξ and the index ι, labelling the fields. Let the algebra (C 1) be supplied with the symmetric bilinear form

$$D_{IJ} = \langle A_I A_J \rangle , \tag{C 2}$$

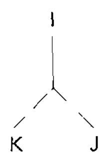
which is no other than a set of all two-point correlation functions Let us introduce the form

$$C_{IJK} = \sum_{K'} D_{KK'} C_{IJ}^{K'} . \tag{C 3}$$

and assume that this is a symmetric function of the indices I, J, K Evidently, (C 3) coincides with the three-point correlation function

$$C_{IJK} = \langle A_I A_J A_K \rangle , \tag{C 4}$$

and it can be conveniently represented by the "vertex" diagram

$C_{IJK} =$ 

(C 5)

Also introduce the diagram

$D^{IJ} =$ 

(C 6)

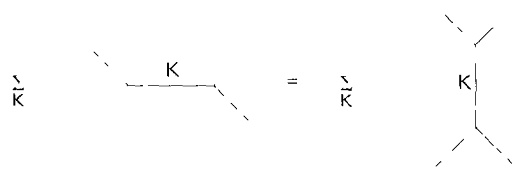
for the "inverse propagator" D^{IJ} defined by the equation

$$\sum_K D^{IK} D_{KJ} = \delta_J^I . \tag{C 7}$$

The associativity condition of the algebra (C 1)

$$\sum_K C_{IJ}^K C_{KL}^M = \sum_K C_{IK}^M C_{JL}^K \tag{C 8}$$

can be represented by the diagrammatic equation



(C 9)

which coincides with the "crossing symmetry" condition by the four-point functions

$$\langle A_I A_J A_L A_M \rangle \tag{C 10}$$

Appendix D

In this appendix we shall derive the differential equation satisfied by the correlation function

$$\langle \psi(z)\phi_1(z_1) \quad \phi_N(z_N)\rangle, \tag{D 1}$$

where $\psi(z)$ denotes any of the degenerate fields $\psi_{(1,3)}(z)$ and $\psi_{(3,1)}(z)$, whereas $\phi_i(z)$ are arbitrary primary fields with the dimensions Δ_i, $i = 1, 2, \quad, N$ First of all, note that the state

$$|\chi_3\rangle = \left[(\Delta+2)L_{-3} - 2L_{-1}L_{-2} + \frac{1}{(\Delta+1)}L_{-1}^3\right]|\Delta\rangle, \tag{D 2}$$

(where $|\Delta\rangle$ is the primary state with the dimension Δ) is the null vector (with the dimension $\Delta + 3$), provided Δ takes any of the values $\Delta_{(1\ 3)}$ or $\Delta_{(3\ 1)}$, i.e

$$\Delta = \tfrac{1}{6}\left[7 - c \pm \sqrt{(1-c)(25-c)}\right] \tag{D 3}$$

The equivalent statement is that the operator

$$\chi_{\Delta+3}(z) = (\Delta+2)\psi^{(-3)}(z) - 2\frac{\partial}{\partial z}\psi^{(-2)}(z) + \frac{1}{\Delta+1}\frac{\partial^3}{\partial z^3}\psi(z) \tag{D 4}$$

is the null field of the dimension $\Delta + 3$ In (D.4) are the secondaries of the degenerate field $\psi(z)$ $(=\psi_{(1,3)}$ or $\psi_{(3\ 1)})$ and Δ is given by (D 3) The differential equation for the correlation function (D 1) follows from the condition

$$\chi_{\Delta+3} = 0. \tag{D.5}$$

It follows that

$$\langle \psi^{(-2)}(z)\phi_1(z_1) \quad \phi_N(z_N)\rangle$$

$$= \left\{ \sum_{i=1}^{N} \frac{\Delta_i}{(z-z_i)^2} + \sum_{i=1}^{N} \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right\} \langle \psi(z)\phi_1(z_1) \quad \phi_N(z_N)\rangle, \qquad \text{(D.6)}$$

$$\langle \psi^{(-3)}(z)\phi_1(z_1). \quad \phi_N(z_N)\rangle$$

$$= -\left\{ \sum_{i=1}^{N} \frac{2\Delta_i}{(z-z_i)^3} + \sum_{i=1}^{N} \frac{1}{(z-z_i)^2} \frac{\partial}{\partial z_i} \right\} \langle \psi(z)\phi_1(z_1) \quad .\phi_N(z_N)\rangle$$

(D 7)

Substituting (D 4) into (D 5) and taking into account (D.6) and (D 7), one gets the third order differential equation

$$\left\{ \frac{1}{\Delta+1} \frac{\partial^3}{\partial z^3} - \sum_{i=1}^{N} \frac{2\Delta\Delta_i}{(z-z_i)^3} - \sum_{i=1}^{N} \frac{\Delta}{(z-z_i)^2} \frac{\partial}{\partial z_i} \right.$$

$$\left. - \sum_{i=1}^{N} \frac{2\Delta_i}{(z-z_i)^2} \frac{\partial}{\partial z} - \sum_{i=1}^{N} \frac{2}{z-z_i} \frac{\partial^2}{\partial z\, \partial z_i} \right\} \langle \psi(z)\phi_1(z_1) \; ..\phi_N(z_N)\rangle = 0 \qquad \text{(D 8)}$$

In the particular case $N = 3$, the derivatives can be excluded by means of the projective Ward identities (A 7) Simple calculations lead to the following ordinary differential equation

$$\left\{ \frac{1}{\Delta+1} \frac{d^3}{dz^3} + \sum_{i=1}^{3} \frac{1}{z-z_i} \frac{d^2}{dz^2} + \sum_{i=1}^{3} \frac{\Delta - 2\Delta_i}{(z-z_i)^2} \frac{d}{dz} \right.$$

$$- \sum_{i=1}^{3} \frac{2\Delta\Delta_i}{(z-z_i)^3} + \sum_{i<j}^{3} \frac{2\Delta + 2 + \Delta_{ij}}{(z-z_i)(z-z_j)}$$

$$\left. + \sum_{i<j}^{3} \frac{\Delta + \Delta_{ij}}{(z-z_i)(z-z_j)} \left(\frac{1}{(z-z_i)} + \frac{1}{(z-z_j)} \right) \right\} \langle \psi(z)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = 0,$$

(D 9)

where

$$\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3 \quad \text{etc}$$

Appendix E

As is well known (see, for instance, [15] and references therein), the two-dimensional Ising model is equivalent to the theory of free Majorana fermions In the continuous limit this theory is described by the lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\psi\frac{\partial}{\partial\bar{z}}\psi + \tfrac{1}{2}\bar{\psi}\frac{\partial}{\partial z}\bar{\psi} + m\bar{\psi}\psi\,. \tag{E.1}$$

where m is the mass parameter, proportional to $T - T_c$, and $(\psi, \bar{\psi})$ is the two-component Majorana field* In what follows we shall consider the critical point only, where this field is massless.

$$m = 0 \tag{E.2}$$

According to (E.1), in this case the fields $\psi, \bar{\psi}$ satisfy the equation of motion

$$\frac{\partial}{\partial\bar{z}}\psi = 0\,, \qquad \frac{\partial}{\partial z}\bar{\psi} = 0\,, \tag{E 3}$$

and therefore these fields are analytic functions of the variables z and $\bar{z}$, respectively We shall write

$$\psi = \psi(z)\,, \qquad \bar{\psi} = \bar{\psi}(\bar{z}) \tag{E 4}$$

The stress-energy tensor corresponding to this theory can be computed directly In the case (E 2) it is traceless and the components (2.5) are given by the formulae

$$T(z) = -\tfrac{1}{2}\ \psi(z)\frac{\partial}{\partial z}\psi(z)\,,$$

$$\bar{T}(\bar{z}) = -\tfrac{1}{2}.\bar{\psi}(\bar{z})\frac{\partial}{\partial\bar{z}}\bar{\psi}(\bar{z})\,. \tag{E 5}$$

It can be easily verified that the fields (E 5) satisfy the Virasoro algebra (2 21), the central charge c being

$$c = \tfrac{1}{2} \tag{E 6}$$

The fundamental fields ψ and $\bar{\psi}$ satisfy the relations (1 16), i e these fields are primary. The dimensions of the field $\psi(z)$ $(\bar{\psi}(\bar{z}))$ are $\Delta = \frac{1}{2}$, $\bar{\Delta} = 0$ $(\Delta = 0,\ \bar{\Delta} = \frac{1}{2})$ It can be shown that four conformal families $[I], [\psi], [\bar{\psi}], [\ \bar{\psi}\psi\ ]$ contitute a complete set of fields $\{A_j\}$, forming the operator algebra (1.6)

* The field $\bar{\psi}$ is an independent component but in general it is not the complex conjugated value of the field ψ

Let us take, for instance, the field $\psi(z)$ This primary field proves to coincide with the degenerate field $\psi_{(2,1)}(z)$ (see (6.13)) Actually, the operator product expansion for $T(\zeta)\psi(z)$ (which is easily computed if (E.5) is employed) is given (up to the first three terms) by the formula

$$T(\zeta)\psi(z)=\frac{1}{2}\frac{1}{(\zeta-z)^2}\psi(z)+\frac{1}{\zeta-z}\frac{\partial}{\partial z}\psi(z)+\frac{3}{4}\frac{\partial^2}{\partial z^2}\psi(z)+O(\zeta-z), \quad (E\,7)$$

which shows that the secondary field (5.2) vanishes. Therefore, the correlation functions, involving the degenerate field $\psi(z)$, satisfy the differential equation

$$\left\{\frac{3}{4}\frac{\partial^2}{\partial z^2}-\sum_{i=1}^{N}\frac{\Delta_i}{(z-z_i)^2}-\sum_{i=1}^{N}\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right\}\langle\psi(z)\phi_1(z_1)\quad\phi_N(z_N)\rangle=0, \quad (E\,8)$$

where $\phi_i(z)$ are arbitrary primary fields (which are local themselves but not necessarily local with respect to $\psi(z)$) In particular, the correlation functions

$$\langle\psi(z)\psi(z_1)\,..\,\psi(z_N)\rangle, \quad (E\,9)$$

(which can be computed if the Wick rules are used) satisfy (E 6)

On the other hand, the critical Ising model can be described in terms of either the order-parameter field $\sigma(z,\bar{z})$ or the disorder-parameter field $\mu(z,\bar{z})$* Obviously, the fields σ and μ are primary. These fields have zero spins, i e $\Delta_\sigma=\bar{\Delta}_\sigma$, $\Delta_\mu=\bar{\Delta}_\mu$ and in virtue of the Krammers-Wanier symmetry, have the same scale dimensions

$$\Delta_\sigma=\Delta_\mu=\Delta. \quad (E\,10)$$

The fields $\sigma(z,\bar{z})$ and $\mu(z,\bar{z})$ are neither local with respect to the fields $\psi(z)$ and $\bar{\psi}(\bar{z})$ nor mutually local. In fact, the correlation function

$$\langle\psi(z)\sigma(\xi_1)\;.\;\sigma(\xi_{2N-1})\mu(\xi_{2N})\;.\;\mu(\xi_{2M})\rangle \quad (E\,11)$$

is a double-valued analytic function of z which acquires the phase factor (-1) after the analytical commutation around any of the singular points $z_k=\xi_k^1+i\xi_k^2$, $k=1,\;,2M$ It follows from the definition that the products $\psi(\zeta)\sigma(z,\bar{z})$ and $\psi(\zeta)\mu(z,\bar{z})$ can be expanded as

$$\psi(\zeta)\sigma(z,\bar{z})=(\zeta-z)^{-1/2}\{\mu(z,\bar{z})+O(\zeta-z)\},$$

$$\psi(\zeta)\mu(z,\bar{z})=(\zeta-z)^{-1/2}\{\sigma(z,\bar{z})+O(\zeta-z)\} \quad (E\,12)$$

* The fields σ and μ are the scaling limit of the lattice spin $\sigma_{n\,m}$ and the dual spin $\mu_{n+1\,2\,m+1\,2}$, respectively See ref [15] for the detailed definition

Substituting these expansions into the differential eq (E.8), one gets the characteristic equation, determining the parameter Δ

$$\Delta = \tfrac{1}{16} \tag{E.13}$$

in agreement with the known value of the scale dimension of the spin field $d_\sigma = 2\Delta = \frac{1}{8}$ [15] So, the differential eq. (E 8) together with the qualitative properties (E 12) of the operator algebra enables one to compute exactly the dimension of the field $\sigma(z, \bar{z})$

Now we are to compute the correlation functions of the order and disorder fields

$$\langle \sigma(\xi_1) \quad \sigma(\xi_{2N})\mu(\xi_{2N+1}) \ldots \mu(\xi_{2M})\rangle \tag{E.14}$$

Note that the double-valued function (E 11) can be represented by

$$\langle \psi(z)\sigma(\xi_1) \ . \ \mu(\xi_{2M})\rangle = \prod_{i=1}^{2M} (z - z_i)^{-1/2} P(z|z_i, \bar{z}_i), \tag{E 15}$$

where $P(z|z_i, \bar{z}_i)$ is a polynomial in z:

$$P(z|z_i, \bar{z}_i) = \sum_{k=0}^{2M-1} (z - z_{2N})^k G_k(z_i, \bar{z}_i) \tag{E 16}$$

The order $2M - 1$ of this polynomial is determined by the asymptotic condition

$$\psi(z) \sim z^{-1}, \qquad z \to \infty \tag{E 17}$$

The coefficients G_k are some functions of $z_1, \quad , z_{2M}, \bar{z}_1, \quad , \bar{z}_{2M}$ In virtue of (E 12), the coefficient $G_0(z_i, \bar{z}_i)$ coincides with the correlation function (E 14) Substituting (E 15) into the differential eq (E.8), one gets the differential equations for the coefficients $G_k(z_i, \bar{z}_i)$ which enables one to compute the correlation function (E 14)

In fact, the differential equations for the correlation functions (E 14) can be obtained in a simpler way Note that comparing (E 13) with (6 13), the field $\sigma(z, \bar{z})$ is the degenerate field $\psi_{(1\ 2)}$ with respect to the both variables z and $\bar{z}$ The same is valid for the field $\mu(z, \bar{z})$ Therefore, the correlation functions (E 14) satisfy the differential equations

$$\left\{ \frac{4}{3} \frac{\partial^2}{\partial z_i^2} - \sum_{j \neq i}^{2M} \frac{\frac{1}{16}}{(z_i - z_j)^2} - \sum_{j \neq i}^{2M} \frac{1}{z_i - z_j} \frac{\partial}{\partial z_j} \right\}$$
$$\times \langle \sigma(z_1, \bar{z}_1) \quad \sigma(z_{2N}, \bar{z}_{2N})\mu(z_{2N+1}, \bar{z}_{2N+1}) \quad \mu(z_{2M}, \bar{z}_{2M})\rangle = 0, \tag{E 18}$$

(where $i = 1, 2, \quad , 2M$) and the differential equations obtained from (E 18) by the substitution $z_i \to \bar{z}_i$

Let us consider, for example, the four-point correlation function

$$G(\xi_1,\xi_2,\xi_3,\xi_4)=\langle\sigma(\xi_1)\sigma(\xi_2)\sigma(\xi_3)\sigma(\xi_4)\rangle$$
$$=\left[(z_1-z_3)(z_2-z_4)(\bar{z}_1-\bar{z}_3)(\bar{z}_2-\bar{z}_4)\right]^{-1/8}Y(x,\bar{x}), \tag{E 19}$$

where $Y(x,\bar{x})$ is some function of the anharmonic quotients

$$x=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}, \qquad \bar{x}=\frac{(\bar{z}_1-\bar{z}_2)(\bar{z}_3-\bar{z}_4)}{(\bar{z}_1-\bar{z}_3)(\bar{z}_2-\bar{z}_4)}, \tag{E 20}$$

(we took into account (A.8)). In this case the differential eq (E 18) is reduced to the following form.

$$\left\{\frac{4}{3}\frac{d^2}{dx^2}-\frac{1}{16}\left[\frac{1}{x^2}+\frac{1}{(x-1)^2}\right]+\frac{1}{8}\frac{1}{x(x-1)}+\left[\frac{1}{x}+\frac{1}{x-1}\right]\frac{d}{dx}\right\}Y(x,\bar{x})=0 \tag{E 21}$$

The same equation with respect to $\bar{x}$ is also valid Substituting

$$Y(x,\bar{x})=\left[x\bar{x}(1-x)(1-\bar{x})\right]^{-1/8}u(x,\bar{x}), \tag{E 22}$$

one gets the following equation for

$$\left\{x(1-x)\frac{\partial^2}{\partial x^2}+\left(\tfrac{1}{2}-x\right)\frac{\partial}{\partial x}+\tfrac{1}{16}\right\}u(x,\bar{x})=0 \tag{E 23}$$

The change of variables

$$x=\sin^2\theta, \qquad \bar{x}=\sin^2\bar{\theta}, \tag{E 24}$$

reduces (E 23) to

$$\left(\frac{\partial^2}{\partial\theta^2}+\tfrac{1}{4}\right)u(\theta,\bar{\theta})=0 \tag{E 25}$$

The equation obtained from (E 25) by the substitution $\theta\to\bar{\theta}$ is also valid Therefore, the general solution of these differential equations has the form

$$u(\theta,\bar{\theta})=u_{11}\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\bar{\theta}+u_{12}\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\bar{\theta}$$
$$+u_{21}\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\bar{\theta}+u_{22}\sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\bar{\theta}, \tag{E 26}$$

where $u_{\alpha\beta}$ $(\alpha,\beta=1,2)$ are arbitrary constants

Note that two independent solutions of (E.21) coincide with the conformal blocks (see (B 9))

$$\mathcal{F}(\tfrac{1}{16},0,x)=[x(1-x)]^{-1/8}\cos\tfrac{1}{2}\theta,$$

$$\mathcal{F}(\tfrac{1}{16},\tfrac{1}{2},x)=[x(1-x)]^{-1/8}\sin\tfrac{1}{2}\theta, \qquad \text{(E 27)}$$

and therefore the formula (E.26) can be considered as the decomposition (4.11), the coefficients $u_{\alpha\beta}$ being the structure constants

Since the field $\sigma(z,\bar{z})$ is local, the correlation function (E 20) should be single-valued in the euclidean domain

$$\bar{x}=x^*, \qquad \text{(E.28)}$$

where the asterisk denotes complex conjugation. As it is clear from (E 24), the analytical continuation of the variables x and $\bar{x}$ around the singular point $x=\bar{x}=0$ corresponds to the substitution

$$\theta\to-\theta,\qquad \bar{\theta}\to-\bar{\theta} \qquad \text{(E 29)}$$

The function (E 26) is unchanged under this transformation provided

$$u_{12}=u_{21}=0. \qquad \text{(E.30)}$$

The same investigation of the singular point $x=\bar{x}=1$ (or, equivalently, imposing the crossing-symmetry condition) leads to the relation

$$u_{11}=u_{22} \qquad \text{(E 31)}$$

The overall factor in (E 26) depends on the σ-field normalization We shall normalize this field so that

$$\langle\sigma(z,\bar{z})\sigma(0,0)\rangle=[z\bar{z}]^{-1/8} \qquad \text{(E 32)}$$

Then

$$u(\theta,\bar{\theta})=\cos\tfrac{1}{2}(\theta-\bar{\theta}) \qquad \text{(E 33)}$$

The four-point function given by the formulae (E.20), (E.22) and (E 33) is in agreement with the previous result (see ref. [16]) obtained by a different method

Note that in virtue of (E.27) the four-point function (E 20) can be represented as

$$G=\mathcal{F}(\tfrac{1}{16},0,x)\bar{\mathcal{F}}(\tfrac{1}{16},0,\bar{x})+\mathcal{F}(\tfrac{1}{16},\tfrac{1}{2},x)\bar{\mathcal{F}}(\tfrac{1}{16},\tfrac{1}{2},\bar{x}) \qquad \text{(E 34)}$$

It is evident from this formula that only two conformal families contribute to the operator product expansion of $\sigma(\xi)\sigma(0)$ The corresponding primary fields have the dimensions $\Delta = \bar{\Delta} = 0$ and $\Delta = \bar{\Delta} = \frac{1}{2}$ The first of them is obviously identified with the identity operator I whereas the second is known as the energy density field

$$\varepsilon(z, \bar{z}) = \bar{\psi}(\bar{z})\psi(z) \tag{E 35}$$

The four-point correlation function

$$H(\xi_1, \xi_2, \xi_3, \xi_4) = \langle\sigma(\xi_1)\mu(\xi_2)\sigma(\xi_3)\mu(\xi_4)\rangle \tag{E.36}$$

can be represented in the form

$$H = \left[(z_1 - z_3)(z_2 - z_4)(\bar{z}_1 - \bar{z}_3)(\bar{z}_2 - \bar{z}_4)\right]^{-1/8}\tilde{Y}(x, \bar{x}), \tag{E 37}$$

where the function $\tilde{Y}$ satisfies the same differential equation (E 21) The investigation similar to the one performed above leads to the result

$$\tilde{Y}(x, \bar{x}) = [x\bar{x}(1 - x)(1 - \bar{x})]^{-1/8}\sin\tfrac{1}{2}(\theta + \bar{\theta}) \tag{E 38}$$

Therefore the function (E 36) is

$$H = \mathcal{F}(\tfrac{1}{16}, 0, x)\bar{\mathcal{F}}(\tfrac{1}{16}, \tfrac{1}{2}, \bar{x}) + \mathcal{F}(\tfrac{1}{16}, \tfrac{1}{2}, x)\bar{\mathcal{F}}(\tfrac{1}{16}, 0, \bar{x}) \tag{E 39}$$

This formula corresponds to the following operator product expansion

$$\sigma(z, \bar{z})\mu(0,0) = z^{3/8}\bar{z}^{-1/8}\{\Psi(z) + O(z, \bar{z})\} + z^{-1/8}\bar{z}^{3/8}\{\bar{\Psi}(\bar{z}) + O(z, \bar{z})\}, \tag{E 40}$$

which is in accordance with the idea of the field ψ as the regularized product $\sigma\mu$

To avoid misunderstanding, let us stress that there are three different sets of fields

$$\begin{aligned}\{A_j\} &= \{[I], [\Psi], [\bar{\Psi}], [\varepsilon]\}, \\ \{B_j\} &= \{[I], [\sigma], [\varepsilon]\}, \\ \{C_j\} &= \{[I], [\mu], [\varepsilon]\}.\end{aligned} \tag{E 41}$$

Each of these sets forms the closed operator algebra and it is appropriate to describe the critical Ising field theory All the fields entering the same set are mutually local whereas the fields entering different sets are in general nonlocal with respect to each other

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