CHAPTER 3

Linear Programming

Linear programming, like its nonlinear counterpart, is a method for making decisions based on solving a mathematical optimization problem. The general field of linear programming has been a major area of applied mathematical research in the last 50 years. A combination of new algorithms, e.g., the simplex method, and widely available computing power now make this an indispensable tool for the mathematical modeler.

We begin our discussion of linear programming by presenting the basic mathematical formulation and terminology in general terms. We will follow this with a number of examples of problems that may be formulated in terms of linear programs. Our goal here is to obtain an abstract understanding of what a linear program is and to develop an intuition that will assist the modeler in assessing whether linear programming is the right tool for a given problem.

Consider a linear function of the variables $(x_1, \ldots, x_n)$,

$$F(x_1, \ldots, x_n) = f_1 x_1 + f_2 x_2 + \cdots + f_n x_n$$

where the parameters f_i are known. We seek to pick the values of all the x_i, referred to as *decision variables*, so as to maximize $F(x_1, \ldots, x_n)$ which is referred to as the *objective function*. Clearly picking each $x_i = \infty$ (or even just one) would provide a maximum, albeit meaningless. The interest arises when the values of the x_i are constrained, e.g.,

$$a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1$$

Based on the application many constraints are possible so we write

$$a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$$

for $i = 1, \ldots, m$. Note that these constraints are also linear in the decision variables. We may interpret this system of constraints geometrically as defining a region, i.e., a continuum of points such that all the constraints are simultaneously satisfied. This region is referred to as the *feasible set* S. So we may view the optimization problem as one to find the maximum value of the objective function over the feasible set S.

We now formulate this optimization problem in terms of vectors and matrices. Let $x = (x_1, \ldots, x_n)^T$ be the (column) vector of the unknown variables, and let $f = (f_1, \ldots, f_n)^T$ be the vector of coefficients of the objective function, $F(x) = f^T x$. We also introduce the $m \times n$ matrix A whose entries are the coefficients in the inequality constraints, $(A)_{ij} = a_{ij}$. If a and b are vectors of the same length then we write $a \geq b$ if $a_i \geq b_i$ holds for all components.

Definition **1**. A linear program associated with f, A, and b is the minimum problem

$$\min_x f^T x$$

or the maximum problem

$$\max_x f^T x$$

subject to the constraint

$$Ax \leq b.$$

3.1 EXAMPLES OF LINEAR PROGRAMS

In this section we survey a variety of applications that fit exactly into the formulation of the abstract linear program.

3.1.1 Red or White?

A winemaker would like to decide how many bottles of red wine and how many bottles of white wine to produce. Given his expertise is in red wine making he can sell a bottle of red wine for $12 while he can only sell a bottle of white wine for $7. Clearly the winemaker would seek to maximize his profits, and, having recently completed a course in mathematical modeling, proceeds to construct the objective function

$$F(x_1, x_2) = 12x_1 + 7x_2$$

where the decision variables are the number of bottles of red wine to produce x_1 and the number of bottles of white wine to produce, i.e., x_2.

Aging wine in wooden or glass-lined vats is an integral component of the production process, but due to limited space the wine must be aged for a limited time. The wine maker has determined that red wine should be aged two years per bottle and white wine one year per bottle and his facilities allow that each batch is limited to 10,000 bottle-years (5 bottles of red and 3 bottles of white require a total of 13 bottle years ripening time). Thus the winemaker formulates a constraint

$$2x_1 + x_2 \leq 10000$$

Also the volume of grapes that may be processed is limited and it takes 3 gallons of grapes to make a bottle of red wine and two gallons of grapes to make a bottle of white wine. Furthermore, the winery can only process a total of 16,000 gallons of grapes for each batch. Thus, the winemaker produces the additional constraint

$$3x_1 + 2x_2 \leq 16000$$

Now the winemaker would like to determine how many bottles of each wine to produce as well as how much money he will expect to make. Note that we must also require that negative bottles of wine are not allowed so

$$x_1 \geq 0$$

and

$$x_2 \geq 0$$

3.1.2 How Many Fish?

A child with a new 29 gallon fish tank asks her daddy to put as many fish in the tank as possible. Sensing that too many fish is not a good thing, the dad asks the pet shop owner how many fish can go into a tank. The answer was more complex than anticipated. "You can put one inch of fish in per gallon of water." The little girl then added that she wanted only the big orange fish (Gouramis) and the small stripy fish (Zebra Danios).

As the child seeks to maximize the total number of fish her objective function is

$$F(x_1, x_2) = x_1 + x_2$$

where x_1 is the number of Gouramis and x_2 is the number of Zebra Danios.

Additionally, a full grown Gourami is two inches long while a Danio is just one inch long. The constraint of not exceeding 29 inches of total fish length can now be written

$$2x_1 + x_2 \leq 29$$

Danios are very active fish and actually require twice as much food as Gouramis. Each Danio eats 4 grams/day of fish flakes while the slower Gourami eats 2 grams/day. The dad decides that he would prefer not to go broke buying fish food and thus wants to limit the tank to 50 grams/day. Thus, we have the constraint

$$2x_1 + 4x_2 \leq 50$$

The pet shop owner adds, by the way, that Danios need to live in schools of at least 5 fish or they don't do well. Thus

$$x_2 \geq 5$$

Additionally, the little girl stipulates that she must have at least two Gouramis as they are known to kiss (hence the term Kissing Gouramis) so we add

$$x_1 \geq 2$$

How many Gouramis and Danios can the little girl have in her tank?

3.2 GEOMETRIC SOLUTION OF A 2D LINEAR PROGRAM

Let us now solve the winemaker's linear programming problem using graphical techniques. Recalling the problem:

- Objective function: $F(x_1, x_2) = 12x_1 + 7x_2$
- Constraint 1: $3x_1 + 2x_2 \leq 16000$
- Constraint 2: $2x_1 + x_2 \leq 10000$
- Constraint 3: $x_1 \geq 0$
- Constraint 4: $x_2 \geq 0$

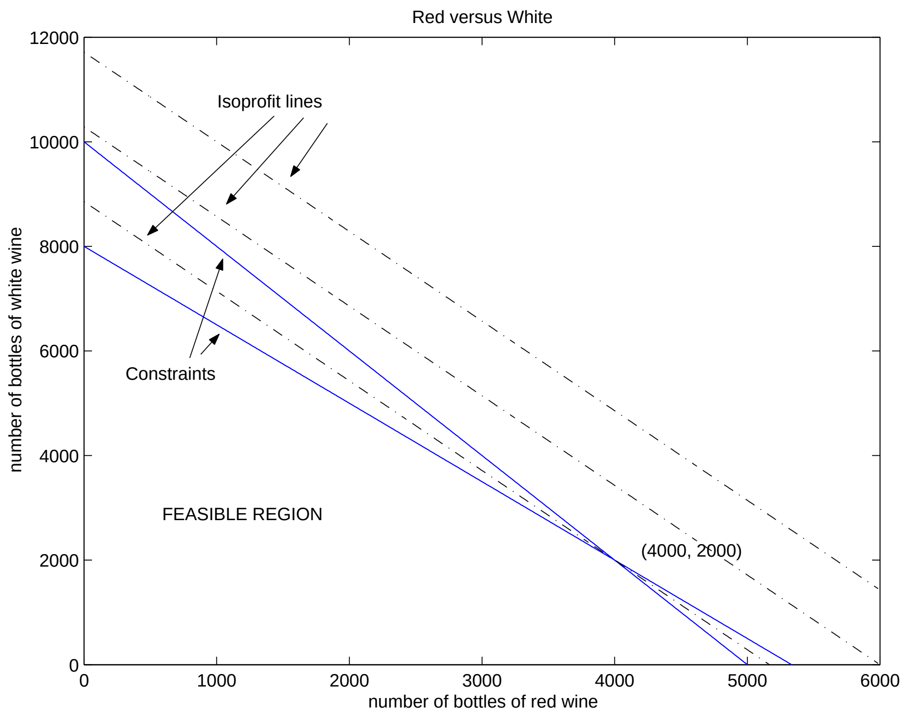


FIGURE 3.1: Geometric picture of the linear programming problem.

First, let us identify the feasible set. Again, this is the intersection of all the regions defined by the constraints. (Note that this set is independent of the objective function.) The boundary of the first constraint is defined by the equality

$$x_2 = 8000 - \frac{3}{2}x_1$$

The constraint may be viewed as a half-plane with this line dividing the region of allowed points from the unallowed points. It is easy to identify which region is the allowed region by considering a single point. For example, is the origin a point that satisfies the first constraint? Since the answer is clearly yes we know that the set of points that satisfies constraint 1 consists of the halfplane defined by $x_2 = 8000 - \frac{3}{2}x_1$ that contains origin.

Similarly, the second constraint defines a halfplane of points containing the origin and bounded by the line

$$x_2 = 10000 - 2x_1$$

The intersection of constraints 3 and 4 is the first quadrant of the x_1x_2 plane.

The intersection of all of these constraints as shown in Figure 3.1 constitutes the feasible set. Now we must pick the point in the feasible set that maximizes the objective function.

We can define an *isoprofit line* to be

$$12x_1 + 7x_2 = c$$

For all points on this line the profit is the same. We can see that as c decreases the line shifts towards the origin. So the goal is to pick the isoprofit line with the largest value of c such that x_1, x_2 is a point in the feasible set. Graphically we see that the first point the descending isoprofit line will touch is the vertex of the intersection of constraints 1 and 2. This is easily calculated to be $(4000, 2000)$.

Thus, the solution to the winemaker's linear programming problem is that he should produce 4000 bottles of red and 2000 bottles of white and that this will lead to a maximum profit of $62,000.

3.3 SENSITIVITY ANALYSIS

Often the coefficients in a linear programming model are known only approximately. Thus, it is interesting to know what the impact of modifying the terms present in the model. How is the objective function impacted? How does the optimal solution change? These questions are the subject of *sensitivity analysis*.

3.3.1 Price Sensitivity

First we examine how changing the price of a bottle of white wine impacts the optimal solution. Letting the price of the white wine be a variable w we now have the linear program

- Objective function: $F(x_1, x_2) = 12x_1 + wx_2$
- Constraint 1: $3x_1 + 2x_2 \leq 16000$

- Constraint 2: $2x_1 + x_2 \leq 10000$
- Constraint 3: $x_1 \geq 0$
- Constraint 4: $x_2 \geq 0$

From our graphical solution we know that any isoprofit line with slope between -2 and -3/2 will produce the same optimal solution of $(4000, 2000)$. Since the slope of the isoprofit line is $-12/w$ this condition is

$$-2 < \frac{12}{w} < -\frac{3}{2}$$

from which we conclude that the price of the white wine may vary as

$$6 < w < 8$$

with the solution unchanged as $(4000, 2000)$. Further examination produces Table 3.1. The double arrows here mean that any point on the isoprofit curve containing these points produces the same profit.

cost of white wine	optimal solution
$6 < w < 8$	(4000, 2000)
$w = 6$	$(4000, 2000) \leftrightarrow (5000, 0)$
$w = 8$	$(4000, 2000) \leftrightarrow (0, 8000)$
$w < 6$	(5000, 0)
$w > 8$	(0, 8000)

TABLE 3.1: The effect of pricing the white wine on the optimal solution.

3.3.2 Resource Sensitivity

Now we let the number of gallons of grapes, α, and the number of bottle-years storage capacity, β, be variable. Now the linear program becomes

- Objective function: $F(x_1, x_2) = 12x_1 + 7x_2$
- Constraint 1: $3x_1 + 2x_2 \leq \alpha$
- Constraint 2: $2x_1 + x_2 \leq \beta$
- Constraint 3: $x_1 \geq 0$
- Constraint 4: $x_2 \geq 0$

The relative values of α and β determine the geometry of the solution. For example, if $\alpha/2 > \beta$ then constraint 1 becomes irrelevant. When the intersection of constraints 1 and 2 determines the optimal solution it is readily shown that

$$x_1 = -\alpha + 2\beta$$

and

$$x_2 = 2\alpha - 3\beta$$

Hence the optimal solution to the objective function can be expressed as

$$f(x_1, x_2) = 2\alpha + 3\beta$$

Consequently, if α is increased by one unit then $f(x_1, x_2)$ is increased by 2, while if β is increased by one unit then $f(x_1, x_2)$ is increased by 3. So if a winemaker considers expanding his winery he realizes that the cost of increasing grape processing must be less than \$2 and the expense of increasing wine storage must be less than \$3. Otherwise expansion will lose money.

3.3.3 Constraint Coefficient Sensitivity

Now we consider the problem of adjusting one of the coefficients in one of the constraint equations. In particular consider the amount of time γ we age a bottle of red wine to be allowed to vary.

- Objective function: $F(x_1, x_2) = 12x_1 + 7x_2$
- Constraint 1: $3x_1 + 2x_2 \leq 16000$
- Constraint 2: $\gamma x_1 + x_2 \leq 10000$
- Constraint 3: $x_1 \geq 0$
- Constraint 4: $x_2 \geq 0$

To simplify the discussion, let's examine the effect of reducing the amount of time we age the red wine from 2 years to 1.95 years. The solution to the resulting linear program suggests that now 4444 bottles of red can be sold while 1333 bottles of white can be sold for a total profit of \$62,659, increasing the income by almost \$700. Of course, for this to be advisable it must be true that all the bottles of this "younger" red wine can still be sold at the same price, i.e., the taste has not suffered enough to reduce its popularity.

3.4 LINEAR PROGRAMS WITH EQUALITY CONSTRAINTS

In the examples treated so far the constraints defining the feasible sets have been inequalities. However, in practice it is often the case that further constraints in the form of equalities have to be met.

DEFINITION **2**. Let f be a column vector of length n, b a column vector of length m, and b_{eq} a column vector of length k. Let further A and A_{eq} be $m \times n$ and $k \times n$ matrices, respectively. A linear program associated with f, A, b, A_{eq} and b_{eq} is the minimum problem

$$\min_x f^T x \qquad (3.1)$$

or the maximum problem

$$\max_x f^T x \qquad (3.2)$$

subject to the constraints

$$\begin{aligned} Ax &\le b \\ A_{eq}x &= b_{eq}. \end{aligned} \tag{3.3}$$

3.4.1 A Task Scheduling Problem

A steel manufacturer produces four different sizes S_i, $1 \le i \le 4$ (small, medium, large, and extra large), of beams. These beams can be produced on any one of three machines M_j, $1 \le j \le 3$. Machine M_j produces l_{ij} feet of the beams of size S_i per hour. Each machine can be used up to 50 hours per week and the hourly operating cost of machine M_j is \$$c_j$. The manufacturer has to produce k_i feet of beams of size S_i per week. We assume that l_{ij}, c_j and k_i are given numbers.

Clearly the manufacturer wants to minimize the total operating costs. To formulate this minimization problem as a linear program, let x_{ij} be the number of hours per week machine M_j produces the beams of size S_i. The total operating costs are

$$F(x) = \sum_{j=1}^{3}\sum_{i=1}^{4} c_j x_{ij} = \left\{ \begin{array}{l} c_1(x_{11}+x_{21}+x_{31}+x_{41}) \\ +c_2(x_{12}+x_{22}+x_{32}+x_{42}) \\ +c_3(x_{13}+x_{23}+x_{33}+x_{43}) \end{array} \right. \tag{3.4}$$

and this function has to be minimized subject to the following constraints:

- Each machine can operate at most 50 hours per week. Thus the variables x_{ij} have to satisfy the inequalities

$$x_{1j}+x_{2j}+x_{3j}+x_{4j} \le 50 \quad (1 \le j \le 3). \tag{3.5}$$

- Since x_{ij} cannot be negative we have to introduce twelve further inequality constraints

$$-x_{ij} \le 0 \quad (1 \le i \le 4,\ \ 1 \le j \le 3). \tag{3.6}$$

- The number of feet of the beams of size S_i produced per week by machine M_j is $l_{ij}x_{ij}$. Thus the total number of feet of this size produced in a week is $\sum_j l_{ij}x_{ij}$, and this must be equal to

$$l_{i1}x_{i1}+l_{i2}x_{i2}+l_{i3}x_{i3} = k_i \quad (1 \le i \le 4). \tag{3.7}$$

We now have a linear program with fifteen inequality constraints and four equality constraints.

To match the steel manufacturer problem to Definition 2, we write the twelve variables in a column vector,

$$x = [x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}, x_{13}, x_{23}, x_{33}, x_{43}]^T.$$

The inequality constraints (3.5) and (3.6) have to be written in matrix vector form as $Ax \le b$. Let us denote by A_1 and b_1 the 3×12–matrix and the column vector of length 3, respectively, such that the inequalities (3.5) take the form $A_1 x \le b_1$, i.e.

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 50 \\ 50 \\ 50 \end{bmatrix}.$$

The inequalities (3.6) can be written as $A_2 x \le b_2$, where $A_2 = -I$ with I the 12×12–identity matrix, and b_2 the column vector of length twelve whose entries are all zero. Thus the diagonal entries of A_2 are -1 and the other entries are zero.

The full 15×12–matrix A is then obtained by appending the twelve rows of A_2 below the three rows of A_1 and similarly for b,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 50 \\ 50 \\ 50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Likewise, setting

$$A_{eq} = \begin{bmatrix} l_{11} & 0 & 0 & 0 & l_{12} & 0 & 0 & 0 & l_{13} & 0 & 0 & 0 \\ 0 & l_{21} & 0 & 0 & 0 & l_{22} & 0 & 0 & 0 & l_{23} & 0 & 0 \\ 0 & 0 & l_{31} & 0 & 0 & 0 & l_{32} & 0 & 0 & 0 & l_{33} & 0 \\ 0 & 0 & 0 & l_{41} & 0 & 0 & 0 & l_{42} & 0 & 0 & 0 & l_{43} \end{bmatrix}, \quad b_{eq} = \begin{bmatrix} k_1 \\ k_1 \\ k_3 \\ k_4 \end{bmatrix},$$

the equality constraints (3.7) can be written in the form $A_{eq} x = b_{eq}$.

3.4.2 Transportation Problems

Transportation problems are typical applications of linear programming. Assume a company has storage depots at m different locations $A_1, \ldots, A_m$ in which k different products $P_1, \ldots, P_k$ are stored. Let M_{ij} be the total amount of product P_j stored in depot A_i. The company has customers $C_1, \ldots, C_r$ in r different cities and has to deliver the amount N_{lj} of product P_j to customer C_l. We assume fixed transportation costs T_{ilj} per unit amount of product P_j if transported to customer C_l from storage deposit A_i.

Let x_{ilj} be the amount of product P_j delivered to customer C_l from deposit

A_i. The problem is to minimize the total transportation costs

$$\sum_{i=1}^{m}\sum_{l=1}^{r}\sum_{j=1}^{k} T_{ilj}x_{ilj} = \min$$

subject to the constraints

$$x_{ilj} \geq 0 \qquad \text{for} \quad 1 \leq l \leq r, \quad 1 \leq j \leq k, \quad 1 \leq i \leq m \tag{3.8}$$

$$\sum_{l=1}^{r} x_{ilj} \leq M_{ij} \qquad \text{for} \quad 1 \leq i \leq m, \quad 1 \leq j \leq k \tag{3.9}$$

$$\sum_{i=1}^{m} x_{ilj} = N_{lj} \qquad \text{for} \quad 1 \leq l \leq r, \quad 1 \leq j \leq k. \tag{3.10}$$

This is clearly a linear programming problem with inequality constraints (3.8) and (3.9) and equality constraints (3.10). If m, k, r and the numbers T_{ilj}, M_{ij}, N_{lj} are given, the vectors and matrices f, A, b, A_{eq}, b_{eq} can be constructed similarly as in Subsection 3.4.1.

3.5 A TARGETING PROBLEM

Consider the following problem of launching a rocket to a fixed altitude h in a given time T, while expending a minimum amount of fuel. Let $a(t)$ be the acceleration exerted, $y(t)$ the rocket altitude, and $v(t)$ the rocket velocity at time t. The problem can be formulated as follows.

$$\begin{array}{ll} \text{Minimize} & \int_0^T |a(t)|dt \\ \\ \text{Subject to} & \frac{dv(t)}{dt} = a(t) - g, \quad \frac{dx(t)}{dt} = v(t) \\ & y(T) = h \\ & y(t) \geq 0 \quad (0 \leq t \leq T) \\ & y(0) = 0, \quad v(0) = 0 \\ & |a(t)| \leq a_0 \quad (0 \leq t \leq T), \end{array} \tag{3.11}$$

where a_0 is the maximal acceleration that can applied due to power limitations. Clearly in order that the rocket can leave the ground a_0 must be greater than the earth acceleration g.

Note that the maximum altitude h_{max} to which the rocket can be launched is reached if $a(t) = a_0$ for $0 \leq t \leq T$. If $h > h_{max}$ then (3.11) has no solution. By integrating the equations for $dv(t)/dt$ and $dy(t)/dt$ in (3.11) we find

$$h_{max} = (a_0 - g)T^2/2.$$

3.5.1 Discretization and Solution of the Equations of Motion

Equation (3.11) belongs to the class of *continuous optimization problems* which does not fit a priori into the class of linear programming problems. In order to make the problem amenable to linear programming, we discretize time and assume that

$$a(t) = a_i = \mathit{const} \quad \text{for} \quad t_{i-1} < t < t_i, \tag{3.12}$$

where

$$t_i = i\tau \quad \tau = T/n,$$

and n is a positive integer. The discretized problem is described by n variables $(a_1, \ldots, a_n)$ which have to be determined.

Within each of the n sub-intervals into which the interval $0 \le t \le T$ is divided, the rocket encounters a constant acceleration,

$$\frac{dv(t)}{dt} = a_i - g, \quad \frac{dx(t)}{dt} = v(t) \quad \text{if} \quad t_{i-1} \le t \le t_i. \tag{3.13}$$

After integration these equations lead to the well known linear and quadratic time dependence of velocity and altitude in each sub-interval,

$$v(t) = (a_i - g)(t - t_{i-1}) + v(t_{i-1}) \tag{3.14}$$

$$y(t) = \frac{1}{2}(a_i - g)(t - t_{i-1})^2 + v(t_{i-1})(t - t_{i-1}) + y(t_{i-1}). \tag{3.15}$$

We now set

$$v_i = v(t_i), \quad y_i = y(t_i) \quad (1 \le i \le n),$$

and evaluate the equations (3.14) and (3.14) at $t = t_i$ to obtain

$$\begin{aligned} v_i &= (a_i - g)\tau + v_{i-1} \\ y_i &= \tfrac{1}{2}(a_i - g)\tau^2 + v_{i-1}\tau + y_{i-1}. \end{aligned} \tag{3.16}$$

Equation (3.16) is a linear system of first order difference equations for the (v_i, y_i). The initial values are $(v_0, y_0) = (0, 0)$. Methods for solving difference equations are discussed in Chapter 6, and we will show there that the solution of (3.16) is given by

$$v_i = \tau\Big(\sum_{j=1}^{i} a_j - ig\Big) \tag{3.17}$$

$$y_i = \tau^2\Big(\sum_{j=1}^{i}(\frac{1}{2} + i - j)a_j - \frac{i^2 g}{2}\Big). \tag{3.18}$$

These equations form the solution of the discretized equations of motion for any given set of acceleration values $(a_1, \ldots, a_n)$.

3.5.2 Formulation as Linear Program

Now we formulate the discretized optimization problem as linear programming problem with inequality and equality constraints. The equations of motion

$$\frac{dv(t)}{dt} = a(t) - g, \quad \frac{dx(t)}{dt} = v(t), \quad y(0) = 0, \; v(0) = 0 \tag{3.19}$$

have been solved already, so we only need to consider the equality and inequality constraints

$$y(T) = h, \quad |a(t)| \le a_0, \quad y(t) \ge 0 \quad (0 < t < T).$$

From equation (3.18) we infer that the discretized forms of the equality and inequality constraints for $y(t)$ (note that $y(T) = y_n$) can be written as

$$\sum_{j=1}^{n}(\frac{1}{2}+n-j)a_j = \frac{n^2 g}{2}+\frac{h}{\tau^2} \tag{3.20}$$

$$\sum_{j=1}^{i}(\frac{1}{2}+i-j)a_j \geq \frac{i^2 g}{2}, \quad (1 \leq i \leq n-1), \tag{3.21}$$

and the constraint for $a(t)$ becomes

$$|a_i| \leq a_0 \quad (1 \leq i \leq n). \tag{3.22}$$

The objective function which has to be minimized in the discretized problem is

$$\sum_{i=1}^{n} |a_i| = \min, \tag{3.23}$$

and the minimization is subject to the constraints (3.20)– (3.22).

Note that (3.22) and (3.23) involve the absolute values of the variables a_i and hence are not described by linear functions. For inequalities this is not a problem, however there is no way to rewrite the objective function (3.23) as a linear function $\sum_i f_i a_i$. To solve this problem we treat the absolute values as extra variables. Our minimization problem then depends on $2n$ unknown variables which we write again in a column vector

$$x = [x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}]^T,$$

where

$$x_i = a_i, \quad x_{i+n} = |a_i| \quad (1 \leq i \leq n).$$

The objective function is now a linear function of x,

$$F(x) = \sum_{i=n+1}^{2n} x_i = \min\,. \tag{3.24}$$

In order that the conditions $x_{i+n} = |x_i|$ are met we have to introduce additional constraints. Since $a_i \leq |a_i|$ and $-a_i \leq |a_i|$ we impose

$$\left.\begin{array}{rcl} x_i & \leq & x_{i+n} \\ -x_i & \leq & x_{i+n} \end{array}\right\} \quad \text{for} \quad 1 \leq i \leq n. \tag{3.25}$$

Clearly the inequalities (3.25) are not equivalent to the condition $x_{i+n} = |x_i|$. However it can be shown that the solution of any linear programming problem is located on the boundary of the feasible set, and for our problem this necessarily implies that for each i one of the two inequalities in (3.25) turns into an equality if x is an optimal solution.

The inequality and equality constraints (3.20)–(3.22) are now rewritten in terms of the x_i as

$$x_{n+i} \leq a_0 \quad \text{for} \quad 1 \leq i \leq n \tag{3.26}$$

$$-\sum_{j=1}^{i}(\frac{1}{2}+i-j)x_j \leq -\frac{i^2 g}{2} \quad \text{for} \quad 1 \leq i \leq n-1 \tag{3.27}$$

$$\sum_{j=1}^{n}(\frac{1}{2}+n-j)x_j = \frac{n^2 g}{2}+\frac{h}{\tau^2}. \tag{3.28}$$

The discretized optimization problem is now to minimize the objective function $F(x)$ in (3.24) subject to the inequality constraints (3.25)–(3.27) and the equality constraint (3.28). The objective function as well the inequality and equality constraints are formulated in terms of linear functions of x and so match the abstract problem (3.1) subject to the constraints (3.3) of Definition 2. The matrix A is a $(4n-1) \times 2n$ matrix and A_{eq} is a $1 \times 2n$ matrix, i.e. a row vector.

Numerical Solutions. In Figure 3.2 we show the optimal acceleration function $a(t)$ obtained by numerical solution of (3.24)–(3.28) together with $v(t)$ and $y(t)$ for $n = 5$ (Figure 3.2 (a)) and $n = 25$ (Figure 3.2 (b)), and $h = 300$ and $h = 700$. The other paramters are fixed at $g = 32$, $T = 10$, and $a_0 = 48$. The optimal solution has been computed using the *linprog* command of Matlab. The procedure for generating the plots in Figure 3.2 can be summarized as follows.

- Generate the matrices and vectors f, A, b, A_{eq}, and b_{eq} of the linear program according to equations (3.24)–(3.28).
- Apply a numerical solver to find the solution vector x. The first n components of x are the optimal acceleration values $(a_1, \ldots, a_n)$.
- Apply equations (3.17) and (3.18) to compute the velocity and altitude vectors $(v_1, \ldots, v_n)$ and $(y_1, \ldots, y_n)$.
- Use equations (3.12), (3.14) and (3.15) to compute the piecewise constant, piecewise linear and piecewise quadratic functions $a(t)$, $v(t)$ and $y(t)$.
- Plot $a(t)$, $v(t)$, $y(t)$.

As can be seen in Figure 3.2 the optimal acceleration and altitude functions show some distinct features. The acceleration function $a(t)$ starts with a_0 and stays there over a certain number of sub–intervals, then it decreases in the next sub–interval, and after that $a(t)$ is zero. The altitude function $y(t)$ is monotonically increasing and reaches the target altitude from below for larger values of h, whereas for smaller values of h it passes through a maximum and reaches the target altitude from above. In Section 3.6 we will see that the optimal solution to the discretized targeting problem is the best approximation of a known analytical solution to (3.11).

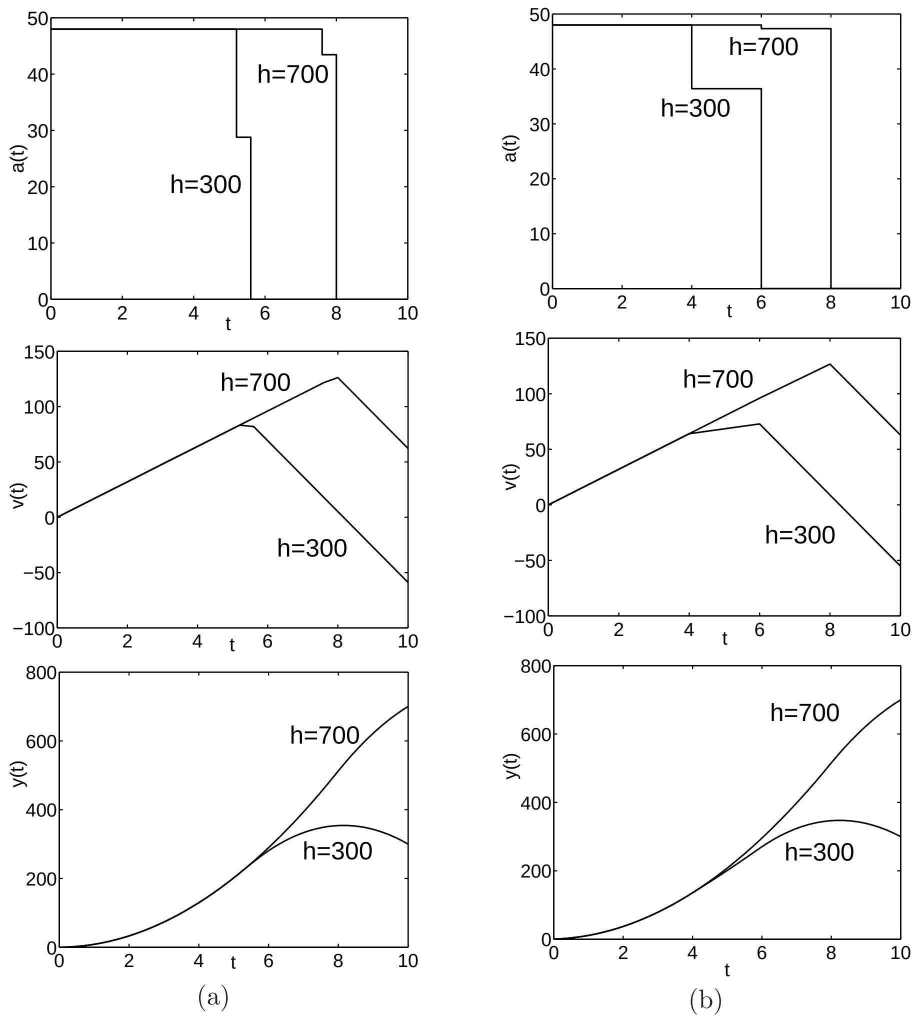


FIGURE 3.2: Graphs of $a(t), v(t), y(t)$ obtained by numerical solution of the linear program (3.24)–(3.28) for $g = 32$, $T = 10$, $a_0 = 48$, and two heights $h = 300$ and $h = 700$. (a): $n = 25$, (b): $n = 5$.

3.5.3 Targeting Problem with Air Resistance

Air resistance is modeled by a friction force $F_d(v)$. Since linear programming requires a linear model we assume $F_d(v)/m = -kv$, where k is a friction coefficient in which the mass is absorbed. The problem (3.11) remains the same except that the equation for $dv(t)/dt$ is now replaced by

$$\frac{dv(t)}{dt} = a(t) - g - kv(t). \tag{3.29}$$

Discretization and Solution of the Equations of Motion. Equation (3.29) is a linear first order differential equation for $v(t)$. In a later chapter we will see that the solution of (3.29) in the interval $t_{i-1} \le t \le t_i$, where $a(t) = a_i = const$, is given by

$$v(t) = \frac{a_i - g}{k} + \big(v(t_{i-1}) - \frac{a_i - g}{k}\big)e^{-k(t-t_{i-1})}. \tag{3.30}$$

The altitude $y(t)$ still satisfies $dy(t)/dt = v(t)$ and so can be found by integration of (3.30),

$$y(t) = y(t_{i-1}) + \frac{a_i - g}{k}\big(t - t_{i-1}\big) + \frac{1}{k}\big(v(t_{i-1}) - \frac{a_i - g}{k}\big)\big(1 - e^{-k(t-t_{i-1})}\big). \tag{3.31}$$

Evaluating (3.30) and (3.31) at $t = t_i$ and letting again $v_i = v(t_i)$, $y_i = y(t_i)$ yields

$$\begin{aligned} v_i &= pa_i - gp + qv_{i-1} \\ y_i &= ra_i - gr + pv_{i-1} + y_{i-1}, \end{aligned} \tag{3.32}$$

where we have set

$$q = e^{-k\tau}, \quad p = (1-q)/k, \quad r = (\tau - p)/k.$$

Equations (3.32) form again a linear system of first order difference equations. This system is more complicated than (3.16), but still can be solved using the methods of Chapter 6. The solution is

$$v_i = p\sum_{j=1}^{i} q^{i-j} a_j - \frac{gp(1-q^i)}{1-q} \tag{3.33}$$

$$y_i = \sum_{j=1}^{i}\Big(r + \frac{p^2(1-q^{i-j})}{1-q}\Big)a_j - \frac{gp^2(i-1-iq+q^i)}{(1-q)^2} - igr. \tag{3.34}$$

Formulation as Linear Program. The formulation of the discretized targeting problem with friction as linear program proceeds in the same way as in Subsection 3.5.2. We introduce the vector x of variables $x_i = a_i$ and $x_{i+n} = |a_i|$ $(1 \le i \le n)$, the objective function (3.24), and the inequality constraints (3.25)–(3.27). The constraints $y_i \ge 0$ for $1 \le i \le n-1$ and $y_n = h$ become

$$-\sum_{j=1}^{i}\Big(r + \frac{p^2(1-q^{i-j})}{1-q}\Big)x_j \le -\frac{gp^2(i-1-iq+q^i)}{(1-q)^2} - igr \tag{3.35}$$

$$\sum_{j=1}^{n}\Big(r + \frac{p^2(1-q^{n-j})}{1-q}\Big)x_j = \frac{gp^2(n-1-nq+q^n)}{(1-q)^2} + ngr + h. \tag{3.36}$$

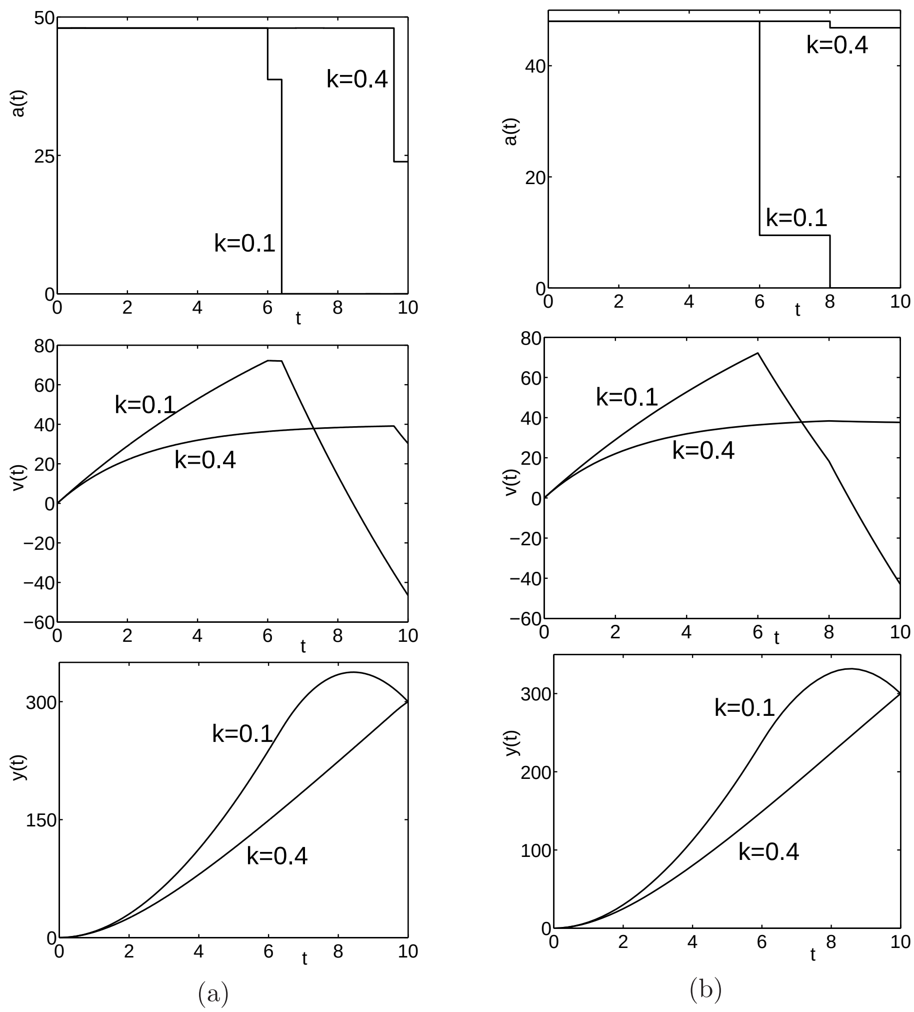


FIGURE 3.3: Graphs of $a(t), v(t), y(t)$ computed from numerical solutions of (3.24)–(3.27), (3.35)–(3.35) for $g = 32$, $T = 10$, $a_0 = 48$, $h = 300$, $k = 0.4$ and $k = 0.1$, and (a): $n = 25$, (b): $n = 5$.

The linear program for the discretized target problem with friction is now to minimize (3.24) subject to the constraints (3.25)–(3.27) and (3.35)–(3.36).

As in the case without friction the maximum possible altitude h_{max} is reached if $a_i = a_0$ for all i. From (3.31) we find

$$h_{max} = \frac{a_0 - g}{k^2}\left(kT - 1 + e^{-kT}\right),$$

and the problem has no solution if $h > h_{max}$.

Numerical Solutions. In Figure 3.3 we show the graphs of $a(t)$, $v(t)$, and $y(t)$ computed from numerical solutions of the linear program (3.24)–(3.27), (3.35)–(3.36) for $g = 32$, $T = 10$, $a_0 = 48$, $h = 300$, $k = 0.4$ and $k = 0.1$, and $n = 25$ and $n = 5$. The solutions are similar to those for the problem without friction, and clearly the greater k the greater is the fuel consumption.

3.5.4 Additional Constraints

There is no problem to impose further conditions on the optimal solution of the targeting problem (with or without friction), provided these conditions can be formulated as linear equality or inequality constraints. We describe two such conditions.

Soft Landing. Soft landing means that the target altitude is reached with velocity $v(T) = 0$. This condition can be build into the linear program by imposing the additional equality constraint

$$v_n = 0,$$

where v_n is represented in terms of the $a_j = x_j$ through equations (3.17) for $k = 0$ or (3.33) for $k > 0$. The vector b_{eq} then becomes a vector of length 2, and accordingly A_{eq} is a $2 \times 2n$–matrix.

Upper Bound for the Velocity. To avoid damage it may be necessary to restrict also the magnitude of the velocity to $|v(t)| \le v_0$. For the discretized problem this requires that $|v_i| \le v_0$ for $1 \le i \le n$. This inequality is equivalent to the two linear inequalities

$$v_i \le v_0, \quad -v_i \le v_0.$$

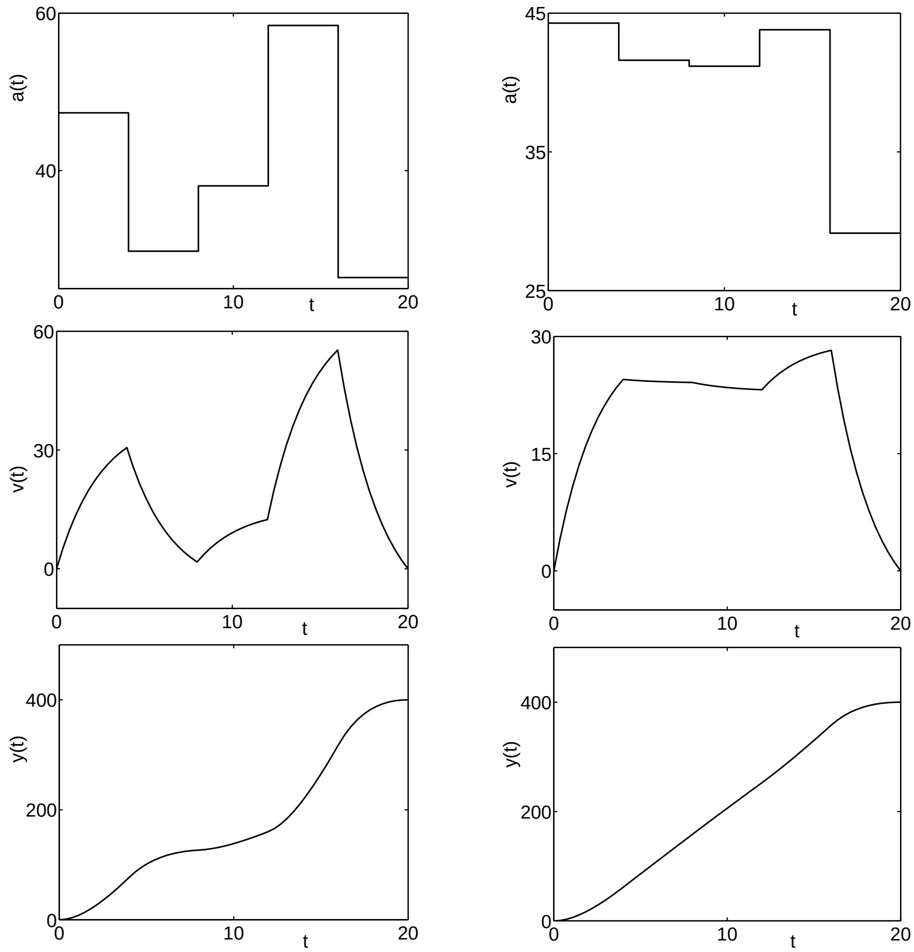
When the v_i are represented in terms of the x_i, these conditions take the form of $2n$ additional linear inequality constraints imposed on x. The vector b is then extended to a vector of length $6n-1$, and accordingly A is extended to a $(6n-1)\times 2n$–matrix.

Numerical Solutions. In Figure 3.4 the graphs of $a(t)$, $v(t)$, and $y(t)$ computed from the optimal solution of the targeting problem with the condition of soft landing are shown for $g = 32$, $T = 20$, $a_0 = 80$, $h = 400$, $k = 0.4$, and $n = 5$. Figure 3.4 (b) was obtained with the additional inequality constraint $|v(t)| \le 30$.

We note that the targeting problem with one of the additional constraints considered in this subsection does not admit easily accessible analytical solutions. In contrast, without these additional constraints analytical solutions can be easily found as will be shown in the next section.

3.6 ANALYSIS OF THE TARGETING PROBLEM

In this section we study the targeting problem (3.11) analytically. The numerical solutions shown in Figure 3.2 suggest that the optimal acceleration function $a(t)$ is maximal in a certain initial interval $0 \le t \le T_1$ and zero for $T_1 < t \le T$, where T_1 is adjusted such that the target altitude h is reached in time T. We will see that for this acceleration function the fuel consumption is indeed minimal.



(a) (b)

FIGURE 3.4: Graphs $a(t), v(t), y(t)$ computed from the optimal solution of the discretized targeting problem for $g = 32$, $T = 20$, $a_0 = 80$, $h = 400$, $k = 0.4$, and $n = 5$, with additional constraints (a): $v(T) = 0$, (b): $v(T) = 0$ and $|v(t)| \leq 30$.

3.6.1 Analytical Solution

Let $a(t)$ be an acceleration function of the form

$$a(t) = \begin{cases} a_0 & \text{if} \quad 0 \leq t \leq T_1 \\ 0 & \text{if} \quad t > T_1, \end{cases} \tag{3.37}$$

where $a_0 > g$ and $T_1 > 0$ are given numbers. For this form the solution of the equations of motion (3.19) is given by (Exercise 3.12 (a))

$$v(t) = \begin{cases} (a_0 - g)t & \text{if} \quad 0 \leq t \leq T_1 \\ a_0 T_1 - gt & \text{if} \quad t \geq T_1, \end{cases} \tag{3.38}$$

$$y(t) = \begin{cases} (a_0 - g)t^2/2 & \text{if} \quad 0 \leq t \leq T_1 \\ -a_0 T_1^2/2 + a_0 T_1 t - gt^2/2 & \text{if} \quad t \geq T_1. \end{cases} \tag{3.39}$$

Consider then the problem of launching the rocket to a prescribed altitude h in a given time T. The condition $y(T) = h$ leads to the quadratic equation

$$-\frac{1}{2} a_0 T_1^2 + a_0 T_1 T - \frac{1}{2} g T^2 = h$$

for T_1. The solution with $T_1 \leq T$ is

$$T_1 = T\bigl(1 - \sqrt{1 - (g + 2h/T^2)/a_0}\bigr), \tag{3.40}$$

and in order that the expression under the square root be positive we have to require that

$$h \leq (a_0 - g)T^2/2. \tag{3.41}$$

If T_1 and a_0 are related by (3.40), the fuel consumption measured by $C = \int_0^T a(t)dt = a_0 T_1$ is

$$C = a_0 T\bigl(1 - \sqrt{1 - (g + 2h/T^2)/a_0}\,\bigr). \tag{3.42}$$

The following theorem (Exercise 3.12 (c)) shows that C is the minimal fuel consumption that can be achieved if $|a(t)|$ is bounded by a_0.

> Theorem **3**. Let $a(t)$ be an arbitrary piecewise constant acceleration function such that $y(T) = h$ for the solution of (3.19), and assume that $|a(t)| \leq a_0$. Then (3.41) is satisfied, and
>
> $$\int_0^T |a(t)|dt \geq C,$$
>
> where C is given by equation (3.42).

Thus the solution of the original (not discretized) targeting problem (3.11) is given by (3.37) with T_1 and a_0 related by (3.40), provided the inequality (3.41) is satisfied. The expression $(a_0 - g)T^2/2$ on the right hand side of this inequality is the maximal altitude to which the rocket can be launched in time T if $|a(t)|$ is bounded by a_0. This altitude is reached if $T_1 = T$, i.e. for the uniform acceleration $a(t) = a_0$ for

$0 \le t \le T$. If $h > (a_0 - g)T^2/2$ then a solution to the targeting problem (3.11) does not exist.

When a numerical solver is applied to the linear program of Subsection 3.5.2, the solver seeks to find the best approximation to the analytical solution (3.37), (3.40). The best approximation is

$$a(t) = \begin{cases} a_0 & \text{if} \quad 0 \le t < m\tau \\ a_1 < a_0 & \text{if} \quad m\tau \le t \le (m+1)\tau \\ 0 & \text{if} \quad t \ge (m+1)\tau, \end{cases}$$

where m is the largest integer for which $m\tau \le T_1(a_0)$. The value of a_1 is adjusted such that the altitude h is reached from the initial data $(v(m\tau), y(m\tau))$ within time $(n-m)\tau$. In the unlikely case that T_1/τ is an integer, the discrete optimal solution coincides with the exact optimal solution.

3.6.2 Dimensionless Variables

Equation (3.40) depends on the physical variables T_1, T, h, g, a_0. We could apply dimensional analysis to reduce the number of variables, but there is a simpler way to identify the relevant dimensionless combinations. If (3.40) is divided by T, the equation can be rewritten as

$$\theta = 1 - \sqrt{1 - 1/\beta}, \tag{3.43}$$

where

$$\theta = \frac{T_1}{T} \le 1, \quad \beta = \frac{a_0}{g + 2h/T^2} \ge 1. \tag{3.44}$$

The variable θ is the ratio of T_1 and T and so $\theta \le 1$. The denominator in β is the uniform acceleration (active for $0 \le t \le T$) through which the rocket is launched to the target altitude h in time T. According to (3.41) $h + gT^2/2 \le a_0$, hence $\beta \ge 1$.

A natural dimensionless variable in terms of which the fuel consumption can be measured is the ratio

$$\gamma = C/C_0, \tag{3.45}$$

where C_0 is the fuel consumption for the uniform acceleration $g + 2h/T^2$,

$$C_0 = (g + 2h/T^2)T = a_0 T/\beta. \tag{3.46}$$

After dividing equation (3.42) by C_0 we obtain

$$\gamma = \beta - \sqrt{\beta^2 - \beta}, \tag{3.47}$$

hence the dimensionless acceleration time θ and fuel consumption γ both depend only on the single dimensionless variable β that measures a_0 in units of $g + 2h/T^2$.

When β increases from $\beta = 1$ towards ∞, γ and θ decrease monotonically from 1 to the limiting values $\gamma_\infty = 1/2$ and $\theta_\infty = 0$, respectively, see Figure 3.5. Consequently the greater β the smaller are θ and γ. In the limit $\beta \to \infty$ and hence $a_0 \to \infty$, the accelerating force becomes an impulsive force that instantaneously, in an infinitesimal time interval, brings the velocity from $v(0-) = 0$ to $v(0+) = v_0$.

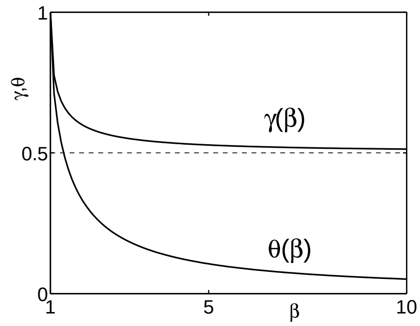


FIGURE 3.5: Graphs of θ and γ versus β, equations (3.43) and (3.47).

Then for $t > 0$ the trajectory of the rocket is $y(t) = v_0 t - gt^2/2$ and v_0 is determined by $y(T) = h$, whence

$$v_0 = \frac{1}{2}(g + 2h/T^2)T = \frac{1}{2}C_0.$$

The limiting value $\lim_{a_0 \to \infty} C(a_0) = C_0/2$ is the minimal fuel consumption if there is no constraint on $|a(t)|$.

3.6.3 Maximum Altitude

Now we address the question when the rocket reaches the target height from above or from below. The altitude function $y(t)$ has a maximum $h_m = y(T_m)$ at time $t = T_m$ determined by $v(t) = 0$,

$$T_m = \frac{a_0}{g} T_1, \quad h_m = \frac{a_0}{2} T_1^2 \big(\frac{a_0}{g} - 1\big). \tag{3.48}$$

Since now h and a_0 have to be treated independently of each other, we introduce the dimensionless variables

$$\alpha = \frac{a_0}{g}, \quad \xi = \frac{2h}{gT^2}, \quad \theta_m = \frac{T_m}{T}, \tag{3.49}$$

and note that $\beta = \alpha/(1+\xi)$. The condition that the maximum of $y(t)$ is attained in the range $0 \le t \le T$ is $\theta_m \le 1$. From (3.40) and (3.48) we find that

$$\theta_m = \alpha - \sqrt{\alpha^2 - (1+\xi)\alpha}, \tag{3.50}$$

and this is less than one if

$$\alpha \ge \frac{1}{1-\xi}. \tag{3.51}$$

Moreover, the condition for a solution to exist at all is $\beta \ge 1$. In terms of α and ξ this condition becomes

$$\alpha \ge 1 + \xi. \tag{3.52}$$

The boundary lines $\alpha = 1/(1-\xi)$ and $\alpha = 1 + \xi$ separate the (α, ξ)–plane into three regions I, II, and III as shown in Figure 3.6. In regions I and II the rocket

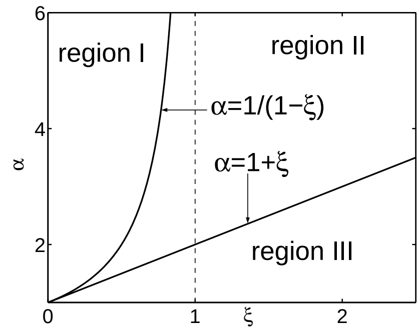


FIGURE 3.6: Regions I, II, III in the (ξ, α)–plane.

reaches h from above and below, respectively. In region III the targeting problem has no solution. We summarize this in terms of the physical variables a_0, h, T:

- If $h < gT^2(1-g/a_0)/2$ then the rocket reaches the target altitude from above.
- If $gT^2(1-g/a_0)/2 < h \leq (a_0 - g)T^2/2$ then the rocket reaches the target altitude from below.
- If $h > (a_0 - g)T^2/2$ then the targeting problem (3.11) has no solution.

PROBLEMS

3.1. Think of an optimization problem that can be written as a linear program with two decision variables. Specify the objective function as well as the constraints. Avoid constructing a problem the solution to which is such that one of the decision variables is zero. Can you extend your problem to more decision variables and constraints?

3.2. Solve graphically the question of how many Zebra Danios and Gouramis should be purchased for the fish tank modeled in section 3.1.2.

3.3. A new burger chain, the EcoliExpress, has has two new products: the large 1/3 pound "Big Whoopie" burger and the smaller 1/4 pound "Wimpy Whoopie" burger. It has been determined in test market trials that the Big Whoopie can be sold at a profit of 45 cents per burger and the Wimpy Whoopie at a profit of 25 cents. Furthermore, a chain knows that it can sell all its burgers if it uses 100 pounds of meat per week. In addition, the preparation time for a Big Whoopie is two minutes and for a Wimpy Whoopie is one minute and the chain has one employee working 40 hours per week preparing both types of Whoopies. Assuming the owner of the EcoliExpress wishes to maximize profits formulate a solution using linear programming. Using this model, answer the following:

(a) How many Big Whoopies and Wimpy Whoopies should be sold?

(b) Assuming the unit profit of the Big Whoopie is fixed at 45 cents, for what range of prices of the Wimpy Whoopie is the solution in a) optimal?

(c) Assuming the unit profit of the Wimpy Whoopie is fixed at 25 cents, how large does the unit profit for the Big Whoopie have to be to justify making only this type of burger?

(d) What should the cost of meat be (per pound) to justify purchasing additional quantities? Hint: the profit must increase.

In Exercises 3.4–3.8 first formulate the problem as linear program. Then use a linear program solver such as the *linprog* function of Matlab to find the optimal solution.

3.4. An agricultural mill manufactures feed for cattle, sheep and chickens. This is done by mixing the following ingredients: corn, limestone, soybeans, and fish meal. These ingredients contain the following nutrients: vitamins, protein, calcium, and crude fat. The contents of the nutrients in each kilogram of the ingredients is summarized in Table 3.4. The mill contracted to produce 10, 8, and 8

Ingredient	Vitamins	Protein	Calcium	Crude Fat
Corn	8	10	6	8
Limestone	6	5	10	6
Soybeans	10	12	6	6
Fish Meal	4	8	6	9

TABLE 3.2:

(metric) tons of cattle feed, sheep feed, and chicken feed. Because of shortages, a limited amount of the ingredients is available, namely 6 tons of corn, 10 tons of limestone, 4 tons of soybeans, and 5 tons of fish meal. The price per kilogram of these ingreients is \$0.20, \$0.12, \$0.24, and \$0.12. The minimal and maximal units of the various nutrients that are permitted is summarized in Table 3.4 for a kilogram of the cattle feed, the sheep feed, and the chicken feed. Formulate this mixed–feed problem as a linear program so that the total costs are minimized.

3.5. A tractor factory has supply depots in three cities C_1, C_2, C_3. Two traders T_1 and T_2 order 22 and 28 tractors of a certain special kind, respectively. The

Product	Vitamins Min	Vitamins Max	Protein Min	Protein Max	Calcium Min	Calcium Max	Crude Fat Min	Crude Fat Max
Cattle Feed	6	∞	6	∞	7	∞	4	8
Sheep Feed	6	∞	6	∞	6	∞	4	6
Chicken Feed	4	6	6	∞	6	∞	4	6

TABLE 3.3:

transportation costs per tractor (in dollars) from each of the three depots to the locations of the traders and the total number N of available tractors in each depot are summarized in Table 3.4. How many tractors should be delivered from each of the three cities to each of the two traders in order that the total transportation costs are minimized?

	C_1	C_2	C_3
T_1	250	80	400
T_2	300	100	200
N	15	25	25

TABLE 3.4:

3.6. Solve the scheduling problem of Subsection 3.4.1 for the following data

$$(l_{ij}) = \begin{bmatrix} 300 & 600 & 880 \\ 250 & 400 & 700 \\ 200 & 350 & 600 \\ 100 & 200 & 300 \end{bmatrix}, \quad (c_j) = \begin{bmatrix} 30 \\ 50 \\ 80 \end{bmatrix}, \quad (k_i) = \begin{bmatrix} 10000 \\ 8000 \\ 6000 \\ 6000 \end{bmatrix}.$$

3.7. A confectioner manufactures two kinds of candy bars: “ProteinPlus”, that has no carbohydrates, and “SugarPlus”, with no fat. ProteinPlus sells for a profit of 40 cents per bar, and SugarPlus sells for a profit of 50 cents per bar. The candy is proccessed in three main operations: blending, cooking and packaging. The following table records the average time in minutes required by each bar for each of the processing operations:

	Blending	Cooking	Packaging
ProteinPlus	1	5	3
SugarPlus	2	4	1

During each production run the blending equipment is available for a maximum of 12 machine hours, the cooking equipment is available for at most 30 machine hours, and the packaging equipment for no more than 15 hours. If this machine time can be allocated to the making of either candy type at all times that is available, the confectioner wants to know how many boxes of each type should be produced in order to realize the maximum profit.

Formulate this problem as a linear program. Sketch the feasible region and the optimal isoprofit line, and find the optimal solution.

3.8. Paul has 2200 per year to invest over the next five years. At the beginning of each year he can invest in one–, two–, and three–year deposits at interest rates of 8%, 17% (total) and 27% (total), respectively. If Paul reinvests his money available each year, how much should he invest in each of the three deposits each year so that his total cash at the end of the five years is a maximum?

The following exercises deal with the targeting problem of Sections 3.5 and 3.6.

3.9. Without using software, solve the optimization problem

$$a_1 + a_2 + a_3 = \min$$

subject to the inequality constraints

$$\begin{aligned} 32 \le a_1 \le a_0 \\ 0 \le a_2 \le a_0 \\ 0 \le a_3 \le a_0 \\ 3a_1 + a_2 \ge 128, \end{aligned}$$

and the equality constraint

$$5a_1 + 3a_2 + a_3 = 336,$$

for

(a) $a_0 = 40$,
(b) $a_0 = 64$,
(c) $a_0 = 96$.

Hint: Solve the equality constraint for a_3 and substitute this into the objective function and the inequality constraints to find a problem with only two variables a_1, a_2. Solve this two–variable problem graphically.

3.10. Consider the linear program (3.24)–(3.28) with the additional constraints $v_n = 0$ and $|v_i| \le v_0$ for $1 \le i \le n$ (see Subsection 3.5.4).

(a) Identify the vectors and matrices f, A, b, A_{eq}, b_{eq}. For example write $f_i = p_1$ for $1 \le i \le n$, $f_i = p_2$ for $n + 1 \le i \le 2n$, with p_1, p_2 to be determined.
(b) Write a Matlab function that receives g, a_0, T, h, v_0 as input and generates the matrices in (a) as output.

3.11. Let $g = 32$, $T = 20$, $a_0 = 80$, $h = 100$, and $n = 25$. Use a linear program solver to find the optimal acceleration values $(a_1, \ldots, a_n)$ for the discretized targeting problem with friction constant k and the given additional constraints. If the solver fails to find a solution explain why. If it finds a solution plot the acceleration function $a(t)$, the velocity $v(t)$, and the altitude $y(t)$. Comment on these plots.

(a) $k = 0$, no additional constraint.
(b) $k = 2$, no additional constraint.
(c) $k = 0.4$, no additional constraint.
(d) $k = 0.4$, additional constraint $|v(t)| \le 30$ for $0 \le t \le T$.
(e) $k = 0.4$, additional constraint $v(T) = 0$.
(f) $k = 0.4$, additional constraints $v(T) = 0$ and $|v(t)| \le 30$ for $0 \le t \le T$.

3.12. In this exercise you work out some of the details of the analysis of Section 3.6.

(a) Verify equations (3.38) and (3.39).
(b) Verify equation (3.48).
(c) Prove Theorem 3 by induction on n.