An introduction to idioms

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This document is the first in the series [1] – [7], concerned with the use of lattices as a tool for analysing modules and spaces. This particular document is concerned with the general basic lattice theoretic background needed to analyse certain ranking techniques as given in [2, 3, 4]. The later documents [5]–[7] are concerned with specific topics in the analysis. The central idea is for each module M its family $\Lambda = \mathcal{S}ub(M)$ of submodules is a complete lattice of a certain kind, and for each topological space S its topology $\Lambda = \mathcal{O}S$ is also a similar kind of complete lattice. Several aspects of M and can be gleaned by a lattice theoretic analysis of Λ. In this document I describe the background lattice theory needed for the analysis. Some of this is well known and some not. Also some of the material is more general than is technically needed for the applications. However, that extra generality will help to place the material in its appropriate context.

As with all the documents in the series this one is written as a teaching document rather than a research document. In other words, the development is quite slow, and no doubt there are some parts than you can omit because you already know that material. However, as mentioned above, there will be places where I refer to one of [1, 2] so that certain details of some proofs need not be repeated.

Of course, the word 'I' refers to myself, but when I say 'we' I mean me and you.

Contents

1 Basic notions

We are concerned here with the category ***Idm*** of idioms and its subcategory ***Frm*** of frames. Each is a certain kind of complete lattice. Before we define these let us recall some standard material, the lattice aspects and the completeness properties.

1.1 DEFINITION. A lattice is a structure

$$(\Lambda, \leq, \wedge, \top, \vee, \bot)$$

where $(\Lambda, \leq)$ is a partially ordered set with a top $\top$ and a bottom $\bot$, and for each pair of elements $a, b \in \Lambda$ there is a

$$\text{meet } a \wedge b \qquad \text{join } a \vee b$$

within Λ.

A lattice morphism

$$\Lambda \xrightarrow{\quad f \quad} B$$

between two lattices is a function f which is monotone

$$(\forall x, y \in \Lambda)[x \le y \implies f(x) \le f(y)]$$

with

$$f(\top) = \top \qquad f(\bot) = \bot$$

and, more importantly, respects the two binary operations, that is

$$f(x \wedge y) = f(x) \wedge f(y) \qquad f(x \vee y) = f(x) \vee f(y)$$

for each $x, y \in \Lambda$. ■

Notice that technically this is a *bounded* lattice since it has a top and a bottom. Every lattice we look at is bounded, so we don't need to keep mentioning that. I should point out that being monotone is a consequence of the other morphism properties. However, when checking that a function is a morphism it is useful to verify the monotone property first. This is always the case when monotonicity is involved.

There are several textbooks on lattices, but perhaps the most comprehensive is [22].

There are two properties that a lattice may or may not have. These play a part here.

1.2 DEFINITION. Let Λ be a lattice.

The lattice Λ is modular if

$$(a \vee x) \wedge b = a \vee (xlabel001 - fwedgeb)$$

for all $a, x, b \in \Lambda$ with $a \le b$.

The lattice Λ is distributive if

$$(a \vee x) \wedge b = (a \wedge b) \vee (x \wedge b)$$
$$(a \wedge x) \vee b = (a \vee b) \wedge (x \vee b)$$

for all $a, x, b \in \Lambda$. ■

Being distributive seems to require two conditions. However, each of the two *universally quantified* conditions implies the other. (If you ever have to teach a bit of lattice theory then set that equivalence as an exercise. They will forget the quantifiers.) Notice that the modular condition requires a comparison $a \le b$ on the two outside elements. A simple exercise show that each distributive lattice is modular. Of course, not every lattice is modular, and not every modular lattice is distributive. A lantern and a pentagon will help you through that.

Being modular or being distributive are important restrictions on some of the lattices we look at. The need for modularity is not apparent in this document, but it does seem to be important for some of the later developments. There are also other conditions, weaker than modularity, that I suspect could be come important for this topic, but they are not needed here.

That is one aspect we use. We also need certain completeness properties.

1.3 DEFINITION. A poset Λ is

$$\bigwedge\text{-complete} \qquad \bigvee\text{-complete}$$

if each subset $X \subseteq \Lambda$ has a

$$\text{infimum (greatest lower bound) } \bigwedge X \qquad \text{supremum (least upper bound) } \bigvee X$$

respectively. ■

Of course, a poset is $\bigwedge$-complete precisely when it is $\bigvee$-complete, but when we look at possible morphisms we need to distinguish between the two properties.

1.4 DEFINITION. A $\bigvee$-morphism

$$\Lambda \xrightarrow{f} B$$

between two complete posets is a function f which is monotone and respects suprema, that is

$$f(\bigvee X) = \bigvee f[X]$$

for each subset $X \subseteq \Lambda$.

Let ***Sup*** be the category of complete posets and $\bigvee$-morphisms. ■

In this definition we write $f[\cdot]$ for the direct image function across f, that is

$$f[X] = \{f(x) \mid x \in X\}$$

for $X \subseteq \Lambda$.

A $\bigvee$-morphism need not preserve arbitrary infima. In fact, a $\vee$-morphism need not be a $\wedge$-morphism. You might like to think of an example of this. There is a corresponding category ***Inf*** of complete lattices and $\bigwedge$-morphisms. The two categories have the same objects but different arrows. However, we don't need ***Inf*** here.

In this document we look at the basic properties of the category ***Idm*** of idioms. This is a subcategory of ***Sup***. Many of the construction we require are first carried out in ***Sup*** and then modified to deal with ***Idm***. Section 3 is a nice example of this technique.

We are almost ready to introduce the main notion of this document, that of an idiom. We need just one more observation.

We consider certain complete lattices. Thus for each subset X both the supremum and the infimum

$$\bigvee X \qquad \bigwedge X$$

exists. However, as indicated above, these play very different roles. Given a complete lattice there is a possible distributive aspect

$$\text{(DL)} \qquad a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$$

for certain elements a and certain subsets X. We may ask that this law holds for all elements a and all subsets X of a certain kind. For instance, this law for finite subsets X is just the standard distributive law.

Recall that a subset X of a poset is directed if it is non-empty and for each $x, y \in X$ there is some $z \in X$ with $x, y \le z$.

1.5 DEFINITION. An idiom is a complete lattice Λ for which the distributive law (DL) holds for all *directed* sets X.

A frame is a complete lattice Λ for which (DL) holds for *all* sets X.

An idiom morphism between two idioms is a lattice morphism that also preserves arbitrary suprema. Let ***Idm*** be the category of idioms and idiom morphism.

A frame morphism between two frames is an idiom morphism. Let ***Frm*** be the category of frames and frame morphisms. ■

An idiom is sometimes said to be upper continuous and sometimes $\wedge$-continuous. I find the name 'idiom' quite convenient. The two books [14] and [28] are concerned with the use of lattices to analyse certain aspect of modules. You should look at them in conjunction with this and the related documents cited above.

Let's take a closer look at the idiom distributive law. Of course, the comparison

$$a \wedge \bigvee X \geq \bigvee \{a \wedge x \mid x \in X\}$$

holds in every complete lattice. Thus the content of the idiom distributive law is the converse comparison.

By definition a frame is a certain kind of idiom. Which kind?

1.6 LEMMA. *Let Λ be an idiom. Then Λ is a frame precisely when it is distributive (in the finitary sense).*

Proof. Trivially, if Λ is a frame then it is distributive. Thus we need the converse.

Suppose Λ is distributive and consider an arbitrary subset $Y \subseteq \Lambda$. Consider any $a \in \Lambda$. We require

$$a \wedge \bigvee Y \leq \bigvee \{a \wedge y \mid y \in Y\}$$

since the converse comparison always holds. Let X be the set of all the joins of finite subsets of Y. Thus a typical element $x \in X$ has the form

$$y_1 \vee \cdots \vee y_n$$

for $y_1, \ldots, y_n \in Y$. This set X is closed under joins and hence is directed. Also $Y \subseteq X$ so that $\bigvee Y \leq \bigvee X$. Thus

$$a \wedge \bigvee Y \leq a \wedge \bigvee X \leq \bigvee \{a \wedge x \mid x \in X\}$$

where a use of the idiom distributive law gives the second comparison. We have

$$x = y_1 \vee \cdots \vee y_n$$

so that

$$a \wedge x = a \wedge y_1 \vee \cdots \vee a \wedge y_n \leq \bigvee \{a \wedge y \mid y \in Y\}$$

since Λ is distributive. Thus

$$\bigvee \{a \wedge x \mid x \in X\} \leq \bigvee \{a \wedge y \mid y \in Y\}$$

to give the required result. ■

The idiom distributive law looks one sided, but it does have a two sided consequence.

1.7 LEMMA. *Let Λ be an arbitrary idiom. Then*

$$(\bigvee X) \wedge (\bigvee Y) = \bigvee \{x \wedge y \mid x \in X, y \in Y\}$$

for each pair X, Y if directed subsets.

Proof. Let

$$a = \bigvee X \qquad b = \bigvee Y$$

be the two separate suprema. Since Y is directed we have

$$a \wedge b = a \wedge \bigvee Y = \bigvee \{a \wedge y \mid y \in Y\}$$

by a use of the idiom distributive law. For each $y \in Y$ we have

$$a \wedge y = (\bigvee X) \wedge y = \bigvee \{x \wedge y \mid x \in X\}$$

by another use of the idiom distributive law. Thus

$$a \wedge b = \bigvee \{a \wedge y \mid y \in Y\} = \bigvee \{\bigvee \{x \wedge y \mid x \in X\} \mid y \in Y\} = \bigvee \{x \wedge y \mid x \in X, y \in Y\}$$

as required. ■

We observed above that each version of the finitary distributive law implies the other. This is not the case with the idiom distributive law. There is a frame with an element $a \neq \top$ and a linearly ordered set X of elements with

$$\bigwedge X = \bot \quad \text{and} \quad a \vee x = \top$$

for each $x \in X$. Thus

$$a \vee \bigwedge X \neq \bigwedge \{a \vee x \mid x \in X\}$$

in this frame. Can you think of such a frame?

Let's now look at some canonical examples of idioms, frames, and morphisms.

Consider any module M (over some ring R) and any topological space S. Each of these has an associated lattice

$$\mathcal{S}ub(M) \qquad \mathcal{O}S$$

the lattice of submodules of M and the topology on S, the lattice of open sets. Each of these is a complete lattice. Infima in $\mathcal{S}ub(M)$ are computed as intersections, and suprema in $\mathcal{O}S$ are computed as unions. It doesn't take long to see that $\mathcal{O}S$ is a frame. It is a subframe of the power set of S. A short calculation shows that $\mathcal{S}ub(M)$ is a modular idiom, but in general it is not a frame.

The notion of a frame was invented in Paris in the late 1950s as a tool to analyse the algebraic properties of a topology without mentioning the points of the space. For that reason the study of frames is sometimes referred to as point-free topology. However, in some ways that view is too narrow. Many aspects of the study of frames have little to do with point-sensitive (point set) topology. A rather old account of frames can be found in [23]. A more recent survey is given in [32], with a detailed account in [30].

Idioms (without the name) have been used for several years, either explicitly or implicitly, to analyse certain aspects of modules. Quite a lot of module theory can be done in terms of idioms without mentioning the elements of the modules. The two books [14] and [28] are concerned with this idea.

In this and related documents [2] – [7] I show there are some quite strong parallels between the study of certain properties of modules and spaces. I also believe the study of idioms in their own right is interesting.

As an example of a frame morphism, consider a continuous map

$$T \xrightarrow{\ \phi\ } S$$

between two topological spaces. We know that each of the topologies $\mathcal{O}S$ and $\mathcal{O}T$ is a frame, and a simple exercise shows that the inverse image function

$$\mathcal{O}S \xrightarrow{\ \phi^{\leftarrow}\ } \mathcal{O}T$$

is a frame morphism. However, in general, a module morphism does not give an idiom morphism between the two idioms of submodules.

To conclude this section we obtain a result that might look a little strange, or even irrelevant. However, it will be useful in Section 4.

Lemma 1.6 characterizes the class of frames withing the class of idioms. There is also a characterizations of the class of frames within the class of complete lattices. To describe that we need a preliminary notion.

1.8 DEFINITION. Let Λ be any lattice. An implication on Λ is a 2-placed operation $(\cdot \succ \cdot)$ such that

$$x \leq (b \succ a) \iff b \wedge x \leq a$$

for all $a, b, x \in \Lambda$. ■

Trivially, any lattice can carry at most one implication. The standard example of an implication is that carried by a boolean algebra. There are also other examples. You might like to describe the implication carried by a topology (but you may find it helpful to work in terms of closed sets rather that open sets).

As the following shows, having an implication is an important distinguishing property.

1.9 THEOREM. *Let Λ be a complete lattice. Then Λ carries an implication precisely when Λ is a frame.*

Proof. Suppose first that Λ is a frame. We check that for $a, b \in \Lambda$

$$b \succ a = \bigvee \{x \mid b \wedge x \leq a\}$$

defines an implication. The condition

$$x \leq (b \succ a) \Leftarrow b \wedge x \leq a$$

is immediate, and the converse is a simple consequence of the frame distributive law.

Conversely, suppose Λ carries an implication operation. We require

$$b \wedge \bigvee X \leq \bigvee \{b \wedge x \mid x \in X\}$$

for $b \in \Lambda$ and $X \subseteq \Lambda$. Let a be the supremum on the right hand side. Then

$$x \in X \implies b \wedge x \leq a \implies x \leq (b \succ a)$$

so that

$$\bigvee X \leq (b \succ a)$$

to give the required comparison. ■

As we will see in Section 4 using an implication is a useful little tick.

2 Inflators and nuclei

In this section we look at various kinds of functions carried by an idiom. You may think this is a bit of a distraction, but in due course we that these functions are important calculating devices.

2.1 DEFINITION. Let Λ be an arbitrary idiom.

An inflator on Λ is a function $f : \Lambda \longrightarrow \Lambda$ that is inflationary and monotone, that is

$$x \leq f(x) \qquad x \leq y \implies f(x) \leq f(y)$$

for $x, y \in \Lambda$.

A closure operation on Λ is an inflator f that is also idempotent, that is $f^2 = f$.

A nucleus on Λ is a closure operation f such that

$$f(a) \wedge f(b) \leq f(a \wedge b)$$

for all $a, b \in \Lambda$.

A binary pre-nucleus on Λ is an inflator f such that

$$f(a) \wedge f(b) \leq f(a \wedge b)$$

for all $a, b \in \Lambda$.

A unary pre-nucleus on Λ is an inflator f such that

$$f(a) \wedge b \leq f(a \wedge b)$$

for all $a, b \in \Lambda$.

A derivative on Λ is an inflator f such that the closure f^{∞} is a nucleus. ■

A few words about this terminology won't go amiss.

Functions that are both inflationary and monotone occur quite a lot in this topic and other places. There doesn't seem to be a common name for such a function, but I find 'inflator' quite useful.

The name 'closure operation' is standard for this kind of function. However, note that this need not be a *topological* closure operation.

As we will see in Section 3 the nuclei on an idiom capture the kernels of the morphism from that idiom. When the morphism is a quotient, surjective, the nucleus controls everything that is going one. Hence the name. Notice that since a nucleus f is inflationary its characteristic property can be improve to

$$f(a) \wedge f(b) = f(a \wedge b)$$

for elements a, b.

There are two kinds of nuclei which control much more complicated gadgets, namely G-topologies. Gabriel topologies on a module category produce the localizations of that category. These are equivalent to a certain family of nuclei that pass through the category. Grothendieck topologies on a presheaf category over a base category are equivalent to a certain kind of nucleus on the classifier of the category. A description and comparison of this machinery is given in [43]. Because of this connection we often write j for a typical nucleus. Why 'j' you might ask. I could tell you but I won't.

As we will see pre-nuclei are useful gadgets to help us deal with nuclei. As indicated, there are two kinds of pre-nuclei, and these have been around for many years. Usually, in any one document, only one kind is needed and this is referred to as a pre-nucleus. However, when we start to develop ranking techniques we find that both kinds are needed, so I have introduced a distinguishing terminology. In fact, in this document we could get by with just the unary version, but we might as well get used to the other kind now. Also, here we don't need derivatives, but we will later and it is useful to have these related notions defined together in one place.

The family of all inflators on Λ is partially ordered by the pointwise comparison. Thus

$$f \leq g \iff (\forall x \in \Lambda)[f(x) \leq g(x)]$$

for inflators f and g. In due course we see that each of these families of functions is a complete lattice, and infima are computed pointwise. Furthermore, the family $U\Lambda$ of all unary pre-nuclei is itself an idiom. More importantly, the family $N\Lambda$ of all nuclei is a frame, and is a quotient of $U\Lambda$. This with the previous observations indicates why these notions are helpful.

Notice that the composite $g \circ f$ of two inflators is itself an inflator. If each is a pre-nucleus of either kind then so is the composite. However, the composite of two closure operations need not be a closure operation. It is merely an inflator. Similarly, the composite of two nuclei need not be a nucleus. It is merely a binary pre-nucleus.

The idea of a derivative and the notation f^∞ needs a bit more explanation. It is concerned with the ordinal iterates of an inflator.

Let

$$\mathbb{O}\mathrm{rd}$$

be the ordinals, or as many of them as we need to analyse the idiom Λ. The composite of two inflators on Λ is itself an inflator. We extend that idea by iteration.

2.2 DEFINITION. Let Λ be an arbitrary idiom, and let f be an inflator on Λ. The ordinal iterates

$$(f^\alpha \mid \alpha \in \mathbb{O}\mathrm{rd})$$

of f are generated by

$$f^0 = \boldsymbol{id} \qquad f^{\alpha+1} = f \circ f^\alpha \qquad f^\lambda = \bigvee \{f^\alpha \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. At the limit jump the pointwise supremum is use, that is

$$f^{\lambda}(a) = \bigvee \{f^{\alpha}(a) \mid \alpha < \lambda\}$$

for each $a \in \Lambda$. ■

Notice how the completeness of Λ is used to pass across the limit jumps. This kind of construction works for any complete poset, but that generality is not needed here.

2.3 LEMMA. *Let Λ be an arbitrary idiom, with f an inflator on Λ. Each ordinal iterate of f is an inflator, and the whole family of iterates is an ascending chain of inflators.*

Proof. We show that each function f^{α} is an iterator by induction on the ordinal α. The base case is trivial. Since the composite of two inflators is an inflator, the successor step is immediate. The jump to a limit ordinal λ follows by a couple of simple calculations. For instance, consider elements $x \leq y$ of Λ. By the induction hypothesis we have

$$f^{\alpha}(x) \leq f^{\alpha}(y)$$

for each ordinal $\alpha < \lambda$. This gives

$$f^{\alpha}(x) \leq f^{\lambda}(y)$$

by the construction of f^{λ}, and hence

$$f^{\lambda}(x) \leq f^{\lambda}(y)$$

by a second use of that construction.

For the second part we require

$$f^{\beta}(x) \leq f^{\alpha}(x)$$

for each pair of ordinals $\beta \leq \alpha$ and each element x. This follows by a simple induction on α with β and x held fixed. ■

Consider this chain of inflators.

$$\boldsymbol{id} = f \leq f^2 \leq \cdots \leq f^{\alpha} \leq \cdots \quad \alpha \in \mathbb{O}\mathrm{rd}$$

On set-theoretic grounds at some stage this chain will stabilize. There is some ordinal θ such that

$$f^{\theta} = f^{\alpha}$$

for all larger ordinals α. The value of θ depends on the parent idiom Λ and the particular inflator f. Different examples can give very different values. In [2] – [4] we see that this idea is the basis of several standard ranking techniques.

Here we are not concerned with the value of θ, so we write

$$f^{\infty}$$

for the stable limit of the ascending chain. This is an inflator since it is in the chain, and it is idempotent by the stability. Thus f^{∞} is a closure operation. A few moment's thought shows that f^{∞} is the least closure operation above f. This closure operation f^{∞} may or may not be a nucleus, but often we want it to be. By definition, an inflator f is a derivative precisely when its closure f^{∞} is a nucleus.

How might we recognize that an inflator is a derivative? This is where pre-nuclei are helpful.

2.4 LEMMA. *Let Λ be an idiom and consider an inflator f on Λ.*

(a) If f is a binary pre-nucleus then each ordinal iterate f^α is also a binary pre-nucleus. In particular, the closure f^∞ is a nucleus.

(b) If f is a unary pre-nucleus then each ordinal iterate f^α is also a unary pre-nucleus, and each limit ordinal iterate f^λ is a binary pre-nucleus. In particular, the closure f^∞ is a nucleus.

(c) Each nucleus is a binary pre-nucleus, each binary pre-nucleus is a unary pre-nucleus, and each unary pre-nucleus is a derivative.

Proof. (a) Suppose f is a binary pre-nucleus. For fixed $a, b \in \Lambda$ we show

$$f^\alpha(a) \wedge f^\alpha(b) \leq f^\alpha(a \wedge b)$$

by induction on α.

The base case, $\alpha = 0$, is trivial.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$f^{\alpha+1}(a) \wedge f^{\alpha+1}(b) = f\big(f^\alpha(a)\big) \wedge f\big(f^\alpha(b)\big) \leq f\big(f^\alpha(a) \wedge f^\alpha(b)\big) \leq f\big(f^\alpha(a \wedge b)\big) = f^{\alpha+1}(a \wedge b)$$

as required. Here the second step uses the given pre-nucleus property of f, and the third uses the induction hypothesis and the monotone property of f.

For the induction leap to a limit ordinal λ we have

$$\begin{aligned}
f^\lambda(a) \wedge f^\lambda(b) &= \big(\bigvee \{f^\alpha(a) \mid \alpha < \lambda\}\big) \wedge \big(\bigvee \{f^\beta(a) \mid \beta < \lambda\}\big) \\
&\leq \bigvee \{f^\alpha(a) \wedge f^\beta(b) \mid \alpha, \beta < \lambda\} \\
&\leq \bigvee \{f^\gamma(a) \wedge f^\gamma(b) \mid \gamma < \lambda\} \\
&\leq \bigvee \{f^\gamma(a \wedge b) \mid \gamma < \lambda\} \\
&= f^\lambda(a \wedge b)
\end{aligned}$$

for the required result. Here the second step uses the double distributive law (as given by Lemma 1.7), the third step uses the increasing property of the generated chain, and the fourth uses the induction hypothesis.

(b) The first part is proved by an ordinal induction by a modified version of the argument of part (a).

For the second part consider any limit ordinal λ and any pair a, b of elements. We have

$$f^\lambda(a) \wedge f^\lambda(b) \leq \bigvee \{f^\alpha(a) \wedge f^\beta(b) \mid \alpha, \beta \leq \lambda\}$$

by a use of the dual idiom distributive law as in part (a). But now two uses of the known unary pre-nucleus property gives

$$\begin{aligned}
f^\lambda(a) \wedge f^\lambda(b) &\leq \bigvee \{f^\alpha\big(a \wedge f^\beta(b)\big) \mid \alpha, \beta < \lambda\} \\
&\leq \bigvee \{f^\alpha\big(f^\beta(a \wedge b)\big) \mid \alpha, \beta < \lambda\} \\
&\leq f^{\beta+\alpha}(a \wedge b) \mid \alpha, \beta < \lambda\} \\
&\leq f^\lambda(a \wedge b)
\end{aligned}$$

for the required result. Here the third step is a standard result concerning the use of ordinals as iteration gadgets, and the fourth holds since $\beta + \alpha < \lambda$.

(c) This is more or less trivial. ■

That is all we need here about these notions. We can now show why the are relevant.

3 Kernels and the factorization property

To motivate the content of this section we consider a bit of algebra which every undergraduate should know. Consider the category of groups and group morphisms or the category of rings and ring morphisms. For each algebra A of one of these kinds there is a notion of a kernel subset K of A. When A is a group the set K is a normal subgroup, and when A is a ring the set K is an ideal (a 2-sided ideal). For each such pair $K \subseteq A$ there is a construction which produces a quotient

$$A \longrightarrow A/K$$

an algebra A/K of the same kind together with a surjective morphism. Furthermore, every quotient from A occurs in this form (up to a canonical equivalence).

What about the corresponding notions and results for ***Idm*** and ***Frm***? This is where the bigger category ***Sup*** is useful. We first go through the analysis for ***Sup*** and then modify this material to deal with ***Idm*** and ***Frm***.

Before we look at these three cases lets consider the finitary version of semilattices, or more generally other kinds of algebraic structures. For such an algebra there isn't a sensible notion of a kernel *subset*. We have to resort to the notion of a congruence, an equivalence relation on A which respects the carried structure in an appropriate fashion. We then prove three results.

(a) For each morphism

$$A \xrightarrow{\ f\ } B$$

the kernel relation on A given by

$$x \equiv y \iff f(x) = f(y)$$

(for $x, y \in A$) is a congruence. We may call $\equiv$ the kernel congruence of f.

(b) For each congruence $\approx$ on A the set of blocks (equivalence classes) $A/\approx$ of the equivalence relation can be furnished as an algebra such that the canonical function

$$\begin{array}{ccc} A & \xrightarrow{\ h\ } & A/\approx \\ a & \longmapsto & [a] \end{array}$$

(which sends each element to the block in which it lives) is a morphism with $\approx$ as its kernel congruence.

(c) Consider any congruence $\approx$ on A and any morphism f from A with its kernel congruence, as above. Suppose $\approx$ is smaller that $\equiv$, that is

$$x \approx y \implies x \equiv y$$

for $x, y \in A$. Then there is a unique morphism $f^\sharp$ such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] & & B \\ & A/\approx \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes.

These result holds in any universal algebraic context, not just in the finitary situations. There is almost nothing in the proofs beyond a long series of simple observations and a few calculations. Result (c) is the fundamental factorization result, although often it is not stated in this form. If you have never done it before it is a good exercise to compare these statements with the usual statement of the corresponding results for groups and rings. They are just the same results.

We now want versions of these results for ***Sup*** and ***Idm***. Of course, we could simple use these results with the appropriate notions of congruence. However, calculations with congruences can get messy. We will see that for ***Sup*** and ***Idm*** there is a much neater way of handling these notions.

In this section we look first at the ***Sup*** context. We then modify the methods to deal with ***Idm***.

3.1 The kernel of a ***Sup*** arrow

In this block we show that for ***Sup*** objects the three basic results (a, b, c) can be obtained directly in terms of closure operations. To do that we first look at a result that might seem a bit unnecessary. However, later we find that it is useful.

Consider a pair of complete posets Λ, Γ and a monotone function between them.

$$\Lambda \xrightarrow{f^* = f} \Gamma$$

This may not be a $\bigvee$-morphism. The following characterization explains why we use this odd notation.

3.1 LEMMA. *Consider a monotone function f^* between two complete posets Λ, Γ, as above. Then f^* is a **Sup**-morphism precisely when it has a right adjoint, that is a monotone function*

$$\Lambda \xleftarrow[f_*]{} \Gamma$$

in the opposite direction such that

$$f^*(a) \leq b \iff a \leq f_*(b)$$

for all $a \in \Lambda, b \in \Gamma$.

Proof. If f^* is a ***Sup***-morphism then

$$f_*(b) = \bigvee \{x \in \Lambda \mid f^*(x) \leq b\}$$

give the right adjoint.

Conversely, suppose a right adjoint f_* does exist and consider any subset $X \subseteq \Lambda$. Let

$$a = \bigvee X \qquad b = \bigvee f^*[X]$$

so that $f^*(a) = b$ is required. A comparison $f^*(a) \leq b$ will suffice since the converse comparison is immediate. We have

$$(\forall x \in X)[f^*(x) \leq b]$$

by the construction of b. Thus

$$(\forall x \in X)[x \le f_*(b)]$$

by the adjunction property, and hence

$$a \le f_*(b)$$

by the construction of a. A second use of the adjunction property gives the required result. ■

We have just gone through a bit of miniature category theory. We can think of a poset as a category with just one object. A monotone function is then a functor. A pair of monotone functions $f^* \dashv f_*$ is an adjunction, which explains some of the terminology.

3.2 DEFINITION. The kernel of a ***Sup*** morphism

$$\Lambda \xrightarrow{\ f\ } \Gamma$$

is the unique function k on Λ such that

$$x \le k(a) \iff f(x) \le f(a)$$

for all $a, x \in \Lambda$. ■

Clearly, for a given ***Sup*** morphism f there is at most one such function k. To show there is such a function and to extract some of its properties we use the adjunction of f.

3.3 LEMMA. *Let*

$$\Lambda \xrightarrow{\ f\ } \Gamma$$

*be an arbitrary **Sup** morphism. Then f does have a kernel k, this kernel is a closure operation, and $f(k(a)) = f(a)$ for each $a \in A$.*

Proof. We know that f has at most one kernel k. Let $f^* = f$, let f_* be the right adjoint, and let $k = f_* \circ f^*$ to obtain a function on Λ. For $x, a \in A$ we have

$$x \le k(a) = f_*(f^*(a)) \iff f(x) = f^*(x) \le f^*(a) = f(a)$$

to show that k is the kernel.

Since $f(a) \le f(a)$ we see that k is inflationary. Since f is monotone, so is k. Since $k(a) \le k(a)$ we have $f(k(a)) \le f(a)$. The converse comparison is immediate to give the required equality.

Finally, with $x = k^2(a)$ we have

$$f(x) = f\big(k(k(a))\big) = f\big(k(a)\big) = f(a)$$

and hence

$$k^2(a) = x \le k(a)$$

to show that k is a closure operation on Λ. ■

This result is the analogue of result (a) given above. We now look at the analogue of (b). This is more interesting.

3.4 DEFINITION. Let j be a closure operation on the $\bigvee$-semilattice Λ. We let

$$\Lambda_j = \{x \in A \mid j(x) = x\} = j[\Lambda]$$

to obtain the fixed set of j. ■

Since j is a closure operation the fixed set Λ_j is just the set $j[\Lambda]$ of outputs of j. Since Λ_j is a subset of Λ it is partially ordered by the restriction of the comparison on Λ.

3.5 LEMMA. *Let j be a closure operation on the $\bigvee$-semilattice Λ. The fixed set Λ_j is closed under arbitrary infima as calculated in Λ. In particular, Λ_j is complete.*

Proof. Consider any subset $X \subseteq \Lambda_j$, and let $a = \bigwedge X$ as calculated in Λ. We required $a \in \Lambda_j$. For each $x \in X$ we have $a \leq x$, so that $j(a) \leq j(x) = x$, and hence $j(a) \leq a$. The converse comparison holds since j is inflationary. ■

This shows that Λ_j is also a ***Sup*** object. However, it is not a sub-object of Λ. We show that it is a *quotient* object. To do that we need to see how suprema are calculated in Λ_j.

3.6 LEMMA. *Let j be a closure operation on the $\bigvee$-semilattice Λ. For each subset $X \subseteq \Lambda_j$ the element*

$$\overset{j}{\bigvee} X = j\big(\bigvee X\big)$$

is the supremum of X in Λ_j.

Proof. Let

$$a = \overset{j}{\bigvee} X$$

so we have an element of Λ_j, and clearly a is an upper bound of X. We must show that it is the least upper bound in Λ_j. Consider any element $b \in \Lambda_j$ which is an upper bound for X. Then $\bigvee X \leq b$, so that

$$\overset{j}{\bigvee} X = j\big(\bigvee X\big) \leq j(b) = b$$

to give the required result. ■

With this we can obtain the analogue of (b).

3.7 THEOREM. *Let j be a closure operation on the $\bigvee$-semilattice Λ. The assignment*

$$\begin{array}{ccc} A & \xrightarrow{\ j^*\ } & \Lambda_j \\ a & \longmapsto & j(a) \end{array}$$

*is a **Sup** morphism, and its kernel is j.*

Proof. Trivially, this assignment j^* is monotone, so it suffices to show it respects suprema, that is

$$j^*(\bigvee X) = \bigvee^{j} j^*[X]$$

for each subset $X \subseteq A$. This unravels to

$$j(\bigvee X) = j\big(\bigvee j[X]\big)$$

and the comparison $\leq$ is more or less immediate. We look at the converse comparison.

For each $x \in X$ we have

$$\bigvee X \geq x \quad \text{and hence} \quad j\big(\bigvee X\big) \geq j(x)$$

since j is monotone. Thus

$$j(\bigvee X) \geq \bigvee j[X]$$

by releasing x. Since j is monotone and idempotent this gives

$$j(\bigvee X) = j^2(\bigvee X) \geq j\big(\bigvee j[X]\big)$$

as required.

Let k be the kernel of j^*. Thus

$$x \leq k(a) \iff j^*(x) \leq j^*(a) \iff j(x) \leq j(a)$$

for all $x, a \in A$. With $x = k(a)$ then

$$k(a) = x \leq j(x) \leq j(a)$$

to give $k \leq j$. With $x = j(a)$ we have

$$j(x) = j^2(a) = j(a) \quad \text{so that} \quad j(a) = x \leq k(a)$$

to give $j \leq k$. Thus $k = j$. ■

Observe that although the have the same behaviour we must distinguish between the two functions j^* and j. The function j^* is a morphism from Λ, but the function j is a closure operation on Λ.

Since j^* is a morphism, it has a right adjoint j_*. You might like work out what it is.

This gives us the result (b) in terms of closure operations. We now look at the factorization result (c).

3.8 THEOREM. *Consider any closure operation j on the $\bigvee$-semilattice Λ and any $\bigvee$-morphism*

$$\Lambda \xrightarrow{f} \Gamma$$

with kernel k. Suppose $j \leq k$. Then there is a unique morphism $f^\sharp$ such that

$$\begin{tikzcd} \Lambda \arrow[rr, "f"] \arrow[dr, "j^*"'] & & \Gamma \\ & \Lambda_j \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes.

Proof. Since j^* is surjective there can be at most one such morphism $f^\sharp$. Thus is suffices to exhibit an example of such a morphism.

For $a \in \Lambda$ we have
$$f(a) \leq f(j(a)) \leq f(k(a)) = f(a)$$
where the last equality holds by Lemma 3.3. Thus these three elements of Γ are equal, and so $f \circ j = f$.

Since, as sets, we have $\Lambda_j \subseteq \Lambda$, we may set
$$f^\sharp(a) = f(a)$$
for $a \in \Lambda_j$. This gives a function from Λ_j to Γ. Also, by the previous observation, for each $a \in \Lambda$ we have
$$(f^\sharp \circ j^*)(a) = f(j(a)) = f(a)$$
so the triangle commutes as the ***Set*** level. Trivially, $f^\sharp$ is monotone, so it suffices to show is respects suprema. For this we require
$$f^\sharp\big(\overset{j}{\bigvee} X\big) = \bigvee f^\sharp[X]$$
for arbitrary $X \subseteq \Lambda_j$. This is
$$f\big(j(\bigvee X)\big) = \bigvee f[X]$$
by unravelling the constructions involved. Since f is a morphism, this holds by the first observation. ■

In this block we have obtained the analogues of results (a, b, c) for the category ***Sup*** but using closure operations in place of congruences. To conclude this block let's see why the two versions are equivalent.

We need to recall what a congruence for ***Sup*** is.

3.9 DEFINITION. Let Λ be a $\bigvee$-semilattice. A congruence on Λ (that is a $\bigvee$-congruence on Λ) is an equivalence $\approx$ such that for each similarly indexed subsets of Λ
$$X = \{x_i \mid i \in I\} \qquad Y = \{y_i \mid i \in I\}$$
the implication
$$(\forall i \in I)[x_i \approx y_i] \implies \bigvee X \approx \bigvee Y$$
holds. ■

In other words the congruence $\approx$ can be passed across arbitrary suprema.

Since such a congruence is an equivalence relation we may form the *set*
$$\Lambda/{\approx}$$
of blocks (equivalence classes). Each such block has the form
$$[a] = \{x \in \Lambda \mid x \approx a\}$$
where a is a representing element of the block. As usual, when manipulating such a block we choose a representative, but take care that what we do is independent of the particular choice. Here we can always choose a special representative.

3.10 LEMMA. *Let Λ be any $\bigvee$-semilattice, let $\approx$ be any congruence on Λ, and let X be any block of this congruence. Then*

$$a = \bigvee X$$

is the unique maximum member of X.

Proof. It suffices to show that $a \in X$. Consider any $b \in X$. We require $a \approx b$. With the given block and $Y = \{b\}$ we have

$$(\forall x \in X, y \in Y)[x \approx y]$$

so that the congruence property gives

$$a = \bigvee X \approx \bigvee Y = b$$

as required. ■

Since each block of a congruence has a special member, it maximum member, it seems a good idea to select this whenever possible. In fact that is what we have been doing.

3.11 THEOREM. *Let Λ be an arbitrary $\bigvee$-semilattice. There is a canonical bijection*

$$\approx \longleftrightarrow j$$

between the congruences $\approx$ on Λ and the closure operations j on Λ. This is given by

$$(\approx \longmapsto j) \qquad j(a) = \bigvee \{x \in \Lambda \mid x \approx a\}$$
$$(\approx \longleftarrow\!\shortmid\, j) \qquad a \approx b \Longleftrightarrow j(a) = j(b)$$

for $a, b \in \Lambda$.

The proof of this is a routine collection of small steps. I suggest you look at some of these. More importantly, observe how calculations with closure operations are easier than the corresponding calculations with congruences.

As an example of this think of producing the join of two congruences to obtain a congruence. It's a bit messy. But now suppose we look at the two closure operations j, k corresponding to the congruences. We want the smallest closure operation l above both of them. Either of the composites

$$f = j \circ k \qquad f = k \circ j$$

is an inflator above j, k. Also

$$f \leq l^2 = l$$

and hence $l = f^\infty$ is the closure operation we want.

3.2 The kernel of a *Idm* arrow

We now want the analogues of results (a, b, c) for idioms. All we have to do is go through the material of the previous block and add a bit of extra material at each step. We are looking for a special kind of closure operation. Guess which kind.

3.12 LEMMA. *When viewed as a **Sup** morphism, the kernel of a **Idm** morphism*

$$\Lambda \xrightarrow{\quad f \quad} \Gamma$$

is a nucleus.

Proof. Let k be the kernel of f. By Lemma 3.3 we know that k is a closure operation with

$$x \le k(a) \iff f(x) \le f(a)$$

for each $a, x \in A$. Thus we require

$$k(a_1) \wedge k(a_2) \le k(a_1 \wedge a_2)$$

for arbitrary $a_1, a_2 \in \Lambda$. Since f respects binary meets, a second use of Lemma 3.3 gives

$$f\big(k(a_1) \wedge k(a_2)\big) = f\big(k(a_1)\big) \wedge f\big(k(a_2)\big) = f(a_1) \wedge f(a_2) = f(a_1 \wedge a_2) = f\big(k(a_1 \wedge a_2)\big)$$

and hence the characteristic property of k gives the required result. ■

This result is the analogue of general result (a) for idioms. Now we look at the analogue of (b). To do that we extend Theorem 3.7.

3.13 THEOREM. *Let j be a nucleus on the idiom Λ. The complete lattice Λ_j is an idiom, and if Λ is a frame then so is Λ_j. The assignment*

$$\begin{array}{ccc} A & \xrightarrow{\quad j^* \quad} & \Lambda_j \\ a & \longmapsto & j(a) \end{array}$$

is an idiom morphism, and its kernel is j.

Proof. We know that Λ_j is a complete lattice. We must show that the assumed distributive law on Λ passes down to Λ_j. Consider any subset $X \subseteq \Lambda_j$, and for the time being suppose that X is directed in Λ_j. Observe that $X \subseteq \Lambda$ and X is directed in Λ. Recall also that a meet calculated in Λ_j is the same as that calculated in Λ.

We require

$$a \wedge \overset{j}{\bigvee} X = \overset{j}{\bigvee} \{a \wedge x \mid x \in X\}$$

for arbitrary $a \in \Lambda_j$. This is

$$a \wedge j\big(\bigvee X\big) = j\big(\bigvee \{a \wedge x \mid x \in X\}\big)$$

by the construction of the supremum in Λ_j. But $a \in \Lambda_j$, so that $a = j(a)$, and hence

$$j(a) \wedge j\big(\bigvee X\big) = j\big(\bigvee \{a \wedge x \mid x \in X\}\big)$$

is the requirement. Since j is a nucleus this is

$$j\big(a \wedge \bigvee X\big) = j\big(\bigvee\{a \wedge x \mid x \in X\}\big)$$

which holds by the corresponding property in Λ. When Λ is just an idiom this calculation works for directed X. When Λ is a frame it works for arbitrary X.

By Theorem 3.7 we know that j^* is a $\bigvee$-morphism with j as its kernel. Thus it suffices to show that j^* passes across meets. Since j is a nucleus, this is immediate. ■

Finally we look at result (c). To do that we modify Theorem 3.8. As with the previous proof there is not a lot left to be done.

3.14 THEOREM. *Consider any nucleus j on the idiom Λ and any idiom morphism*

$$\Lambda \xrightarrow{\ f\ } \Gamma$$

with kernel k. Suppose $j \leq k$. Then there is a unique idiom morphism $f^\sharp$ such that

$$\begin{tikzcd}
\Lambda \arrow[rr, "f"] \arrow[dr, "j^*"'] & & \Gamma \\
& \Lambda_j \arrow[ur, "f^\sharp"'] &
\end{tikzcd}$$

commutes.

Proof. By Theorem 3.8 we know that the only possible morphism is given by

$$f^\sharp(a) = f(a)$$

for $a \in \Lambda_j$. Theorem 3.8 ensures that this $f^\sharp$ is a $\bigvee$-morphism. Thus it suffices to show that $f^\sharp$ passes across meets. Since f is given as an idiom morphism, this is immediate. ■

These results show why nuclei are important. Clearly, we ought to start to gather more information about the whole family of nuclei on an idiom. That is the topic of the next section.

4 The assembly of an idiom

We know that the inflators on an idiom Λ can be partially ordered by the pointwise comparison. In the same way we can partially order the pre-nuclei and the nuclei on Λ. In this section we look at the some of the properties of these posets. In particular, we investigate the nature of the assembly of Λ, the poset of all nuclei.

4.1 DEFINITION. For an arbitrary idiom Λ we let

$$I\Lambda \qquad U\Lambda \qquad B\Lambda \qquad N\Lambda$$

be the poset of all

inflators   unary pre-nuclei   binary pre-nuclei   nuclei

on Λ (under the pointwise comparison). ■

This gives us a short chain

$$N\Lambda \subseteq B\Lambda \subseteq U\Lambda \subseteq I\Lambda$$

of posets associated with Λ. However, we will see that this is the wrong way to think of these posets.

Our first job is to show that each of these associated posets is a complete lattice, and infima are calculated in a simple way.

4.2 DEFINITION. Let Λ be an arbitrary idiom, and let F be any set of inflators on Λ. The pointwise infimum $\bigwedge F$ of F is the function on Λ given by

$$\left(\bigwedge F\right)(a) = \bigwedge \{f(a) \mid f \in F\}$$

for each $a \in \Lambda$. ■

Observe that, almost trivially, this function $\bigwedge F$ is an inflator on Λ, and is a lower bound for F in $I\Lambda$. Of course, we can do better than that, and remove any confusion that could be caused by the terminology.

4.3 LEMMA. *Let Λ be an arbitrary idiom, and let F be any set of inflators on Λ.*
The pointwise infimum $\bigwedge F$ is the actual infimum of F in $I\Lambda$.
If $F \subseteq U\Lambda$ then $\bigwedge F \in U\Lambda$ and is the actual infimum in $U\Lambda$.
If $F \subseteq B\Lambda$ then $\bigwedge F \in B\Lambda$ and is the actual infimum in $B\Lambda$.
If $F \subseteq N\Lambda$ then $\bigwedge F \in N\Lambda$ and is the actual infimum in $N\Lambda$.

Proof. Consider any inflator g which is a lower bound for F. Thus

$$g(a) \le f(a)$$

for each $a \in \Lambda$ and each $f \in F$. But then

$$g(a) \le \bigwedge \{f(a) \mid f \in F\} = \left(\bigwedge F\right)(a)$$

and hence

$$g \le \bigwedge F$$

to give the first required result.

Suppose $F \subseteq U\Lambda$. Consider any $a, b \in \Lambda$. For each $f \in F$ we have

$$\left(\bigwedge F\right)(a) \wedge b \le f(a) \wedge b \le f(a \wedge b)$$

and hence

$$\left(\bigwedge F\right)(a) \wedge b \le \left(\bigwedge F\right)(a \wedge b)$$

to deal with the unary case.

The binary case follows by a similar argument.

Suppose $F \subseteq N\Lambda$. It suffices to show that $g = \bigwedge F$ is idempotent. Consider any $a \in \Lambda$ and any $f \in F$. Then $g \le f$ so that

$$g^2(a) \le f(g(a)) \le f^2(a) = f(a)$$

and hence $g^2 \leq g$, as required. ■

Before we continue let's consider the particular case where $F = \emptyset$. On general grounds $\bigwedge \emptyset$ is the top of $I\Lambda$, namely, that inflator $\boldsymbol{tp}$ with

$$\boldsymbol{tp}(a) = \top$$

for each $a \in \Lambda$. The bottom of $I\Lambda$ is the identity function, that function $\boldsymbol{id}$ with

$$\boldsymbol{id}(a) = a$$

for each $a \in \Lambda$. Each of $\boldsymbol{tp}, \boldsymbol{id}$ is a nucleus.

Lemma 4.3 show that each of $I\Lambda, U\Lambda, B\Lambda, N\Lambda$ is a complete lattice and infima are easy to compute. What about suprema?

4.4 DEFINITION. Let Λ be an arbitrary idiom, and let F be any *non-empty* set of inflators on Λ. The pointwise supremum $\bigvee F$ of F is the function on Λ given by

$$\left(\bigvee F\right)(a) = \bigvee \{f(a) \mid f \in F\}$$

for each $a \in \Lambda$. ■

Note that we use this construction only on a non-empty set of inflators. The pointwise supremum of the empty set $\emptyset$ of inflators is given by

$$\left(\bigvee \emptyset\right)(a) = \bigvee \emptyset = \bot$$

which does not produce an inflator. The least inflator is $\boldsymbol{id}$, the identity function.

4.5 LEMMA. *Let Λ be an arbitrary idiom, and let F be any non-empty set of inflators on Λ. Then the pointwise supremum $\bigvee F$ is the actual infimum of F in $I\Lambda$.*

Now suppose F is directed.

If $F \subseteq U\Lambda$ then $\bigvee F \in U\Lambda$ and is the actual supremum in $U\Lambda$.

If $F \subseteq B\Lambda$ then $\bigvee F \in B\Lambda$ and is the actual supremum in $B\Lambda$.

Proof. Let g be this pointwise supremum. Since F is non-empty there is at least one inflator $f \in F$. Thus for each $a \in \Lambda$ we have

$$a \leq f(a) \leq g(a)$$

to show that g is inflationary. A similar calculation shows that g is monotone, and hence is an inflator. In other words, we have $g \in I\Lambda$.

Trivially, g is an upper bound for F. We show it is the least upper bound. To this end consider any $h \in I\Lambda$ which is an upper bound for F, that is $f \leq h$ for each $f \in F$. For each $a \in \Lambda$ we have $f(a) \leq h(a)$ so that

$$g(a) = \bigvee \{f(a) \mid f \in F\} \leq h(a)$$

to show that $g \leq h$, for the required result.

Now suppose F is directed. Observe that for each $a \in \Lambda$ the set

$$\{f(a) \mid f \in F\} \leq h(a)$$

is directed in Λ.

Suppose that each $f \in F$ is a unary pre-nucleus. For each $a, b \in \Lambda$ we have

$$\begin{aligned} g(a) \wedge b &= \left(\bigvee \{f(a) \mid f \in F\}\right) \wedge b \\ &= \bigvee \{f(a) \wedge b \mid f \in F\} \\ &\leq \bigvee \{f(a \wedge b) \mid f \in F\} = g(a \wedge b) \end{aligned}$$

to show that g is a unary pre-nucleus. The second step holds by the idiom distributive property of Λ, and the third holds since each f is a unary pre-nucleus.

Suppose that each $f \in F$ is a binary pre-nucleus. For each $a, b \in \Lambda$ we have

$$\begin{aligned} g(a) \wedge g(b) &= \left(\bigvee \{f_1(a) \mid f_1 \in F\}\right) \wedge \left(\bigvee \{f_2(b) \mid f_2 \in F\}\right) \\ &= \bigvee \{f_1(a) \wedge f_2(b) \mid f_1, f_2 \in F\} \\ &\leq \bigvee \{f(a) \wedge f(b) \mid f \in F\} \\ &\leq \bigvee \{f(a \wedge b) \mid f \in F\} = g(a \wedge b) \end{aligned}$$

to show that g is a binary pre-nucleus. The second step holds by the extended idiom distributive property of Λ as given by Lemma 1.7, the third holds since F is directed, that is for each $f_1, f_2 \in F$ there is some $f \in F$ with $f_1, f_2 \leq f$, and the fourth step holds since each f is a binary pre-nucleus. ■

This result does not extend to sets of nuclei. The pointwise supremum of a directed set of nuclei is certainly a binary pre-nucleus, but it need not be idempotent. We look at suprema of nuclei at the end of this section.

Being a complete lattice is interesting but not surprising. The next result is a hint that there is something more important going on.

4.6 THEOREM. *For each idiom Λ the complete lattice $U\Lambda$ is itself an idiom.*

Proof. We must show that $U\Lambda$ satisfies the idiom distributive law. Thus we require

$$f \wedge \bigvee G = \bigvee \{f \wedge g \mid g \in G\}$$

for each $f \in U\Lambda$ and each directed $G \subseteq U\Lambda$. We recall that these two suprema and all infima are computed pointwise. We evaluate at an arbitrary element $a \in \Lambda$. Thus

$$\begin{aligned} \left(f \wedge \bigvee G\right)(a) &= f(a) \wedge \left(\bigvee G\right)(a) \\ &= f(a) \wedge \bigvee \{g(a) \mid g \in G\} \\ &= \bigvee \{f(a) \wedge g(a) \mid g \in G\} \\ &= \bigvee \{(f \wedge g)(a) \mid g \in G\} = \left(\bigvee \{f \wedge g \mid g \in G\}\right)(a) \end{aligned}$$

to give the required result. The second step holds since G is directed, and this produces a directed subset of Λ. The third step holds by the idiom distributive law. The final step holds since the involved subset of $U\Lambda$ is directed. ■

Consider any unary pre-nucleus f on Λ. From Section 2 we know that we may iterate f though the ordinals to produce

$$f^\alpha$$

and ascending chain on unary pre-nuclei. The stable limit

$$f^\infty$$

is the least nucleus above f. A few moment's thought shows that the operation $(\cdot)^\infty$ is a closure operation on $U\Lambda$. And there is more.

4.7 THEOREM. *Let Λ be an arbitrary idiom. The closure operation $(\cdot)^\infty$ on the idiom $U\Lambda$ is actually a nucleus. It fixed set is precisely the assembly $N\Lambda$ of Λ.*

Proof. Since $(\cdot)^\infty$ is a closure operation on $U\Lambda$ it suffices to show it is a unary pre-nucleus on $U\Lambda$, that is

$$f^\infty \wedge g \le (f \wedge g)^\infty$$

for arbitrary $f, g \in U\Lambda$. To do that us we show

$$f^\alpha \wedge g \le (f \wedge g)^\alpha$$

for each ordinal α, and then take α sufficiently large. To help with this let

$$h = f \wedge g$$

so that our objective is

$$f^\alpha(a) \wedge g(a) \le h^\alpha(a)$$

for arbitrary $a \in \Lambda$. We proceed by induction on α with allowable variation of a.

The base case, $\alpha = 0$, is immediate.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$f^{\alpha+1}(a) \wedge g(a) = f\big(f^\alpha(a)\big) \wedge g(a) \le f\big(f^\alpha(a) \wedge g(a)\big) \le f\big(h^\alpha(a)\big)$$

where the second step holds since f is a unary pre-nucleus, and the third holds by the induction hypothesis. This gives

$$f^{\alpha+1}(a) \wedge g(a) \le f\big(h^\alpha(a)\big) \wedge g(a) \le f\big(h^\alpha(a)\big) \wedge g\big(h^\alpha(a)\big) = h^{\alpha+1}(a)$$

to conclude this step.

For the induction leap to a limit ordinal λ, for arbitrary $a \in \Lambda$ we have

$$\begin{aligned} f^\lambda(a) \wedge g(a) &= \bigvee \{f^\alpha(a) \mid \alpha < \lambda\} \wedge g(a) \\ &= \bigvee \{f^\alpha(a) \wedge g(a) \mid \alpha < \lambda\} \\ &\le \bigvee \{h^\alpha(a) \mid \alpha < \lambda\} \qquad = h^\lambda(a) \end{aligned}$$

as required. Here the second step uses the idiom distributive law on Λ, and the third uses the induction hypothesis.

Finally, the fixed set property

$$f^{\infty} = f \iff f \text{ is a nucleus}$$

is immediate. ■

By the analysis of Section 3 these two results give the following.

4.8 COROLLARY. *For each idiom Λ each of $U\Lambda, N\Lambda$ is an idiom and*

$$\begin{array}{rcl} U\Lambda & \longrightarrow & N\Lambda \\ f & \longmapsto & f^{\infty} \end{array}$$

is a canonical quotient with $(\cdot)^{\infty}$ as the kernel nucleus.

This result can be improved but to do that it seems that we need to use a different tactic. We invoke Theorem 1.9. To do that we need a preliminary. The following result is a generalization of Lemma 3.1 of [34].

4.9 LEMMA. *Let Λ be an arbitrary idiom, let j be a nucleus, and let k be any inflator on Λ. Let F be the set of unary pre-nuclei f with $f \wedge k \leq j$. The following hold.*

(i) F is closed under composition.

(ii) F is directed.

(iii) F has a unique maximum member.

(iv) This maximum member is a nucleus.

Proof. (i). Consider $f_1, f_2 \in F$ and let $f = f_1 \circ f_2$. For each $x \in \Lambda$ we have

$$f(x) \wedge k(x) = f_1(f_2(x)) \wedge k(x) \leq f_1(f_2(x) \wedge k(x)) \leq f_1(j(x))$$

where the second step holds since f_1 is unary, and the third step holds since $f_2 \in F$. But now, since $f_1 \in F$ we have

$$f(x) \wedge k(x) \leq f_1(j(x)) \wedge k(x) \leq f_1(j(x)) \wedge k(j(x)) \leq j^2(x) = j(x)$$

to show that $f \in F$.

(ii). For $f_1, f_2 \in F$ we have $f_1, f_2 \leq f_1 \circ f_2 \in F$, to give the required result.

(iii). Consider the pointwise supremum

$$g = \bigvee F$$

of F given by

$$g(a) = \bigvee \{f(a) \mid f \in F\}$$

for each $a \in \Lambda$. Since F is directed we see that f is an inflator. A use of upper continuity shows that g is a unary pre-nucleus. Thus it suffices to show that $g \in F$.

For each $x, y \in \Lambda$ we have

$$g(x) \wedge y = \bigvee \{f(x) \mid f \in F\} \wedge y = \bigvee \{f(x) \wedge y \mid f \in F\}$$

where the second step holds by a use of the upper continuity (since the supremum is directed). In particular, we have

$$g(x) \wedge k(x) = \bigvee \{f(x) \wedge k(x) \mid f \in F\} \leq j(x)$$

to show that $g \in F$.

(iv). We have $g \in F$ and hence $g^2 \in F$ by part (i). But now $g^2 \leq g$ to show that g is a nucleus. ■

With this we can obtain the main result of this section.

4.10 THEOREM. *Let Λ be an idiom. Then the assembly $N\Lambda$ is a frame.*

Proof. We know that $N\Lambda$ is a complete lattice so, by Theorem 1.9 it suffices to show that $N\Lambda$ carries an implication. Consider any nuclei $j, k \in N\Lambda$. By Lemma 4.9 there is a nucleus $l \in N\Lambda$ such that

$$f \wedge k \leq j \iff f \leq l$$

for each inflator f. In particular, this is holds for all $f \in N\Lambda$ to show that l is the required implication $k \succ j$. ■

This result explains the title of this section.

4.11 DEFINITION. For each idiom Λ the assembly is the frame $N\Lambda$ of all nuclei on Λ. ■

We know that $N\Lambda$ is a complete lattice, so each set J of nuclei has a supremum in $N\Lambda$. To compute that we modify a trick given in Section 3. Let J° be the family of all composites

$$j_1 \circ \cdots \circ j_m$$

for $j_1, \ldots, j_m \in J$. (If J is empty then $J^\circ = \{\boldsymbol{id}\}$.) This is a directed set of binary pre-nuclei, and each lies below any nucleus that is an upper bound for J. By Lemma 4.3, and by Theorem 4.7 its closure is a nucleus. This more or less gives the following.

4.12 THEOREM. *Let Λ be an arbitrary idiom and let J° be the family of all composites of members of J. Then the pointwise supremum*

$$\left(\bigvee J^\circ\right)^\infty$$

is the supremum of J in $N\Lambda$.

A long term project is to understand the structure of this assembly $N\Lambda$. Believe me, some weird things happen.

5 Some final remarks

This document is the first in a series [1]–[7]. In this final section a say a few words about the aim of that series, especially [2, 3, 4].

The general aim is to investigate certain ranking techniques for modules M and spaces S. Each of these has an associated modular idiom

$$\Lambda = \mathcal{S}ub(M) \qquad \Lambda = \mathcal{O}S$$

(and, in fact, $\mathcal{O}S$ is a frame). The idea is to investigate ranking machinery for an arbitrary modular idiom as a combined approach to the methods used for modules and spaces.

The general idea is to find a derivative f on a idiom Λ, an inflator for which the closure f^∞ is a nucleus. The length of the iteration from f to f^∞ is then the rank of Λ relative to f. (For some historical examples the rank is called the dimension or length.)

The document [2] sets up the basic machinery and then looks at two particular examples, the socle $\boldsymbol{soc}$ and the CB-derivative $\boldsymbol{cbd}$ (which originated in the study of modules and spaces, respectively). The general idea is to select a set $\mathcal{B}$ of intervals of the parent idiom Λ and look for the smallest nucleus for which the corresponding quotient collapses of these intervals. The two particular examples are given by the simple intervals and the complemented intervals, respectively. To find the collapsing nucleus we first convert $\mathcal{B}$ into a derivative f and then iterate f. The process $\mathcal{B} \longmapsto f$ is not obvious, and the central theme of [2] is to describe the technique used. Almost always f is a unary pre-nucleus, but on the better occasions it a binary.

Each nucleus j on the idiom Λ gives a quotient idiom Λ_j, and this carries its own ranking devices. We may lift any of these up to Λ to obtain a j-relative device on Λ. In particular, we have $\boldsymbol{soc}_j$ and $\boldsymbol{cbd}_j$, the j-socle and the j-CB-derivative on Λ. Precisely how we do this is not immediately obvious, and [3] is concerned with the details of these two examples. It turns out that there is quite a lot of similarity between these two examples, and there are also some significant differences.

Each of $\boldsymbol{soc}_j$ and $\boldsymbol{cbd}_j$ is a pre-nucleus on Λ, and so we obtain a pair of operations

$$j \longmapsto \boldsymbol{soc}_j^\infty \qquad j \longmapsto \boldsymbol{cbd}_j^\infty$$

on $N\Lambda$, the assembly of Λ. It turns out that each of these is a binary pre-nucleus on $N\Lambda$, and have historical analogues in module theory. They are the

Gabriel Boyle

derivatives, respectively. The document [4] is an analysis of these devices.

An idea is to set

$$\boldsymbol{Gab}(j) = \boldsymbol{soc}_j^\infty \qquad \boldsymbol{Boy}(j) = \boldsymbol{cbd}_j^\infty$$

to obtain the first level pre-nuclei. Suppose now we are trying to rank Λ by $\boldsymbol{soc}$. If $\boldsymbol{soc}^\infty$ is the top of $N\Lambda$ then we have a measure for the whole of Λ. But what if $\boldsymbol{soc}^\infty = \boldsymbol{Gab}(\boldsymbol{id})$ is too small? We then move up a level and iterate $\boldsymbol{Gab}$. If $\boldsymbol{Gab}^\infty(\boldsymbol{id})$ is the top of $N\Lambda$ then we have a measure of $N\Lambda$ and of Λ. But what if $\boldsymbol{Gab}^\infty(\boldsymbol{id})$ is too small? We still have the higher level assemblies $N^2\Lambda, N^3\Lambda, \ldots$ we could use. However, that is 'work in progress', and as far as I am aware this idea has no historical antecedent.

The documents [5, 6, 7, 41] look at more specific topics. Perhaps I should mention [6] in which I show how that classical Gabriel dimension of a module M is determined entirely by the structure of the idiom $\mathcal{S}ub(M)$. This is not unknown but I believe I bring out some new aspects.

This document is item [1] in a series of *Notes*, [1] – [7], concerned with the use of lattices as a tool for analysing modules and, to some extent, spaces. This bibliography is for the whole of the series. Not all the items listed are needed for this document. I also have a series of documents on Frames (distributive idioms). I hope to release these soon.

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