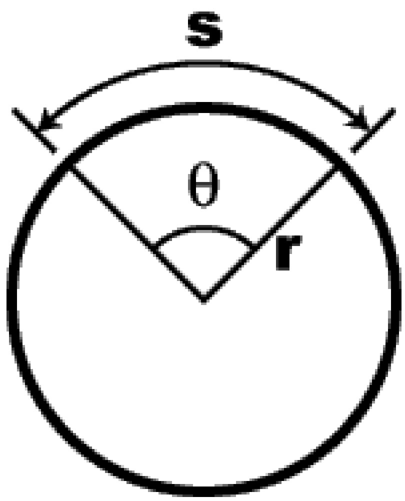
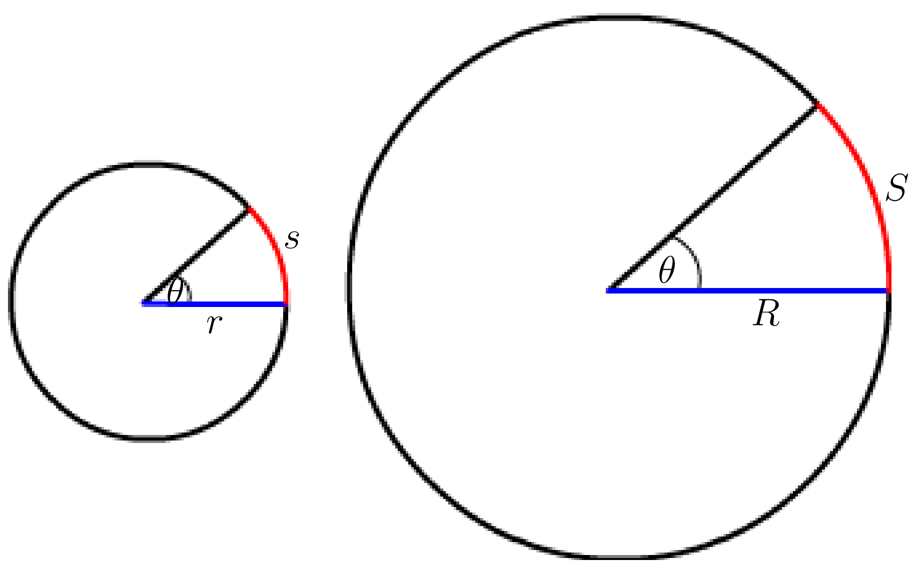
Radian Measure

Given any circle with radius r, if θ is a central angle of the circle and s is the length of the arc *sustained* by θ, we define the **radian measure** of θ by:

$\theta = \frac{s}{r}$

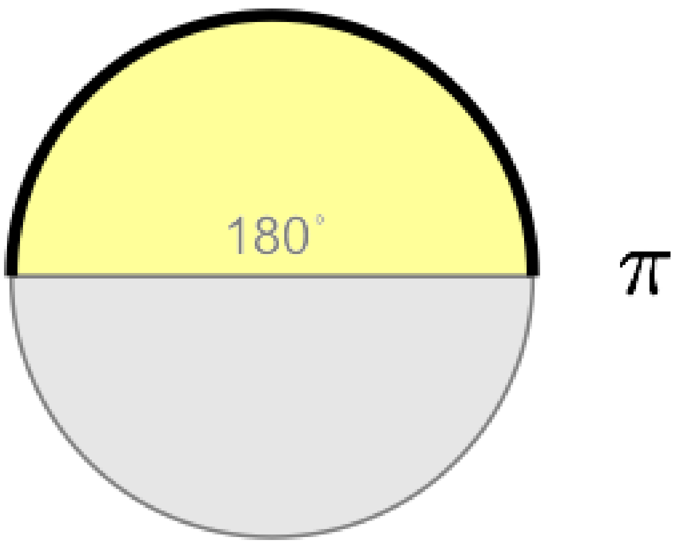


This defintion of an angle in radian measure is independent of the radius of the circle. A larger circle with a longer radius, R, will also sustain a longer arc, S, and the ratio, $\frac{S}{R}$ will be the same as $\frac{s}{r}$. In other words, $\theta = \frac{s}{r} = \frac{S}{R}$



For a semi-circle with radius r, its circumfrence is πr, so the radian measure of a semi-circle (a straight line) is

$\theta = \dfrac{\pi r}{r} = \pi$



Since a (semi-circle) straight angle has measure $180°$, π radian is equivalent to $180°$.

Given an angle measurement in degree, multiply that number by $\dfrac{\pi}{180°}$ to find the radian measure.

Given an angle measurement in radian, multiply that number by $\dfrac{180°}{\pi}$ to find the degree measure.

Example: What is $55°$ in radian?

Ans: $55° \cdot \dfrac{\pi}{180°} = \dfrac{55}{180}\pi = \dfrac{11}{36}\pi \approx 0.31\pi \approx 0.96$

Example: What is $45°$ in radian?

Ans: $45° \cdot \dfrac{\pi}{180°} = \dfrac{45}{180}\pi = \dfrac{1}{4}\pi = \dfrac{\pi}{4}$

Unless a decimal approximation is desired, we should always leave the number in exact format. That is, $\dfrac{11}{36}\pi$ or $\dfrac{11\pi}{36}$ is the most desirable way of writing the above angle measure in radian.

You should note that **radian measure is a (real) number**, and is more canonical than degree measure when used in working with mathematical functions of real numbers.

If an angle measurement is written *without* the $\circ$ symblo in the upper right, it is a radian measure. If the $\circ$ is present, it is a degree measure.

Distingush the difference between $1°$ and 1 (radian).

Example: What is 1° in radian?

Ans: $1^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{180} \approx 0.0056$ (rad)

Example: What is 1 (rad) in degree?

Ans: $1 \cdot \frac{180^\circ}{\pi} \approx 57.3^\circ$

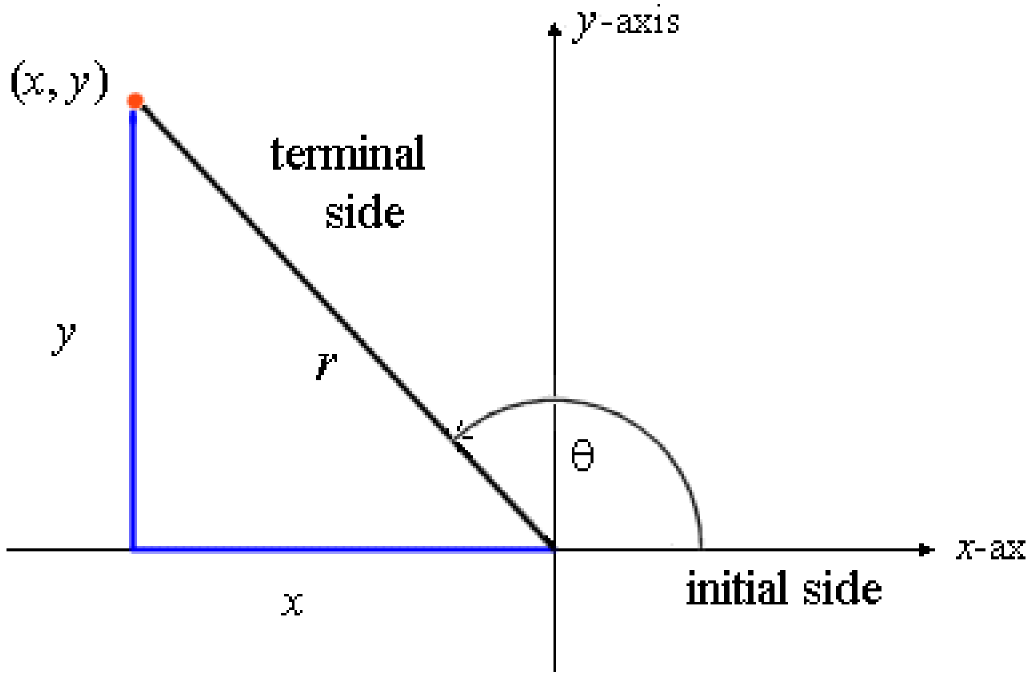
Example: What is $\frac{2\pi}{3}$ radian expressed in degree measure?

Ans: $\frac{2\pi}{3} \cdot \frac{180^\circ}{\pi} = \frac{360^\circ}{3} = 120^\circ$

Example: what is $\frac{\pi}{6}$ radian in degree?

Ans: $\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{180^\circ}{6} = 30^\circ$

With the Cartesian plane, we define an angle in **Standard Position** if it has its vertex on the origin and one of its sides (called the **initial side**) is always on the positive side of the $x-$axis. If we obtained the other side (Called the **Terminal Side**) of the angle via a counter-clockwise rotation, we have a positive angle. If the *terminal side* of the angle is obtained via a clockwise rotation, we have a negative angle.



Using this definition, it is possible to define an angle of any (positive or negative) measurement by recognizing how its terminal side is obtained.

E.g. The terminal side of $\frac{\pi}{4}$ is in the first quadrant.

E.g. The terminal side of $-\frac{\pi}{6}$ is in the fourth quadrant.

E.g. The terminal side of $\frac{2\pi}{3}$ is in the second quadrant.

E.g. The terminal side of $\frac{4\pi}{3}$ is in the third quadrant.

E.g. The terminal side of $\frac{16\pi}{7}$ is in the first quadrant.

E.g. The terminal side of π is the negative $x-$axis.

E.g. The terminal side of $-\frac{\pi}{2}$ is the negative $y-$axis.

Two angles are **co-terminal angles** if they have the same terminal side.

E.g. The two angles $\frac{\pi}{3}$ and $-\frac{5\pi}{3}$ are **co-terminal**.

Notice that if θ is any angle, then the angles $\theta + 2\pi, \theta + 4\pi, \theta + 6\pi, \theta + 8\pi, \cdots$ are all co-terminal angles

Given a **Circle with radius r centered at the origin** (The equation of this circle is $x^2 + y^2 = r^2$), we define the **terminal point** of an angle θ to be the point of intersection of the circle with the terminal side of θ.

E.g. The terminal point P of $\theta = \dfrac{\pi}{6}$ is in the first quadrant.

If the radius of the circle is $r = 1$, then the coordinate of this terminal point P is $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$

If the radius of the circle is $r = 3$, then the coordinate of this terminal point P is $\left(\dfrac{3\sqrt{3}}{2}, \dfrac{3}{2}\right)$

E.g. The terminal point of $\theta = \dfrac{\pi}{2}$ is on the positive side of the $y-$axis. If the radius of the circle is $r = 1$, then the coordinate of the terminal point of θ is $(0, 1)$.

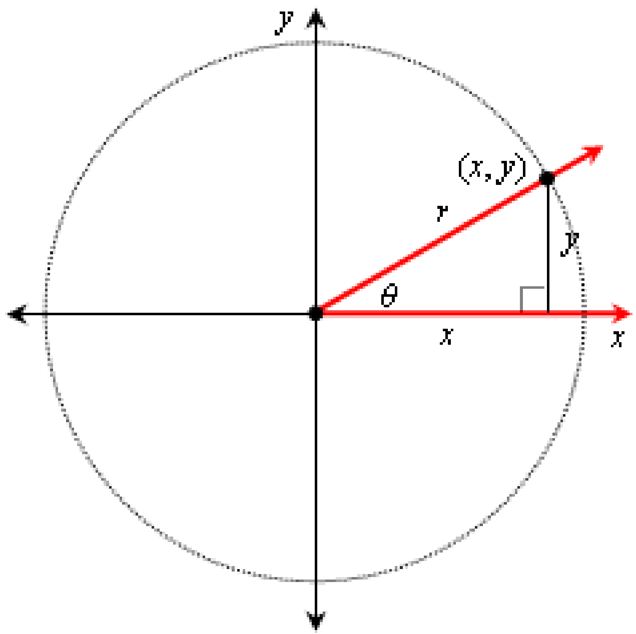
Note:

Any co-terminal angle has the same terminal point.

If θ is an angle with terminal point P, then any angle of the form $\theta + 2\pi, \theta + 4\pi, \theta + 6\pi, \theta + 8\pi, \cdots$ all have P as the terminal point.

To find the coordinate of the terminal point, we need to know the radius of the circle and the measurement of the angle.

Definition of the Trigonometric Functions Using circle:



Given an angle θ in standard position, on a circle with radius r, and the terminal point of θ is P, and (x, y) is the coordinate of P, we define the six trigonometric functions of θ by:

$\cos\theta = \frac{x}{r}$ This is the **cosine** function.

$\sin\theta = \frac{y}{r}$ This is the **sine** function.

$\tan\theta = \frac{y}{x}$ This is the **tangent** function.

$\sec\theta = \frac{r}{x}$ This is the **secant** function.

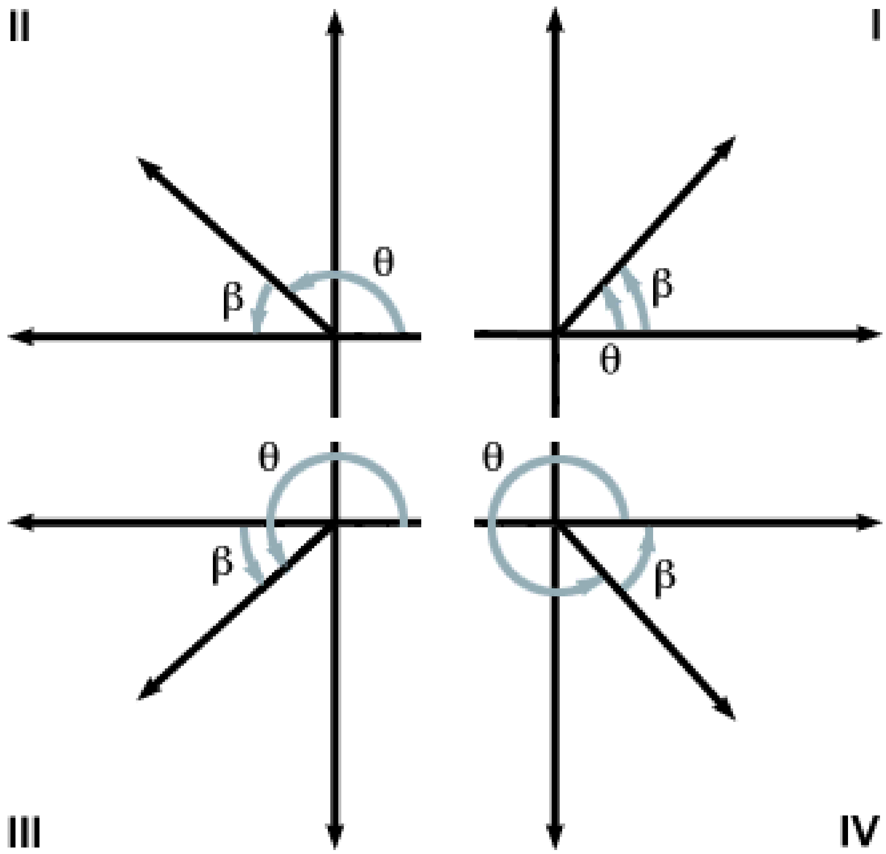
$\csc\theta = \frac{r}{y}$ This is the **cosecant** function.

$\cot\theta = \frac{x}{y}$ This is the **cotangent** function.

Sometimes, for convenience, we assume a circle of radius $r = 1$, called a **unit circle**, when defining or evaluating the values of the trigonometric functions.

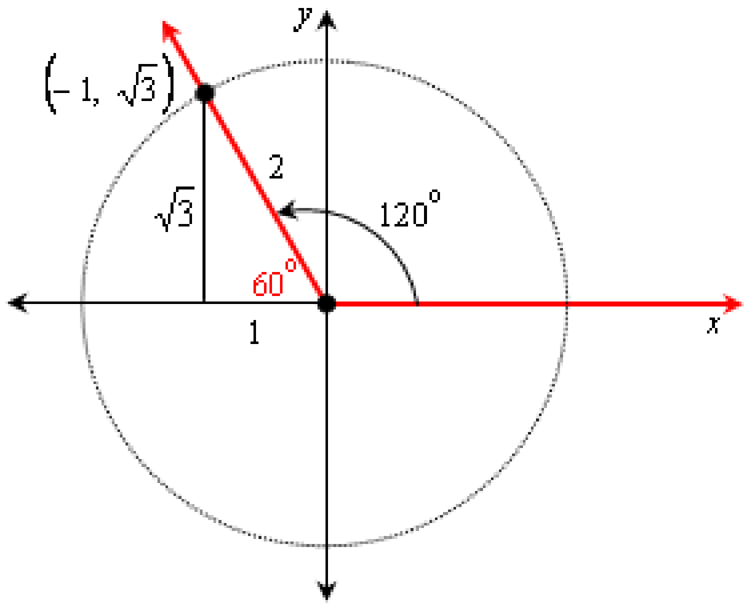
Note that we only need the measurement of the angle (we do *not* need to know the radius of the circle) to find the values of the six tri functions (because the r cancles out when we take the ratio of x or y with each other or with r.

Given an angle θ in standard position, the **reference angle** of θ is the **acute angle** that the terminal side of θ makes with the $x-$axis. To find the coordinate of the terminal point, it is most often easier to consider the length of the sides of the right triangle formed by the terminal side of θ and the $x-$axis with the reference angle of θ being one of the interior angles.

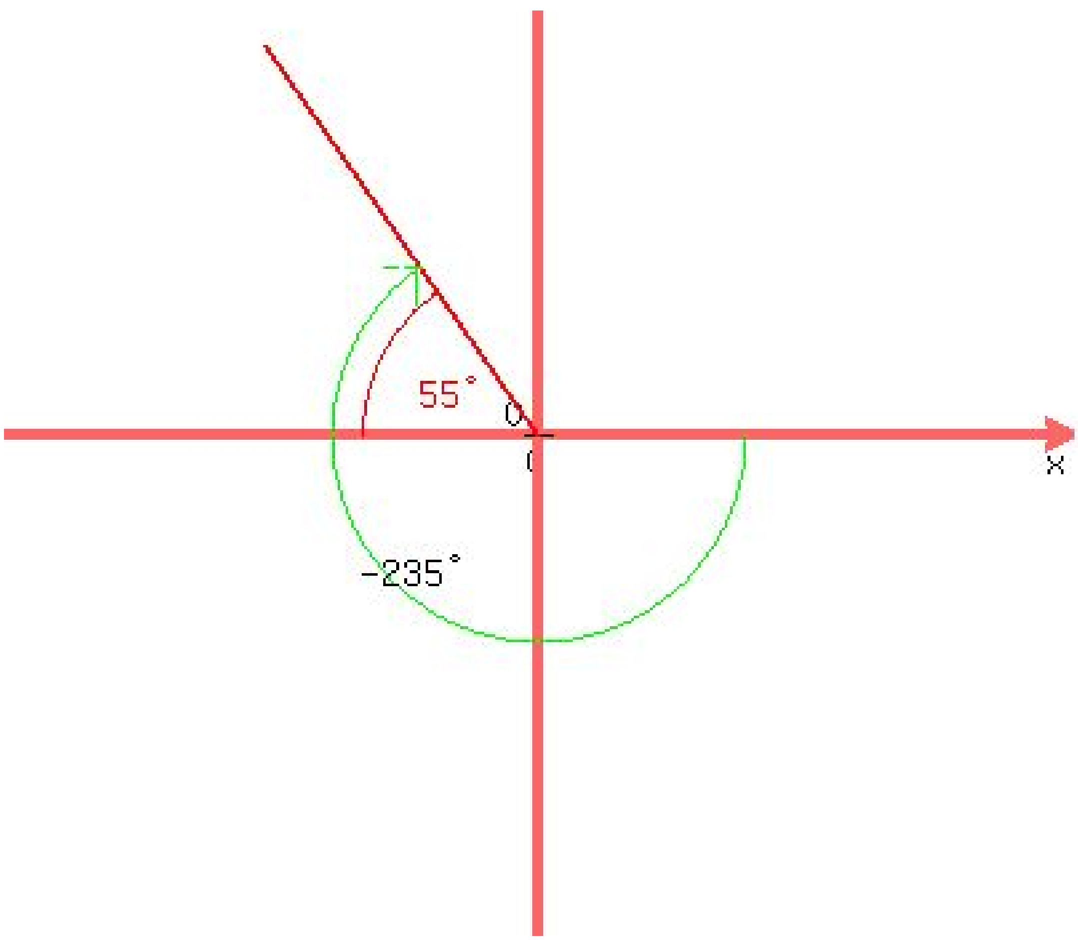


In the above picture, $\boldsymbol{\theta}$ is the angle in standard position and $\boldsymbol{\beta}$ is the reference angle. Note that the reference angle **must** be angle between 0 to $\dfrac{\pi}{2}$ (0 to 90°).

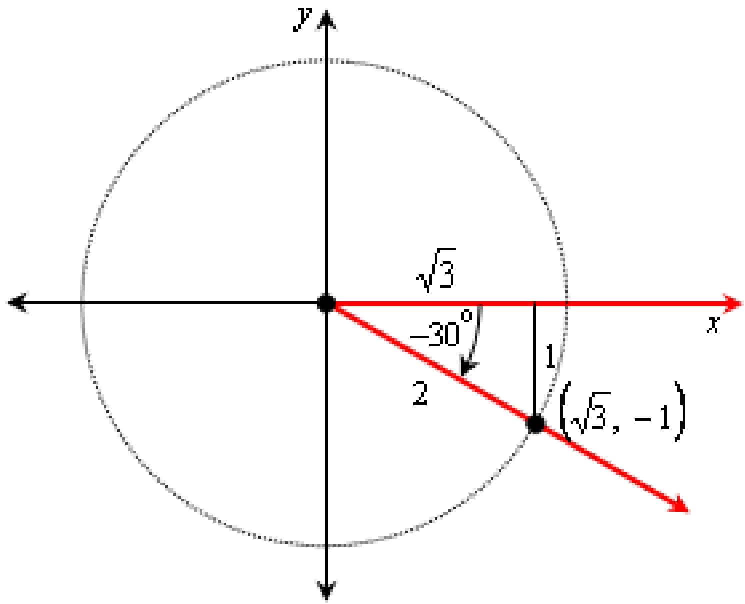
Example: The reference angle of a 120° angle is a 60° angle.



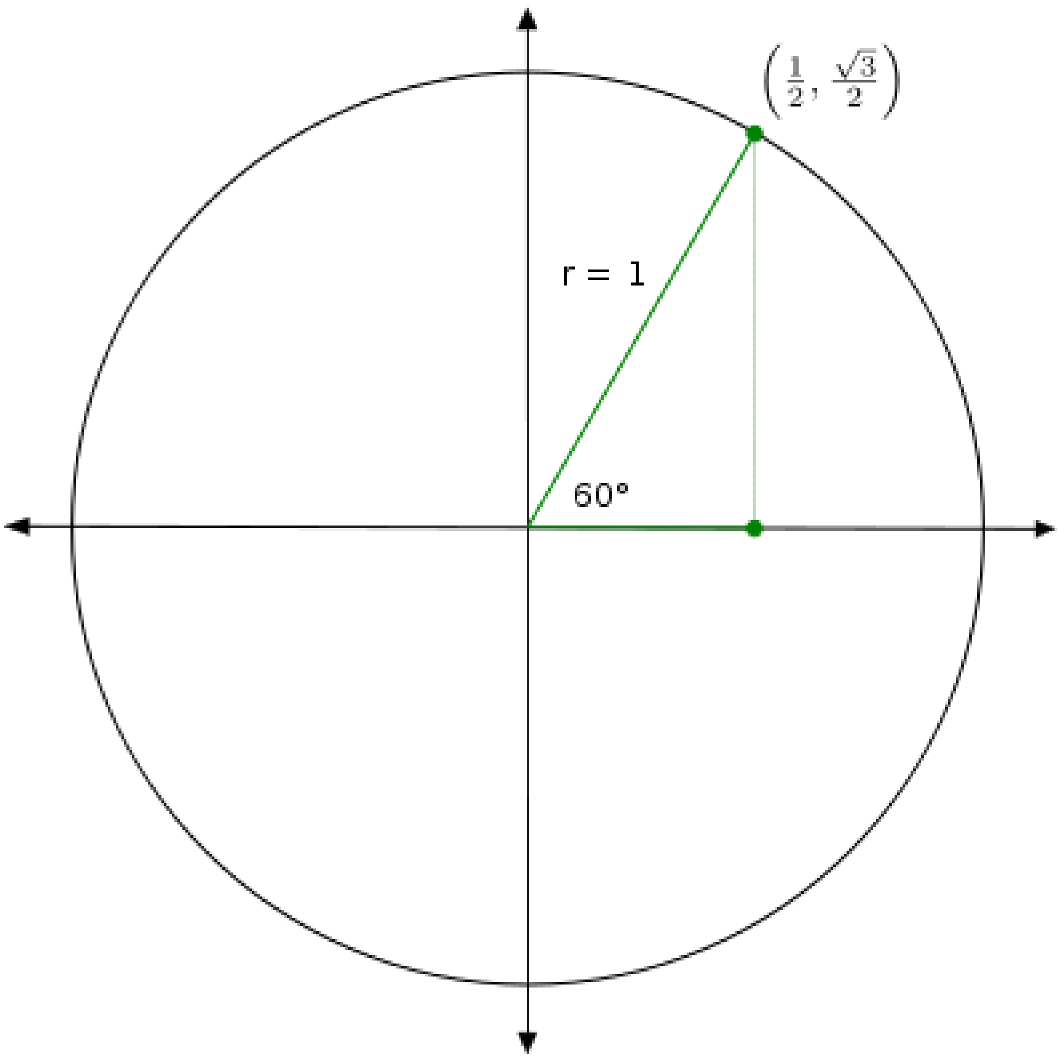
Example: The reference angle of a −235° angle is a 55° angle.



Example: The reference angle of a -30° angle is a 30° angle.



Example: For $\theta = \dfrac{\pi}{3}$, find the values of the six tri functions of θ.



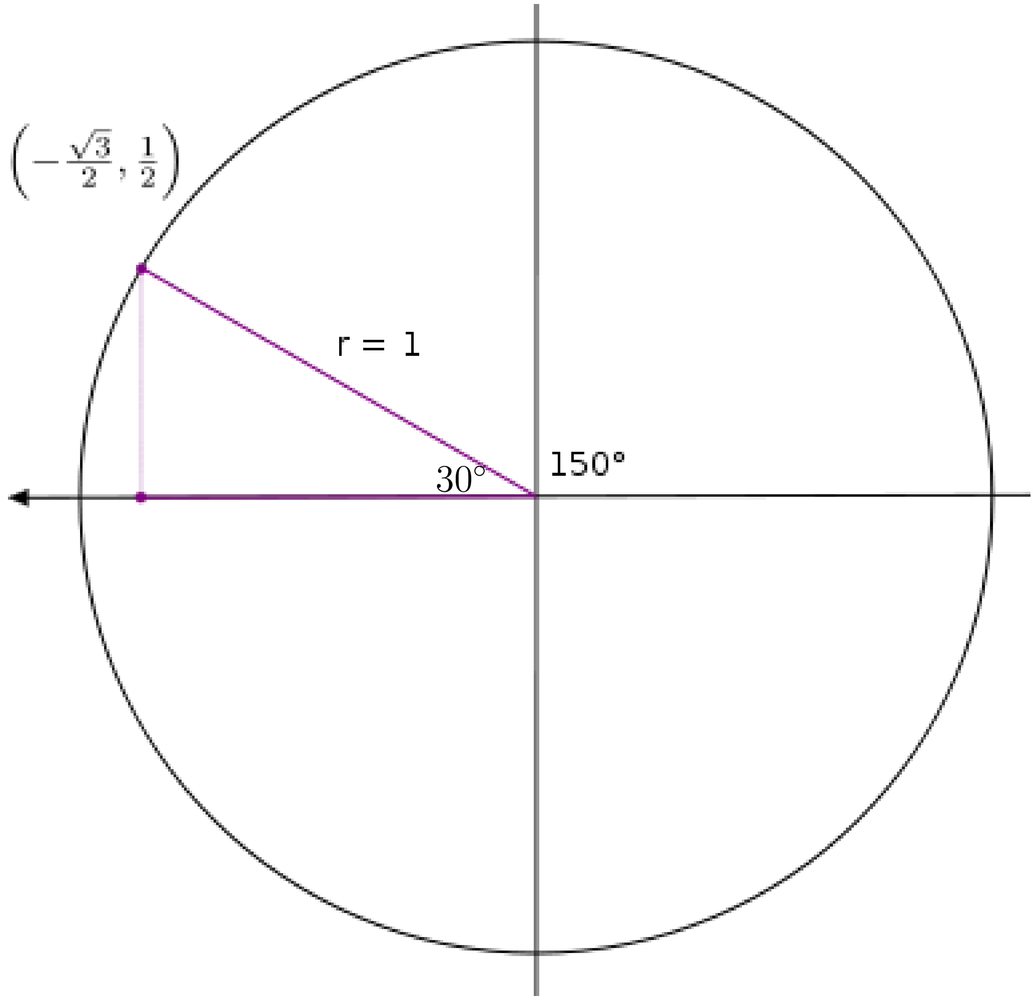
Notice that $\dfrac{\pi}{3} = 60°$. Assuming a circle of radius 1, the terminal point P of θ has coordinate $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$, so the values of the trig functions would be:

$$\cos\theta = \frac{x}{r} = \frac{\frac{1}{2}}{1} = \frac{1}{2} \qquad \sin\theta = \frac{y}{r} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$$

$$\tan\theta = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \qquad \cot\theta = \frac{x}{y} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{1}{\frac{1}{2}} = 2 \qquad \csc\theta = \frac{r}{y} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Example:



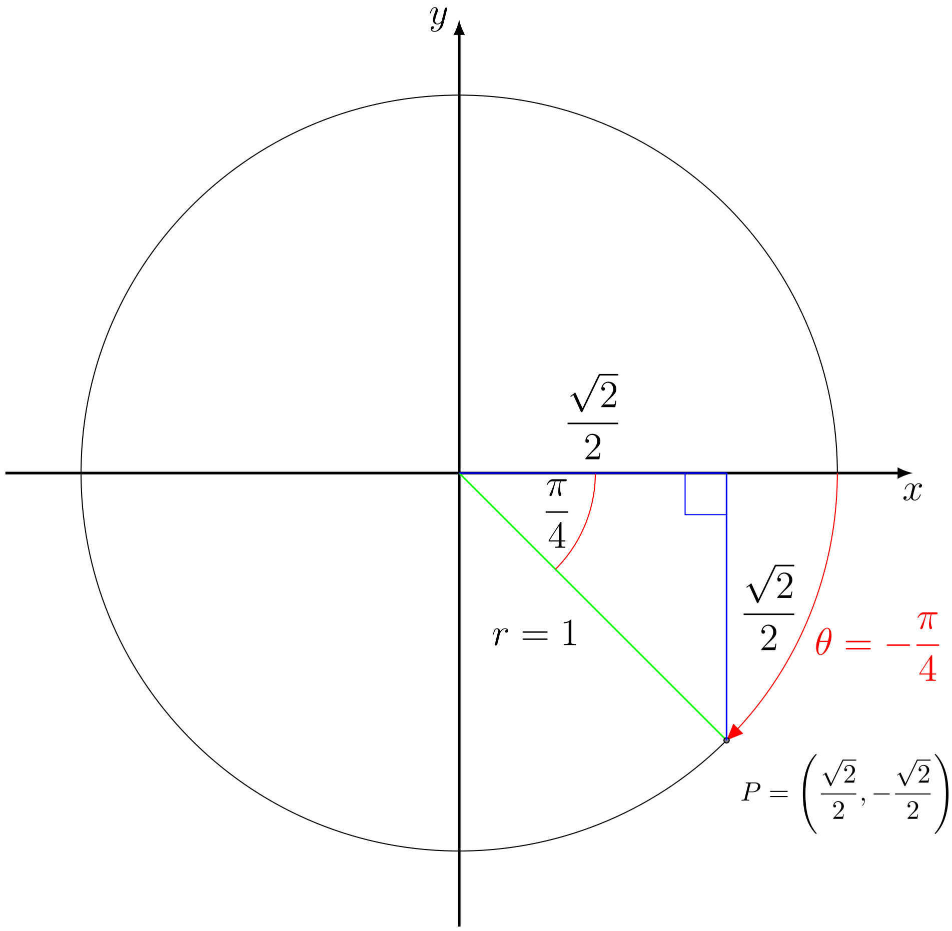
Assuming a unit circle, if $\theta = \frac{5\pi}{6} = 150°$, its *reference angle* is $\frac{\pi}{6} = 30°$, and the terminal point P of θ has coordinate $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Therefore,

$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$\tan\left(\frac{5\pi}{6}\right) = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

Example:



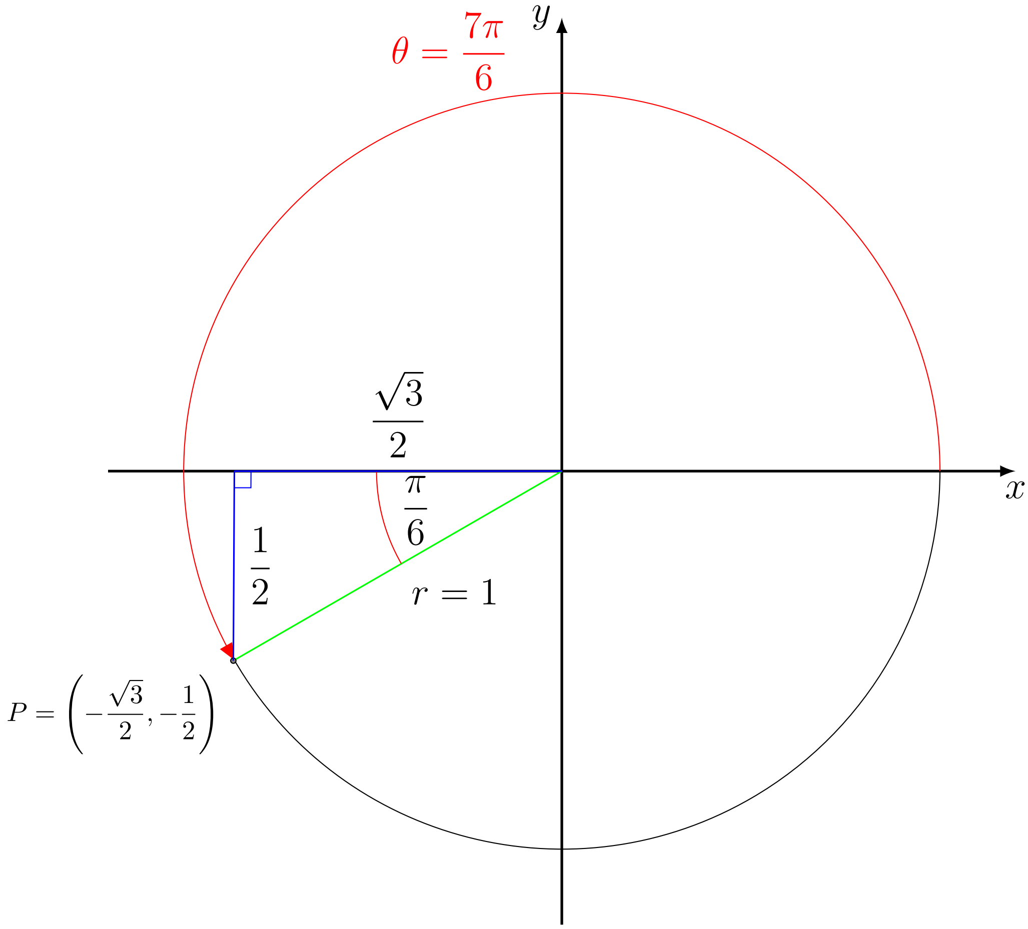
If we use a unit circle, when $\theta = -\frac{\pi}{4}$ (45°), then its **reference angle** is $\frac{\pi}{4}$, and the terminal point P of θ has coordinate $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Therefore,

$$\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\tan\left(-\frac{\pi}{4}\right) = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$$

Example:



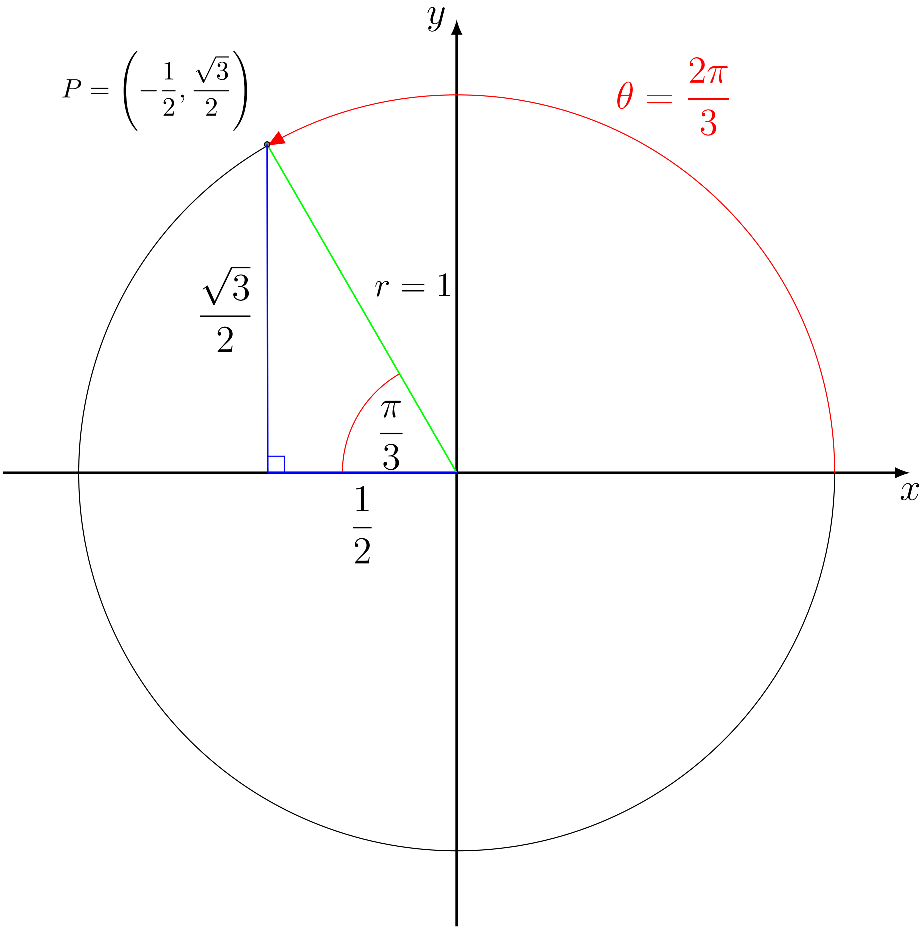
If we use a unit circle, when $\theta = \frac{7\pi}{6}$ (210°), then its **reference angle** is $\frac{\pi}{6}$ (30°), and the terminal point P of θ has coordinate $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$. Therefore,

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\tan\left(\frac{7\pi}{6}\right) = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

Example:



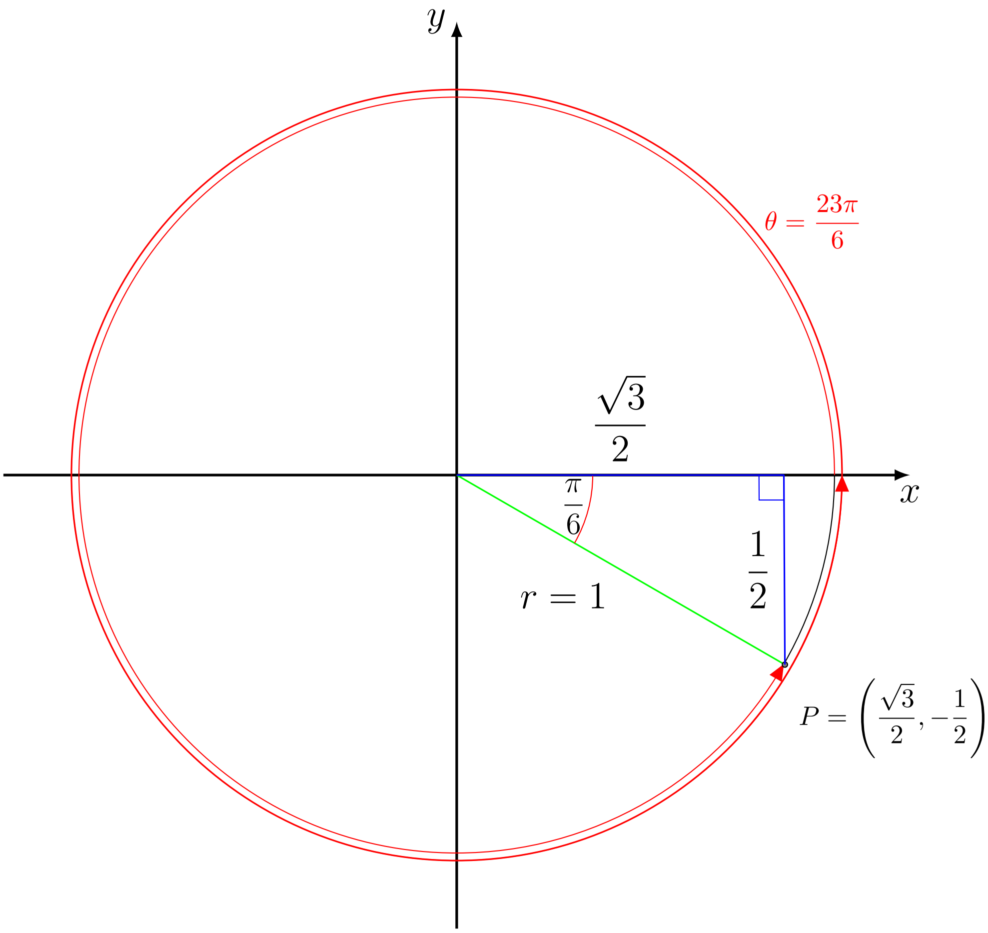
If we use a unit circle, when $\theta = \frac{2\pi}{3}$ (120°), then its **reference angle** is $\frac{\pi}{3}$ (60°), and the terminal point P of θ has coordinate $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore,

$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\tan\left(\frac{2\pi}{3}\right) = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

Example:



If we use a unit circle, when $\theta = \dfrac{23\pi}{6}$ (690°), then its **reference angle** is $\dfrac{\pi}{6}$, and the terminal point P of θ has coordinate $\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$. Therefore,

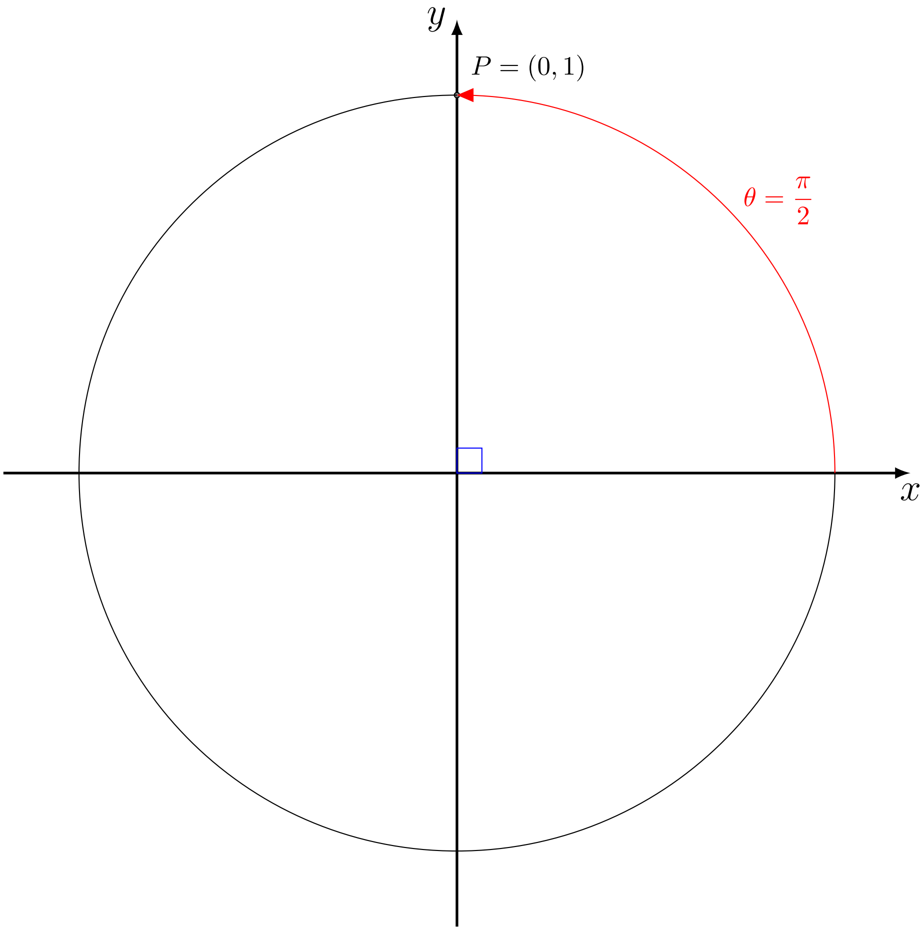
$$\sin\left(\frac{23\pi}{6}\right) = -\frac{1}{2}$$

$$\cos\left(\frac{23\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\tan\left(\frac{23\pi}{6}\right) = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

Notice that $\theta = \dfrac{23\pi}{6} = 2\pi + \dfrac{11\pi}{6}$ is greater than 2π, therefore when we draw the terminal point of θ in standard position, we completed one cycle (2π), then rotated another $\dfrac{11\pi}{6} = 330°$ to reach the terminal point P.

Example:



If we use a unit circle, when $\theta = \dfrac{\pi}{2}$ $(90°)$, then its terminal point lies on the $y-$axis and has coordinate $(0, 1)$. Therefore,

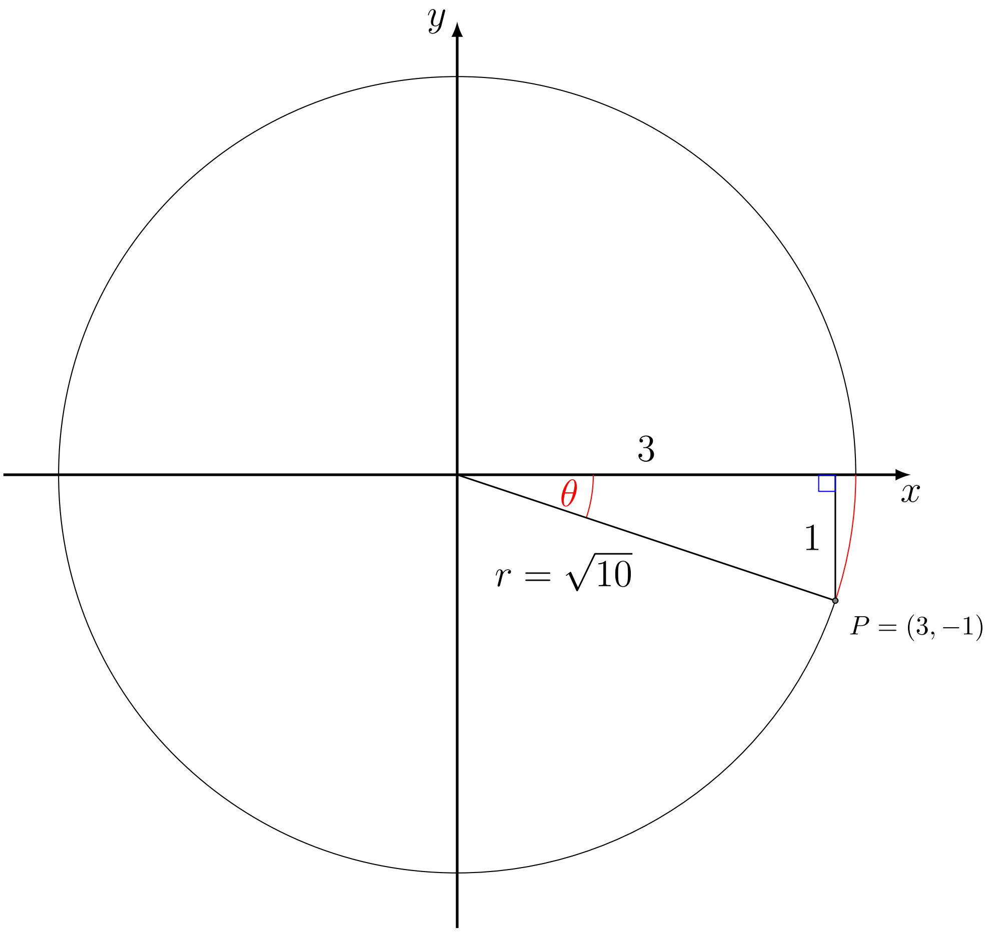
$\sin\left(\dfrac{\pi}{2}\right) = 1$

$\cos\left(\dfrac{\pi}{2}\right) = 0$

$\tan\left(\dfrac{\pi}{2}\right) = \dfrac{1}{0} =$ undefined.

While in all the examples we did we assumed a circle of radius 1, it is important for you to know that, in order to find the value of the trig functions on an angle θ, we can use a circle of **any** radius. We used 1 out of convenience, but our answer would be the same if we had used a circle of radius 2, or 0.5, or any other positive number.

Example: The terminal point of angle θ has coordinate $(3, -1)$, find the value of the six trigonometric functions of θ.



In this example, it would be more convenient to solve the problem if we use a circle of radius $\sqrt{10}$ (why?), then according to the picture,

$$\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{10}} = -\frac{1}{\sqrt{10}}$$

$$\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{10}}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{3} = -\frac{1}{3}$$

Notice that in solving this problem, we do not need to know the value of θ. In fact, we do not even know if θ is positive or negative. The only information we have (and it is the only information we need) is the coordinate of the terminal point of θ.

Example:

Find the *exact value* of the other trigonometric functions of θ if

$\sin\theta = -\dfrac{2}{5}$ and the terminal point of θ is in the third quadrant.

Ans: It would be more convenient if we use a circle of radius 5. Since the terminal point of θ is in the third quadrant, the coordinate of the terminal point P of θ is $P = (-\sqrt{21}, -2)$. We get:

$$\cos\theta = -\frac{\sqrt{21}}{5}$$

$$\tan\theta = \frac{-2}{-\sqrt{21}} = \frac{2}{\sqrt{21}}$$

$$\csc\theta = \frac{1}{\sin\theta} = -\frac{5}{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{\sqrt{21}}$$

$$\cot\theta = \frac{-\sqrt{21}}{-2} = \frac{\sqrt{21}}{2}$$

Fundamental Properties of The Trigonometri Functions:

sin and csc are reciprocal functions of each other, that is:

$$\csc(x) = \frac{1}{\sin(x)}$$

cos and sec are reciprocal functions of each other:

$$\sec(x) = \frac{1}{\cos(x)}$$

tan and cot are reciprocal functions of each other:

$$\cot(x) = \frac{1}{\tan(x)}$$

In addition, tan is the quotient of sin and cos:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

The following equation can be derived from the pythagorean theorem, and is called the **Pythagorean Identity**: For all real numbers x, we have:

$$\sin^2(x) + \cos^2(x) = 1$$

Note: $\sin^2(x)$ means $(\sin x)^2$. In general, to represent $(\sin x)^n$, we write $\sin^n(x)$. This notation applies to other tri functions too.

More properties of the tri functions:

sin (and its reciprocal, csc), is an **odd** function, that is,

$\sin(-x) = -\sin(x)$ for all real numbers x

cos (and its reciprocal, sec), is an **even** function, that is,

$\cos(-x) = \cos(x)$ for all real numbers x

The product (and quotient) of an odd function with an even function is odd, so tan and cot are both odd functions.

Periodic Function

A function f is said to be **periodic** is there exists a positive number p such that $f(x+p) = f(x)$ for all real number x in the domain of f. The smallest of such number is called the *period* of f.

All six of the tri functions are *periodic*. Intuitively, a periodic function repeats itself in behavior. The period of sin, cos, csc, and sec is 2π, and the period of tan and cot is π. Therefore, we have:

$\sin(x+2\pi) = \sin(x)$ for all x in the domain of sin

$\cos(x+2\pi) = \cos(x)$ for all x in the domain of cos

$\tan(x+\pi) = \tan(x)$ for all x in the domain of tan

$\csc(x+2\pi) = \csc(x)$ for all x in the domain of csc

$\sec(x+2\pi) = \sec(x)$ for all x in the domain of sec

$\cot(x+\pi) = \cot(x)$ for all x in the domain of cot

Domain and Range of Tri Functions

The domain of both the sin and cos functions is all real numbers. I.e., sin and cos are defined everywhere.

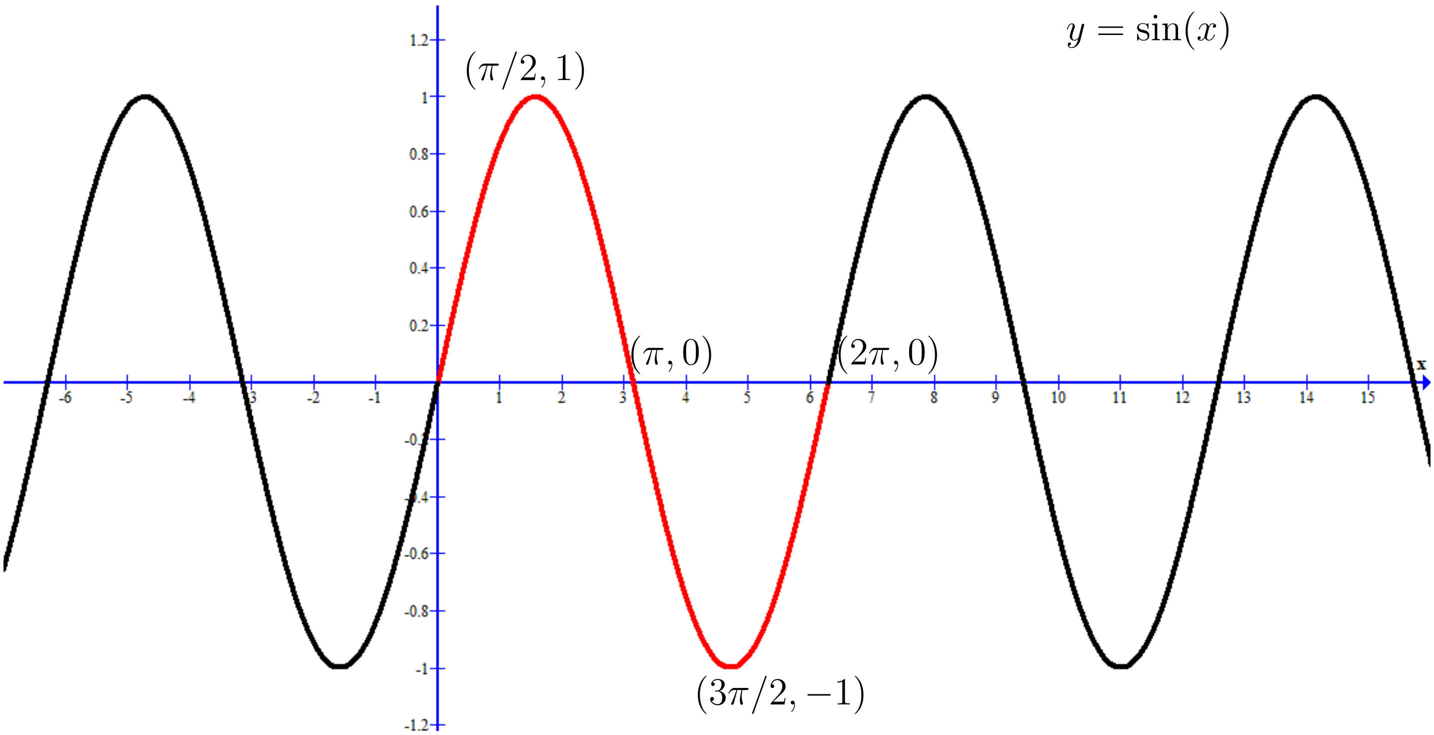
The range of both sin and cos is the closed interval $[-1, 1]$

The domain of the other four trigonometric functions are not as easily determined, and we will discuss about that later.

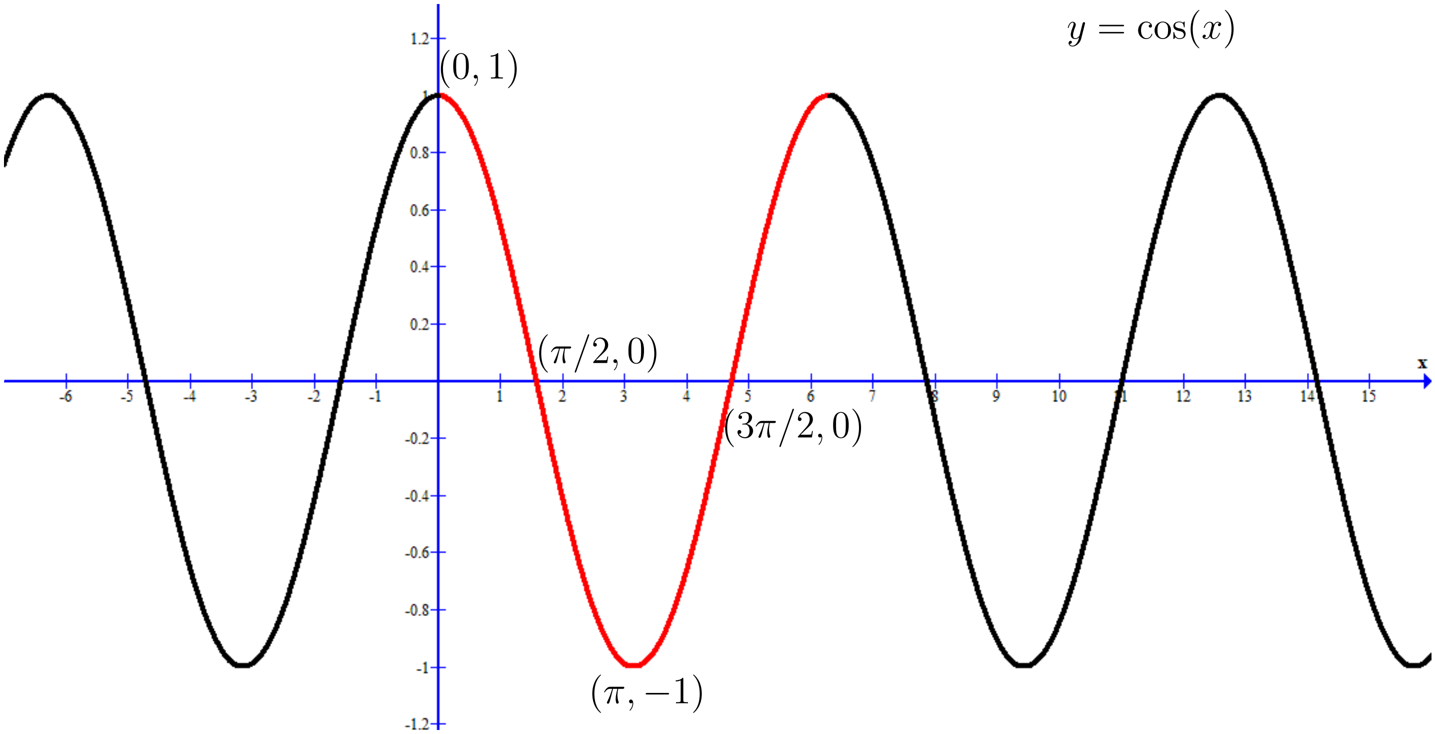
Graphing

To draw the graph of $f(x) = \sin(x)$, we start from $x = 0$ and go counter-clockwise (positive x) for a 2π period. Notice that $\sin(x)$ moves from 0 up to 1 when x moves from 0 to $\frac{\pi}{2}$, then $\sin(x)$ moves from 1 back to 0 as x moves from $\frac{\pi}{2}$ moves to π, then $\sin(x)$ becomes negative and moves from 0 to -1 as x moves from π to $\frac{3\pi}{2}$, and $\sin(x)$ moves from -1 back to 0 as x moves from $\frac{3\pi}{2}$ to 2π, completing the cycle.

Once one cycle of the sin graph is drawn, one need only to *copy and paste* this shape onto the rest of the x-axis to complete the graph of sin, as we know that sin is periodic.



A similar technique can be used to draw the basic graph of the cosine function.



$y = \cos(x)$
$(0, 1)$
$(\pi/2, 0)$
$(3\pi/2, 0)$
$(\pi, -1)$
x
1.2
1
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-6
-5
-4
-3
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Graphing Trigonometric Functions in General Form

Graph $f(x) = A\sin(Bx + C) + k$, where A, B, C, and k are real constants, and $B > 0$

We want to form a rectangle (the *envelop*) which encloses one cycle of this function. We need the following information of the rectangle:

1. The **Amplitude** (the distance from the middle of the envelop to the top or bottom) of the rectangle is $|A|$.

2. The *starting point* of the envelop is at $-\dfrac{C}{B}$ (This is called the **phase shift**)

3. The **period,** p**,** of the function (length of the rectangle) is $\dfrac{2\pi}{|B|}$

The **frequency,** f**,** of a periodic function is the reciprocal of its period, that is, $f = \dfrac{|B|}{2\pi}$

4. The **end point** of the rectangle is at *starting point* + *period*, i.e. *end point* $= -\dfrac{C}{B} + \dfrac{2\pi}{|B|}$

5. The **intersections** of the function f with the middle line (x-axis if $k = 0$) occurs at the points:

starting point + $\dfrac{p}{4}$

starting point + $\dfrac{p}{2}$

starting point + $\dfrac{3p}{4}$

where p is the period.

6. The rectangle is moved **up** k units if $k > 0$ and moved **down** $|k|$ units if $k < 0$.

E.g. Graph $f(x) = 3\cos(2x - 1) - 2$

Ans:

We have $A = 3$, $B = 2$, $C = -1$, and $k = -2$.

$|A| = |3| = 3$. This is the amplitude.

Setting $2x - 1 = 0$ and solve for x, we get $x = \frac{1}{2}$. This is our starting point. (Phase shift).

$p = \frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$. The period of f is π.

End point $=$ *starting point* $+$ *period* $= \frac{1}{2} + \pi$.

$\frac{p}{4} = \frac{\pi}{4}$, so the intersections of the function with the middle line occurs at $\frac{1}{2} + \frac{\pi}{4}$, $\quad \frac{1}{2} + \frac{\pi}{2}$, $\quad \frac{1}{2} + \frac{3\pi}{4}$

$k = -2$, so the graph is moved 2 units *down*.

Don't forget to copy and paste this rectangle to the whole $x-$axis.

Graphing Tangent and Cotangent functions:

The domain of tangent is all real numbers x such that $x \neq \dfrac{\pi}{2} + n\pi$, where n is any integer.

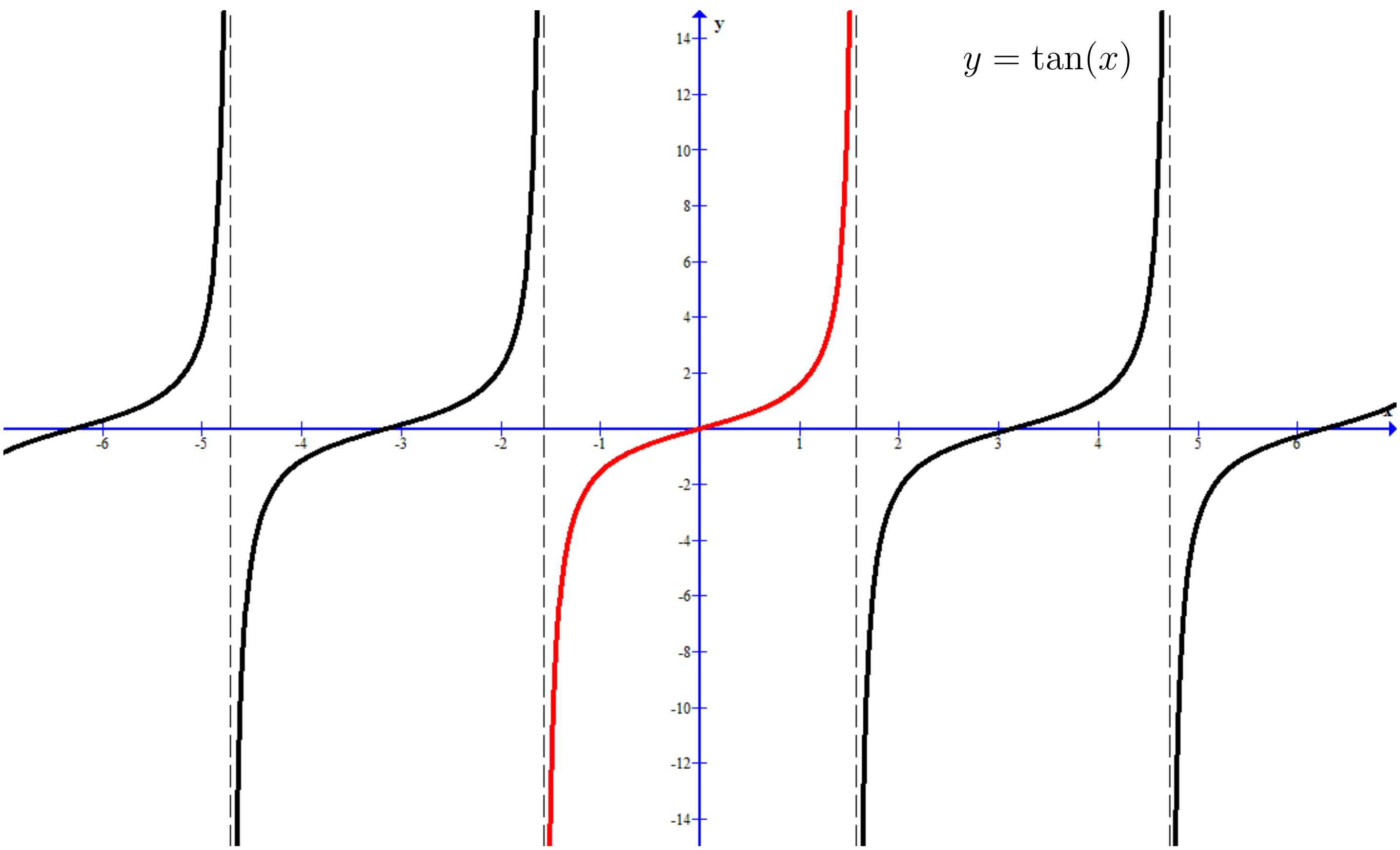
Suppose θ is an angle in standard position, $P = (x, y)$ is the terminal point of θ, we want to find out the behavior of tangent for θ between 0 to π.

$\tan(0) = 0$, and as θ moves from 0 to $\dfrac{\pi}{2}$, the value of y becomes larger and larger while the value of x becomes smaller and smaller, so the ratio $\dfrac{y}{x}$ becomes larger and larger. In fact, $\dfrac{y}{x} \to \infty$ as $\theta \to \dfrac{\pi}{2}$

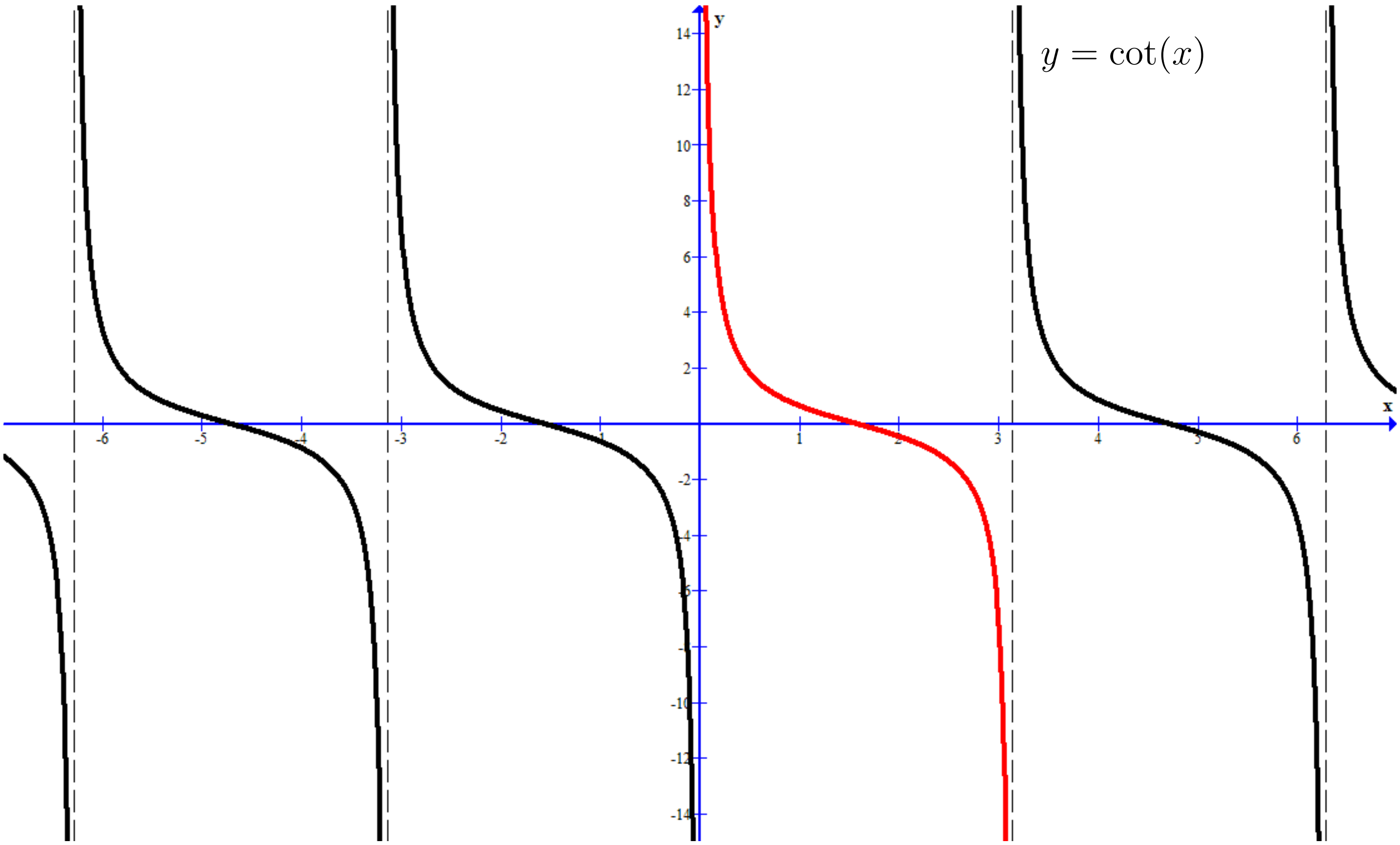
At $\theta = \dfrac{\pi}{2}$, $x = 0$, so $\tan(\theta) = \dfrac{y}{x}$ is undefined. Tangent has a **vertical asymptote** at the point $\theta = \dfrac{\pi}{2}$.

As θ moves from $\dfrac{\pi}{2}$ toward π, y becomes smaller and smaller in value, as x increases in magnitude. But notice that x is now negative, so $\dfrac{y}{x}$ is now a negative number, and it becomes closer and closer to zero as θ moves toward π. At $\theta = \pi$, $y = 0$ and $\tan(\theta) = \dfrac{y}{x} = 0$.

It is generally more natural to consider that one cycle of tangent starts from $-\dfrac{\pi}{2}$ and ends at $\dfrac{\pi}{2}$, as this gives us one connected piece of the tangent function.



We can also draw the graph of the cotangent function in a similar way:



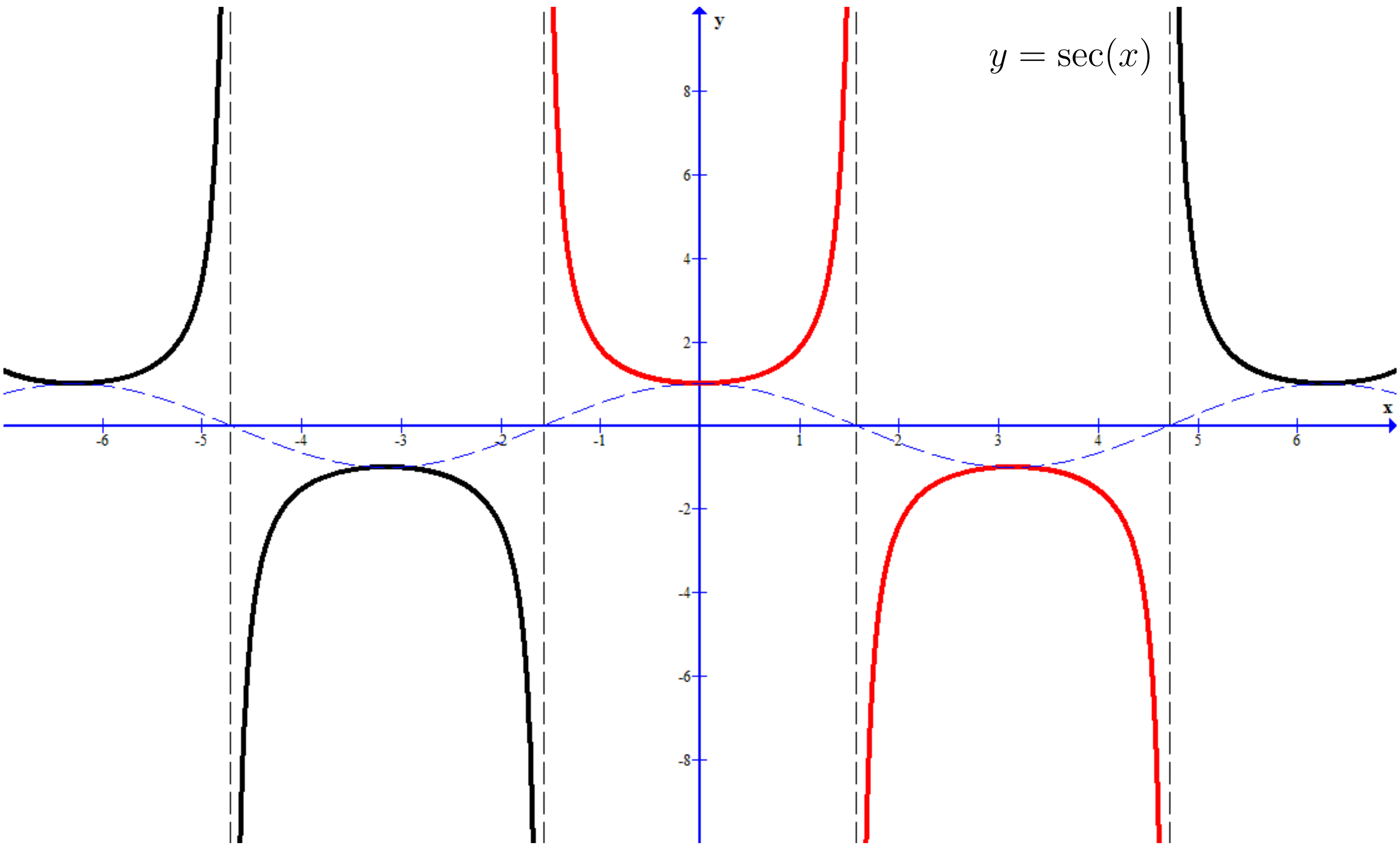
For a general graph of the form: $f(x) = A\tan(Bx + C)$, we set $Bx + C = -\frac{\pi}{2}$ to solve for x, this again gives us the *starting point* of one cycle of tangent.

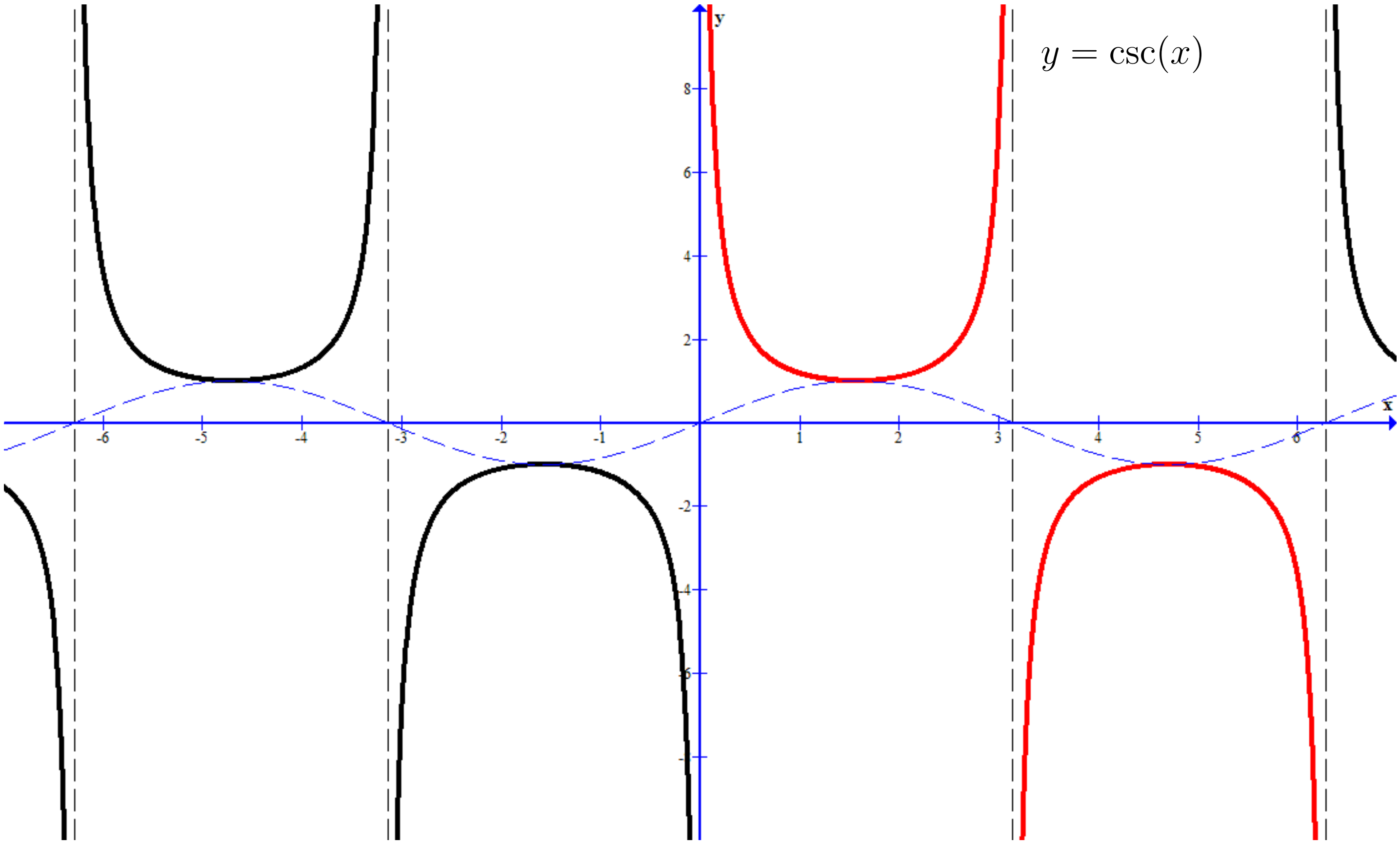
The *period* of f is $\frac{\pi}{|B|}$. Notice that we are taking π divided by $|B|$, not 2π, as the natural period for tan is π.

Tangent does not have an amplitude since it does not have a lowest or highest point. A determines the *vertical stretch* of the function.

To draw the graph of $f(x) = A\csc(Bx+C)$ or $f(x) = A\sec(Bx+C)$, we draw the corresponding sin or cos function and use that as a guideline. The x intercepts of sin and cos will be where the vertical asymptotes of the csc and cos is. Notice that the domain of csc is all real numbers x where $\sin(x) \neq 0$, and since $\sin(n\pi) = 0$ for all integer n, the domain of csc is all real numbers x such that $x \neq n\pi$, n an integer.

Similarly, the domain of sec is all real numbers x such that $x \neq \dfrac{\pi}{2} + \pi n$





$y = \csc(x)$
y
x
8
6
4
2
-2
-4
-6
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6

Trigonometric Identities:

We have seen some of the identities of the trigonometri functions. We will introduce more:

Fundamental Identities:

(*) $\csc(x) = \dfrac{1}{\sin(x)}$

(*) $\sec(x) = \dfrac{1}{\cos(x)}$

(*) $\cot(x) = \dfrac{1}{\tan(x)}$

(*) $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

(*) $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

Even Odd properties:

(*) $\sin(-x) = -\sin(x)$

(*) $\tan(-x) = -\tan(x)$

(*) $\cos(-x) = \cos(x)$

Pythagorean Identities:

(*) $\sin^2(x) + \cos^2(x) = 1$

$\tan^2(x) + 1 = \sec^2(x)$

$1 + \cot^2(x) = \csc^2(x)$

Cofunction Identities:

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos(x)$

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin(x)$

Sum (and difference) of Angles Identities:

(*) $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

(*) $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\tan(x+y) = \dfrac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$

(*) $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

(*) $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$$\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

Double Angle Identities:

(*) $\sin 2x = 2\sin x\cos x$

(*) $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Half Angle Identities:

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

Power Reduction Identity :

$$\sin^2 x = \frac{1}{2}\left(1-\cos(2x)\right)$$

$$\cos^2 x = \frac{1}{2}\left(1+\cos(2x)\right)$$

Product to Sum and Sum to Product Identities:

$$\sin x\cos y = \frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$

$$\sin x\sin y = \frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$$

$$\cos x\cos y = \frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Verifying Trig Identities:

Using the known trigonometric identities, we can transform a trigonometric expression written in one form into another. We need to use rules of algebra and other known tri identities to *verify* that the two sides of the equal sign are equal to each other.

It is important to note that, while verifying a tri-identity, one **may not assume** the identity as equal. Instead, you must start with one side of the identity (either the left or right hand side is fine) and, by using correct algebra and other tri identites, change the expression to look the same as the other side. You **may not** work the expression as if you are solving an equation.

Example

Verify

$$\frac{\sin t}{\csc t} + \frac{\cos t}{\sec t} = 1$$

Ans: We may start with either the left or right hand side, but for this example, the right hand side is too simple and does not allow for any clue as to where we should go, it is easier that we start with the left hand side:

$$\text{L.H.S} = \frac{\sin t}{\csc t} + \frac{\cos t}{\sec t} = \sin t \cdot \frac{1}{\csc t} + \cos t \cdot \frac{1}{\sec t}$$

$$= \sin t \sin t + \cos t \cos t = \sin^2 t + \cos^2 t = 1 = \text{R.H.S.}$$

Example: Verify

$$\frac{1 - \cos x}{1 + \cos x} = \frac{\sec x - 1}{\sec x + 1}$$

Ans: Both sides are equally complicated. Let's start with the left hand side:

$$\text{L.H.S.} = \frac{1 - \cos x}{1 + \cos x} = \frac{(1 - \cos x)(\sec x)}{(1 + \cos x)(\sec x)} = \frac{\sec x - \cos x \sec x}{\sec x + \cos x \sec x}$$

$$= \frac{\sec x - 1}{\sec x + 1} = \text{R.H.S.}$$

An example of an **incorrect** way of trying to "*verify*" the above identity would be something like this:

$$\frac{1 - \cos x}{1 + \cos x} = \frac{\sec x - 1}{\sec x + 1}$$

$$(1 - \cos x)(\sec x + 1) = (\sec x - 1)(1 + \cos x)$$

$$\sec x + 1 - \cos x \sec x - \cos x = \sec x + \sec x \cos x - 1 - \cos x$$

$$\sec x + 1 - 1 - \cos x = \sec x + 1 - 1 - \cos x$$

$\sec x - \cos x = \sec x - \cos x$

In doing the *cross multiply* and whatever steps follow, you have already *assumed* that the two sides are equal to each other. This is logically invalid and does *not* constitute a valid argument for varifying anything. (In logic, this is called a *circular argument*)

Example: Verify the identity:

$(\sin x + \cos x)^2 = 1 + \sin 2x$

Ans: We start with the left hand side:

L.H.S $= (\sin x + \cos x)^2 = \sin^2 x + 2 \sin x \cos x + \cos^2 x$

$= \sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + 2 \sin x \cos x = 1 + \sin 2x =$ R.H.S.

Example: Verify the identity:

$$\cot x - \tan y = \frac{\cos(x + y)}{\sin x \cos y}$$

This time, it is easier to start with the right hand side:

$$\text{R.H.S.} = \frac{\cos(x + y)}{\sin x \cos y} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y}$$

$$= \frac{\cos x \cos y}{\sin x \cos y} - \frac{\sin x \sin y}{\sin x \cos y} = \frac{\cos x}{\sin x} - \frac{\sin y}{\cos y} = \cot x - \tan y = \text{L.H.S.}$$

Inverse Trigonometric Functions:

The *inverse sine function*, denoted by $f(x) = \arcsin(x)$ or $f(x) = \sin^{-1}(x)$ is defined by:

$y = \sin^{-1}(x)$ if and only if $\sin(y) = x$ **and** $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

I.e., the *range* of $f(x) = \arcsin(x)$ is all real numbers y such that $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

The *inverse cosine function*, denoted by $f(x) = \arccos(x)$ or $f(x) = \cos^{-1}(x)$ is defined by:

$y = \cos^{-1}(x)$ if and only if $\cos(y) = x$ **and** $0 \leq y \leq \pi$

I.e., the *range* of $f(x) = \arccos(x)$ is all real numbers y such that $0 \leq y \leq \pi$

The *inverse tangent function*, denoted by $f(x) = \arctan(x)$ or $f(x) = \tan^{-1}(x)$ is defined by:

$y = \tan^{-1}(x)$ if and only if $\tan(y) = x$ **and** $-\frac{\pi}{2} < y < \frac{\pi}{2}$

I.e., the *range* of $f(x) = \arctan(x)$ is all real numbers y such that $-\frac{\pi}{2} < y < \frac{\pi}{2}$

The inverse trigonometric functions are the inverse functions of the trigonometric functions. They *undo* the effect of the original tri functions. Keep in mind that since a trigonometric function takes an angle measurement (in radian) as input and gives a real number as output, an inverse tri function (such as arccos or arcsin takes a real number as input and produces an angle measurement (in radian) as output.

The *domain* of the *sine* function is all real numbers. Ideally, to *undo* the effect of sine, we would like the *range* of the $f(x) = \arcsin(x)$ function to be all real numbers too. Unfortunately, since *sine* is **NOT** a one-to-one function, this is not possible.

More precisely, since:

$\sin(0) = 0$

$\sin(\pi) = 0$

$\sin(2\pi) = 0$

We would like

$\arcsin(0) = 0$

$\arcsin(0) = \pi$

$\arcsin(0) = 2\pi$

But this would make $f(x) = \arcsin(x)$ fails to be a function. As a result, we need to *restrict* the range for arcsin such that, within this range, arcsin returns only one output for each input. We make the choice based on convenience. As a result, we *chose* the range of arcsin to be $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Using similar guidelines, we *defined* the range of the other inverse trigonometric functions the way we did.

Example:

Find the value of $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Ans: Let $\theta = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, then by definition, $\sin(\theta) = -\frac{\sqrt{2}}{2}$ **and** $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. The only angle that satisfies both of these requirements is $\theta = -\frac{\pi}{4}$.

Therefore, $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

Example:

Find $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Ans: Let $\theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, by definition, we want

$\cos(\theta) = -\frac{\sqrt{3}}{2}$ **and** $0 \leq \theta \leq \pi$. The only angle that satisfies both of these requirements is $\theta = \frac{5\pi}{6}$.

Therefore, $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

Example:

Find $\arcsin\left(\sin\left(\frac{31\pi}{4}\right)\right)$

Ans: $\frac{31\pi}{4} = 7\pi + \frac{3\pi}{4}$.

Since sine is periodic, $\sin\left(7\pi + \frac{3\pi}{4}\right) = \sin\left(\pi + \frac{3\pi}{4}\right)$.

Let $\theta = \arcsin\left(\sin\left(\frac{31\pi}{4}\right)\right)$. We want:

$\sin(\theta) = \sin\left(\pi + \frac{3\pi}{4}\right)$ **and** $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

The terminal side of $\left(\pi + \frac{3\pi}{4}\right)$ is in the fourth quadrant, therefore, $\sin\left(\pi + \frac{3\pi}{4}\right)$ is negative. We need $\sin(\theta)$ to be negative **and** $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. This tells us that θ must be an angle in the fourth quadrant and the only angle θ that satisfies this requirement is $\theta = -\frac{\pi}{4}$.

Example:

Find $\arccos(\cos(23\pi + 3))$

Ans: $(23\pi + 3 = 22\pi + \pi + 3)$

Since cosine is periodic, $\cos(23\pi + 3) = \cos(\pi + 3)$

Let $\theta = \arccos(\cos(23\pi + 3))$. We want:

$\cos(\theta) = \cos(\pi + 3)$ **and** $0 \leq \theta \leq \pi$

The terminal side of $(\pi + 3)$ is in the fourth quadrant (why?), which makes $\cos(\pi + 3)$ positive. Therefore, $\cos(\theta)$ must also be positive **and** $0 \leq \theta \leq \pi$. The terminal side of θ must be in the first quadrant and the only angle θ that satisfies this requirement is $\theta = \pi - 3$ (why)?

Example:

Find $\cos\left(\arctan\left(-\frac{5}{7}\right)\right)$

Ans: Let $\theta = \arctan\left(-\frac{5}{7}\right)$, we have

$\tan(\theta) = -\frac{5}{7}$ **and** $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. If we draw θ in standard position, the terminal side of θ is in quadrant IV. Using the Pythegorean Theorem, we see that the coordinate of the terminal point **P** of θ is $P = (7, -5)$ where the radius of the circle we are using is $r = \sqrt{74}$.

Using this information, $\cos\left(\arctan\left(-\frac{5}{7}\right)\right) = \cos\theta = \frac{7}{\sqrt{74}}$

Example:

Find $\sin\left(\arccos\left(-\frac{2}{3}\right)\right)$

Ans: Let $\theta = \arccos\left(-\frac{2}{3}\right)$, we have

$\cos(\theta) = -\frac{2}{3}$ **and** $0 \le \theta \le \pi$. If we draw θ in standard position, the terminal side of θ is in quadrant II. Using the Pythegorean Theorem, we see that the coordinate of the terminal point **P** of θ is $P = \left(-2, \sqrt{5}\right)$ where the radius of the circle we are using is $r = 3$.

Using this information, $\sin\left(\arccos\left(-\frac{2}{3}\right)\right) = \sin\theta = \frac{\sqrt{5}}{3}$

Example:

Express $\tan(\arcsin(x))$ in terms of x so that the expression is free of any trigonometric function. Assume that x is a positive number.

Ans: Let $\theta = \arcsin(x)$, we have

$\sin(\theta) = x$ and since x is positive, the terminal side of θ must be in the first quadrant. Using the Pythegorean Theorem, we see that the coordinate of the terminal point **P** of θ is $P = \left(\sqrt{1-x^2}, x\right)$ where the radius of the circle we are using is $r = x$.

Using this information, $\tan(\arcsin x) = \tan(\theta) = \frac{x}{\sqrt{1-x^2}}$

Solving Trigonometric Equations To solve equations that involve trigonometric functions, one wants to isolate the (hopefully one) tri function then use special triangle or the inverse tri functions to isolate the variable. Note that since a trigonometric function is not an algebraic function (i.e. to find $\sin(x)$ we cannot find the value by performing algebraic operations on x), we cannot isolate x by performing algebraic operations. We can only use our knowledge of special triangles, or using the inverse functions.

Example:

Solve the equation:

$\sin(x) = \frac{\sqrt{3}}{2}$

Ans: There are two angles in $[0, 2\pi)$ that solve the equation, namely $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

In addition, since sine is periodic with a period of 2π, if x is a solution to the equation, then $x + 2\pi$, $x + 4\pi$, $x + 6\pi$, $x - 2\pi$, $x - 4\pi$, ... are all solutions to the equation. Therefore, our complete solution set will be:

$x = \frac{\pi}{3} + 2\pi n$ or $x = \frac{2\pi}{3} + 2\pi n$

where n is an integer.

Example:

Solve the equation:

$\tan(x) = -1$

Ans: Tangent is negative in the second and fourth quadrant. In the second quadrant the angle that would solve this equation is $x = \frac{3\pi}{4}$

Since tan has a period of π instead of 2π, we don't need to add 2π to the angle to find all the solutions. Instead, we have:

$x = \frac{3\pi}{4} + \pi n$, where n is an integer.

Example:

Solve the equation:

$\cos(2x + 1) = -\frac{\sqrt{3}}{2}$

Ans: We treat $2x - 1$ as a single quantity. I.e., substitue $y = 2x - 1$ and we are looking at the equation $\cos(y) = -\frac{\sqrt{3}}{2}$.

The two angles in $[0, 2\pi)$ that solves the equation is:

$y = \frac{5\pi}{6}$ and $y = \frac{7\pi}{6}$

Since we are solving for x, not for y, we must back substitue x into the equation:

$$2x + 1 = \frac{5\pi}{6}$$

$$2x + 1 = \frac{7\pi}{6}$$

Since cos is a periodic function with period of 2π, any integral multiple of 2π added to the angles will solve the equation too, so the complete solutions will be:

$$2x + 1 = \frac{5\pi}{6} + 2\pi n$$

$$2x + 1 = \frac{7\pi}{6} + 2\pi n$$

where n is an integer.

We still need to isolate x in the above equations:

$$x = \frac{5\pi}{12} - \frac{1}{2} + \pi n$$

or

$$x = \frac{7\pi}{12} - \frac{1}{2} + \pi n$$

Note from the example that you add the $2\pi n$ to the angles **immediately** after you isolated the argument $(2x + 1)$ of cos, but **before** you isolated x. Some people, not understanding the reason for adding the $2\pi n$, would just always add the $2\pi n$ after they isolated x. This is incorrect. The angles that cosine must take to produce a value of $-\frac{\sqrt{3}}{2}$ is $\left(\frac{5\pi}{6} + 2\pi n\right)$ or $\left(\frac{7\pi}{6} + 2\pi n\right)$.

Since $(2x + 1)$ is the argument to cosine in the equation, $(2x + 1)$ must equal to $\left(\frac{5\pi}{6} + 2\pi n\right)$ or $\left(\frac{7\pi}{6} + 2\pi n\right)$, not x itself. We will, instead, isolate x using algebraic means after we solved for $2x + 1$.

Example: Solve: $\cos(2x) - 3\cos(x) - 1 = 0$

Ans: We use the double angle formula to change $\cos(2x)$ to

$\cos(2x) = 2\cos^2(x) - 1$.

We have:

$$\left(2\cos^2(x) - 1\right) - 3\cos(x) - 1 = 0$$

$$2\cos^2(x) - 3\cos(x) - 2 = 0$$

This is a quadratic equation in $\cos(x)$, factoring we get:

$(2\cos(x)+1)(\cos(x)-2)=0$

This gives:

$2\cos(x)+1=0$

or

$\cos(x)-2=0$

We can ignore the second equation as it will not give us any solution. (why?) The first equation gives us:

$\cos(x)=-\frac{1}{2}$

The two angles in $[0,2\pi)$ that solves the equation is:

$x=\frac{2\pi}{3}$ and $x=\frac{4\pi}{3}$

Adding any integral multiple of 2π will give us the same solutions too, so the complete solution set will be:

$x=\frac{2\pi}{3}+2\pi n$

or

$x=\frac{4\pi}{3}+2\pi n$

where n is an integer

Example: Solve for exact value:

$\sin(x)=-\frac{1}{3}$, $0\leq x<2\pi$

Ans: Since 1/3 is the not the ratio of a special triangle, we have to use $\sin^{-1}$. Notice that

$x=\sin^{-1}\left(-\frac{1}{3}\right)$ is an angle between $-\frac{\pi}{2}$ and 0 (why?). This value is outside of the interval where we want our solution for x to be. Instead, note that sin will be negative in the third and fourth quadrant. In the third quadrant, the angle that is within the desired interval is:

$x=\pi-\sin^{-1}\left(-\frac{1}{3}\right)$.

The other angle that would work is the angle in the fourth quadrant, $x=2\pi+\sin^{-1}\left(-\frac{1}{3}\right)$.

The solution set is thus:

$x = \pi - \sin^{-1}\left(-\frac{1}{3}\right)$ or $x = 2\pi + \sin^{-1}\left(-\frac{1}{3}\right)$.

Example: Solve for exact value:

$\cos(x) = -\frac{1}{5}$

Ans: Another ratio that does not come from a special triangle. We use:

$x = \cos^{-1}\left(-\frac{1}{5}\right)$

This will produce the angle in the second quadrant. Since cos is an even function,

$x = -\cos^{-1}\left(-\frac{1}{5}\right)$ will also work. Adding our $2\pi n$ gives the complete solution:

$x = \pm\cos^{-1}\left(-\frac{1}{5}\right) + 2\pi n$