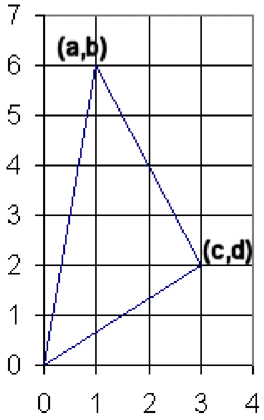
The area of a triangle formed by vectors

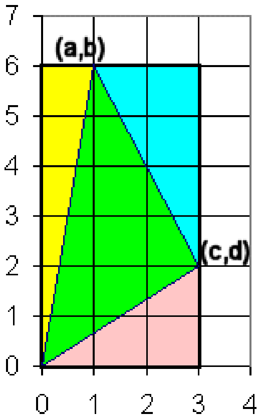
Start with the vectors (a,b) and (c,d) and draw them so that they both start from the origin (0,0). The end of the vectors are joined to form a triangle. What is the area of this triangle?

K = |(ad-bc)|/2

Here's why. Consider this triangle:



Draw a rectangle around this triangle, so that the rectangle is divided into four regions:



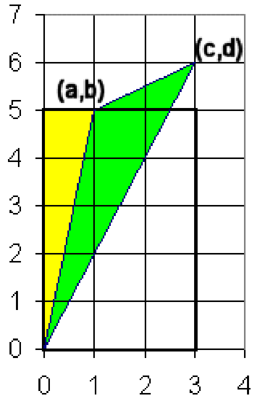
The area of the triangle is the area of the rectangle minus the three new triangles.

triangle = rectangle - yellow - pink - blue:

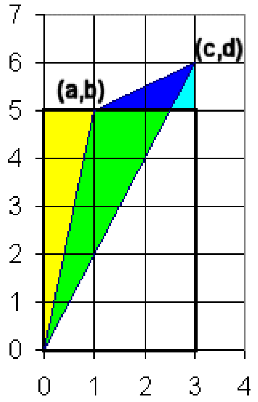
bc - ab/2 - cd/2 - (a-c)(d-b)/2
bc - ab/2 - cd/2 - (ad-ab-cd+bc)/2
bc - ab/2 - cd/2 - ad/2 + ab/2 + cd/2 - bc/2
bc/2 + bc/2 - ab/2 - cd/2 - ad/2 + ab/2 + cd/2 - bc/2
bc/2 - ad/2
(bc - ad)/2

The drawing, above, shows b>d and c>a. What if these relationships don't exist? Let's consider the case where d>b and c>a.

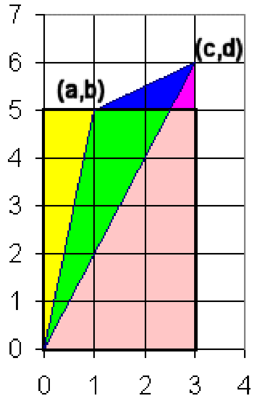
We will color this diagram the same way as the previous one, but it's a little more complicated because the colored regions will overlap. First, we make the rectangle the same way -- its height is b, and its width is c. Then we color the green and yellow regions as before.



Then we color the blue region as before, but this time we will add it to the area of the rectangle because it is outside the rectangle. The blue region partially overlaps the green triangle. The overlap is shown as a darker blue.



Then we color the pink region as before, showing the partial overlap with the blue region as a darker pink:



triangle = rectangle - yellow - pink + blue

bc - ab/2 - cd/2 + (c-a)(d-b)/2
bc - ab/2 - cd/2 + (cd-bc-ad+ab)/2
bc - ab/2 - cd/2 + cd/2 - bc/2 - ad/2 + ab/2
bc/2 + bc/2 - ab/2 - cd/2 + cd/2 - bc/2 - ad/2 + ab/2
bc/2 - ad/2
(bc - ad)/2

So either way, the area is the same: (bc-ad)/2. Sometimes this ends up being a negative number, and the area can't be negative, so one final adjustment is needed:

K = |(bc-ad)|/2 which is the same as K = |(ad-bc)|/2.

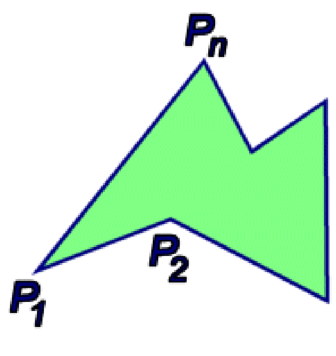
Determinant area of polygon

The signed area of a polygon is given by

$$A = (1/2) \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{vmatrix} = (^1/_2)\,(x_1y_2 + x_2y_3 + \ldots + x_ny_1 - x_2y_1 - x_3y_2 - \ldots - x_1y_n)$$

The proof of this can be obtained by induction. We <u>know</u> the formula is true for a triangle (n=3). Let's assume it's true for any n, and show it's true for n+1.

Here is our polygon with n points, labeled counterclockwise, P_1, P_2, ..., P_n.

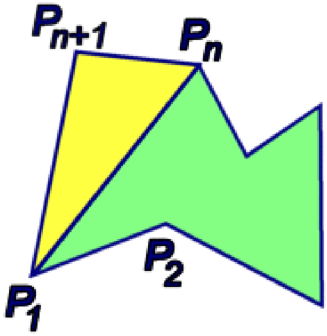


Its area is given by the formula above,

$$A_n = {}^1/_2(x_1y_2 + x_2y_3 + \ldots + x_ny_1 - x_2y_1 - x_3y_2 - \ldots - x_1y_n)$$

Remember, this is the "signed area", which is positive if the vertices are given in counterclockwise order. When we add the n+1st point, we will change the area of the polygon, and the amount (and sign) of the change is exactly the area of the newly formed triangle

$P_n(x_n,y_n)$
$P_{n+1}(x_{n+1},y_{n+1})$
$P_1(x_1,y_1)$



The area of the yellow triangle, A_t, is given by the same formula:

$A_t = {}^1/_2(x_n y_{n+1} + x_{n+1} y_1 + x_1 y_n - x_{n+1} y_n - x_1 y_{n+1} - x_n y_1)$

Here is the new polygon with n+1 points, labeled counterclockwise, P_1, P_2, ..., P_n, P_{n+1}.

Its area, A_{n+1}, is the sum of the areas of the original n-gon, A_n, and the new triangle, A_t.

$A_{n+1} = A_n + A_t$

$A_{n+1} = {}^1/_2(x_1 y_2 + x_2 y_3 + ... + x_n y_1 - x_2 y_1 - x_3 y_2 - ... - x_1 y_n + x_n y_{n+1} + x_{n+1} y_1 + x_1 y_n - x_{n+1} y_n - x_1 y_{n+1} - x_n y_1)$

$A_{n+1} = {}^1/_2 (x_1 y_2 + x_2 y_3 + ... + x_n y_{n+1} + x_{n+1} y_1 - x_2 y_1 - x_3 y_2 - ... - x_{n+1} y_n - x_1 y_{n+1})$,

which proves, by induction, that the formula is right.

What if the new point is *inside* the original polygon, you might ask. In that case, the new triangle, Pn, Pn+1, P1, has its vertices named in clockwise order, not counterclockwise. So the area of the new triangle, as calculated by the formula, is negative. The algebraic addition of the area of the triangle to the area of the original n-gon results in the new, smaller, area of the n+1-gon.