Weak Interactions Made Simple

P. R. Silva – Retired associated professor – Departamento de Física – ICEx – Universidade Federal de Minas Gerais – email: prsilvafis@gmail.com

ABSTRACT

A very simplified way of calculating the lifetime of some elementary particles decaying through the weak interaction is presented. The method makes use of a non-linear version of the Klein-Gordon-Yukawa equation, combined with the time-energy uncertainty principle. Although the analytical relations evaluated in this work do not always reproduce those usually found in literature, numerical estimates of them agree with their respective experimental values.

1 – INTRODUCTION

As was pointed out by Perkins [1], the weak interactions take place between all the quark and lepton constituents, each of them endowed with a "weak charge". This interaction is so feeble, so that it manifests itself only in the absence of the much stronger electromagnetic and strong interactions, when some conservation rule is at work. Yet according Perkins [1], the observable weak interactions therefore either involve neutrinos (not endowed with electric or strong charges) or quarks with a flavor change, which is forbidden for strong or electromagnetic interactions. In this work we want to discuss some examples of weak decay in a very simplified way.

2 – THE MUON DECAY: FIRST APPROXIMATION

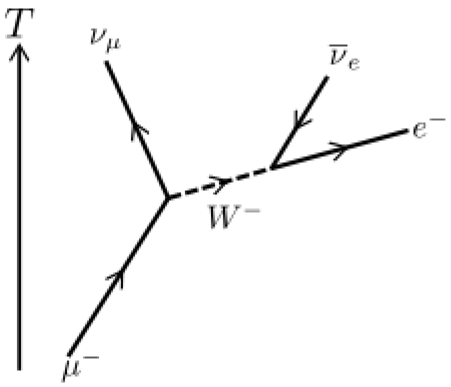

Figure 1 – Feynman diagram for the muon decay

Figure 1 shows a Feynman diagram for the muon beta decay. The interaction happens with the exchange of a massive W-boson, responsible for the short-range character of the weak force. But as was pointed out by Griffiths [2] in the Fermi's original theory of the beta decay (1933) there was no W-boson: the interaction was supposed to be a direct four-particle coupling, the broken line representing the W-boson was absent and the remaining lines of the diagram representing the muon, the electron and their respective neutrino and anti-neutrino are jointed in a single vertex.

In this first approximation we intend to obtain the muon lifetime in a more simplified way than that considered in the Fermi's original approach.

Inspired in the Newton's theory of gravitation, we can write

$$G_F = m_\mu^2\, c^4 / (hc)^2 = m_\mu\, c^2\, R. \qquad (1)$$

Relation (1) proposes that the mass-energy of the muon can be computed by taking in account the muon self-interaction, where G_F is the Fermi constant of the weak interaction and R is some characteristic radius related to the range of the weak interaction. Solving (1) for R, we obtain

$$R = G_F\, m_\mu\, c^2 / [(2\pi)^2\, (\hbar c)^2]. \tag{2}$$

To compute the muon lifetime it is worth to evaluate the cross-section σ, related to the process responsible for the muon decay. We have

$$\sigma = ⅓\, \pi\, R^2 = ⅓\, \pi\, G_F^2\, m_\mu^2\, c^4 / \; [(2\pi)^4\, (\hbar c)^4]. \tag{3}$$

The factor one over three which appears in eq.(3), is a way to express the radius of the cross-sectional area in terms of R. Now let us consider the phenomenology of the molecular diffusion and write

$$\sigma\, (c\tau)\, n = 1. \tag{4}$$

In (4), $c\tau$ is the mean free path and n is the number of targets per unit of volume, and c is the speed of light. To make the adaptation of eq.(4) in order to the determine the muon lifetime we assume that

$$n = m_\mu^3\, c^3 / [(4/3)\pi\, \hbar^3], \tag{5}$$

where we have just one muon occupying a sphere which radius is equal to its reduced Compton wavelength. Inserting the information given by (3) and (5) into (4) and solving for τ, we get

$$\tau = (64\, \pi)\, \pi^3\, \hbar^7 / [G_F^2\, m_\mu^5\, c^4]. \tag{6}$$

The above result for the muon lifetime must be compared with that which is usually found in the literature (please see Griffiths [2] and Rohlf [3]), namely

$$\tau|_{\text{literature}} = 192\, \pi^3\, \hbar^7 / [G_F^2\, m_\mu^5\, c^4]. \tag{7}$$

Therefore the result here presented differs from that found in the literature by a factor of $3/\pi$.

3 – THE MUON DECAY: SECOND APPROXIMATION

As was pointed out by Griffiths [2], from the modern perspective , Fermi's theory combined the W propagator with two vertex factors in the diagram of figure 1, to make an effective four-particle constant G_F.

Meanwhile if we look at relation (1), and by taking in account the usual way of dealing with the coupling constants it is possible to write

$$G_F E^2 = \xi \hbar c \alpha_W, \tag{8A}$$

$$G_F M_W^2 c^4 = \xi \hbar c \alpha. \tag{8B}$$

In (8), ξ is a constant, E is the variable energy and M_W is the W-boson mass. Dividing (8A) by (8B) we get

$$\alpha_W = \alpha [E/(M_W c^2)]^2. \tag{9}$$

We observe from (9) that, when the energy E equals the W-boson mass-energy value, we have the identification between the weak coupling α_W and the electromagnetic coupling α.

Now if we think of the possibility of create a muon-antimuon pair through the weak interaction we write

$$2 m_\mu c^2 = (\alpha_W \hbar c)/R, \tag{10}$$

where α_W must be evaluated at the energy scale of the mass-energy of the muon, namely

$$\alpha_W = \alpha (m_\mu/M_W)^2. \tag{11}$$

Solving (10) for R, and working in an analogous way we have done in a first approximation, and after to consider the first equality in eq.(3), and eqs. (4), (5) and (11), we obtain

$$\tau = [(16\, \hbar)/(\alpha^2\, c^2)]\, (M_W{}^4/m_\mu{}^5). \qquad (12)$$

4 – MUON DECAY: THIRD APPROXIMATION

In this third approximation we intend to look at the muon decay by using a non-linear wave equation (a non-linear Klein-Gordon-Yukawa equation).

It is worth to point out that this equation would play a similar role of that represented by the use of the W-boson propagator. We write

$$\Delta\Psi - (1/c^2)\, \partial^2 \Psi/\partial t^2 = (m_\mu{}^2\, c^4/\hbar^2)\, \Psi - (M_W{}^2\, c^4/\hbar^2)\, \Psi^3. \qquad (13)$$

We look for the "stationary" condition by making the two sides of eq.(13) equal to zero. This gives

$$\Psi^2 = (m_\mu/M_W)^2. \qquad (14)$$

We will call the square of the Ψ^2-term in (14), namely the Ψ^4-term: "the geometric factor". This geometric interpretation comes from the fact that if we consider in the four-dimensional momentum space a hypercube of edge $M_W\, c$, the fraction of volume occupied by a tinny cube of edge $m_\mu\, c$ is given by Ψ^4.

Now from the –energy form of the uncertainty principle we have

$$\upsilon = 1/\Delta t = m_\mu\, c^2/h. \qquad (15)$$

Each vertex of the Feynman diagram (fig.1) contributes with $\sqrt{\alpha}$ for the amplitude, so that the overall contribution will be proportional to α^2. Therefore the transition rate or the total decay rate Γ is

$$\Gamma = ⅓\, \upsilon\, \alpha^2\, \Psi^4. \qquad (16)$$

The factor ⅓ which appears in (16) has the same origin as that of eq. (3). Thus according the third approximation the muon lifetime can be written as

$$\tau = 1/\Gamma = [(6\pi\hbar)/(\alpha^2 c^2)]\,(M_W^{\,4}/m_\mu^{\,5}). \qquad (17)$$

By comparing (17) with (7) we get

$$G_F = (4\sqrt{2}\pi\hbar^3\alpha)/(M_W^{\,2} c). \qquad (18)$$

The Fermi constant G_F given by (18) agrees with that estimated by Rohlf [3]. We also verify that the muon lifetime evaluated in the second ((eq.(12)) and third (eq.(17)) approximations differ by a numerical factor close to the unity.

5 – THE NEUTRON BETA DECAY

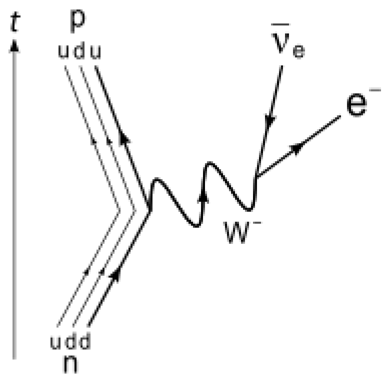

Figure 2 - The Feynman diagram for beta-minus decay of a neutron into a proton, electron and electron anti-neutrino, via an intermediate heavy W− boson

In the present study of the neutron weak decay, we intend to proceed in a similar way we have done in the muon weak decay case. First let us look at the geometric factor. We write

$$\Delta\Psi - (1/c^2)\,\partial^2\Psi/\partial t^2 = [(m_n - m_p)^2 c^4/\hbar^2]\,\Psi - (M_W^{\,2} c^4/\hbar^2)\,\Psi^3. \qquad (19)$$

Here we are essentially considering the three legs at the left of the Feynman diagram exhibited in figure 2. Indeed, the neutron and proton legs, with their constituents' quarks are "three-bone" legs. The $(m_n - m_p)$-term stands for the neutron-proton mass difference.

The search for the "stationary regime" is reached by putting left and right sides of eq. (19) equals to zero, and solving for Ψ^2, we find

$$\Psi^2 = (m_n - m_p)^2 / M_W^2. \tag{20}$$

Now, looking at the left vertex of figure 2, we observe that although neutron decay is governed by the weak interaction, it seems that in the process leading from neutron to proton "transmutation", the strong interaction is also at work. A basic feature present in QCD description of the strong interaction is its non-Abelian character. We intend to mimic this fact through the action A ($\hbar$=c=1).

$$A = \int d^4 \mathbf{x}\, [\partial_\mu \Phi\, \partial^\mu \Phi - \kappa\, \Phi^4]. \tag{21}$$

In a paper dealing with the critical behavior of the Ising model, C. J. Thompson [4,5] wrote a action as a means to study its critical behavior, supposed to be within the same class of universality of the Φ^4-theory.

Inspired in the Thompson work [4], we assume that each term of the action given by (21) is separately equal to the unity.

Applying Thompson's recipe to the first term of (21) we get

$$\left|\int d^4 \mathbf{x}\, (\partial_\mu \Phi\, \partial^\mu \Phi)\right| = 1, \tag{22}$$

which implies that

$$\Phi^2 = 1 / x^2. \tag{23}$$

Applying the same recipe to the second term of (21), we have

$$\left|\int d^4 \mathbf{x}\, (\kappa\, \Phi^4)\right| = \left|\int (\kappa / x^4)\, x^3 dx\right| = 1. \tag{24}$$

In the next step we will put $\langle\kappa\rangle$ outside the integral symbol, where $\langle\kappa\rangle$ stands for averaged "κ- coupling" and perform the integration between the low and high energies cutoff, after making the change of variable $E \equiv 1/x$. Doing this we have

$$\langle\kappa\rangle \int_L^H d(\ln E) = 1, \tag{25}$$

and after solving for $\langle\kappa\rangle$, we get

$$\langle\kappa\rangle^{-1} = \ln[M_W/(m_n - m_p)], \tag{26}$$

where we have identified $M_W c^2$ and $(m_n - m_p) c^2$ with the high(H) and low(L) energies cutoff, respectively.

In the next step of evaluating the neutron decay we take in account the uncertainty principle. Also looking at the left vertex of figure 2, we may consider that the intermediate time where neutron and proton undergo a virtual interaction, we have a typical problem of a two-body interaction of nearly equal mass. Therefore we have from the uncertainty relation

$$\upsilon = 1/\Delta t = ½\, m_n c^2/h. \tag{27}$$

In (27), $½\, m_n$ is the approximated reduced mass of a proton-neutron virtual pair.

Therefore the transition rate for the neutron weak decay is given by

$$\Gamma_n = ⅓\, \upsilon\, \alpha^2 \langle\kappa\rangle^2\, \Psi^4, \tag{28}$$

and the neutron lifetime $\tau_n = (\Gamma_n)^{-1}$ reads

$$\tau_n = [(12\pi \hbar)/(\alpha^2 m_n c^2)]\, [M_W/(m_n - m_p)]^4 \{\ln [M_W/(m_n - m_p)]\}^2. \tag{29}$$

We observe that the electron mass does not appears in relation (29) for the neutron lifetime. Despite this, the right side vertex of figure 2, can be considered the only possible outcome consistent with energy, electric charge and lepton number conservations which must be obeyed by any physical process, including the neutron weak decay.

6 – PION WEAK DECAY

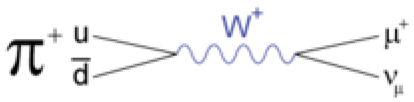

Figure 3 - π+ decay through the weak interaction

Figure 3 shows the Feynman diagram for the pion(+) decay. As a means to study the charged pion decay let us write again a non-linear Klein-Gordon-Yukawa equation, namely

$$\Delta\Psi - (1/c^2)\,\partial^2\,\Psi/\partial\,t^2 = [(m_\pi - m_\mu)^2\,c^4/\hbar^2]\,\Psi - (M_W^2\,c^4/\hbar^2)\,\Psi^3. \qquad (30)$$

The “stationary” solution of (30) is given by

$$\Psi^2 = [(m_\pi - m_\mu)/M_W]^2. \qquad (31)$$

Now in order to determine the geometric factor we must remember the pion as being a bound state of two quarks, therefore having a linear axis connecting them. With this idea in mind we can imagine an anisotropic prism in the four-dimensional momentum space, with the axis of this four-dimensional prism essentially infinite ($p_L \rightarrow \infty$). If we consider in the four-dimensional momentum space a hyper-prism of perpendicular edge M_W c, the fraction of volume occupied by a tinny prism of perpendicular edge ($m_\pi - m_\mu$) c can be written as

$$[(m_\pi - m_\mu)^3\,c^3\,p_L]/[M_W^3\,c^3\,p_L] \equiv \Psi^3. \qquad (32)$$

The time-energy uncertainty principle for this process implies in

$$\upsilon = 1/\Delta t = m_\pi\,c^2/h. \qquad (33)$$

Therefore the transition rate Γ_π of the pion weak decay is

$$\Gamma_\pi = \tfrac{1}{3}\, \upsilon\, \alpha^{2}\, \Psi^{3}, \qquad (34)$$

And the charged pion lifetime $\tau_\pi \equiv \Gamma_\pi^{-1}$ reads

$$\tau_\pi = [(6\pi\, \hbar)/(\alpha^2\, m_\pi\, c^2)]\, [M_W/(\, m_\pi - m_\mu)]^{3}. \qquad (35)$$

7- CHARGED KAON DECAY

The decay of the charged Kaon, despite the mesonic character of this particle, will be treated on equal footing with the neutron decay. We can imagine a pair of particles K^+ and K^- forming a bound state having total electric charge equals to zero, and with total mass close to the neutron mass. We propose that in an analogous way to the neutron beta decay case, in the charged Kaon decay, both the strong and the weak forces are at work. Therefore the expression for the charged Kaon decay can be written by adapting the expression which was obtained in the neutron decay (please see (29)). We write

$$\tau_K = [(6\pi\, \hbar)/(\alpha^2\, 2m_K\, c^2)]\, [M_W/(\, m_K - m_\mu)]^4\, \{\ln\, [M_W/(\, m_K - m_\mu)]\}^2. \quad (36)$$

In writing (36) we had in mind the reaction

$$K^{\pm} \rightarrow \mu^{\pm} + \underline{\upsilon}_\mu. \qquad (37)$$

8- DISCUSSION

In this work we are proposing a novel way to calculate a particle lifetime which decays through the weak interaction. The analytical expressions here obtained differ from those usually found in the literature. However numerical estimates of the lifetimes agree approximately with their respective experimental determinations. In this way by taking numbers from the Particle Data Group [6], as quoted in Blin-Stoyle's book [7], namely

$M_W = 80.49$ GeV; $m_n = 939.57$ MeV; $m_p = 938.28$ MeV;

$M_K = 493.65$ MeV; $m_\pi = 139.57$ MeV; $m_\mu = 105.66$ MeV;

$m_e = 0.511$ MeV,

we get the numerical estimates:

$$\tau_\pi = 2.23 \text{ x } 10^{-8} \text{ s}; \quad \tau_n = 916 \text{ s}; \quad \tau_K = 1.24 \text{ x } 10^{-8} \text{ s}. \tag{38}$$

These numbers must be compared with their respective lifetime experimental determinations, namely:

$2.60 \text{ x } 10^{-8}$s, for the charged pion; 899.7 ± 8.9 s, for the neutron and $1.24 \text{ x } 10^{-8}$ s, for the charged kaon. An interesting discussion about the difficulties to calculate the neutron decay lifetime can be found in Griffiths [2]

We must to stress that we was unable to determine the branching ratio between the two possibilities of the charged pion decay, namely the relation between line widths resulting in electrons or muons and their respective neutrinos.

Finally we would like to point out that a previous paper related to present one was published by the author in 1995 [8].

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