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The Mathematics of Charles Sanders Peirce

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I. Introduction

This essay explores the Mathematics of Charles Sanders Peirce. We concentrate on his notational approaches to basic logic and his general ideas about Sign, Symbol and diagrammatic thought.

In the course of this paper we discuss two notations of Peirce, one of Nicod and one of Spencer-Brown. Needless to say, a notation connotes an entire language and these contexts are elaborated herein. The first Peirce notation is the portmanteau (see below) Sign of illation. The second Peirce notation is the form of implication in the existential graphs (see below). The Nicod notation is a portmanteau of the Sheffer stroke and an (overbar) negation sign. The Spencer-Brown notation is in line with the Peirce Sign of illation. It remained for Spencer-Brown (some fifty years after Peirce and Nicod) to see the relevance of an arithmetic of forms underlying his notation and thus putting the final touch on a development that, from a broad perspective, looks like the world mind doing its best to remember the significant patterns that join logic, speech and mathematics. The movement downward to the Form ("we take the form of distinction for the form."[9, Chapter 1, page 1]) through the joining together of words into archetypal portmanteau Signs can be no accident in this process of return to the beginning.

We study a system of logic devised by Peirce based on a single Sign for inference that he calls his 'Sign of illation". We then turn to Peirce's Existential Graphs. The Existential Graphs lead to a remarkable connection between the very first steps in Logic and mirror plane symmetries of a "Logical Garnet" [30] in three dimensional space. Peirce's ideas about these graphs are related to his ideas about infinity and infinitesimals, and with his more general philosophy that regards a human being as a Sign. It is the intent of this paper to bring forth these themes in both their generality and their particularity.

It is amazing that three dimensional geometry is closely allied to the first few distinctions of Logic. It is my intent in this paper to make that aspect crystal clear.

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[1] Department of Mathematics, University of Illinois, Chicago. Email: kauffman@uic.edu. I would like to take this opportunity to thank Diane Slaviero, David Solzman, Jim Flagg , G. Spencer-Brown, Annetta Pedretti and Kyoko Inoue for many conversations real and imaginary related to this paper. I wish to dedicate this paper to the memory of Milton Singer and to our many meetings in the Piccolo Mondo Restaurant in Hyde Park, Chicago in the 1990's.

We also clarify the relationships of Peirce's Mathematics in other ways that are described below. These clarifications are the specific content of the paper. Their purpose is to shed light on the beautiful philosophical generality of Peirce's work, and to encourage the reader to look at this in his or her own context and in the contexts of semiotics and second order cybernetics.

We begin with a detective story about the design of a Sign for inference that Peirce calls the "Sign of illation." Peirce designed this Sign as a "portmanteau", a combination of two Signs with two meanings! The plot thickens as we find that the very same design idea was independently taken up by two other authors (Nicod and Spencer-Brown). This is treated in Section 2 of this paper. The Sign of illation is a convenient Sign for inference. A very similar combination Sign was independently devised by Nicod, using the Sheffer stroke and overbar negation. I illustrate these notations and how they are related. The way that both Peirce and Nicod arrive at a portmanteau symbol is part of a movement that goes beyond either of them to unify logic and logical notation into a simpler structure.

The double meaning of the portmanteau is a precursor to the interlock of syntax and semantics that led to Gödel's work on the incompleteness of formal systems. See Section 11 for a discussion of this theme.

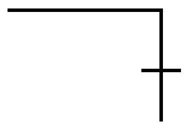
Peirce also had another approach to basic logic. This is his theory of Existential Graphs. Peirce's Existential Graphs are an economical way to write first order logic in diagrams on a plane, by using a combination of alphabetical symbols and circles and ovals. Existential graphs grow from these beginnings and become a well-formed two dimensional algebra. I make the following observation: *There is a natural combinatorial "arithmetic" of circles and ovals that underlies the Peircian Existential Graphs.* The arithmetic of circles is a formal system that is interpreted in terms of itself. It is a calculus about the properties of the distinction made by any circle or oval in the plane, and by abduction it is about the properties of any distinction. This circle arithmetic in relation to existential graphs is discussed in Section 6 where we show that it is isomorphic with the Calculus of Indications of G. Spencer-Brown. Spencer-Brown's work can be seen as part of a continuous progression that began with Peirce's Existential Graphs. In essence what Spencer-Brown adds to the existential graphs is the use of the unmarked state. That is, he allows the use of empty space in place of a complex of Signs. This makes a profound difference and reveals a beautiful and simple calculus of indications underlying the existential graphs. Indeed Spencer-Brown's true contribution is that he added Nothing to the Peirce theory!

In Section 7 we discuss how the "Logical Garnet" of Shea Zellweger fits into this picture. Zellweger discovered that the sixteen binary connectives studied by Peirce fit naturally on the vertices of a rhombic dodecahedron (plus a new central vertex) in such a way that symmetries of these connectives correspond to mirror symmetries of this polyhedron in three dimensional space. The Logical Garnet fits perfectly into the context of the existential graphs.

In Section 8 we discuss Peirce's ideas about continuity and infinitesimals, and relate this to extensions of existential graphs to infinite graphs. In Section 9 we quote a famous passage in Peirce about a "Sign of itself" and discuss this passage in terms of topology and self-reference. This passage and the remarks on infinitesimals show how Peirce's thought reaches far beyond the specific formalisms that he produced, and that his intuition was right on target with respect to much of the subsequent (and future!) development of mathematics and logic. Section 10 continues the discussion of Section 9 in the context of second order cybernetics. Section 11 is an epilogue and a reflection on the theme of the portmanteau word.

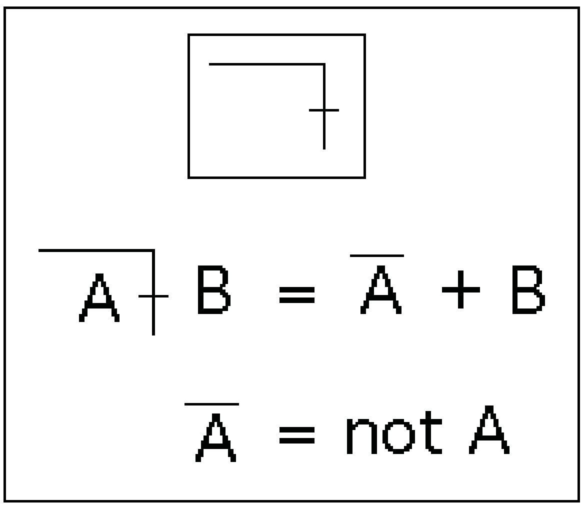
II. The Sign of Illation

Peirce wrote a remarkable essay [1] on the Boolean mathematics of a Sign that combines the properties of addition and negation. It is a portmanteau Sign in the sense of Lewis Carroll (See below and Section III). We do not have the capabilities to typeset the Peirce Sign of illation, but see right for a rendition of it.



Instead, I shall use the following version in this text: [a]b. When you see [a]b in the text you are to imagine that a horizontal bar has been placed over the top of the letter "a", and a vertical bar, crossed with a horizontal bar (very like a plus sign) has been placed just to the right of the "a" in such a way that the vertical bar and the horizontal overbar share a corner. In this way [a]b forms the Peirce Sign of illation, and we see that this Sign has been created by fusing a horizontal bar with a plus sign. The horizontal bar can be interpreted as negating the Sign beneath it.

Peirce went on to write an essay on the formal properties of his *Sign of illation* and how it could be used in symbolic logic. Here is the dictionary definition of the word illation. Note that we have taken the "Sign of Illation" as the title of this section of the paper.



> **il . la`tion.** [L. *illation,* fr. *illatus*, used as past participle of *inferre*, to carry or bring in] Inference from premises of reasons; hence that which is inferred or deduced. [2]

The Sign of illation enables a number of notational conveniences, not the least of which is that the implication "A implies B" usually written as " A -> B" is expressed as "[A]B " using the Sign of illation.

The Sign of illation is a *portmanteau Sign* in the sense of Lewis Caroll [3], [4] who created that concept in his poem "Jabberwocky" where one encounters a bestiary of words like “slithy” – a combination of lithe and slimy. A portmanteau is literally a coat and hat rack (also a suitcase), an object designed to hold a multiplicity of objects. Just so, a portmanteau word is a holder of two or more words, each justly truncated to fit with the truncate of the other. A modern version to contemplate is the word “smog” a combination of “smoke” and “fog”.

It is this fitting together of the two words that is so characteristic of the portmanteau. It recalls the amazing doublings of nature that would use a mouth for eating and speaking, a throat for breathing and drinking, and the amazing multiple use of the DNA at biology’s core. In Peirce’s case of his portmanteau Sign of illation there is no truncation, but rather a perfect fit at the corner of the horizontal overbar as sign of negation, and the vertical plus sign as sign of “logical or”. The two fit into one Sign that can then hold neatly yet another meaning as a Sign of implication.

It is the meaning of the Sign of illation as implication that Peirce takes as primary. In his essay [1], he begins with this interpretation, deduces many properties of the Sign from this interpretation, and only in the last paragraph does he reveal that his Sign can be taken apart into a plus sign and overbar (interpreted as a negation). In the beginning he writes [1]

> This symbol must signify the relation of antecedent to consequent. In the form I would propose for it, it takes the shape of a cross placed between antecedent and consequent with a sort of streamer extending over the former. Thus, “if a then b” would be written [a]b.
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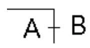
> 
>
> From a, it follows that if b then c”, would be written [a][b]c.
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> 
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> “From ‘if a then b.’ follows c,” would be written [[a]b]c.
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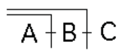
> 
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> To say that a is false, is the same as to say that from a as an antecedent follows any consequent that we like. This is naturally shown by leaving a blank space for the consequent, which may be filled in at pleasure. That is, we may write “a is false” as [a],
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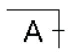
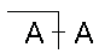
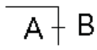
> 
>
> implying that from a, every consequence may be drawn without passing from a true antecedent to a false consequent, since a is not true.” [We are still quoting from [1].]

At this point, Peirce has partially let the cat out of the bag by noting that with the use of a blank space for a variable, his Sign can express negation.

He then introduces signs 0, 1, to stand for falsity (absence) and truth (presence) respectively. The symbol 1 is taken to stand in for the expression [a]a for any a (as ‘a implies a” is true for any a).



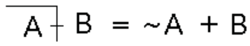
One of the charming features of the essay is that he deduces many formal properties of this symbolism wholly conceptually, based on this interpretation of the Sign of illation as implication. In fact, he remarks [1] " Here then we have a written language for relations of dependence. We have only to bear in mind the meaning of the symbol



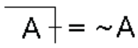
(not by translating it into if and then, but by associating it directly with the conception of the relation it signifies), in order to reason as well in this language as in the vernacular, - and indeed much better."

At the end of the essay [1], he writes

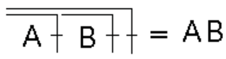
> We now have a complete algebra for qualitative reasoning concerning individuals. But it is not yet a very commodious calculus. To render it so, we introduce certain abbreviations which make it identical with the algebra of Boole … Namely, we first separate the streamer of the Sign of illation from the cross, and in place of [a]b write ~a + b.



> Second,
> whenever the Sign of illation is followed by a blank we omit the cross, and thus in place of [a], write ~a.



> Third,
> as the sign of the simultaneous truth of a and b, instead of writing [[a][b]], we simply write ab." [1]



In the end, it is important that the portmanteau Sign can be decomposed back into its component parts, for this allows the translation between Peirce's thought and the Boolean algebra. It is these issues of translation, from one formalism to another and from meaning in natural language to meaning in the formalism, that he holds with great sensitivity.

III. Nicod and the Sheffer Stroke

A remarkable paper by Nicod [5], creates a portmanteau Sign for implication almost identical to that of Peirce. Nicod wrote in the context of the Sheffer stroke **a|b**

$$A|B = \text{not}(A \text{ and } B)$$

that represents "not both a and b". Nicod noticed that he could create a Sign for implication by putting a negation bar over one of the variables of the stroke. Thus "not both a and not b" is logically equivalent to "a implies b".

$$A|\overline{B} = \text{not}(A \text{ and not } B)$$
$$= A \text{ implies } B$$

Thus for Nicod

$$A|\overline{B}$$

stood for "A implies B" and became a "convenient sign for implication". The decomposition is nearly the same as in the Peirce Sign of illation; the meanings have shifted, and the Cheshire cat is smiling in the background.

What we are witnessing here is the peculiar relationship between spoken and written language (ordinary language) and symbolic logic.

We think naively that there is not much more to reasoning than the simplest properties of inference. That all you need to know to reason is that if A implies B and B implies C, then A implies C. And so one might start down the road to symbolic logic by a convenient sign for implication. This is just what Peirce did in his essay [1] that we discussed in the previous section. And yet when you think about the matter of when an implication is true and when it is false and how it interfaces with the meanings of "and" and "or" a complexity arises. This complexity expands to the curious intricacy of first order symbolic logic, and then it seems like a breathe of fresh air to find the symbols of the logical system combining (almost of their own accord) to assemble a sign for illation. It is clear that this experience occurred to both Peirce and to Nicod quite separately, and we shall see that they were not alone!

I believe that it is quite significant to see the sign of implication as a complex sign composed of other logical signs. This places it in its proper context. Implication itself is not simple, yet something simple underlies it. Inference is a portmanteau, a gluing of separate meanings into a coherent whole.

IV. Pivot and Portmanteau

Along with the concept of a portmanteau word or symbol there is a notion that I like to call a "pivot duality". A portmanteau word is a combination of separate meanings such that their signs can be fitted together. In a pivot duality a word or symbol can be interpreted in more than one way, and this multiplicity of interpretation gives rise to a pivot, or translation, between the different contexts of these interpretations.

Pivot duality is the essence of multiplicity of interpretation, while pormanteau is the exemplar of the condensation of a multiplicity of meanings into a single sign. A portmanteau always has an associated pivot, but a pivot need not be a portmanteau. By bringing forth the pivot, we can expand the context of the consideration of the multiplicity of meanings associated with Signs. This has direct bearing on the understanding of the use of Signs in Peirce and in language as a whole.

A good example of pivot duality is the simple Feynman diagram

that can be interpreted (with time's arrow going up the page) as two particles interacting by the exchange of a photon (the horizontal line), and can also be interpreted (with time's arrow going from left to right) as a particle and an antiparticle annihilating to produce a photon that then momentarily decays into a new particle pair. Here we have two completely different (yet related) interpretations of the same bit of formalism. The formalism seems to point to a deeper reality, beyond the particular way that the physicist observer decomposes process into space and time.

There is an affinity between the portmanteau symbol and a pivot. The portmanteau is a single word that holds two meanings. The pivot is a word or symbol or text that is subject to a multiplicity of interpretations. We make them both because these makings are the essence of the condensation of meaning into Signs and the use of Signs in the expansion of meaning. If the meaning of a Sign is its use, then the meaning of the Sign is not one but many, according to its uses, and yet one according to the unity that these uses find in the Sign itself (as a complex of Signs fully embedded in language). A hammer makes a good example, being one tool and yet being capable of both the impulsive insertion of the nail and the levering extraction of the nail. Two meanings pivot over the hammer. The combination of claw and hammerhead makes the tool itself a portmanteau of these two actions.

The reason, I believe, that portmanteau and pivot are so important to find in looking at formal systems, and in particular symbolic logic, is that the very attempt to make formal languages is fraught with the desire that each term shall have a single well assigned meaning. It cannot be! The single well-assigned meaning is against the nature of language itself. All the formal system can actually do is choose a line of development that calls some entities elementary (they are not) and builds other entities from them. Eventually meanings and full relationships to ordinary language emerge. The pattern of pivot and portmanteau is the clue to this robust nature of the formal language in relation to human thought and to the human as a Sign for itself.

The grin of the Cheshire cat is the quintessential pivot, yet it is not a portmanteau. To quote Martin Gardner in his comment on Alice's encounter with the Cheshire Cat [4, p. 91],

> The phrase 'grin without a cat' is not a bad description of pure mathematics. Although mathematical theorems can often be usefully applied to the structure of the external world, the theorems themselves are abstractions that belong in another realm 'remote from human passions,' as Bertrand Russell once put it in a memorable passage, 'remote even from the pitiful facts of Nature…an ordered cosmos where pure thought can dwell as in its natural home.'

In mathematics the grin without the cat is often obtained through a process of distillation. The structure is traversed again and again and each time the inessential is thrown away. At last only a small and potent pattern remains. This is the grin of the cat. That grin is a pattern that fits into many contexts, a key to many doors. It is this multiplicity of uses for a single symbolic form that makes mathematics useful. It is the search for such distillation of pattern that is the essence of mathematical thought.

V. Peirce's Existential Graphs

We now turn to a development of Peirce for logic that is closely related to the Sign of illation. These are his existential graphs [6], [7], [8]. In this development, Peirce takes the operations "and", "not", and a space in which they are represented as fundamental. He develops logic from that ground. It is important to see this development both for the structure of basic logic and for the view that it gives of Peirce's thought as he examines the same(!) subject from a different angle and finds that it is a different subject.

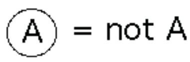
The first stage of the Existential Graphs are called the "alpha" graphs. These alpha graphs are concerned with the logic of implication, and we shall concentrate on their structure.

Here is a quick description of the context for the Existential Graphs.

We are given a plane on which to make inscriptions. If we place a graph or symbol on the plane, then the proposition corresponding to this symbol is asserted. If we place two disjoint complexes of symbols on the plane, call them A and B, then we are asserting the conjunction "A and B".

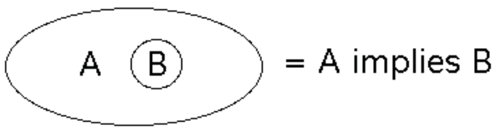
A B

A circle (or simple closed curve) drawn around a symbol changes the assertion to the negative. Thus a circle around A asserts the negation of A.



For ease of notation, we shall make an algebraic version of the existential graphs where AB denotes "A and B" and (A) denotes a circle around A. Hence (A) denotes "not A".

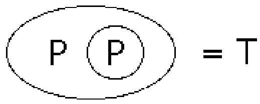
Since "A implies B" is logically equivalent to "not (A and not B)" we see that "A implies B" has an existential graph consisting of a big circle that contains both A and a circle around B. Algebraically this is (A(B)) for "A implies B".



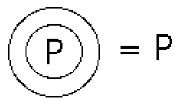
Peirce worked out a number of rules for manipulating these graphs by different patterns of substitution and replacement. With such rules in place, the graphs become an arena for analyzing basic arguments and tautologies in logic. Note that the idea behind the existential graphs and the Sign of illation is essentially the same, although the underlying model for implication is "(not a) or b" in the case of the Sign of illation.

It is interesting to see how these graphs work. To this end let us set up some rules for manipulating the graphs. We must remark that in the discussion that follows I will consider only transformations that preserve the full logical structure of the Existential Graphs. Peirce considers transformations on the graphs that preserve truth in one direction (e.g. see [7]). That is he allows two graphs X and Y to be transformed one to another so long as whenever X is true then Y is true. Here we require that X is true if and only if Y is true. Since this is the usual notion of equality of logical expressions, I believe its use will introduce a measure of clarity in the study of the existential graphs.

1. Since "P implies P" is always true, we see that any graph having the pattern (P(P)) will have the truth value true.

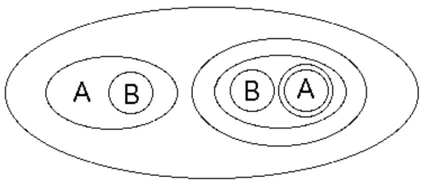


2. Two circles around any P has the same truth value as P: ((P)) = P.

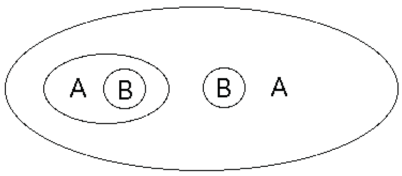


3. PQ = QP because they are both true exactly when A is true and B is true.

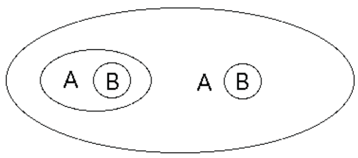
With the help of these patterns, we can see how various tautologies arise. For example, "A implies B implies [not B implies not A] " is transcribed to:



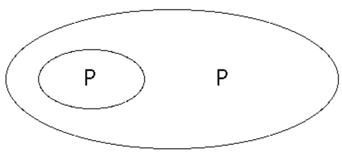
This reduces to:



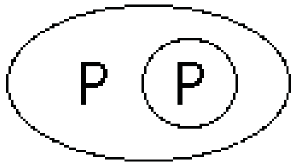
Which is equal to:



And this has the form:



This is equivalent to:



which is always true since it expresses the truism “P implies P”. In this way the existential graphs give access to the interior structure of the tautologies.

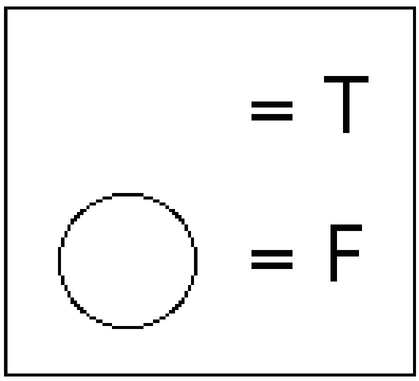
This is the mystery of elementary logic: *the interior structure of the tautologies.* What forms of utterance are necessarily true, which are contingent on circumstance and which are simply false? All logical systems aim at clarifying this matter. What is important about the existential graphs is that they allow the visual manipulation of complexes of Signs to arrive at the desired answers. A visual language for logic emerges from the existential graphs.

Extra decorations on the Existential Graphs allow them to include quantifiers and modal logic as well. What is quite fascinating in reading Peirce on these developments is his maintenance of a clear conceptual line connecting spoken and written language on the one hand, and diagrammatic and written formalisms on the other. The places where these domains can touch are sometimes sparse and delicate. A good example is implication: “not (A and not B)” is a denial in language drawing the precise boundary that defines implication. The logically equivalent statement “(not A) or B” is puzzling in ordinary language, requiring an analysis to ferret out its meaning. There is a subtle difference in the use of “or” as opposed to “and” in ordinary language. In ordinary language A “or” B usually means “A or B and not both”. In standard logic it is “A or B or both” that is intended.

A minimal formalism may not be the most effective interface with speech and word, and yet the mathematician will continue to search for these least structures for the sake of economy, elegance and computational effectiveness. Peirce walks the creative middle road. Elegance and economy emerge nonetheless.

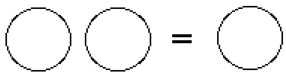
VI. Existential Graphs and Laws of Form

We now make a descent into the internal structure of the existential graphs in a direction that Peirce apparently did not take to its full conclusion. What is the truth value of the empty existential plane? In that plane nothing is asserted to exist. The truth value is True ,T. An empty circle encloses empty space and so negates it, giving rise to the value False, F.

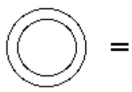


With this convention, we can evaluate patterns of adjacent and nested circles in the plane.

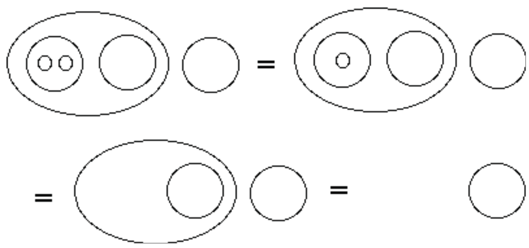
0. “nothing” = T
1. () = F
2. False and False = False. Thus () () = ().



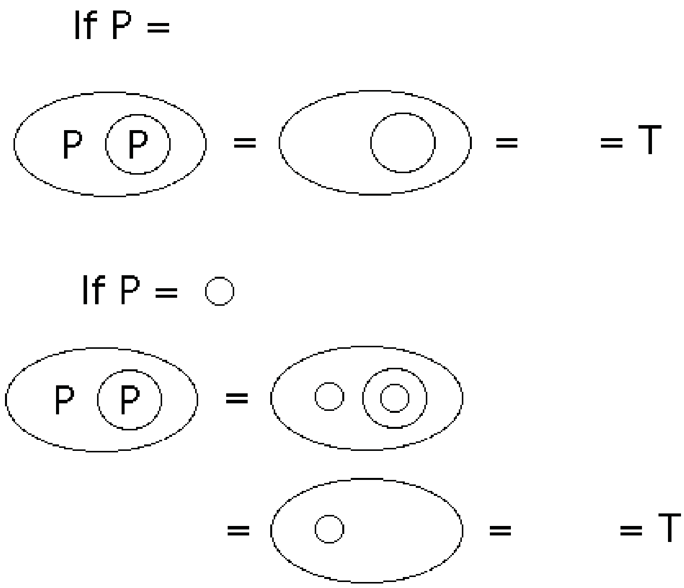
3. (()) = (F) = not F = T. Thus (()) = “nothing”.



More complex expressions can be simplified uniquely by the successive application of these rules. For example



With this arithmetic of circles we can handle the evaluation of existential graphs by direct substitution. For example,



Thus (P(P)) is true in all cases, as it should be, since (P(P)) expresses “P implies P” in the existential graphs.

Something else is going on here. While efficiently calculating the truth tables, the circles hold a simpler and wider meaning of their own. Each circle makes a distinction between its inside and its outside. It is this calculus of distinctions that handles the tautologies.

The two existential graphs are equivalent, exactly when they have the same circle evaluations for all possible substitutions of circles or blanks into the variables in the two graphs.

This arithmetic of circles, implicit in C.S. Peirce's existential graphs, is isomorphic with the primary arithmetic (calculus of indications) discovered by G. Spencer Brown in his lucid book " Laws of Form" [9].

The basic symbol in Laws of Form is a right angle bracket, rather than a circle, but its use is just the same (as an enclosure) and the primary algebra of Spencer-Brown is also isomorphic with the existential graphs themselves. In the usual interpretation for logic in Laws of Form one takes juxtaposition of forms to be "or" rather than "and" thus getting a dual calculus where (A)B stands for " A implies B" and () stands for "True".

Access to the primary arithmetic adds an extra dimension to the structure of the existential graphs.

This primary arithmetic of circles (or brackets) is a fundamental pattern underlying first order logic. First order logic is the mathematical pattern that emerges from the primary arithmetic. It requires some time to get used to this very different point of view about logic. We all know that logic is basically simple, and yet viewed from "and" and "or" and "implies" it has a curious complexity that baffles the intuition. Yet logic is nothing more than the properties of the act of distinction! At the level of the primary arithmetic we are returned to this enlightened state. Then we have to work hard to reestablish connection with the complex world that has been left behind.

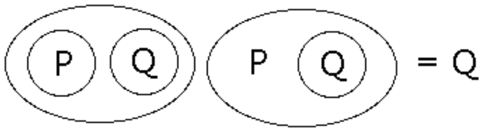
We can use the primary arithmetic to verify many graphical identities, just as we verified (P)P = () above by considering the different substitutions of P as marked or unmarked.

Here is a list of some basic identities that can be checked in this fashion:

1. PP =P
2. () P = ()
3. ((P)) = P
4. (P)Q = (PQ)Q
5. (P(P)) =
6. ((P)(Q))R = ((PR)(QR))
7. ((P) (Q)) (P (Q)) = Q

With the help of these identities it is easy to decide on the validity of expressions in symbolic logic that are expressed in terms of the existential graphs. Identities 4. and 5. (plus the commutativity that is implicit in a planar representation) are sufficient to derive all valid equational identities in this system.

With more work one can in fact use identity number 7. as a basis for the logical algebra. To prove this was a difficult problem in the original context of Boolean algebra where no symbol could be unmarked. (In the usual notation the unmarking of a symbol could lead to unintelligible expressions such as a + ‘.) The problem was solved by Huntington in 1936 [19]. The question is easier when we allow an unmarked symbol. See [15], [16], [17], [18]. Since the seventh equation above is sufficient for the whole alpha theory of existential graphs, it is interesting to consider its interpretation in terms of illation (inference).



This equivalence states that

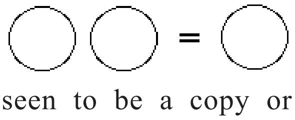
“[[not P] implies Q] and [P implies Q]” is equivalent to “Q”.

Once again, the tautology is understandable in the realm of ordinary language (it is an expression of the law of the excluded middle). It is remarkable that this single identity can be taken as the foundation for the theory of Peirce’s alpha graphs.

It is important to note that with the primary arithmetic, Spencer-Brown was able to turn the epistemology around so that one could start with the concept of a distinction and work outwards to the patterns of first order logic. The importance of this is that the simplicity of the making (or imagining) of a distinction is always with us, in ordinary language and in formal systems. Once it is recognized that the elementary act of discrimination is at the basis of logic and mathematics, many of the puzzling enigmas of passing back and forth from formal to informal language are seen to be nothing more than the inevitable steps that occur in linking the simple and the complex. The elementary act, the deep structure, is not simple. The locutions of ordinary language are in fact quite simple but elaborate. These locutions enable us to speak without thinking. Yet they enable us to speak thoughtfully. It is in the articulation of careful thought that the descent into basic discrimination is called for without compromise.

In this view, the empty circle is not just a contracted or abstracted notation, but rather *an iconic representative of an elemental distinction*. The primary arithmetic is a mathematical language that is based on that distinction and incorporates it into its own symbol system.

In that view the identity ()() =()



states (when one circle is seen to be a copy or symbol for the other) the redundancy of naming the distinction with a copy of itself.

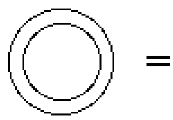
Here we make contact with the elements of Peirce’s trichotomy

Sign (Representamen)/Signified (Object)/Interpretant.

The circle is a Sign for itself where "itself" is a distinction drawn in the plane and the Sign is also a circle drawn in the plane. The circle signifies a distinction and the distinction that is signified is made by the circle itself! Thus in the primary arithmetic or calculus of indications, we have a minimal form of that Peircian semiotic epistemology. The circle refers to itself, but that self is a Sign in a context of Signs, and so the circle can refer to other Signs individually indistinct from itself and yet distinguished from the original circle by the context of this community of Signs. In this community, an individual circle can appear or disappear, yet the identity of the circle as a Sign is inviolate. Here the interpretant may at first seem to be the formal system of the circles themselves, but this widens to include the person making these distinctions, and widens further to include the entire Sign complex that constitutes that person and is reflected in the arithmetic itself. All this remains true for numerical arithmetic and beyond. In the primary arithmetic the relation of Sign and distinction is transparent.

In Chapter 12 of Laws of Form [9], Spencer-Brown writes " We see now that the first distinction, the mark and the observer are not only interchangeable, but, in the form, identical. " The mathematician is not distinguished from the system that he/she is making.

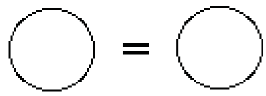
In that view the identity (())= "nothing"



is interpreted as the statement

"A crossing from the marked state yields the unmarked state".

And the unwritten identity () = ()



is interpreted as the statement

"A crossing from the unmarked state yields the marked state".

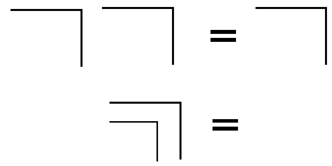
In both cases the circle is viewed as either a noun (name of the outside of the distinction) or a verb (crossing from the state indicated on the inside of the circle). I would summarize what we have just said by the following sentence.

> In descending to the primary arithmetic, one enters a natural world of (formal) speech that has its own meaning in relation to a distinction (made by that speech itself), a meaning that informs and underlies the logic of language and mathematics.

This description marks the beginning of seeing Peirce's existential graphs in this light.

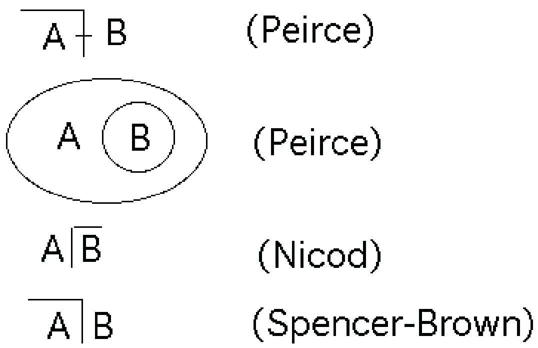
Remark on Notation.
We remark that in Spencer-Brown's Laws of Form [9], the notation for an enclosure is not a circle, but a right angle bracket.

Thus the laws of calling and crossing as we have drawn them in circles become the following patterns in the right angle bracket:



As we go from arithmetic to algebra and logic, Spencer-Brown makes the choice that AB (A juxtaposed with B) represents "A or B" rather than "A and B" as we have seen in the existential graphs and with the Sign of illation. However, with the marked state interpreted as "True" and the unmarked state interpreted as "False", implication in the Spencer-Brown algebra is given by the form shown below.

This puts implication in Laws of Form in exactly the same pattern as in Peirce's Sign of illation! In fact, now we have the following curious rogues gallery of notations for implication:



The first Peirce notation is the portmanteau Sign of illation. The second Peirce notation is the form of implication in the existential graphs. The Nicod notation is a portmanteau of the Sheffer stroke and an (overbar) negation sign. The Spencer-Brown notation is in line with the Peirce Sign of illation. It remained for Spencer-Brown (some fifty years after Peirce and Nicod) to see the relevance of an arithmetic of forms underlying his notation and thus putting the final touch on a development that, from a broad perspective, looks like the world mind doing its best to remember the significant patterns that join logic, speech and mathematics. The movement downward to the Form ("we take the form of distinction for the form."[9, Chapter 1, page 1]) through the joining together of words into archetypal portmanteau Signs can be no accident in this process of return to the beginning.

VII. The Logical Garnet

The purpose of this section is to point out a remarkable connection between Laws of Form, the Existential Graphs of Peirce, polyhedral geometry, mirror symmetry and the work of Shea Zellweger [30].

Zellweger did an extensive study of the sixteen binary connectives in Boolean logic (“and”, “or” and their relatives — all the Boolean functions of two variables), starting from Peirce’s own study of these patterns. He discovered a host of iconic notations for the connectives and a way to map them and their symmetries to the vertices of a four dimensional cube and to a three dimensional projection of that cube in the form of a rhombic dodecahedron. Symmetries of the connectives become, for Zellweger, mirror symmetries in planes perpendicular to the axes of the rhombic dodecahedron. See Figure 2. Zellweger uses his own iconic notations for the connectives to label the rhombic dodecahedron, which he calls the “Logical Garnet”.

This is a remarkable connection of polyhedral geometry with basic logic. The meaning and application of this connection is yet to be fully appreciated. It is a significant linkage of domains. On the one hand, we have logic embedded in everyday speech. One does not expect to find direct connections of the structure of logical speech with the symmetries of Euclidean Geometry. It is the surprise of this connection that appeals to the intuition. Logic and reasoning are properties of language/mind in action. Geometry and symmetry are part of the mindset that would discover eternal forms and grasp the world as a whole. To find, by going to the source of logic, that we build simultaneously a world of reason and a world of geometry incites a vision of the full combination of the temporal and the eternal, a unification of action and contemplation. The relationship of logic and geometry demands a deep investigation. This investigation is in its infancy.

In this section I will exhibit a version of the Logical Garnet (Figure 2) that is labeled so that each label is an explicit function of the two Boolean variables A and B. A list of these functions is given in Figure 1. We will find a new symmetry between the Marked and Unmarked states in this representation. In this new symmetry the mirror is a Looking Glass that has Peirce on one side and Spencer-Brown on the other!

Before embarking on Figure 1, I suggest that the reader look directly at Figure 2. That Figure is a depiction of the Logical Garnet. Note the big dichotomies across the opposite vertices. These are the oppositions between Marked and Unmarked states, the opposition between A and not A, and the opposition between B and not B. If you draw a straight line through any pair of these oppositions and consider the reflection in the plane perpendicular to this straight line, you will see one of the three basic symmetries of the connectives. Along the A/not A axis the labels and their reflections change by a cross around the letter A. Along the B/not B axis, the labels change by a cross around the letter B. These reflections correspond to negating A or B respectively. Along the Marked/Unmarked axis, the symmetry is a bit more subtle. You will note the corresponding formulas differ by a cross around the whole formula and that both variables have been negated (crossed). Each mirror plane performs the corresponding symmetry through reflection. In the very center of the Garnet is a double labeled cube, labeled with the symbols “A S B” and “A Z B”. These stand for “Exclusive Or” and its negation. We shall see why S and Z have a special combined symmetry under these operations. The rest of this section provides the extra details of the discussion.

Let us summarize. View Figure 2. Note that in this three-dimensional figure of the Logical Garnet there are three planes across which one can make a reflection symmetry. Reflection in a horizontal plane has the effect of changing B to its crossed form in all expressions. Reflection in a vertical plane that is transverse to projection plane of the drawing, interchanges A and its crossed form. Finally, reflection in a plane parallel to the projection plane of the drawing interchanges marks with unmarks. We call this the Marked/Unmarked symmetry.

On first pass, the reader may wish to view Figure 2 directly, think on the theme of the relationship of logic and geometry, and continue into the next section. The reader who wishes to see the precise and simple way that the geometry and logic fit together should read the rest of this section in detail.

Figure 1 is a list of the sixteen binary connectives given in the notation of Laws of Form. Each entry is a Boolean function of two variables. In the first row we find the two constant functions, one taking both A and B to the marked state, and one taking both A and B to the unmarked state (indicated by a dot). In row two are the functions that ignore either A or B. The remaining rows have the functions that depend upon both A and B. The reader can verify that these are all of the possible Boolean functions of two variables. The somewhat complicated looking functions in the last row are “Exclusive Or” , A *S* B and its negation A *Z* B.

In order to discuss these functions in the text, and in order to discriminate between the Existential Graphs and the Laws of Form notations, I will write <A> for the Laws of Form mark around A. Thus, in contrast, (A) denotes the Existential graph consisting in a circle around A. The Spencer-Brown mark itself is denoted by <>, while the circle in the Peirce graphs is denoted by (). Exclusive Or and its negation are given by the formulas:

A *S* B = <<A>B><A<B>>

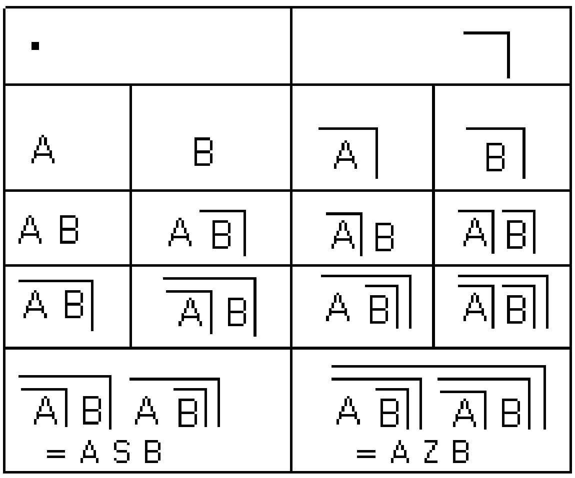


Figure 1. The Sixteen Binary Boolean Connectives

Here we have used Zellweger's alphabetic iconics for Exclusive Or with the letters *S* and *Z* topological mirror images of each other.

Exclusive Or is actually the simplest of the binary connectives, even though it looks complex in the chart in Figure 1. Let "Light" denote the unmarked state and "Dark" denote the marked state. Then the operation of Exclusive Or is as follows:

Dark *S* Dark = Light,
Dark *S* Light = Dark,
Light *S* Dark = Dark,
Light *S* Light = Light.

Imagine two dark regions, partially superimposed upon one another. Where they overlap, the darknesses cancel each other, and a light region appears. This is the action of Exclusive Or. Darkness upon darkness yields light, while darkness can quench the light, and light combined with light is light. In other words, Exclusive Or is the connective closest to the simple act of distinction itself, and it is closest to the mythologies of creation of the world (heaven and earth, darkness and light) than the more complex movements of "and" and "or". Exclusive Or and its negation sit at the center of the logical garnet, unmoved by the symmetries that interchange the other connectives.

The operation of Exclusive Or on the marked state is the same as negation (darkness cancels darkness to light) and the operation of Exclusive Or on the unmarked state is the identity operation that makes no change.

A S<> = <A>
while
A S . = A

The symmetries of Exclusive Or are very simple. If we change one of the variables to its negation we just switch from S to Z! That is,

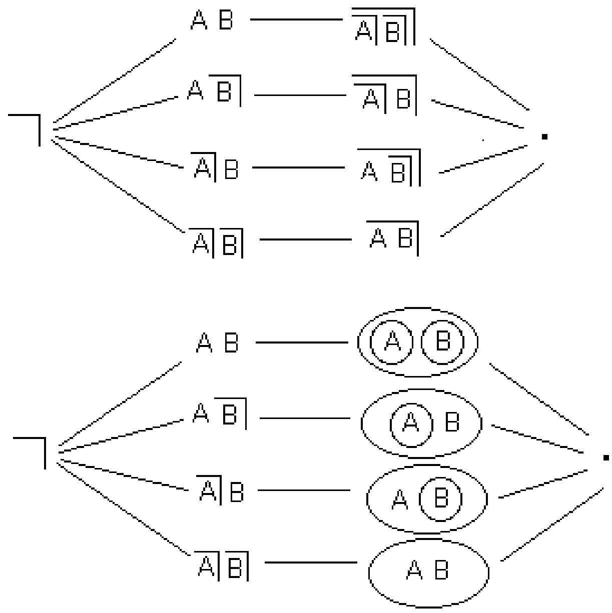
<A>S B = A S<B> = A Z B.

As a result, A S B and A Z B together are invariant under the symmetries induced by the A/not A and B/ not B polarity.

We now discuss the Marked/Unmarked symmetry in the Garnet. This symmetry is available on a look at the Logical Garnet (Figure 2). For reference I have also shown the corresponding terms below (Diagram 1). Note that the horizontal lines in the middle of each diagram are not edges in the Garnet, but they do connect terms that correspond to one another under the mirror symmetry.

We see from Diagram one that the correspondence of terms in the Marked/Unmarked symmetry on the Logical Garnet is exactly the translation that we have described in this paper between the Laws of Form representation and Peirce's Existential Graphs! This last symmetry is actually a translation between two closely related languages for basic logic. This translation is a translation that interchanges "and" and "or". In fact, the reader will note that each of these terms occurs in between two single letters (possibly with one or both negated) on the Garnet. We have arranged the labels on the Garnet so that the front plane compound terms are the "or" of the adjacent vertices and the back plane terms are the "and" of the adjacent vertices.

Diagram 1 - Marked - Unmarked Symmetry



Note also that the central vertex (cube) in the Garnet (labeled with A *S* B and A *Z* B) is connected to the eight compound terms on the periphery of the Garnet. These terms are the terms that arise from Exclusive Or and its Complement when we take it apart. For example

A *S* B = <<A>B><A<B>>

and we can take this apart into the two terms

<<A>B> and <A<B>>,

while

<A> *S* B = <AB><<A><B>>

and we can take this apart into the two terms

<AB> and <<A><B>>

The reader will enjoy looking at the geometry of the way the central and simple operation of Exclusive Or is taken apart into the more complex versions of “and” and “or” and how the Geometry holds all these patterns together.

As for the periphery of the Garnet, it is useful to diagram this as a plane graph with the corresponding labels shown upon it. The illustration below (diagram 2) exhibits this graph of the rhombic dodecahedron.

The rhombic dodecahedron itself does not have a central vertex and the graph below shows precisely the actual vertices of the rhombic dodecahedron and their labels. By comparing with Figure 2, one can see how to bring this graph back into the third dimension. Note how we have all the symmetries apparent in this planar version of the rhombic dodecahedron, but not yet given by space reflection. It requires bringing this graph up into space to realize all its symmetries in geometry.

Diagram 2 - Planar Graph of the Rhombic Dodecahedron

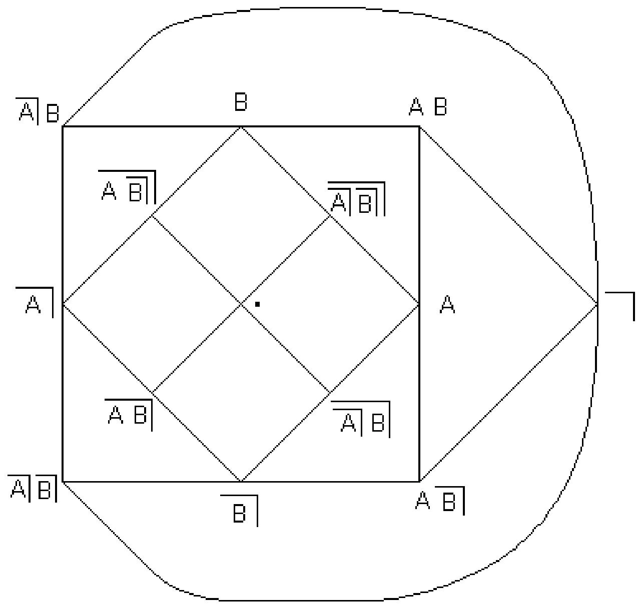
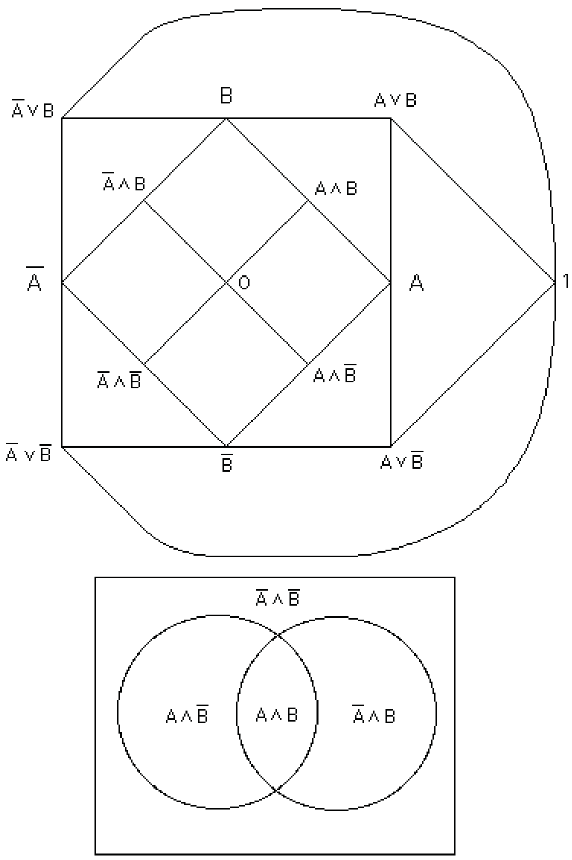


Diagram 3 - The Rhombic Dodecahedron as the Lattice Associated with a Two Circle Venn Diagram



Looking at this peripheral structure, we see the genesis of the pattern of the rhombic dodecahedron in relation to the connectives.

This graphical pattern can be viewed as the lattice of inclusions of these functions regarded as subsets of a universal set. To see this clearly, view the next diagram where we have labeled the vertices of the graph in standard notation with an upward pointing wedge denoting intersection ("and"), a downward pointing wedge denoting union ("or"), 0 denoting empty set and 1 denoting the universe. Then, going outward from 0, pairs of vertices are connected to vertices denoting the union of their labels until we reach the whole universe which is denoted by 1. This lattice is exactly the graph of the rhombic dodecahedron (Diagram 3).

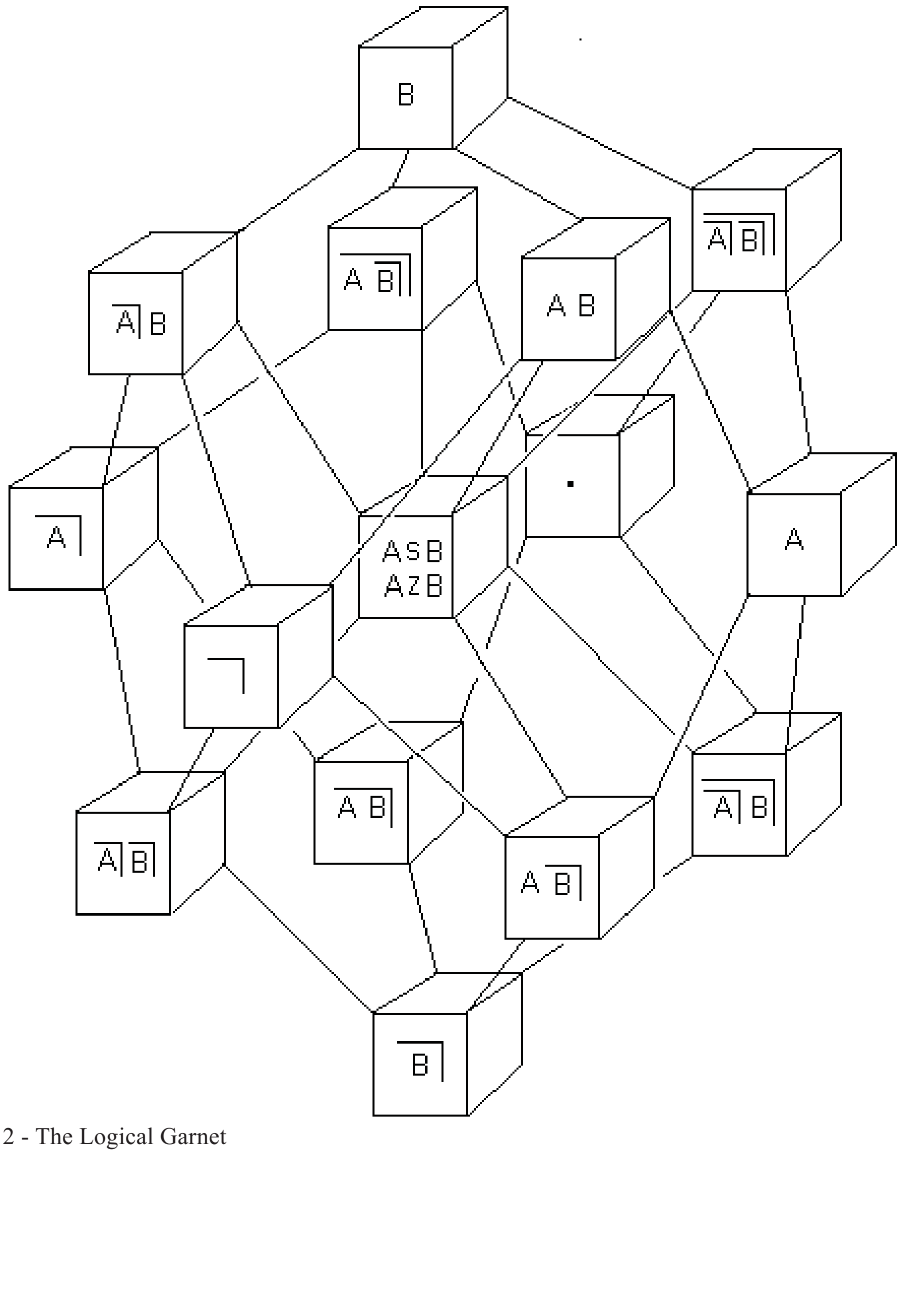


Figure 2 - The Logical Garnet

In this section we have exhibited a version of the Logical Garnet that intermediates between Peirce's Existential Graphs and the dual approach of Laws of Form. This appearance of significant Geometry at the very beginning of Logic deserves deeper investigation. The diagrammatic investigations of Peirce, Venn, Carroll, Nicod and Spencer-Brown are all ways of finding geometry in logic, but in Zellweger's Logical Garnet classical three-dimensional geometry appears, and this is an indication that one should think again on the relationship of logic and mathematics.

VIII. Infinity and Infinitesimals, Recursive Domains

Peirce was an advocate of the notion that infinitesimal numbers were as natural as the concepts of infinity and infinite numbers.

That there was a controversy over this point is a consequence of the history of the calculus where, at first, Newton and Leibniz both used infinitesimals freely. Later as a critical period set in, mathematicians decided to keep track of all approximations as precisely as possible and the concept of limits was born. With this, a direct need for infinitesimals vanished and the machinery of mathematical scholarship kept these "ghosts of departed quantities" in the background. Peirce was one of the few mathematical people who advocated infinitesimals in the early part of the twentieth century.

The situation did not begin to clear up until the 1970's when the logician Abraham Robinson [20] published his beautiful work showing how to work with infinitesimals in their full subtlety. Later developments produced different models of numbers that included infinitesimals with less formal machinery than the Robinson theory [20]. For example, there are the surreal numbers of John Conway [21], the square zero infinitesimals of Lawvere [27] and Bell [22], the sequence infinitesimals of Henle [23]. It will help this discussion to consider the concept of infinitesimal in an informal way, and then to compare with what Peirce said about them.

We imagine a new sort of positive number d that is not zero, and is nevertheless "smaller" than any ordinary positive number that you can name. This infinitesimal d is in itself a generator of other infinitesimals. Thus d+d = 2d is also infinitesimal and larger than d, while d x d = d^2 is smaller than d. If we take the reciprocal 1/d we obtain a number that is "larger" than any standard number. This means that 1/d is a kind of "infinite number", but 1/d is not the same as the reciprocal of 0 (and we do not allow 1/0 in our calculations since it leads to contradictions). The addition of d and its powers and reciprocals to the number system actually does not , if handled correctly, lead to any contradictions. Adding d to the numbers is quite analogous to extending the number system to include the square root of minus one (to create the complex numbers). The extension of numbers to include infinitesimals can be accomplished, and once it is done, one can do calculus without using limits at the fundamental level. The basic idea is that with an infinitesimal one can study how a function changes "instantaneously". That is we can form the ratio $(f(x+d)-f(x))/d$ and call the standard part (the non-infinitesimal part) of this quotient the *derivative* of the function f at the point x.

For example, if $f(x) = xx$ (the product of x with itself), then

$$(f(x+d)-f(x))/d = ((x+d)(x+d) - xx)/d = (2xd - dd)/d = 2x - d.$$

Since 2x is the non-infinitesimal part of this difference quotient, we conclude that the derivative of xx is 2x.

The concept of infinitesimal is closely tied with the concept of continuity. Infinitesimals seem to form a glue that holds the points of the line together. These sorts of intuitions were at the core of Peirce's discussion of infinitesimals. Here is his voice:

> It is singular that nobody objects to the square root of minus one as involving any contradictions, nor, since Cantor, are infinitely great quantities much objected to, but still the antique prejudice against infinitesimally small quantities remains. [6, Vol. 3, p. 123].

A little later he continues with arguments relating this to our understanding of consciousness.

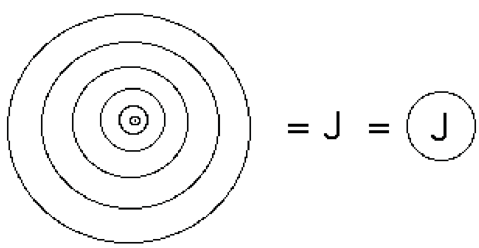
> It is difficult to explain the fact of memory and our apparently perceiving the flow of time, unless we suppose immediate consciousness to extend beyond a single instant. Yet if we make such a supposition we fall into grave difficulties unless we suppose the time of which we are immediately conscious to be strictly infinitesimal. [6, Vol. 3, p. 124]

In this way, Peirce identifies the infinitesimal with the consciousness of the immediate moment.

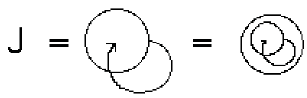
> We are conscious only of the present time, which is an instant, if there be any such thing as an instant. But in the present we are conscious of the flow of time. There is no flow in an instant. Hence the present is not an instant. [6, Vol. 3, p. 126]

By taking the stance that that there can be no movement in an instant, Peirce argues that the present (infinitesimal) moment cannot be an instant. Along with this argument for the notion that the perception of the present is not a point but rather an infinitesimal, Peirce takes the stance that the continuum of the line is not made of points and that any attempt to analyze the line into points will lead to higher and higher orders of infinity for the number of points on that line. This idea is precisely born out in the surreal numbers of John Conway [21]. See also discussion along a similar vein in the paper by Robin Robertson [34] in this volume.

It is interesting to speculate whether Peirce took these ideas of infinitesimals, continuity and infinity into the arena of his existential graphs. If so, he might have considered infinite graphs such as the one shown below.



Peirce's view of an inexhaustible infinity is closely related conceptually with the reflexivity embodied in the self-containing form J depicted above. If we say that J is identical to J with a circle around it, then this identification is really the step of adding one more circle to the pattern. It is a conceptual recognition of the potential endlessness of the series of nested circles. Just as Peirce's work with the existential graphs skirted close to the underlying structure of the primary arithmetic of the circles themselves, so does his work on continuity skirt close to the paradoxical and topological patterns of reflexivity. These patterns have flowered in modern logic through the work of Church and Curry [25] (lambda calculus), Gödel [31], [32], [33] (self-reference and incompleteness), and Turing [33] (recursive instruction and ideal computers). In the tradition of existential graphs we might write a new graph just designed to indicate this reflexive reentry of the form into itself that takes place not in an instant but in a time rendered infinitesimally small.



While it is certainly speculation to imagine that Peirce entertained the formalisms of self-reference and re-entry, nevertheless the context for such constructions is indeed not far from his point of view about the nature of mathematics. It is clear from his writings that he regarded mathematics as a creative enterprise where one could make a hypothesis, draw a figure, visualize a pattern and follow out the consequences of this activity. If it should happen that the assumptions lead to a contradiction, then that is a result in itself. Thus he would take infinitesimals as innocent before proven guilty. And in this way, he managed to foretell the fate of these structures as they were indeed pronounced innocent by the court of Abraham Robinson [20].

This human and constructive attitude (innocent before proven guilty as in the last paragraph) toward mathematics would have made Peirce quite receptive to the reflexive domains of the untyped lambda calculus [25] where the fixed points and self-references can exist. A lambda domain in this sense is a class of objects that can act on one another to form new objects of the same kind.

The action of x on y is denoted by the juxtaposition xy. The characteristic of a lambda domain is that functional operations on the domain acquire names and become objects in the domain. Thus if we decide that F will operate on the domain via Fx=(xy)x for some fixed y, then this definition of F is sufficient to allow it membership in the domain. A lambda domain is like a computer language where you can add new words to the language by defining them as actions on previously created words, and allowing these actions to extend to the new words themselves so that a word can act on itself. This dictum has recursive consequences as we shall see in a moment, but my point is that Peirce's view of mathematics as a whole really is that Mathematics is a lambda domain managed (gardened) by the judgment of the human beings who sign, signify and make references in that garden. This means that the function of the mathematician is not to determine the eternal nature of the objects in the garden, but rather to find that they and the mathematician himself (or herself) are all Signs, growing together in the expansion of Language.

The reflexive and recursive nature of lambda domains was recognized most clearly by Church and Curry [25] who proved the

Theorem. For every element **F** in a lambda domain, there is a **J** in that domain such that **F(J) = J**.

This is the *Fixed Point Theorem of Church and Curry.*

Proof. Let **G** be defined by **Gx = F(xx)** .
G itself is in the lambda domain. Thus we can form **GG**.
But **GG = F(GG)** since **Gx=F(xx)** for any **x**.
Therefore take **J=GG** and conclude that **J=F(J)** as desired. //

This fixed point theorem is in the very center of modern logic, mathematics and computer science. It encodes most of the known paradoxes and the form of Gödel's Incompleteness Theorem, the essence of recursion and the sizes of infinity [27] in a guise of extreme simplicity. Peirce would have approved.

It would take us too far afield to mark out just how this one fixed point theorem is the core of so many apparently diverse matters. Some of these themes have already been taken up in the author's columns for Cybernetics and Human Knowing [35]. The article [27] by Lawvere is also a useful introduction. The relation with paradox is easy and will give the flavor of the matter: Let **F** be the operation of negation **~.** Then the Theorem supplies us with J such that **~J = J.** An entity that is equal to its own negation is the same as an entitity that asserts "I am a liar." Thus the famous Paradox of the Liar is a direct production of the Fixed Point Theorem.

Lambda domains have been formalized and partially tamed for the sake of mathematical foundations and computer science. Philosophical biologists and cyberneticians such as F. Varela [10], H. Maturana and F. Varela [11] L. H. Kauffman and F. Varela [12], Heinz von Foerster [13] and L. H. Kauffman [14] have written eloquently of the basic nature of reflexive domains in relation to the biological and linguistic view of Nature inseparable from her sentient creations.

Leibniz, seventeenth century philosopher, logician and mathematician dreamed of a logical calculus of thought that would allow persons inclined to investigate a topic or settle a dispute to simply sit down and calculate together to come to agreement and knowledge. The active possibility of a symbolic logic with the power to encompass Leibniz's dream of a *calculus ratiocinator* was present to Peirce and the logicians of his generation with many hoping to create that language in thought and diagrams. The dream has now expanded into the present world of recursive complexity. The dream has not disappeared.

IX. A Sign of Itself

There is clearly much more to be done in this arena of investigation and speculation into the nature and structure of the mathematics of Charles Sanders Peirce. I believe that the key to understanding Peirce on mathematics is his view of the nature of a human being as a Sign. In this view there can be no essential separation of the human being and the mathematics or language of that being. This may seem a radical stance in this antiseptic age. Here is Peirce himself, speaking about "a Sign of itself":

> "But in order that anything should be a Sign it must 'represent', as we say, something else called its *Object,* although the condition that a Sign must be other than its Object is perhaps arbitrary, since, if we insist upon it we must at least make an exception in the case of a Sign that is part of a Sign. Thus nothing prevents an actor who acts a character in a an historical drama from carrying as a theatrical 'property' the very relic that article is supposed merely to represent, such as the crucifix that Bulwer's Richelieu holds up with such an effort in his defiance. On a map of an island laid down upon the soil of that island there must, under all ordinary circumstances, be some position, some point, marked or not, that represents *qua* place on the map the very same point *qua* place on the island…
>
> If a Sign is other than its Object, there must exist, either in thought or in expression, some explanation or argument or other context, showing how – upon what system or for what reason the Sign represents the Object or set of Objects that it does. Now the Sign and the explanation make up another Sign, and since the explanation will be a Sign, it will probably require an additional explanation, which taken together with the already enlarged Sign will make up a still larger Sign; and proceeding in the same way we shall, or should ultimately reach a Sign of itself, containing its own explanation and those of all its significant parts; and according to this explanation each such part has some other part as its Object." [24]

In this passage Peirce speaks as a topologist. He tells us that if we overlay or in any (continuous) way place the map of a territory upon that territory then there must be a point on the map that coincides with the corresponding point on the territory. At first this statement might seem quite astonishing, but it is indeed true and it is the content of the famous

Brouwer Fixed Point Theorem. If **F** is any continuous mapping of a disk **D** to itself, then there exists a point **p** in the disk **D** that is left fixed by the mapping**: F(p) =p**.

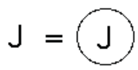
The proof of this Theorem is illuminating and we refer the reader to its exposition in [26]. Presumably Peirce is assuming that his territory is in the topological shape of a disk. Otherwise the result is false. Imagine a world in the shape of a donut and a map of that world just the size of the world itself. Rotate the map a small amount in both of the turns available on a donut and every point of the map will move away from itself. In this example of a world in the shape of a donut, there are mappings that do not have fixed points.

In a universe in the shape of a disk, let the map be of the same size and shape as the disk itself. If we rotate that map about the center point of the disk by a small angle, then the center point of the map will coincide with the center point of the territory, and this will be the only point with this property.

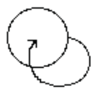
But of course Peirce is not just assuming that the territory mapped is in the shape of a disk. He is using topology as an amplifier for our thought about self-reference. We are all familiar with the situation of superimposing a map on its territory. We have all been in a park and encountered a map with a marked point signifying "You are here." We have all seen that the orientation of the map itself in the space may not match the actual directions, but the truth of the self-locator on the map is still most useful.

In fact, Peirce in this passage is coming very close to the message of the Fixed Point Theorem of Church and Curry that we discussed in the last section. When he says of the place of coincidence of map and territory "we shall reach, or should, ultimately reach a Sign of itself, containing its own explanation and those of all its significant parts: and according to this explanation each such part has some other part as its Object" he is describing a Sign that refers to a significant part of itself and through that to itself. The Sign should "contain its own explanation". This is the reflexive or recursive nature of the reentrant or self-referential form.

Compare this discussion with the reentrant J of the previous section. The equation



asserts the reentry of J into its own indicational space, and it exhibits J as a "part of itself". The equation is the explanation of the nature of J as reentrant and can be taken as a description of the recursive process that generates an infinite nest of circles. It is only *J as an equation* that yields J as a Sign of itself. If we wish to embody the equation in the Sign itself then we need to allow the Sign to indicate its own reentry as we did in the last section with the symbol shown below.



This symbol does "contain its own explanation" in the sense that we interpret the arrow as an instruction to reenter the form inside the circle (ad infinitum). Self-reference is infinity in finite guise.

It has been said that "the map is not the territory" and this is indeed correct. But the most interesting terrain is that territory where we have no choice but to use the territory in the course of the construction of the map. And this is exactly what is done in mathematics, linguistics and science. In order to study language one must use language. In order to study mathematics one must use mathematics, and indeed we use mathematics to elucidate mathematics. Map and territory grow and evolve together in the course of time. In this view it is obvious that any attempt to fully explain anything will cause the map and territory to expand into a new domain in which further explanation will be needed. As Spencer-Brown says ([9] p.106) "In this sense, in respect of its own information, the universe must expand to escape the telescopes through which we, who are it, are trying to capture it, which is us."

The lambda domain of the previous section is a territory that is susceptible to self-evolution. Functions and descriptions of the lambda domain are also elements of that domain. And even though a lambda domain is not a topological disk, there will (via the fixed point theorem) be necessary points of coincidence between the points in the territory and the descriptions (maps) of that territory. It is an abstract model of language as a texture not of just words, but speakers. Each speaker is both noun (person, listener) and verb (person, speaker). Each person is his/her own explanation, but that explanation is a function of the entire domain of language, including the explanation itself.

X. Peirce and Second Order Cybernetics

Peirce, in speaking of the necessary occurrence of a "Sign for itself" in the relation of map and territory is referring, through a topological metaphor, to the reflexive nature of the domain of human discourse. Here is the conduit between Peirce and second order cybernetics.

In the last part of the quoted passage in the last section Peirce speaks of a hierarchy of Signs and explanations leading eventually to the Sign of itself " containing its own explanation and those of all its significant parts." The hierarchy occurs each time one looks into the context for the explanation of the given Sign. Sign and explanation form a new Sign to be explained ad infinitum (or in a circular network of explanations of explanations).

In the case of the lambda calculus or the simple infinite nest of circles we see images of this process of enfoldment where a larger external context is kept in the background. It is important to realize the extent to which we will keep such a background hidden for our own convenience! "I am the one who says I" and indeed the Sign "I" is a Sign for itself, but the full context is the entire English language and all its speakers, each of whom says "I". In a restricted context, one may manage without being engulfed by the language as a whole, and this is indeed the game played by a mathematician (or Humpty Dumpty! [3]) who would have words mean what he wants them to mean in a special context. The cost to Humpty Dumpty is well known; the cost to the mathematician is the emergence of paradox and complexity.

To avoid paradox in the lambda calculus, just such a (restricted) hierarchy was constructed by Dana Scott [28]. Scott used topology and recursive construction to create a reflexive space where every homeomorphism (continuous self-mapping with continuous inverse) of the Scott Space corresponds to a point in the space itself. (See [29].) Unbeknownst (perhaps!) to Scott, Peirce had laid down the program for such a construction many years before in his theory of Signs. As we have seen before, Peirce is a presence in the background of Modern Logic.

In reflexivity and in second order cybernetics Signs and their Objects become inextricably interlinked. Here is how Peirce puts the matter:

> According to this, every Sign has a *Precept* of explanation according to which it is understood to be a sort of emanation, so to speak, of its Object. (If the Sign be an Icon, a scholastic might say that the 'species' of the Object emanating from it found its matter in the Icon. If the Sign be an Index, we may think of it as a fragment torn away from the Object, the two in their Existence being one whole or a part of such a whole. If the Sign is a Symbol, we may think of it as embodying the 'ratio' or reason, of the Object that has emanated from it. These of course are mere figures of speech; but that does not render them useless.) [24]

Here Peirce speaks of the interlocking relationship of Sign and Object. An example of the use of this concept in mathematics is the notion of *Gödel numbering* where the Sign for a text is a code number assigned to that text (by a definite procedure specified beforehand). The text is the Object and its Indexical Sign is the Gödelian code number. The reason for the use of such coding is that it then becomes possible for sentences in a formal system to refer to themselves by referring to their own(!) code numbers. This form of controlled self-reference was used by Gödel to prove that sufficiently rich formal systems are either inconsistent or incomplete. His Theorem shows that mathematics can not be encompassed by any single formal system.

Here follows a sketch of how this self-reference is accomplished. In it we shall see an astonishing interlock of Sign and Object.

In the formal context of this Gödelian work, a text can have a free variable that refers to a numerical value. Thus we may write

g ->T(u)

denoting the text by T(u) with its free variable u, and g is the Gödel number of the text. The interlocking relationship between Sign and Object is specifically the fact that the text can use numbers and these numbers can be code numbers for other texts. In particular one can define a function from code numbers to code numbers (from Signs to Signs) as follows

#g is equal to the Gödel number of the text T(g) (with g substituted for u) when g is the Gödel number of T(u).

Thus if

g -> T(u)

then

#g -> T(g).

The movement from the Sign g to the Sign #g is a shift of reference in which the original "name" g is now inherent in the Object T(g) and that Object T(g) has acquired the new name #g. As Peirce says, "If the Sign be an Index, we may think of it as a fragment torn away from the Object, the two in their Existence being one whole or a part of such a whole." Here #g is the "fragment" (disguised by the coding method) of the Object T(g) that holds g within it.

Once this formality of the Gödel numbering gets underway, it is possible to have the actual situation of a fragment in the sense that the text speaks directly about #G, and #G is the Sign of that text. Gödel's trick is to first consider a Sign and Object in the form

G ->T(#u)

Here G is the Sign of the Object T(#u). If we apply the shift to this pair we obtain

#G -> T(#G)

and the Sign is indeed a fragment of the Object. Since the Object is itself a referential text discussing #G, this text discusses its own Sign.

It is through this interlock of Sign and Object that Gödel constructs a text that asserts its own unprovability in the given formal system. See [18].

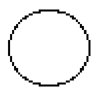
The interlocking relationship of Sign and Object was already well understood by Peirce. The mathematical ingredient for Gödel is the careful use of a restricted context (the given formal system).

The Gödelian result is the ultimate inability of such restricted formal systems to express the full range of mathematical truth. The full Sign of itself lies beyond such restrictions. Just so, we (who are the embodiments of the Sign of itself) can prove the Gödelian Theorem.

It is at the level of the Sign of itself that the Theorem, unprovable in the restricted system, can be proved. The human level, transcending the level of the restricted formal system, is the level of a Sign that is a Sign for itself.

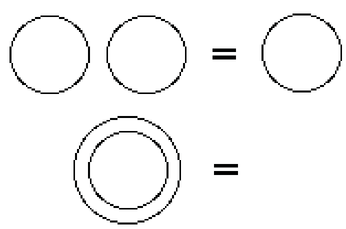
Let us not forget the circle.

As we saw in descending from Peirce's existential graphs to the calculus of indications (by allowing a variable to take the unmarked (true) state), the circle



lives in a language where it is a sign of itself. That language, the calculus of indications, unfolds the patterns of the existential graphs and marks a larger unfolding of language, mathematics and logic as a patterning of possible distinctions.

This point has been discussed at greater length earlier in this paper. We bring it up again here to remind the reader of the essential reflexivity of the basic language in which each circle, seen as a distinction, refers to any other circle seen as a distinction. The two equations of the calculus of indications are each self-referential in this sense. In the first equation (see below) either circle can be regarded as the name of the other circle. In the second equation, one circle acts as instruction to cross from the marked state that is indicated by the other circle. All expressions in the calculus of indications are self-referential. The more objective forms of logic and communication are based on this ground of circularity!



As a Sign of itself, the circle has only itself as a part. That part, equal to the whole, makes the distinction that is the referent of the Sign.

The explanation of this Sign is the Sign itself. The explanation of the circle is what it does in the plane upon which it is drawn. And that doing is the separation and joining of the inside of the circle with its outside. The circle is its own explanation. The circle is a Sign of itself that has no proper part.

And lest the tale of the circle seem too mystical or too particular, let us not forget that this circle is seen to do all that it does by an Observer who is the constructing of the circle.

The Observer is the Object that is the emanation of the Sign of Distinction.

The Distinction is the Object that is the emanation of the Sign of the Observer.

The act of drawing the circle is the motion of a point outward from an original location, only to return to that source in the primordial act of self-reference.

There is no plane, no circle but only an act that moves outward from self-identity and returns to identity.

That act is its own explanation.

XI. Epilogue

We have seen that Logic can pack a double meaning, that Logic could be an encoded form of Geometry. Peirce's portmanteau Sign has expanded to a vastness of multiple interpretations. This was implicit in Boole's original symbolic logic. He borrowed the symbolism of ordinary algebra and invited that symbolism to carry the structure of class and inference. The fit of a good portmanteau operates like a key in a lock, opening a connection between the apparently separate domains that compose it.

Another portmanteau lives in Logic. It is the Gödelian sentence that asserts its own unprovability. This sentence carries a double meaning. Inside the formal system, it is a statement about properties of certain integers. Within the formal system there is no hint that the Gödel sentence has any other meaning. From outside the formal system, the sentence is seen to assert its own unprovability. These two meanings interlock in the compound Sign that is the Gödel sentence, to form a portmanteau that has forever changed our understanding of the nature of formal systems. This understanding is already present in Peirce through his view of the nature of the perceiving consciousness as a Sign for itself.

Examine Charles Peirce's Sign of illation in the light of these reflections. The Sign lives in its own system of inference and needs no hint of the encoding that translates it to the world of Boolean algebra. The Sign has its own life in one language, and a description in terms of component parts in the other. Which is syntax and which is semantics? The portmanteau Sign is born of a linking of syntax and semantics - a portent of the future of language, logic and mathematics, as we have outlined in the body of this paper with the connections with Gödel's Theorem, lambda calculus and the calculus of indications of G. Spencer-Brown.

We ourselves are portmanteau Signs of a complex order. We are packing cases of multiple meaning large enough to make a human being a Sign of itself.

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