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Euler's Formula

Math 220

Complex numbers

A **complex number** is an expression of the form

$$x + iy$$

where x and y are real numbers and i is the "imaginary" square root of -1. For example, $2 + 3i$ is a complex number. Just as we use the symbol $\mathbb{R}$ to stand for the set of real numbers, we use $\mathbb{C}$ to denote the set of all complex numbers. Any real number x is also a complex number, $x + 0i$; that is, the set of real numbers is a subset of the set of complex numbers, or, in set notation, $\mathbb{R} \subset \mathbb{C}$.

If $z = x + iy$, then x is called the **real part** of z and y is called the **imaginary part** of z. This is written

$$x = \mathrm{Re}(z) \qquad \text{and} \qquad y = \mathrm{Im}(z).$$

Two complex numbers are equal if and only if both their real parts are equal and their imaginary parts are equal. The **conjugate** of z is the complex number

$$\bar{z} = x - iy$$

and the **absolute value** of z is

$$|z| = \sqrt{x^2 + y^2}.$$

Note that when $y = 0$, this is the same as the absolute value formula for real numbers x. Note also that, since $(x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2$,

$$z\bar{z} = |z|^2.$$

Complex Arithmetic

You can add, subtract, multiply, and divide complex numbers using the usual rules of algebra, keeping in mind that $i^2 = -1$.

Example 1: *Find the sum:*

$$(2 + 3i) + (4 - i) = 6 + 2i$$

end Example 1

Example 2: *Find the product:*

$$\begin{aligned}(4 - 7i)(2 + 3i) &= 8 - 14i + 12i - 21i^2 \\ &= 8 - (-1)21 - 2i \\ &= 29 - 2i\end{aligned}$$

end Example 2

Writing a quotient in $x+iy$ form requires the use of the conjugate, as the next example demonstrates.

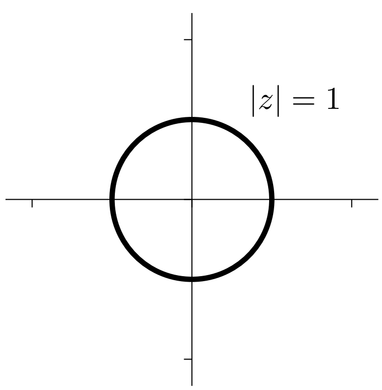
Example 3: *Find the quotient:*

$$\frac{4-7i}{2-3i} = \left(\frac{4-7i}{2-3i}\right)\left(\frac{2+3i}{2+3i}\right) = \frac{29-2i}{4+9} = \frac{29}{13} - \frac{2}{13}i$$

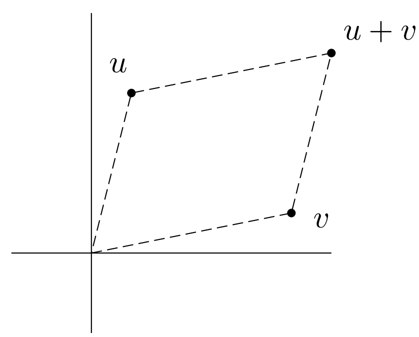
end Example 3

The Complex Plane

Just as we think of a real number x as a point on the number line, we can think of a complex number $z = x + iy$ as a point (x, y) on the plane[1]. Doing so makes for some neat geometric interpretations of complex arithmetic. For instance, if $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$ is the distance from z to the origin. If u and v are two complex numbers, then $|u - v|$ is the distance between u and v in the complex plane. The equation $|z| = 1$ describes the **Unit Circle**, that is, the circle centered at the origin with radius 1. The real number line is the x-axis, i.e., those complex numbers for which $y = 0$. Addition of complex numbers obeys the **Parallelogram Law**: the sum $u + v$ of complex numbers u and v is the fourth vertex of the parallelogram whose other three vertices are u, v, and the origin.



The Unit Circle.



The Parallelogram Law.

Euler's Formula

Most of the functions with domain $\mathbb{R}$ that we use in calculus can be meaningfully extended to the larger domain $\mathbb{C}$. For polynomials and rational functions, for instance, it's clear how to plug in complex numbers.

Example 4: *Find $p(1 + 4i)$ if $p(x) = x^2 + 3x$.*

[1] Calculators sometimes display complex numbers in the form (x, y). To see how your calculator displays the complex number $2 + 3i$, enter $2 + \sqrt{-9}$.

$$p(1+4i) = (1+4i)^2 + 3(1+4i) = 1 + 8i + 16i^2 + 3 + 12i = -12 + 20i.$$

end Example 4

Example 5: *Evaluate $r(5-2i)$ if $r(x) = (x-1)/(x+2)$.*

$$r(5-2i) = \frac{5-2i-1}{5-2i+2} = \left(\frac{4-2i}{7-2i}\right)\left(\frac{7+2i}{7+2i}\right) = \frac{32}{53} - \frac{6}{53}i.$$

end Example 5

But what of other functions? What does it mean, for instance, to evaluate the exponential function at the complex number $x+iy$? For the laws of exponents to still be true with complex numbers, we need that

$$e^{x+iy} = e^x e^{iy}.$$

Since x and y are real, the only expression on the right side that's new is e^{iy}. It turns out that the natural interpretation of this is given by **Euler's Formula**:

$$\boxed{(1) \qquad e^{i\theta} = \cos\theta + i\sin\theta}$$

(Euler is pronounced "Oil-er.") That is, $e^{i\theta}$ is the point on the unit circle θ radians from the positive real axis.

The best explanation of why Euler's Formula is true involves power series, a topic to be covered later in this course[2]. See page A63 of our text for a proof. In the meantime, it may be illuminating to note that Euler's formula is consistent with two fundamental rules of trigonometry.

Start with

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta). \tag{2}$$

According to the laws of exponents, we can rewrite the left side as

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &\quad + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \end{aligned} \tag{3}$$

Equating real and imaginary parts of (2) and (3), we see

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \text{ and} \\ \sin(\alpha+\beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta, \end{aligned} \tag{4}$$

[2] Power series also allow us to define $f(x+iy)$ for many familiar functions $f(x)$.

the sum formulas for sine and cosine.

Euler's formula allows us to rewrite exponentials in terms of trigonometric functions. It is also useful to be able to go the other way: write trigonometric functions in terms of exponentials. To derive the necessary formula, note that, since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$,

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \quad \text{and} \\ e^{-i\theta} &= \cos\theta - i\sin\theta. \end{aligned}$$

That is, $e^{i\theta}$ and $e^{-i\theta}$ are conjugates[3]. By adding or subtracting these equations, and dividing by 2 or $2i$, we obtain

$$\boxed{(5) \qquad \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta \qquad \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta}$$

Note the similarity to the definitions of cosh and sinh[4].

Using equations (1) and (5) often makes it unnecessary to remember tricks involving trigonometric identities. We next look at two examples of indefinite integrals that, without Euler's formula, would require use of the sum and difference formulas for sine and cosine.

Example 6: *Integrate:* $\int \sin^2 x\, dx$.

Before integrating, it is necessary to rewrite the integrand. By (5),

$$\begin{aligned} \sin^2 x &= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 \\ &= \frac{-1}{4}(e^{i2x} + e^{-i2x} - 2e^0) \\ &= \frac{-1}{4}(2\cos 2x - 2). \end{aligned}$$

Now we can integrate:

$$\begin{aligned} \int \sin^2 x\, dx &= \int \frac{-1}{4}(2\cos 2x - 2)\, dx \\ &= \frac{-1}{4}(\sin 2x - 2x) + C \\ &= \frac{-1}{4}\sin 2x + \frac{1}{2}x + C. \end{aligned}$$

(Compare with the solution in Example 3, page 461, which uses the half-angle identities.)

end Example 6

[3] Another familiar trigonometry formula follows from these two equations: $\cos^2\theta + \sin^2\theta = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = e^{i\theta}e^{-i\theta} = e^0 = 1$.

[4] Equations (5) are special cases of the formula $\frac{z+\bar{z}}{2} = \operatorname{Re}(z)$ and $\frac{z-\bar{z}}{2i} = \operatorname{Im}(z)$.

Functions of the form

$$A\sin(Bx+C)+D \quad \text{or} \quad A\cos(Bx+C)+D$$

are called **sinusoidal** functions. Euler's formula allows us to rewrite products of sinusoidals as sums of other sinusoidals which we can then integrate. (Usually, C is zero in all our examples.)

Example 7: *Integrate:* $\int \sin 4x \cos 5x \, dx$.

$$\begin{aligned}
\sin 4x \cos 5x &= \left(\frac{e^{i4x}-e^{-i4x}}{2i}\right)\left(\frac{e^{i5x}+e^{-i5x}}{2}\right) \\
&= \frac{e^{i9x}-e^{-i9x}-e^{ix}+e^{-ix}}{4i} \\
&= \frac{1}{2}\left(\frac{e^{i9x}-e^{-i9x}}{2i}-\frac{e^{ix}-e^{-ix}}{2i}\right) \\
&= \frac{1}{2}(\sin 9x - \sin x).
\end{aligned}$$

Now integrating is straightforward:

$$\begin{aligned}
\int \sin 4x \cos 5x \, dx &= \frac{1}{2}\int (\sin 9x - \sin x)\, dx \\
&= \frac{-1}{18}\cos 9x + \frac{1}{2}\cos x + C
\end{aligned}$$

(Compare with the solution in Example 9, page 465, which relies on the three identities on that page.)

end Example 7

Generally, when presented with an integral of the form

$$(6) \qquad \int \sin \alpha x \sin \beta x \, dx \qquad \int \sin \alpha x \cos \beta x \, dx \qquad \int \cos \alpha x \cos \beta x \, dx$$

or

$$(7) \qquad \int \sin^n x \cos^m x \, dx$$

you can use Euler's formula to rewrite the integrand as a sum of sinusoidals so as to make the integration simple. (When either n or m is odd, it's easier to integrate (7) with a substitution as in §7.2.)

The Binomial Theorem and Pascal's Triangle

For integrals of the last kind, it is helpful to be able to quickly expand powers of the form $(x+y)^n$. The expansion must look something like

$$(x+y)^n = ?x^n + ?x^{n-1}y + ?x^{n-2}y^2 + \cdots + ?x^2y^{n-2} + ?xy^{n-1} + ?y^n. \tag{8}$$

Setting x or y equal zero tells us that the first and last coefficients must both be 1, but what about the others?

The **Binomial Theorem** tells us that the missing constants in (8), called the **binomial coefficients**, are found in the nth row of **Pascal's Triangle**[5]:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

(Pascal's Triangle has infinitely many rows. We refer to the top row as its 0th row.) For instance, the 2nd row, "1 2 1," and the 3rd row, "1 3 3 1," tell us that

$$(x+y)^2 = x^2 + 2xy + y^2, \text{ and}$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

To generate the next row, begin and end with a 1, and then add the two elements above to find the next entry. For example, since $1+3=4$, the next row will begin

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & & & & & 1 \end{array}$$

Finish the row with the sums $3+3$, and $3+1$:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Consequently,

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

[5] Like many old objects in mathematics, this triangle isn't named for any of the various people who first discovered it, including mathematicians in China, Persia, and ancient India. See articles on Pascal's triangle at britannica.com, encyclopediaofmath.org, and wikipedia.org

To see why this works, consider the problem of expanding $(x+y)^3$. You could find $(x+y)^3$ by multiplying $(x+y)^2$ by $(x+y)$:

$$\begin{aligned}(x+y)^3 &= (x^2+2xy+y^2)(x+y) \\ &= x^3+2x^2y+xy^2 \\ &\qquad +x^2y \;\; +2xy^2+y^3 \\ &= x^3+3x^2y+3xy^2+y^3.\end{aligned}$$

Similarly, you could find $(x+y)^4$ by multiplying $(x+y)^3$ by $(x+y)$:

$$\begin{aligned}(x+y)^4 &= (x^3+3x^2y+3xy^2+y^3)(x+y) \\ &= x^4+3x^3y+3x^2y^2+xy^3 \\ &\qquad +x^3y+3x^2y^2+3xy^3+y^4 \\ &= x^4+4x^3y+6x^2y^2+4xy^3+y^4.\end{aligned}$$

Notice that each coefficient in the expansion of $(x+y)^3$ or of $(x+y)^4$, with the exception of the beginning and ending 1s, is the sum of the two neighboring coefficients in the row above. When we use one row of the Pascal's triangle to generate the next, we're just performing this process without all the symbols.

Example 8: *Integrate:* $\int \cos^6 x\,dx$.

Start by using Euler's formula to rewrite the integrand as a sum of sinusoids.

$$\cos^6 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^6.$$

After some addition, we find the 6th row of Pascal's Triangle to be

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

and so

$$\begin{aligned}\left(\frac{e^{ix}+e^{-ix}}{2}\right)^6 &= \frac{1}{64}\big(e^{i6x}+6e^{i4x}+15e^{i2x}+20+15e^{-i2x}+6e^{-i4x}+e^{-i6x}\big) \\ &= \frac{1}{32}(\cos 6x+6\cos 4x+15\cos 2x+10)\end{aligned}$$

Now we can integrate: $\int \cos^6 x\,dx = \frac{1}{32}\left(\frac{1}{6}\sin 6x+\frac{3}{2}\sin 4x+\frac{15}{2}\sin 2x+10x\right)+C$.

end Example 8

Example 9: *Integrate:* $\int \cos^4 x \sin^4 x\,dx$.

Rewrite the integrand:

$$\cos^4 x \sin^4 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^4 \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^4.$$

The 4th row of Pascal's Triangle is $1 \quad 4 \quad 6 \quad 4 \quad 1$, so

$$\begin{aligned}
&\left(\frac{e^{ix}+e^{-ix}}{2}\right)^4\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^4 \\
=&\frac{1}{2^8}\left(e^{i4x}+4e^{i2x}+6+4e^{-i2x}+e^{-i4x}\right)\left(e^{i4x}-4e^{i2x}+6-4e^{-i2x}+e^{-i4x}\right)
\end{aligned}$$

Next, carefully multiply out this product.

$$\begin{aligned}
\frac{1}{2^8}(e^{i8x}&+4e^{i6x}+6e^{i4x}+4e^{i2x}+1 \\
&-4e^{i6x}-16e^{i4x}-24e^{i2x}-16-4e^{-i2x} \\
&+6e^{i4x}+24e^{i2x}+36+24e^{-i2x}+6e^{-i4x} \\
&-4e^{i2x}-16-24e^{-i2x}-16e^{-i4x}-4e^{-i6x} \\
&1+4e^{-i2x}+6e^{-i4x}+4e^{-i6x}+e^{-i8x})
\end{aligned}$$

Collect up like terms, and the integrand is

$$\begin{aligned}
=&\frac{1}{2^8}\left(e^{i8x}-4e^{i4x}+6-4e^{-i4x}+e^{-i8x}\right) \\
=&\frac{1}{2^7}\left(\frac{e^{i8x}+e^{-i8x}}{2}-4\frac{(e^{i4x}e^{-i4x})}{2}+3\right) \\
=&\frac{1}{2^7}(\cos 8x-4\cos 4x+3)
\end{aligned}$$

which is straightforward to integrate. The integration is left to the reader.

end Example 9

It helps to keep on eye out for times when the integrand can be simplified using the double angle formulas or the pythagorean identities. For instance, let's look once more at the integral from the previous example.

Example 10: *Integrate:* $\int \cos^4 x \sin^4 x\, dx$.

This particular integral can be made a lot easier by first using the double angle formula for sine: $\sin 2x = 2\sin x\cos x$. Rewrite the integrand as

$$\cos^4 x\sin^4 x = (\frac{1}{2}\sin 2x)^4 = \frac{1}{16}\sin^4 2x$$

and then substitute $u = 2x$ (so that $\frac{1}{2}du = dx$). The integral then becomes $\int \frac{1}{32}\sin^4 u\, du$.

end Example 10

Exercises

1. Write in $x + iy$ form:

a. $3 + 2i + 2(1 - i)$	b. $3(4 - 5i) - (2 - 4i)$	c. $(i + 1)(i - 1)$
d. $(6 + 3i)(\frac{1}{3} - 2i)$	e. $(4 + i) \div (1 - 8i)$	f. $(3 + 2i) \div 2(1 - i)$
g. $e^{i\pi/4}$	h. $e^{-5i\pi/6}$	i. e^{2-3i}

2. Prove that, if $u = a + ib$ and $v = c + id$, then

$$\overline{u + v} = \overline{u} + \overline{v} \qquad \text{and} \qquad \overline{uv} = \overline{u}\,\overline{v}.$$

3. Plot 0, u, v, and $u + v$ in the complex plane. Compare with Parallelogram Law.

a. $u = 3 + 2i,\ v = 2 - i$	b. $u = 1 + i,\ v = -1 + 2i$	c. $u = 1,\ v = 2i$

4. Plot z and $\bar{z}$. In general, how are z and $\bar{z}$ related geometrically?

a. $z = 4 + 2i$	b. $z = -1 - 2i$	c. $z = 3$

5. Plot the solution set: it might help you to first write the equation in terms of x and y, the real and imaginary parts of z.

a. $\lvert z\rvert = 2$	b. $\lvert z - 1\rvert = 1$	c. $\lvert z - 1 + i\rvert = 1/2$

6. Prove DeMoivre's formula: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

7. Use (5) to "derive" the rules $\dfrac{d}{dx}\sin x = \cos x$ and $\dfrac{d}{dx}\cos x = -\sin x$.

8. Use Pascal's Triangle to expand the binomial:

a. $(x + y)^7$	b. $(x - y)^7$	c. $(x - y)^8$	d. $\left(a - \frac{1}{a}\right)^8$
e. $\left(a + \frac{1}{a}\right)^9$	f. $(e^x - e^{-x})^9$	g. $(1 - v^2)^8$	h. $(u^2 + 1)^9$

9. Rewrite as a sum of sinusoidal functions:

a. $\sin 3x \cos 5x$	b. $\cos 3x \cos 4x$	c. $\sin 2x \sin 4x$	d. $\cos^4 x$
e. $\sin^6 x$	f. $\sin^8 x$	g. $\cos^2 x \sin^4 x$	h. $\sin^6 x \cos^4 x$
i. $\cos^8 x$	j. $\cos^3 x \sin^5 x$	k. $\sin^3 x$	l. $\sin^7 x$

10. Integrate:

a. $\int \sin 3x \sin 5x\, dx$	b. $\int \cos 2x \cos 4x\, dx$	c. $\int \sin 2x \cos 4x\, dx$
d. $\int \sin^4 x\, dx$	e. $\int \sin^6 x\, dx$	f. $\int \sin^2 x \cos^2 x\, dx$
g. $\int \cos^2 x - \sin^2 x\, dx$	h. $\int \cos^4 x - \sin^4 x\, dx$	i. $\int \cos 2x \sin 2x\, dx$
j. $\int_0^{\pi/2} \cos^4 x\, dx$	k. $\int_0^{\pi} \cos 3x \cos 5x\, dx$	l. $\int_0^{\pi/4} \cos^2 x \sin^2 x\, dx$
m. $\int \cos^8 x\, dx$	n. $\int \cos^4 x \sin^2 x\, dx$	o. $\int \frac{\tan^4 x}{\sec^4 x}\, dx$

Selected Answers

1. b. $10-11i$; c. -2; f. $\frac{1}{4}+\frac{5}{4}i$; h. $-\frac{\sqrt{3}}{2}-\frac{1}{2}i$; i. $e^2\cos 3 - ie^2\sin 3$.
2. $\overline{u+v} = \overline{a+ib+c+id} = \overline{(a+c)+i(b+d)} = (a+c)-i(b+d) = a-ib+c-id = \overline{u}+\overline{z}$. 4. $\bar{z}$ is the reflection of z across the real axis. 5. a. circle of radius 2, centered at origin. c. circle of radius $\frac{1}{2}$, centered at $(1,-1)$. 6. $(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$. 8. b. $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$; d. $a^8 - 8a^6 + 28a^4 - 56a^2 + 70 - 56a^{-2} + 28a^{-4} - 8a^{-6} + a^{-8}$; f. $e^{9x} - 9e^{7x} + 36e^{5x} - 84e^{3x} + 126e^x - 126e^{-x} + 84e^{-3x} - 36e^{-5x} + 9e^{-7x} - e^{-9x}$. h. $u^{18} + 9u^{16} + 36u^{14} + 84u^{12} + 126u^{10} + 126u^8 + 84u^6 + 36u^4 + 9u^2 + 1$. 9. a. $\frac{1}{2}(\sin 8x - \sin 2x)$; b. $\frac{1}{2}(\cos 7x + \cos x)$; d. $\frac{1}{8}(\cos 4x + 4\cos 2x + 3)$; f. $\frac{1}{2^7}(\cos 8x - 8\cos 6x + 28\cos 4x - 56\cos 2x + 35)$; h. $-\frac{1}{2^9}(\cos 10x - 2\cos 8x - 3\cos 6x + 8\cos 4x + 2\cos 2x - 6)$; j. $\frac{1}{2^7}(\sin 8x - 2\sin 6x - 2\sin 4x + 6\sin 2x)$; l. $-\frac{1}{2^6}(\sin 7x - 7\sin 5x + 21\sin 3x - 35\sin x)$; 10. a. $\frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C$; c. $\frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C$; d. $\frac{1}{8}(\frac{1}{4}\sin 4x - 2\sin 2x + 3x) + C$; e. $\frac{-1}{192}\sin 6x + \frac{3}{64}\sin 4x - \frac{15}{64}\sin 2x + \frac{5}{16}x + C$; f. $\frac{1}{8}x - \frac{1}{32}\sin 4x + C$. g. $\sin x\cos x + C$; h. same as g.; i. $-\frac{1}{8}\cos 4x + C$ j. $3\pi/16$; k. 0; l. $\pi/32$; m. $\frac{1}{2^7}(\frac{1}{8}\sin 8x + \frac{4}{3}\sin 6x + 7\sin 4x + 27\sin 2x + 70x) + C$; n. $\frac{1}{2^5}(-\frac{1}{6}\sin 6x - \frac{1}{2}\sin 4x + \frac{1}{2}\sin 2x + 4x) + C$; o. same as d.