Chapter 11

Heron's formula for the area of a triangle

11.1 Heron: *Metrica* I.8

[1]

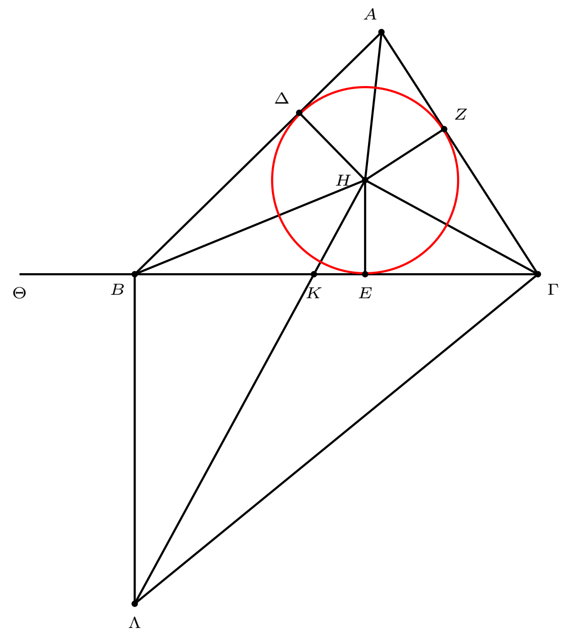
There is a general method for finding, without drawing a perpendicular, the area of any triangle whose three sides are given.
For example, let the sides of the triangle be 7, 8 and 9.
Add together 7, 8 and 9; the result is 24.
Take half of this, which gives 12.
Take away 7; the remainder is 5.
Again, from 12 take away 8; the remainder is 4.
And again 9; the remainder is 3.
Multiply 12 by 5; the result is 60.
Multiply this by 4; the result is 240.
Multiply this by 3; the result is 720.
Take the **square root** of this and it will be the area of the triangle.

Let $AB\Gamma$ be the given triangle, and let each of AB, $B\Gamma$, ΓA be given; to find the area. Let the circle ΔEZ be inscribed in the triangle with center H (*Euclid* IV.4), and let AH, BH, ΓH, EH, ZH be joined. Then

$$\begin{aligned} B\Gamma \cdot EH &= 2 \cdot \text{triangle } BH\Gamma, \\ \Gamma A \cdot ZH &= 2 \cdot \text{triangle } AH\Gamma, \\ AB \cdot \Delta H &= 2 \cdot \text{triangle } ABH. \end{aligned}$$

Therefore the rectangle contained by the perimeter of the triangle $AB\Gamma$ and EH, that is the radius of the circle ΔEZ, is double of the triangle $AB\Gamma$. Let ΓB

[1] See Thomas, Greek Mathematical Works, II, 470–473. The dates of Heron are very uncertain, between 150 B.C. and A.D. 250. From Arabic sources, it is known that Heron's formula had been discovered by Archimedes.



be produced and let $B\Theta$ be placed equal to $A\Delta$; then $\Gamma B\Theta$ is half of the perimeter of the triangle $AB\Gamma$ because $A\Delta = AZ$, $\Delta B = BE$, $Z\Gamma = \Gamma E$ (by *Euclid* III.17). Therefore,

$$\Gamma\Theta \cdot EH = \text{triangle } AB\Gamma.$$

But

$$\Gamma\Theta = \sqrt{\Gamma\Theta^2 \cdot EH^2};$$

therefore

$$(\text{triangle } AB\Gamma)^2 = \Gamma\Theta^2 \cdot EH^2.$$

Let $H\Gamma$ be drawn perpendicular to ΓH and $B\Lambda$ perpendicular to ΓB, andlet ΓA be joined. Then, since each of the angles $\Gamma H\Lambda$, $\Gamma H\Lambda$ is right, a circle can be described about the quadrilateral $\Gamma HB\Lambda$ (*Eucl.* III.31); therefore the angles ΓHB,$\Gamma\Lambda B$ are together equal to two right angles (*Eucl.* III.22). But the angles ΓHB, $AH\Delta$ are together equal to two right angles because the angles at H are bisected by AH, BH, ΓH and the angles ΓHB, $AH\Delta$ together with $AH\Gamma$, ΔHB are equal to four right angles; therefore the angle $AH\Delta$ is equal to the angle ΓAB. But the right angle $A\Delta H$ is equal to the right angle $\Gamma B\Lambda$; therefore the triangle $AH\Delta$ is similar to the triangle $\Gamma B\Lambda$. Therefore,

$$\begin{aligned} B\Gamma : B\Lambda =& A\Delta : \Delta H \\ =& B\Theta : EH, \end{aligned}$$

and *permutando*,

$$\begin{aligned} \Gamma B : B\Theta =& B\Lambda : EH \\ =& BK : KE, \end{aligned}$$

because $B\Lambda$ is parallel to EH, and *componendo*,

$$\Gamma\Theta : B\Theta = BE : EK;$$

therefore,

$$\begin{aligned}\Gamma\Theta^2 =& BE \cdot E\Gamma : \Gamma E \cdot EK \\ =& BE \cdot E\Gamma : EH^2,\end{aligned}$$

for in a right-angled triangle EH has been drawn from the right angle perpendicular to the base; therefore $\Gamma\Theta^2 \cdot EH^2$, whose square root is the area of the triangle $AB\Gamma$, is equal to $(\Gamma\Theta \cdot \Theta B)(\Gamma E \cdot EB)$. And each of $\Gamma\Theta$, ΘB, BE, ΓE is given; for $\Gamma\Theta$ is half of the perimeter of the triangle $AB\Gamma$, while $B\Theta$ is the excess of half the perimeter over ΓB, BE is the excess of half the perimeter over $A\Gamma$, and $E\Gamma$ is the excess of half the perimeter over AB, inasmuch as $E\Gamma = \Gamma Z$, $B\Theta = A\Delta = AZ$. Therefore the area of the triangle $AB\Gamma$ is given.

11.1.1 Heron's formula

Denote the side lengths of a triangle by a, b, and c. Compute the *semiperimeter* $s = \frac{1}{2}(a + b + c)$. The area of the triangle is then given by the *Heron formula*:

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

11.1.2 Heron's approximation of square root

…Since 720 has not a rational square root,
we shall make a close approximation to the root in this manner.
Since the square nearest to 720 is 729, having a root 27,
divide 27 into 720; the result is $26\frac{2}{3}$;
add 27; the result is $53\frac{2}{3}$.
Take half of this; the result is $26\frac{1}{2} + \frac{1}{3} (= 26\frac{5}{6})$.
Therefore the square root of 720 will be very nearly $26\frac{5}{6}$.
For $26\frac{5}{6}$ multiplied by itself gives $720\frac{1}{36}$,
so that the difference is $\frac{1}{36}$.
If we wish to make the difference less than $\frac{1}{36}$,
instead of 729 we shall take the number now found, $720\frac{1}{36}$,
and by the same method we shall find an approximation
differing by much less than $\frac{1}{36}$.

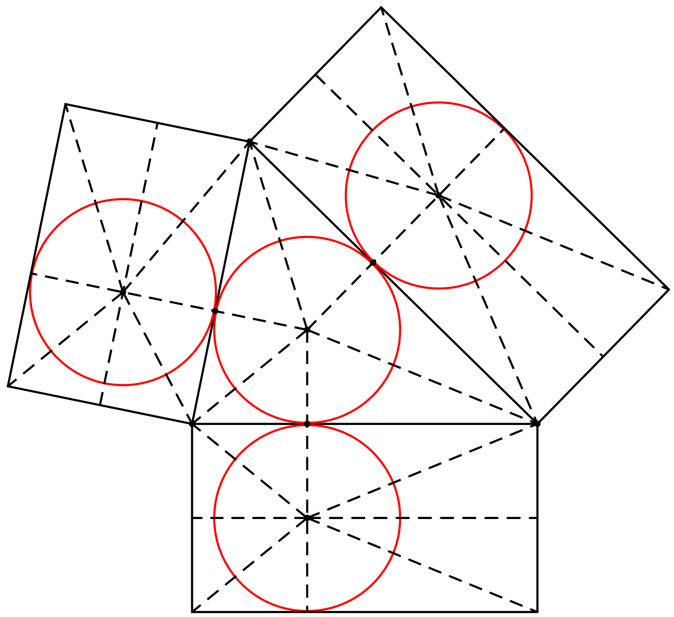
11.2 Other proofs of Heron's formula

11.2.1 Newton and Euler

Newton [2] and Euler [3] have both given proofs of the Heron formula, essentially the same as Heron's. See also Heath, *Euclid's Elements*, volume 2, pp. 87–88, and Dunham, *Journey through Genius*, Chapter 5.

11.2.2 MEI Wending

MEI Wending's (1633–1721) early work *Elements of Plane Trigonometry* [4] contains a proof of the formula that the area of a triangle is the product of the inradius and the semiperimeter, and the area formula in details.



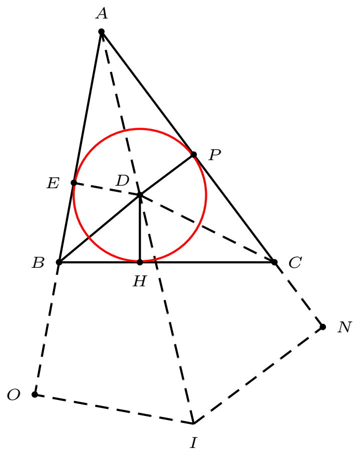
11.2.3 A 13-th century Chinese example

Qin Jiushao (1202–1261) *Shushu jiuzhang*
III.2: A sand field has three sides:
the shortest 13 *li*,
the median one 14 *li*,
and the longest 15 *li*.

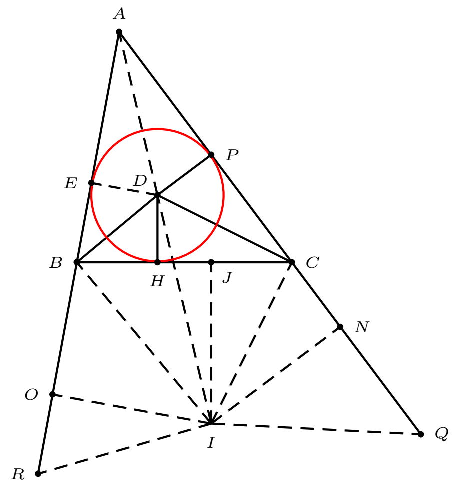
[2]D.T. Whiteside, *The Mathematical Papers of Issac Newton, V, 1683 – 1684*, pp. 50 – 53. This is part of Newton's preliminary notes and drafts for his *Arithmetica*. See also Problem 23 of his *Lectures on Algebra*, ibid. pp.224 – 227.

[3]Variae Demonstrationes Geometriae, *Opera Omnia*, ser 1. vol. 26, pp.

[4]*Ping sanjiao juyao*.



() Figure 3A



() Figure 3B

What is the area?

Answer: 84 square *li*
(= 84×300^2 square *bu* = 7560000 *square bu* = 31500 *mu* = 315 *qing*).

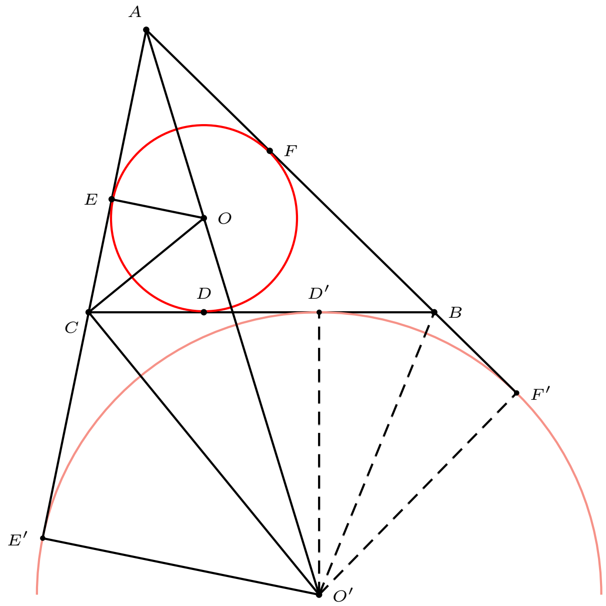
Method: Use the method of *shaoguang*.
Find the square of the shortest side,
add the square of the longest side,
subtract from the sum the square of median side.
Half the remainder, and square to form a number.
Multiply the square of the shortest side to the square of the longest side. Subtract the previously found number, and divide by 4.
Extract the square root.
That is the area.

$$\sqrt{\frac{1}{4}\left(c^2a^2 - \left(\frac{a^2+c^2-b^2}{2}\right)^2\right)}$$

11.2.4 Casey's: *A Sequel to Euclid*

Casey's proof makes use of an excircle. According to Court,[5] "interest in the escribed circles awakened only in the beginning of the 19th century.

[5] *College Geometry*, p.298.



11.3 Heron's Formula in Modern Textbooks

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With usual notation, $\triangle = \frac{1}{2}bc\sin A$,

$$\begin{aligned}
\triangle^2 =& \frac{1}{4}b^2c^2\sin^2 A = \frac{1}{4}b^2c^2(1-\cos^2 A)\\
=& \frac{1}{4}b^2c^2(1+\cos A)(1-\cos A)\\
=& \frac{1}{4}b^2c^2\cdot\frac{2bc+(b^2+c^2-a^2)}{2bc}\cdot\frac{2bc-(b^2+c^2-a^2)}{2bc}\\
=& \frac{1}{4}b^2c^2\cdot\frac{(b+c)^2-a^2}{2bc}\cdot\frac{a^2-(b-c)^2}{2bc}\\
=& \frac{1}{16}[(b+c)^2-a^2][a^2-(b-c)^2]\\
=& \frac{1}{16}[(b+c)+a][(b+c)-a][a-(b-c)][a+(b-c)].
\end{aligned}$$

This result becomes much simpler if we employ the abbreviation s for the semi-

[6]L.E. Dickson, *Plane Trigonometry*, (1921), pp. 128 – 129, Chelsea Reprint. For an interesting derivation making use of the symmetry of the formula in the sides of the triangle, see Alperin, *College Math. Journal*, vol 18 (1987) p.137.

perimeter. Then

$$\begin{aligned}a+b+c =&2s,\\ b+c-a =&(b+c+a)-2a = 2(s-a),\\ a-b+c =&2(s-b),\\ a+b-c =&2(s-c).\end{aligned}$$

Thus we obtain Heron of Alexandria's formula

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

Chapter 12

Archimedes' Book of Lemmas

Proposition 4

If AB be the diameter of a semicircle and N any point on AB, and if semicircles be described within the first semicircle and having AN, BN as diameters respectively, the figure included between the circumferences of the three semicircles is "what Archimedes called an arbelos"; and its area is equal to the circle on PN as diameter, where PN is perpendicular to AB and meets the original semicircle in P.

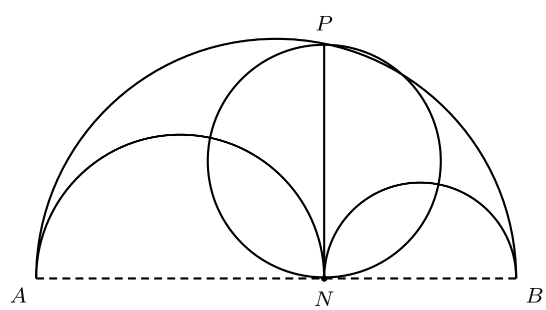


Figure 12.1:

Proposition 5

Let AB be the diameter of a semicircle C any point on AB, and CD perpendicular to it, and let semicircles be described within the first semicircle and having AC, AB as diameters. Then, if two circles be drawn touching CD on different sides and each touching two of the semicircles, the circles so drawn will be equal.

Let one of the circles touch CD at E, the semicircle on AB in F, and the semicircle on AC in G.

Draw the diameter EH of the circle, which will accordingly be perpendicular to CD and therefore parallel to AB.

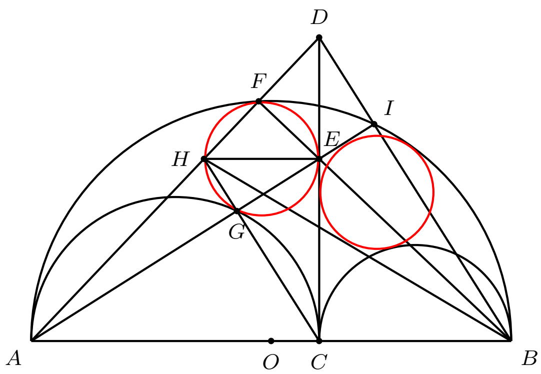


Figure 12.2:

Join FH, HA, and FE, FB. Then, by Proposition 1, [1] FHA, FEB are both straight lines, since EH, AB are parallel.

For the same reason AGE, CGH are straight lines.

Let AF produced meet CD in D, and let AE produced meet the outer semicircle in I. Join BI, ID.

Then since the angles AFB, ACD are right, the straight lines AD, AB are such that the perpendiculars on each from the extremity of the other meet in the point E. Therefore, by the properties of triangles, AE is perpendicular to the line joining B to D.

But AE is perpendicular to BI.

Therefore BID is a straight line.

Now, since the angles at G, I are right, CH is parallel to BD.

Therefore,

$$AB : BC = AD : DH = AC : HE,$$

so that

$$AC \cdot CB = AB \cdot HE.$$

In like manner, if d is the diameter of the other circle, we can prove that

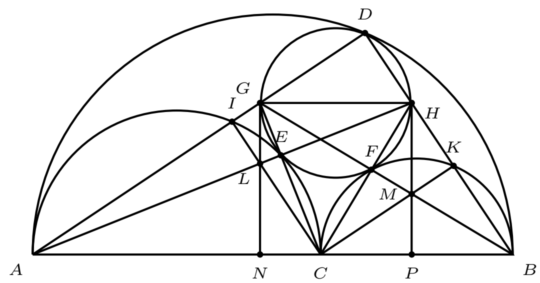
$$AC \cdot CB = AB \cdot d.$$

Therefore, $d = HE$, and the circles are equal.

Proposition 6

Let AB, the diameter of a semicircle, be divided at C so that $AC = \frac{2}{3}CB$ [or in any ratio]. Describe semicircles within the first semicircle and on AC, CB as diameters, and suppose a circle drawn touching all three semicircles. If GH be the diameter of this circle, to find the relation between GH and AB.

[1] Proposition 1: IF two circles touch at A, and if BD, EF be parallel diameters in them, ADF is a straight line.



Let GH be that diameter of the circle which is parallel to AB, and let the circle touch the semicircles on AB, AC, CB in D, E, F respectively.

Join AG, GD and BH, HD. Then by Proposition 1, AGD, BHD are straight lines.

For like reason AEH, BFG are straight lines, as also are CEG, CFH.

Let AD meet the semicircle on AC in I, and let BD meet the semicircle on CB in K. Join CI, CK meeting AE, BF respectively in L, M, and let GL, HM produced meet AB in N, P respectively.

Now, in the triangle AGC, the perpendiculars from A, C on the opposite sides meet in L. Therefore, by the properties of triangles, GLN is perpendicular to AC.

Similarly, HMP is perpendicular to CB.

Again, since the angles at I, K, D are right angles, CK is parallel to AD, and CI to BD.

Therefore,

$$AC : CB = AL : LH = AN : NP,$$

and

$$BC : CA = BM : MG = BP : PN.$$

Hence,

$$AN : NP = NP : PB,$$

or AN, NP, PB are in continued proportion.

Now, in the case where $AC = \frac{3}{2}CB$,

$$AN = \frac{3}{2}NP = \frac{9}{4}PB,$$

whence $BP : PN : NA : AB = 4 : 6 : 9 : 19$. Therefore,

$$GH = NP = \frac{6}{19}AB.$$

And similarly GH can be found when $AC : CB$ is equal to any other given ratio.

Chapter 13

Diophantus

13.1 Diophantus of Alexandria

Problem (II.8)**.** To divide a given square number into two squares.

Diophantus' solution. Given square number 16.
x^2 one of the required squares. Therefore $16 - x^2$ must be equal to a square.
Take a square of the form $(mx - 4)^2$, m being any integer and 4 the number which is the square root of 16, *e.g.* take $(2x - 4)^2$, and equate it to $16 - x^2$.
Therefore $4x^2 - 16x + 16 = 16 - x^2$,
or $5x^2 = 16x$, and $x = \frac{16}{5}$.
The required squares are therefore $\frac{256}{25}$, $\frac{144}{25}$.

Remark. This Diophantine problem is the historical origin of the famous Fermat last theorem. Fermat appended here, in his copy of *Arithmetica*, his famous note

> On the other hand it is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally *any power except a square into two powers with the same exponent*. I have discovered a truly marvelous proof of this, which however the margin is not large enough to contain.

Thus, $x^n + y^n = z^n$ has no nonzero integer solutions. This was finally proved by Andrew Wiles in 1995.

Problem (II.9)**.** To divide a given number which is the sum of two squares into two other squares.

Diophantus' solution:
Given $13 = 2^2 + 3^2$.
As the roots of these squares are 2, 3, take $(x+2)^2$ as the first square and $(mx-3)^2$ as the second (where m is an integer), say $(2x - 3)^2$.
Therefore $(x^2 + 4x + 4) + (4x^2 + 9 - 12x) = 13$,

or $5x^2 + 13 - 8x = 13$.
Therefore $x = \frac{8}{5}$ and the required squares are $\frac{324}{25}$, $\frac{1}{25}$.

Remark. This solution can be generalized to find rational points on a conic which is known to contain one rational point. If (x_0, y_0) is a rational point on a conic is represented by a quadratic equation in x and y with rational coefficients, then for an arbitrary rational number m, the line through (x_0, y_0) with slope m will intersect the conic again at a point whose x coordinate is a rational number. This is because the substitution $(x, y) = (x + x_0, mx + y_0)$ into the equation of the conic leads to, after simplification, a simple (linear) equation in x. For this value of x, $(x+x_0, mx+y_0)$ is a rational point on the conic.

Proposition. If a conic represented by an equation with rational coefficients contains a rational point, then there is a parametrization of its rational points.

Problem (IV.24)**.** To divide a given number into two parts such that the product is a cube minus its side.

Diophantus' solution. Given number 6. First part x; therefore second = $6 - x$, and $6x - x^2$ = a cube minus its side. Form a cube from a side of the form $mx - 1$, *say*, $2x - 1$, and equate $6x - x^2$ to this cube minus its side. Therefore, $8x^3 - 12x^2 + 4x = 6x - x^2$. Now, if the coefficient of x were the same on both sides, this would reduce to a simple equation, and x would be rational. In order that this may be the case, we must put m for 2 in our assumption, where $3m - m = 6$ (the 6 being the given number in the original hypothesis). Thus, $m = 3$. We therefore assume $(3x-1)^3 - (3x-1) = 6x - x^3$, or $27x^3 - 27x^2 + 6x = 6x - x^2$, and $x = \frac{26}{27}$. The parts are $\frac{26}{27}$ and $\frac{136}{27}$.

Remark. This problem seeks *rational* solutions of the equation $y^3 - y = 6x - x^2$. Clearly, $(0, -1)$ is a rational solution. The line through $(0, -1)$ with slope m (assumed rational) has equation $y = mx - 1$. This line cuts the curve $E : y^3 - y = x^2 - 6x$ at 3 points, one of which is $(0, -1)$. In general, there is no guarantee that any of the remaining two points is rational. However, if this line is tangent to E at $(0, 1)$, then $(0, 1)$ being counted twice, it is clear that the remaining point is rational. The tangent to E at $(0, -1)$ turns out to be $y = 3m - 1$, and we obtain $(\frac{26}{27}, \frac{136}{27})$ for the third point.

Problem (V.29)**.** To find three squares such that the sum of their squares is a square.

Diophantus' solution. Let the squares be x^2, 4, 9 respectively.
Therefore $x^4 + 97 = \Box = (x^2 - 10)^2$, say;
whence $x^2 = \frac{3}{20}$.
If the ratio of 3 to 20 were the ratio of a square to a square, the problem would be solve; but it is not.
Therefore I have to find two squares (p^2, q^2, say) and a number (m say) such that $m^2 - p^4 - q^4$ has to $2m$ the ratio of a square to a square.

Let $p^2 = z^2$, $q^2 = 4$ and $m = z^2 + 4$.
Therefore, $m^2 - p^4 - q^4 = (z^2+4)^2 - z^4 - 16 = 8z^2$.
Hence, $\dfrac{8z^2}{2z^2+8}$ or $\dfrac{4z^2}{z^2+4}$ must be the ratio of a square to a square.
Put $z^2 + 4 = (z+1)^2$, say;
therefore, $z = 1\frac{1}{2}$, and the squares are $p^2 = 2\frac{1}{4}$, $q^2 = 4$, while $m = 6\frac{1}{4}$;
or, if we take 4 times each, $p^2 = 9$, $q^2 = 16$, $m = 25$.
Starting again, we put for the square x^2, 9, 16;
then the sum of the squares $= x^4 + 337 = (x^2 - 25)^2$, and $x = \frac{12}{5}$.
The required squares are $\frac{144}{25}$, 9, 16.

Remark. This is the Diophantine equation $x^4 + y^4 + z^4 = w^2$.

Euler had conjectured that $x^4 + y^4 + z^4 = w^4$ has no nontrivial integer solution. It was disproved by Noam Elkies in 1988, who found, among other things,

Exercise

Diophantus II.21. To find two numbers such that the square of either *minus* the other number gives a square. [1]

Diophantus II.22. To find two numbers such that the square of either added to the sum of both gives a square. [2]

Diophantus II.23. To find two numbers such that the square of either *minus* the sum of both gives a square. [3]

Diophantus II.24. To find two numbers such that either added to the square of their sum gives a square. [4]

Diophantus II.25. To find two numbers such that the square of their sum *minus* either number gives a square. [5]

Proposition (III.19)**.** *To find four numbers such that the square of their sum plus or minus any one singly gives a square.*

Since, in any right - angled triangle,

(sq. on hypotenuse) $\pm$ (twice product of perpendiculars) = a square,

we must seek four right - angled triangles [in rational numbers] having the same hypotenuse,

or we must find a square which is divisible into two squares in four different ways; and we saw how to divide a square into two squares in an infinite number of ways. [II.8]

[1]Diophantus gave $\frac{8}{5}$, $\frac{11}{5}$.
[2]Diophantus gave $\frac{2}{8}$, $\frac{10}{8}$.
[3]Diophantus gave $2\frac{1}{2}$, $3\frac{1}{2}$.
[4]Diophantus gave $\frac{3}{121}$, $\frac{8}{121}$.
[5]Diophantus gave $\frac{192}{361}$, $\frac{112}{361}$.

Take right - angled triangle in the smallest numbers, (3,4,5) and (5,12,13); and multiply the sides of the first by the hypotenuse of the second and *vice versa*.

This gives the triangle (39,52,65) and (25,60,65); thus 65^2 is split up into two squares in *two* ways.

Again, 65 is naturally divided into two squares in two ways, namely into 7^2+4^2 and 8^2+1^2, which is due to the fact that 65 is the product of 13 and 5, each of which numbers is the sum of two squares.

Form now a right - angled triangle from 7, 4. The sides are $(7^2-4^2, 2\cdot 7\cdot 4, 7^2+4^2)$ or (33,56,65).

Similarly, forming a right - angled triangle from 8, 1, we obtain $(2\cdot 8\cdot 1, 8^2-1^2, 8^2+1^2)$, or (16,63,65).

Thus, 65 is split into two squares in four ways.

Assume now as the sum of the numbers $65x$ and
as first number $2\cdot 39\cdot 52x^2 = 4056x^2$,
as second number $2\cdot 25\cdot 60x^2 = 3000x^2$,
as third number $2\cdot 33\cdot 56x^2 = 3696x^2$,
as fourth number $2\cdot 16\cdot 63x^2 = 2016x^2$,
the coefficients of x^2 being four times the areas of the four right - angled triangles respectively.

The sum $12768x^2 = 65x$, and $x = \frac{65}{12768}$. The numbers are

$$\frac{17136600}{163021824}, \quad \frac{12675000}{163021824}, \quad \frac{15615600}{163021824}, \quad \frac{8517600}{163021824}.$$

Proposition (VI.14)**.** *To find a right - angled triangle such that its area minus the hypotenuse or minus one of the perpendiculars gives a square.*

Let the triangle be $(3x, 4x, 5x)$.
Therefore $6x^2-5x$, $6x^2-3x$ are both squares.
Making the latter a square ($= m^2x^2$), we have $x = \frac{3}{6-m^2}$, ($m^2 < 6$).
The first equation then gives

$$\frac{54}{m^4-12m^2+36} - \frac{15}{6-m^2} = \text{ a square}$$

or

$$15m^2 - 36 = \text{ a square.}$$

This equation we cannot solve because 15 *is not the sum of two squares.* Therefore we must change the assumed triangle.

Now (with reference to the triangle 3,4,5), $15m^2 =$ the continued product of a square less than the area, the hypotenuse and one perpendicular;

while 36 = the continued product of the area, the perpendicular, and the difference between the hypotenuse and the perpendicular.

Therefore we have to find a right - angled triangle $(h, p, b$, say) and a square (m^2) less than 6 such that

$$m^2 hp - \frac{1}{2}pb \cdot p \text{ is a square.}$$

The problem can be solved if X_1, X_2 are "similar plane numbers."

From the auxiliary triangle from similar plane numbers accordingly, say, 4, 1 [The conditions are then satisfied].

[The equation for m then becomes

$$8 \cdot 17m^2 - 4 \cdot 15 \cdot 8 \cdot 9 = \text{ a square,}$$

or

$$136m^2 - 4320 = \text{ a square.]}$$

Let $m^2 = 36$.[This satisfies the equation, and 36 $<$ area of triangle.]

The triangle formed from 4, 1 being (8,15,17), we assume $8x, 15x, 17x$ for the original triangle.

We now put $60x^2 - 8x = 26x^2$, and $x = \frac{1}{3}$.

The required triangle is therefore $(\frac{8}{3}, 5, \frac{17}{3})$.

Exercise

Diophantus VI.16. To find a right - angled triangle such that the number representing the (portion intercepted within the triangle of the) bisector of an acute angle is rational. [6]

Exercise

(Heron) [7] In a right - angled triangle the sum of the area and the perimeter is 280 feet; to separate the sides and find the area. [8]

13.2 Arithmetica, Book VI

To find a (rational) right-angled triangle such that

1. the hypotenuse minus each of the sides gives a cube,
2. the hypotenuse added to each side gives a cube,
3. its area added to a given number makes a square,

[6] Diophantus gave the triangle (28,96,100) and bisector 35.
[7] *Geometrica 24,10.*; see Thomas, II, pp. 506–509.
[8] Answer: (20,21,29;210).

4. its area minus a given number makes a square,
5. if its area be subtracted from a given number, the remainder is a square,
6. its area be added to one of the perpendiculars makes a given number,
7. its area minus one of the perpendiculars is a given number,
8. the area added to the sum of the perpendiculars makes a given number,
9. the area minus the sum of the perpendiculars is a given number,
10. the sum of its area, the hypotenuse, and one of the perpendiculars is a given number,
11. its area minus the sum of the hypotenuse and one of the perpendiculars is a given number,
12. the area added to either of the perpendiculars gives a square,
13. the area minus either perpendicular gives a square,
14. its area minus the hypotenuse or minus one of the perpendiculars gives a square,
15. the area added to either the hypotenuse or one of the perpendiculars gives a square,
16. the [length of] bisector of an acute angle is rational,
17. the area added to the hypotenuse gives a square, while the perimeter is a cube,
18. the area added to the hypotenuse gives a cube, while the perimeter is a square,
19. its area added to one of the perpendiculars gives a square, while the perimeter is a cube,
20. the sum of its area and one perpendicular is a cube, while its perimeter is a square,
21. its perimeter is a square, while its perimeter added to its area gives a cube,
22. its perimeter is a cube, while the perimeter added to the area gives a square,
23. the square of its hypotenuse is also the sum of a different square and the side of the square, while the quotient obtained by dividing the square of the hypotenuse by one of the perpendiculars of the triangle is the sum of a cube and the side of the cube,
24. one perpendicular is a cube, the other is the difference between a cube and its side, and the hypotenuse is the sum of a cube and its side.