The First 150 Years of the Riemann Zeta-Function

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I. Synopsis of Riemann's paper

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse

(On the number of primes less than a given magnitude)

Figure: Riemann

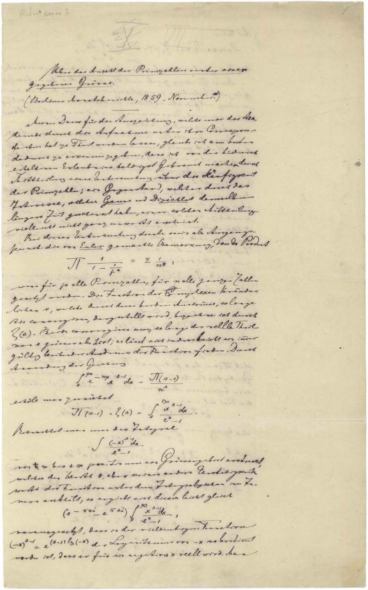

Figure: First page of Riemann's paper

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- $\zeta(s)$ has an **analytic continuation** to $\mathbb{C}$, except for a simple pole at $s = 1$. The only zeros in $\sigma < 0$ are simple zeros at $s = -2, -4, -6, \ldots$
- $\zeta(s)$ has a **functional equation**

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$$

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Here ρ runs over the nontrivial zeros of $\zeta(s)$.

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  Let $\Lambda(n) = \log p$ if $n = p^k$ and 0 otherwise. Then

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)}$$

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Note that from this one can see why the Prime Number Theorem,

$$\psi(x) \sim x$$

might be true.

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Conjecture (The Riemann Hypothesis)

All the zeros $\rho = \beta + i\gamma$ *in the critical strip lie on the line* $\sigma = 1/2$.

II. Early developments after the paper

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To do this, they both needed to prove that

$$\zeta(1 + it) \neq 0$$

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This required him to prove that there is a *zero-free* region

$$\sigma < 1 - \frac{c_0}{\log t}$$

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$$\text{RH} \Longrightarrow \psi(x) = x + O(x^{1/2} \log^2 x)$$

III. The order of $\zeta(s)$ in the critical strip

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- answers to other arithmetical questions depend on it.

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$$\log\left(\frac{R^n}{|z_1 z_2 \cdots z_n|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|.$$

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$$\sum_{n \le x} d_k(n) = xP_{k-1}(\log x) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta^k(s)\frac{x^s}{s}ds.$$

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and this implied that the O-term in the PNT is $\ll xe^{-\sqrt{c_1 \log x}}$.

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This is a so called *convexity bound* .

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Conjecture (Lindelöf)

$\mu(\sigma) = 0$ *for* $\sigma \geq 1/2$. *That is,* $\zeta(1/2 + it) \ll |t|^{\epsilon}$ *for t large*

What we expect the order to be

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Which bound, the upper or the lower, is closest to the truth is one of the important open questions.

IV. Mean value theorems

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- mean values are easier to prove than point wise bounds.
- the techniques developed to treat them have proved important in other contexts.

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For this H-L developed the **approximate functional equation**

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- Keating and Snaith used random matrix theory to conjecture the value of g_k for every value of $k > -1/2$.
- Soundararajan has recently shown that on RH

$$\int_0^T |\zeta(1/2+it)|^{2k} dt \ll T \log^{k^2+\epsilon} T.$$

V. Zero-density estimates

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this says the proportion of zeros to the right of $\sigma > 1/2$ tends to 0 as $T \to \infty$.

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LH implies $N(\sigma, T) \ll T^{2(1-\sigma)+\epsilon}$.

VI. The distribution of *a*-values of $\zeta(s)$

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- show that $\zeta(\sigma + it) \approx \prod_{p \leq N}(1 - p^{-\sigma - it})^{-1}$ for most t.
- use Kronecker's theorem to find a t so that the numbers p^{-it} point in such a way that $\prod_{p \leq N}(1 - p^{-\sigma - it})^{-1} \approx a$.

The distribution of *a*-values of $\zeta(s)$

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As a second result, let $N_a(\sigma_1, \sigma_2, T)$ be the number of solutions of $\zeta(s) = a$ in the rectangular area $\sigma_1 \leq \sigma \leq \sigma_2,\ 0 \leq t \leq T$.

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Notice that this is quite different from the case $a = 0$, because modern zero-density estimates imply

$$N_0(\sigma_1, \sigma_2, T) \ll T^{\theta} \qquad (\theta < 1).$$

VII. Number of zeros on the line as $T \to \infty$

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These all rely heavily on mean value estimates.

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would be the same size as $T \to \infty$. But they are not.

VIII. Calculations of zeros on the line

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IX. More recent developments

Pair correlation

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- From 1980 on Odlyzko did a vast amount of numerical calculation that strongly supported Montgomery's conjecture.

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- there are large and small gaps between consecutive zeros.
- over 70% of the zeros are simple.

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- Hughes, Keating, O'Connell used it to conjecture the discrete means $\sum_{0<\gamma\leq T} |\zeta'(\rho)|^{2k}$
- Mezzadri used it to study the distribution of the zeros of $\zeta'(s)$.

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(Conrey, Farmer, Keating, Rubenstein, Snaith, Zirnbauer, ...)

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It also explains why the constant in the moment splits as $\frac{a_k g_k}{\Gamma(k^2+1)}$.

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Arguments from the hybrid model suggest that the 2 should be dropped.

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$$\sqrt{1/2}(1+o(1)) \leq_{i.o.} \frac{\log|\zeta(1/2+it)|}{\sqrt{\log t\ \log\log t}} \leq \sqrt{1/2}(1+o(1)).$$