Hyperbolic functions

The hyperbolic functions have similar names to the trigonmetric functions, but they are defined in terms of the exponential function. In this unit we define the three main hyperbolic functions, and sketch their graphs. We also discuss some identities relating these functions, and mention their inverse functions and reciprocal functions.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- define the functions $f(x) = \cosh x$ and $f(x) = \sinh x$ in terms of the exponential function, and define the function $f(x) = \tanh x$ in terms of $\cosh x$ and $\sinh x$,
- sketch the graphs of $\cosh x$, $\sinh x$ and $\tanh x$,
- recognize the identities $\cosh^2 x - \sinh^2 x = 1$ and $\sinh 2x = 2 \sinh x \cosh x$,
- understand the meaning of the inverse functions $\sinh^{-1} x$, $\cosh^{-1} x$ and $\tanh^{-1} x$ and specify their domains,
- define the reprocal functions $\operatorname{sech} x$, $\operatorname{csch} x$ and $\coth x$.

Contents

1. Introduction

In this video we shall define the three hyperbolic functions $f(x) = \sinh x$, $f(x) = \cosh x$ and $f(x) = \tanh x$. We shall look at the graphs of these functions, and investigate some of their properties.

2. Defining $f(x) = \cosh x$

The hyperbolic functions $\cosh x$ and $\sinh x$ are defined using the exponential function e^x. We shall start with $\cosh x$. This is defined by the formula

$$\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\,.$$

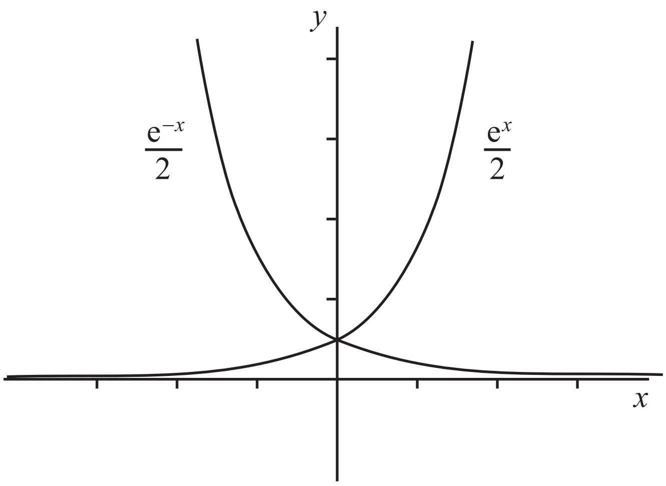
We can use our knowledge of the graphs of e^x and e^{-x} to sketch the graph of $\cosh x$. First, let us calculate the value of $\cosh 0$. When $x = 0$, $\mathrm{e}^x = 1$ and $\mathrm{e}^{-x} = 1$. So

$$\cosh 0 = \frac{\mathrm{e}^0 + \mathrm{e}^{-0}}{2} = \frac{1+1}{2} = 1\,.$$

Next, let us see what happens as x gets large. We shall rewrite $\cosh x$ as

$$\cosh x = \frac{\mathrm{e}^x}{2} + \frac{\mathrm{e}^{-x}}{2}\,.$$

To see how this behaves as x gets large, recall the graphs of the two exponential functions.



As x gets larger, e^x increases quickly, but e^{-x} decreases quickly. So the second part of the sum $\mathrm{e}^x/2 + \mathrm{e}^{-x}/2$ gets very small as x gets large. Therefore, as x gets larger, $\cosh x$ gets closer and closer to $\mathrm{e}^x/2$. We write this as

$$\cosh x \approx \frac{\mathrm{e}^x}{2} \qquad \text{for large } x.$$

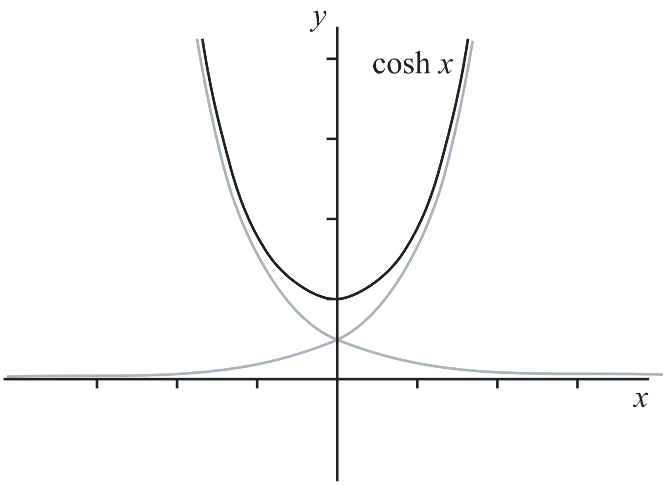
But the graph of $\cosh x$ will always stay above the graph of $\mathrm{e}^x/2$. This is because, even though $e^{-x}/2$ (the second part of the sum) gets very small, it is always greater than zero. As x gets larger and larger the difference between the two graphs gets smaller and smaller.

Now suppose that $x < 0$. As x becomes more negative, e^{-x} increases quickly, but e^x decreases quickly, so the first part of the sum $e^x/2 + e^{-x}/2$ gets very small. As x gets more and more negative, $\cosh x$ gets closer and closer to $e^{-x}/2$. We write this as

$$\cosh x \approx \frac{e^{-x}}{2} \qquad \text{for large negative } x.$$

Again, the graph of $\cosh x$ will always stay above the graph of $e^{-x}/2$ when x is negative. This is because, even though $e^x/2$ (the first part of the sum) gets very small, it is always greater than zero. But as x gets more and more negative the difference between the two graphs gets smaller and smaller.

We can now sketch the graph of $\cosh x$. Notice the graph is symmetric about the y-axis, because $\cosh x = \cosh(-x)$.



Key Point

The hyperbolic function $f(x) = \cosh x$ is defined by the formula

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

The function satisfies the conditions $\cosh 0 = 1$ and $\cosh x = \cosh(-x)$. The graph of $\cosh x$ is always above the graphs of $e^x/2$ and $e^{-x}/2$.

3. Defining $f(x) = \sinh x$

We shall now look at the hyperbolic function $\sinh x$. In speech, this function is pronounced as 'shine', or sometimes as 'sinch'. The function is defined by the formula

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\,.$$

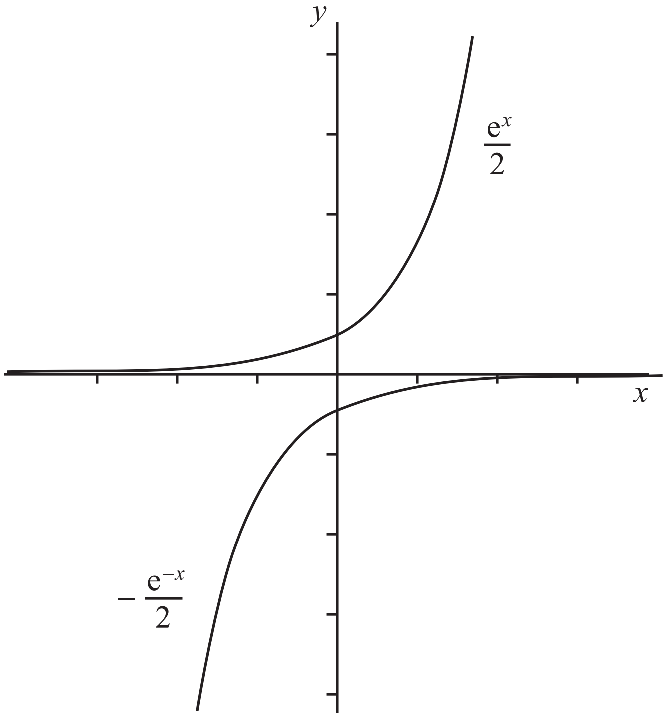
Again, we can use our knowledge of the graphs of e^x and e^{-x} to sketch the graph of $\sinh x$. First, let us calculate the value of $\sinh 0$. When $x = 0$, $\mathrm{e}^x = 1$ and $\mathrm{e}^{-x} = 1$. So

$$\sinh 0 = \frac{\mathrm{e}^0 - \mathrm{e}^{-0}}{2} = \frac{1-1}{2} = 0\,.$$

Next, let us see what happens as x gets large. We shall rewrite $\sinh x$ as

$$\sinh x = \frac{\mathrm{e}^x}{2} - \frac{\mathrm{e}^{-x}}{2}\,.$$

To see how this behaves as x gets large, recall the graphs of the two exponential functions.



As x gets larger, e^x increases quickly, but e^{-x} decreases quickly. So the second part of the difference $\mathrm{e}^x/2 - \mathrm{e}^{-x}/2$ gets very small as x gets large. Therefore, as x gets larger, $\sinh x$ gets closer and closer to $\mathrm{e}^x/2$. We write this as

$$\sinh x \approx \frac{\mathrm{e}^x}{2} \qquad \text{for large } x.$$

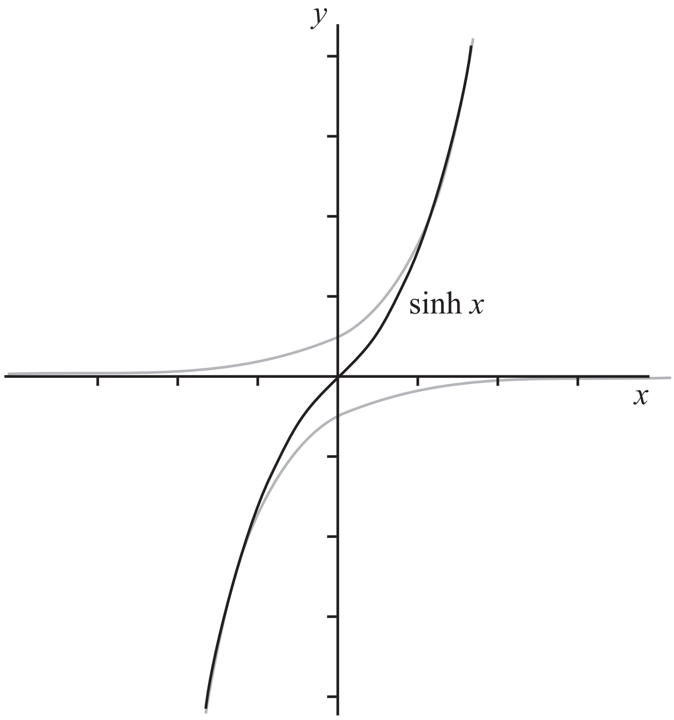
But the graph of $\sinh x$ will always stay below the graph $\mathrm{e}^x/2$. This is because, even though $-\mathrm{e}^{-x}/2$ (the second part of the difference) gets very small, it is always less than zero. As x gets larger and larger the difference between the two graphs gets smaller and smaller.

Next, suppose that x is negative. As becomes more negative, $-e^{-x}$ becomes large and negative very quickly, but e^x decreases very quickly. So as x becomes more negative, the first part of the difference $e^x/2 - e^{-x}/2$ gets very small. So $\sinh x$ gets closer and closer to $-e^{-x}/2$. We write this as

$$\sinh x \approx \frac{-e^{-x}}{2} \qquad \text{for large negative } x.$$

Now the graph of $\sinh x$ will always stay above the graph of $e^{-x}/2$ when x is negative. This is because, even though $e^x/2$ (the first part of the difference) gets very small, it is always greater than zero. But as x gets more and more negative the difference between the two graphs gets smaller and smaller.

We can now sketch the graph of $\sinh x$. Notice that $\sinh(-x) = -\sinh x$.



Key Point

The hyperbolic function $f(x) = \sinh x$ is defined by the formula

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

The function satisfies the conditions $\sinh 0 = 0$ and $\sinh(-x) = -\sinh x$. The graph of $\sinh x$ is always between the graphs of $e^x/2$ and $e^{-x}/2$.

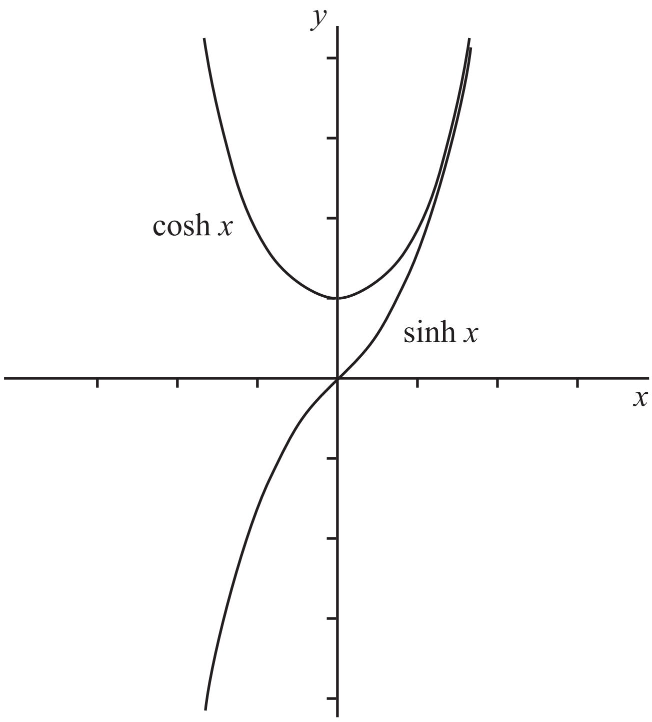
We have seen that $\sinh x$ gets close to $\mathrm{e}^x/2$ as x gets large, and we have also seen that $\cosh x$ gets close to $\mathrm{e}^x/2$ as x gets large. Therefore, $\sinh x$ and $\cosh x$ must get close together as x gets large. So

$$\sinh x \approx \cosh x \qquad \text{for large } x.$$

Similarly, we have seen that $\sinh x$ gets close to $-\mathrm{e}^{-x}/2$ as x gets large and negative, and we have seen that $\cosh x$ gets close to $\mathrm{e}^{-x}/2$ as x gets large and negative. Therefore, $\sinh x$ and $-\cosh x$ must get close together as x gets large and negative. So

$$\sinh x \approx -\cosh x \qquad \text{for large negative } x.$$

We can see this by sketching the graphs of $\sinh x$ and $\cosh x$ on the same axes.



Key Point

For large values of x the graphs of $\sinh x$ and $\cosh x$ are close together. For large negative values of x the graphs of $\sinh x$ and $-\cosh x$ are close together.

4. Defining $f(x) = \tanh x$

We shall now look at the hyperbolic function $\tanh x$. In speech, this function is pronounced as 'tansh', or sometimes as 'than'. The function is defined by the formula

$$\tanh x = \frac{\sinh x}{\cosh x}\,.$$

We can work out $\tanh x$ out in terms of exponential functions. We know how $\sinh x$ and $\cosh x$ are defined, so we can write $\tanh x$ as

$$\tanh x = \frac{e^x - e^{-x}}{2} \div \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{e^x + e^{-x}}\,.$$

We can use what we know about $\sinh x$ and $\cosh x$ to sketch the graph of $\tanh x$. We first take $x = 0$. We know that $\sinh 0 = 0$ and $\cosh 0 = 1$, so

$$\tanh 0 = \frac{\sinh 0}{\cosh 0} = \frac{0}{1} = 0\,.$$

As x gets large, $\sinh x \approx \cosh x$, so $\tanh x$ gets close to 1:

$$\tanh x \approx 1 \qquad \text{for large } x.$$

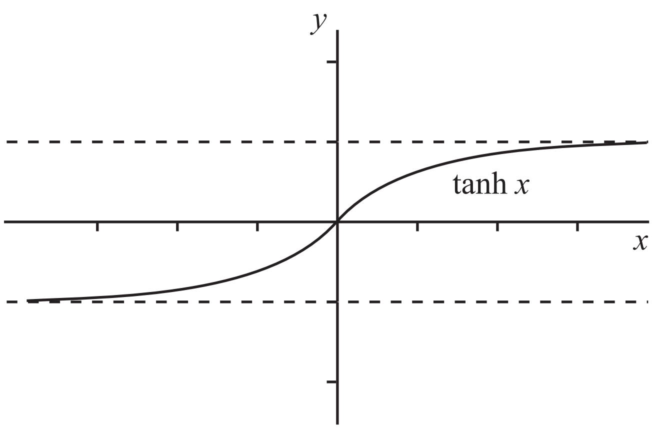
But $\sinh x$ is always less than $\cosh x$, so $\tanh x$ is always slightly less than 1. It gets close to 1 as x gets very large, but never reaches it.

As x gets large and negative, $\sinh x \approx -\cosh x$, so $\tanh x$ gets close to -1:

$$\tanh x \approx -1 \qquad \text{for large negative } x.$$

But $\sinh x$ is always greater than $-\cosh x$, so $\tanh x$ is always slightly greater than -1. It gets close to -1 as x gets very large and negative, but never reaches it.

We can now sketch the graph of $\tanh x$. Notice that $\tanh(-x) = -\tanh x$.



5. Identities for hyperbolic functions

Hyperbolic functions have identities which are similar to, but not the same as, the identities for trigonometric functions. In this section we shall prove two of these identities, and list some others.

The first identity is

$$\cosh^2 x - \sinh^2 x = 1\,.$$

To prove this, we start by substituting the definitions for $\sinh x$ and $\cosh x$:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\,.$$

If we expand the two squares in the numerators, we obtain

$$\begin{aligned}(e^x + e^{-x})^2 &= e^{2x} + 2(e^x)(e^{-x}) + e^{-2x} \\ &= e^{2x} + 2 + e^{-2x}\end{aligned}$$

and

$$\begin{aligned}(e^x - e^{-x})^2 &= e^{2x} - 2(e^x)(e^{-x}) + e^{-2x} \\ &= e^{2x} - 2 + e^{-2x}\,,\end{aligned}$$

where in each case we use the fact that $(e^x)(e^{-x}) = e^{x+(-x)} = e^0 = 1$. Using these expansions in our formula, we obtain

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}\,.$$

Now we can move the factor of $\frac{1}{4}$ out to the front, so that

$$\cosh^2 x - \sinh^2 x = \tfrac{1}{4}\left((e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right)\,.$$

If, finally, we remove the inner brackets and simplify, we obtain

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}) \\ &= \tfrac{1}{4} \times 4 \\ &= 1\,,\end{aligned}$$

which is what we wanted to prove.

Here is another identity involving hyperbolic functions:

$$\sinh 2x = 2\sinh x \cosh x\,.$$

On the left-hand side we have $\sinh 2x$ so, from the definition,

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}\,.$$

We want to manipulate the right-hand side to achieve this. So we shall start by substituting the definitions of $\sinh x$ and $\cosh x$ into the right-hand side:

$$2\sinh x\cosh x = 2\left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right).$$

We can cancel the 2 at the start with one of the 2's in the denominator, and then we can take the remaining factor of $\frac{1}{2}$ out to the front. We get

$$2\sinh x\cosh x = \tfrac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})(\mathrm{e}^x + \mathrm{e}^{-x}).$$

Now we can multiply the two brackets together. This gives us

$$2\sinh x\cosh x = \tfrac{1}{2}(\mathrm{e}^{2x} + 1 - 1 - \mathrm{e}^{-2x}).$$

Cancelling the ones finally gives us

$$2\sinh x\cosh x = \tfrac{1}{2}(\mathrm{e}^{2x} - \mathrm{e}^{-2x}) = \sinh 2x,$$

which is what we wanted to achieve.

There are several more identities involving hyperbolic functions:

$$\begin{aligned}
\cosh 2x &= (\cosh x)^2 + (\sinh x)^2 \\
\sinh(x+y) &= \sinh x\cosh y + \sinh y\cosh x \\
\cosh(x+y) &= \cosh x\cosh y + \sinh x\sinh y \\
\cosh^2\frac{x}{2} &= \frac{1+\cosh x}{2} \\
\sinh^2\frac{x}{2} &= \frac{\cosh x - 1}{2}
\end{aligned}$$

If you know the trigonometric identities, you may notice that these hyperbolic identities are very similar, although sometimes plus signs have become minus signs and vice versa. In fact the hyperbolic functions are very closely related to the trigonometric functions, and $\sinh x$ and $\cosh x$ are sometimes called the hyperbolic sine and hyperbolic cosine functions. If you go on to study complex numbers then you might learn more about how these functions are related.

6. Other related functions

Finally, we shall look at some other functions that are related to the three hyperbolic functions we have just seen. These are the inverse functions, and the reciprocal functions. It is important to understand the notation for these types of function, as it can sometimes be confusing. For example, the function $f(x) = \sinh^2 x$ refers to the square of the function $f(x) = \sinh x$, so that

$$\sinh^2 x = (\sinh x)^2,$$

whereas the function $f(x) = \sinh^{-1} x$ does not refer to the reciprocal of the function $f(x) = \sinh x$, so that

$$\sinh^{-1} x \neq (\sinh x)^{-1} = \frac{1}{\sinh x}.$$

Instead, $\sinh^{-1} x$ means the 'inverse function'. This means that $f^{-1}(x) = y$ whenever $f(y) = x$. So, for instance,

$$\sinh^{-1} x = y \qquad \text{whenever} \qquad \sinh y = x\,.$$

This inverse function is defined for all values of x. We can also define the inverse functions for $\cosh x$ and $\tanh x$. We define

$$\cosh^{-1} x = y \qquad \text{whenever} \qquad \cosh y = x\,,$$

and this function is valid for $x \geq 1$. We also define

$$\tanh^{-1} x = y \qquad \text{whenever} \qquad \tanh y = x\,,$$

and this function is valid for $-1 < x < 1$.

We have also mentioned the reciprocal functions, and these have special names related to the names of the trigonometric reciprocal functions. They are

$$\operatorname{sech} x = \frac{1}{\cosh x}\,, \qquad \operatorname{csch} x = \frac{1}{\sinh x}\,, \qquad \coth x = \frac{1}{\tanh x}\,.$$

Exercises

1.

(a) Simplify $\cosh x + \sinh x$ and $\cosh x - \sinh x$.
(b) Use the answer to part (a) to give an alternative proof that $\cosh^2 x - \sinh^2 x = 1$.

2. Find the domain and range of the following functions:

(a) $\sinh^{-1} x$, (b) $\cosh^{-1} x$, (c) $\tanh^{-1} x$, (d) $\operatorname{sech} x$, (e) $\operatorname{csch} x$, (f) $\coth x$.

Answers

1.

(a) $\cosh x + \sinh x = \mathrm{e}^x$ and $\cosh x - \sinh x = \mathrm{e}^{-x}$.
(b) $(\cosh x + \sinh x) \times (\cosh x - \sinh x) = \cosh^2 x - \sinh^2 x$, whereas $\mathrm{e}^x \times \mathrm{e}^{-x} = 1$.

2.

(a) domain: all real x, range: all real y;
(b) domain: $x \geq 1$, range: $y \geq 0$;
(c) domain: $-1 < x < 1$, range: all real y;
(d) domain: all real x, range: $0 < y < 1$;
(e) domain: $x \neq 0$, range: $y \neq 0$;
(f) domain: $x \neq 0$, range: $y < -1$ or $y > 1$.