Unfalsifiable Conjectures in Mathematics

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It is generally accepted among scientists that an unfalsifiable theory, a theory which can never conceivably be proven false, can never have any use in science. In this paper, we shall address the question, "Can an unfalsifiable conjecture ever have any use in mathematics?"

1 Introduction

It is generally accepted among scientists that an unfalsifiable theory, a theory which can never conceivably be proven false, can never have any use in science. As the philosopher Karl Popper said, "the criterion of the scientific status of a theory is its falsifiability, or refutability, or testability" [4]. In this paper, we shall address the question, "Can an unfalsifiable conjecture ever have any use in mathematics?" First, we shall present a famous mathematical conjecture and prove that it is unfalsifiable. Next, we shall discuss the implications of proving that a mathematical conjecture is unfalsifiable. And finally, we shall present some open problems.

2 An unfalsifiable conjecture

Landau's fourth problem is to prove that there are an infinite number of prime numbers of the form n^2+1, where $n \in \mathbb{N}$ [6]. We shall call this conjecture the $n^2 + 1$-Conjecture. And we shall prove that the n^2+1-Conjecture is unfalsifiable, i.e., that its negation is unprovable in any reasonable axiom system:

Theorem: The $(n^2 + 1)$-Conjecture is unfalsifiable.

Proof: Suppose there exists a proof that there are only a finite number of primes of the form $n^2 + 1$. Then there would exist an $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ in which $n > N$, $n^2 + 1$ would be composite; thus, one could deduce that $n^2 + 1$ is composite from only the assumption that $n - N \in \mathbb{N}$. But this is impossible, since the polynomial $n^2 + 1$ is irreducible over the integers. Hence, it is impossible to prove that there are only a finite number of primes of the form $n^2 + 1$. So the $n^2 + 1$-conjecture is unfalsifiable. □

3 Implications

Let us assume that the ZFC axioms are consistent [10]. Then what are the implications of proving that a mathematical conjecture is unfalsifiable? The answer is that even though an unfalsifiable conjecture might not be true, there is still no harm in assuming that it is true, since there is no chance that one could derive any provably false statements from it; if one could derive any provably false statements from an unfalsifiable conjecture, this would imply that the conjecture is falsifiable, which is a contradiction.

For example, there is a probabilistic heuristic argument that the $n^2 + 1$-Conjecture is true [3]. This implies that all statements which can be derived from the $n^2 + 1$-Conjecture are almost certainly true. Since our theorem above says that the $n^2 + 1$-Conjecture is unfalsifiable, there is no chance that any of these statements could be proven false.

As a different type of example, in 2006 the author showed that the Riemann Hypothesis is unprovable in any reasonable axiom system [1]. This implies that the negation of the Riemann Hypothesis is unfalsifiable, so one might conjecture that the Riemann Hypothesis is false. However, there is a probabilistic heuristic argument that the Riemann Hypothesis is true [2]; therefore, if one were to assume that the Riemann Hypothesis is false, one could derive statements which are almost certainly false from this assumption. However, these statements could never be proven false, since the negation of the Riemann Hypothesis is unfalsifiable.

4 Open problems

Can the following famous conjectures also be proven to be unfalsifiable?

1. There are an infinite number of pairs of primes which differ by two. These are called twin primes [9].
2. There are an infinite number of primes of the form $2^p - 1$, where p is also prime. These are called Mersenne primes [7].
3. There are an infinite number of primes p, where $2p + 1$ is also prime. These are called Sophie Germain primes [8].
4. There are an infinite number of primes of the form $2^{2^n} + 1$. These are called Fermat primes [5].

5 Conclusion

An unfalsifiable theory can never have any use in science; however, an unfalsifiable conjecture can be very useful in mathematics: When an unfalsifiable conjecture is difficult to prove, one can assume that the conjecture is true and not have to worry about deriving any provably false statements from it, assuming that the ZFC axioms are consistent.

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