Power Optimal Solution of the Magnetic Inverse Problem for Heteropolar Magnetic Bearings

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Abstract: The relationship between current and force in magnetic bearings is typically inverted via bias current linearization. This work considers an alternate inverse strategy based on minimizing the power required to produce a given force while accommodating slew rate limitations of the actuator. Continuation techniques are used to obtain an inverse mapping between forces and currents. Specifically considered is an 8-pole magnetic bearing with an independently controlled coil on each leg. The result of this method is decreased power consumption by the bearing. A method of electrically decoupling the coils, crucial to the practical implementation of independent coil control, is also considered.

1 Introduction

Many magnetic bearings employ a bias linearization scheme to invert the relationship between current applied to the bearing's coils and resulting force [1]. However, requiring a linear relationship between desired force and currents is overly restrictive; an actuator need not have a linear inverse for all desired forces to be realizable. In addition, bias linearization does not necessarily yield an inverse with optimal performance in terms of maximizing bearing load capacity or minimizing resistive power losses.

An alternate philosophy for choosing a particular inverse is to select the solution that optimizes some measure of performance while also realizing the desired forces. For the inverse to be realizable with a finite current slew rate, it should also have the following properties [2]:

- *All currents must go to a nominal bias value when the force requested is zero.* This requirement avoids the slew rate limiting problem at low force levels if the bias currents are appropriately selected [3].

- *Coil currents should be a continuous function of force.* This requirement avoids jumps in required currents that would cause slew rate limiting problems away from $f = 0$.

- *The algorithm* [should be] *computationally quick and simple.* For a magnetic actuator to have adequate bandwidth, the throughput rate must be fast. The time spent solving the magnetic inverse problem should therefore not take up a large portion of the sampling interval in a digital controller implementation. An inverse computed off-line and stored in a look-up table for real-time used is assumed to be adequate.

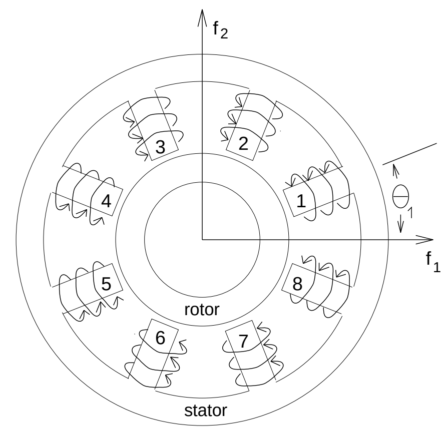

Figure 1: 8-pole heteropolar radial magnetic bearing

2 Current-to-Force Relations

The class of actuators considered by this work is characterized by current-to-force relationships of the form:

$$\begin{aligned} F_1 &= I' M_1 I \\ &\vdots \\ F_k &= I' M_k I \end{aligned} \tag{1}$$

where $I \in \Re^n$ is a vector of applied currents and $M_j \in \Re^{n\times n}$ is an indefinite matrix relating current to force F_j in the j^{th} direction. Any active magnetic actuator without permanent magnet biasing can be reduced to this form via a magnetic circuit analysis [1].

Of particular interest is the case of heteropolar radial magnetic bearings with an independently controlled coil on each leg of the bearing. By convention, each pole is wound with N right-hand turns of wire, and positive flux flows from the rotor to the stator, as pictured in Figure 1. For symmetric n-pole bearings, it is convenient to work in terms of non- dimensional current, air gap flux density, and force, denoted *i*, *b*, and f respectively:

$$\begin{aligned} b &= \frac{B}{B_{sat}} \\ i &= \left(\frac{\mu_o N}{g_o B_{sat}}\right) I \\ f &= \left(\frac{\mu_o}{a B_{sat}^2}\right) f \end{aligned} \tag{2}$$

where a is the pole area, g_o is the nominal air gap, and B_{sat} is the saturation flux density. If the reluctance of iron parts is neglected, the applied current vector can be related to air gap flux density by

$$b = V\,i \tag{3}$$

where V is an $n \times n$ matrix whose diagonal elements are $\frac{n-1}{n}$ and off-diagonal elements are $-\frac{1}{n}$. Using Maxwell's stress tensor, the flux in the air gap is related to force on the rotor by

$$\begin{aligned} f_1 &= b'\Lambda_1 b \\ f_2 &= b'\Lambda_2 b \end{aligned} \tag{4}$$

where

$$\begin{aligned} \Lambda_1 &= \tfrac{1}{2}\mathrm{diag}\{\cos\Theta_1,\ldots,\cos\Theta_n\} \\ \Lambda_2 &= \tfrac{1}{2}\mathrm{diag}\{\sin\Theta_1,\ldots,\sin\Theta_n\} \end{aligned} \tag{5}$$

and Θ_j describes the placement of the j^{th} pole. Combining (3) and (4) yields the general form of the current-to-force relationship:

$$\begin{aligned} f_1 &= i'M_1 i \\ f_2 &= i'M_2 i \end{aligned} \tag{6}$$

where $M_j = V'\Lambda_j V$.

3 Formulation of the Power Optimal Inverse Problem

A natural candidate for a cost function to optimize is $i'\mathsf{Q}\,i$ where Q is a positive definite matrix used for weighting the currents relative to one another. Minimizing this cost would give, in a sense, the smallest current necessary to realize a given force. Another interpretation is that this cost function minimizes the resistive power losses necessary to produce a given force. Using this quadratic cost function, the definition of the power optimal inverse problem is:

$$\min_i J(i) \equiv i'\mathsf{Q}\,i \tag{7}$$

$$\text{subject to} \quad \begin{aligned} i'M_1 i &= f_1 \\ &\vdots \\ i'M_k i &= f_k \end{aligned}$$

However, this formulation has an immediately apparent problem. Consider the change in force, $\frac{df}{dt}$, produced by some change in current, $\frac{di}{dt}$:

$$\begin{aligned} \frac{df_1}{dt} &= 2i'M_1\frac{di}{dt} \\ &\vdots \\ \frac{df_k}{dt} &= 2i'M_k\frac{di}{dt} \end{aligned} \tag{8}$$

At zero force, $i = 0$ satisfies the constraints while at the same time producing $J = 0$. A current of $i = 0$ is clearly the optimal solution at $f = 0$. However, $i = 0$ makes the right-hand side of (8) equal to zero for any finite $\frac{di}{dt}$; any requested change of force about $f = 0$ cannot be realized by an amplifier with a finite switching voltage. This phenomenon is known as slew rate limiting [3].

Consider instead the cost function

$$\min_i J(i) \equiv (i-i_o)'\mathsf{Q}\,(i-i_o) \tag{9}$$

$$\text{subject to} \quad \begin{aligned} i'M_1 i &= f_1 \\ &\vdots \\ i'M_k i &= f_k \end{aligned}$$

where i_o is a bias current vector that satisfies:

$$i_o'M_j i_o = 0 \;\; \forall \;\; j = 1,\ldots,k \tag{10}$$

and the matrix $H[i_o]$ is of rank k where

$$H[i_o] \equiv \begin{bmatrix} i_o'M_1 \\ \vdots \\ i_o'M_k \end{bmatrix} \tag{11}$$

As shown in [4], it is possible to control an actuator with finite current slew rate if and only if an i_o fulfilling these conditions exists. For this cost function, $f = 0$ and $J = 0$ at $i = i_o$; current vector i_o therefore must be the optimal solution of i at $f = 0$. Away from $f = 0$, i_o becomes increasingly insignificant in comparison to i. As i gets larger,

$$(i-i_o)'\mathsf{Q}\,(i-i_o) \approx i'\mathsf{Q}\,i \tag{12}$$

The modified cost converges to the power-optimal cost for large i.

The problem defined by (9) may be adequate if there is a way of solving (9) that yields a smooth inverse mapping. Perhaps the best way to produce an inverse mapping in this case is through a continuation (or homotopy) approach. The optimal solution is known at $f = 0$. The idea is then to make small changes to i that produce a non-zero force but still are optimal in the sense of (9). Similar techniques have been used in the literature, particularly in the area of optimal power system studies [5] [6] [7].

The first step in developing this approach is to combine the desired force constraints into the cost function via scaling by Lagrange multipliers, denoted by λ [8]:

$$\hat{J} \equiv (i-i_o)'\mathsf{Q}\,(i-i_o) + \lambda'(H[i]i - f_j) \tag{13}$$

The Lagrange multipliers can be thought of heuristically as representing a relative cost of satisfying the constraints. For an optimum, the partial derivatives of $\hat{J}$ with respect to both i and λ must be equal to zero:

$$\begin{aligned} 2\mathsf{Q}\,(i-i_o) + 2H'[i]i &= 0 \\ H[i]i - f &= 0 \end{aligned} \tag{14}$$

If a small change in force is desired, i should change in such a way that the change in force is realized while still satisfying the optimality conditions. Let s denote the distance along an arbitrarily chosen continuous trajectory originating at $f = 0$ in the space of desired forces. A small change in forces can be represented now by df/ds.

For the optimality conditions to be satisfied for a given df/ds, the total derivative of (14) with respect to s must be zero:

$$2\begin{bmatrix} \mathsf{Q} + \sum_{j=1}^{k}\lambda_j M_j & H'[i] \\ H[i] & 0 \end{bmatrix}\begin{Bmatrix} \frac{di}{ds} \\ \frac{d\lambda}{ds} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{df}{ds} \end{Bmatrix} \tag{15}$$

Equation (15) is a system of ordinary differential equations in s. On the right hand side, df/ds is specified by the choice of path through the k-dimensional space of f. The left-hand side can then be inverted at any particular i and λ to yield the change in currents and Lagrange multipliers that correspond to any df/ds. An exposition by Bryson and Ho [9] indicates that this integration yields the same i and λ for a given f regardless of path as long as the left-hand side of (15) is always non-singular and the initial condition is itself a minimum.

Initial conditions must be supplied so that (15) can be integrated. The initial condition on current is $i = i_o$ at $f = 0$, since i_o is the optimal solution to (9) at zero force. However, the Lagrange multipliers, λ, are also functions of s, and an appropriate condition on λ must also be supplied at $f = 0$. The value of λ can be determined by considering the conditions (14) at the $f = 0$ point. Substituting $f = 0$ and $i = i_o$ into (14) yields

$$\begin{aligned} 2H'[i_o]\lambda &= 0 \\ 0 &= 0 \end{aligned} \tag{16}$$

The constraint equations in (14) are satisfied at $f = 0$ by definition of i_o. Recall that another condition on i_o is that $H[i_o]$ must be of rank k. An equivalent condition is that the columns of $H'[i_o]$ are linearly independent. Since the columns of $H'[i_o]$ must be linearly independent, no non-zero combination of columns can add up to zero; only $\lambda = 0$ will satisfy (16). The correct initial condition on λ is therefore $\lambda = 0$ at $f = 0$ so that the manifold tracked out of the zero force solution is an optimum. If some other initial condition is used for λ, a manifold will result that satisfies the constraint equations; however, a manifold produced by $\lambda[0] \neq 0$ will not be optimal in the sense of (9).

An optimal inverse mapping is created by integrating (15) numerically along many different paths heading out of the origin, using $i = i_o$, $\lambda = 0$ as the initial condition at $f = 0$. For example, in a 2-force actuator, f can be parameterized in terms of s and an angle θ as

$$\begin{aligned} f_1 &= s\cos\theta \\ f_2 &= s\sin\theta \end{aligned} \tag{17}$$

The path is chosen so that the choice of θ corresponds to the direction of the force, and s corresponds to the magnitude of the force along that direction. To create an inverse mapping, (15) would be integrated from $s = 0$ to some desired maximum force at a great enough number of θ's so that the inverse is suitably defined in the range of forces of interest.

This method relies on the fact that the inverse has a finite slope with respect to s to compute the inverse; therefore, any inverse obtained by this method will have the desired property of smoothness along each integration path. Unfortunately, it is not clear that the left hand side of (15) will always be non-singular for every possible set of M's and i_o's. However, as shown in a subsequent example, this method can give smooth inverse mappings in the practically important case of 8-pole radial magnetic bearings.

4 One Degree of Freedom Example

As an example of the method, consider the 1 d.o.f. problem resulting from two opposed horsehoe magnets acting upon a mass, as illustrated in Figure 2. The current-to-force relations

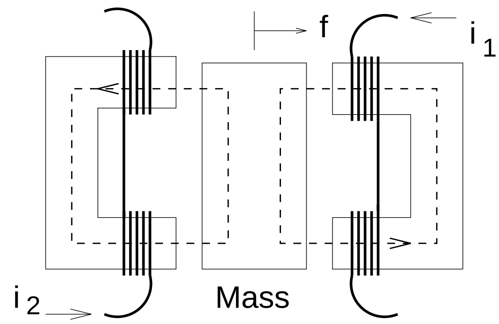

Figure 2: One degree of freedom actuator.

for this actuator are characterized by:

$$f = \{\, i_1 \quad i_2 \,\} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix} \tag{18}$$

The solution that optimizes $i_1^2 + i_2^2$ is

$$\begin{aligned} i_1 = \sqrt{f} \quad &; \quad i_2 = 0 \quad ; \quad f > 0 \\ i_1 = 0 \quad &; \quad i_2 = \sqrt{f} \quad ; \quad f < 0 \end{aligned} \tag{19}$$

However, this solution prescribes zero current at zero force, leading to slew rate limiting at low force levels. Instead, the continuation approach can be applied to yield a low-power solution that avoids slew rate problems. For this example, one can choose

$$i_o = c \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{20}$$

as a biasing vector where c is a constant that scales the magnitude of the vector. The cost function to be optimized is given by (13):

$$\hat{J} = (i_1 - c)^2 + (i_2 - c)^2 + \lambda(i_1^2 - i_2^2 - f) \tag{21}$$

Taking derivatives with respect to i_1, i_2 and λ yields the optimality conditions:

$$\begin{aligned} 2(i_1 - c) + 2\lambda i_1 &= 0 \\ 2(i_2 - c) - 2\lambda i_2 &= 0 \qquad (22) \\ i_1^2 - i_2^2 - f &= 0 \qquad (23) \end{aligned}$$

Define f to be linear with s:

$$f[s] = s \tag{24}$$

Now, taking the total derivative of the optimality conditions with respect to s yields:

$$2 \begin{bmatrix} (1+\lambda) & 0 & i_1 \\ 0 & (1-\lambda) & -i_2 \\ i_1 & -i_2 & 0 \end{bmatrix} \begin{Bmatrix} \frac{di_1}{ds} \\ \frac{di_2}{ds} \\ \frac{d\lambda}{ds} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{25}$$

This system of ordinary differential equations is then integrated numerically, using $i_1 = i_2 = c; \lambda = 0$ as the initial condition at $s = 0$.

The resulting currents for i_1 are shown in Figure 3. The required i_2 is the same plot reflected about $f = 0$. Several different magnitudes of c are considered between 0.05 and

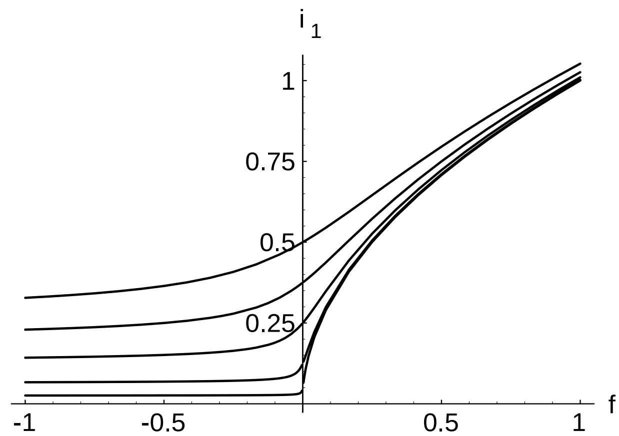

Figure 3: Solution for 1-d example at different bias magnitudes.

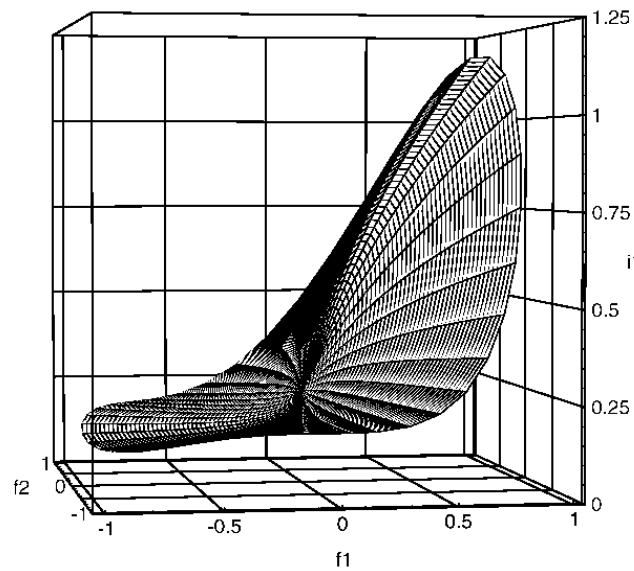

Figure 4: Eight pole power optimal force-to-current mapping.

0.5. As c goes to zero, the solution converges to (19), the optimal solution disregarding slew rate considerations. As the magnitude of i_o increases, the high slopes around $f = 0$ are smoothed out, yielding solutions that require greater current but do not lead to slew rate limiting.

5 Eight Pole Bearing Example

Of practical interest is the performance of the direct optimal solution on the 8-pole symmetric bearing presented in §2. In this instance, two obvious candidates for i_o are of the form $\{1,-1,1,-1,1,-1,1,-1\}^T$ and $\{1,1,-1,-1,1,1,-1,-1\}^T$ which correspond to the NSNS and NNSS biasing schemes typically used in 8-pole bearings. Of these two options, the NSNS scheme has been observed to yield consistently lower power losses and maximum flux densities in the stator when used as i_o; therefore, the NSNS will be exclusively considered here.

For this example, the particular i_o is chosen to be

$$i_o = \frac{1}{4}\{1,-1,1,-1,1,-1,1,-1\}^T \tag{26}$$

and weighting matrix Q is chosen to be the identity matrix. For this case, the ordinary differential equations to be integrated are:

$$2\begin{bmatrix} \mathsf{Q}+\lambda_1 M_1+\lambda_2 M_1 & M_1 i & M_2 i \\ i' M_1 & 0 & 0 \\ i' M_2 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{di}{ds} \\ \frac{d\lambda_1}{ds} \\ \frac{d\lambda_2}{ds} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{Bmatrix} \tag{27}$$

starting from the initial condition $i = i_o$, $\lambda = 0$. Due to symmetry, the inverse mapping from force to current for each leg has the same form, so it is sufficient to display the inverse mapping for just one leg. The inverse mapping for leg "1" as depicted in Figure 1 is displayed in Figure 4. Note that the 8-pole inverse has the same qualitative properties as the one degree-of-freedom inverse: currents in the pole nearest to the force direction go roughly with the square-root of force; currents for poles opposite the force direction are approximately zero; the inverse solution is smooth and non-zero at zero force.

For comparison, the usual opposed-horseshoe bias linearization scheme prescribes:

$$i = \left(\frac{1}{2}\right)\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ f_1 \sec\frac{\pi}{8} \\ f_2 \sec\frac{\pi}{8} \end{Bmatrix} \tag{28}$$

This current set represents biasing with flux levels in the legs half-way to saturation. If load capacity is defined as the largest magnitude force that can be produced for every direction without causing saturation in the bearing, this bias level yields the largest possible load capacity for the bias linearization scheme.

If slew rate limitations allow a small magnitude biasing vector to be chosen for the power optimal scheme, decreased power consumption can be realized relative to the bias linearization solution. Figure 5 compares the power dissipated with the bias linearization scheme in (28) to the power optimal inverse using i_o as defined in (26). The power optimal inverse yields power dissipation that starts at a low level and increases linearly with force magnitude. Conversely, the bias linearization scheme starts at a high power dissipation, and power increases with the square of force magnitude.

6 Electrical Decoupling of the Coils

In most magnetic bearings, each pair of adjacent coils is wound in reverse series. The result is a bearing that is effectively composed of independent horseshoes that are not coupled by mutual inductance. The current in any horseshoe can then be controlled via a PWM amplifier, since each horseshoe is an first-order single input-single output system. However, the same is not the case with independently controlled coils

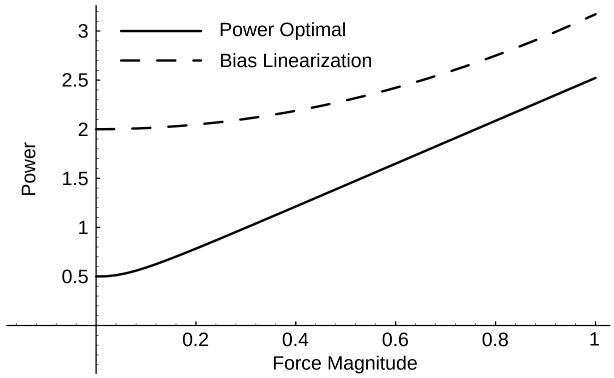

Figure 5: Power optimal and bias linearization power consumption.

on each bearing leg. A current in one coil produces return flux through all other coils, fundamentally coupling all of the electrical circuits through mutual inductance. Consider, for example, a 4-pole bearing with independently wound coils. The electric circuit equations are:

$$L\frac{di}{dt}+ri=v \tag{29}$$

where

$$L=\left(\frac{N^2 a\mu_o}{g_o}\right)\begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \tag{30}$$

and v a vector of amplifier voltages. Inductance matrix L is singular; its eigenvalues are $(\frac{N^2 a\mu_o}{g_o})\{1,1,1,0\}$. Since L is singular, there are only 3 states to the system, even though there are four currents and four inputs. Writing this system in standard form via a singular value decomposition yields:

$$\frac{dx}{dt}=-(\frac{g_o r}{N^2 a\mu_o})x+(\frac{g_o r}{N^2 a\mu_o})\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} v \tag{31}$$

$$i=\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} x+\left(\frac{1}{4r}\right)\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} v \tag{32}$$

Any component of v along $\{1,1,1,1\}$ is fed instantaneously into i. If switching amplifiers are used to produce v, half of the possible switching states have a component along $\{1,1,1,1\}$; exciting the zero-inductance vector is unavoidable with switching amplifiers. Problems in tracking the desired coil currents will result because part of the bearing is, in effect, a purely resistive load.

One solution to these problems is to add extra inductance to the electrical circuit equations associated only with the component of i along the null space of L. The result is that the electrical circuit equations associated with each coil become

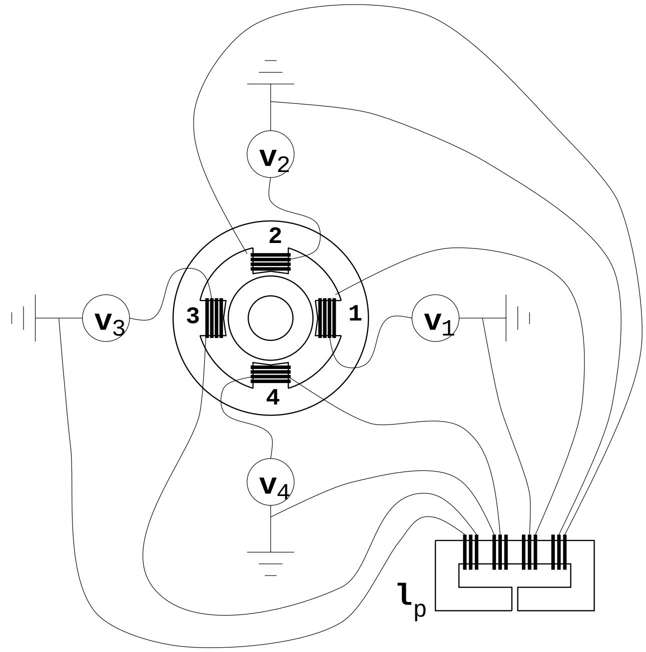

Figure 6: Circuits including l_p to cancel mutual inductance.

decoupled; the same decentralized current control scheme used for horseshoes can then be used for the independent coil actuator. To achieve this decoupling, each bearing coil is also attached in series to windings around a laminated slotted ring. Each electric circuit has the same number of turns wound in the same direction around the ring; therefore, flux is only induced in the ring if i has a component along $\{1,1,\ldots,1\}$. Schematically, the arrangement is illustrated in Figure 6. If the ring is designed so that the self-inductance of the ring for each electric circuit, l_p, has a value of

$$l_p=\frac{1}{n}\left(\frac{a\mu_o N^2}{g_o}\right) \tag{33}$$

the negative off-diagonal mutual inductances in L from the bearing are exactly canceled out by the positive mutual inductances from the ring. The electric circuit equations become:

$$\left(\frac{a\mu_o N^2}{g_o}\right)\frac{di}{dt}+ri=v \tag{34}$$

Although the inductance matrix has been changed with respect to the electric circuit, the bearing still has all of the coupled magnetic properties that allow low power loss performance. Since the ring only adds inductance along the null vector of L, adding this extra ring does not adversely influence slew rate.

7 Conclusions

The magnetic inverse problem in magnetic actuators has been considered from the perspective of minimizing the 2-norm of the currents needed to produce a given force. A problem definition was presented that forces the inverse to go to a set of biasing currents at zero force so that slew rate limiting problems are avoided. At high force levels, this formulation yields currents that are, in a sense, the smallest possible currents needed to realize a given force. The power-optimal inverse formulation leads to a set of ordinary differential equations that must be integrated numerically to yield an inverse mapping between desired forces and required currents. Compared to the usual bias current linearization solution to the magnetic inverse problem, the present method requires significantly less power to produce a given force.

A method for electrically decoupling the coils in a bearing with independent coil control has also been considered. This method consists of winding turns of each coil's electric circuit around an extra iron ring to cancel out the mutual inductance coupling from the bearing. With this modification, the desired currents in each coil can be tracked by using the same decentralized scheme that is currently used in horseshoe-type actuators. The extra decoupling ring does not adversely affect the slew rate performance of the actuator, since all inductance is added along a current vector that is orthogonal to the currents that actually produce force in the bearing.

References

[1] E. H. Maslen and D. C. Meeker. Fault-tolerance of magnetic bearings by generalized bias current linearization. *IEEE Transactions on Magnetics*, MAG-31(3):2304–2314, 1995.

[2] E. W. Green. Control of magnetic actuators for increased force capability. Master's thesis, University of Virginia, 1996.

[3] E. H. Maslen, P. E. Allaire, M. D. Noh, and C. K. Sortore. Magnetic bearings for reduced power consumption. *Journal of Tribology*, page to appear, July 1996.

[4] D. C. Meeker. *Optimal solutions to the inverse problem in quadratic magnetic actuators*. PhD thesis, University of Virginia, 1996.

[5] M. Huneault *et al.* The continuation method in power system optimization: applications to economy-security functions. *IEEE Transactions on Power Apparatus and Systems*, 104:114–123, 1985.

[6] M. Huneault and F. D. Galiana. An investigation of the solution to the optimal power flow problem incorporating continuation methods. *IEEE Transactions on Power Systems*, 5:103–110, 1990.

[7] R. A. Ponrajah and F. D. Galiana. The minimum cost optimal power flow problem solved via the restart homotopy continuation method. *IEEE Transaction on Power Systems*, 4:138–148, 1992.

[8] R. L. Fox. *Optimization Methods for Engineering Design*. Addison-Wesley, 1971.

[9] A. Bryson and Y. Ho. *Applied optimal control*. Blaisdell Publishing, 1969.