A New Method for Computing Pseudo-Time for Real Gas Flow Using the Material Balance Equation

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Abstract

Analytical solutions in well-test analysis are oriented towards fluids with a constant viscosity and compressibility in a porous medium with a constant porosity. To use these solutions in gas-flow situations, one needs to apply the pseudo-pressure and pseudo-time transformations. Thus, the non-linear diffusivity equation for gas flow is transformed into a linear one, allowing one to use the solutions for fluids with constant compressibility, viscosity, and porosity. Although the computation of pseudo-pressure is reasonably accurate, the conventional computation of pseudo-time by direct integration of the compressibility-viscosity function over time can result in significant errors when modelling gas reservoirs with residual fluid saturation, rock compressibility, and a large degree of depletion. Errors in calculating the pseudo-time result in substantial errors in the material balance, which has an adverse impact on reservoir modelling and production forecasting.

In this study, a new method for computing pseudo-time is presented. This method is based on the material balance equation that considers the rock and fluid compressibility. This formulation honours the material balance equation in all situations. Examples are presented to show that the problems with computing pseudo-time, using the traditional definition, can be resolved when the new method is used. Accurate computations of pseudo-time allow one to use the solutions for fluids with constant compressibility and viscosity for modelling and forecasting gas production.

Introduction

Analytical solutions are generally used to analyze and model well test and production data. However, these solutions have been developed for fluids with a constant viscosity and compressibility and for formations with a constant porosity. The governing diffusivity equation and its boundary conditions are linear when expressed in terms of pressure, space, and time variables. The analytical solutions to these equations are reasonably accurate for the liquid-flow situations. In contrast, the diffusivity equation for gas flow and its boundary conditions are non-linear when expressed in terms of pressure, space, and time variables. As no analytical solutions to these non-linear equations are available, one needs to apply pseudo-pressure and pseudo-time transformations in order to use the analytical solutions for liquid flow in gas-flow situations. This approach introduces two variables in the diffusivity equation for gas flow—pseudo-pressure (ψ) as the dependent variable, and pseudo-time (t_a) as an independent variable. As a result, the diffusivity equation for gas flow is transformed into a linear one, allowing one to use the slightly compressible fluid (liquid) solutions.

In this study, we are considering a single-phase gas flow situation in the presence of residual fluid saturation and a compressible formation. Here gas is the only mobile phase, while oil and water phases are immobile, if there are any. The pseudo-variables (pseudo pressure and pseudo-time) can be defined as:

Pseudo-Pressure[1]

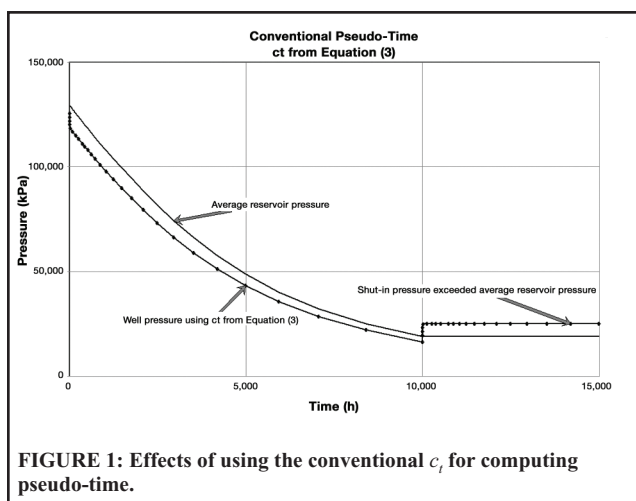$$\psi(p) = 2 \int_0^p \frac{p\,dp}{\mu Z} \quad\quad (1)$$

and

Pseudo-Time[2]

$$t_a(t) = \int_0^t \frac{dt}{\mu c_t} \quad\quad (2)$$

The rationale for defining the above pseudo-variables is demonstrated in Appendix A. Martin[3] made the first systematic attempt to define the total system compressibility in multi-phase conditions, neglecting the rock compressibility. Later, Ramey[4] followed Martin's lead and included the formation compressibility in defining the total system compressibility, c_t, as:

$$c_t = c_f + S_{oi}c_o + S_{wi}c_w + S_{gi}c_g \quad\quad (3)$$

Although neither Martin nor Ramey was explicit about what pressure level the fluid saturations should be considered in defining the total system compressibility, most reservoir calculations consider these saturations at the initial pressure, as used in Equation (3). The definition of c_t as in Equation (3) has been generally accepted[5, 6] for computing pseudo-time since the work of Agarwal[2]. The computation of pseudo-pressure from the direct integration of Equation (1) is reasonably accurate for all practical purposes. But, the computation of pseudo-time from the direct integration of Equation (2), with c_t as defined in Equation (3), has been challenging, especially in gas reservoirs with compressible formations, residual fluid saturations, and huge depletions. The definition of c_t in Equation (3) is not rigorous, as it fails to honour the material balance equation in some instances. In these situations, the objective of transforming the gas flow equations into linear ones may not succeed, and there can be considerable errors in the computed values of pseudo-time. The computation of pseudo-time using the total system compressibility defined in Equation (3) is henceforth referred to as

FIGURE 1: Effects of using the conventional c_t for computing pseudo-time.

k, mD	14.8
h, m	3.05
ϕ_i, %	6
S_{gi}, %	80
S_{wi}, %	20
c_{ti}, kPa^{-1}	2.33e-6
p_i, kPa absolute	129,600
G, 10^6m^3	185.7
q, 10^3m^3/day	281.8 for $0 < t < = 10,000$ hrs (0 for $t > 10,000$ hrs)
G_p, 10^6m^3	117.4 (63.2% depletion)

conventional pseudo-time. The conventional pseudo-time calculations are not rigorous due to the following assumptions:

a) Fixed fluid saturations over time; and,

b) Fixed porosity with pressure (even though c_f has been included in c_t).

Whereas pseudo-pressure accounts for the variations of gas compressibility factor and viscosity with pressure, pseudo-time is intended to unify and take care of the effects of the following variables with time (thus, with pressure) in the diffusivity equation:

a) Time;

b) Gas viscosity;

c) Gas compressibility;

d) Porosity (due to formation compressibility); and,

e) Fluid saturations.

The effect of using inaccurate pseudo-time is illustrated in Figure 1. The parameters of this case are presented in Table 1. The gas reservoir has been subject to a 63.2% depletion of the 185.7 10^6m^3 original gas-in-place during the drawdown period from an initial pressure of 129,600 kPa absolute. Conventional pseudo-time is used to calculate the well pressures in this gas-flow situation. After the well has been shut in ($t > 10,000$ hours), the well pressure is supposed to gradually catch up to the average reservoir pressure of 18,800 kPa absolute. Instead, the build-up pressure has risen up to 25,000 kPa absolute, which is greater than the average reservoir pressure by about 6,200 kPa absolute. This anomaly is directly due to the use of the definition of c_t in Equation (3) when calculating pseudo-time. Moreover, drawdown pressure calculations are affected as well. This matter will be discussed further after the rigorous definition of c_t has been introduced.

Note that c_t presented in Equation (A-5) in Appendix A is rigorous. But, the calculation of c_t does not appear to be very practical as it needs the updating of saturations, saturation gradients, and gas compressibility. Therefore, we need to get the c_t values from a different source that will be rigorous and convenient to calculate pseudo-time and will also honour the material balance equation at all times.

Rigorous Pseudo-Time

As presented earlier, the accuracy of pseudo-time is dependent on the following steps:

a) How the total system compressibility, c_t, is defined and calculated; and,

b) How the integral in Equation (2) is evaluated numerically.

From our practical experience, we have found that the calculated pseudo-time can be inaccurate due to the inaccuracies in either or both of the above steps. Also, any errors accumulated due to the inaccurate c_t values cannot be compensated through a rigorous numerical procedure for evaluating the integral. Thus, if there is a choice, one needs to be more diligent in Step a than in Step b. The following sub-sections show that both of these steps can be

made rigorous and convenient by manipulating the material balance equation.

Rigorous and Convenient Definition of c_t

As shown in Appendix B, c_t can be expressed rigorously but conveniently from the material balance equation for the purpose of calculating the pseudo-time. This is:

$$c_t = c_{ti}\left[1 - c_g\left(p_i - p\right)\right] + S_{gi}\left[c_g - c_{gi} + c_g c_{gi}\left(p_i - p\right)\right] \quad \text{........................(4)}$$

where:

$$c_{ti} = c_f + S_{oi}c_o + S_{wi}c_w + S_{gi}c_{gi} \quad \text{........................(5)}$$

As expressed above, c_t is a function of pressure for a given reservoir system, and c_{ti} is constant for a given initial pressure. The variation of c_t is due to the variation of the gas compressibility, c_g, and pressure only, as the other parameters are fixed for a given reservoir system.

Equation (4) is identical to Equation (A-5). This is not evident by a simple comparison of these two equations, but it is easily demonstrated by numerical evaluation of each equation for any specific reservoir. However, the expression for c_t in Equation (4) is convenient to evaluate as this expression requires only the updating of gas compressibility and pressure. Thus, the expression of c_t in Equation (4) is both rigorous and convenient for computing the pseudo-time. Using this definition of the total system compressibility to calculate pseudo-time with Equation (2) is henceforth referred to as rigorous pseudo-time.

Computation of Pseudo-Time

Since we have now established the rigorous definition of c_t [Equation (4)], one can use Equation (2) for computing the pseudo-time by numerical integration. A suitable technique (e.g., trapezoidal rule, Simpson's rule, etc.) can be employed to evaluate the integral. In most cases, the errors incurred in integration are acceptable, depending on the time step.

Drawdown or Injection Situations

There are occasions when the errors due to numerical integration in calculating pseudo-time are beyond tolerable limits. We recognize that during drawdown or injection, the average pressure (or average pseudo-pressure) changes with time. When the drawdown or injection occurs under a variable rate scenario, the pseudo-time can be calculated alternatively from the average pseudo-pressure data. The basis of this approach is derived in Appendix C. Using Equation (C-2), pseudo-time t_a can be calculated for any number of successive rate variations during the drawdown or injection period. Computing the pseudo-time from pseudo-pressures works only when the rate at a given time is non-zero, because the average reservoir pressure p does not change (neither does average pseudo-pressure, ψ) when the wells are shut in. As the definition in Equation (2) suggests, the pseudo-time should always be advancing with real time, even during a build-up period. But the

pseudo-time cannot be calculated from the pseudo-pressure formulation of Equation (C-2) during a build-up period.

Drawdown Build-up or Injection Fall-off Situations

As discussed earlier, the pseudo-time calculated from Equation (C-1) or Equation (C-2) does not advance when the wells are not producing (the reservoir pressure is not changing). Thus, it is recommended that one switch back to the numerical integration approach of Equation (2) for the purpose of calculating pseudo-time during the build-up or fall-off period. In order to do so, the total system compressibility c_t needs the current values of gas compressibility and the average reservoir pressure (or wellbore pressure depending on the specific situation as discussed later), keeping the c_o and c_w values at a constant reference pressure. Thus, a general formula for calculating pseudo-time can be written for n rates, including occasional build-up or fall-off periods, as:

$$t_a(t) = \sum_{j=1}^{n} \left\{ \delta_j \int_{t_j}^{t_{j+1}} \frac{dt}{\mu(p_j)\, c_t(p_j)} + (1-\delta_j)\frac{G\, Z_i}{2\, p_i\, S_{gi}}\left[\frac{(\psi_{j-1} - \psi_j)}{q_j}\right] \right\} \quad \ldots\ldots (6)$$

where $\delta_j = 1$ when $q_j = 0$, and $\delta_j = 0$ when $q_j \neq 0$. Keep in mind that the pseudo-pressure ψ_j is due to the average reservoir pressure, p_j. The first term (involving integral) on the right hand side in

FIGURE 2: Average pseudo-pressure variations with the stepped-rate variations.

FIGURE 3: Implementation of new formulation eliminates the problem with calculating well pressure.

Equation (6) vanishes when it belongs to a drawdown or injection period. Also, the second term (involving pseudo-pressures) on the right hand side in Equation (6) vanishes when it belongs to a build-up (or fall-off) period. Consider a seven-rate situation, including two build-up periods, as illustrated in Figure 2. As $q_3 = q_6 = 0$, the pseudo-pressures have not changed during these periods. Therefore, $\psi_2 = \psi_3$ and $\psi_5 = \psi_6$. Thus, the pseudo-time for this case with $q = q_7$ at time t is given by:

$$t_a(t) = \frac{G\, Z_i}{2\, p_i\, S_{gi}}\left\{ \frac{[\psi_i - \psi_1]}{q_1} + \frac{[\psi_1 - \psi_2]}{q_2} + \frac{[\psi_3 - \psi_4]}{q_4} + \frac{[\psi_4 - \psi_5]}{q_5} + \right.$$

$$\left. \frac{[\psi_6 - \psi(t)]}{q_7} \right\} + \int_{t_2}^{t_3} \frac{dt}{\mu(p_3)\, c_t(p_3)} + \int_{t_5}^{t_6} \frac{dt}{\mu(p_6)\, c_t(p_6)} \quad \ldots\ldots (7)$$

The last two terms involving integrals on the right hand side of Equation (7) are due to the build-up periods. This approach for dealing with multiple drawdown build-up periods is applicable in using the production data for rate-transient analysis where the reservoir behaviour dominates the long-term production data. In such situations, the viscosity-compressibility components associated with build-up or fall-off periods [in integrals in Equation (7)] should be evaluated at the average reservoir pressure. However, for analyzing well test data, one requires special consideration during build-up or fall-off, treating the drawdown periods identical to the approach discussed above. As the fluid properties at the wellbore during build-up or fall-off are dependent mostly on the wellbore pressure, the viscosity-compressibility component in the integrals of Equation (6) should be evaluated at the respective wellbore pressure, instead of the average reservoir pressure. This is essentially true with the wellbore storage effect. Therefore, the modified version of Equation (7) for the well test analysis can be written as:

$$t_a(t) = \frac{G\, Z_i}{2\, p_i\, S_{gi}}\left\{ \frac{\psi_i - \psi_1}{q_1} + \frac{\psi_1 - \psi_2}{q_2} + \frac{\psi_3 - \psi_4}{q_4} + \frac{\psi_4 - \psi_5}{q_5} + \frac{\psi_6 - \psi[p(t)]}{q_7} \right\}$$

$$+ \int_{t_2}^{t_3} \frac{dt}{\mu(p_w)\, c_t(p_w)} + \int_{t_5}^{t_6} \frac{dt}{\mu(p_w)\, c_t(p_w)} \quad \ldots\ldots (8)$$

The approach taken to derive Equations (6) through (8) can be regarded as a *combination method* which uses both numerical integration and pseudo-pressure techniques.

Figure 3 presents the identical case of Figure 1 (parameters in Table 1), but with the pseudo-time as calculated with rigorous c_t [defined in Equation (4)]. Since this example consists of a drawdown and a build-up period, the approach of calculating t_a is taken as Equation (8). This shows that the build-up pressure merges steadily with the average reservoir pressure of 25,000 kPa absolute, and the inconsistencies observed in Figure 1 do not exist any
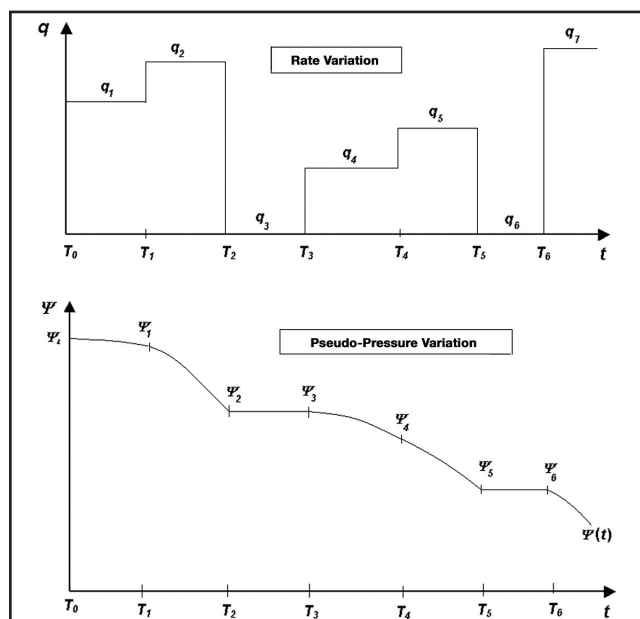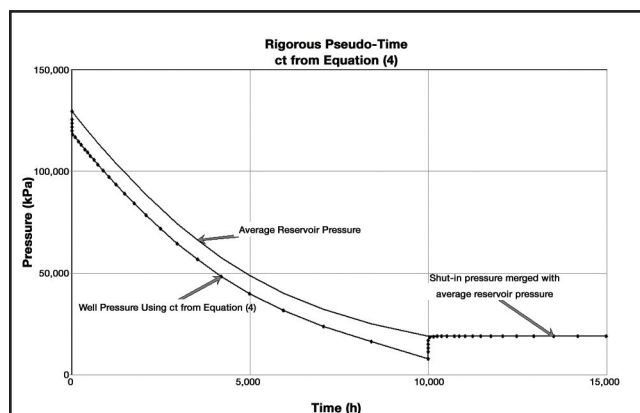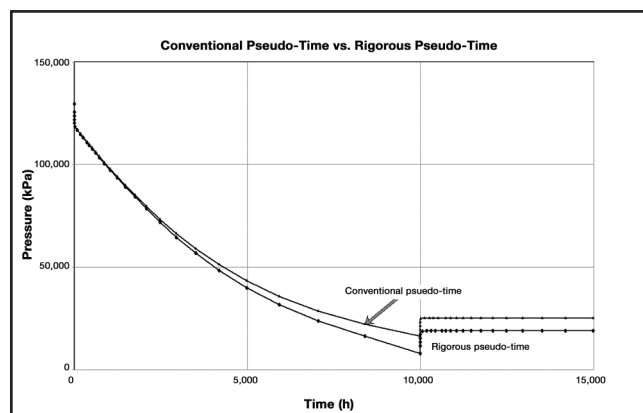

FIGURE 4: Comparison of pressure profiles calculating pseudo-times from conventional and rigorous methods.

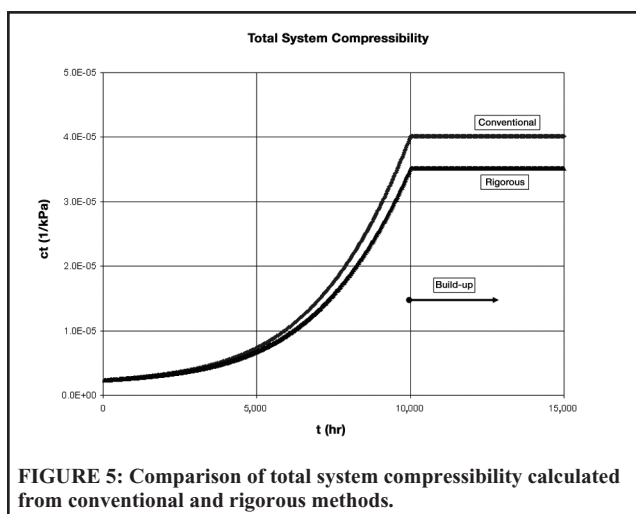**FIGURE 5: Comparison of total system compressibility calculated from conventional and rigorous methods.**

FIGURE 6: Comparison of normalized pseudo-time calculated from conventional and rigorous methods.

longer. This is obviously due to the fact that calculated pseudo-time honours the material balance. The wellbore pressure profiles from Figure 1, with conventional pseudo-time, and from Figure 3, with rigorous pseudo-time, are superposed in Figure 4 for comparison. This figure demonstrates that both the drawdown and the build-up profiles have changed due to using the rigorous pseudo-time. The pressure at the end of the drawdown period is 16,150 kPa absolute with the conventional pseudo-time, while it is 7,900 kPa absolute with the rigorous pseudo-time.

Let us look at two diagnostic plots for the case presented earlier with Figures 1 and 3. Figure 5 shows the total system compressibility calculations with time, using both conventional and rigorous formulae. The conventional formula over-estimates the total compressibility and the difference in the c_t values increases during the drawdown period. During the build-up period, the difference stays constant as the compressibility in each method is calculated at the average reservoir pressure which does not change during this time. Figure 6 shows the effects of the difference in the c_t calculations on the pseudo-time calculations. For convenience, the normalized pseudo-time vs. time is plotted. The relationship between normalized pseudo-time and pseudo-time is given by:

$$t_a' = \mu_i c_{ti} t_a \qquad (9)$$

An over-estimation of c_t during the drawdown period has resulted in an under-estimation of the values of t_a (and of t_a'). This is obvious due to the nature of Equation (2) for calculating pseudo-time. Also shown further in Figure (6), the pseudo-time is increasing linearly with time during the build-up period. This is due to the fact that pseudo-time advances due to the advancing real time only.

Computation of Pseudo-Time in Other Scenarios

The formulation for pseudo-time presented earlier was derived for volumetric gas reservoirs governed by the Material Balance Equation (B-1). These reservoirs have residual liquid saturations and a compressible formation. A formulation for pseudo-time for water-drive systems can be derived from the appropriate material balance equation by following a similar approach.

Procedure for Computing Rigorous Pseudo-Time

Step 1

History With Drawdown or Injection

This approach is applicable when the production history involves drawdown or injection only. Use the gas material balance
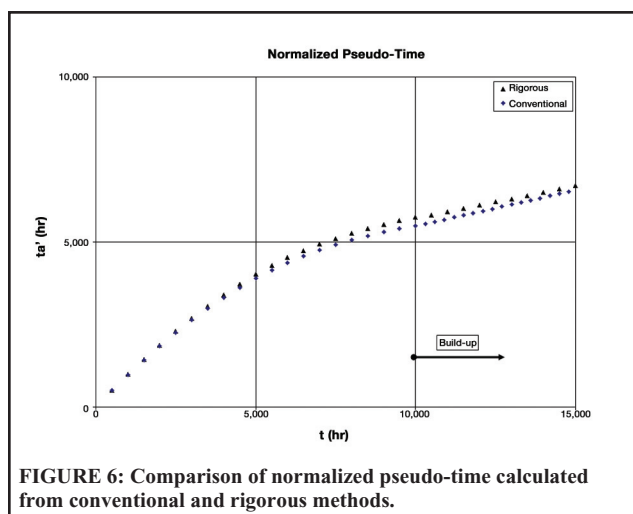
equation [e.g., Equation (B-1)] to calculate p at a given time t. Calculate the corresponding ψ value. The ψ values at the times when the rates have changed and at current time are stored for the next step. Compute pseudo-time, t_a, from an appropriate form of Equation (C-1) [e.g., Equation (C-2)]. When errors due to numerical integration are within tolerable limits, one may choose to compute pseudo-time from Equation (2) by numerical integration instead.

History With Drawdown Build-up or Injection Fall-off

If there is at least one build-up or fall-off period in the history and pseudo-time, t_a, needs to be calculated during a current drawdown or injection period, one may choose either the numerical integration method or the combination method. As mentioned earlier, during build-up or fall-off periods, one must compute t_a from a numerical integration of Equation (2), using c_t from Equation (4). According to Equation (4), updating of c_t is required through the changing gas compressibility and pressure with time. Use one of the following approaches in updating c_t during a build-up or fall-off period:

a) Build-up or Fall-off With Wellbore Storage:
   Total system compressibility, c_t, during this period will change due to the changing gas compressibility only, while the wellbore pressure varies. Thus, t_a, will advance due to the changing real time, viscosity, and gas compressibility, which are evaluated at the changing well pressure. Use an appropriate equation, similar to Equation (8), for computing t_a. This approach is particularly important for analyzing and modelling the well test data; and,

b) Build-up or Fall-off Without Wellbore Storage:
   Total system compressibility and viscosity during this period will not change, as these values are taken at the average reservoir pressure which is not varying. However, t_a will advance due to the advancing real time only. Use an appropriate equation, similar to Equation (7), for computing t_a. This approach is suitable for rate transient analysis and modelling.

Step 2

Calculate the dimensionless time, t_D, corresponding to t_a, using the following equation:

$$t_D = \frac{C_2 k\, t_a}{\phi_i\, r_w^2} \qquad (10)$$

Step 3

Use the principle of superposition, incorporating the rate variations, to calculate the dimensionless pseudo-pressure, $\psi_D(t_D)$, from an analytical solution derived from the diffusivity equation for liquid flow.

Advantages of New Formulation

1) Pseudo-pressure data can be used to calculate the pseudo-time in drawdown or injection periods; therefore, numerical errors due to performing an integration of Equation (2) can be avoided. In other words, any propagation and growth of errors in calculating the pseudo-time through the process of numerical integration can be restricted. The errors in performing the integration in the build-up or fall-off periods are minimal and do not significantly affect the overall results. From numerical experiments, it has been found that the errors due to integration are not significant even during the drawdown or injection periods as c_t, defined in Equation (4), and has a tendency to honour the material balance equation at all times.

2) The pseudo-time calculated from the new formulation can be used in conjunction with the principle of superposition to account for any rate changes. This flexibility is very desirable from the standpoint of practical application.

3) This formulation is general in nature; thus, it holds true even for the cases when the pore volume is entirely occupied by gas ($S_{gi} = 1$) and/or when the rock compressibility is negligible ($c_f = 0$).

Applications of Rigorous Pseudo-Time

Examples illustrating the modelling capability with rigorous pseudo-time have been presented earlier. This section will highlight the extended applicability of rigorous pseudo-time in forecasting gas production and analyzing production data with material balance pseudo-time.

Forecasting

Here we consider the same reservoir parameters as those in Table 1. A forecast of cumulative gas production over the next 12 months (starting from the time when the average reservoir pressure is 18,800 kPa absolute) for different specified wellbore pressures of 6,900, 11,300, and 13,800 kPa absolute have been calculated using both the conventional and rigorous pseudo-times. Table 2 presents the comparative summary of the forecasted cumulative gas production at the end of 12 months. This shows that the differences in the forecasts are substantial.

Rate Transient Analysis

Palacio and Blasingame[7] have demonstrated that the constant-pressure solution can be approximated by the constant-rate solution when the normalized transient rates and the normalized rate integrals are compared at a given material balance time. Thus, the field data with variable rate and variable bottomhole pressure can be analyzed with the constant rate solutions. Alternatively speaking, this approach, also known as the rate transient analysis, is particularly applicable for analyzing long-term production data. More illustrations on this matter are available in the work of Agarwal et al.[8]. The reason for introducing the material balance pseudo-time at this point is that calculating the pseudo-time is a necessary step for calculating the material balance pseudo-time. The relationship between the normalized material balance pseudo-time and the pseudo-time has been derived in Appendix D as:

$$t'_{ca}(t) = \frac{\mu_i c_{ti} \displaystyle\int_0^{t_a(t)} q(t_a)\, dt_a}{q(t)} \quad\text{..(11)}$$

Production data (rates and flowing bottomhole pressure) can be analyzed using the normalized material balance pseudo-time. Details of this analysis technique are outlined in the work of Blasingame et al.[9]. The original gas-in-place can be estimated from this

TABLE 2: Comparison of forecast recovery with conventional and rigorous pseudo-times.

p_{wf} (kPa absolute)	Forecast Recovery With Conventional Pseudo-Time (10^6m^3)	Forecast Recovery With Rigorous Pseudo-Time (10^6m^3)	Percentage Difference (%)
6,900	34.0	41.5	18.1
11,300	25.5	31.1	18.0
13,800	15.4	18.7	17.8

analysis. It has been observed that the conventional pseudo-time has a tendency to under-estimate the reserve.

Conclusions

This study presents the following conclusions:

1) The conventional definition of the total system compressibility disregards the variations of fluid saturations and formation porosity with pressure. In the rigorous definition, these variations have been considered and they honour the material balance equation;

2) The definition of the total system compressibility for gas reservoirs with residual fluid saturation and compressible formations can be rigorously defined from the respective material balance equation. With this rigorously calculated total system compressibility, one can calculate pseudo-time rigorously;

3) The pseudo-time can be calculated from the pseudo-pressure data in multiple-rate drawdown or injection situations;

4) The new method proposed in this study to calculate pseudo-time for a production history with multiple drawdown build-up or injection fall-off periods can be used for well test and rate-transient analyses; and,

5) The conventional pseudo-time generally under-estimates the forecast rates and cumulative productions in comparison to the rigorous pseudo-time.

Acknowledgements

NOMENCLATURE

A = reservoir area, m^2
B_g = formation volume factor of gas, m^3/m^3
B_o = formation volume factor of oil, m^3/m^3
B_w = formation volume factor of water, m^3/m^3
C_1 = constant for field units, 1.295
C_2 = constant for metric units, (3.6e-3)(24)
c_e = effective compressibility, kPa^{-1}
c_f = formation compressibility (constant), kPa^{-1}
c_g = gas compressibility at p, kPa^{-1}
c_{gi} = gas compressibility at p_i, kPa^{-1}
c_o = oil compressibility (constant), kPa^{-1}
c_w = water compressibility (constant), kPa^{-1}
c_t = total system compressibility at p, kPa^{-1} [Equation (4)]
c_t' = expression in Equation (A-3)
c_{ti} = total system compressibility at p_i, kPa^{-1} [Equation (5)]
G = original gas-in-place, 10^3m^3
G_p = cumulative gas production, 10^3m^3
G_{pa} = pseudo-cumulative gas production, 10^3m^3
h = pay thickness, m
j = refers to j^{th} rate in production rate sequence
k = reservoir permeability, mD
n = number of rates in a production (or injection) rate sequence

p = average reservoir pressure, kPa
p_i = initial reservoir pressure, kPa
p_j = average reservoir pressure at end of jth rate, q_j, kPa absolute
p_{st} = standard pressure, 101.325 kPa absolute
p_w = wellbore pressure, kPa absolute
Q = cumulative liquid produced, m^3
q = production rate (can be a function of time), $10^3 m^3$/day
q_j = j^{th} rate in a rate sequence (+ve for production, and –ve for injection), $10^3 m^3$/day
r_w = wellbore radius, r_w, m
S_g = gas saturation at p
S_o = oil saturation at p
S_w = water saturation at p
S_{gi} = gas saturation at p_i
S_{oi} = oil saturation at p_i
S_{wi} = water saturation at p_i
t = real time, day
t_a = pseudo-time, day-kPa/μPa.s
t_a' = normalized pseudo-time [Equation (9)], day
t_c = material balance time for liquid, day
t_{ca} = material balance pseudo-time for gas, day-kPa/μPa.s
t_{ca}' = normalized material balance pseudo-time for gas, day
t_D = dimensionless time [Equation (10)]
t_{DA} = dimensionless time based on area,
  $\dfrac{C_2 k t_a}{\phi_i A}$
t_j = starting time of the q_{j+1}^{th} production rate (for $j = 0, 1, \ldots., n$), day
T = temperature, K
T_{st} = standard temperature, 288.15 K
Z = gas compressibility factor at p
Z_i = gas compressibility factor at p_i
ϕ = porosity at p
ϕ_i = porosity at p_i
∇ = Laplacian operator
δ_j = parameter, equal to 1 when $q_j = 0$, and 0, when $q_j \neq 0$
μ = gas viscosity at p, μPa.s
μ_i = gas viscosity at p_i, μPa.s
ψ = pseudo-pressure corresponding to p, kPa^2/μPa.s
ψ_D = dimensionless pseudo-pressure corresponding to ψ,
  $\dfrac{kh(\psi_i - \psi)}{C_1 q T}$
ψ_i = pseudo-pressure corresponding to p_i, kPa^2/μPa.s
ψ = pseudo-pressure corresponding to p, kPa^2/μPa.s

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Appendix A: Basis of Pseudo-Variables

The rationale of the forms of pseudo-pressure and pseudo-time as described in Equations (1) and (2) can be appreciated from the following discussion.

Rigorous diffusivity equation for gas flow in the presence of residual fluid saturation in a compressible formation can be derived as[3]:

$$\nabla \cdot \left(\frac{1}{\mu B_g} \nabla p \right) = \frac{1}{C_2 k} \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} + \frac{\phi S_w}{B_w} + \frac{\phi S_g}{B_g} \right) \quad \text{(A-1)}$$

Applying the pseudo-pressure transformation from Equation (1) in Equation (A-1), it follows that:

$$\nabla^2 \psi = \frac{\phi \mu c_t'}{C_2 k} \frac{\partial \psi}{\partial t} \quad \text{(A-2)}$$

where c_t' is defined as:

$$c_t' = \frac{B_g S_o}{B_o}\left(c_f + c_o\right) + \frac{B_g S_w}{B_w}\left(c_f + c_w\right) + B_g S_g \left(c_f + c_o\right) +$$

$$\frac{B_g}{B_o}\frac{\partial S_o}{\partial p} + \frac{B_g}{B_w}\frac{\partial S_w}{\partial p} + \frac{\partial S_g}{\partial p} \quad \text{(A-3)}$$

Notice that Equation (A-2) is not truly linear, since each of ϕ, μ, and c_t' on the right hand side of Equation (A-2) is a variable of time (and of pressure). The variation of porosity with pressure can be expressed using the formation compressibility as:

$$\phi = \phi_i \left[1 - c_f \left(p_i - p\right)\right] \quad \text{(A-4)}$$

In Equation (A-2), if the porosity ϕ is replaced by Equation (A-4), c_t' can be replaced by c_t, as follows:

$$c_t = c_t'\left[1 - c_f \left(p_i - p\right)\right] \quad \text{(A-5)}$$

Combined with Equations (2), (A-3), (A-4), and (A-5), Equation (A-2) becomes:

$$\nabla^2 \psi = \frac{\phi_i}{C_2 k} \frac{\partial \psi}{\partial t_a} \quad \text{(A-6)}$$

Equation (A-6) is an effective linear form of the diffusivity equation for gas flow with residual fluid saturation in a compressible porous medium. Therefore, the definitions of pseudo-variables presented in Equations (1) and (2) can effectively linearize the diffusivity equation for gas flow.

Appendix B: Mathematical Development of Rigorous c_t

This section shows how the pseudo-time can be defined with rigorous c_t, while honouring the material balance equation at all times.

An average reservoir pressure can be calculated from the Ramagost and Farshad gas material balance equation[10] which is:

$$\frac{p}{Z}\left[1 - c_e\left(p_i - p\right)\right] = \frac{p_i}{Z_i}\left(1 - \frac{G_p}{G}\right) \quad \text{(B-1)}$$

where:

$$c_e = \frac{c_f + S_{oi}c_o + S_{wi}c_w}{S_{gi}} \quad \text{(B-2)}$$

Note that the compressibilities of the formation and of the residual fluids (oil and water) and the initial fluid saturations are constant in the material balance equation. This equation certainly accounts for variations in the porosity and fluid saturations with depletion. The average reservoir pressure, p, obtained from Equation (B-1), is only meaningful when the compressibility and saturation values are consistent with the condition that:

$$c_e\left(p_i - p\right) \leq 1 \quad \text{(B-3)}$$

Differentiating Equation (B-1) partially with respect to real time, t, one gets:

$$\frac{\partial}{\partial t}\left(\frac{p}{Z}\right) = -\frac{p_i}{Z_i}\frac{\left(q/G\right)\left[1 - c_e\left(p_i - p\right)\right] + c_e\left(1 - \frac{G_p}{G}\right)\frac{\partial p}{\partial t}}{\left[1 - c_e\left(p_i - p\right)\right]^2} \quad \text{(B-4)}$$

where:

$$q(t) = \frac{dG_p(t)}{dt} \quad \text{(B-5)}$$

Consider the left hand side of Equation (B-4) as:

$$\frac{\partial}{\partial t}\left(\frac{p}{Z}\right) = \left[\frac{\partial}{\partial p}\left(\frac{p}{Z}\right)\right]\frac{\partial p}{\partial t} = \frac{p\,c_g}{Z}\frac{\partial p}{\partial t} \quad \text{(B-6)}$$

From Equations (B-4) and (B-6), one can write:

$$\frac{\partial p}{\partial t} = -\frac{q\left[1 - c_e\left(p_i - p\right)\right]}{\left(G - G_p\right)\left\{c_e + c_g\left[1 - c_e\left(p_i - p\right)\right]\right\}} \quad \text{(B-7)}$$

From Equation (B-1), one recognizes:

$$\frac{1 - c_e\left(p_i - p\right)}{G - G_p} = \frac{p_i Z}{G p Z_i} \quad \text{(B-8)}$$

Substituting Equation (B-8) into Equation (B-7), it follows:

$$\frac{\partial p}{\partial t} = -\frac{p_i q Z}{G p Z_i\left\{c_e + c_g\left[1 - c_e\left(p_i - p\right)\right]\right\}} \quad \text{(B-9)}$$

Similarly from partially differentiating Equation (1) with respect to p, one gets:

$$\frac{\partial \psi}{\partial p} = \frac{2p}{\mu Z} \quad \text{(B-10)}$$

Now, using the chain rule for partial differentiation:

$$\frac{\partial \psi}{\partial t} = \frac{\partial p}{\partial t} \cdot \frac{\partial \psi}{\partial p} \quad \text{(B-11)}$$

with Equations (B-9) and (B-10), it follows:

$$\frac{\partial \psi}{\partial t} = -\frac{2p_i q}{G Z_i \mu\left\{c_e + c_g\left[1 - c_e\left(p_i - p\right)\right]\right\}} \quad \text{(B-12)}$$

Now consider the dimensionless form of the gas flow equation for pseudo-steady state:

$$\psi_D = 2\pi t_{DA} \quad \text{(B-13)}$$

Equation (B-13) is indeed a form of the material balance equation based on pseudo-pressure and pseudo-time, even with rock compressibility and residual fluid saturation. Differentiating the dimensional version of Equation (B-13) partially with respect to pseudo-time t_a, one obtains:

$$\frac{\partial \psi}{\partial t_a} = -\frac{2p_i q S_{gi}}{G Z_i} \quad \text{(B-14)}$$

recognizing that:

$$G = \frac{(1.0e - 3)\phi_i A h S_{gi} p_i T_{st}}{Z_i p_{st} T} \quad \text{(B-15)}$$

Equation (B-14) demonstrates the fundamental relationship between pseudo-pressure and pseudo-time. This also establishes a linear relationship between the pseudo-variables for a constant rate of production. We will use Equation (B-14) later to determine the relationship between the real time and the pseudo-time while honouring the material balance equation. Now, consider the following chain rule:

$$\frac{\partial t_a}{\partial t} = \left(\frac{\partial \psi}{\partial t}\right)\bigg/\left(\frac{\partial \psi}{\partial t_a}\right) \quad \text{(B-16)}$$

with Equations (B-12) and (B-14), to find:

$$\frac{\partial t_a}{\partial t} = \frac{1}{\mu\left\{c_e + c_g\left[1 - c_e\left(p_i - p\right)\right]\right\}S_{gi}} \quad \text{(B-17)}$$

Equation (B-17) can be simplified to:

$$\frac{\partial t_a}{\partial t} = \frac{1}{\mu c_t} \quad \text{(B-18)}$$

if we substitute:

$$c_t = c_{ti}\left[1 - c_g\left(p_i - p\right)\right] + S_{gi}\left[c_g - c_{gi} + c_g c_{gi}\left(p_i - p\right)\right] \quad \text{(B-19)}$$

and:

$$c_{ti} = c_f + c_o S_{oi} + c_w S_{wi} + c_{gi} S_{gi} \quad \text{(B-20)}$$

Integrating Equation (B-18) for pseudo-time, it follows:

$$t_a(t) = \int_0^t \frac{dt}{\mu\, c_t}$$

.... (2)

Thus, Equation (2) constitutes the definition of pseudo-time. Also, c_t is defined rigorously in Equation (B-19) with Equation (B-20), honouring the material balance equation. These two equations are referred to as Equations (4) and (5), respectively, in the main body of this paper. In other words, the pseudo-time can be calculated by numerical integration with Equation (2). It is obvious that c_t is a function of the average reservoir pressure, p. Note that the compressibilities of formation, oil, and water are considered to be constant in the material balance Equation (B-1).

Appendix C: Computation of Pseudo-Time From Pseudo-Pressure Data

Besides evaluating the numerical integration of Equation (2), pseudo-time can be calculated another way from pseudo-pressures by manipulating Equation (B-14) in Appendix B. Separating the variables in Equation (B-14) and integrating the resulting equation with the appropriate limits, one finds:

$$t_a(t) = -\frac{G\, Z_i}{2\, p_i S_{gi}} \int_{\psi_i}^{\psi} \frac{d\psi}{q} = -\frac{A\, h\, \phi_i T_{st}}{2\, p_{st} T} \int_{\psi_i}^{\psi} \frac{d\psi}{q}$$

.... (C-1)

Thus, Equation (C-1) forms the basis of computing pseudo-time, t_a, from pseudo-pressure. As an example, for a three-rate drawdown situation with rates q_1, q_2, and q_3, Equation (C-1) takes the form as:

$$t_a(t) = \frac{G\, Z_i}{2\, p_i\, S_{gi}} \left[\frac{(\psi_i - \psi_1)}{q_1} + \frac{(\psi_1 - \psi_2)}{q_2} + \frac{(\psi_2 - \psi_3)}{q_3} \right] =$$

$$\frac{A\, h\, \phi_i T_{st}}{2\, p_{st} T} \left[\frac{(\psi_i - \psi_1)}{q_1} + \frac{(\psi_1 - \psi_2)}{q_2} + \frac{(\psi_2 - \psi_3)}{q_3} \right]$$

.... (C-2)

where ψ_i, ψ_1, and ψ_2 are the pseudo-pressure values, corresponding to the average reservoir pressures at the beginning of rates q_1, q_2, and q_3, and ψ_3 can be considered as the current pseudo-pressure at the current rate q_3 at time t for which t_a is being calculated. This formulation is very useful because it allows one to calculate the pseudo-time from average reservoir pseudo-pressure, especially when it becomes difficult to calculate the same with a reasonable degree of accuracy by the numerical integration of Equation (2). In other words, the procedure with numerical integration to calculate pseudo-time can be avoided under certain conditions. The above example of a three-rate drawdown situation can be extended to any number of rate variations in drawdown or injection.

Appendix D: Relationship Between Pseudo-Time and Material Balance Pseudo-Time

For liquids, the calculation of material balance time is straightforward as:

$$t_c(t) = \frac{Q(t)}{q(t)}$$

.... (D-1)

However, the above formula does not work for gas reservoirs due to the variation of gas properties with pressure. The

normalized material balance pseudo-time for gas can be defined as follows:

$$t'_{ca}(t) = \frac{G_{pa}(t)}{q(t)}$$

.... (D-2)

where G_{pa} is the pseudo-cumulative gas production at a given time (this is different from the cumulative gas production), which can be calculated by using the following relationship:

$$G_{pa}(t) = \mu_i c_{ti} \int_0^{t_a(t)} q(t_a)\, dt_a$$

.... (D-3)

From Equations (D-2) and (D-3), it is obvious that the following general relationship exists between normalized material balance pseudo-time and pseudo-time as:

$$t'_{ca}(t) = \frac{\mu_i c_{ti} \displaystyle\int_0^{t_a(t)} q(t_a)\, dt_a}{q(t)}$$

....(11)

For a constant-rate production or injection case, the relationship between normalized material balance pseudo-time and pseudo-time turns out to be:

$$t'_{ca}(t) = \mu_i c_{ti} t_a(t)$$

.... (D-4)

Following the general relationships of Equations (B-14) and (11), one can find the expression for the normalized material balance pseudo-time as:

$$t'_{ca}(t) = \frac{GZ_i \mu_i c_{ti} \left[\psi_i - \psi(t) \right]}{2 p_i S_{gi} q(t)}$$

.... (D-5)

Note that Equation (D-5) can be derived independently for gas reservoirs with residual fluid saturation in a compressible formation by extending the approach of Palacio and Blasingame[7]. This confirms the validity of the relationship between pseudo-time and normalized material balance pseudo-time as presented above in Equation (11).

There exists a relationship between the normalized material balance pseudo-time and the material balance pseudo-time as:

$$t'_{ca}(t) = \mu_i c_{ti} t_{ca}(t)$$

.... (D-6)

Using Equations (D-6) and (11), one can derive the relationship between pseudo-time and material balance pseudo-time for real gas as:

$$t_{ca}(t) = \frac{\displaystyle\int_0^{t_a(t)} q(t_a)\, dt_a}{q(t)}$$

.... (D-7)

For a constant-rate drawdown or injection case, Equation (D-7) becomes:

$$t_{ca}(t) = t_a(t)$$

.... (D-8)

Thus, for a constant-rate drawdown or injection case, the pseudo-time and the material balance pseudo-time at a given time are identical.

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