**Determinant $|A|$**

*To evaluate the determinant $|A|$, select any row (or column) of $|A|$ and then multiply each element in the row (or column) by its own cofactor and then perform a summation of these products. This is called expansion by cofactors.*

For example, by selecting the first row of $|A|$, the determinant is obtained from the expansion

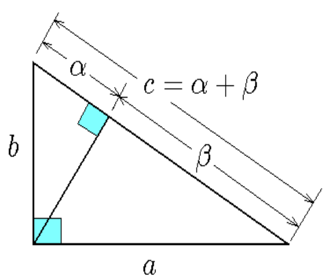
$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n} = \sum_{k=1}^{n} a_{1k}C_{1k} \tag{2.47}$$

In the special case of a $3\times 3$ determinant the expansion by cofactors along the first row gives

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{2.48}$$

For determinants of order $n > 4$ the method of cofactor expansion for the evaluation of a determinant is not recommended because it is too time consuming. Evaluation of cofactors requires the evaluation of approximately $n!$ arithmetic operations. Use instead elementary row operations to reduce the matrix of the determinant to a diagonal form. The determinant is then the product of the diagonal elements.

## Pythagorean Theorem




**Theorem** Let $a$ and $b$ denote the legs of a right triangle and let $c$ denote the hypotenuse, then $a^2 + b^2 = c^2$.

**Proof:** For the right triangles illustrated, show using similar triangles $\frac{b}{c} = \frac{\alpha}{b}$ and $\frac{a}{c} = \frac{\beta}{a}$ or $b^2 = \alpha c$ and $a^2 = \beta c$ and consequently by addition $b^2 + a^2 = c(\alpha + \beta) = c \cdot c = c^2$

Certain special right triangles having integer values for their sides have been found in Egyptian and Babylonian documents. However, there is no written record of any proofs of the Pythagorean theorem from these cultures. Pythagoras is thought to be the first European to have proved this result, giving us the theorem bearing his name. However, the Chinese knew a geometric proof of this theorem many hundreds of years before Pythagoras.

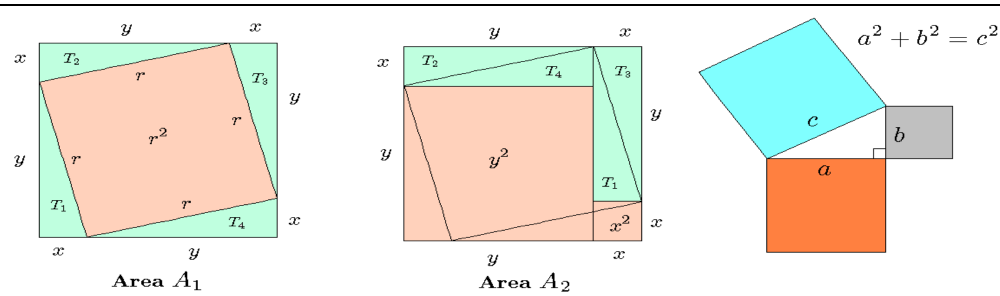


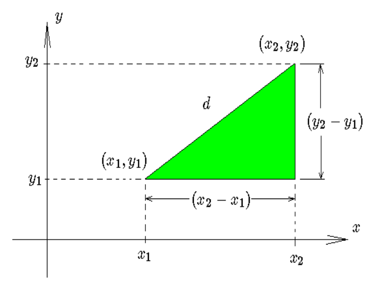

**Figure 2-3** Chinese proof of the Pythagorean theorem

The Chinese proof is illustrated in the figure 2-3 where a square of side $r$ is constructed inside a larger square with sides of length $x+y$, and there are four congruent right triangles, with legs $x$ and $y$, labeled $T_1, T_2, T_3, T_4$, also within the larger square. Now move the triangles $T_1$ and $T_4$ to the positions indicated in the middle sketch in figure 2-3 and note that now $x^2+y^2=r^2$. Equating area $A_1$ to area $A_2$, $A_2=(x+y)^2=x^2+y^2+4\left(\frac{1}{2}xy\right)=r^2+4\left(\frac{1}{2}xy\right)=A_1$which implies $x^2+y^2=r^2$. This was the Chinese proof of the Pythagorean theorem. It was proved by Pythagoras using the ratio of areas of similar triangles. It is estimated that there are over 400 different proofs of the Pythagorean theorem. The right-hand sketch in figure 2-3 is a geometric representation of the Pythagorean theorem.

There are special right triangles having integer values for the sides. The integer values are known as Pythagorean triples. The Euclid formula for Pythagorean triples associated with a right triangle with sides $x$ and $y$ and hypotenuse $r$ is written

$$x=m^2-n^2, \quad y=2mn, \quad r=m^2+n^2 \quad \text{which satisfies} \quad x^2+y^2=r^2 \tag{2.49}$$

where $m$ and $n$ are integers such that $x$ is positive. For example, with $m=2$ and $n=1$ there results the Pythagorean triple $(x,y,r)=(3,4,5)$. Other selected Pythagorean triples are $(5,12,13)$, $(7,24,25)$, $(9,40,41)$, $(8,15,17)$, and $(11,60,61)$.

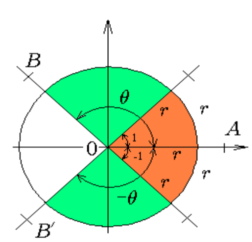



An application of the Pythagorean theorem is used to calculate the distance between two points in the plane. With reference to the accompanying figure, if $(x_1, y_1)$ and $(x_2, y_2)$ are the coordinates of two points in the plane, then by connecting these two points with a straight line and drawing horizontal lines $y=y_1$ and $y=y_2$ together with the vertical lines $x=x_1$ and $x=x_2$ one can construct the right triangle illustrated having sides with distances $(x_2-x_1)$ and $(y_2-y_1)$ and hypotenuse $d$. The Pythagorean theorem requires that $d^2=(x_2-x_1)^2+(y_2-y_1)^2$, from which there results the formula for distance between two points in the plane

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

## Trigonometry

The word trigonometry comes from the Greek language and means 'measure of triangles'. Plane trigonometry deals with triangles in a plane and spherical trigonometry deals with spherical triangles on a sphere.




In plane trigonometry if a line $\overline{0A}$ is drawn through the origin of a coordinate system and then rotated counterclockwise about the origin to the position $\overline{0B}$, then the positive angle $\angle A0B=\theta$ is said to have been generated. If $\overline{0B}'$ is the reflection of $\overline{0B}$ about the $x$-axis, then when the line $\overline{0A}$ is rotated in a clockwise direction to the position $\overline{0B}'$, a negative angle $-\theta$ is said to be generated. The situation is illustrated in the accompanying figure.

By definition if two angles add to $90°$, then they are called complementary angles and if two angles add to $180°$, then the angles are called supplementary. Angles are sometimes measured in degrees (°), minutes (′), and seconds (″) where there are 360° in a circle, 60 minutes in one degree and 60 seconds in one minute. Another unit of measurement for the angle is the radian. One radian is the angle subtended by an arc on a circle which has the same length as the radius of the circle as illustrated in the above figure. The circumference of a circle is given by the formula $C = 2\pi r$, where $r$ is the radius of the circle. In the special case $r = 1$,

$$\begin{array}{llll} (a) & 2\pi\,\text{radians} = 360\,^\circ & (c) & 1\,\text{radian} = \dfrac{180}{\pi}^\circ \\ (b) & \pi\,\text{radians} = 180\,^\circ & (d) & \dfrac{\pi}{180}\,\text{radians} = 1\,^\circ \end{array} \tag{2.50}$$

Thus, to convert $30\,^\circ$ to radians, just multiply both sides of equation (2.50) (d) by 30 to get the result that $30\,^\circ = \pi/6\,\text{radians}$.

An angle is classified as acute if it is less than $90\,^\circ$ and called obtuse if it lies between $90\,^\circ$ and $180\,^\circ$. An angle $\theta$ is called a right angle when it has the value of $90\,^\circ$ or $\frac{\pi}{2}$ radians.

In scientific computing one always uses the radian measure in the calculations. Also note that most hand-held calculators have a switch for converting from one angular measure to another and so owners of such calculators must learn to set them appropriately before doing any calculations.

## Trigonometric Functions

The first recorded trigonometric tables comes from the Greeks around 150 BCE. However, it is inferred that trigonometry was used in sailing, astronomy and construction going back thousands of years before this time, but only limited records are available.

The ratio of sides of a right triangle are used to define the six trigonometric functions associated with one of the acute angles of the right triangle. These definitions can then be extended to apply to positive and negative angles associated with a point moving on a unit circle.

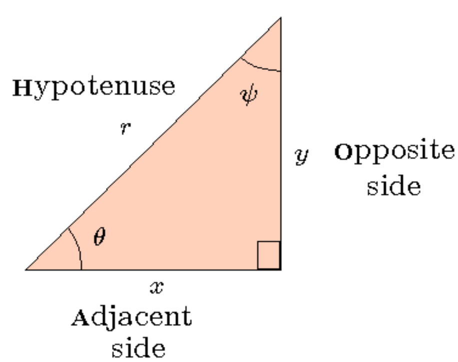
The six trigonometric functions associated with a right triangle are

| | | |
|---|---|---|
| sine | tangent | secant |
| cosine | cotangent | cosecant |

which are abbreviated respectively as

sin, tan, sec, cos, cot , and csc

and are defined



$$\sin\theta = \frac{y}{r} = \frac{\textbf{O}\text{pposite side}}{\textbf{H}\text{ypotenuse}} \qquad \cos\theta = \frac{x}{r} = \frac{\textbf{A}\text{djacent side}}{\textbf{H}\text{ypotenuse}} \qquad \tan\theta = \frac{y}{x} = \frac{\textbf{O}\text{pposite side}}{\textbf{A}\text{djacent side}}$$
$$\csc\theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite side}} \qquad \sec\theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{adjacent side}} \qquad \cot\theta = \frac{x}{y} = \frac{\text{adjacent side}}{\text{opposite side}} \tag{2.51}$$

A mnemonic devise to remember the sine, cosine and tangent definitions is the expression "**S**ome **O**ld **H**orse **C**ame **A** **H**opping **T**owards **O**ur **Alley** "

Observe that the following functions are reciprocals

$$\sin\theta \quad \text{and} \quad \csc\theta, \qquad \cos\theta \quad \text{and} \quad \sec\theta, \qquad \tan\theta \quad \text{and} \quad \cot\theta \tag{2.52}$$

and satisfy the relations $\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}$ (2.53)

In addition to the above 6 trigonometric functions, the functions versine, coversine and haversine are sometimes used when dealing with trigonometric relations. These functions do not have standardize abbreviations and can be found under different names in different books. These functions are defined

$$\begin{aligned}
\text{versed sine } \theta &= \text{versine}\,\theta = \text{vers}\,\theta = \text{ver}\,\theta = 1 - \cos\theta \\
\text{versed cosine } \theta &= \text{coversed sine}\,\theta = \text{coversine}\,\theta = \text{covers}\,\theta = \text{cov}\,\theta = 1 - \sin\theta \\
\text{haversine } \theta &= \text{hav}\,\theta = \frac{1}{2}\text{ver}\,\theta = \frac{1}{2}(1 - \cos\theta)
\end{aligned} \tag{2.54}$$

The figure 2-4 gives a graphical representation of the above trigonometric functions in terms of distances associated with the unit circle.

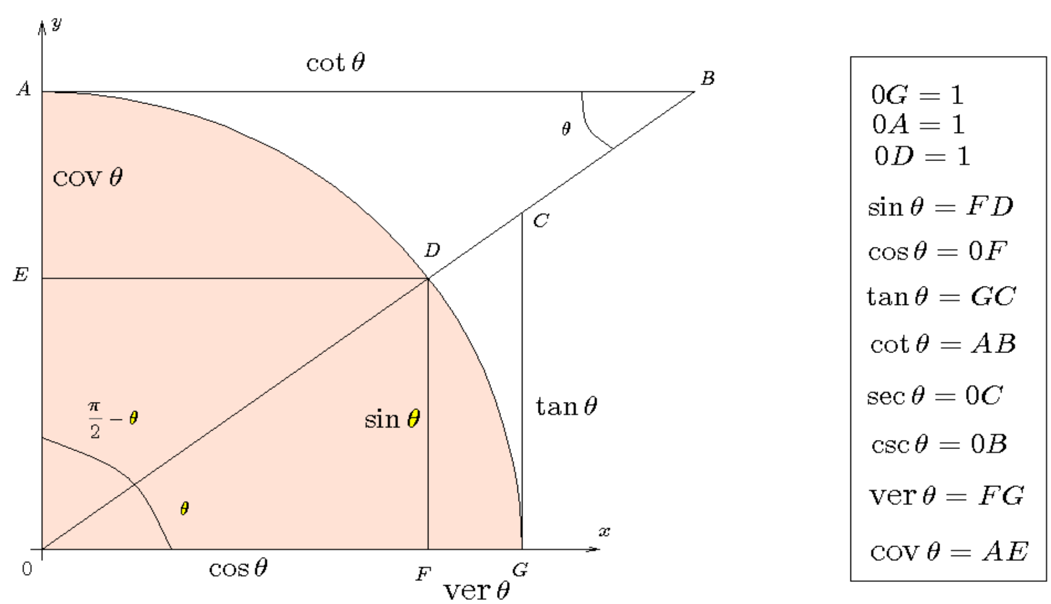
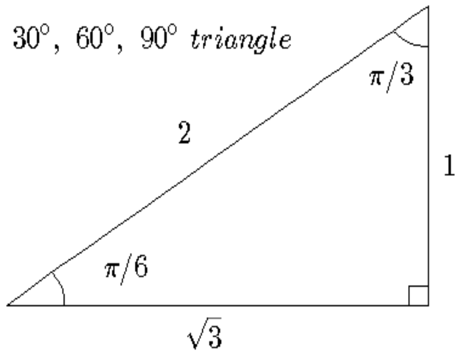
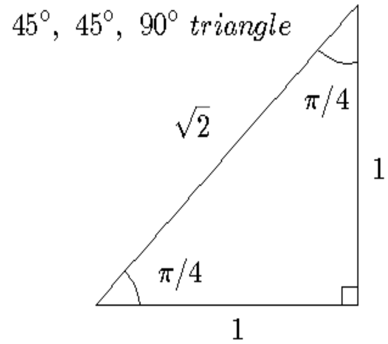



**Figure 2-4.** Trigonometric functions in terms of distances associated with the unit circle.

Consider the special right triangles illustrated

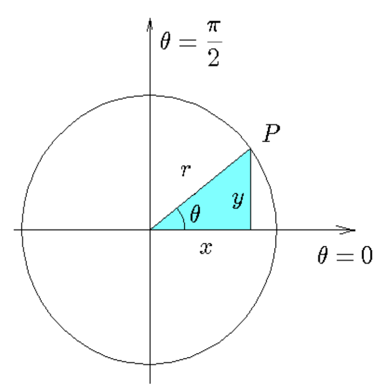



Note that in the 45 degree triangle, the sides opposite the 45 degree angles are equal and so any convenient length can be used to represent the equal sides. If the number 1 is used, then the hypotenuse has the value $\sqrt{2}$. Similarly, in the 30-60-90 degree right triangle, the side opposite the 30 degree angle is always one-half the hypotenuse and so by selecting the value of 2 for the hypotenuse one obtains the sides $2, 1, \sqrt{3}$ as illustrated. By using the trigonometric definitions given by the equations (2.51) together with the special 30-60-90, and 45 degree right triangles, the following table of values result.

| Angle $\theta$ in degrees | Angle $\theta$ in radians | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\cot\theta$ | $\sec\theta$ | $\csc\theta$ |
|---|---|---|---|---|---|---|---|
| $0^\circ$ | 0 | 0 | 1 | 0 | undefined | 1 | undefined |
| $30^\circ$ | $\pi/6$ | $1/2$ | $\sqrt{3}/2$ | $\sqrt{3}/3$ | $\sqrt{3}$ | $2\sqrt{3}/3$ | 2 |
| $45^\circ$ | $\pi/4$ | $\sqrt{2}/2$ | $\sqrt{2}/2$ | 1 | 1 | $\sqrt{2}$ | $\sqrt{2}$ |
| $60^\circ$ | $\pi/3$ | $\sqrt{3}/2$ | $1/2$ | $\sqrt{3}$ | $\sqrt{3}/3$ | 2 | $2\sqrt{3}/3$ |
| $90^\circ$ | $\pi/2$ | 1 | 0 | undefined | 0 | undefined | 1 |

The values for $\theta = 0$ and $\theta = 90$ degrees are special cases which can be examined separately by imagining the triangle changing as the point $P$ moves around a circle having radius $r$. The 45 degree triangle being a special case when $x = 1$, $y = 1$ and $r = \sqrt{2}$. The 30-60-90 degree triangle is a special case when there is a circle with $x = 1$, $r = 2$ and $y = \sqrt{3}$. The trigonometric functions associated with the limiting values of $\theta = 0$ and $\theta = 90$ degrees can be obtained from these special circles by examining the values of $x$ and $y$ associated with the point $P$ for the limiting conditions $\theta = 0$ and $\theta = \pi/2$.



## Cofunctions

In the definitions of the trigonometric functions certain combinations of these functions are known as cofunctions. For example, the sine and cosine functions are known as cofunctions. Thus the cofunction of the sine function is the cosine function and the cofunction of the cosine function is the sine function. Similarly, the functions tangent and cotangent are cofunctions and the functions secant and cosecant are cofunctions.

**Theorem** *Each trigonometric function of an acute angle $\theta$ is equal to the cofunction of the complementary angle $\psi = \frac{\pi}{2} - \theta$.*

The above theorem follows directly from the definitions of the trigonometric functions giving the results

$$\begin{array}{lll} \sin\theta = \cos(\frac{\pi}{2} - \theta) & \tan\theta = \cot(\frac{\pi}{2} - \theta) & \sec\theta = \csc(\frac{\pi}{2} - \theta) \\ \cos\theta = \sin(\frac{\pi}{2} - \theta) & \cot\theta = \tan(\frac{\pi}{2} - \theta) & \csc\theta = \sec(\frac{\pi}{2} - \theta) \end{array} \tag{2.65}$$

The above results are known as the cofunction formulas.

## Trigonometric Functions Defined for Other Angles

Consider a circle drawn with respect to some $xy$ coordinate system and let $P$ denote a point on the circle which moves around the circle in a counterclockwise direction. The figure 2-5 illustrates the point $P$ lying in various quadrants which are denoted by the Roman numerals I,II,III,IV. Figure 2-5(a) denotes the point $P$ lying in the first quadrant with figures 2-5(b)(c) and (d) illustrating the point $P$ lying in the second, third and fourth quadrant respectively.

Let $\theta$ denote the angle swept out as $P$ moves counterclockwise about the circle and define the six trigonometric functions of $\theta$ as follows

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}, \quad \cot\theta = \frac{x}{y}, \quad \sec\theta = \frac{r}{x}, \quad \csc\theta = \frac{r}{y} \tag{2.66}$$

Here $y$ denotes the ordinate of the point $P$, $x$ denotes the abscissa of the point $P$ and $r$ denotes the polar distance of the point $P$ from the origin. These distances are illustrated in the figures 2-5 (a)(b)(c) and (d). Note that these definitions imply that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta} \tag{2.67}$$

Write the Pythagorean theorem, $x^2 + y^2 = r^2$ for the right triangle in each quadrant of figure 2-5. The Pythagorean theorem can be written in any of the alternative forms

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1, \qquad 1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2, \qquad \left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

which by the above trigonometric definitions become

$$\cos^2\theta + \sin^2\theta = 1, \qquad 1 + \tan^2\theta = \sec^2\theta, \qquad \cot^2\theta + 1 = \csc^2\theta \tag{2.68}$$

These are fundamental relations between the trigonometric functions and are known as trigonometric identities. The above identities are sometimes called the Pythagorean identities.

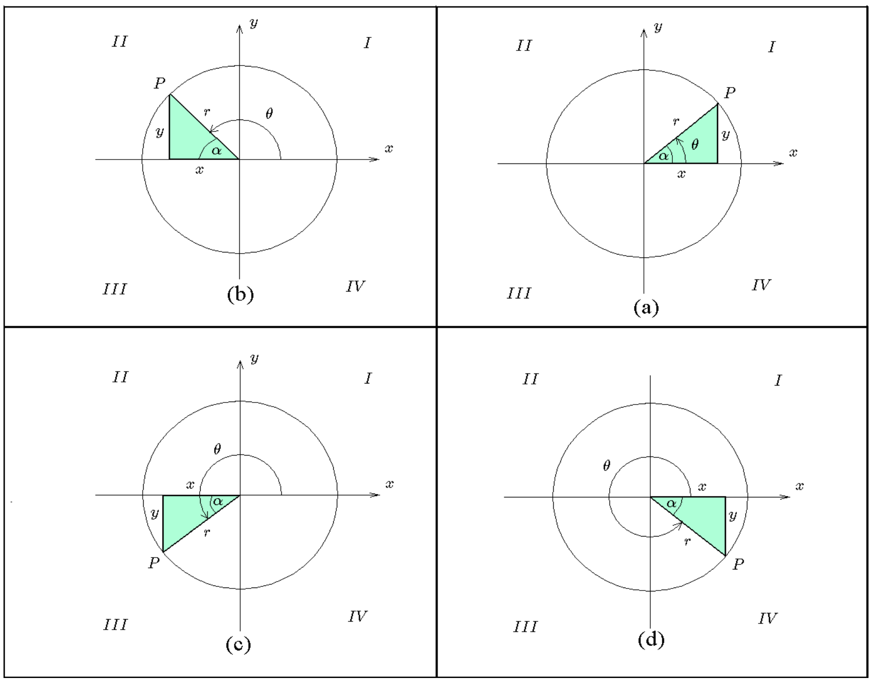



**Figure 2-5.** Point $P$ on circle of radius $r$ where $P$ is in quadrant I, quadrant II, quadrant III and quadrant IV.

## Sign Variation of the Trigonometric Functions

In the figure 2-5 make note of the following.

(i) The six trigonometric functions of $\theta$, as the point $P$ moves about the circle, are sometimes referred to as circular functions.

(ii) The ray $\overline{0P}$ defines not only the angle $\theta$ but also the angles $\theta \pm 2m\pi$ where $m$ is a positive integer or zero.

(iii) As $P$ moves around the circle, the radial distance $r = \overline{0}P$ always remains constant, but the $x$ and $y$ values change sign as $P$ moves through the different quadrants. An analysis of these sign changes produces the table of signs given in the figure 2-6.

(iv) The six trigonometric functions take on special values whenever $x = 0$ or $y = 0$. One of these special values will occur whenever $\theta$ is equal to some multiple of $\frac{\pi}{2}$.

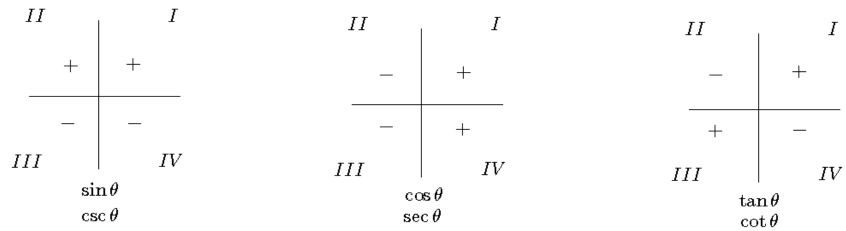
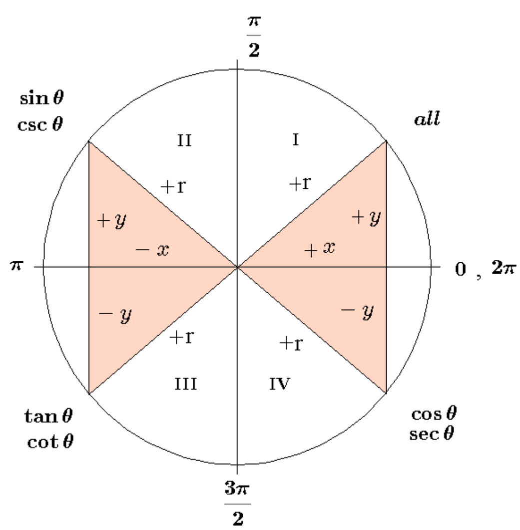


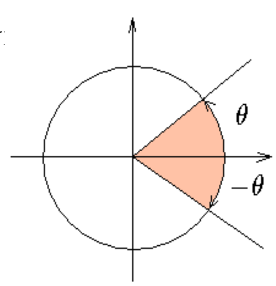

**Figure 2-6** Sign variation for the trigonometric functions by quadrant.

(v) The angle $\alpha$ in the figures 2-5 is the smallest nonnegative angle between the line $\overline{0P}$ and the $x$-axis. Limiting values for $\alpha$ are $\frac{\pi}{2}$ and $0$ radians. The angle $\alpha$ is called a reference angle. If the reference angle is different from 0 or $\frac{\pi}{2}$, then it can be viewed as an acute positive angle in the first quadrant. Note that there is then a definite relationship between the six trigonometric functions of $\theta$ and the six trigonometric functions of the reference angle $\alpha$. The six trigonometric functions of the reference angle $\alpha$ are all positive and so one need only add the appropriate sign change to obtain the six trigonometric functions of $\theta$. The appropriate sign changes are given in the figure 2-6 and are used to construct the relations given in the table 2.1.

(vi) The figures 2-5 and 2-6 can be combined into one figure so that by using the definition of the trigonometric functions, the correct sign of a trigonometric function can be determined corresponding to $\theta$ in any quadrant. For example, in quadrant II the functions $\sin\theta$ and $\csc\theta$ are positive. In quadrant III the functions $\tan\theta$ and $\cot\theta$ are positive and in quadrant IV, the functions $\cos\theta$ and $\sec\theta$ are positive.




(v




If the angle $\theta$ is a positive acute angle, then $-\theta$ lies in the fourth quadrant. The reference angle is $\alpha = \theta$ is positive and gives the results

$$\begin{aligned} &\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = \tan\theta \\ &\csc(-\theta) = -\csc(\theta) \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta \end{aligned} \tag{2.69}$$

(viii) Recall that any function $f(\theta)$ satisfying $f(-\theta) = f(\theta)$ is called an even function of $\theta$ and functions satisfying $f(-\theta) = -f(\theta)$ are called odd functions of $\theta$. The above arguments show that the functions sine, tangent, cotangent and cosecant are odd functions of $\theta$ and the functions cosine and secant are even functions of $\theta$. These results are sometimes referred to as even-odd identities.

Special Values

| Angle $\theta$ degrees | Angle $\theta$ radians | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\cot\theta$ | $\sec\theta$ | $\csc\theta$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | *undefined* | 1 | *undefined* |
| 90 | $\pi/2$ | 1 | 0 | *undefined* | 0 | *undefined* | 1 |
| 180 | $\pi$ | 0 | $-1$ | 0 | *undefined* | $-1$ | *undefined* |
| 270 | $3\pi/2$ | $-1$ | 0 | *undefined* | 0 | *undefined* | $-1$ |
| 360 | $2\pi$ | 0 | 1 | 0 | *undefined* | 1 | *undefined* |

Quadrant II

| Angle $\theta$ degrees | Angle $\theta$ radians | Reference angle $\alpha$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\cot\theta$ | $\sec\theta$ | $\csc\theta$ |
|---|---|---|---|---|---|---|---|---|
| 120 | $2\pi/3$ | $\pi/3$ | $\sqrt{3}/2$ | $-1/2$ | $-\sqrt{3}$ | $-\sqrt{3}/3$ | -2 | $2\sqrt{3}/3$ |
| 135 | $3\pi/4$ | $\pi/4$ | $\sqrt{2}/2$ | $-\sqrt{2}/2$ | $-1$ | $-1$ | $-\sqrt{2}$ | $\sqrt{2}$ |
| 150 | $5\pi/6$ | $\pi/6$ | $1/2$ | $-\sqrt{3}/2$ | $-\sqrt{3}/3$ | $-\sqrt{3}$ | $-2\sqrt{3}/3$ | 2 |

Quadrant III

| Angle $\theta$ degrees | Angle $\theta$ radians | Reference angle $\alpha$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\cot\theta$ | $\sec\theta$ | $\csc\theta$ |
|---|---|---|---|---|---|---|---|---|
| 210 | $7\pi/6$ | $\pi/6$ | $-1/2$ | $-\sqrt{3}/2$ | $\sqrt{3}/3$ | $\sqrt{3}$ | $-2\sqrt{3}/3$ | $-2$ |
| 225 | $5\pi/4$ | $\pi/4$ | $-\sqrt{2}/2$ | $-\sqrt{2}/2$ | 1 | 1 | $-\sqrt{2}$ | $-\sqrt{2}$ |
| 240 | $4\pi/3$ | $\pi/3$ | $-\sqrt{3}/2$ | $-1/2$ | $\sqrt{3}$ | $\sqrt{3}/3$ | $-2$ | $-2\sqrt{3}/3$ |

Quadrant IV

| Angle $\theta$ degrees | Angle $\theta$ radians | Reference angle $\alpha$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\cot\theta$ | $\sec\theta$ | $\csc\theta$ |
|---|---|---|---|---|---|---|---|---|
| 300 | $5\pi/3$ | $\pi/3$ | $-\sqrt{3}/2$ | $1/2$ | $-\sqrt{3}$ | $-\sqrt{3}/3$ | 2 | $-2\sqrt{3}/3$ |
| 315 | $7\pi/4$ | $\pi/4$ | $-\sqrt{2}/2$ | $\sqrt{2}/2$ | $-1$ | $-1$ | $\sqrt{2}$ | $-\sqrt{2}$ |
| 330 | $11\pi/6$ | $\pi/6$ | $-1/2$ | $\sqrt{3}/2$ | $-\sqrt{3}/3$ | $-\sqrt{3}$ | $2\sqrt{3}/3$ | $-2$ |

**Table 2.1** Special values for the Trigonometric Functions