LAB 7
PID Control

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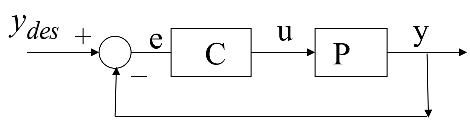
REVISITING P-CONTROL



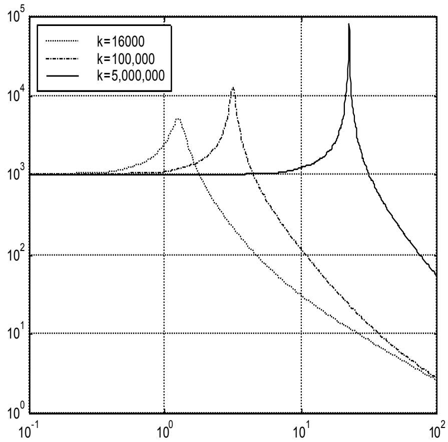
- Closed-loop transfer function is

$$\frac{Y(s)}{Y_{des}(s)} = \frac{\dfrac{K_p K_m}{T_m}}{s^2 + \dfrac{1}{T_m}s + \dfrac{K_p K_m}{T_m}}$$

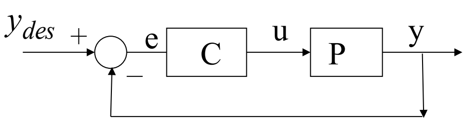
- For good damping $\xi \geq 1 \;\; \Rightarrow \;\; K_p \leq \dfrac{1}{4K_m T_m}$

REVISITING P-CONTROL

- For good bandwidth $\omega_n = \sqrt{\frac{K_p K_m}{T_m}} \Rightarrow K_p$ needs to be large



PD CONTROL



$$u = K_p e + K_d \dot{e}$$

$$P = \frac{K_m}{s(T_m s + 1)}$$

$$C = K_p + K_d s$$

$$\frac{Y(s)}{Y_{des}(s)} = \frac{PC}{1+PC} = \frac{K_p K_m + K_m K_d s}{T_m s^2 + s + K_m K_d s + K_p K_m}$$

PD CONTROL

- Draw Bode plot on blackboard

$$\frac{Y(s)}{Y_{des}(s)} = \frac{PC}{1+PC} = \frac{K_p K_m + K_m K_d s}{T_m s^2 + s + K_m K_d s + K_p K_m}$$

$$= \frac{\frac{K_p K_m}{T_m}\left(1 + \frac{K_d}{K_p} s\right)}{s^2 + \frac{1}{T_m}(1 + K_m K_d)s + \frac{K_p K_m}{T_m}}$$

PD CONTROL

- Independent control of ξ and ω_n .
  - Damping ratio $2\xi\omega_n = \frac{1}{T_m}(1 + K_m K_d)$
  - Bandwidth $\omega_n = \sqrt{\frac{K_m K_p}{T_m}}$
  - High bandwidth and good damping
- The derivative feedback (D-term) provides more damping, better stability and more speed of response.

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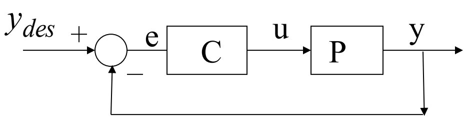
IMPLEMENTING PD CONTROL

```
{
error = ref_position - encoder;
error_derivative = (error - previous_error)/Ts;
previous_error = error;
control_voltage =       Kp*error
                      + Kd* error_derivative ;
}
```


PID CONTROL

- Proportional-integral-derivative feedback

$$u = K_p e + K_i \int e \, \mathrm{dt} + K_d \dot{e}$$

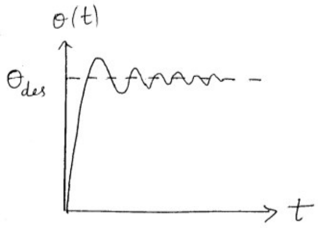


$$P = \frac{K_m}{s(T_m s + 1)}$$

$$C = K_p + K_i \frac{1}{s} + K_d s$$

MOTIVATION FOR INTEGRAL TERM

- In the case of P-control, the steady state error is zero, if the desired signal $y_{des}(t)$ is a step signal.



- However, the steady state error is not zero, if the desired signal $y_{des}(t)$ is a ramp signal, or if dry friction is present in the motor.

MOTIVATION FOR INTEGRAL TERM

- Consider the presence of dry friction in the motor

  Original plant model: $P = \dfrac{K_m}{s(T_m s + 1)} \Rightarrow T_m\ddot{\theta} + \dot{\theta} = K_m V$

  Modified plant model: $T_m\ddot{\theta} + \dot{\theta} + F_f = K_m V$

  Assuming dry friction is a constant opposing force and the sign of angular velocity remains constant, in the Laplace domain

$$\theta = \frac{K_m}{s(T_m s + 1)} V - \frac{1}{s(T_m s + 1)} \frac{F_o}{s}$$

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FINAL VALUE THEOREM

- The Final Value Theorem can be used to calculate steady state error
- Let $Y(s)$ be the Laplace transform of y(t)
- If y(t) has a steady state value, then the steady state value is given by $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$

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STEADY STATE ERROR: P-CONTROL

- Error is

$$\frac{E(s)}{Y_{des}(s)} = \frac{1}{1+PC} \quad \text{or} \quad \frac{E(s)}{Y_{des}(s)} = \frac{1}{1+\dfrac{K_m K_p}{s(T_m s+1)}}$$

- Desired value of y is a step signal

$$\left.\begin{aligned} Y_{des}(s) &= \frac{A}{s} \\ E(s) &= \frac{1}{1+\dfrac{K_m K_p}{s(T_m s+1)}}\frac{A}{s} \end{aligned}\right\} \quad \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{A}{1+\dfrac{K_m K_p}{s(T_m s+1)}} = 0$$

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MOTIVATION FOR INTEGRAL TERM

- Consider the presence of dry friction in the motor

$$\theta = \theta_{des} - E(s) = \frac{K_m}{s(T_m s+1)} V - \frac{1}{s(T_m s+1)} \frac{F_o}{s}$$

$$\theta_{des} - E(s) = \frac{K_m}{s(T_m s+1)} K_p E - \frac{1}{s(T_m s+1)} \frac{F_o}{s}$$

$$\Rightarrow \left[1 + \frac{K_m K_p}{s(T_m s+1)}\right] E = \theta_{des} + \frac{1}{s(T_m s+1)} \frac{F_o}{s}$$

$$\Rightarrow E = \frac{s(T_m s+1)}{T_m s^2 + s + K_m K_p} \frac{\theta_o}{s} + \frac{1}{T_m s^2 + s + K_m K_p} \frac{F_o}{s}$$

MOTIVATION FOR INTEGRAL TERM

- Steady State Error

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s(T_m s+1)\theta_o}{T_m s^2 + s + K_m K_p} + \frac{F_o}{T_m s^2 + s + K_m K_p} = \frac{F_o}{K_m K_p}$$

- Hence steady state error is not zero in the presence of dry friction

$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{F_o}{K_m K_p}$$

MOTIVATION FOR INTEGRAL TERM

- Dry friction in the motor, with PI control

$$\theta = \theta_{des} - E(s) = \frac{K_m}{s(T_m s+1)} V - \frac{1}{s(T_m s+1)} \frac{F_o}{s}$$

$$\theta_{des} - E(s) = \frac{K_m}{s(T_m s+1)} \left(K_p E + \frac{K_I}{s} E \right) - \frac{1}{s(T_m s+1)} \frac{F_o}{s}$$

$$\Rightarrow E = \frac{s^2 (T_m s+1)}{T_m s^3 + s^2 + K_m K_p s + K_m K_I} \frac{\theta_o}{s} + \frac{s}{T_m s^3 + s^2 + K_m K_p s + K_m K_I} \frac{F_o}{s}$$

MOTIVATION FOR INTEGRAL TERM

- Steady State Error, with PI control

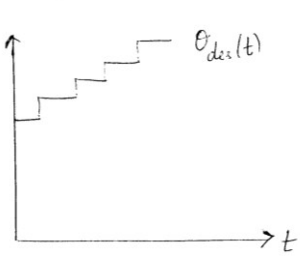
$$e_{ss} = \lim_{s \to 0} sE(s)$$

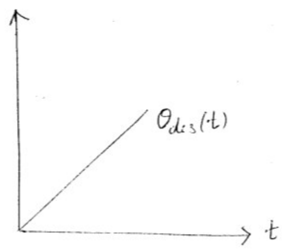
$$= \lim_{s \to 0} \frac{s^2 (T_m s+1)\theta_o}{T_m s^3 + s^2 + K_m K_p s + K_m K_I} + \frac{sF_o}{T_m s^3 + s^2 + K_m K_p s + K_m K_I} = 0$$

- Hence steady state error is zero with PI control in the presence of dry friction, even without knowledge of the magnitude of dry friction.

MOTIVATION FOR INTEGRAL TERM

- Consider a case where the desired value of signal keeps changing
- Alternatively, consider the case where the desired signal is a ramp signal





STEADY STATE ERROR: P-CONTROL

- Desired value of y is a ramp signal

$$Y_{des}(s) = \frac{A}{s^2}$$

$$E(s) = \frac{1}{1 + \dfrac{K_m K_p}{s(T_m s + 1)}} \frac{A}{s^2}$$

Steady state error

$$\lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{1}{1 + \dfrac{K_m K_p}{s(T_m s + 1)}} \frac{A}{s} = \frac{A}{K_p K_m} \neq 0$$

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STEADY STATE ERROR: PID CONTROL

- Desired value of y is a ramp signal

$$Y_{des}(s) = \frac{A}{s^2} \qquad \frac{E(s)}{Y_{des}(s)} = \frac{1}{1+PC}$$

$$E(s) = \frac{1}{1+PC} \; \frac{A}{s^2}$$

$$E(s) = \frac{1}{1+\frac{K_m}{s(T_m s+1)}\left(K_p + \frac{K_i}{s} + K_d s\right)} \; \frac{A}{s^2}$$

STEADY STATE ERROR: PID-CONTROL

Steady state error

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+\frac{K_m}{s(T_m s+1)}\left(K_p + \frac{K_i}{s} + K_d s\right)} \; \frac{A}{s^2}$$

or

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s^3(T_m s+1)}{s(T_m s+1) + K_m\left(K_p s + K_i + K_d s^2\right)} \; \frac{A}{s^2} = 0$$

Hence steady state error is zero without requiring large K_p

PID GAINS: ZIEGLER-NICHOLS RULES

$$C(s) = K_p\left(1 + T_d s + \frac{1}{T_i s}\right)$$

Controller	K_p	T_i	T_d
P	$0.5K_{cr}$	-	-
PI	$0.45K_{cr}$	$P_{cr}/1.2$	-
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

IMPLEMENTING PID CONTROL

```
{
error = ref_position - encoder;
error_derivative = (error - previous_error)/Ts;
previous_error = error;
error_integral += Ts*error;
control_voltage =       Kp*error
                      + Kd* error_derivative
                      + Ki*error_integral;
}
```

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TASKS IN LAB

- Task 1
  - P Control. Use Kp = 0.004
  - Reference displacement is a sine wave of magnitude 250 counts
  - Various frequencies 4 Hz, 8, 10, 12, 14, 16, 20 Hz
  - Find output/input ratio
  - Estimate the bandwidth of the closed-loop controller from the experimental data
  - Estimate bandwidth from theoretical Bode plots using Matlab

TASKS IN LAB

- Task 2
  - Implement PI control.
  - Plot experimental step response
- Task 3
  - Implement PID control.
  - Plot experimental step response
- Post-Lab
  - Compare step responses with P, PI and PID controllers
  - Comment on the steady state error value, oscillatory behavior and speed of response.

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