Tutorial on Geometric Calculus

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The book *Clifford Algebra to Geometric Calculus* is the first and still the most complete exposition of **Geometric Calculus (GC)**. But it is more of a reference book than a textbook, so can it be a difficult read for beginners. This tutorial is a guide for serious students who want to dig deeply into the subject. It presents helpful background and aims to clarify objectives, important results and methods in the book. It is supplemented by a hot-linked annotated bibliography of papers elaborating on various aspects of Geometric Algebra and Calculus.

Objectives of this Tutorial

Only a handful of people have mastered *Clifford Algebra to Geometric Calculus* [2] to the point of using it in their research. Among the first are Chris Doran, Steve Gull and Anthony Lasenby. I am proud to say that it helped them produce some truly innovative theoretical physics, most notably, an improvement of *General Relativity* called *Gauge Theory Gravity* [4, 5]. The book [2] has many ideas and results that remain to be exploited, but first it is necessary to master the core concepts and mathematical tools.

Readers of this tutorial are presumed to be familiar with the basics of **Geometric Algebra (GA)**, so we can concentrate on more ambitious objectives. The tutorial emphasizes the following fundamental concepts of Geometric Calculus, explaining their unique features and advantages:

- **Universal Geometric Algebra** – arbitrary dimension and signature
- **Linear** and **multilinear algebra** (tensors, determinants)
- **Vector manifolds** – for representing any manifold
- **Directed integrals** and **differential forms**
- **Vector derivative** and the **fundamental theorem** of calculus
- **Differentials** and **codifferentials** for mappings and fields
- Coordinate-free **differential geometry**
- **Lie groups** as **Spin groups**

Origins of the Geometric Calculus book

Though **Geometric Calculus (GC)** had important precursors [6], its systematic development as a unified language for mathematics and physics began in 1966 with publication of the book **SpaceTime Algebra (STA)** [1]. That book shows how STA provides compact, coordinate-free formulations for basic equations of physics that provide new insights into their geometric structure. Specifically,

- Maxwell's four electromagnetic equations are reduced to a single equation: $\nabla F = J$
- Dirac's electron equation is given a new manifestly geometrical form.
- Einstein's General Relativity is formulated with a spinor form of the *Principle of Local Relativity.*

However, to solve and apply these equations by standard mathematical methods at the time, it was necessary to translate them into standard formulations, which detracted from their compact and elegant structure. To take full advantage of the new formulations, new computational tools and methods or, at least, coordinate-free reformulations and adaptations of old methods were needed.

Creation of new and more powerful mathematical tools began immediately with extraction of the concepts of **vector derivative** and **directed integral** from [1] and further development in mathematical papers in {3, 4}. [Note: Papers available for direct online access are numbered in curly brackets {…} and commented on below before the references, while book references are numbered in square brackets […].]

This led immediately to reduction of the integral theorems of Gauss, Stokes, Green and Cauchy into a single formula, and, more remarkably, to generalization of Cauchy's integral theorem to arbitrary dimension. In this way, it unified real and complex variable theory. Moreover, it raised the questions about how GC relates to the Calculus of Differential forms, in particular, with respect to transformations (change of variables) in integrals. I was fortunate to have a capable student, Garret Sobczyk [7] to help me answer this.

When Sobczyk completed his thesis in 1971, I combined it with ideas of my own into a series of three papers submitted for publication in mathematics journals. He went off on his own, ending up as a postdoc in Poland. The papers were rejected by three different journals in three successive years, but each with the recommendation that they be published as a book. As I had a lot more to say on the subject, I was not averse to writing a book, though I thought it was premature. Clifford Truesdell directed me to Mario Bunge, then editor of an advanced book series published by D. Reidel Company, who accepted my book plan immediately. That was the easy part. Little did I know that it would take a decade to get the book in print.

Writing was the easy part. In those days before desktop publishing technical manuscripts were written by hand and then typed by a secretary. Arizona State University had only one technical typist for the departments of physics, mathematics and chemistry, so it took three years to get my manuscript typed. Anyway, the manuscript was finished by 1976 and shipped off to Reidel for publication, only to be rejected by the publisher some six months later. I had made the mistake of submitting directly to the publisher instead of going through the editor Bunge. Back to square one! I wasn't worried though, because I was confident of the book's quality.

In 1978 I was pleased to get a letter from the distinguished mathematician Gian-Carlo Rota requesting a copy of my book *SpaceTime Algebra* [1]. I looked him up shortly thereafter when I attended a Maximum Entropy Conference at MIT and asked him why he was interested. He gave me copies of several papers of his on Invariant Theory. I was astounded by how close it was to my treatment of GA identities in the GC book, so in just a few days I was able to work it into my treatment of determinants in Chap.1. That was the last change made to the manuscript. It opened up rich opportunities for integrating GA with Invariant Theory that are yet to be fully exploited.

Rota agreed to consider my manuscript for publication by Addison-Wesley, for which he was editor of the Encyclopedia of Mathematics series. He requested six copies of the manuscript to be sent to reviewers. What happened thereafter is too involved to recount in detail. The net result was many delays. After several years, Rota surprised me one Saturday morning with the most gratifying phone call of my life: First, he explained that I had been unable to reach him because he had been in the hospital for the better part of a year, but he had already strongly recommended that my book be published. Then, he spent the better part of an hour praising the book in great detail!

On the strength of Rota's recommendation, I was awarded a signed contract. But that hardly mattered! When I contacted Addison-Wesley expecting immediate publication, I was told that the manuscript had been sent out for further review, which took two years and turned out to be a fiasco, because the publisher did not have sufficient mathematical competence to evaluate the results. Shortly thereafter, all my queries about the status of my book were ignored. In desperation, after more than a year of that, I asked Rota to find out what was going on. He reported that Addison-Wesley had hired a new publisher for mathematics, and "He has something against you!" Then I realized that he was the guy who had rejected my book for Reidel years earlier. So I resubmitted my book to Reidel, where it was reviewed favorably by Asim Barut and published, all within six months. The delay due to the publisher had been eight years.

Now we turn to brief commentaries on each chapter of the book, with slides from my lecture summarizing important definitions and results.

Chapter 1: Geometric Algebra

This chapter presents basic definitions and derives a large number of useful algebraic identities involving inner and outer products. See Fig. 1 for a selective summary. Careful readers have noticed a circularity in the definitions, but that is easily rectified by choice of starting point. In fact, several alternative definitions are presented, each with its distinct advantages, but that is of little consequence to the whole system. And slightly different notations are used in the slides.

Universal Geometric Algebra

Real Vector Space: $\mathbb{V}^{r,s} = \{a, b, c, \ldots\}$ dimension $r+s = n \to \infty$

Geometric product: $a^2 = \pm |a|^2$ nondegenerate signature $\{r, s\}$

generates Real GA: $\mathbb{G}^{r,s} = \mathbb{G}(\mathbb{V}^{r,s}) = \{A, G, M \ldots\} = \{\text{Multivectors}\}$

Inner product: $a \cdot b \equiv \frac{1}{2}(ab + ba)$ Outer product: $a \wedge b \equiv \frac{1}{2}(ab - ba)$

$\Rightarrow$ $ab = a \cdot b + a \wedge b$ $\quad a \wedge A_k \equiv \frac{1}{2}\left(aA_k + (-1)^k A_k a\right)$

k-blade: $a_1 \wedge a_2 \wedge \ldots \wedge a_k = \langle a_1 a_2 \ldots a_k \rangle_k \equiv A_k$ $\Rightarrow$ k-vector

$$a \cdot (a_1 \wedge a_2 \wedge \ldots \wedge a_k) = \sum_{j=1}^{k} (-1)^{j+1} a \cdot a_j (a_1 \wedge \ldots \wedge \breve{a}_j \wedge \ldots \wedge a_k)$$

Graded algebra: $\mathbb{G}^{r,s} = \sum_{k=0}^{n} \mathbb{G}^{r,s}_k = \left\{ A = \sum_{k=0}^{n} \langle A \rangle_k \right\}$

Reverse: $(a_1 \wedge a_2 \wedge \ldots \wedge a_k)^{\sim} = a_k \wedge \ldots \wedge a_2 \wedge a_1$ $\quad \tilde{A} = \sum_{k=0}^{n} \langle \tilde{A} \rangle_k = \sum_{k=0}^{n} (-1)^{k(k-1)/2} \langle A \rangle_k$

Unit pseudoscalar: $\mathrm{I} = \langle \mathrm{I} \rangle_n$ $\quad \mathrm{I}\tilde{\mathrm{I}} = (-1)^s$ $\quad a \wedge \mathrm{I} = 0$ defines subalgebra!

Fig. 1 Dual: $A^* \equiv A\mathrm{I}$ $\quad$ Thm: $a \cdot A^* = a \cdot (A\mathrm{I}) = (a \wedge A)\mathrm{I}$

Readers need not go through all the calculations in Chap. 1; it is advisable just to sample the calculations and proofs to see how they work. The rest can be left for reference as needed. Some years after writing this chapter, when Grassmann's original work was translated into English, I learned that he had derived the main identities a hundred years before.

An important feature of *Universal Geometric Algebra* defined in the book is that it is generated by an infinite dimensional vector space. All finite dimensional vector spaces and their geometric algebras are then defined by choice of a pseudoscalar, as indicated in the next to last line of Fig. 1. The linear structure of such algebras is schematized in Fig. 2.

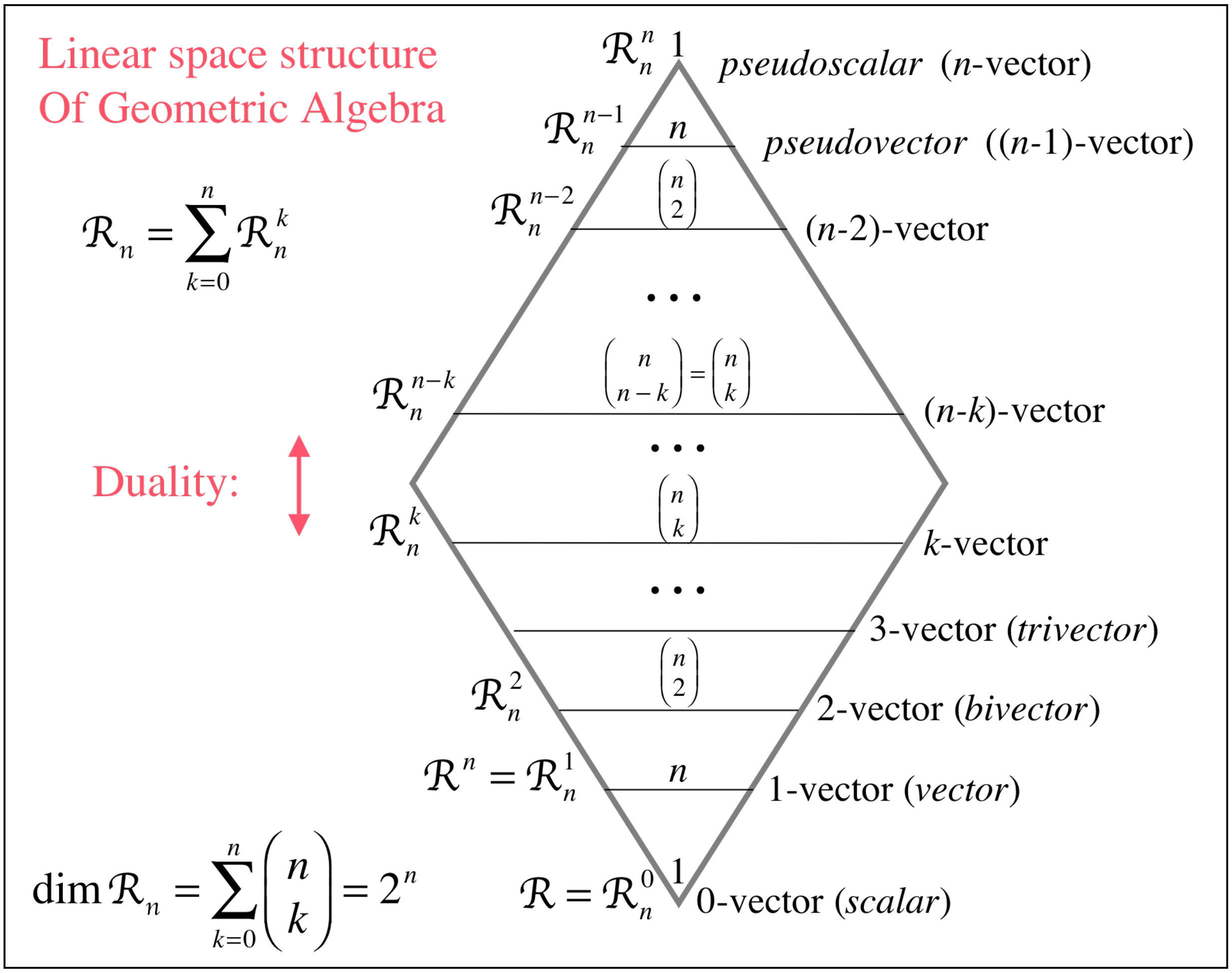


Fig. 2

The book starts out by assuming a Euclidean signature for the algebra, because that was done in the three original mathematical papers from which Chap. 1 is mostly composed. I wasn't sure that all the proofs would carry over to non-Euclidean signatures, and I didn't want to take the trouble to check, so I introduced non-Euclidean signatures only at the end of Chap. 1.

This brings up a common misconception about GA, namely, that use of an inner product limits its applicability to metric spaces. On the contrary, as shown in Fig. 3, the inner product can serve as a contraction without any notion of metric. Thus it can be seen that the infinite dimensional GA is equivalent to the algebra of fermion operators in quantum field theory. The formulation of finite dimensional GA with all possible signatures is schematized in Fig. 4.

Quadratic forms vs. contractions

Claim: *Linear forms* on a vector space can be represented by inner products in a geometric algebra *without assuming a metric*.

$\mathcal{V}^n$ a real vector space spanned by $\{w_i\}$

Dual space $\mathcal{V}^{*n}$ of linear forms spanned by $\{w_j^*\}$

And defined by $w_i^*(w_j) = \frac{1}{2}\delta_{ij}$ or $w_i^* \cdot w_j = \frac{1}{2}\delta_{ij}$

The associative outer product $w_i \wedge w_j = -w_j \wedge w_i$ generates the Grassmann algebra :

$$\Lambda_n = \Lambda_n^0 + \Lambda_n^1 + \ldots + \Lambda_n^n = \sum_{k=0}^{n} \Lambda_n^k \qquad \Lambda_n^0 = \mathcal{R}, \quad \Lambda_n^1 = \mathcal{V}^n$$

Likewise, the dual space generates the dual algebra $\Lambda_n^* = \sum_{k=0}^{n} \Lambda_n^{*k}$

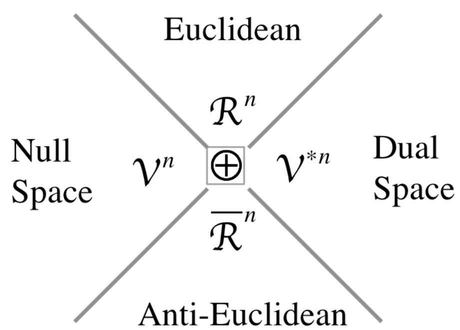
Assume the null metric $w_i \cdot w_j = 0 = w_i^* \cdot w_j^*$ so

$\mathcal{V}^n$ has geometric product $w_i w_j = w_i \wedge w_j = -w_j w_i$

Now define $w_i w_j^* + w_j^* w_i = \delta_{ij}$ $\qquad$ $w_i^2 = 0 = w_i^{*2}$

The algebra of fermion creation and annihilation operators!!

Fig. 3



Mother algebra

$\mathcal{R}_{n,n} = \mathcal{G}(\mathcal{R}^{n,n})$

Covers all!

$\mathcal{R}^{n,n} = \mathcal{V}^n \oplus \mathcal{V}^{*n} = \mathcal{R}^n \oplus \overline{\mathcal{R}}^n$ $\qquad$ $\mathcal{R}_{n,n} = \Lambda_n \otimes \Lambda_n^* = \mathcal{R}_n \otimes \overline{\mathcal{R}}_n$

Euclidean basis: $e_i = w_i + w_i^*$ spans $\mathcal{R}^n$

Anti-Euclidean basis: $\bar{e}_i = w_i - w_i^*$ spans $\mathcal{R}^n$

$$e_i \cdot e_j = \delta_{ij}, \quad e_i \cdot \bar{e}_j = 0, \quad \bar{e}_i \cdot \bar{e}_j = -\delta_{ij}$$

$\mathcal{R}_{p,q} \subset \mathcal{R}_{n,n}$ $(p+q=n)$ All signatures in subalgebras

Ideal arena for linear algebra! $\qquad \mathcal{R}_{\infty,\infty}$ $\qquad$ All dimensions!

Fig. 4

Not only does GA enable calculations without matrices, it facilitates calculations with matrices, as shown in Fig. 5.

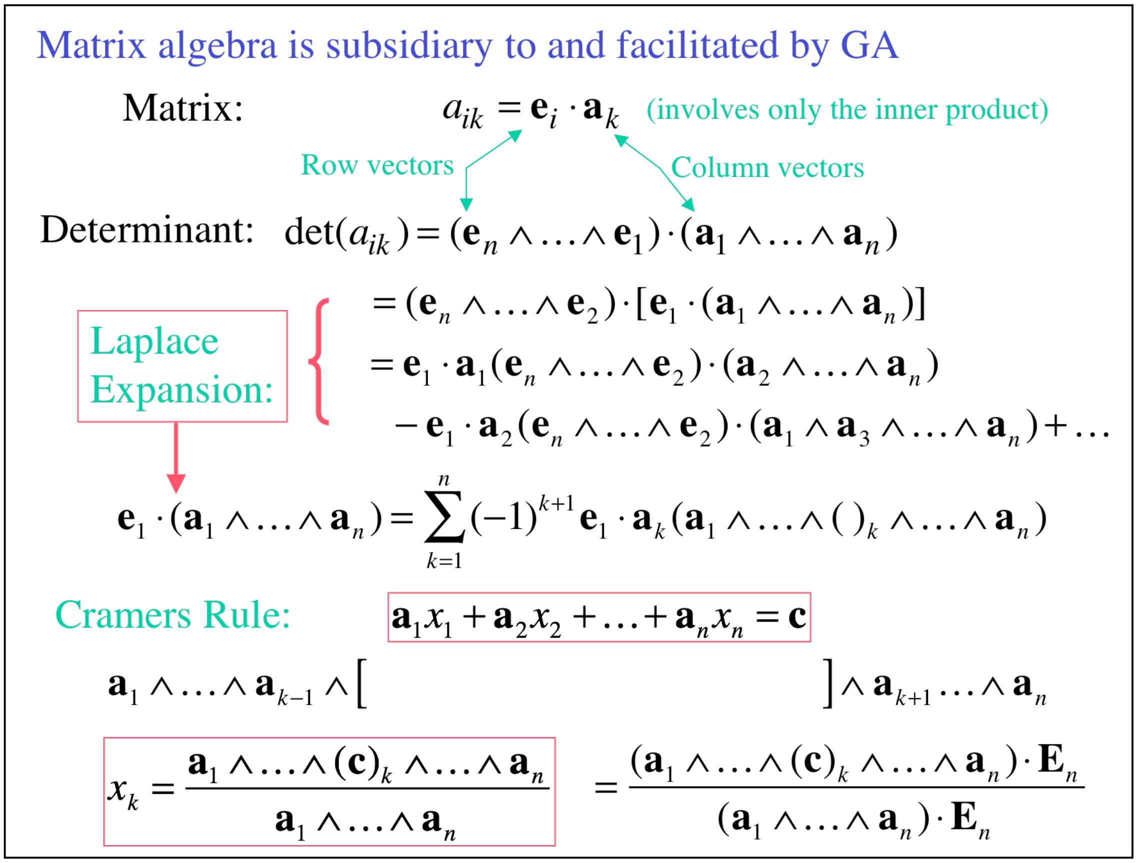


Fig. 5

There is very much more to the theory of GA identities than determinants. Subsequent research suggests that some, if not all, theorems of classical geometry can be formulated as GA identities, so their proofs can be reduced to proofs of algebraic identities. Hongbo Li has pushed this subject a long way [8,9], with intriguing results and great promise for more. I regard this as an extension of classical Invariant Theory.

Chapter 2: Differentiation

The vector derivative is the central object in geometric calculus. It can be defined in terms of the conventional partial derivative, as shown in Fig. 6. As in standard scalar calculus, one does not refer to the definition of derivative for calculations. Rather, one has in hand a small catalog of elementary derivatives and formulas, from which more complex derivatives can be calculated. Such a catalog for the vector derivative is given in Fig. 6. To make contact with standard vector calculus, a term in one formula is expressed in terms of the vector cross product, so it applies only in the case $n = 3$. With an easy extension to include delta functions, this catalog suffices for calculating all derivatives in classical electrodynamics.

Chap. 2 generalizes the vector derivative to define the derivative with respect to any multivector variable and derives many formulas for applications. At the time, I thought this concept was too esoteric to be of much interest, so I avoided it in subsequent work. Much to my surprise a few years later, the Cambridge group used it for elegant derivations of conservation laws in Lagrangian field theory. More recently, it has been applied to robotics by Lasenby and Doran as well as Valkenburg and Alwesh in [9].

Chap. 2 also introduces a concept of "simplcial derivative," but my current opinion is that its applications are better done by other means.

Vector derivatives in $\mathcal{R}^n$

Rectangular coordinates: $x^k = x^k(\mathbf{x}) = \boldsymbol{\sigma}_k \cdot \mathbf{x} \qquad \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k = \delta_{jk}$

Position vector: $\mathbf{x} = x^k \boldsymbol{\sigma}_k = x^1\boldsymbol{\sigma}_1 + x^2\boldsymbol{\sigma}_2 + \ldots + x^n\boldsymbol{\sigma}_n$

Vector derivative: $\nabla = \partial_{\mathbf{x}} = \boldsymbol{\sigma}_k \partial_k \qquad \partial_k = \dfrac{\partial}{\partial x^k} = \boldsymbol{\sigma}_k \cdot \nabla$

Basic derivatives for routine calculations:

$$\partial_k \mathbf{x} = \boldsymbol{\sigma}_k \qquad \nabla \mathbf{x} = \boldsymbol{\sigma}_k \partial_k \mathbf{x} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2\boldsymbol{\sigma}_2 + \ldots + \boldsymbol{\sigma}_n\boldsymbol{\sigma}_n = n$$

$$\mathbf{r} = \mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{x}', \qquad r = |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|$$

$$\Rightarrow \quad \nabla \mathbf{r} = n \quad \Rightarrow \quad \nabla \cdot \mathbf{r} = n \qquad \nabla \wedge \mathbf{r} = 0$$

$$\nabla r = \hat{\mathbf{r}} \qquad \nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a} \cdot \nabla \mathbf{r} = \mathbf{a} \quad \text{for constant } \mathbf{a}$$

$$\nabla \hat{\mathbf{r}} = \tfrac{2}{r} = \nabla r^2 \qquad \nabla \times (\mathbf{a} \times \mathbf{r}) = \nabla \cdot (\mathbf{a} \wedge \mathbf{r}) = (n-1)\mathbf{a}$$

$$\nabla r^k = k r^{k-2}\mathbf{r} \qquad \nabla \tfrac{\mathbf{r}}{r^k} = \tfrac{n-k-1}{r^k}$$

Chain rule

Fig. 6

Chapter 3: Linear and Multilinear Functions

Geometric Algebra introduces the powerful new concept of "outermorphism" to the field of linear algebra. The *outermorphism* of a linear function defined on a vector space is a unique linear extension to the entire geometric algebra of the vector space (as defined in Fig. 2). In Chap. 3 the simplicial derivative is used to define it, but I now prefer the alternative approach used in {6}, which also extends the treatment of linear algebra. This material deserves to be extended to a complete book on linear algebra. The typical reader is advised to peruse the chapter to get a sense of the approach, and then to refer back to it for details as needed.

Chapter 4: Calculus on Vector Manifolds

The preceding definition of vector derivative presumes the vector variable is defined on a vector space. This chapter removes that restriction by defining the concept of *vector manifold*, and that enables us in subsequent chapters to create a completely coordinate-free approach to differential geometry. The concept of vector manifold is schematized in Fig. 7, and its relation to the use of coordinates is outlined in Fig. 8.

What is a manifold? $\mathcal{M}^m$ of dimension m

— a set on which differential and integral calculus is well-defined!

- Calculus done indirectly by local mapping to $\mathcal{R}^m = \mathcal{R} \otimes \mathcal{R} \cdots \otimes \mathcal{R}$
- Proofs required to establish results independent of coordinates.

Geometric Calculus defines a manifold as any set isomorphic to a vector manifold

Vector manifold $\mathcal{M}^m = \{x\}$ is a set of vectors in GA that generates at each point x a tangent space with pseudoscalar $I_m(x)$

Advantages:

- Manifestly coordinate-free vs. nominally coordinate-free!
- Calculus done directly with algebraic operations on points
- Geometry completely determined by derivatives of $I_m(x)$.

Remark: It is unnecessary to assume that $\mathcal{M}^m$ is embedded in a vector space, though embedding theorems can be proved.

Fig. 7

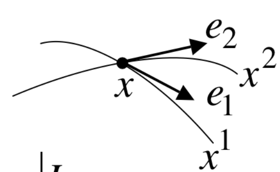
How GA facilitates use of coordinates on a vector manifold

Patch of $\mathcal{M}^m$ parametrized by coordinates: $x = x(x^1, x^2, \ldots x^m)$

Inverse mapping by coordinate functions: $x^\mu = x^\mu(x)$

Coordinate frame $\{e_\mu = e_\mu(x)\}$ defined by

(Cartan) $\boxed{e_\mu = \partial_\mu x} = \dfrac{\partial x}{\partial x^\mu} = \lim \dfrac{\Delta x}{\Delta x^\mu}$



With pseudoscalar: $e_{(m)} = e_1 \wedge e_2 \wedge \ldots \wedge e_m = |e_{(m)}| I_m$

Reciprocal frame $\{e^\mu\}$ implicitly defined by $e^\mu \cdot e_\nu = \delta^\mu_\nu$

with solution: $e^\mu = (e_1 \wedge \ldots \wedge (\;)_\mu \wedge \ldots \wedge e_m) e_{(m)}^{-1}$

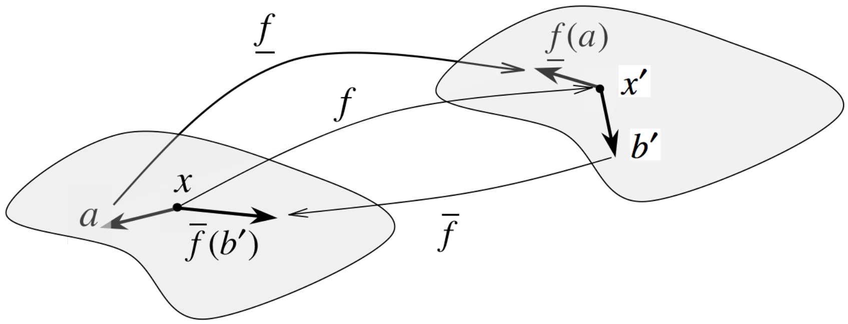
Vector derivative: $\partial = \partial_x = e^\mu \partial_\mu$ $\qquad \partial_\mu = e_\mu \cdot \partial = \dfrac{\partial}{\partial x^\mu}$

$\Rightarrow$ $\boxed{e^\mu = \partial x^\mu}$ (gradient)

Problem: How define vector derivative without coordinates?

Fig. 8

Mappings **of** & Transformations **on** Vector Manifolds



diffeomorphism: $f: x \rightarrow x' = f(x)$ $\quad x = f^{-1}(x')$

Induced transformations of vector fields (active)

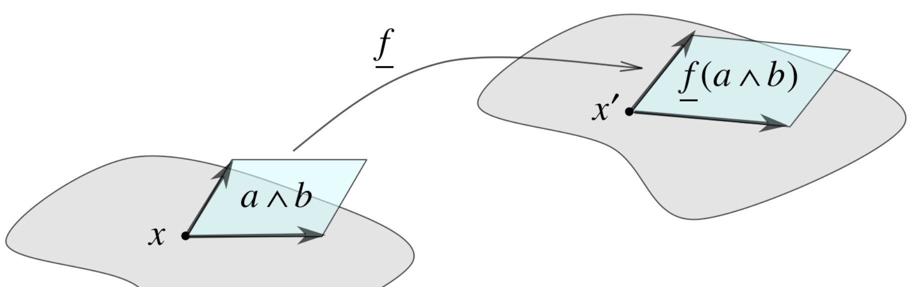
differential: $\underline{f}: a = a(x) \rightarrow a' = \boxed{\underline{f}(a) \equiv a \cdot \nabla f}$ $\quad a = \underline{f}^{-1}(a')$

adjoint: $\bar{f}: b' = b'(x') \rightarrow b = \boxed{\bar{f}(b') \equiv \grave{\nabla} \grave{f} \cdot b' = \partial_x f(x) \cdot b'}$

Tensor fields: $T(a,b')$ *covariant:* $\bar{f}(b')$, *contravariant:* $\underline{f}(a)$

Fig. 9 *Theorem*: $\boxed{\overline{f^{-1}} = \bar{f}^{-1}}$: $\quad b(x) \rightarrow b'(x') = \overline{f^{-1}}\left[b\left(f(x')\right)\right]$

outermorphism: $\underline{f}: a \wedge b \rightarrow \boxed{\underline{f}(a \wedge b) = \underline{f}(a) \wedge \underline{f}(b)}$



$i^2 = -1$

Jacobian: $\underline{f}: i \rightarrow \underline{f}\, i = J_f\, i' \Rightarrow \boxed{J_f = \det \underline{f} = -i' \underline{f}\, i}$

Chain rule: (induced mapping of differential operators)

$$\bar{f}: \nabla' \rightarrow \boxed{\nabla = \bar{f}\, \nabla'} \qquad \text{or} \qquad \partial_x = \bar{f}(\partial_{x'})$$

$$\Rightarrow \; a \cdot \nabla = a \cdot \bar{f}(\nabla') = \underline{f}(a) \cdot \nabla' = a' \cdot \nabla'$$

Fig. 10

$$\left.\begin{array}{l} x = x(\tau) \\ \dot{x} = \dfrac{dx}{d\tau} \end{array}\right\} \Rightarrow \boxed{\frac{d}{d\tau} = \dot{x} \cdot \nabla} = \dot{x} \cdot \bar{f}(\nabla') = \underline{f}(\dot{x}) \cdot \nabla' = \dot{x}' \cdot \nabla'$$

As illustrated in Fig. 9, a *diffeomorphism* (map) on a vector manifold induces a transformation of the tangent space at each point called the *differential* (also called the "push-forward"). The *adjoint* of this transformation (also called the "pull-back") goes in the opposite direction. Note that the differential and adjoint are defined, respectively, by the *directional derivative* and the *gradient* of the map. As indicated in Fig. 10, the differential and adjoint transformations of tangent vectors are extended to the entire tangent algebra by outermorphism. The outermorphism of the pseudoscalar gives the Jacobian of the map, notably, without introducing a local coordinate system.

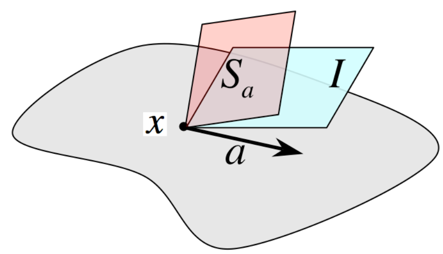
This coordinate-free treatment of diffeormorphism has been applied to great effect in defining the concept of *position gauge transformation*, a keystone in the Gauge Theory of Gravity of Lasenby, Doran and Gull [4,5]. That resolves a longstanding problem of providing a precise definition of Einstein's General Relativity Principle [5].

Note that the whole apparatus of differential outermorphism applies equally well to linear algebra. Indeed, a map is linear if it is equal to its differential.

Chapters 5 & 6: Differential Geometry

Now we have the necessary tools for addressing coordinate-free differential geometry. There are two distinct methods, each with distinct advantages. The *method of mobiles* (comoving frames in Chap. 6 is originated in the 19th century, but GA greatly enhances it with a spinor treatment of rotations and their derivatives. The method of *sliding pseudoscalar* in Chap. 5 is reviewed in {9} and summarized in Fig. 11.

Thm I: The shape bivector S_a is the rotational velocity of the pseudoscalar I as it slides along the manifold!!



$$S_a = I^{-1} a\cdot\partial I \quad = \langle S_a \rangle_2 \qquad a = a(x)$$

$$a\cdot\partial I = IS_a \quad = \underbrace{I\cdot S_a}_{0} + I\times S_a + \underbrace{I\wedge S_a}_{0} \qquad A\times B \equiv \frac{1}{2}(AB - BA)$$

$$\partial I = \partial\wedge I = S(I) = -NI \quad \Rightarrow \quad \partial\cdot I = N\cdot I = 0$$

Thm II: The shape commutator gives the curvature!

$$\textit{Curvature}: \quad C(a\wedge b) \equiv S_a \times S_b \quad = \underbrace{P(S_a\times S_b)}_{\text{intrinsic}} + \underbrace{P_\perp(S_a\times S_b)}_{\text{extrinsic}}$$

$$\textit{Intrinsic (Riemann) curvature}: \quad R(a\wedge b) \equiv P(S_a\times S_b)$$

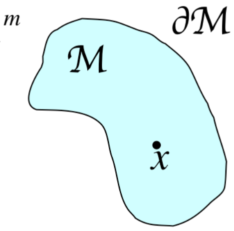
Fig. 11

Chapter 7: Directed Integration Theory

The *directed integral* is defined in Fig. 12 in terms of the more familiar multiple integral. Its power is shown by the formulation of the *Fundamental Theorem of Geometric Calculus* in Fig. 13.

Directed integrals in GA

$F = F(x)$ = multivector-valued function on $\mathcal{M} = \mathcal{M}^m$

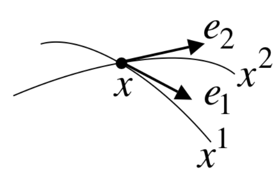


$d^m x = |d^m x| I_m(x)$ = directed measure on $\mathcal{M}$

In terms of coordinates:

$$d^m x = d_1 x \wedge d_2 x \wedge \ldots \wedge d_m x = e_1 \wedge e_2 \wedge \ldots \wedge e_m \, dx^1 dx^2 \ldots dx^m$$

where $d_\mu x = e_\mu(x)\, d^\mu x$ (no sum)



$|d^m x| = |e_{(m)}| dx^1 dx^2 \ldots dx^m$ = *Volume element*

Directed Integral $\Big\}$ $\int_{\mathcal{M}} d^m x \, F = \int_{\partial\mathcal{M}} e_{(m)} F \, dx^1 dx^2 \ldots dx^m$

expressed as a standard multiple integral

Fig. 12

Fundamental Theorem of Geometric Calculus

For $\mathcal{M} = \mathcal{M}^m \subset \mathcal{V}^n$ = vector space, field $F = F(x)$

∇ = vector derivative on $\mathcal{V}^n$

$$\nabla F = \underbrace{\nabla \cdot F}_{\text{div}} + \underbrace{\nabla \wedge F}_{\text{curl}}$$

$$\int_{\mathcal{M}} (d^m x) \cdot \nabla F = \int_{\partial\mathcal{M}} d^{m-1} x \, F$$

$\partial = \partial_x = I_m^{-1}(I_m \cdot \nabla)$ = vector derivative on $\mathcal{M}$

$$\Rightarrow \quad d^m x \, \partial = (d^m x) \cdot \partial = (d^m x) \cdot \nabla$$

✪ $$\boxed{\int_{\mathcal{M}} d^m x \, \partial F = \int_{\partial\mathcal{M}} d^{m-1} x \, F}$$

Inspires coordinate-free definition for the

tangential derivative: $\partial = \partial_x$ = derivative by x on $\mathcal{M}$

✪✪ $$\boxed{\partial F = \lim_{d\omega \to 0} \frac{1}{d\omega} \oint d\sigma F}$$ $\qquad d\omega = d^m x$, $\quad d\sigma = d^{m-1} x$

Fig. 13

As shown in Fig. 14, the fundamental Theorem in GC unifies many different integral theorems.

Fundamental Theorem of Geometric Calculus

For $\mathcal{M} = \mathcal{M}^m \subset \mathcal{V}^n$ = vector space, field $F = F(x)$

∇ = vector derivative on $\mathcal{V}^n$ $\qquad \nabla F = \underbrace{\nabla \cdot F}_{\text{div}} + \underbrace{\nabla \wedge F}_{\text{curl}}$

$$\int_{\mathcal{M}} (d^m x) \cdot \nabla F = \int_{\partial\mathcal{M}} d^{m-1}x\, F$$

$\partial = \partial_x = I_m^{-1}(I_m \cdot \nabla)$ = vector derivative on $\mathcal{M}$

$$\Rightarrow \quad d^m x\, \partial = (d^m x)\cdot\partial = (d^m x)\cdot\nabla$$

✪ $$\int_{\mathcal{M}} d^m x\, \partial F = \int_{\partial\mathcal{M}} d^{m-1}x\, F$$

Inspires coordinate-free definition for the

tangential derivative: $\partial = \partial_x$ = derivative by x on $\mathcal{M}$

✪✪ $$\partial F = \lim_{d\omega \to 0} \frac{1}{d\omega} \oint d\sigma F$$

$d\omega = d^m x$

$d\sigma = d^{m-1}x$

Fig. 14

Theory of differential forms generalized by GA

$L(d^k x, x)$ = multivector-valued k-form

= linear function of k-vector $d^k x$ at each point x.

e.g.: $L = d^k x$ = k-vector-valued k-form

Exterior differential of k-form L:

$$dL \equiv \dot{L}(d^{k+1}x \cdot \dot{\partial}) = L(d^{k+1}x \cdot \dot{\partial}, \dot{x})$$

Fundamental Theorem:

(most general form) $$\int_{\mathcal{M}} dL = \oint_{\partial\mathcal{M}} L$$

Special cases: $L = d^k x F(x)$ $\qquad L = \langle d^{m-1}x\, F\rangle$

$$dL = \langle d^m x\, \partial F\rangle = \langle d^m x\, \partial \wedge F\rangle$$

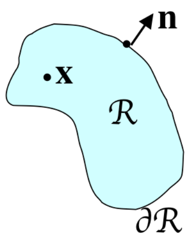
$$= (d^m x)\cdot(\partial \wedge F) \quad \text{if} \quad F = \langle F\rangle_{m-1}$$

Advantages over standard theory:

- Cauchy Theorem: $\partial F = 0 \Leftrightarrow \oint d^k x F = 0$
- Cauchy Integral Theorem

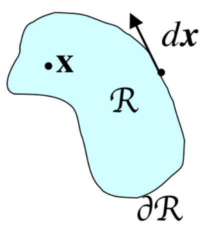
Fig. 15

The most general form of the Fundamental Theorem is given in Fig. 15. It reduces precisely to the standard theory of differential forms when the integrands are scalar-valued. But non-scalar integrands lead to Cauchy's Theorem and Cauchy's Integral Theorem in complex variable theory and their generalizations to higher dimensions, as first demonstrated in {4}. This is illustrated in Fig. 16, which gives the integral form of Maxwell's equation $\nabla \mathbf{E} = \rho$ for a static electric field with charge density $\rho = \rho(\mathbf{x})$. This, of course, is a solution of Maxwell's equation if the charge density inside the region and the electric field on the boundary are given. Note that in the 2D case the formula reduces to a generalization of Cauchy's integral formula that includes an area integral. That generalization was first given by Pompieu in 1910, but is seldom mentioned in books on complex variable theory, which are hung up on the notion that complex line integrals are something special.



dim $\mathcal{R} = n = 3$:

$$\mathbf{E}(\mathbf{x}) = \int_{\mathcal{R}} \frac{d^3x'}{4\pi} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\, \rho(\mathbf{x}') + \int_{\partial\mathcal{R}} \frac{d^2x'}{4\pi} \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\, \mathbf{n}\mathbf{E}(\mathbf{x}')$$



dim $\mathcal{R} = n = 2$: $d\mathbf{x}' = \mathbf{i}\mathbf{n}dx'$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathcal{R}} d^2x' \frac{1}{\mathbf{x}-\mathbf{x}'}\rho(\mathbf{x}') + \frac{1}{2\pi\mathbf{i}}\int_{\partial\mathcal{R}} \frac{1}{\mathbf{x}-\mathbf{x}'}\, d\mathbf{x}'\,\mathbf{E}(\mathbf{x}')$$

Complex variable: $z = \mathbf{a}\mathbf{x}$ $\quad d\mathbf{x} = \mathbf{a}dz$

Fig 16 *Generalised Cauchy Integral Formula*

$$F(z) = \frac{1}{2\pi}\int_{\mathcal{R}} d^2x' \frac{1}{z-z'}\rho(z') + \frac{1}{2\pi\mathbf{i}}\int_{\partial\mathcal{R}} \frac{dz'}{z-z'}F(z')$$

A more detailed discussion of differential forms in Geometric Calculus is given in {7}, and a valuable alternative approach is given in {8}.

Chapter 8: Lie Groups and Lie Algebras

This chapter is a prospectus for incorporating the theory of Lie Groups and Lie Algebras. It was originally proposed as a doctoral thesis for a student of mine, but he got engaged in other things, so I wrote it up myself. Incomplete as it is, it has served well as a stimulus for further developments. For example, I worked out an explicit representation for the conformal group, which has played a key role in developing *Conformal Geometric Algebra* [6,9]. In another direction, it stimulated me to represent generators of the symplectic group as bivectors in phase space, and Chris Doran extended that to a complete treatment of the classical groups {10}. The general idea is summarized in Fig. 17. The crystallographic point groups and space groups are fully worked out in {11,12}.

Group Theory with Geometric Algebra

odd/even parity

Versor (of order k): $G = n_k \ldots n_2 n_1$ $\quad G^{-1} = n_1^{-1} n_2^{-1} \ldots n_k^{-1}$ $\quad G^{\#} = (-1)^k G$

Groups: $\mathrm{Pin}(r, s) = \left\{ G : GG^{-1} = 1 \right\} \supset \mathrm{Spin}(r, s) = \left\{ G : G = G^{\#} \right\}$

on vectors: $\mathrm{O}(r, s) = \left\{ \underline{G} : \underline{G}(a) = G^{\#} a G^{-1} \right\} \supset \mathrm{SO}(r, s) \underset{2}{\cong} \mathrm{Spin}(r, s)$

Advantages over matrix representations:

- Coordinate-free
- Simple composition laws: $G_2 G_1 = G_3 \qquad \underline{G}_2 \underline{G}_1 = \underline{G}_3$
- Reducible to multiplication and reflection by vectors:
- Reflection in a hyperplane in $\mathbb{V}_{r,s}$ with normal n_i: $\underline{G}_i(a) = -n_i a n_i^{-1}$

$\Rightarrow$ *Cartan-Dieudonné Thm* (Lipschitz, 1880): $\underline{G} = \underline{G}_k \cdots \underline{G}_2 \underline{G}_1$

For example: All the classical groups!

In particular: Conformal group: $\mathrm{C}(r, s) \cong \mathrm{O}(r+1, s+1)$

Hence define: Conformal GA: $\mathbb{G}^{r+1,s+1}$

Fig. 17

Selected online papers

Here is a brief description of selected papers available online that elaborate the fundamental concepts of GA or treat them at a more elementary level. The first two papers {1, 2} develop the fundamentals for undergraduate physics majors. The first published papers on Geometric Calculus {3, 4} referred to the subject more modestly as Multivector Calculus, because development was not yet sufficient to claim it as a universal mathematical language. That claim could be confidently made in {5}, because adequate foundations had been laid in the books [1, 2, 3].

Of the remaining papers listed below, the most recent {9} reviews the powerful approach to differential geometry using the Shape Operator and suggests directions for further research. Comments on the other papers are given in the chapter discussions above.

(Note: titles are linked to web pages on which the papers can be found.)

{1} Synopsis of Geometric Algebra

Summarizes and extends some of the basic ideas and results of GA. To make the summary self-contained, all essential definitions and notations are explained, and geometric interpretations of algebraic expressions are reviewed.

{2} Geometric Calculus (Fundamentals)

A thorough treatment of differentiation and integration with respect to vector variables sufficient for most applications in undergraduate physics and engineering.

{3} Multivector Calculus

Shows how differential and integral calculus in many dimensions can be greatly simplified by using geometric algebra. The necessary notations, definitions, and fundamental theorems are developed to make the calculus ready for use.

{4} Multivector Functions

Employs geometric calculus to derive some powerful theorems that generalize well-known theorems of potential theory and the theory of functions of a complex variable. *Analytic multivector functions* on *Euclidean n-space* are defined and shown to be appropriate generalizations of analytic functions of a complex variable. Some of their basic properties are pointed out. These results have important applications to physics.

{5} A Unified Language for Mathematics and Physics

Makes the case for adopting Geometric Calculus as a unified language. That case has been abundantly validated and strengthened by many publications to the present day.

{6} The Design of Linear Algebra and Geometry

Improves and extends the treatment of linear algebra in Ref. [2].

{7} Differential Forms in Geometric Calculus

Reviews the rationale for embedding differential forms in the more comprehensive system of Geometric Calculus. The most significant application of the system is to relativistic physics where it is referred to as Spacetime Calculus. The fundamental integral theorems are discussed along with applications to physics, especially electrodynamics.

{8} Simplicial calculus with geometric algebra

Develops geometric calculus on an oriented *k*-surface embedded in Euclidean space by utilizing the notion of an oriented *k*-surface as the limit set of a sequence of *k*-chains. This method provides insight into the relationship between the vector derivative, and the Fundamental Theorem of Calculus and Residue Theorem. It should be of practical value in numerical finite difference calculations with integral and differential equations in Clifford algebra.

{9} The Shape of Differential Geometry in Geometric Calculus.

Reviews foundations for coordinate-free differential geometry in Geometric Calculus. In particular, it shows how both extrinsic and intrinsic geometry of a manifold can be characterized by a single bivector-valued one-form called the *Shape Operator*.

{10} Lie Groups as Spin Groups

Shows how the computational power of Geometric Algebra simplifies analysis and applications of Lie groups and Lie algebras. Representation of Lie algebras as bivector algebras enables representation of Lie groups as spin groups. Spin representations of the classical groups are worked out.

{11} Point and Space Groups in Geometric Algebra

A detailed treatment of the 3D point groups and crystal classes with GA. Minimal background with GA required.

{12} The Crystallographic Space Groups in Geometric Algebra

Complete treatment of the 17 different 2D space groups and 230 different 3D space groups demonstrating the considerable advantages of formulation with Conformal GA.

Books about Geometric Algebra

[1] D. Hestenes, *Space-Time Algebra*, Gordon & Breach, New York, (1966).

[2] D. Hestenes and G. Sobczyk, *CLIFFORD ALGEBRA to GEOMETRIC CALCULUS, A Unified Language for Mathematics and Physics,* Kluwer: Dordrecht/Boston (1984), paperback (1985). Fourth printing 1999. {Reviewed by James S. Marsh (1984), *American Journal of Physics* **53**(5): 510-11.)

[3] D. Hestenes, *New Foundations for Classical Mechanics*, Kluwer: Dordrecht/Boston (1986), paperback (1987). Second Edition (1999).

[4] C. Doran and A. *Lasenby. Geometric Algebra for Physicists*. Cambridge: The University Press, 2003.

[5] D. Hestenes, Gauge Theory Gravity with Geometric Calculus, *Foundations of Physics* **36**, 903-970 (2005).

[6] D. Hestenes, Grassmann's Legacy. In H-J. Petsche, A. Lewis, J. Liesen, S. Russ (Eds.) *From Past to Future: Grassmann's Work in Context* (Birkhäuser: Berlin, 2011).

[7] G. Sobszyk, Mappings of Surfaces in Euclidean Space using Geometric Algebra, Thesis, Arizona State University (1971).

[8] H. Li, *Invariant Algebras and Geometric Reasoning*. (Beijing: World Scientific, 2008)

[9] L. Dorst and J. Lasenby (Eds.) *Guide to Geometric Algebra in Practice* (Springer: London, 2011).

[10] L. Dorst, D. Fontijne, and S. Mann. *Geometric Algebra for Computer Science*