Cylinders and quadratic surfaces (Sect. 12.6)

- Cylinders.
- Quadratic surfaces:
  - Spheres, $\dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} + \dfrac{z^2}{r^2} = 1.$
  - Ellipsoids, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$
  - Cones, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0.$
  - Hyperboloids, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1, \qquad -\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$
  - Paraboloids, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z}{c} = 0.$
  - Saddles, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z}{c} = 0.$

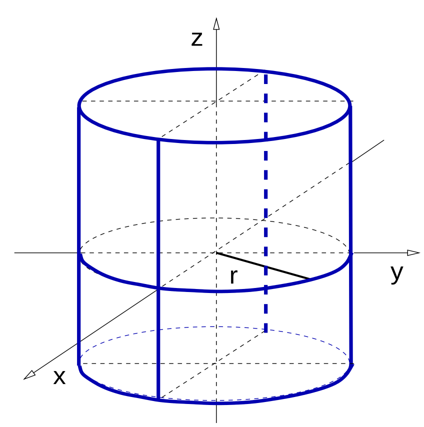
Cylinders

Definition

Given a curve on a plane, called the *generating curve*, a *cylinder* is a surface in space generating by moving along the generating curve a straight line perpendicular to the plane containing the generating curve.

Example

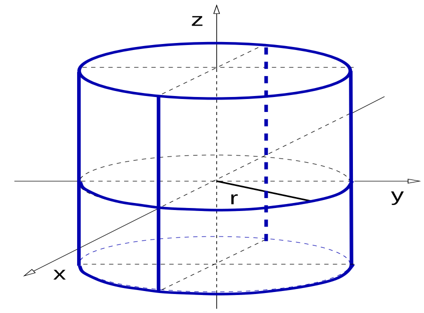
A *circular cylinder* is the particular case when the generating curve is a circle. In the picture, the generating curve lies on the xy-plane. ◁



Cylinders

Example

Find the equation of the cylinder given in the picture.



Solution:

The intersection of the cylinder with the $z = 0$ plane is a circle with radius r, hence points of the form $(x, y, 0)$ belong to the cylinder iff $x^2 + y^2 = r^2$ and $z = 0$.

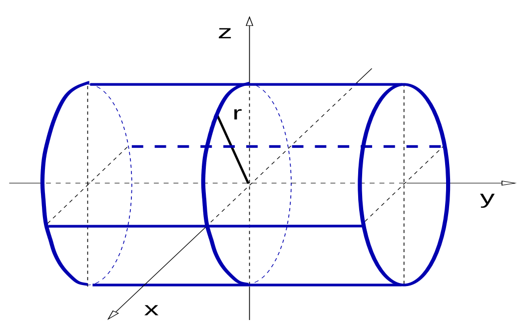
For $z \neq 0$, the intersection of horizontal planes of constant z with the cylinder again are circles of radius r, hence points of the form (x, y, z) belong to the cylinder iff $x^2 + y^2 = r^2$ and z constant.

Summarizing, the equation of the cylinder is $x^2 + y^2 = r^2$. The coordinate z does not appear in the equation. The equation holds for every value of $z \in \mathbb{R}$. $\triangleleft$

Cylinders

Example

Find the equation of the cylinder given in the picture.



Solution:

The generating curve is a circle, but this time on the plane $y = 0$. Hence point of the form $(x, 0, z)$ belong to the cylinder iff $x^2 + z^2 = r^2$.

We conclude that the equation of the cylinder above is

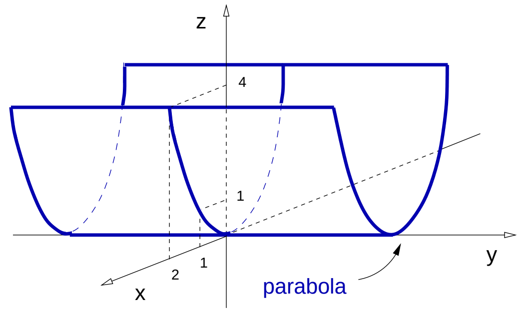
$$x^2 + z^2 = r^2, \qquad y \in \mathbb{R}.$$

The coordinate y does not appear in the equation. The equation holds for every value of $y \in \mathbb{R}$. $\triangleleft$

Cylinders

Example

Find the equation of the cylinder given in the picture.



Solution:

The generating curve is a parabola on planes with constant y.

This parabola contains the points $(0, 0, 0)$, $(1, 0, 1)$, and $(2, 0, 4)$.

Since three points determine a unique parabola and $z = x^2$ contains these points, then at $y = 0$ the generating curve is $z = x^2$.

The cylinder equation does not contain the coordinate y. Hence,

$$z = x^2, \qquad y \in \mathbb{R}.$$

$\triangleleft$

Cylinders and quadratic surfaces (Sect. 12.6)

- Cylinders.
- **Quadratic surfaces:**
  - Spheres.
  - Ellipsoids.
  - Cones.
  - Hyperboloids.
  - Paraboloids.
  - Saddles.

Quadratic surfaces

Definition

Given constants a_i, b_i and c_1, with $i = 1, 2, 3$, a *quadratic surface* in space is the set of points (x, y, z) solutions of the equation

$$a_1\, x^2 + a_2\, y^2 + a_3\, z^2 + b_1\, x + b_2\, y + b_3\, z + c_1 = 0.$$

Remarks:

- There are several types of quadratic surfaces.
- We study only quadratic surfaces given by

$$a_1\, x^2 + a_2\, y^2 + a_3\, z^2 + b_3\, z = c_2. \tag{1}$$

- The surfaces below are rotations of the one in Eq. (1),

$$a_1\, z^2 + a_2\, x^2 + a_3\, y^2 + b_3\, y = c_2,$$
$$a_1\, y^2 + a_2\, x^2 + a_3\, x^2 + b_3\, x = c_2.$$

Cylinders and quadratic surfaces (Sect. 12.6)

- Cylinders.
- Quadratic surfaces:
  - **Spheres.** $\quad \dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} + \dfrac{z^2}{r^2} = 1.$
  - Ellipsoids.
  - Cones.
  - Hyperboloids.
  - Paraboloids.
  - Saddles.

Spheres

Recall: We study only quadratic equations of the form:

$$a_1 x^2 + a_2 y^2 + a_3 z^2 + b_3 z = c_2.$$

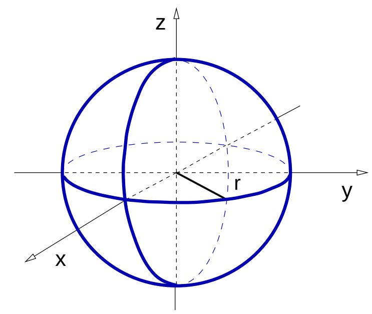
Example

A *sphere* is a simple quadratic surface, the one in the picture has the equation

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$$

($a_1 = a_2 = a_3 = 1/r^2$, $b_3 = 0$ and $c_2 = 1$.)

Equivalently, $\quad x^2 + y^2 + z^2 = r^2.$ ◁



Spheres

Remark: Linear terms move the sphere around in space.

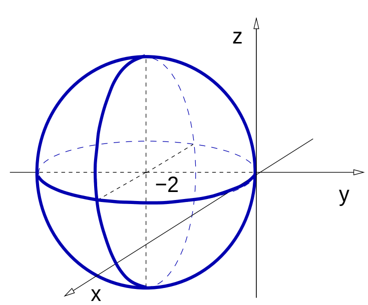
Example

Graph the surface given by the equation $x^2 + y^2 + z^2 + 4y = 0$.

Solution: Complete the square:

$$x^2 + \left[y^2 + 2\left(\frac{4}{2}\right) y + \left(\frac{4}{2}\right)^2\right] - \left(\frac{4}{2}\right)^2 + z^2 = 0.$$

Therefore, $x^2 + \left(y + \frac{4}{2}\right)^2 + z^2 = 4$. This is the equation of a sphere centered at $P_0 = (0, -2, 0)$ and with radius $r = 2$. ◁



Cylinders and quadratic surfaces (Sect. 12.6)

- Cylinders.
- Quadratic surfaces:
  - Spheres, $\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$
  - **Ellipsoids**, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$
  - Paraboloids.
  - Cones.
  - Hyperboloids.
  - Saddles.

Ellipsoids

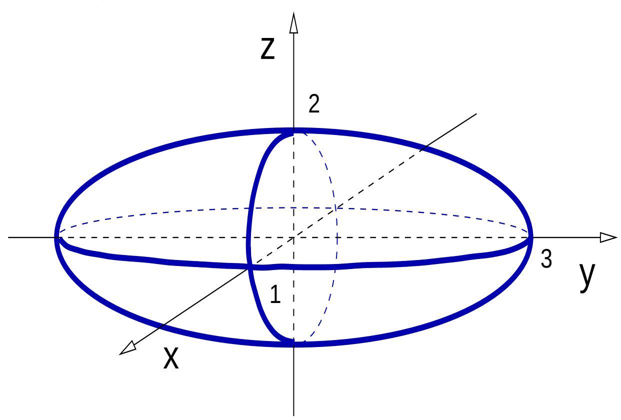
Definition

Given positive constants a, b, c, an *ellipsoid* centered at the origin is the set of point solution to the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Example

Graph the ellipsoid,
$x^2 + \dfrac{y^2}{3^2} + \dfrac{z^2}{2^2} = 1.$ ◁



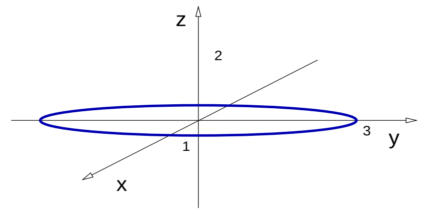
Ellipsoids

Example

Graph the ellipsoid, $x^2 + \dfrac{y^2}{3^2} + \dfrac{z^2}{2^2} = 1.$

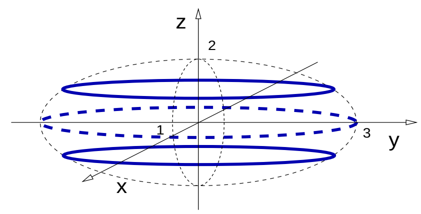
Solution:

On the plane $z = 0$ we have the ellipse $x^2 + \dfrac{y^2}{3^2} = 1.$



On the plane $z = z_0$, with $-2 < z_0 < 2$ we have the ellipse $x^2 + \frac{y^2}{3^2} = \left(1 - \frac{z_0^2}{2^2}\right).$

Denoting $c = 1 - (z_0^2/4)$, then $0 < c < 1$, and $\dfrac{x^2}{c} + \dfrac{y^2}{3^2 c} = 1.$ $\lhd$



Cylinders and quadratic surfaces (Sect. 12.6)

- Cylinders.
- Quadratic surfaces:
  - Spheres, $\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$
  - Ellipsoids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - **Cones**, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0.$
  - Hyperboloids.
  - Paraboloids.
  - Saddles.

Cones

Definition

Given positive constants a, b, a *cone* centered at the origin is the set of point solution to the equation

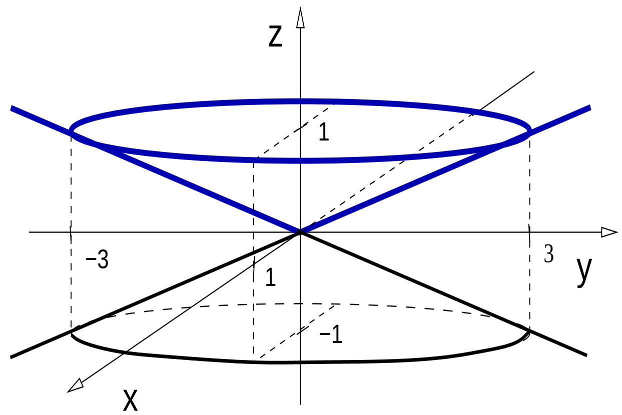
$$z = \pm\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}.$$

Example

Graph the cone,

$$z = \sqrt{x^2 + \frac{y^2}{3^2}}.$$

◁



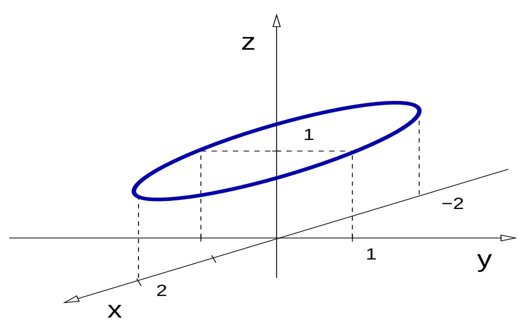
Cones

Example

Graph the cone, $z = +\sqrt{\frac{x^2}{2^2} + y^2}$.

Solution:

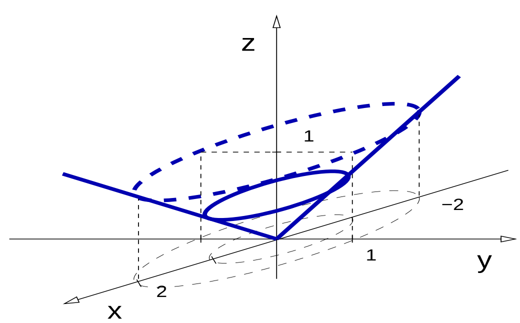
On the plane $z = 1$ we have the ellipse $\dfrac{x^2}{2^2} + y^2 = 1$.



On the plane $z = z_0 > 0$ we have the ellipse $\dfrac{x^2}{2^2} + y^2 = z_0^2$, that is,

$$\frac{x^2}{2^2 z_0^2} + \frac{y^2}{z_0^2} = 1.$$

◁



Cylinders and quadratic surfaces (Sect. 12.6)

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  - Ellipsoids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Cones, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$
  - **Hyperboloids**, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Paraboloids.
  - Saddles.

Hyperboloids

Definition

Given positive constants a, b, c, a *one sheet hyperboloid* centered at the origin is the set of point solution to the equation

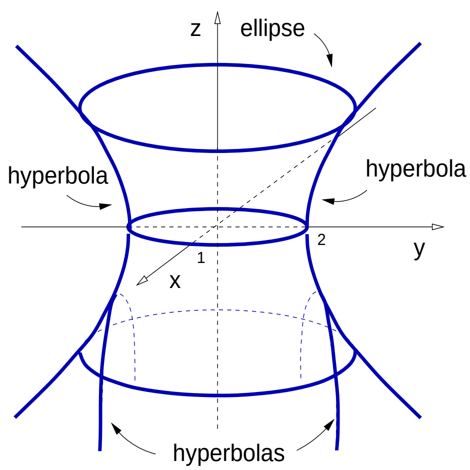
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

(One negative sign, one sheet.)

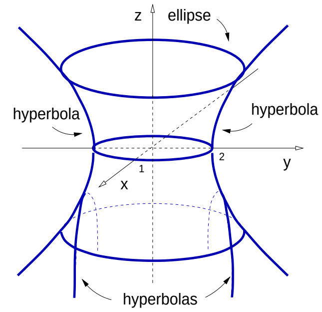
Example

Graph the hyperboloid,

$x^2 + \frac{y^2}{2^2} - z^2 = 1.$ ◁



Hyperboloids



Example

Graph the hyperboloid $x^2 + \frac{y^2}{2^2} - z^2 = 1$.

Solution: Find the intersection of the surface with horizontal and vertical planes. Then combine them into a qualitative graph.

- On horizontal planes, $z = z_0$, we obtain ellipses $x^2 + \frac{y^2}{2^2} = 1 + z_0^2$.
- On vertical planes, $y = y_0$, we obtain hyperbolas $x^2 - z^2 = 1 - \frac{y_0^2}{2^2}$.
- On vertical planes, $x = x_0$, we obtain hyperbolas $\frac{y^2}{2^2} - z^2 = 1 - x_0^2$.

◁

Hyperboloids

Definition

Given positive constants a, b, c, a *two sheet hyperboloid* centered at the origin is the set of point solution to the equation

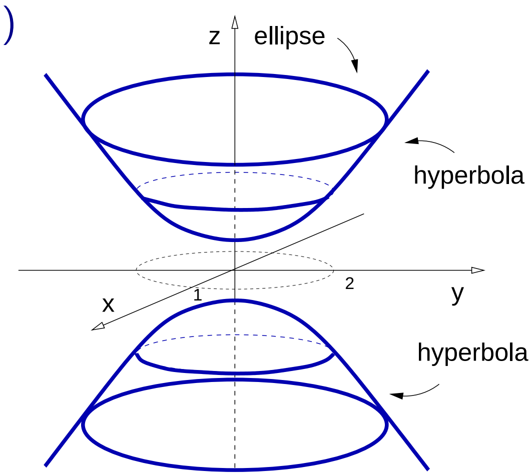
$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(Two negative signs, two sheets.)

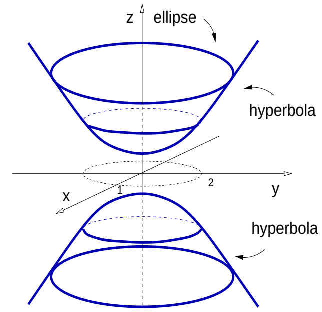
Example

Graph the hyperboloid,

$-x^2 - \frac{y^2}{2^2} + z^2 = 1$. ◁



Hyperboloids



Example

Graph the hyperboloid $-x^2 - \frac{y^2}{2^2} + z^2 = 1$.

Solution:

Find the intersection of the surface with horizontal and vertical planes. Then combine all these results into a qualitative graph.

- On horizontal planes, $z = z_0$, with $|z_0| > 1$, we obtain ellipses $x^2 + \frac{y^2}{2^2} = -1 + z_0^2$.
- On vertical planes, $y = y_0$, we obtain hyperbolas $-x^2 + z^2 = 1 + \frac{y_0^2}{2^2}$.
- On vertical planes, $x = x_0$, we obtain hyperbolas $-\frac{y^2}{2^2} + z^2 = 1 + x_0^2$. ◁

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  - Ellipsoids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
  - Cones, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.
  - Hyperboloids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
  - **Paraboloids**, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0$.
  - Saddles.

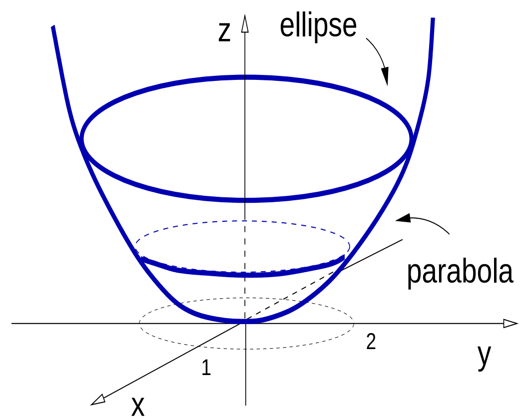
Paraboloids

Definition
Given positive constants a, b, a *paraboloid* centered at the origin is the set of point solution to the equation

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

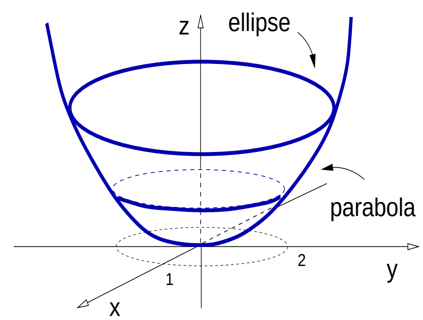
Example
Graph the paraboloid,

$z = x^2 + \dfrac{y^2}{2^2}.$ ◁



Paraboloids.

Example
Graph the paraboloid $z = x^2 + \dfrac{y^2}{2^2}.$



Solution:
Find the intersection of the surface with horizontal and vertical planes. Then combine all these results into a qualitative graph.

- On horizontal planes, $z = z_0$, with $z_0 > 0$, we obtain ellipses $x^2 + \dfrac{y^2}{2^2} = z_0$.
- On vertical planes, $y = y_0$, we obtain parabolas $z = x^2 + \dfrac{y_0^2}{2^2}$.
- On vertical planes, $x = x_0$, we obtain parabolas $z = x_0^2 + \dfrac{y^2}{2^2}$.

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  - Hyperboloids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,\quad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$
  - Paraboloids, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0.$
  - **Saddles**, $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0.$

Saddles, or hyperbolic paraboloids

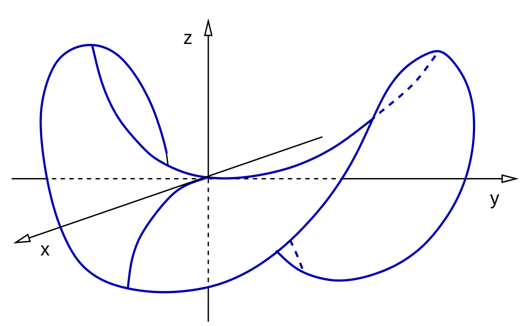
Definition

Given positive constants a, b, c, a *saddle* centered at the origin is the set of point solution to the equation

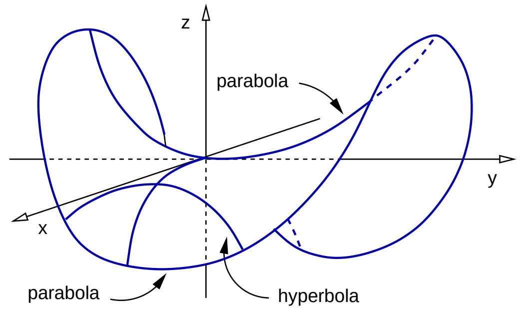
$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Example

Graph the paraboloid, $z = -x^2 + \frac{y^2}{2^2}$. $\triangleleft$



Saddles



Example

Graph the saddle

$z = -x^2 + \frac{y^2}{2^2}.$

Solution:

Find the intersection of the surface with horizontal and vertical planes. Then combine all these results into a qualitative graph.

- On planes, $z = z_0$, we obtain hyperbolas $-x^2 + \frac{y^2}{2^2} = z_0$.
- On planes, $y = y_0$, we obtain parabolas $z = -x^2 + \frac{y_0^2}{2^2}$.
- On planes, $x = x_0$, we obtain parabolas $z = -x_0^2 + \frac{y^2}{2^2}$.