On the Bilateral Laplace Transform of the positive even functions and proof of the Riemann Hypothesis

Seong Won Cha Ph.D.

Seongwon.cha@gmail.com

Remark

This is not an official paper, rather a brief report. I am now trying to prove the Riemann hypothesis in a simple way and I intend writing a full paper when I finish or fail to prove it. Because of lack of time, I wrote this report in a hurry, but I tried to explain all the theorems as clear as possible. Therefore, I think there is any problem to understand.

1. The bilateral Laplace transform

Definition: The bilateral Laplace transform

For a real function $\mathrm{f(t)}$, the bilateral Laplace transform is defined as follows:

$$\mathrm{F(z)} \equiv \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt$$

and the inverse transform:

$$\mathrm{f(t)} = \frac{1}{i2\pi} \int_{x-i\infty}^{x+i\infty} F(z) \cdot e^{zt} dz$$

where $z = x + iy$.

If $\mathrm{f(t)}$ is even, then

$$\mathrm{F(z)} = \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt = \int_{-\infty}^{\infty} f(t) \cdot e^{zt} dt = 2\int_{0}^{\infty} f(t) \cdot \cosh(zt) dt = \int_{-\infty}^{\infty} f(t) \cdot \cosh(zt) dt \qquad (1)$$

Since $\mathrm{F(-z)} = \mathrm{F(z)}$, $\mathrm{F}(iy)$ is real-valued for all y.

Now, we consider a function $\mathrm{f(t)}$, which is even and positive for all t.

Theorem:

If $\mathrm{f(t)}$ is even and positive for all t, then its bilateral Laplace transform $\mathrm{F(z)}$ is transcendental.

It is not always easy to find the bilateral Laplace transform of a function in the closed form, but we can get the series using the definition of the Laplace transform. Consider the series:

$$\mathrm{F(z)} = \sum_{n=0}^{\infty} a_n \cdot z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

The coefficients can be found using the definition, that is

$$a_n = \frac{1}{n!}\int_{-\infty}^{\infty} t^n f(t)dt$$

Moreover, if the function $\mathrm{f(t)}$ is even and positive, all odd terms are vanished and all coefficients are positive. Hence the series will be like that:

$$\mathrm{F(z)} = \sum_{n=0}^{\infty} a_{2n} \cdot z^{2n} = a_0 + a_2 z^2 + a_4 z^4 + \cdots$$

where $a_{2n} = \frac{1}{(2n)!}\int_{-\infty}^{\infty} t^{2n} f(t)dt$

To converse the series, the sequence of coefficients should be rapidly decreased and therefore $f(t)$ as well. Also, we can note from the series that $\mathrm{F}(\bar{z}) = \bar{F}(z)$, where the bar denotes the conjugate.

Since $f(t)$ is even and positive, $\mathrm{F(z)}$ is also even, that is, $\mathrm{F(-z) = F(z)}$. Moreover, $\mathrm{F}(-\bar{z}) = \mathrm{F}(\bar{z}) = \bar{F}(z)$ because $\mathrm{F(z)}$ is even and the property of $\mathrm{F}(\bar{z}) = \bar{F}(z)$. Hence we have $|\mathrm{F(z)}| = |\mathrm{F(-z)}| = |\mathrm{F}(\bar{z})| = |\mathrm{F}(-\bar{z})|$. This means $|\mathrm{F(z)}|$ is even at iy-axis as well as x-axis.

2. Convex functions

Definition:

For a real function $\mathrm{f(x)}$ and $x_1, x_2 \in \mathcal{R}$ and $\lambda \in [0,1]$, then $\mathrm{f(x)}$ is convex if and only if

$$f[\lambda x_1 + (1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda)f(x_2).$$

Similarly, $\mathrm{f(x)}$ is strictly convex if and only if

$$f[\lambda x_1 + (1-\lambda)x_2] < \lambda f(x_1) + (1-\lambda)f(x_2).$$

Definition.

$\mathrm{f(x)}$ is a midpoint convex if

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$$

Definition:

A continuous function$f(x)$ is multiplicatively convex if and only if

$$f\left(\sqrt{x_1 x_2}\right) \leq \sqrt{f(x_1)f(x_2)}$$

A multiplicatively convex function is which is increasing convex.

Theorem: Hardy-Littlewood

Every polynomial $f(x) = \sum_{k=0}^{n} c_k x^k$ with non-negative coefficients is multiplicatively convex on $(0,\infty)$. Moreover $f(x) = \sum_{k=0}^{\infty} c_k x^k$ for $c_k \geq 0$ is strictly multiplicatively convex which is also increasing and strictly convex.

By the theorem 2, $\mathrm{F(z)}$ is increasing and strictly convex on the real line and since $\mathrm{F(-x) = F(x)}$, $\mathrm{F(x)}$ is symmetric at iy-axis.

3. The positive definite functions

Definition:

A function $f(x)$ is positive-definite if and only if

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n \overline{c_k} f(x_n - x_k) \geq 0$$

for any $c_n \in \mathbb{C}$ and $x_n \in \mathcal{R}$.

Similarly, $f(x)$ is strictly positive-definite if and only if

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n \overline{c_k} f(x_n - x_k) > 0$$

Theorem: Bochner

For any function $f(t) \geq 0$ for all t, then its Fourier transform $F(i\omega)$ is strictly positive-definite.

Definition:

A complex function $f(z)$ if complex-valued positive definite if

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n \overline{c_k} f(z_n - \overline{z_k}) \geq 0$$

4. The co-positive definite functions

Definition:

A complex function $f(z)$ if complex-valued co-positive definite if

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n \overline{c_k} f(z_n + \overline{z_k}) \geq 0$$

Similarly, $f(z)$ is strictly complex-valued co-positive definite if

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n \overline{c_k} f(z_n + \overline{z_k}) > 0$$

Theorem:

If a complex function $f(z)$ is (strictly) complex-valued co-positive definite, then the real-valued function $f(x)$ is also (strictly) complex-valued co-positive definite.

Theorem:

A real function $f(t) \geq 0$ for all t, then its bilateral Laplace transform is strictly complex-valued co-positive definite.

Proof:

From the definition $\mathrm{F(z)} = \int_{-\infty}^{\infty} f(t)\cdot e^{-zt}dt$

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n\bar{c}_k F(z_n+\overline{z_k}) = \sum_{n=1}^{N}\sum_{k=1}^{N} c_n\bar{c}_k \int_{-\infty}^{\infty} f(t)\cdot e^{-(z_n+\overline{z_k})t}dt$$
$$= \int_{-\infty}^{\infty} f(t)\cdot\sum_{n=1}^{N}\sum_{k=1}^{N} c_n\bar{c}_k\, e^{-(z_n+\overline{z_k})t}dt = \int_{-\infty}^{\infty} f(t)\cdot\sum_{n=1}^{N} c_n\, e^{-z_n t}\sum_{k=1}^{N} c_k\, e^{-\overline{z_k}t}dt$$
$$= \int_{-\infty}^{\infty} f(t)\cdot\left|\sum_{n=1}^{N} c_n\, e^{-z_n t}\right|^2 dt > 0$$

Theorem:

Let $\mathrm{F(z)}$ be the bilateral Laplace transform of $f(t)\geq 0$, then $|\mathrm{F(z)}|^2$ is strictly complex-valued co-positive definite.

Proof:

From the definition $G(z)\equiv|\mathrm{F(z)}|^2 = \mathrm{F(z)}\cdot\mathrm{F}(\bar{z}) = \left[\int_{-\infty}^{\infty} f(t)\cdot e^{-zt}dt\right]\cdot\left[\int_{-\infty}^{\infty} f(t)\cdot e^{-\bar{z}t}dt\right]$ and , $\sum_{n=1}^{N}\sum_{k=1}^{N} c_n\bar{c}_k G(z_n+\overline{z_k})$:

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n\bar{c}_k F(z_n+\overline{z_k})\cdot\overline{F(z_n+\overline{z_k})} = \sum_{n=1}^{N}\sum_{k=1}^{N}\left[\int_{-\infty}^{\infty} f(t)\cdot e^{-(z_n+\overline{z_k})t}dt\right]\cdot\left[\int_{-\infty}^{\infty} f(t)\cdot e^{-\overline{(z_n+\overline{z_k})}t}dt\right]$$
$$= \sum_{n=1}^{N}\sum_{k=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cdot f(\tau)\cdot e^{-(z_n+\overline{z_k})t}\cdot e^{-\overline{(z_n+\overline{z_k})}\tau}d\tau dt$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cdot f(\tau)\cdot\sum_{n=1}^{N}\sum_{k=1}^{N} c_n\bar{c}_k\, e^{-(z_n t+\overline{z_n}\tau)}\cdot e^{-(\overline{z_k}t+z_k\tau)}d\tau dt$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cdot f(\tau)\cdot\left|\sum_{n=1}^{N} c_n\bar{c}_k\, e^{-(z_n t+\overline{z_n}\tau)}\right|^2 d\tau dt > 0$$

Theorem:

If $F(z)$ is co-positive definite, then $F(0)$ is real and $F(0)>0$.

Proof:

From Eq. (2), let N to 1, then $c_1\bar{c}_1 F(x_1+x_1)>0$. Let $\mathrm{x} = x_1+x_1$ and we have $|c_1|^2F(x)>0$. Hence $F(x)$ is real and $F(x)>0$ for all real x and therefore $F(0)$ is real and $F(0)>0$.

Theorem:

If $F(z)$ is co-positive definite, then $F^{(2n)}(z)$ is also co-positive definite for positive integer n.

Proof:

From the definition of the bilateral Laplace transform, we differentiate $2n$ times:

$$\int_{-\infty}^{\infty} t^n f(t)\cdot e^{-zt}dt$$

Since $f(t) > 0$ for all t and even, and therefore $t^n f(t) > 0$ and $t^n f(t)$ is even. Hence co-positive definte.

Theorem:

If y is fixed, say $\mathrm{y} = y_0$ and $\mathrm{z} = \mathrm{x} + \mathrm{i}y_0$, then any complex-valued co-positive definite function $F(z)$ is co-positive definite for x.

Proof:

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n \bar{c}_k F(z_n + \overline{z_k}) =$$

$$= \int_{-\infty}^{\infty} f(t) \cdot \sum_{n=1}^{N}\sum_{k=1}^{N} c_n \bar{c}_k \, e^{-(z_n + \overline{z_k})t} dt$$

$$= \int_{-\infty}^{\infty} f(t) \cdot \sum_{n=1}^{N}\sum_{k=1}^{N} c_n \bar{c}_k \, e^{-(x_n + iy_0 + x_k - iy_0)t} dt = \int_{-\infty}^{\infty} f(t) \cdot \sum_{n=1}^{N}\sum_{k=1}^{N} c_n \bar{c}_k \, e^{-(x_n + x_k)t} dt$$

$$= \int_{-\infty}^{\infty} f(t) \cdot \left| \sum_{n=1}^{N} c_n \, e^{-x_n t} \right|^2 dt > 0$$

This is clear since

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n \bar{c}_k F(z_n + \overline{z_k}) = \sum_{n=1}^{N}\sum_{k=1}^{N} c_n \bar{c}_k F(x_n + x_k) > 0$$

All the y_0 are cancelled out.

We can show that $|\mathrm{F(z)}|^2$ has the same property:

$$\sum_{n=1}^{N}\sum_{k=1}^{N} c_n \bar{c}_k F(z_n + \overline{z_k}) \cdot \overline{F(z_n + \overline{z_k})} = \sum_{n=1}^{N}\sum_{k=1}^{N} \left[\int_{-\infty}^{\infty} f(t) \cdot e^{-(z_n + \overline{z_k})t} dt \right] \cdot \left[\int_{-\infty}^{\infty} f(t) \cdot e^{-\overline{(z_n + \overline{z_k})}t} dt \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot f(\tau) \cdot \sum\nolimits_{n=1}^{N} \sum\nolimits_{k=1}^{N} c_n \bar{c}_k \, e^{-(x_n t + x_n \tau)} \cdot e^{-(x_k t + x_k \tau)} d\tau dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cdot f(\tau) \cdot \left| \sum\nolimits_{n=1}^{N} c_n \bar{c}_k \, e^{-(x_n t + x_n \tau)} \right|^2 d\tau dt > 0$$

Now, we consider the series expansion of the bilateral Laplace transform of a positive even function $f(t)$.

$$\mathrm{F(z)} = \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt = \sum_{n=0}^{\infty} a_{2n} \cdot z^{2n}$$

Let $\mathrm{G(z)} = |F(z)|^2 = F(z) \cdot \overline{F(z)} = F(z) \cdot F(\bar{z}) = [\sum_{n=0}^{\infty} a_{2n} \cdot z^{2n}] \cdot [\sum_{n=0}^{\infty} a_{2n} \cdot \bar{z}^{2n}]$. Expanding the the multiplication of two series, we have

$$\mathrm{G(z)} = |F(z)|^2 = C_0 + \sum_{n=1}^{\infty} C_{2n} \cdot (z^{2n} + \bar{z}^{2n})$$

where $C_0 = \sum_{k=0}^{\infty} a_{2k}^2 \, |z|^{4k}$, $C_{2n} = \sum_{k=0}^{\infty} a_{2k} \cdot a_{2k+2n} \, |z|^{4k}$

Since

$$|z|^{4k} = (x^2 + y^2)^{2k} = \sum\nolimits_{j=0}^{k} \binom{2k}{j} y^{4k-2j} x^{2j}$$

and

$$z^{2n} + \bar{z}^{2n} = 2 \sum\nolimits_{m=0}^{n} (-1)^{n-m} \binom{2n}{2m} y^{2n-2m} x^{2m},$$

we arrange them in the terms of x:

$$\mathrm{G(z)} = |F(z)|^2 = \sum_{j=0}^{\infty} A_j \cdot x^{2j} + \sum_{j=0}^{\infty} B_{j,m} \cdot x^{2j+2m}$$

where

$$A_j = \sum_{k=j}^{\infty} a_{2k}^2 \binom{2k}{j} y^{4k-2j} + 2 \sum_{n=1}^{\infty} (-1)^n \sum_{k=j}^{\infty} a_{2k} a_{2k+2n} \binom{2k}{j} y^{2n+4k-2j}$$

$$B_{j,m} = \sum_{k=j}^{\infty} \sum_{n=m}^{\infty} (-1)^{n-m} a_{2k} a_{2k+2n} \binom{2n}{2n-2m} \binom{2k}{j} y^{2n-2m} y^{4k-2j}$$

By arranging in the terms of x^{2p}

$$\mathrm{G(z)} = |F(z)|^2 = A_0 + \sum_{p=1}^{\infty} A_p \cdot x^{2p}$$

where

$$A_0 = \sum_{k=0}^{\infty} a_{2k}^2 y^{4k} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (-1)^n a_{2k} a_{2k+2n} y^{2n+4k}$$

$$A_p = \sum_{k=p}^{\infty} \left[a_{2k}^2 + 2 \sum_{n=1}^{\infty} (-1)^n a_{2k} a_{2k+2n} y^{2n} \right] \binom{2k}{p} y^{4k-2p}$$
$$+ 2 \sum_{j=0}^{p-1} \sum_{m=1}^{p-j} \sum_{k=j}^{\infty} \sum_{n=m}^{\infty} (-1)^{n-m} a_{2k} a_{2k+2n} \binom{2n}{2n-2m} \binom{2k}{j} y^{2n-2m} y^{4k-2j}$$

When y is fixed, A_0 and A_p are constants and $\mathrm{G(z)} = |F(z)|^2$ are a function of only x, that is, we have a function $\mathrm{G(x)}$ which is lying on the horizontal line at $\mathrm{y} = y_0$. We showed that if y is fixed, $\mathrm{G(z)}$ is co-positive definite for x. Therefore $\mathrm{G(x)}$ is more accurate than $\mathrm{G(z)}$ if y is fixed.

We also showed that if $\mathrm{G(x)}$ is co-positive definite, then $\mathrm{G}^{(2n)}(\mathrm{x})$, that is, even-time derivative of $\mathrm{G(x)}$ is also co-positive definite and $\mathrm{G}(0) > 0$.

Now we look at the relationship:

$$\mathrm{G(z)} = |F(z)|^2 = A_0 + \sum_{p=1}^{\infty} A_p \cdot x^{2p}$$

Since we assume that y is fixed and $\mathrm{G(x)} = |F(x)|^2$ is co-positive definite and therefore $\mathrm{G}(0) > 0$. Hence $A_0 > 0$. We differentiate $\mathrm{G(x)}$ twice, that is, $\mathrm{G''(x)}$. We know $\mathrm{G''(x)}$ is still co-positive definite. Letting $\mathrm{x} = 0$, we have $2 \cdot 3A_2 > 0$ and $A_2 > 0$. By deviating four times and letting $\mathrm{x} = 0$, we have $A_4 > 0$, and so on, which means that all A_p are positive. By Hardy-Littlewood, $\mathrm{G(x)}$ is a strictly multiplicatively convex function on $(0, \infty)$, which is strictly increasing convex. Since all the orders of x are even, $\mathrm{G(x)}$ is an even function as expected, that is, symmetric at iy-axis and has a unique minimum at $\mathrm{x} = 0$.

Conclusion

1. Let $F(z)$ be the bilateral Laplace transform of a positive and even function, then if y is fixed, $|F(z)|^2$ is strictly multiplicatively convex for $0 \le x < \infty$ on the line $x + i\mathrm{y}_0$. If $F(z)$ is not entire, $|F(z)|^2$ is multiplicatively convex in ROC.
2. $|F(z)|^2$ is symmetric by iy_0 and therefore $|F(z)|^2$ has a unique minimum at $x = 0$.
3. Since $|F(z)|^2$ has a unique minimum at $x = 0$, all zeros of $|F(z)|^2$ locate at iy-axis and so $|F(z)|$.
4. Since the zeros of $|F(z)|$ locate at iy-axis, the zeros of $F(z)$ locate only at iy-axis, if $F(z)$ has any zeros.

Replacing $\mathrm{z} = iz$, we have

$$\mathrm{F(z)} = \int_{-\infty}^{\infty} f(t) \cdot e^{-izt} dt$$

It can be shown that if $f(t) \ge 0$ for all t, $\mathrm{F(z)}$ is complex-valued positive definite, meaning

$$\sum_{n=1}^{N} \sum_{k=1}^{N} c_n \overline{c_k} F(z_n - \overline{z_k}) \ge 0$$

It is the generalized Bochner's theorem. The positive definite functions have similar properties, which the co-positive definite functions have. For example, if $\mathrm{F(z)}$ is positive definite, $(-1)^n F^{(2n)}(z)$ is also positive definite and $\mathrm{F}(0) \ge 0$ and therefore $\mathrm{F}(0)$ is real.

If $f(t)$ is positive and even, we have:

$$\mathrm{F(z)} = \int_{-\infty}^{\infty} f(t) \cdot e^{-izt} dt = \mathrm{F(z)} = 2\int_{0}^{\infty} f(t) \cdot \cos(zt) dt = \int_{-\infty}^{\infty} f(t) \cdot \cos(zt) dt$$

Like the Laplace transform of $f(t)$, we can expand it as a series.

$$\mathrm{F(z)} = \int_{-\infty}^{\infty} f(t) \cdot \cos(zt) dt = \sum_{n=0}^{\infty} (-1)^n \cdot a_{2n} \cdot z^{2n} = a_0 - a_2 z^2 + a_4 z^4 - \cdots$$

With the series and the properties of the positive definite functions, we can derive similar properties like the Laplace transform that we have proved. The differences are that for the fixed x, $\mathrm{F(z)}$ has a unique minimum at x=0 and only real zeros.

5. The Riemann hypothesis

The Riemann zeta function $\zeta(s)$ is defined as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

where $s = \sigma + i\omega$

The Riemann zeta function has only one pole at $s = 1$ and infinitely many zeros. It can be shown that all the zeros are located on the stripe of $0 < \sigma < 1$. Riemann conjectured that all zeros of zeta function would be located at $\sigma = \frac{1}{2}$, so called Riemann hypothesis.

Riemann proved the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

Using the functional equation, he derived xi function $\xi(s)$ which is symmetric at $\sigma = \frac{1}{2}$, defined

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} s(s-1)\Gamma(s/2)\zeta(s)$$

The unique pole of the zeta function at $s = 1$ is cancelled out because of $(s-1)$. Hence $\xi(s)$ is an entire function but the zeros of $\xi(s)$ locate at the same position of the zeta function, that is, on the stripe of $0 < \sigma < 1$ and since $\xi(s)$ is symmetric at $\sigma = \frac{1}{2}$, $\xi\left(\frac{1}{2} + i\omega\right)$ is real. As mentioned, the zeros of $\xi(s)$ locate at the same position of the zeta function $\zeta(s)$. Hence, if we can prove that all zeros of $\xi(s)$ locate only at $\sigma = \frac{1}{2}$, then the Riemann hypothesis is true.

Using the Hankel contour, Poisson summation formula and Mellin transform, the inverse Fourier transform at $\sigma = \frac{1}{2}$ can be derived which is:

$$\varphi(t) = 2\sum_{n=1}^{\infty} e^{-\pi n^2 e^{2t}} \left(2\pi^2 n^4 e^{\frac{9}{2}t} - 3\pi n^2 e^{\frac{5}{2}t}\right)$$

Since $\varphi(t)$ is the inverse Fourier transform of $\xi\left(\frac{1}{2} + i\omega\right)$, $\varphi(t)$ is even and positive for all t. and the relationship between $\xi(s)$ and $\varphi(t)$ is as follows

$$\xi(s) = \int_{-\infty}^{\infty} \varphi(t) \cdot e^{\left(s-\frac{1}{2}\right)t} dt = \sum_{n=0}^{\infty} h_{2n} \left(s - \frac{1}{2}\right)^{2n}$$

Letting $z = s - \frac{1}{2}$, that is, shifted by $\frac{1}{2}$ and therefore the zeros now locate on the stripe of $-\frac{1}{2} < x < \frac{1}{2}$ and we have

$$\Phi(z) = \int_{-\infty}^{\infty} \varphi(t) \cdot e^{zt} dt$$

Moreover, since $\varphi(t)$ is an even function

$$\Phi(z) = \int_{-\infty}^{\infty} \varphi(t) \cdot e^{-zt} dt$$

which is the bilateral Laplace transform of the positive and even function $\varphi(t)$. We know that all zeros of the bilateral Laplace transform of a positive and even functions locate at $x = 0$. Hence all the zeros of $\xi(s)$ must locate at $\sigma = \frac{1}{2}$ and therefore $\zeta(s)$ too, which means the Riemann hypothesis is true.

Riemann also defined a function named “big xi-function” $\Xi(z)$, which is:

$$\Xi(z) = 2\int_{0}^{\infty} \varphi(t) \cdot \cos(zt) \cdot dt = \int_{-\infty}^{\infty} \varphi(t) \cdot \cos(zt) \cdot dt$$

If the Riemann hypothesis is true, $\Xi(z)$ has only real zeros.

As mentioned, this function is nothing but a positive definite function and by the similar way, we can prove that all the zeros of $\Xi(z)$ are real and therefore the Riemann hypothesis is true.