Conformal Gravity, Maxwell and Yang-Mills Unification in $4D$ from a Clifford Gauge Field Theory

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Abstract

A model of Emergent Gravity with the observed Cosmological Constant from a BF-Chern-Simons-Higgs Model is revisited which allows to show how a Conformal Gravity, Maxwell and $SU(2) \times SU(2) \times U(1) \times U(1)$ Yang-Mills Unification model in *four* dimensions can be attained from a Clifford Gauge Field Theory in a very natural and geometric fashion.

Keywords: C-space Gravity, Clifford Algebras, Grand Unification.

1 Emergent Gravity and Cosmological Constant from a BF-Chern-Simons-Higgs Model

In this introduction we shall review how Einstein Gravity with the observed Cosmological Constant emerges from a BF-Chern-Simons-Higgs Model [1]. The $4D$ action is inspired from a BF-CS model defined on the boundary of the open $5D$ tubular region $D^2 \times R^3$, where D^2 is the open domain of the two-dim disk. For instance, AdS_4 has the topology of $S^1 \times R^3$ which can be seen as the *lateral* boundary of the tubular region $D^2 \times R^3$. The upper/lower boundaries at $\pm\infty$ of the open tubular region have a topology of $D^2 \times S^2$. The relevant BF-CS-Higgs *inspired* action is based on the isometry group of AdS_4 space given by $SO(3,2)$, that also coincides with the conformal group of the 3-dim *projective* boundary of AdS_4 of topology $S^1 \times S^2$. The action involves the $SO(3,2)$ valued gauge fields A_μ^{AB} and a family of Higgs scalars ϕ^A that are $SO(3,2)$ vector-valued 0-forms and the indices run from $A = 1,2,3,4,5$. The action is comprised of an integral associated with the open tubular 5-dim region M_5 and an integral associated with the 4-dim boundary M_4. It can be written in a compact notation using gauge-covariant differential forms as

$$S_{BF-CS-Higgs} = \int_{M_4} \phi\ F \wedge F + \phi\ d_A\phi \wedge d_A\phi \wedge d_A\phi \wedge d_A\phi \quad -$$

$$\int_{M_5} V_H(\phi)\ d_A\phi \wedge d_A\phi \wedge d_A\phi \wedge d_A\phi \wedge d_A\phi. \tag{1}$$

Strictly speaking, because we are using a *covariantized* exterior differential $d_A = d + A$ operator, we don't have the standard BF-CS theory. For this reason we use the terminology BF-CS-Higgs inspired model. The $5D$ origins of the BF-CS inspired action is due to the correspondence

$$\int_{D^2\times R^3} d\phi \wedge F \wedge F \ \longleftrightarrow \ \int_{D^2\times R^3} B \wedge F_4. \quad B = d\phi. \quad F_4 = F \wedge F. \tag{2}$$

and

$$\int_{D^2\times R^3} d\phi \wedge F \wedge F \ = \ \int_{M_4} \phi\ F \wedge F$$

$$\int_{D^2\times R^3} d\phi \wedge d\phi \wedge d\phi \wedge d\phi \wedge d\phi \ = \ \int_{M_4} \phi\ d\phi \wedge d\phi \wedge d\phi \wedge d\phi. \tag{3}$$

after an integration by parts when d is the *ordinary* exterior differential operator obeying $d^2 = 0$ and $F = dA$. When one uses the gauge-covariant exterior differential $d_A = d + A$, F and $F \wedge F$ fields satisfy the Bianchi-identity:

$$F = d_A A = dA + A \wedge A. \quad d_A^2\phi = F\phi \neq 0. \quad d_A^2 A = d_A F = 0 \Rightarrow d_A(F \wedge F) = 0. \tag{4}$$

The Higgs-like potential is:

$$V_H(\phi) \ = \ \kappa\ (\eta_{AB}\phi^A\phi^B - v^2)^2; \quad \eta_{AB} = (+,+,+,-,-). \quad \kappa = constant. \tag{5}$$

The gauge covariant exterior differential is defined: $d_A = d + A$ so that $d_A\phi = d\phi + A \wedge \phi$ and the $SO(3,2)$-valued field strength $F = dA + A \wedge A$ corresponds to the $SO(3,2)$-valued gauge fields in the adjoint representation

$$A_\mu^{AB} = A_\mu^{ab}\ ;\ A_\mu^{5a};\ a, b = 1, 2, 3, 4. \tag{6}$$

which, after symmetry breaking, will be later identified as the Lorentz spin connection ω_μ^{ab} and the vielbein field respectively: $A_\mu^{5a} = \lambda\ e_\mu^a$ where λ is the inverse scale of the throat of AdS_4. Notice that the scalars Φ^A are *dimensionless* and so is the parameter κ, compared to the usual Higgs scalars in $4D$ of dimensions of *mass*. Also, the action (1) does not have the standard kinetic terms $g^{\mu\nu}(D_\mu\varphi)(D_\nu\varphi)$.

The Lie algebra $SO(3,2)$ generators obey the commutation relations:

$$[M_{AB}, M_{CD}] = \eta_{BC}M_{AD} - \eta_{AC}M_{BD} + \eta_{AD}M_{BC} - \eta_{BD}M_{AC}. \tag{7}$$

We will show next how gravitational actions with the observed cosmological constant can be obtained from an action *inspired* from a BF-CS-Higgs theory. If one writes the action (1) explicitly in terms of coordinates, one can see that it is spacetime covariant since the metric factors in the products of the covariant epsilon symbol and measures $[\sqrt{|g|}\ d^n x]\ [\frac{\epsilon^{\mu_1\mu_2.....\mu_n}}{\sqrt{|g|}}]$ cancel out as they should. In this sense one may view the action (1) as being "topological" due to the fact that the metric does not appear explicitly. Different actions where the scalars play the role of a Jacobian-like measure have been proposed by [2]. Before we continue with our derivation we must emphasize that our action (1) (and procedure) is not the *same* as the action studied by [3]; we have a covariantized Chern-Simons term instead of a Jacobian-squared expression and it is not *necessary* to choose a preferred volume, leaving a residual invariance under volume-preserving diffeomorphisms, in order to retrieve the MacDowell-Mansouri-Chamseddine-West (MMCW) action for gravity [4], [5].

We shall perform a separate minimization of the $4D$ and $5D$ terms. The Higgs-like potential is minimized at tree level when the *vev* (vacuum expectation values) are

$$< \phi^5 > \ = \ v. \quad < \phi^a > \ = \ 0. \quad a = 1, 2, 3, 4.. \tag{8}$$

which means that one is freezing-in at each spacetime point the internal 5 direction of the internal space of the group $SO(3,2)$. Using these conditions (8) in the definitions of the gauge covariant derivatives acting on the internal $SO(3,2)$-vector-valued spacetime scalars $\phi^A(x)$, we have that at tree level:

$$\nabla_\mu \phi^5 \ = \ \partial_\mu \phi^5 + A_\mu^{5a}\ \phi^a = 0; \quad \nabla_\mu \phi^a \ = \ \partial_\mu \phi^a + A_\mu^{ab}\ \phi^b + A_\mu^{a5} \phi^5 \ = \ A_\mu^{a5}\ v. \tag{9}$$

A variation of the action (1) w.r.t the scalars ϕ^a yields the zero torsion condition after imposing the results (8, 9) *solely after* the variations have been taken place. Therefore it is not necessary to impose by hand the zero torsion condition like in the MMCW procedure. Despite that the v.e.v of ϕ^a ($a = 1, 2, 34$) are 0 one must not forget the constraint equations which arise from their variation. Thus, varying the action w.r.t the ϕ^a yields the $SO(3,2)$-covariantized Euler-Lagrange equations that lead naturally to the zero torsion $T^a_{\mu\nu}$ condition (without having to impose it by hand)

$$\frac{\delta S}{\delta \phi^a} \ - \ d_A \frac{\delta S}{\delta(d_A \phi^a)} \ = \ 0 \ \Rightarrow \ F^{5b} \ \wedge F^{cd}\ \epsilon_{abcd} \ = \ 0 \ \Rightarrow$$

$$F^{5b}_{\mu\nu} \ = \ T^b_{\mu\nu} \ = \ \partial_\mu e^b_\nu + \omega^{bc}_\mu e^c_\nu \ - \ \mu \leftrightarrow \nu \ = \ 0$$

$$\Rightarrow \ \omega^{bc}_\mu \ = \ \omega^{bc}_\mu(e^a_\mu) \ \sim \ e^{\nu b}\ \partial_\nu\ e^c_\mu \ - \ e^{\nu c}\ \partial_\nu\ e^b_\mu. \tag{10}$$

and one recovers the standard Levi-Civita (spin) connection in terms of the (vielbein) metric. A variation w.r.t the remaining ϕ^5 scalar yields after using the relation $A^{a5}_\mu = \lambda e^a_\mu$:

$$F^{ab}_{\mu\nu}\ F^{cd}_{\rho\tau}\ \epsilon_{abcd5}\ \epsilon^{\mu\nu\rho\tau} \ + \ 5\lambda^4 v^4\ e^a_\mu\ e^b_\nu\ e^c_\rho\ e^d_\tau\ \epsilon_{abcd5}\ \epsilon^{\mu\nu\rho\tau} = 0 \ \Rightarrow$$

$$-\frac{1}{5}\phi^5 \; F^{ab}_{\mu\nu} \; F^{cd}_{\rho\tau} \; \epsilon_{abcd5} \; \epsilon^{\mu\nu\rho\tau} \;\; =_{on-shell} \;\; \phi^5 \; \nabla_\mu \phi^a \; \nabla_\nu \phi^b \; \nabla_\rho \phi^c \; \nabla_\tau \phi^d \; \epsilon_{abcd5} \; \epsilon^{\mu\nu\rho\tau} \quad (11)$$

The origins of the crucial factor 5 in (11) arises from the variation w.r.t ϕ^5 of the terms in the action (1)

$$\phi^5 \; \nabla_\mu \phi^a \; \nabla_\nu \phi^b \; \nabla_\rho \phi^c \; \nabla_\tau \phi^d \; \epsilon_{5abcd} \; \epsilon^{\mu\nu\rho\tau} + \phi^a \; \nabla_\mu \phi^5 \; \nabla_\nu \phi^b \; \nabla_\rho \phi^c \; \nabla_\tau \phi^d \; \epsilon_{a5bcd} \; \epsilon^{\mu\nu\rho\tau} +$$

$$....... \; + \; \phi^a \; \nabla_\mu \phi^b \; \nabla_\nu \phi^c \; \nabla_\rho \phi^d \; \nabla_\tau \phi^5 \; \epsilon_{abcd5} \; \epsilon^{\mu\nu\rho\tau}. \quad (12)$$

Using these last equations (8-11), after the minimization procedure, will allows us to *eliminate* on-shell all the scalars ϕ^A from the action (1) furnishing the MacDowell-Mansouri-Chamseddine-West action for gravity as a result of an spontaneous symmetry breaking of the internal $SO(3,2)$ gauge symmetry due to the Higgs mechanism leaving unbroken the $SO(3,1)$ Lorentz symmetry:

$$S_{MMCW} \;=\; \frac{4}{5} \; v \; \int d^4x \; F^{ab}_{\mu\nu} \; F^{cd}_{\rho\tau} \; \epsilon_{abcd5} \; \epsilon^{\mu\nu\rho\tau}. \quad (13)$$

with the main advantage that it is no longer necessary to impose by hand the zero Torsion condition in order to arrive at the Einstein-Hilbert action. On the contrary, the zero Torsion condition is a direct result of the spontaneous symmetry breaking and the dynamics of the orginal BF-CS inspired action. Upon performing the decomposition

$$A^{ab}_\mu = \omega^{ab}_\mu. \quad A^{a5}_\mu = \lambda e^a_\mu. \quad (14a)$$

where λ is the inverse length scale of the model (like the AdS_4 throat), taking into account that $\eta_{55} = -1$, the antisymmetry $A^{a5} = -A^{5a}$, and inserting these relations (14a) into the definition

$$F^{ab} \;=\; dA^{ab} \;+\; A^{ac} \wedge A^{cb} \;-\; A^{a5} \wedge A^{5b} \;=$$

$$d\omega^{ab} \;+\; \omega^{ac} \wedge \omega^{cb} \;+ \lambda^2 \; e^a \wedge e^b \;=\; R^{ab} \;+ \lambda^2 \; e^a \wedge e^b. \quad (14b)$$

leads to the MMCW action (13) comprised of the Einstein-Hilbert action, the cosmological constant term (vacuum energy density) plus the Gauss-Bonnet Topological invariant in $D = 4$, respectively

$$S = \frac{8}{5}\lambda^2 v \int R \wedge e \wedge e \;+ \frac{4}{5}\lambda^4 v \int e \wedge e \wedge e \wedge e \;+ \frac{4}{5} v \int R \wedge R. \quad (15)$$

which implies that the gravitational constant $G = L^2_{Planck}$ (in natural units of $\hbar = c = 1$) and the vacuum energy density ρ are fixed in terms of the throat-size of the AdS_4 space $(\lambda)^{-1}$ and $|v|$ as

$$\frac{8}{5} \; \lambda^2 \; |v| \;=\; \frac{1}{16\pi G} \;=\; \frac{1}{16\pi L^2_P}; \quad |\rho| \;=\; \frac{4}{5} \; \lambda^4 \; |v|. \quad (16)$$

Eliminating the vacuum expectation value (vev) value v from eq-(16) yields a geometric mean relationship among the three scales:

$$\frac{\lambda^2}{32\pi}\ \frac{1}{L_P^2}\ =\ |\rho|. \tag{17}$$

By setting the throat-size of the AdS_4 space $(1/\lambda) = R_H$ to coincide precisely with the Hubble radius $R_H \sim 10^{61}\ L_P$, the relation (17) furnishes the observed vacuum energy density [1]

$$|\rho|\ =\ \frac{1}{32\pi}\ \frac{1}{R_H^2}\ \frac{1}{L_P^2}\ \sim\ (\frac{L_P}{R_H})^2\ \frac{1}{L_P^4}\ \sim\ 10^{-122}\ (M_{Planck})^4. \tag{18}$$

A value of $\lambda^{-1} = l = L_p$ in (17) would yield a huge vacuum energy density (cosmological constant). The (Anti) de Sitter throat size must be of the order of the Hubble scale. The reason one can obtain the correct numerical value of the cosmological constant is due to the *key* presence of the numerical factor $< \phi^5 > \ = \ v$ in (16) and whose value is *not* of the order of unity which would have led to $\lambda^{-1} \sim L_P$ and a huge cosmological constant. On the contrary, its v.e.v value is of the order of $|v| \sim (R_H/L_p)^2 \sim 10^{122}$. The results here also apply to the de Sitter case with positive cosmological constant after replacing the AdS_4 gauge group $SO(3,2)$ with the dS_4 group $SO(4,1)$ and breaking the symmetry $SO(4,1) \rightarrow SO(3,1)$.

2 Conformal Gravity and Yang-Mills from Gauge Field Theory based on Clifford Algebras

Let $\eta_{ab} = (+,-,-,-)$, $\epsilon_{0123} = -\epsilon^{0123} = 1$, the Clifford $Cl(1,3)$ algebra associated with the tangent space of a $4D$ spacetime $\mathcal{M}$ is defined by $\{\Gamma_a, \Gamma_b\}\ =\ 2\eta_{ab}$ such that

$$[\Gamma_a, \Gamma_b] = 2\Gamma_{ab},\ \ \Gamma_5\ =\ -i\ \Gamma_0\ \Gamma_2\ \Gamma_3\ \Gamma_4,\ \ (\Gamma_5)^2\ =\ 1;\ \ \ \{\Gamma_5, \Gamma_a\}\ =\ 0; \tag{19}$$

$$\Gamma_{abcd}\ =\ \epsilon_{abcd}\ \Gamma_5;\quad \Gamma_{ab} = \frac{1}{2}\left(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a\right). \tag{20a}$$

$$\Gamma_{abc}\ =\ \epsilon_{abcd}\ \Gamma_5\ \Gamma^d;\quad \Gamma_{abcd} = \epsilon_{abcd}\ \Gamma_5. \tag{20b}$$

$$\Gamma_a\ \Gamma_b\ =\ \Gamma_{ab} + \eta_{ab},\quad \Gamma_{ab}\ \Gamma_5\ =\ \frac{1}{2}\epsilon_{abcd}\ \Gamma^{cd}, \tag{21a}$$

$$\Gamma_{ab}\ \Gamma_c\ =\ \eta_{bc}\ \Gamma_a - \eta_{ac}\ \Gamma_b + \epsilon_{abcd}\ \Gamma_5\ \Gamma^d \tag{21b}$$

$$\Gamma_c\ \Gamma_{ab}\ =\ \eta_{ac}\ \Gamma_b - \eta_{bc}\ \Gamma_a + \epsilon_{abcd}\ \Gamma_5\ \Gamma^d \tag{21c}$$

$$\Gamma_a\ \Gamma_b\ \Gamma_c\ =\ \eta_{ab}\ \Gamma_c + \eta_{bc}\ \Gamma_a - \eta_{ac}\Gamma_b + \epsilon_{abcd}\ \Gamma_5\ \Gamma^d \tag{21d}$$

$$\Gamma^{ab}\ \Gamma_{cd}\ =\ \epsilon^{ab}{}_{cd}\ \Gamma_5\ - 4\delta^{[a}_{[c}\ \Gamma^{b]}{}_{d]}\ - 2\delta^{ab}_{cd}. \tag{21e}$$

$$\delta^{ab}_{cd} \; = \; \frac{1}{2} \; (\delta^a_c \; \delta^b_d \; - \; \delta^a_d \; \delta^b_c \;). \tag{22}$$

the generators $\Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}$ are defined as usual by a signed-permutation sum of the anti-symmetrizated products of the gammas. A representation of the $Cl(1,3)$ algebra exists where the generators $\mathbf{1}, \Gamma_0, \Gamma_5, \Gamma_i\Gamma_5, i = 1,2,3$ are chosen to be Hermitian; while the generators $-i\;\Gamma_0 \equiv \Gamma_4$; Γ_a, Γ_{ab} for $a,b = 1,2,3,4$ are chosen to be anti-Hermitian. For instance, the anti-Hermitian generators Γ_k for $k = 1,2,3$ can be represented by 4×4 matrices, whose block diagonal entries are 0 and the 2×2 block off-diagonal entries are comprised of $\pm\sigma_k$, respectively, where σ_k, are the 3 Pauli's spin Hermitian 2×2 matrices obeying $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$. The Hermitian generator Γ_0 has zeros in the main diagonal and $-\mathbf{1}_{2\times 2}, -\mathbf{1}_{2\times 2}$ in the off-diagonal block so that $-i\;\Gamma_0 = \Gamma_4$ is anti-Hermitian. The Hermitian Γ_5 chirality operator has $\mathbf{1}_{2\times 2}, -\mathbf{1}_{2\times 2}$ along its main diagonal and zeros in the off-diagonal block. The unit operator $\mathbf{1}_{4\times 4}$ has 1 along the diagonal and zeros everywhere else.

Using eqs-(19-22) allows to write the $Cl(1,3)$ algebra-valued one-form as

$$\mathbf{A} \; = \; \left(i \; a_\mu \; \mathbf{1} + i \; b_\mu \; \Gamma_5 \; + \; e^a_\mu \; \Gamma_a \; + i \; f^a_\mu \; \Gamma_a \; \Gamma_5 \; + \; \frac{1}{4} \omega^{ab}_\mu \; \Gamma_{ab} \right) \; dx^\mu. \tag{23}$$

the anti-Hermitian gauge field obeys the condition $(\mathcal{A}_\mu)^\dagger \; = \; - \; \mathcal{A}_\mu$.

The Clifford-valued anti-Hermitian gauge field A_μ transforms according to $A'_\mu \; = \; U^{-1} \; A_\mu \; U + U^{-1}\partial_\mu U$ under Clifford-valued gauge transformations. The anti-Hermitian Clifford-valued field strength is $F = dA + [A, A]$ so that F transforms covariantly $F' = U^{-1} \; F \; U$. Decomposing the anti-Hermitian field strength in terms of the Clifford algebra anti-Hermitian generators gives

$$F_{\mu\nu} \; = \; i \; F^1_{\mu\nu} \; \mathbf{1} \; + i \; F^5_{\mu\nu} \; \Gamma_5 \; + \; F^a_{\mu\nu} \; \Gamma_a \; + i \; F^{a5}_{\mu\nu} \; \Gamma_a \; \Gamma_5 \; + \; \frac{1}{4} F^{ab}_{\mu\nu} \; \Gamma_{ab}. \tag{24}$$

where $F = \frac{1}{2} \; F_{\mu\nu} \; dx^\mu \wedge dx^\nu$. The field-strength components are given by

$$F^1_{\mu\nu} \; = \partial_\mu a_\nu - \partial_\nu a_\mu \tag{25a}$$

$$F^5_{\mu\nu} \; = \; \partial_\mu b_\nu - \partial_\nu b_\mu + 2 e^a_\mu f_{\nu a} - 2 e^a_\nu f_{\mu a} \tag{25b}$$

$$F^a_{\mu\nu} \; = \; \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{ab}_\mu e_{\nu b} - \omega^{ab}_\nu e_{\mu b} + 2 f^a_\mu b_\nu - 2 f^a_\nu b_\mu \tag{25c}$$

$$F^{a5}_{\mu\nu} \; = \; \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \omega^{ab}_\mu f_{\nu b} - \omega^{ab}_\nu f_{\mu b} + 2 e^a_\mu b_\nu - 2 e^a_\nu b_\mu \tag{25d}$$

$$F^{ab}_{\mu\nu} \; = \; \partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu \omega_{\nu c}{}^{\;b} + 4 \left(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu \right) - \mu \longleftrightarrow \nu. \tag{25e}$$

A Clifford-algebra-valued dimensionless anti-Hermitian scalar field $\Phi(x^\mu) = \Phi^A(x^\mu) \; \Gamma_A$ belonging to a section of the Clifford bundle in $D = 4$ can be expanded as

$$\Phi \; = \; i \; \phi^{(1)} \; \mathbf{1} \; + \; \phi^a \; \Gamma_a \; + \; \phi^{ab} \; \Gamma_{ab} \; + i \; \phi^{a5} \; \Gamma_a \; \Gamma_5 \; + i \; \phi^{(5)} \; \Gamma_5 \tag{26}$$

so that the covariant exterior differential is

$$d_A \Phi = (d_A \Phi^C) \Gamma_C = \left(\partial_\mu \Phi^C + \mathcal{A}^A_\mu \Phi^B f_{AB}^{\ \ C} \right) \Gamma_C dx^\mu .$$

where

$$[\mathcal{A}_\mu, \Phi] = \mathcal{A}^A_\mu \Phi^B [\Gamma_A, \Gamma_B] = \mathcal{A}^A_\mu \Phi^B f_{AB}^{\ \ C} \Gamma_C . \quad (27)$$

The generalization of the action in section **1** to the full-fledged Clifford-algebra case is given by three terms. The first term is

$$I_1 = \int_{M_4} d^4x \ \epsilon^{\mu\nu\rho\sigma} < \Phi^A F^B_{\mu\nu} F^C_{\rho\sigma} \Gamma_A \Gamma_B \Gamma_C >_0 . \quad (28)$$

where the operation $< .... >_0$ denotes taking the *scalar* part of the Clifford geometric product of $\Gamma_A \ \Gamma_B \ \Gamma_C$. The scalar part of the Clifford geometric product of the gammas is for example

$$< \Gamma_a \Gamma_b > = \delta_{ab}, \quad < \Gamma_{a_1 a_2} \Gamma_{b_1 b_2} > = \delta_{a_1 b_1} \delta_{a_2 b_2} - \delta_{a_1 b_2} \delta_{a_2 b_1}$$

$$< \Gamma_{a_1} \Gamma_{a_2} \Gamma_{a_3} > = 0, \quad < \Gamma_{a_1 a_2 a_3} \Gamma_{b_1 b_2 b_3} > = \delta_{a_1 b_1} \delta_{a_2 b_2} \delta_{a_3 b_3} \pm ......$$

$$< \Gamma_{a_1} \Gamma_{a_2} \Gamma_{a_3} \Gamma_{a_4} > = \delta_{a_1 a_2} \delta_{a_3 a_4} - \delta_{a_1 a_3} \delta_{a_2 a_4} + \delta_{a_2 a_3} \delta_{a_1 a_4}, \ etc ...... \quad (29)$$

The integrand of (28) is comprised of terms like

$$F^{ab} \wedge F^{cd} \phi^{(5)} \epsilon_{abcd}; \quad F^{(1)} \wedge F^{(5)} \phi^{(5)}; \quad F^a \wedge F^{a5} \phi^{(5)};$$
$$2 \ F^a_{\ b} \wedge F^b_{\ a} \phi^{(1)}; \quad F^{(1)} \wedge F^{(1)} \phi^{(1)}; \quad F^{(5)} \wedge F^{(5)} \phi^{(1)};$$
$$F^{(1)} \wedge F^{ab} \phi_{ab}; \quad F^{(1)} \wedge F^{a5} \phi_{a5}; \quad F^{(1)} \wedge F^a \phi_a;$$
$$F^a \wedge F_a \phi^{(1)}; \quad F^{a5} \wedge F_{a5} \phi^{(1)}; \quad F^{ab} \wedge F^c (\eta_{bc}\phi_a - \eta_{ac}\phi_b);$$
$$F^{ab} \wedge F^c \phi^{5d} \epsilon_{abcd}; \quad F^a \wedge F^{b5} \phi^{cd} \epsilon_{abcd}; \ .... \quad (30)$$

The numerical factors and signs of each one of the above terms is determined from the relations in eqs-(19-22). Due to the fact that $\epsilon^{\mu\nu\rho\sigma} = \epsilon^{\rho\sigma\mu\nu}$ the terms like

$$F^a_{\ b} \wedge F^{bc} \phi_{ac} = F^{bc} \wedge F^a_{\ b} \phi_{ac} = F^{cb} \wedge F_b^{\ a} \phi_{ac} =$$
$$F^c_{\ b} \wedge F^{ba} \phi_{ac} = - F^a_{\ b} \wedge F^{bc} \phi_{ac} \Rightarrow F^a_{\ b} \wedge F^{bc} \phi_{ac} = 0$$
$$F^a \wedge F^b \phi_{ab} = 0; \quad F^{a5} \wedge F^{b5} \phi_{ab} = 0; \ F^{a5} \wedge F^{b5} \phi^{cd} \epsilon_{abcd} = 0, \ .... \quad (31)$$

vanish. Thus the action (28) is a generalization of the McDowell-Mansouri-Chamseddine-West action. The Clifford-algebra generalization of the Chern-Simons terms are

$$I_2 = \int_{M_4} < \Phi^E d\Phi^A \wedge d\Phi^B \wedge d\Phi^C \wedge d\Phi^D \Gamma_{[E} \Gamma_A \Gamma_B \Gamma_C \Gamma_{D]} >_0 =$$

$$\int_{M_4} \left(\phi^{(5)} \ d\phi^a \wedge d\phi^b \wedge d\phi^c \wedge d\phi^d \ \epsilon_{abcd} \ - \ \phi^a \ d\phi^{(5)} \wedge d\Phi^b \wedge d\Phi^c \wedge d\Phi^d \ \epsilon_{abcd} \ + \ .... \right). \tag{32}$$

The Clifford-algebra generalization of the Higgs-like potential is given by

$$I_3 \ = \ -\int_{M_5} \ < \ d\Phi^A \wedge d\Phi^B \wedge d\Phi^C \wedge d\Phi^D \wedge d\Phi^E \ \Gamma_{[A} \ \Gamma_B \ \Gamma_C \ \Gamma_D \ \Gamma_{E]} \ >_0 \ V(\Phi) \ =$$

$$- \int_{M_5} \ d\Phi^5 \wedge d\Phi^a \wedge d\Phi^b \wedge d\Phi^c \wedge d\Phi^d \ \epsilon_{abcd} \ V(\Phi) \ + \ ....$$

where

$$V(\Phi) \ = \ \kappa \ \left(\ \Phi_A \ \Phi^A \ - \mathbf{v}^2 \ \right)^2 \tag{33}$$

and

$$\Phi_A \ \Phi^A \ = \ \phi^{(1)} \ \phi_{(1)} + \phi^a \ \phi_a + \phi^{ab} \ \phi_{ab} + \phi^{a5} \ \phi_{a5} + \phi^{(5)} \ \phi_{(5)}. \tag{34}$$

Vacuum solutions can be found of the form

$$< \phi^{(5)} > \ = \ \mathbf{v}; \ \ < \phi^{(1)} > \ = \ < \phi^a > \ = \ < \phi^{ab} > \ = \ < \phi^{a5} > \ = \ 0. \tag{35}$$

Similarly as it occurred in section **1**, a variation of $I_1 + I_2 + I_3$ given by eqs-(28,32,33) w.r.t ϕ^5, following similar steps as in eqs-(9,11,12) and taking into account the v.e.v of eq-(35) which minimize the potential (33) solely *after the variation* w.r.t the scalar fields is taken place, allows to eliminate the scalars on-shell leading to

$$I_1 + I_2 + I_3 \ = \ \frac{4}{5} \ \mathbf{v} \int_M d^4x \ \left(\ F^{ab} \wedge F^{cd} \ \epsilon_{abcd} \ + \ F^{(1)} \wedge F^{(5)} \ + \ F^a \wedge F^{a5} \right) \ =$$

$$\frac{4}{5} \ \mathbf{v} \int_M d^4x \ \left(\ F^{ab}_{\mu\nu} \ F^{cd}_{\rho\sigma} \ \epsilon_{abcd} \ + \ F^{(1)}_{\mu\nu} \ F^{(5)}_{\rho\sigma} \ + \ F^a_{\mu\nu} \ F^{a5}_{\rho\sigma} \ \right) \ \epsilon^{\mu\nu\rho\sigma}. \tag{36}$$

where Einstein's summation convention over repeated indices is implied.

The upshot of having started with the action $I_1 + I_2 + I_3$ involving the three expressions of eqs-(28,32,33) is that one does not have to impose by hand constraints on the field strengths in eq-(36) in order to recover Einstein gravity. Despite that one has chosen the v.e.v conditions (35) on the scalars, one must not forget the equations which result from their variations. Hence, performing a variation of $I_1 + I_2 + I_3$ w.r.t the remaining scalars $\phi^1, \phi^a, \phi^{ab}, \phi^{a5}$, following similar steps as in eqs-(9,11,12) and taking into account the v.e.v of eq-(35) which minimize the potential (33), yields

$$2 \ F^a_{\ b} \wedge F^b_{\ a} \ + \ F^{(1)} \wedge F^{(1)} \ + \ F^{(5)} \wedge F^{(5)} \ + \ F^a \wedge F_a \ + \ F^{a5} \wedge F_{a5} \ = \ 0. \tag{37a}$$

$$F^{(1)} \wedge F^a \ + \ F^{ab} \wedge F^c \ \eta_{bc} \ = \ 0. \tag{37b}$$

$$F^{(1)} \wedge F_{ab} \ + \ F^c \wedge F^{d5} \ \epsilon_{abcd} \ = \ 0. \tag{37c}$$

$$F^{(1)} \wedge F_{a5} \ + \ F^{bc} \wedge F^{d} \ \epsilon_{abcd} \ = \ 0. \tag{37d}$$

From eqs-(37) one can infer that $F^1 = F^a = 0, \ a = 1,2,3,4$ are solutions compatible with eqs-(37b, 37c, 37d), while the non-zero values F^{ab}, F^5, F^{a5} will be constrained to obey

$$2 \ F^a_{\ b} \wedge F^b_{\ a} \ + \ F^{(5)} \wedge F^{(5)} \ + \ F^{a5} \wedge F_{a5} \ = \ 0. \tag{37e}$$

Therefore, when $F^1 = F^a = 0$ the action (36) will then reduce to

$$S = \ \frac{4}{5} \ \mathbf{v} \ \int_M \ d^4x \ \ (\ F^{ab}_{\mu\nu} \ F^{cd}_{\rho\sigma} \ \epsilon_{abcd} \ ) \ \epsilon^{\mu\nu\rho\sigma}. \tag{38}$$

A solution to the the zero torsion condition $F^a = 0$ can be simply found by setting $f^a_\mu = 0$ in eq-(25c), and which in turn, furnishes the Levi-Civita spin connection $\omega^{ab}_\mu(e^a_\mu)$ in terms of the tetrad e^a_μ. Upon doing so, the field strength F^{ab} in eq-(25e) when $f^a_\mu = 0$ and $\omega^{ab}_\mu(e^a_\mu)$ becomes then $F^{ab} = R^{ab}(\omega^{ab}_\mu) + 4e^a \wedge e^b$, where $R^{ab} \ = \ \frac{1}{2} R^{ab}_{\mu\nu} \ dx^\mu \wedge dx^\nu$ is the standard expression for the Lorentz-curvature two-form in terms of the Levi-Civita spin connection. Finally, the action (38) becomes once again the Macdowell-Mansouri-Chamseddine-West action

$$S \ = \ \frac{4}{5} \ \mathbf{v} \int \ d^4x \ (\ R^{ab} \ + 4 \ e^a \wedge e^b \ ) \ \wedge \ (\ R^{cd} \ + 4 \ e^c \wedge e^d \ ) \ \epsilon_{abcd}. \tag{39}$$

comprised of the Gauss-Bonnet term $R \wedge R$; the Einstein-Hilbert term $R \wedge e \wedge e$, and the cosmological constant term $e \wedge e \wedge e \wedge e$.

In order to have the proper dimensions of $(length)^{-2}$ in the above curvature $R + e \wedge e$ terms, one has to introduce the suitable length scale parameter l in the terms $\frac{1}{l^2} e \wedge e$. If we wish to recover the same results as those found in section **1** obtained after the elimination of the v.e.v $< \phi^5 >= v$ in eq-(16), and consistent with the correct value of the observed vacuum energy density one requires to set $l \sim R_H$. A value of $l = L_p$ would yield a huge cosmological constant. The (Anti) de Sitter throat size can be set to the Hubble scale as we explained above in section **1** due to the key presence of the numerical factor $< \phi^5 >= v$ in eq-(16) and whose value is *not* of the order of unity.

At this stage we can also provide the relation of the action (36) to the Conformal Gravity action based in gauging the conformal group $SO(4,2) \sim SU(2,2)$ in $4D$. The anti-Hermitian operators of the Conformal algebra can be written in terms of the Clifford algebra anti-Hermitian generators as [6]

$$P_a \ = \ \frac{1}{2}\Gamma_a \ (1 \ + i \ \Gamma_5); \quad K_a \ = \ \frac{1}{2}\Gamma_a \ (1 \ - i \ \Gamma_5); \quad D \ = \ \frac{i}{2} \ \Gamma_5, \ \ L_{ab} \ = \ \frac{1}{2} \ \Gamma_{ab}. \tag{40}$$

P_a ($a = 1,2,3,4$) are the translation generators; K_a are the conformal boosts; D is the dilation generator and L_{ab} are the Lorentz generators. The total number of generators is respectively $4 + 4 + 1 + 6 = 15$. Having established this, a real-valued tetrad V^a_μ field and its real-valued partner $\tilde{V}^a_\mu$ can be defined in terms of the real-valued gauge fields e^a_μ, f^a_μ, as follows

$$e^a_\mu \ + \ f^a_\mu \ = \ V^a_\mu ; \quad e^a_\mu \ - \ f^a_\mu \ = \ \tilde V^a_\mu . \tag{41}$$

such that the combination

$$e^a_\mu \ \Gamma_a \ + i \ f^a_\mu \ \Gamma_a \Gamma_5 \ = \ V^a_\mu \ P_a \ + \ \widetilde V^a_\mu \ K_a . \tag{42}$$

is anti-Hermitian for real-valued e^a_μ, f^a_μ fields. The components of the torsion and conformal-boost curvature two-forms of conformal gravity are given respectively by the linear combinations of eqs-(25c, 25d)

$$F^a_{\mu\nu} \ + \ F^{a5}_{\mu\nu} \ = \ \widetilde F^a_{\mu\nu}[P]; \quad F^a_{\mu\nu} \ - \ F^{a5}_{\mu\nu} \ = \ \widetilde F^a_{\mu\nu}[K] \ \Rightarrow$$

$$F^a_{\mu\nu} \ \Gamma_a \ + i \ F^{a5}_{\mu\nu} \ \Gamma_a \ \Gamma_5 \ = \ \widetilde F^a_{\mu\nu}[P] \ P_a \ + \ \widetilde F^a_{\mu\nu}[K] \ K_a . \tag{43}$$

The components of the curvature two-form corresponding to the Weyl dilation generator are $F^5_{\mu\nu}$ (25b). The Lorentz curvature two-form is *contained* in $F^{ab}_{\mu\nu} \ dx^\mu \wedge dx^\nu$ (25e) and the Maxwell curvature two-form is $F^1_{\mu\nu} \ dx^\mu \wedge dx^\nu$ (25a). To sum up, the real-valued tetrad gauge field V^a_μ (that gauges the translations P_a) and the real-valued conformal boosts gauge field $\widetilde V^a_\mu$ (that gauges the conformal boosts K_a) of conformal gravity are given, respectively, by the linear combination of the gauge fields $e^a_\mu \ \pm \ f^a_\mu$ associated with the anti-Hermitian $\Gamma_a, \ i \ \Gamma_a \ \Gamma_5$ generators of the Clifford algebra $Cl(1,3)$ of the tangent space of spacetime $\mathcal{M}^4$ after performing a Wick rotation $-i \ \Gamma_0 = \Gamma_4$.

If one wishes to recover ordinary Einstein gravity directly from the action (36) *without* invoking the equations of motion (37) resulting from a *variation* of $I_1 + I_2 + I_3$ w.r.t the scalar components of Φ^A, one would require, firstly, to set the fields $f^a_\mu = 0$ *and* $b_\mu = 0$ in the expressions for the field strengths in eqs-(25). Secondly, by imposing by hand the zero torsion and conformal boost curvature conditions $\widetilde F^a_{\mu\nu}[P] = \widetilde F^a_{\mu\nu}[K] = 0 \Rightarrow F^a_{\mu\nu} = F^{a5}_{\mu\nu} = 0$ in eqs-(25c, 25d), furnish the Levi-Civita spin connection $\omega^{ab}_\mu(e^a_\mu)$, so that F^{ab} in eq-(25e) becomes then $F^{ab} = R^{ab}(\omega^{ab}_\mu) + 4 e^a \wedge e^b$, where $R^{ab} = \frac{1}{2} R^{ab}_{\mu\nu} \ dx^\mu \wedge dx^\nu$ is the standard expression for the Lorentz-curvature two-form in terms of the Levi-Civita spin connection. Since $F^5_{\mu\nu} = 0$ in eq-(25b) when $f^a_\mu = b_\mu = 0$, the remaining nonvanishing terms in the action (36), after setting $\phi^5 = \mathbf{v}$ and $F^a_{\mu\nu} = F^{a5}_{\mu\nu} = F^5_{\mu\nu} = 0$, are comprised once again of the Gauss-Bonnet term $R \wedge R$; the Einstein-Hilbert term $R \wedge e \wedge e$, and the cosmological constant term $e \wedge e \wedge e \wedge e$.

One should emphasize that our results in this section are based on a very $different$ action (28) (plus the terms in eqs-(32,33)) than the invariant gravitational action studied by Chameseddine [7] based on the constrained gauge group $U(2,2)$ broken down to $U(1,1) \times U(1,1)$. In general, our action (28) is comprised of many *more* terms displayed by eq-(30) than the action chosen by Chamseddine

$$I \ = \ \int_M \ Tr \ (\Gamma_5 \ F \ \wedge \ F) . \tag{44}$$

Secondly, as shown in section **1**, our procedure furnishes the correct value of the cosmological constant via the key presence of the v.e.v $< \phi^5 >= v$ in all the terms of the action (15). Thirdly, by invoking the equations of motion (37) resulting from a *variation* of $I_1 + I_2 + I_3$ w.r.t the scalar components of Φ^A, one does *not* need to impose by *hand* the zero torsion constraints as done by [7]. The condition $F^a = 0$ results from solving eqs-(37).

To sum up, ordinary gravity with the correct value of the cosmological constant emerges from a very specific vacuum solution. Furthermore, there are *many* other vacuum solutions of the more fundamental action associated with the expressions $I_1 + I_2 + I_3$ of eqs-(28, 32, 33) and involving *all* of the terms in eq-(30). For example, for *constant* field configurations Φ^A, the inclusion of all the gauge field strengths in eq-(30) *contain* the Euler type terms $F^{ab} \wedge F^{cd} \epsilon_{abcd}$; theta type terms $F^1 \wedge F^1; F^5 \wedge F^5$ corresponding to the Maxwell a_μ and Weyl dilatation b_μ fields, respectively; Pontryagin type terms $F^a_{\ b} \wedge F^b_{\ a}$; torsion squared terms $F^a \wedge F^a$, etc ... all in one stroke.

Tensorial Generalized Yang-Mills in C-spaces (Clifford spaces) based on poly-vector valued (anti-symmetric tensor fields) gauge fields $\mathcal{A}_M(\mathbf{X})$ and field strengths $\mathcal{F}_{MN}(\mathbf{X})$ have ben studied in [6] where $\mathbf{X} = X_M \Gamma^M$ is a C-space poly-vector valued coordinate and $\mathcal{A}_M(\mathbf{X}) = A^I_M(\mathbf{X})\ \Gamma_I$ is a Clifford-value gauge field whose Clifford algebra is spanned by the Γ_I generators. The Clifford-algebra-valued gauge field $\mathcal{A}^I_\mu(x^\mu)\Gamma_I$ in ordinary spacetime is naturally embedded into a far richer object in C-spaces. In order to retrieve (Conformal) Gravity one required earlier to choose the $Cl(1,3)$ tangent spacetime algebra because the chosen signature of the underlying spacetime manifold was chosen to be $(+,-,-,-)$. The advantage of recurring to C-spaces associated with the $4D$ spacetime manifold is that one can have a Conformal Gravity, Maxwell and $SU(2)$ Yang-Mills unification in a very geometric fashion. To briefly illustrate how it can be attained, let us write in $4D$ the several components of the C-space poly-vector valued gauge field $\mathbf{A}(\mathbf{X})$ as

$$A^I_0 = \Phi^I;\ \ \mathcal{A}^I_\mu;\ \ \mathcal{A}^I_{\mu\nu};\ \ \mathcal{A}^I_{\mu\nu\rho}\ =\ \epsilon_{\mu\nu\rho\sigma}\ \widetilde{\mathcal{A}}^I_\sigma;\ \ \mathcal{A}^I_{\mu\nu\rho\sigma}\ =\ \epsilon_{\mu\nu\rho\sigma}\ \widetilde{\Phi}^I. \qquad ()$$

where $\Phi, \widetilde{\Phi}$ correspond to the scalar (pseudo-scalars) components of a poly-vector. Let us freeze all the degrees of freedom of the poly-vector C-space coordinate $\mathbf{X}$ in $\mathbf{A}(\mathbf{X})$ except those of the ordinary spacetime vector coordinates x^μ. As we have shown in this section, Conformal Gravity and Maxwell are encoded in the components of $\mathcal{A}^I_\mu$. The antisymmetric tensorial gauge field of rank three $\mathcal{A}^I_{\mu\nu\rho}$ is dual to the vector $\widetilde{\mathcal{A}}^I_\sigma$ and has 4 independent spacetime components ($\sigma = 1,2,3,4$), the same number as the vector gauge field $\mathcal{A}^I_\mu$. Therefore, the Yang-Mills group $U(2,2)$ is encoded in $\widetilde{\mathcal{A}}^I_\sigma$, it has 16 generators and contains the compact subgroup $U(2) \times U(2) = SU(2) \times SU(2) \times U(1) \times U(1)$ after symmetry breaking. $U(4)$ is not large enough to accommodate the Standard Model Group $SU(3) \times SU(2) \times U(1)$ as its maximally compact subgroup. The GUT group $SU(5)$ is large enough to achieve this goal. In general, the group $SU(m+n)$ has $SU(m) \times SU(n) \times U(1)$ for compact subgroups. Other approaches, for instance, to Grand Unification with Gravity based on C-spaces and Clifford algebras have been proposed by [9] and [10], respectively. In the model by [9] the 16-dim C-space (corresponding to $4D$ Clifford algebra) metric G_{MN} has enough components to accommodate ordinary gravity and Yang-Mills in the decomposition $G_{\mu\nu} = g_{\mu\nu} + A^i_\mu \ A^j_\nu \ g_{ij}$. Furthermore, it is shown how a unified theory of generalized branes coupled to gauge fields, including the gravitational and Kalb-Ramond fields can be attained in C-spaces. A large number of references pertaining the role of Clifford algebras in Geometric Unification models is also provided. The Gravity-Yang-Mills-Maxwell-Matter GUT model in [10] relies on the $Cl(8)$ algebra in $8D$. In forthcoming work we will present further details of the Unification program within the C-space framework. To conclude, Conformal Gravity, Maxwell and $SU(2) \times SU(2)$ Yang-Mills unification can be attained in a very natural and geometric way in $four$ dimensions. To incorporate the $SU(3)$ (QCD) symmetry and the fermion family flavor symmetry requires going to higher dimensions. For instance, the E_8 Geometry of the Clifford Superspace associated with $Cl(16)$ and Conformal Gravity Yang-Mills Grand Unification can be found in [8].

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