1. Lines and planes in $\mathbb{R}^n$.

Definition 1.1. We say a subset L of $\mathbb{R}^n$ is a **line** if there are $\mathbf{r}_0, \mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \neq \mathbf{0}$ and

$$L = \{\mathbf{r}(t) : t \in \mathbb{R}\}$$

where we have set

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} \quad \text{for } t \in \mathbb{R}. \tag{1}$$

Remark 1.1. Thus

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$$

and

$$\mathbf{rng}\,\mathbf{r} = L.$$

The function $\mathbf{r}$ is called a **parameterization of** L**.** It is obviously **univalent**; this means that

$$t_1, t_2 \in \mathbb{R} \text{ and } \mathbf{r}(t_1) = \mathbf{r}(t_2) \;\Rightarrow\; t_1 = t_2.$$

Theorem 1.1. Suppose

(i) L is a line in $\mathbb{R}^n$;
(ii) $\mathbf{a}, \mathbf{b} \in L$ and $\mathbf{a} \neq \mathbf{b}$;
(iii) $\mathbf{r}$ is as in (1) with $\mathbf{v} = \mathbf{b} - \mathbf{a}$.

Then

(iv) $\mathbf{r}$ is a parameterization of L;
(v) all parameterizations of L arise in this way;
(vi) if K is a line in $\mathbb{R}^n$ and $\mathbf{a}, \mathbf{b} \in K$ then $K = L$.

Proof. Once you understand what all this means it's obvious. □

Definition 1.2. We say a subset P of $\mathbb{R}^n$ is a **plane** if there are $\mathbf{r}_0, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{v} \nparallel \mathbf{w}$ and

$$P = \{\mathbf{r}(t, u) : (t, u) \in \mathbb{R}^2\}$$

where we have set

$$\mathbf{r}(t, u) = \mathbf{r}_0 + t\mathbf{v} + u\mathbf{w} \quad \text{for } (t, u) \in \mathbb{R}^2. \tag{2}$$

Remark 1.2. Thus

$$\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^n$$

and

$$\mathbf{rng}\,\mathbf{r} = P.$$

The function $\mathbf{r}$ is called a **parameterization of** P**.** It is obviously univalent; this means that

$$(t_1, u_1), (t_2, u_2) \in \mathbb{R}^2 \text{ and } \mathbf{r}(t_1, u_1) = \mathbf{r}(t_2, u_2) \;\Rightarrow\; (t_1, u_1) = (t_2, u_2).$$

Definition 1.3. We say a set S of points in $\mathbb{R}^n$ is **collinear** if there is a line L in $\mathbb{R}^n$ such that $S \subset L$. We say S is **noncollinear** if S is not **collinear**. Note that a noncollinear set has at least three points.

Remark 1.3. Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Then $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is noncollinear if and only if

$$\begin{aligned} \mathbf{0} &\neq (\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) \\ &= (\mathbf{b}-\mathbf{a}) \times \mathbf{c} - (\mathbf{b}-\mathbf{a}) \times \mathbf{a} \\ &= \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} + \mathbf{a} \times \mathbf{a} \\ &= \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}. \end{aligned}$$

Theorem 1.2. Suppose
- (i) P is a plane in $\mathbb{R}^n$;
- (ii) $\mathbf{a}, \mathbf{b}, \mathbf{c} \in P$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is noncollinear;
- (iii) $\mathbf{r}$ is as in (2) with $\mathbf{v} = \mathbf{b} - \mathbf{a}$ and $\mathbf{w} = \mathbf{c} - \mathbf{a}$.

Then
- (iv) $\mathbf{r}$ is a parameterization of P;
- (v) all parameterizations of P arise in this way;
- (vi) if O is a plane in $\mathbb{R}^n$ and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in O$ then $O = P$.

Proof. Once you understand what all this means it's obvious. □

Remark 1.4. Can you see what how to define higher dimensional analogues of lines and planes?

Definition 1.4. Whenever $S \subset \mathbb{R}^n$ and $\mathbf{a} \in \mathbb{R}^n$ we let

$$\mathbf{a} + S = \{\mathbf{a} + \mathbf{x} : \mathbf{x} \in S\}$$

and call this set the **translation of** S **by** $\mathbf{a}$. We say a pair of lines or planes in $\mathbb{R}^N$ are **parallel** if one is a translation of the other.

Definition 1.5. Suppose S is a line or a plane in $\mathbb{R}^n$. We let

$$S^\perp = \{\mathbf{n} \in \mathbb{R}^n : (\mathbf{x} - \mathbf{y}) \bullet \mathbf{n} = 0 \text{ whenever } \mathbf{x}, \mathbf{y} \in S.$$

We say a vector $\mathbf{n} \in S^\perp$ is **normal to** S. Note the following:
- (i) $\mathbf{0} \in S^\perp$;
- (ii) if $c \in \mathbb{R}$ and $\mathbf{x} \in S^\perp$ then $c\mathbf{x} \in S^\perp$.
- (iii) if $\mathbf{x}, \mathbf{y} \in S^\perp$ then $\mathbf{x} + \mathbf{y} \in S^\perp$.

A set with these three properties is called a **linear subspace of** $\mathbb{R}^n$. Note that a line or a plane is a linear subspace if and only if it contains $\mathbf{0}$.

Theorem 1.3. Two lines or planes in $\mathbb{R}^n$ are parallel if and only if they have the same normal space.

Proof. Exercise for the reader. □

Theorem 1.4. Suppose L is a line in $\mathbb{R}^2$, $\mathbf{r}$ is as in (1) and $\mathbf{n} = \mathbf{v}^\perp$. Then

$$L = \{\mathbf{x} \in \mathbb{R}^2 : (\mathbf{x} - \mathbf{a}) \bullet \mathbf{n} = 0\}$$

and

$$L^\perp = \{t\mathbf{n} : t \in \mathbb{R}\};$$

in particular, $L^\perp$ is a line in $\mathbb{R}^2$.

Moreover, if $\mathbf{n} \in \mathbb{R}^2$, $\mathbf{n} \neq \mathbf{0}$, $c \in \mathbb{R}$ and

$$L = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \bullet \mathbf{n} = c\}$$

then L is a line in $\mathbb{R}^2$; $\mathbf{n}$ is a normal to L; $c\mathbf{n}/|\mathbf{n}|^2 \in L$ and is the closest point in L to $\mathbf{0}$; and $|c|/|\mathbf{n}|$ is the distance from $\mathbf{0}$ to L.

Proof. Not hard at all. We'll do it in class. □

Theorem 1.5. Suppose P is a plane in $\mathbb{R}^3$, $\mathbf{r}$ is as in (2) and $\mathbf{n} = \mathbf{v} \times \mathbf{w}$. Then

$$P = \{\mathbf{x} \in \mathbb{R}^3 : (\mathbf{x} - \mathbf{a}) \bullet \mathbf{n} = 0\}$$

and

$$P^\perp = \{t\mathbf{n} : t \in \mathbb{R}\};$$

in particular, $P^\perp$ is a line in $\mathbb{R}^3$.

Moreover, if $\mathbf{n} \in \mathbb{R}^3$, $\mathbf{n} \neq \mathbf{0}$, $c \in \mathbb{R}$ and

$$P = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \bullet \mathbf{n} = c\}$$

then P is a plane in $\mathbb{R}^3$; $\mathbf{n}$ is a normal to P; $c\mathbf{n}/|\mathbf{n}|^2 \in P$ and is the closest point in P to $\mathbf{0}$; and $|c|/|\mathbf{n}|$ is the distance from $\mathbf{0}$ to P.

Proof. Not hard at all. We'll do it in class. □

Theorem 1.6. Suppose L is a line in $\mathbb{R}^3$ and $\mathbf{r}$ is as in (1). Then

$$L^\perp = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \bullet \mathbf{v} = 0\};$$

in particular, $L^\perp$ is a plane in $\mathbb{R}^3$.

Moreover, if $\mathbf{n}_i \in L^\perp$, $i = 1, 2$, $\mathbf{n}_1 \not\parallel \mathbf{n}_2$ and

$$P_i = \{\mathbf{x} \in \mathbb{R}^3 : (\mathbf{x} - \mathbf{a}) \bullet \mathbf{n}_i = 0\}, \quad i = 1, 2,$$

then P_1 and P_2 are planes in $\mathbb{R}^3$ with normal $\mathbf{v}$ and

$$L = P_1 \cap P_2.$$

Proof. Not hard at all. We'll do it in class. □

Remark 1.5. Suppose

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R}, \tag{3}$$

parameterizes the line L in $\mathbb{R}^3$. Let

$$x_0,\ y_0,\ z_0 \quad \text{and} \quad a,\ b,\ c$$

be scalars such that

$$\mathbf{r}_0 =< x_0, y_0, z_0 > \quad \text{and} \quad \mathbf{v} =< a, b, c > .$$

Let

$$x : \mathbb{R} \to \mathbb{R}, \quad y : \mathbb{R} \to \mathbb{R}, \quad z : \mathbb{R} \to \mathbb{R}$$

be such that

$$\mathbf{r}(t) =< x(t), y(t), z(t) > \quad \text{for } t \in \mathbb{R}.$$

Then (3) amounts to

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt, \quad z(t) = z_0 + at$$

which, when each equation is solved for t, amounts to

$$\frac{x(t) - x_0}{a} = \frac{y(t) - y_0}{b} = \frac{z(t) - z_0}{c}$$

provided none of a, b, c are zero.

Remark 1.6. Suppose a, b, c are scalars, $< a, b > \neq < 0, 0 >$,

$$P = \{(x, y) \in \mathbb{R}^2 : ax + by = c\}.$$

Then P is a line with normal $< a, b >$ at distance

$$\frac{|c|}{\sqrt{a^2 + b^2}}$$

to the origin.

The point here is that if $\mathbf{n} = < a, b >$ then

$$P = \{\mathbf{x} \bullet \mathbf{n} = c\}.$$

Remark 1.7. Suppose a, b, c, d are scalars, $< a, b, c > \neq < 0, 0, 0 >$,

$$P = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = d\}.$$

Then P is a plane with normal $< a, b, c >$ at distance

$$\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

to the origin.

The point here is that if $\mathbf{n} = < a, b, c >$ then

$$P = \{\mathbf{x} \bullet \mathbf{n} = d\}.$$