Precalculus, Quarter 3, Unit 3.1

Unit Circle and Right Triangle Trigonometry

Overview

Number of instructional days: 10 (1 day = 45–60 minutes)

Content to be learned

- Derive radian measure.
- *Convert between degree and radian measure.**
- Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers.
- Utilize special right triangles to determine the values of sine, cosine, and tangent for π/3, π/4, and π/6.
- Use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x 2\pi - x$ in terms of their values for x, where x is any real number.
- Explain symmetry (odd and even) and periodicity of trigonometric functions using the unit circle.

Mathematical practices to be integrated

Model with mathematics.

- Write special right triangles to model real-world problems.

Make sense of problems and persevere in solving them.

- Match common trigonometric values in relation to their respective angles in problems.

Use appropriate tools strategically.

- Change between radian and degree mode on a graphing calculator.
- Use a graphing calculator to calculate the radian and degree or to find the inverse of the trigonometric value.

Look for and make use of structure.

- Prove how the values of the special right triangles allow us to know the trigonometric 3 π/3, π/4, and π/6.

Look for and express regularity in repeated reasoning.

- Recognize the pattern of the unit circle common values and quadrant signs from the 0, 1, 2, 3, 4 rule of square roots.

*Italics are used to indicate that the italicized content statement, practice statement, or essential question does not appear in the Common Core State Standards for Mathematics.

Essential questions

- What are the common trigonometric values, their corresponding angles in radians and degrees, and the pattern for these values?
- What does a radian represent, and how do you change from radian to degree measures?
- What causes the periodicity of trigonometric functions?
- How do the special right triangles help in solving some trigonometric functions?
- What do the values obtained from the various trigonometric functions represent?

Written Curriculum

Common Core State Standards for Mathematical Content

Trigonometric Functions	**F-TF**

Extend the domain of trigonometric functions using the unit circle

F-TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F-TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Common Core State Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Clarifying the Standards

Prior Learning

In eighth grade, students explained a proof of the Pythagorean Theorem and its converse. They also applied the proof to find missing lengths. In Geometry, students learned the definition of trigonometric ratios for acute angles, and they studied the relationship between sine and cosine as well as the sine and cosine of complementary angles. In Algebra II, students understood radian measure as the length of the arc on the unit circle. Algebra II students also used the unit circle to extend the domain of the trigonometric functions and to model periodic phenomena. Students in Algebra II proved the Pythagorean trigonometric identities. Finally, Algebra II students recognized even and odd functions as well as symmetry from their graphs.

Current Learning

Students continue to use radian measure as the length of the arc on the unit circle to calculate trigonometric values. Students extend the special right triangles to common trigonometric values. Students also expand odd and even functions, as well as symmetry, to trigonometric functions.

Future Learning

In Calculus, students will graphically and algebraically analyze the derivation and integration of trigonometric functions. Students will also utilize trigonometric substitutions to complete more complex integration techniques. In engineering, students will use trigonometric functions to model real-world phenomena and calculate solutions or relationships.

Additional Findings

It is most challenging for students to understand the definition of radian measure and recognize the values, with the corresponding angle measures, around the unit circle. Students often struggle with understanding the difference between the value of a trigonometric function and the corresponding angle. Also, students often switch cosine and sine for the appropriate coordinate values. Teachers have difficulty deriving the unit circle and students have difficulty reproducing the unit circle with correct radian measures and values.

Precalculus, Quarter 3, Unit 3.2

Graphing Trigonometric Functions

Overview

Number of instructional days: 10 (1 day = 45–60 minutes)

Content to be learned

- Graph trigonometric functions.
- Understand and identify key features of trigonometric graphs.
- Calculate various aspects of graphs from trigonometric functions.
- Graph trigonometric functions using a graphing calculator.

Mathematical practices to be integrated

Make sense of problems and persevere in solving them.

- Transform from one representation of trigonometric functions to another.

Reason abstractly and quantitatively.

- Identify changes in graphical representations from what the symbolic representations do to the output values produced.

Model with mathematics.

- Graph and produce symbolic representations for real-world phenomena.

Use appropriate tools strategically.

- Utilize a graphing utility to graph more advanced forms.

Essential questions

- What causes vertical and horizontal translations on a trigonometric function?
- What causes a change in the amplitude or period of a trigonometric function?
- What are the importance of key features of trigonometric functions, including period, midline, and amplitude?

Written Curriculum

Common Core State Standards for Mathematical Content

Interpreting Functions	F-IF

Analyze functions using different representations [~~*Logarithmic*~~ *and trigonometric functions*]

F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

e. Graph ~~exponential and logarithmic functions, showing intercepts and end behavior, and~~ trigonometric functions, showing period, midline, and amplitude.

Common Core State Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Clarifying the Standards

Prior Learning

In grade 5, students graphed points on a coordinate plane to solve real-world mathematical problems. In grade 6, students plotted points on a coordinate plane from tables with reference to quadrants and distance between points. In grade 7, students plotted data from a table to determine proportional relationships. In grade 8, students graphed functions on a coordinate plane. In algebra 1, students graphed various types of functions by hand and using technology. In geometry, students performed calculations in coordinate geometry. In algebra 2, students examined the amplitude, frequency, and midline of trigonometric graphs as well as several types of functions and relations.

Current Learning

Students graph trigonometric functions and their transformations by hand and with the graphing calculator. Students also produce key features of trigonometric graphs from their symbolic representations.

Future Learning

In AP Calculus, students will produce trigonometric graphs from their functions to help analyze their derivative and integrals.

Additional Findings

Finding the characteristics of the graph from the symbolic representations can be challenging to students, as well as thinking in radian values with the unit circle to aid in graphing the functions. A misconception students have is identifying an incorrect parent function by switching sine and cosine.

Precalculus, Quarter 3, Unit 3.3

Inverse Trigonometric Functions

Overview

Number of instructional days: 10 (1 day = 45–60 minutes)

Content to be learned

- Find and graph the inverse of a trigonometric function given a table.
- Use a graphing calculator utility to find the inverse of a trigonometric function graphically and numerically.
- Restrict the domain of a trigonometric function to produce an inverse trigonometric function's graph.
- Find the inverse of a trigonometric function using inverse operations.
- Model periodic phenomena using trigonometric functions and inverse trigonometric functions.
- Verify that two trigonometric functions are inverses through composition.
- Apply the horizontal line test and demonstrate how it applies to the inverse.

Mathematical practices to be integrated

Make sense of problems and persevere in solving them.

- Examine trigonometric graphs and functions to write their corresponding inverse functions.

Reason abstractly and quantitatively.

- Utilize the inverse of the table and/or common trigonometric points in order to produce graphs of inverse functions.

Model with mathematics.

- Use inverse trigonometric functions to solve real-world problems involving trigonometric functions.

Essential questions

- How do you find the inverse of a trigonometric function?
- What do the domain and range of an inverse trigonometric function represent?
- Why do you need to restrict the domain of a trigonometric function to be able to produce an inverse?
- To what do you restrict the domain of trigonometric functions to construct an inverse trigonometric function?
- Why is the horizontal test important for inverse trigonometric functions?

Written Curriculum

Common Core State Standards for Mathematical Content

Building Functions	**F-BF**

Build new functions from existing functions.

F-BF.4 Find inverse functions.

- b. (+) Verify by composition that one function is the inverse of another.
- c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
- d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Trigonometric Functions	**F-TF**

Model periodic phenomena with trigonometric functions.

F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

F-TF.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

F-TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★

Common Core State Standards for Mathematical Practice

1 Make sense of problems and persevere in solving them.

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2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

4 Model with mathematics.

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Clarifying the Standards

Prior Learning

In Algebra I, students solved linear inverse functions and were introduced to simple quadratic inverse functions. In Algebra II, students found inverses with basic polynomial functions.

Current Learning

Students calculate inverse trigonometric values and graph inverse trigonometric functions.

Future Learning

In AP Calculus, students will differentiate and integrate inverse trigonometric functions. Students will apply inverse trigonometric functions to calculate values in engineering applications.

Additional Findings

Students find that inverse trigonometric functions are most challenging due to the domain restrictions of sine, cosine, and tangent. In order to overcome misconceptions, students must be proficient in graphing sine, cosine, and tangent.

Precalculus, Quarter 3, Unit 3.4

Proving and Applying Trigonometric Identities

Overview

Number of instructional days: 10 (1 day = 45–60 minutes)

Content to be learned

- Derive various Pythagorean and reciprocal trigonometric identities.
- Utilize trigonometric substitution to prove basic trigonometric identities.
- Use trigonometric identities to solve trigonometric equations with general solutions.
- Use trigonometric identities to solve trigonometric equations with exact solutions for restricted domain intervals.
- Prove sum or difference formulas.
- Solve problems using sum or difference formulas.

Mathematical practices to be integrated

Make sense of problems and persevere in solving them.

- Prove trigonometric identities when using incorrect trigonometric substitutions on previous attempts.
- Solve trigonometric equations upon successive attempts at trigonometric substitutions.

Reason abstractly and quantitatively.

- Derive the formulas for trigonometric identities.

Construct viable arguments and critique the reasoning of others.

- Verify a proof to determine equality.

Essential questions

- How do you derive the Pythagorean identities using the unit circle?
- What are the steps in verifying trigonometric identity equations?
- What is the advantage to using trigonometric identities?
- How can trigonometric identities be used to solve trigonometric equations?

Written Curriculum

Common Core State Standards for Mathematical Content

Trigonometric Functions	F-TF

Prove and apply trigonometric identities

F-TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

F-TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Common Core State Standards for Mathematical Practice

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2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Clarifying the Standards

Prior Learning

In eighth grade, students proved and applied the Pythagorean Theorem. In Geometry, students used similar triangles to prove the Pythagorean Theorem and to solve application problems. In Algebra II, students used the unit circle to extend the domain of trigonometric functions.

Current Learning

Students derive, verify, and solve trigonometric equations using identities. Students solve problems using sum or difference formulas.

Future Learning

In AP Calculus, students will use trigonometric identities to differentiate and integrate. Students will also simplify complex integrals by trigonometric substitution. In engineering, students will use trigonometric identities in design implementation and, in science, in modeling real-world phenomena.

Additional Findings

The most challenging thing for students is to select the correct identity in order to prove various identities. What makes the content difficult for teachers is the many ways to prove that an identity is equal or to solve a trigonometric equation. Students have difficulty because there is more than one way to prove an identity or reduce a trigonometric equation.