Knot physics: Deriving the fine structure constant.

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Abstract

Knot physics describes the geometry of particles and fields. From the geometry of an electron we can construct a mathematical model relating its charge to its spin angular momentum. From experimental data, the spin angular momentum is $\hbar/2$. Therefore the mathematical model provides a comparison of electron charge to Planck's constant, which gives the fine structure constant α. We find that using only electromagnetic momentum to derive the fine structure constant predicts a value for α^{-1} that is about two orders of magnitude too small. However, the equations of knot physics imply that the electromagnetic field cusp must be compensated by a geometric field cusp. The geometric cusp is the source of a geometric field. The geometric field has momentum that is significantly larger than the momentum from the electromagnetic field. The angular momentum of the two fields together predicts a fine structure constant of $\alpha^{-1} \approx 136.85$. Compared to the actual value of $\alpha^{-1} \approx 137.04$, the error is 0.13%. Including the effects of virtual particles may reduce the error further.

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I. BACKGROUND

This paper will use the assumptions from the paper "Knot physics: Spacetime in co-dimension 2" [1] (available at **www.knotphysics.net**), which is necessary background reading. Many mathematical conventions and assumptions will be carried over from that paper. The application of partial derivatives on embedded manifolds, in particular, may be unfamiliar to many readers.

Mathematica documents are also available at **knotphysics.net** that provide mathematical modeling associated with the calculations here.

II. OVERVIEW

A. The fine structure constant

The fine structure constant is a dimensionless number α defined by the relation $\alpha = q^2/(4\pi\epsilon_0\hbar c)$ where q is the charge of the electron and $\hbar$ is Planck's reduced constant $h/(2\pi)$. In our discussion and calculations we will use $\epsilon_0 = \mu_0 = c = 1$. Then

$$\alpha = \frac{q^2}{4\pi\hbar} \tag{1}$$

The charge of the electron is the fundamental unit of charge associated with every elementary particle. Planck's constant is a unit of action that appears in a wide variety of quantum applications. The fact that the charge of every electron is the same and that every elementary particle has a charge which is an integer multiple of electron charge is of great physical significance, but deriving that fact is not obvious. Furthermore, the number α that determines the magnitude of the elementary charge has not previously been shown to have a numerical formula in terms of non-physical constants. There is a very precise experimental measurement of α, but there is no known theoretical calculation that produces the number without experimental data as an input. The purpose of this paper is to show two things. First, we show how the properties of elementary fermions in knot physics imply leptons have unit charge. Second, we approximate the fine structure constant by applying those properties. Showing integer charge for hadrons and deriving an exact number for α are subjects for future work.

B. Planck's constant

Planck's constant is a unit of action that appears in several different quantum calculations. Though it is not necessary for the purposes of this paper, it may help illuminate the calculation to hypothesize an interpretation of Planck's constant.

The spacetime manifold M has metric $g_{\mu\nu} = \rho^2 A_{\alpha,\mu} A^{\alpha}_{\ ,\nu}$. Using that metric, the manifold M is Ricci flat, $\hat{R}^{\mu\nu} = 0$. Even with that constraint, M has degrees of freedom such that the manifold is under-constrained. Therefore, the manifold maximizes entropy. In classical thermodynamics, a system at equilibrium has $(1/2)kT$ of energy for each degree of freedom. Similarly, we hypothesize that the spacetime manifold M has $\hbar$ of action for each degree of freedom. Then M is a branched manifold and for each degree of freedom the branches are randomly distributed with variance corresponding to Planck's constant. For an elementary fermion, the branches of M separate into branches with spin up and branches with spin down. The difference in angular momentum between spin up and spin down is a degree of freedom, therefore the difference in angular momentum is $\hbar$ and each spin orientation has angular momentum of $\hbar/2$.

Regardless of the interpretation, it is known from experimental data that elementary fermions have spin angular momentum $S = \hbar/2$. Assume we have a formula for spin angular momentum as a function of charge squared, $S(q^2) = \hbar/2$. The fine structure constant is

$$\alpha = \frac{q^2}{4\pi\hbar} \tag{2}$$

We choose to invert the equation

$$\alpha^{-1} = \frac{4\pi\hbar}{q^2} = \frac{8\pi S(q^2)}{q^2} \tag{3}$$

where the experimental value of α^{-1} is approximately $\alpha^{-1}_{exp} \approx 137.0$. Deriving the fine structure constant follows from finding the spin angular momentum as a function of charge.

C. Overview of the derivation

The derivation consists almost entirely of deriving the angular momentum as a function of charge on the electron. The electron topology is $\mathbb{R}^3 \# (S^1 \times P^2)$. To find angular momentum on the electron we first derive the Ricci flat electron geometry. Then we use that geometry and the Lagrangian to solve for a generic field $W^{\mu\nu}$ with Lagrangian $L = W^{\alpha\beta}W_{\alpha\beta}$ such that the electron is a field source. Using the electron geometry and the field $W^{\mu\nu}$, we can compare the field strength of $W^{\mu\nu}$ to its angular momentum. If we know the field strength of all the fields on the electron compared to its electromagnetic field then we can solve for their momenta relative to the charge. Using the sum of those momenta, we can solve for the fine structure constant.

Any charged particle has an electromagnetic field. However, Ricci flatness implies that an electromagnetic field cusp requires a corresponding geometric cusp to preserve flatness. The geometric cusp produces a geometric field with the same field equations as the electromagnetic field. To find the momentum from the geometric field, we compare the energy in the geometric field to the energy in the electromagnetic field. The comparison is analogous to Hooke's law $E = (1/2)kx^2$. The electromagnetic field $F^{\mu\nu}$ has the analog of a spring constant k_F and spring extension x_F. The geometric field $C^{\mu\nu}$ has the analog of a spring constant k_C and spring extension x_C. The spring extension of the electromagnetic field x_F is proportional to the radius of the P^2 in the particle topology $S^1 \times P^2$. The spring extension of the geometric field x_C is proportional to the circumference of the P^2, which means that $x_C = 4\pi x_F$. The spring constant for each field is proportional to the number of degrees of freedom in each field. The Lagrangian is not considered a constraint when counting these degrees of freedom. The electromagnetic field is sensitive to the change of A^ν parallel to the manifold, which has 4 dimensions. The electromagnetic field therefore has 4 degrees of freedom. The geometric field is sensitive to the change of A^ν in 5 spatial dimensions, for 5 degrees of freedom. Therefore $k_C = (5/4)k_F$. We can then compare the energies and find that $E_C = (1/2)k_C x_C^2 = (1/2)(5/4)k_F(4\pi)^2 x_F^2 = (20\pi^2)(1/2)k_F x_F^2 = 20\pi^2 E_F$. The energy, and therefore momentum, in the geometric field is $20\pi^2$ times larger than the energy and momentum in the electromagnetic field. The total angular momentum is the sum of the contributions from the geometric and electromagnetic components. From that angular momentum, we solve for the fine structure constant. The result is $\alpha_{calc}^{-1} \approx 136.85$ with 0.13% error. Virtual particles may have differing effect on the electromagnetic field energy compared to the geometric field energy, which may contribute error.

III. COORDINATES

We will use the following three coordinate systems to describe particle geometry.

A. Cylindrical coordinates

The full 6 dimensions of the Minkowski space can be expressed as $(t, r, z, \phi, x^4, x^5)$ using notation that borrows from two different coordinate conventions. These coordinates will typically be used to describe fields and geometry when t, x^4, and x^5 are suppressed. In that case the coordinates are (r, z, ϕ).

B. Toroidal coordinates

$\mathbb{R}^3$ has toroidal coordinates (τ, σ, ϕ) that relate to polar coordinates (r, z, ϕ) as follows

$$
\begin{aligned}
r &= a\frac{sinh\tau}{cosh\tau - cos\sigma} \\
z &= a\frac{sin\sigma}{cosh\tau - cos\sigma}
\end{aligned}
\tag{4}
$$

The sets of constant τ are tori centered around a circle of radius a. At distance zero from the circle $\tau = \infty$. At infinite distance from the circle, $\tau = 0$. The sets of constant σ are spheres such that their intersection with sets of constant τ are orthogonal. Close to the circle, the coordinate σ is a polar angle around the circle. Toroidal coordinates are an orthogonal coordinate system. Their properties assist with field equations.

C. Mapping coordinates

We use a mapping from 3 dimensions to 5 dimensions to describe the electron $S^1 \times P^2$. The coordinates of the 3-space are toroidal coordinates (τ, σ, ϕ) and the coordinates of the 5-space are a mix of toroidal and cartesian coordinates $(\tau, \sigma, \phi, x^4, x^5)$. Let T be the solid torus $\tau > 1$, then we map from $\mathbb{R}^3 - T$ to $\mathbb{R}^5$.

$$
X : (\tau, \sigma, \phi) \rightarrow (\tau/(1-\tau), \sigma, \phi, \tau sin(2\sigma), \tau cos(2\sigma)) \tag{5}
$$

The mapping begins from $\mathbb{R}^3 - T$, which cuts out the solid torus T where $\tau > 1$. Then it stretches $\mathbb{R}^3 - T$ to cover the missing torus using $\tau/(1-\tau)$ and attaches each point on the boundary of T to the point that is diametrically opposite it. Each point on the boundary of T is mapped to a point with coordinates $X(1, \sigma, \phi) = (\infty, \sigma, \phi, sin(2\sigma), cos(2\sigma))$. This is a torus in the 5-space that projects onto the unit circle (∞, σ, ϕ) in 3 dimensions. For $\tau = 1$ and $\phi = \phi_0$, we have $X(1, \sigma, \phi_0)$, which is a circle of radius 1 that is mapped onto twice by the boundary of T. See Fig. 1.

The mapping X produces a $\mathbb{R}^3\#(S^1 \times P^2)$ with amplitude 1. The geometry of the manifold, in particular Ricci flatness, is not accurately expressed by this mapping.

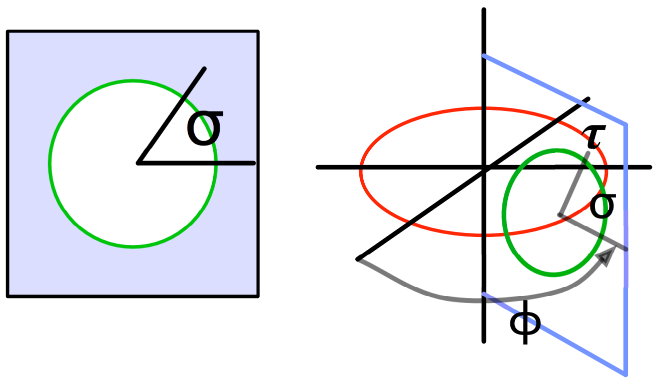

FIG. 1: On the left is $\mathbb{R}^2 - D^2$ with polar angle σ. On the right is $\mathbb{R}^3 - T$ in toroidal coordinates, with a slice at $\phi = \phi_0$. The green circle is $\tau = 1$. The mapping X identifies opposite points on the circumference of the green circle. Using this identification of points makes $\mathbb{R}^2\#P^2$ on the left and $\mathbb{R}^3\#(S^1 \times P^2)$ on the right.

IV. ELECTRON GEOMETRY

Based on the metric $g_{\mu\nu} = \rho^2 A_{\alpha,\mu} A^{\alpha}_{\ ,\nu}$, the manifold M is Ricci flat, $\hat{R}^{\mu\nu} = 0$. If there is no electromagnetic field, then $A^\nu = x^\nu$ and the metric is a conformal scaling $\rho^2 \bar{\eta}_{\mu\nu}$ of the metric $\bar{\eta}_{\mu\nu}$ that is inherited from the Minkowski 6-space.

The electron topology is $\mathbb{R}^3\#(S^1 \times P^2)$. We solve for the conformal scaling ρ and the geometry of the electron that makes it Ricci flat, $\hat{R}^{\mu\nu} = 0$. Then we show how to achieve Ricci flatness on the electron with an electromagnetic field.

A. Quantum branch weight ρ and branching

The metric on the manifold is $g_{\mu\nu} = \rho^2 A_{\alpha,\mu} A^{\alpha}{}_{,\nu}$, which is scaled by the conformal weight ρ. The weight $w = (-det(g))^{1/2} \approx \rho^4$ is a conserved branch weight for the branches associated with the sum-over-histories of quantum mechanics. For the calculations in this paper we only need to consider a single branch. Because the field strengths all scale with ρ in the same way, we can assume that ρ converges to a background value of one, $lim_{r\to\infty}\rho(r) = 1$.

B. Ricci flatness with no electromagnetic field

1. *Flatness in 2 dimensions*

To flatten $\mathbb{R}^3\#(S^1 \times P^2)$ we begin by finding Ricci flat solutions for $\mathbb{R}^2\#P^2$, the 2 dimensional case. Cut $\mathbb{R}^2\#P^2$ so that it has a circular boundary. Call the manifold with boundary M_2. Then we find the Gaussian curvature using the Gauss-Bonnet theorem.

$$\int_{M_2} R dA + \int_{\partial M_2} k_g ds = 2\pi\chi(M_2) \tag{6}$$

From Ricci flatness, $R = 0$ on M_2. The Euler characteristic is $\chi(P^2) = 1$. Therefore the Euler characteristic of M_2 (equivalent to $P^2 - D^2$) is $\chi(M_2) = 0$. Therefore the geodesic curvature $k_g = 0$ at every radius. Therefore the circumference is constant at every radius, the manifold has the same geometry as a cylinder. In the degenerate case that the amplitude of the P^2 goes to zero, M_2 approaches a flat disk and the weight ρ compensates the geometry by $\rho = b/r$, for some constant b. If we increase the amplitude of the P^2 geometry, the manifold remains Ricci flat if the circumference weighted by ρ remains constant for every radius, $C\rho = k$.

This solution is similar to the general solution for Ricci flatness on 2-dimensional planes. In 2 dimensions, for any harmonic function κ, if the metric $\bar{\eta}_{\mu\nu}$ is Ricci flat then the metric $e^{2\kappa}\bar{\eta}_{\mu\nu}$ is also Ricci flat. For multiple source points p_i there is a harmonic function $\kappa(x) = \sum_i -\ln(d(p_i, x))$ where $d(p_i, x)$ is the distance from p_i to x. Then $\rho = e^{\kappa}$ and $\rho^2\bar{\eta}_{\mu\nu}$ is a Ricci flat metric. We can replace any of those points p_i by a degenerate P^2. To expand the geometry of the P^2, we again compensate by reducing the weight ρ such that the circumference is conserved.

We consider the case of $\mathbb{R}^2$ with natural metric $\bar{\eta}_{\mu\nu} = diag(1, 1)$. We use this plane as the slice $\phi = 0, \phi = \pi$ through $\mathbb{R}^3\#(S^1 \times P^2)$, but completely suppress the third dimension for the moment. In this slice there is a P^2 at the point $p_1 = (1, 0)$ and at the point $p_2 = (-1, 0)$. We begin by finding the harmonic function $\kappa(x) = \sum_i -\ln(d(p_i, x))$ and then $\rho = e^\kappa$. Now we use the metric $\rho^2\bar{\eta}_{\mu\nu}$ with degenerate P^2 at each of the points p_i. Then we expand the geometry of the P^2 as desired, compensating the geometry by reducing ρ as needed.

2. *Flatness in 2+1 dimensions*

Now we introduce the time dimension. The inherited metric for a flat manifold is $\bar{\eta}_{\mu\nu} = diag(1, -1, -1)$. Introducing $P^2 \times \mathbb{R}$ on the manifold at $p_1 = (t, 1, 0)$ and $p_2 = (t, -1, 0)$, we can scale the metric as above to get $\rho^2\bar{\eta}_{\mu\nu}$. However, the volume in 3 dimensions scales by ρ^3 and we find that the time dimension makes the conformal scaling no longer Ricci flat. To compensate the time dimension, we use symmetry and motion. Rather than beginning with initial metric $\bar{\eta}_{\mu\nu} = diag(1, -1, -1)$, we have the manifold move such that $(1 - \beta^2)^{1/2} = 1/\gamma = 1/\rho$. Then the inherited metric is $\bar{\eta}_{\mu\nu} = diag(1/\gamma^2, -1, -1) = diag(1/\rho^2, -1, -1)$. Including the scaling by ρ^2 we have the metric $\rho^2\bar{\eta}_{\mu\nu}$ which is Ricci flat. Again, the geometry of the P^2 can be expanded and ρ is reduced to compensate. Now that ρ is linked to motion through $\rho = \gamma$, we see that reducing ρ reduces the velocity. In particular, a P^2 that is fully expanded to $\rho = 1$ has no motion.

3. *Flatness in 3+1 dimensions*

Introducing the third spatial dimension, we use polar coordinates (t, r, z, ϕ) on the flat 4-manifold. For the $\mathbb{R}^3\#(S^1 \times P^2)$, we have a degenerate $S^1 \times P^2$ such that there is a degenerate P^2 in each constant ϕ slice at $r = 1$. We have a harmonic function κ whose source is the degenerate $S^1 \times P^2$. There is a weight $\rho = e^\kappa$. By symmetry of the system, this is like solving the 2-dimensional case if the manifold measure is weighted by $rd\phi$. To keep the geometry Ricci flat, we must use motion to reduce distance in both the ϕ and t direction by $1/\rho$. Then the conformal factor ρ returns those distances to $\rho(1/\rho)rd\phi = rd\phi$ and $\rho(1/\rho)dt = dt$ respectively. The volume of the manifold is dV/γ therefore $\gamma = \rho^2$. The metric is $\rho^2\bar{\eta}_{\mu\nu}$, where $\bar{\eta}_{\mu\nu}$ is the inherited metric and includes the motion. Then this metric is Ricci flat. Again, the P^2 geometry can be expanded and ρ compensates such that circumference is conserved. Likewise, as ρ reduces, $\gamma = \rho^2$ implies that the motion also reduces. If the manifold has the $S^1 \times P^2$ fully expanded such that $\rho = 1$ then the $S^1 \times P^2$ is at rest.

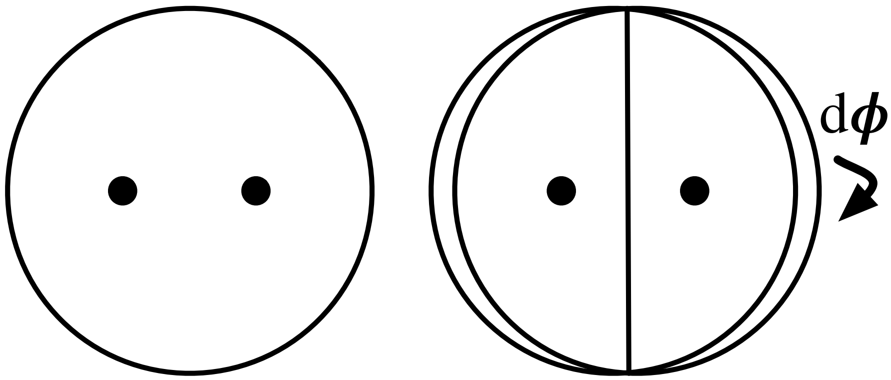

FIG. 2: On the left is $\mathbb{R}^2$ with degenerate P^2 at (-1,0) and (1,0). The harmonic function κ is $\kappa(x) = \sum_i - \ln(d(p_i, x))$. On the right is a $d\phi$ slice from $\mathbb{R}^3$ with corresponding harmonic function κ such that each point is weighted by $rd\phi$.

In 3 spatial dimensions, if the distance from the particle is d, then $lim_{d\to\infty}\kappa = 1/d$. Therefore $lim_{d\to\infty}e^{\kappa} = lim_{d\to\infty}e^{1/d} = 1$. This is in contrast to the 2-dimensional solution where κ scales like $-\ln(r)$ and ρ tends to zero at infinite distance. Therefore, in 3 dimensions it makes sense to say that at infinite distance $\rho = 1$ and $\gamma = \rho^2 = 1$.

C. Fields on the particle

Assume a Lagrangian of the form $L = \rho^4 Y^{\alpha,\beta} Y_{\alpha,\beta}$ on the spacetime manifold with electron topology $\mathbb{R}^3\#(S^1 \times P^2)$. The ρ^4 term is the quantum branch weight and $Y^{\alpha,\beta}$ is an example field. We would like to describe the field $Y^{\alpha,\beta}$ relative to a more manageable field on flat space. From the previous discussion, Ricci flatness requires $\gamma = \rho^2$. The ρ-weighted volume is $\rho^4/\gamma = \rho^2$. Therefore the Lagrangian is $L = (\rho^4/\gamma)Y^{\alpha,\beta}Y_{\alpha,\beta} = \rho^2 Y^{\alpha,\beta}Y_{\alpha,\beta} = \rho Y^{\alpha,\beta}\rho Y_{\alpha,\beta}$ With a source term j^α, the field equation is

$$\nabla_\mu \frac{\partial(\rho Y^{\alpha,\beta}\rho Y_{\alpha,\beta})}{\partial(\rho Y_\alpha{}^{,\mu})} = \nabla_\mu(\rho Y^{\alpha,\mu}) = j^\alpha \tag{7}$$

Let $W^{\nu\mu}$ be a field on flat 3-dimensional space. Assume a source current j^ν on the ρ cusp of the electron and the charge density j^0 is equal to the current density j^ϕ. Using cylindrical coordinates, $j^\nu = \hat{t} + \hat{\phi}$ on the unit circle corresponding to the ρ cusp of the electron and $j^\nu = 0$ everywhere else. Then we define $W^{\nu\mu}$ such that the field goes to zero at infinite distance $lim_{r\to\infty} W^{\nu\mu} = 0$ and

$$\partial_\mu W^{\nu\mu} = j^\nu \tag{8}$$

The $W^{\nu\mu}$ field is invariant in ϕ. To find the divergence $\partial_\mu W^{\nu\mu}$ we can find the integral of the outward facing normal over the boundary of an infintesimal differential element. Distances in that differential element are increased in the r and z direction by the conformal scale ρ. However, in the ϕ direction the conformal length change is negated by the Lorentz contraction of rotational motion. Performing the integral over the boundary of an infinitesimal volume, the lengths in r and z are scaled by ρ. If we scale $W^{\nu\mu}$ to $(1/\rho)W^{\nu\mu}$ then we recover the zero covariant divergence property $\nabla_\mu((1/\rho)W^{\nu\mu}) = 0$ for the conformal metric. This gives a solution to our equation $\nabla_\mu(\rho Y^{\alpha,\mu}) = j^\alpha$, which is $\rho Y^{\nu,\mu} = (1/\rho)W^{\nu\mu}$, or

$$Y^{\nu,\mu} = (1/\rho^2)W^{\nu\mu} \tag{9}$$

Beginning from the electromagnetism Lagrangian $L_F = \rho^4 F^{\alpha\beta}F_{\alpha\beta}$ we conclude that the electromagnetic field is $F^{\mu\nu} = k_F^{1/2}(1/\rho^2)W^{\nu\mu}$ and the Lagrangian is $L = k_F W^{\alpha\beta}W_{\alpha\beta}$ up to multiplication by a scalar k_F.

The Lagrangian, including the geometric term, is

$$L = \rho^4((1/2)F^{\alpha\beta}F_{\alpha\beta} - R) \tag{10}$$

with scalar curvature R. We note that this is scalar curvature R relative to $\bar{\eta}^{\mu\nu}$ and not $\hat{R}$ relative to $g^{\mu\nu}$. A field generated by the R term in the Lagrangian also propagates and has momentum. We show in the next section how the electromagnetic field produces a geometric field.

D. Ricci flatness with weak field at the cusp

For convenience of coordinates we will assume in this section that the calculations are performed at a point on the $S^1 \times P^2$ where the S^1 fiber is in the x_2 direction. This allows us to describe the magnetic field using derivatives of A_2.

The electromagnetic field $F^{\mu\nu}$ comes to maximum at a cusp, which is also the cusp of the quantum branch weight ρ. Even at the cusp, the geometry must be Ricci flat, in local coordinates $\partial^\lambda \partial_\lambda g^{\mu\nu} = 0$. For the terms on the diagonal, g_{jj}, we have $\partial_\lambda \partial^\lambda g_{jj} = \partial_\lambda \partial^\lambda (A_{\alpha,j} A^\alpha{}_{,j}) = 0$. The electric field $A^{0,j}$ comes to a cusp. To preserve flatness, the magnetic field $A^{2,j} = A^{0,j}$ in a neighborhood of the cusp. Then $F^{\alpha\beta} F_{\alpha\beta} = 0$ at the cusp and the current j^ν has $j^0 = j^2 \neq 0$ and $j^3 = j^4 = 0$.

The electric field also affects the component $g^{0\nu} = \rho^2 A^{\alpha,0} A_\alpha{}^{,\nu}$. To preserve $\partial^\lambda \partial_\lambda g^{\mu\nu} = 0$ we have the geometric components of $g^{\mu\nu}$ compensate the electric field components.

$$g^{0\nu} = \rho^2 A^{\alpha,0} A_\alpha{}^{,\nu} = \rho^2 (A^{0,0} A_0{}^{,\nu} + \sum_{\alpha \neq 0} A^{\alpha,0} A_\alpha{}^{,\nu}) \tag{11}$$

We have $A^{0,0} = 1$. Therefore the other terms have divergence equal to $\partial_\lambda \partial^\lambda A^{0,\nu}$. Their divergence is the same as the electric charge j^0.

The magnetic field has a cusp with the same location and magnitude as the electric field. We write out the components for the relationship between magnetic field and geometry as

$$g^{2\nu} = \rho^2 A^{\alpha,2} A_\alpha{}^{,\nu} = \rho^2 (A^{2,2} A_2{}^{,\nu} + \sum_{\alpha \neq 2} A^{\alpha,2} A_\alpha{}^{,\nu}) \tag{12}$$

We have $A^{2,2} = 1$. Therefore the other terms have divergence equal to $\partial_\lambda \partial^\lambda A^{2,\nu}$. Their curl is the same as the current j^2.

Ricci flatness on a $S^1 \times P^2$ requires that $\gamma = \rho^2$, which implies waves that circulate around the $S^1 \times P^2$ as in Fig. 3. If the particle is charged then equations 11 and 12 indicate that $A^{\alpha,\nu}$ must be non-zero, the waves advance in phase approaching the particle cusp, as in Fig. 4. In this way, the geometric effect of the electromagnetic fields from $A^{0,\mu}$ and $A^{2,\mu}$ is negated by the particle geometry to make the Ricci curvature flat at the cusp, $\partial^\lambda \partial_\lambda g^{\mu\nu} = 0$. In particular $\partial^\lambda \partial_\lambda g^{0\nu} = 0$ and $\partial^\lambda \partial_\lambda g^{2\nu} = 0$. See Fig. 5

The geometric cusp that is required to flatten the electromagnetic field is the source for a geometric field. The field equation for that geometric field comes from the scalar curvature term R in the Lagrangian. We describe the geometric field in more detail in the next section.

V. GEOMETRIC ENERGY AND MOMENTUM

The Lagrangian is the entropy of the manifold. The manifold is Ricci flat relative to the metric $g_{\mu\nu}$, meaning $\hat{R}^{\mu\nu} = 0$. The geometry of the manifold is underconstrained by this

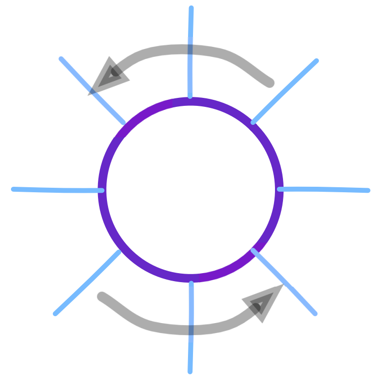

FIG. 3: A neutral $S^1 \times P^2$ with angular momentum has transverse waves that rotate around the particle. The transverse waves have periodic change in the x^4 and x^5 displacement. This is a $\sigma = 0$ slice through a $S^1 \times P^2$. The light blue lines are lines of constant phase in the x^4 and x^5 coordinates with rotation as indicated by the arrows.

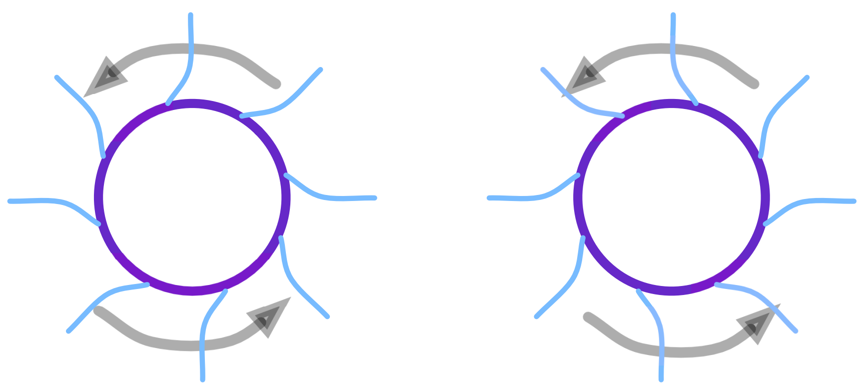

FIG. 4: If the $S^1 \times P^2$ is charged then the electromagnetic field comes to a cusp on the purple circle. The cusp interferes with Ricci flatness. The geometry must compensate to restore $\hat{R}^{\mu\nu} = 0$. To do this, the rotating waves on the particle change phase based on the distance to the cusp. Depending on the sign of the particle charge, the phase either moves forward (as on the left) or backward (as on the right).

flatness, it can wrinkle. Reducing the entropy by pulling the manifold requires force and energy. In the weak field limit, the force and energy are analogous to Hooke's law with $F = -kx$ and $E = (1/2)kx^2$. The scalar curvature relative to the metric $\bar{\eta}_{\mu\nu}$ is R. The entropy of geometric fields is maximized when R is minimized. The Lagrangian of geometric fields is $L = -R$.

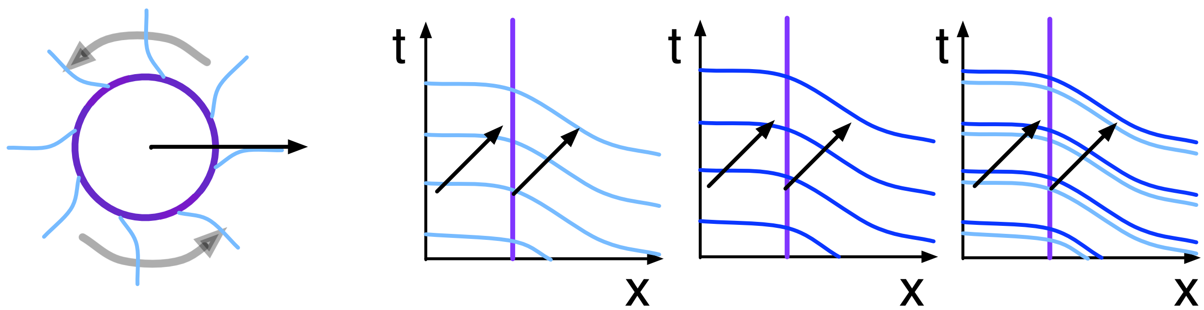

FIG. 5: To see how geometry compensates the electric field, we take a slice of the $S^1 \times P^2$, on the left, in one spatial dimension (as indicated by the arrow) and the time dimension. The light blue lines are lines of constant phase of the waves rotating around the particle. The dark blue lines are lines of constant A^0. Crossing more light blue lines indicates traveling a greater distance, the path has greater change in x^4 and x^5. Crossing more dark blue lines indicates traveling a shorter distance by the metric $g_{\mu\nu} = \rho^2 A^{\alpha,\mu} A_\alpha{}^{,\nu}$ because change of A^0 reduces the effective distance traveled. The number of lines crossed by the arrows indicates the effect on $g_{0\nu}$. If only the electric field contributed to the metric, then $g_{0\nu}$ would change at the cusp in a way that would not be Ricci flat, $\partial_\lambda \partial^\lambda g_{0\nu} \neq 0$. To compensate, the geometry produces an equal but opposite change in $g_{0\nu}$ to restore Ricci flatness.

A. Geometric Fields

When it is necessary to distinguish between the geometric field of gravity and the geometric field that results from the charge cusp, we will refer to the charge-generated field as the geometric charge field.

The geometric charge field on the charged $S^1 \times P^2$ is distinct from the gravitational field. The source for the geometric charge field is the geometric cusp that matches the charge cusp. The source for the gravitational field is the energy-momentum tensor. The geometric charge field does not alter geodesics to produce a gravitational force and it is not additive across multiple particles. The gravitational field is mostly determined by $\bar{\eta}^{00,\nu}$, which is separate from the fields $A^{\alpha,\nu}$ that result from the geometric cusp. However, the gravitational field and the geometric charge field are generated by the same Lagrangian and therefore follow the same field equation. The gravitational field for a steadily rotating mass has a gravitoelectromagnetic field with components E_g and B_g, analogous to the electric and magnetic fields of electromagnetism, that result from the field equation of gravitation. Application of the same field equation to the geometric charge field produces field components E_{gc} and B_{gc}

whose source is the geometric cusp and also obey the same field equations as the electric and magnetic field, respectively. We compose the fields E_{gc} and B_{gc} into a single field $C^{\mu\nu}$. Then the field equation requires that $C^{\mu\nu} = k_C^{1/2}(1/\rho^2)W^{\mu\nu}$, using the field $W^{\mu\nu}$ defined in section IV C and a scalar multiple k_C. This is proportional to the electromagnetic field $F^{\mu\nu} = k_F^{1/2}(1/\rho^2)W^{\mu\nu}$. To compare the momenta of the geometric charge field and the electromagnetic field, we examine the field coupling at the cusp.

B. Field coupling at the cusp

Flatness $\hat{R}^{\mu\nu} = 0$ constrains the shape of the manifold, but because of particle topology the geometry is under-constrained. To see this, use the mapping of the particle

$$X : (\tau, \sigma, \phi) \rightarrow (\tau/(1-\tau), \sigma, \phi, \tau sin(2\sigma), \tau cos(2\sigma)) \tag{13}$$

There are continuous variations of the mapping X and A^α that preserve the metric $g_{\mu\nu}$ therefore the variations also preserve $\hat{R}^{\mu\nu} = 0$. Describing the entropy of the particle's shape is a complicated procedure. However, we can assume that a particle with no field has some maximally entropic geometry and a particle with a field has some other maximally entropic geometry. Comparing the two geometries gives the change in entropy that results from a weak field. If there is no field, then the derivative $A^{\alpha,\nu}$ is non-zero leading up to the geometric cusp and the variation of A^α produces some non-zero P^2 ampliude. The variation in A^α has entropy. If there is charge then Ricci flatness requires that $A^{\alpha,0}A_\alpha{}^{,\nu}$ compensate $A^{0,0}A_0{}^{,\nu}$. This forces change of the A^α geometry as in Fig. 4. By forcing A^α to change in the direction of the S^1 fiber, the effective radius $h(\tau)$ of the P^2 must change by the corresponding amount, which is $A^{0,\mu}$. This affects the circumference of the P^2 by an amount that is proportional to the relation between P^2 circumference and the amplitude at the cusp, $C = 4\pi h(\tau)$. Entropy from geometry obeys Hooke's Law $S = (1/2)kx^2$. Therefore the entropy varies as $S = (1/2)k(4\pi h(\tau))^2$ with displacement $x = 4\pi h(\tau)$. This is larger than the displacement of the $A^{0,\mu}$ field by a factor of 4π and has a larger impact on entropy by a factor of $(4\pi)^2 = 16\pi^2$.

C. Degrees of freedom

Both the electromagnetic field and geometric field are derivatives of A^ν. To compare the microstates in electromagnetism and geometry, we compare the number of degrees of freedom that are part of the electromagnetic field and the geometric field. The field $F^{\mu\nu}$ is determined by derivatives in directions parallel to the manifold M. This gives variation of A^ν in 4 directions, for 4 degrees of freedom. There are 6 total directions that the potential A^ν can vary in $C^{\mu\nu}$. However, the constraint $\partial_\nu A^\nu = -2$ constrains one direction, without loss of generality we can assume constraint of A^0. This leaves a total of 5 degrees of freedom. Therefore, the ratio of degrees of freedom in the geometric field compared to the electromagnetic field is 5/4.

D. Geometric fields summary

A charged particle has an electromagnetic field with a source of divergence that is a field cusp. The field cusp alone does not satisfy Ricci flatness. To make the metric Ricci flat, the geometry compensates the field at the cusp. This produces a field E_{gc} satisfying the same field equations as the electric field E and a field B_{gc} satisfying the same field equations as the magnetic field B. The two fields E_{gc} and B_{gc} combine into a single tensor $C^{\mu\nu}$. The relative magnitude of the momentum from $C^{\mu\nu}$ compared to $F^{\mu\nu}$ comes from the relative effect on entropy. The field $C^{\mu\nu}$ has a displacement that is 4π times larger than the corresponding displacement in $F^{\mu\nu}$, which increases the momentum by a factor of $16\pi^2$. The field $C^{\mu\nu}$ has 5 degrees of freedom and $F^{\mu\nu}$ has 4 degrees of freedom. Therefore the geometric field has momentum that is $(5/4)16\pi^2 = 20\pi^2$ times larger than the electromagnetic field momentum.

E. Energy and momentum on the electron

For a Lagrangian $L = (\rho^4/\gamma)W^{\alpha\beta}W_{\alpha\beta}$ the energy momentum tensor is

$$T^{\mu\nu} = \Big(\frac{\rho^4}{\gamma}\Big)\Big(- W^{\alpha\mu}W_\alpha^{\ \nu} + (1/4)\bar{\eta}^{\mu\nu}W^{\alpha\beta}W_{\alpha\beta}\Big) \tag{14}$$

On an embedded manifold the term $\bar{\eta}^{\mu\nu}W^{\alpha\beta}W_{\alpha\beta}$ can have momentum if there are transverse waves. The transverse waves have velocity $\bar{\eta}^{0\nu}$ that transports energy of the form $\bar{\eta}^{0\nu}W^{\alpha\beta}W_{\alpha\beta}$. However, Lagrangian optimization and energy conservation imply that the effect on $T^{\mu\nu}$ is equivalent whether the field is of the form $-W^{\alpha\mu}W_\alpha^{\ \nu}$ or $\bar{\eta}^{\mu\nu}W^{\alpha\beta}W_{\alpha\beta}$. For that reason we calculate as if all field momentum is of the form $W^{\alpha\mu}W_\alpha^{\ \nu}$.

The spin angular momentum of a particle is $S = \int \vec{r} \times T^{0\mu} dV$ where r is the displacement vector from the axis of rotation. To find the momentum we use the approximation

$$T^{0\mu} = -F^{\alpha\mu}F_\alpha^{\ 0} - C^{\alpha\mu}C_\alpha^{\ 0} \tag{15}$$

where $C^{\alpha\mu}C_\alpha^{\ 0} = (5/4)16\pi^2 F^{\alpha\mu}F_\alpha^{\ 0}$. The total momentum is therefore

$$T^{0\mu} = -F^{\alpha\mu}F_\alpha^{\ 0} - (5/4)16\pi^2 F^{\alpha\mu}F_\alpha^{\ 0} = -(1+20\pi^2)F^{\alpha\mu}F_\alpha^{\ 0} \tag{16}$$

Using the field $W^{\mu\nu}$ defined in section IV C, we have

$$T^{0\mu} = -k_F(1+20\pi^2)(1/\rho^4)W^{\alpha\mu}W_\alpha^{\ 0} \tag{17}$$

VI. MATHEMATICAL MODEL

The mathematical model is available as a Mathematica™ notebook file on **www.knotphysics.net**. The mathematical model generates ρ and the field $W^{\mu\nu}$ and then integrates the angular momentum $r \times p$ to get the spin angular momentum S using the formula

$$S = \int rT^{0\mu}dV = \int r(1+20\pi^2)F^{\alpha\mu}F_\alpha^{\ 0}dV = \int r(1+20\pi^2)(1/\rho^4)k_F W^{\alpha\mu}W_\alpha^{\ 0}dV \tag{18}$$

The metric volume dV is scaled by ρ^4 and is Lorentz contracted by $\gamma = (1/\rho)^2$. Comparing to the flat volume dV_f, we have $dV = \rho^2 dV_f$. Therefore

$$S = \int r(1+20\pi^2)(1/\rho^4)k_F W^{\alpha\mu}W_\alpha^{\ 0}dV = \int r(1+20\pi^2)(1/\rho^4)k_F W^{\alpha\mu}W_\alpha^{\ 0}(\rho^2 dV_f) \tag{19}$$

$$S = \int r(1+20\pi^2)(1/\rho^2)k_F W^{\alpha\mu}W_\alpha^{\ 0}dV_f \tag{20}$$

A. Inputs and calculations

We describe the inputs and calculations done in the Mathematica™ notebook. We use polar coordinates (r, z, phi) and toroidal coordinates (tau, sigma, phi) for the mathematical model. All calculations are done in the (r,z) plane, which is also the (tau, sigma) plane, and extended to ϕ by symmetry.

The following is a list of the functions and inputs in the mathematical model. All functions use toroidal coordinate inputs. The vector valued functions produce vectors that are in toroidal coordinates. Because toroidal coordinates are orthogonal, the cross product $E \times B$ gives the same result whether it is calculated in toroidal coordinates or Cartesian coordinates.

- **radius**: the S^1 radius of the particle. An input. Can be any positive value.
- **charge**: the particle charge. An input. Can be any non-zero value.
- **rco(tau,sigma,phi)**= the value of the r coordinate in cylindrical coordinates. This is necessary to calculate angular momentum $r \times p$
- **dVtor(tau,sigma)**=the volume measure dV_f using toroidal coordinates. This is used to calculate the integral of angular momentum in toroidal coordinates.
- **harmonic(tau,sigma)**=a harmonic function whose source is the circle tau=∞
- **rho(tau,sigma)**=the conformal factor ρ of quantum branch weight $w = \rho^4$
- **DivField(tau,sigma)**=a vector field that has zero curl and divergence that is zero everywhere except tau=∞
- **ScaledDivField(tau,sigma)**= a scaling of DivField so that the divergence is equal to the charge. On flat space, this would be the electric field.
- **StokesCurrent(tau,sigma)**=a function to assist the calculation of CurlField.
- **CurlField(tau,sigma)**=a vector field that has zero divergence and curl is zero everywhere except tau=∞.
- **ScaledCurlField(tau,sigma)**=a scaling of CurlField so that the curl is equal to the current. On flat space, this would be the magnetic field.

We then use **ScaledDivField** and **ScaledCurlField** as components of the flat space field tensor $k_F^{1/2} W^{\mu\nu}$. The product $k_F W^{\alpha\mu} W_\alpha{}^0$ is the cross product of **ScaledDivField** and **ScaledCurlField**. Then the spin angular momentum is $S = \int r(1 + 20\pi^2)(1/\rho^2) k_F W^{\alpha\mu} W_\alpha{}^0 dV_f$, using the above calculated functions. The inverse fine structure constant estimate is $\alpha^{-1} = (8\pi S)(1/q^2)$.

B. Momentum calculations

The calculation for α^{-1} is approximately $\alpha^{-1}_{calc} = 136.854$ compared to the experimental value $\alpha^{-1}_{exp} = 137.036$. The error is -0.18 and the percent error is 0.13%. The calculation was performed, in part, by comparison of the entropy in electromagnetic fields to the entropy in geometry. That entropy comparison follows from comparing the way that fields affect the microstates of electromagnetism and geometry. However, the microstates that were considered all had flat topology. The microstates associated with virtual fermions were not counted. For example, the electric field affects the microstates of virtual electron/positron pairs by increasing the probability that pair production will put the charged particles in opposite alignment to the field. That reduces the entropy of the particle pairs. To increase the accuracy of the calculation, one would need to develop additional methods of accounting for the effect on entropy from electromagnetic and geometric fields. Comparing those entropic effects would then give a comparison relative to particle charge.

VII. PARTICLE DYNAMICS

A. Hadrons

Linked $S^1 \times P^2$ are quarks. Hadrons consist of multiple linked $S^1 \times P^2$. Each hadron preserves Ricci flatness. With multiple linked $S^1 \times P^2$, this may require relative motion of the quarks to compensate the geometric effects of their topology. If the quantum branch weight ρ comes to a single cusp, then there is a single time-invariant Ricci flat solution for ρ. In that case, the metric is isometric to the solution for the electron, to within a scaling of the particle radius. The electromagnetic and geometric fields on the particle therefore also scale in the same way. To show that quarks can have charges of magnitude 1/3 and 2/3 requires showing that the field cusp associated with $\partial_\lambda \partial^\lambda g^{0\nu} = \partial_\lambda \partial^\lambda (A^{0,0} A_0{}^{,\nu})$ can separate according to dimension on each quark. For example, a particular quark might have geometry corresponding to divergence $\partial^2 \partial_2$ but no other direction, giving it a charge of 1/3. Naturally, this is impossible for an individual $S^1 \times P^2$. However, the quarks cannot be separated and it is possible that their masses equilibrate according to this type of geometry.

B. Particle radii

Knot physics models particles as knots in the spacetime manifold. The knots have geometry with non-zero radii. The geometry helps to explain, for example, spin angular momentum and charge. Collisions between particles can give estimates of their charge radii. When particles collide, the distance at which the force between the particles is no longer $1/r^2$ is the distance at which one can assume there is a geometric component to the interaction. With hadrons, the collision data implies a particle with non-zero radius. The electron appears to be nearly point-like in collisions. To explain the difference, we use Ricci flatness to show how electrons and other particles behave at rest and during collisions.

1. Lepton charge radius

For multiple electrons, the quantum branch weight ρ is not additive, it is multiplicative. To find the quantum branch weight for multiple electrons, we use harmonic functions $\kappa_i(x)$, each associated with a particular electron. Then the quantum branch weight for a single electron, as above, is $\rho(x) = e^{\kappa_i(x)}$. The quantum branch weight for multiple electrons is

$$\rho(x) = exp(\sum_i \kappa_i(x)) \tag{21}$$

As two electrons approach each other, their charge is conserved and ρ scales multiplicatively. The result is that the radius of each electron shrinks. As the distance between electrons goes to zero, the radius of each electron also goes to zero. A similar effect occurs for any charged lepton.

2. Hadron charge radius

Hadrons consist of multiple linked charged $S^1 \times P^2$. The proximity of the $S^1 \times P^2$ has the same effect as for multiple electrons; the quantum branch weight scales as $\rho(x) = exp(\sum_i \kappa_i(x))$. If the quarks are closer to each, then ρ is larger and therefore the S^1 radius of each quark is less. The relative motion of the quarks introduces relativistic terms to the Ricci flatness equation that may also affect the S^1 radius. Introducing another particle by collision would have less relative effect on the apparent charge radius for a hadron than for a charged lepton.

C. Higher generations and the Koide formula

The fine structure constant compares charge to spin angular momentum. Spin angular momentum is a degree of freedom, which means it has one $\hbar$ of action. Because of the relationship between charge and angular momentum, there is a relationship between charge and $\hbar$. This relationship holds for all elementary fermions and, in modified form, for quarks. The quantum phase frequency is another degree of freedom and it also has one $\hbar$ of action, of form $E = \hbar\omega$. An elementary fermion has a topology given by the mapping

$$X : (\tau, \sigma, \phi) \rightarrow (\tau/(1-\tau), \sigma, \phi, \tau sin(2\sigma + n\phi), \tau cos(2\sigma + n\phi)) \tag{22}$$

where n determines the generation. Including the time component introduces the σ rotation frequency ω

$$X : (\tau, \sigma, \phi) \rightarrow (\tau/(1-\tau), \sigma, \phi, \tau sin(2\sigma + n\phi + \omega t), \tau cos(2\sigma + n\phi + \omega t)) \tag{23}$$

From the time derivatives dX/dt we see that rotation from the phase frequency is equivalent to rotation in ϕ because of the term $n\phi$. Therefore, the angular momentum and the phase frequency ω are related through n. Angular momentum is fixed at $\hbar/2$, but the energy $E = \hbar\omega$ can change. Because the geometries of the particle generations are related in a simple way, it is possible that their masses would also be related in a simple way. The Koide formula relates the three lepton masses m_e, m_μ, and m_τ by the following formula

$$\frac{m_e + m_\mu + m_\tau}{(m_e^{1/2} + m_\mu^{1/2} + m_\tau^{1/2})^2} \approx 2/3 \tag{24}$$

The formula is accurate to within the experimental accuracy for the particle masses. It may be that the formula is not random and that the geometry of particle generations could be used to derive the relationship between particle masses.

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[1] C. Ellgen **www.knotphysics.net** Knot physics: Spacetime in co-dimension 2