IV	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	X	X	X	X	X	X	X	X	X	X	X								
2	X	X	X	X	X	X	X	X											
3	X	X	X	X	X	X	X												
4	X	X	X	X															
5	X	X	X	X															
6	X	X	X	X															
7	X	X	X	X	X	X	X	X	X	X	X								
8	X	X	X	X	X	X	X	X											

I.1.1

Identify and sketch the set of points satisfying.

(a)	$\lvert z-1-i\rvert = 1$	(f)	$0 < \operatorname{Im} z < \pi$
(b)	$1 < \lvert 2z-6\rvert < 2$	(g)	$-\pi < \operatorname{Re} z < \pi$
(c)	$\lvert z-1\rvert^2 + \lvert z+1\rvert^2 < 8$	(h)	$\lvert\operatorname{Re} z\rvert < \lvert z\rvert$
(d)	$\lvert z-1\rvert + \lvert z+1\rvert \leq 2$	(i)	$\operatorname{Re}(iz+2) > 0$
(e)	$\lvert z-1\rvert < \lvert z\rvert$	(j)	$\lvert z-i\rvert^2 + \lvert z+i\rvert^2 < 2$

Solution

Let $z = x+iy$, where $x, y \in \mathbb{R}$.

(a) Circle, centre $1+i$, radius 1.

$$|z-1-i| = 1 \Leftrightarrow |(x-1) + i\,(y-1)| = 1 \Leftrightarrow (x-1)^2 + (y-1)^2 = 1^2$$

(b) Annulus with centre 3, inner radius 1/2, outer radius 1.

$$1 < |2z-6| < 2 \Leftrightarrow 1 < 2\,|z-3| < 2 \Leftrightarrow$$
$$\Leftrightarrow 1/2 < |z-3| < 1 \Leftrightarrow (1/2)^2 < (x-3)^2 + y^2 < 1^2$$

(c) Disk, centre 0, radius $\sqrt{3}$.

$$|x+iy-1|^2 + |x+iy+1|^2 < 8 \Leftrightarrow$$
$$\Leftrightarrow (x-1)^2 + y^2 + (x+1)^2 + y^2 < 8 \Leftrightarrow x^2 + y^2 < \left(\sqrt{3}\right)^2$$

(d) Interval $[-1, 1]$.

$$|z-1| + |z+1| \leq 2 \Leftrightarrow \sqrt{(x-1)^2 + y^2} \leq 2 - \sqrt{(x+1)^2 + y^2} \Leftrightarrow$$
$$\Leftrightarrow \left(\sqrt{(x-1)^2 + y^2}\right)^2 \leq \left(2 - \sqrt{(x+1)^2 + y^2}\right)^2 \Leftrightarrow$$
$$\Leftrightarrow \sqrt{(x+1)^2 + y^2} \leq x+1 \Leftrightarrow \left(\sqrt{(x+1)^2 + y^2}\right)^2 \leq (x+1)^2 \Leftrightarrow y = 0$$

Now, take $y = 0$ in the inequality, and compute the three intervals

$$\begin{array}{llcccc} x < -1, & \text{then} & |x-1| + |x+1| & = & -(x-1) - (x+1) & = & -2x \geq 2, \\ -1 \leq x \leq 1 & \text{then} & |x-1| + |x+1| & = & -(x-1) + (x+1) & = & 2 \leq 2 \\ x > 1, & \text{then} & |x-1| + |x+1| & = & (x-1) + (x+1) & = & 2x \geq 2. \end{array}$$

(e) Half–plane $x > 1/2$.

$$|z-1| < |z| \Leftrightarrow |z-1|^2 < |z|^2 \Leftrightarrow |x+iy-1|^2 < |x+iy|^2 \Leftrightarrow$$
$$\Leftrightarrow (x-1)^2 + y^2 < x^2 + y^2 \Leftrightarrow x > 1/2$$

(f) Horizontal strip, $0 < y < \pi$.
(g) Vertical strip, $-\pi < x < \pi$.
(h) $\mathbb{C} \backslash \mathbb{R}$.

$$|\mathrm{Re}\, z| < |z| \Leftrightarrow |\mathrm{Re}\,(x+iy)|^2 < |x+iy|^2 \Leftrightarrow x^2 < x^2 + y^2 \Leftrightarrow |y| > 0$$

(i) Half plane $y < 2$.

$$\mathrm{Re}\,(iz+2) > 0 \Leftrightarrow \mathrm{Re}\,(i\,(x+iy)+2) > 0 \Leftrightarrow -y+2 > 0 \Leftrightarrow y < 2$$

(j) Empty set.

$$|z-i|^2 + |z+i|^2 < 2 \Leftrightarrow |x+iy-i|^2 + |x+iy+i|^2 < 2 \Leftrightarrow$$
$$\Leftrightarrow x^2 + (y-1)^2 + x^2 + (y+1)^2 < 2 \Leftrightarrow x^2 + y^2 < 0$$

I.1.2
Verify from the definitions each of the identities
(a) $\overline{z+w} = \bar{z} + \bar{w}$ (b) $\overline{zw} = \bar{z}\bar{w}$ (c) $|\bar{z}| = |z|$ (d) $|z|^2 = z\bar{z}$
Draw sketches to illustrate (a) and (c).

Solution
Substitute $z = x + iy$ and $w = u + iv$, and use the definitions.
(a)

$$\overline{z+w} = \overline{(x+iy)+(u+iv)} = \overline{(x+u)+(y+v)\,i} = \\ = (x+u)-(y+v)\,i = (x-iy)+(u-iv) = \bar{z}+\bar{w}.$$

(b)

$$\overline{zw} = \overline{(x+iy)\,(u+iv)} = \overline{(xu-yv)+(xv+yu)\,i} = \\ = (xu-yv)-(xv+yu)\,i = (x-iy)\,(u-iv) = \bar{z}\bar{w}.$$

(c)

$$|\bar{z}| = \left|\overline{x+iy}\right| = |x-iy| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |x+iy| = |z|\,.$$

(d)

$$|z|^2 = |x+iy|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2 = \\ = x^2 - i^2y^2 = (x+iy)\,(x-iy) = z\bar{z}.$$

I.1.3

Show that the equation $|z|^2 - 2\,\mathrm{Re}\,(\bar{a}z) + |a|^2 = \rho^2$ represents a circle centered at a with radius ρ.

Solution

Let $z = x + iy$ and $a = \alpha + i\beta$, we have

$$\begin{aligned}
&|z|^2 - 2\,\mathrm{Re}\,(\bar{a}z) + |a|^2 = \\
&= x^2 + y^2 - 2\,\mathrm{Re}\,((\alpha - i\beta)(x + iy)) + \alpha^2 + \beta^2 = \\
&= x^2 + y^2 - 2\,\mathrm{Re}\,((\alpha x + \beta y) + i(\alpha y - \beta x)) + \alpha^2 + \beta^2 = \\
&= x^2 + y^2 - 2(\alpha x + \beta y) + \alpha^2 + \beta^2 = \\
&= (x - \alpha)^2 + (y - \beta)^2 .
\end{aligned}$$

Thus the equation $|z|^2 - 2\,\mathrm{Re}\,(\bar{a}z) + |a|^2 = \rho^2$, becomes

$$(x - \alpha)^2 + (y - \beta)^2 = \rho^2,$$

which is the equation for a circle of radius ρ centered at (α, β), which in complex notation is the point $a = \alpha + i\beta$.

I.1.4
Show that $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$, and sketch the set of points for which equality holds.

Solution
Apply triangle inequality to $z = \operatorname{Re} z + i \operatorname{Im} z$, to obtain $|z| \leqslant |\operatorname{Re} z| + |\operatorname{Im} z|$. Now set $z = x + iy$, and see then equality holds

$$|z| = |\operatorname{Re} z| + |\operatorname{Im} z| \Leftrightarrow \sqrt{x^2 + y^2} = \sqrt{x^2} + \sqrt{y^2} \Leftrightarrow \left(\sqrt{x^2 + y^2}\right)^2 = \left(\sqrt{x^2} + \sqrt{y^2}\right)^2 \Leftrightarrow$$
$$\Leftrightarrow x^2 + y^2 = \left(\sqrt{x^2}\right)^2 + \left(\sqrt{x^2}\right)^2 + 2\sqrt{x^2}\sqrt{y^2} \Leftrightarrow x^2y^2 = 0 \Leftrightarrow x = 0 \text{ or } y = 0.$$

Equality holds only when z is real or pure imaginary, which are all the points on the real and imaginary axis.

I.1.5
Show that

$$|\mathrm{Re}\, z| \leq |z| \ , \qquad |\mathrm{Im}\, z| \leq |z| \, .$$

Show that

$$|z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}\,(z\bar{w}) \, .$$

Use this to prove the triangle inequality $|z+w| \leq |z| + |w|$**.**

Solution
Let $z = x + iy$. Then since the square root function is monotone, we have

$$\begin{aligned} |\mathrm{Re}\, z| &= |x| = \sqrt{x^2} \leqslant \sqrt{x^2+y^2} = |z| \, , \\ |\mathrm{Im}\, z| &= |y| = \sqrt{y^2} \leqslant \sqrt{x^2+y^2} = |z| \, . \end{aligned}$$

Now for z, w we have

$$\begin{aligned} &|z+w|^2 = \\ &\qquad = (z+w)\,\overline{(z+w)} = (z+w)\,(\bar{z}+\bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} = \\ &\qquad\qquad = |z|^2 + 2\,\mathrm{Re}\,(z\bar{w}) + |w|^2 \, , \end{aligned}$$

where we have used the fact that $2\,\mathrm{Re}\,(z\bar{w}) = z\bar{w} + \overline{z\bar{w}} = z\bar{w} + \overline{\bar{w}z} = z\bar{w} + w\bar{z}$. We use both of the above facts and the trivial identities $|z| = |\bar{z}|$ and $|zw| = |z|\,|w|$ to prove the triangle inequality for z, w. We have

$$\begin{aligned} |z+w|^2 &= |z|^2 + 2\,\mathrm{Re}\,(z\bar{w}) + |w|^2 \leq |z|^2 + 2\,|\mathrm{Re}\,(z\bar{w})| + |w|^2 \leq \\ &\leq |z|^2 + 2\,|z\bar{w}| + |w|^2 = |z|^2 + 2\,|z|\,|w| + |w|^2 = (|z| + |w|)^2 \, . \end{aligned}$$

The desired inequality now follows by taking square root of both sides.

I.1.6
For fixed $a \in \mathbb{C}$, show that $|z-a| \,/\, |1-\bar{a}z| = 1$ if $|z| = 1$ and $1-\bar{a}z \neq 0$.

Solution
If $|z| = 1$, then $|\bar{z}| = 1$ and $z\bar{z} = 1$. Use this and get

$$|z-a| = |z-a|\,|\bar{z}| = |z\bar{z} - a\bar{z}| = |1-a\bar{z}| = |1-\bar{a}z|\,.$$

We have

$$\frac{|z-a|}{|1-\bar{a}z|} = 1,$$

as was to be shown.

I.1.7

Fix $\rho > 0$, $\rho \neq 1$, and fix $z_0, z_1 \in \mathbb{C}$. Show that the set of z satisfying $|z - z_0| = \rho |z - z_1|$ is a circle. Sketch it for $\rho = \frac{1}{2}$ and $\rho = 2$, with $z_0 = 0$ and $z_1 = 1$. What happens when $\rho = 1$?

Solution

Recall that a circle in $\mathbb{R}^2$ centered at (a, b) with radius r is given by the equation

$$(x-a)^2 + (y-b)^2 = r^2.$$

We manipulate the equation

$$|z - z_0| = \rho |z - z_1|.$$

The solutions set of the equation above remains the same if we square both sides,

$$|z - z_0|^2 = \rho^2 |z - z_1|^2.$$

Let $z = x + iy$, $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$. Thus our equation becomes

$$(x - x_0)^2 + (y - y_0)^2 = \rho^2 \left((x - x_1)^2 + (y - y_1)^2\right).$$

Expanding the squares, and grouping terms, we have

$$\left(1-\rho^2\right) x^2 - 2\left(x_0 - \rho^2 x_1\right) x + \left(x_0^2 - \rho^2 x_1^2\right) + \left(1-\rho^2\right) y^2 - 2\left(y_0 - \rho^2 y_1\right) y + \left(y_0^2 - \rho^2 y_1^2\right) = 0$$

Dividing both sides by $(1 - \rho^2)$, we have

$$x^2 - 2\frac{(x_0 - \rho^2 x_1)}{1-\rho^2} x + \frac{(x_0^2 - \rho^2 x_1^2)}{1-\rho^2} + y^2 - 2\frac{(y_0 - \rho^2 y_1)}{1-\rho^2} y + \frac{(y_0^2 - \rho^2 y_1^2)}{1-\rho^2} = 0$$

Now complete the squares for both the x and y terms. Recall that

$$x^2 - 2ax + b = (x-a)^2 - a^2 + b.$$

So we have

$$\left(x-\frac{x_0-\rho^2 x_1}{1-\rho^2}\right)^2+\frac{\left(1-\rho^2\right)\left(x_0^2-\rho^2 x_1^2\right)-\left(x_0-\rho^2 x_1\right)^2}{\left(1-\rho^2\right)^2}+$$
$$+\left(y-\frac{y_0-\rho^2 y_1}{1-\rho^2}\right)^2+\frac{\left(1-\rho^2\right)\left(y_0^2-\rho^2 y_1^2\right)-\left(y_0-\rho^2 y_1\right)^2}{\left(1-\rho^2\right)^2}=0.$$

This becomes

$$\left(x-\frac{x_0-\rho^2 x_1}{1-\rho^2}\right)^2+\left(y-\frac{y_0-\rho^2 y_1}{1-\rho^2}\right)^2=$$
$$=\frac{\left(x_0-\rho^2 x_1\right)^2+\left(y_0-\rho^2 y_1\right)^2-\left(1-\rho^2\right)\left(x_0^2+y_0^2-\rho^2\left(x_1^2+y_1^2\right)\right)}{\left(1-\rho^2\right)^2}=$$
$$=\frac{\rho^2\left(\left(x_1-x_0\right)^2+\left(y_1-y_0\right)^2\right)}{\left(1-\rho^2\right)^2},$$

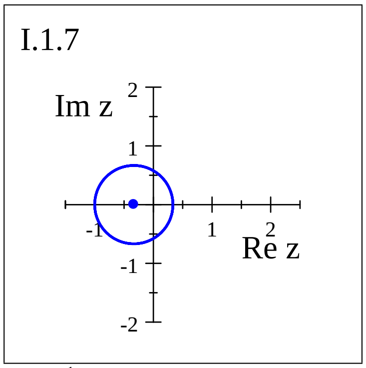
which is the equation for a circle. If $z_0=0$, and $z_1=1$, we have

$$\left(x-\frac{-\rho^2}{1-\rho^2}\right)^2+y^2=\left(\frac{\rho}{1-\rho^2}\right)^2.$$

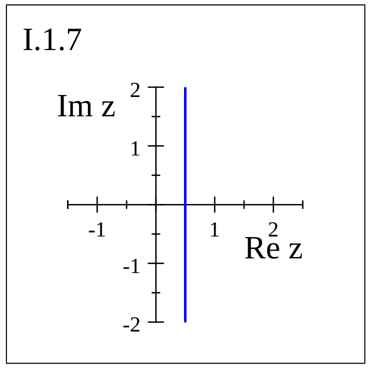
When $\rho=\frac{1}{2}$ we have a circle of radius $\frac{2}{3}$ centered at $\left(-\frac{1}{3},0\right)$, and when $\rho=2$, we have a circle of radius $\frac{2}{3}$ centered at $\left(\frac{4}{3},0\right)$. When $\rho=1$, we have the equation

$$\left|z-z_0\right|=\left|z-z_1\right|,$$

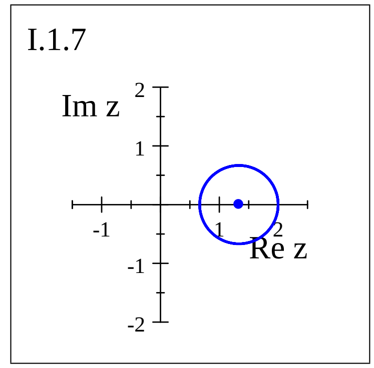
which is the line bisecting the two points. When $z_0=0$, $z_1=1$, this is the line $\operatorname{Re} z=\frac{1}{2}$.



$\rho = \frac{1}{2}$



$\rho = 1$



$\rho = 2$

I.1.8

Let $p(z)$ be a polynomial of degree $n \geq 1$ and let $z_0 \in \mathbb{C}$. Show that there is a polynomial $h(z)$ of degree $n-1$ such that $p(z) = (z-z_0)\,h(z) + p(z_0)$. In particular, if $p(z_0) = 0$, then $p(z) = (z-z_0)\,h(z)$.

Solution

Set

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0.$$

and

$$h(z) = b_{n-1} z^{n-1} + b_{n-2} z^{n-2} + b_{n-3} z^{n-3} + \ldots + b_2 z^2 + b_1 z + b_0.$$

We equate coefficients in the polynomial identity $p(z) = (z-z_0)\,h(z) + p(z_0)$, and get

$$\begin{gathered}
a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_2 z^2 + a_1 z + a_0 = \\
= (z-z_0)\left(b_{n-1} z^{n-1} + b_{n-2} z^{n-2} + b_{n-3} z^{n-3} + \ldots + b_2 z^2 + b_1 z + b_0\right) + p(z_0) = \\
= b_{n-1} z^n + b_{n-2} z^{n-1} + b_{n-3} z^{n-2} + \ldots + b_2 z^3 + b_1 z^2 + b_0 z + p(z_0) - \\
- b_{n-1} z_0 z^{n-1} - b_{n-2} z_0 z^{n-2} - b_{n-3} z_0 z^{n-3} - \ldots - b_2 z_0 z^2 - b_1 z_0 z - b_0 z_0 = \\
b_{n-1} z^n + (b_{n-2} - b_{n-1} z_0)\, z^{n-1} + (b_{n-3} - b_{n-2} z_0)\, z^{n-2} + (b_{n-4} - b_{n-3} z_0)\, z^{n-3} + \cdots \\
\cdots + (b_2 - b_3 z_0)\, z^3 + (b_1 - b_2 z_0)\, z^2 + (b_0 - b_1 z_0)\, z + p(z_0) - b_0 z_0.
\end{gathered}$$

Equate and solve for the b_j´s in terms of a_j´s.

$$\left\{\begin{array}{lcl} a_n & = & b_{n-1} \\ a_k & = & b_{k-1} - b_k z_0, \quad 0 \leq k \leq n-1 \\ a_0 & = & p(z_0) - b_0 z_0 \end{array}\right. \Rightarrow \left\{\begin{array}{lcl} b_{n-1} & = & a_n \\ b_k & = & \sum\limits_{i=0}^{n-k-1} a_{k+1+i} z_0^i, \quad 0 \leq k \leq n-2 \\ p(z_0) & = & \sum\limits_{i=0}^{n} a_i z_0^i \end{array}\right.$$

Proof by induction on degree n of $p(z)$, set

$$p(z) = a_n z^n + a_{n-1} z^{n-1} \ldots + a_0,$$

where $a_n \neq 0$.

Fix z_0 and write

$$p(z) = a_n (z - z_0) z^{n-1} + r(z),$$

where $\deg r(z) \leq n - 1$.
By using the induction hypothesis, we can assume that

$$r(z) = q(z)(z - z_0) + c,$$

where $\deg q(z) \leq \deg r(z)$. Then

$$p(z) = \left(a_n z^{n-1} + q(z)\right)(z - z_0) + c = h(z)(z - z_0) + c$$

Since $\deg q(z) \leq n - 2$, $\deg r(z) \leq n - 1$.
Plug in z_0, get $p(z_0) = c$.

I.1.9

Find the polynomial $h(z)$ in the preceding exercise for the following choices of $p(z)$ and z_0

(a) $p(z) = z^2$ **and** $z_0 = i$

(b) $p(z) = z^3 + z^2 + z$ **and** $z_0 = -1$

(c) $p(z) = 1 + z + z^2 + \cdots + z^m$ **and** $z_0 = -1$

Solution

From the preceding exercise we have

$$p(z) = (z - z_0)\, h(z) + p(z_0).$$

We solve for $h(z)$ then

$$h(z) = \frac{p(z) - p(z_0)}{z - z_0}.$$

(a)

We have that $p(z) = z^2$ and $z_0 = i$, thus $p(z_0) = p(i) = -1$.
Thus

$$h(z) = \frac{p(z) - p(z_0)}{z - z_0} = \frac{z^2 + 1}{z - i} = z + i,$$

and

$$z^2 = (z - i)(z + i) - 1.$$

(b)

We have that $p(z) = z^3 + z^2 + z$ and $z_0 = -1$, thus $p(z_0) = p(-1) = -1$.
Thus

$$h(z) = \frac{p(z) - p(z_0)}{z - z_0} = \frac{z^3 + z^2 + z + 1}{z + 1} = \frac{(z+1)(z^2+1)}{z+1} = z^2 + 1,$$

and

$$z^3 + z^2 + z = (z + 1)\left(z^2 + 1\right) - 1.$$

(c)

We have that $p(z) = 1 + z + z^2 + \cdots + z^m$ and $z_0 = -1$, thus

$$p\left(z_0\right)=p\left(-1\right)=\begin{cases}0, & m \text{ odd}\\ 1, & m \text{ even}\end{cases}$$

Thus

$$h\left(z\right)=\frac{p\left(z\right)-p\left(z_0\right)}{z-z_0}=$$
$$=\begin{cases}\frac{\left(z^{m-1}+z^{m-3}\cdots+z^2+1\right)(z+1)}{z+1}=z^{m-1}+z^{m-3}+\cdots+z^2+1 & \text{if } m \text{ is odd}\\ \frac{\left(z^{m-1}+z^{m-3}\cdots+z^3+z\right)(z+1)+1}{z+1}=z^{m-1}+z^{m-3}+\ldots+z^3+z & \text{if } m \text{ is even}\end{cases}$$

and

$$z^m+z^{m-1}+\cdots z^2+z+1=$$
$$=\begin{cases}\left(z+1\right)\left(z^{m-1}+z^{m-3}+\cdots+z^2+1\right) & \text{if } m \text{ is odd}\\ \left(z+1\right)\left(z^{m-1}+z^{m-3}+\cdots+z^3+z\right)+1 & \text{if } m \text{ is even}\end{cases}$$

I.1.10
Let $q(z)$ be a polynomial of degree $m \geq 1$. Show that any polynomial $p(z)$ can be expressed in the form

$$p(z) = h(z)\,q(z) + r(z),$$

where $h(z)$ and $r(z)$ are polynomials and the degree of the remainder $r(z)$ is strictly less than m.
Hint. **Proceed by induction on the degree of $p(z)$. The resulting method is called the division algorithm.**

Solution
First suppose $p(z)$ is the zero polynomial. (So, the degree of $p(z)$ is $-\infty$.) The degree of $r(z)$ must be less than the degree of $q(z)$. If $h(z) \neq 0$, it follows that the degree of $h(z)\,q(z)$ is greater than the degree of $r(z)$. This then implies that $h(z)\,q(z)+r(z) \neq 0$. So, $h(z)\,q(z)+r(z) = 0$ implies that $h(z) = 0$, and thus $r(z) = 0$. So, the polynomials are $h(z) = 0$ and $r(z) = 0$, and these polynomials are the only ones that satisfy both conditions.

Now assume that the division algorithm is true for all polynomials $p(z)$ of degree less than n. (Where $n \geq 0$.) If the degree of $q(z)$ is greater than the degree of $p(z)$, and $h(z)$ is nonzero, then $h(z)\,q(z)+r(z)$ has degree greater than $p(z)$. So, if the degree of $q(z)$ is greater than the degree of $p(z)$, then $h(z) = 0$ and thus $r(z) = p(z)$. This proves both existence and uniqueness of $h(z)$ and $r(z)$, in this case.

Now, suppose that the degree of $q(z)$ is less than or equal to the degree of $p(z)$. Set

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

and

$$q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0,$$

where $a_n \neq 0$, $b_m \neq 0$ and $m \leq n$.
Let

$$p_1(z) = \frac{a_n}{b_m} z^{n-m} q(z) - p(z),$$

then

$$p_1(z) = \frac{a_n}{b_m} z^{n-m} \left(b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0\right) - \left(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\right).$$

The monomials of degree n cancel, and therefore $p_1(z)$ is a polynomial of degree at most $n-1$. It follows, by assumption, that $p_1(z) = h_1(z)\, q(z) + r_1(z)$, where $h_1(z)$ and $r_1(z)$ are the unique polynomials satisfying the conditions above.
Let

$$h(z) = \frac{a_n}{b_m} z^{n-m} - h_1(z),$$

and let

$$r(z) = -r_1(z).$$

Then

$$\begin{aligned} h(z)\, q(z) + r(z) &= \\ &= \left(\frac{a_n}{b_m} z^{n-m} - h_1(z)\right) q(z) - r_1(z) = \\ &= \frac{a_n}{b_m} z^{n-m} q(z) - \left(h_1(z)\, q(z) + r_1(z)\right) = \\ &= \frac{a_n}{b_m} z^{n-m} q(z) - p_1(z) = \\ &= p(z). \end{aligned}$$

Thus given $p(z)$ and $q(z)$, there exist polynomials $h(z)$ and $r(z)$ satisfying the above two conditions.

I.1.11
Find the polynomials $h(z)$ and $r(z)$ in the preceding exercise for $p(z) = z^n$ and $q(z) = z^2 - 1$.

Solution
Require

$$z^n = h(z)\left(z^2 - 1\right) + r(z), \qquad \deg r(z) \le 1.$$

If n is even

$$z^n = \left(z^{n-2} + z^{n-4} + z^{n-6} + \cdots + z^2 + 1\right)\left(z^2 - 1\right) + 1.$$

If n is odd

$$z^n = \left(z^{n-2} + z^{n-4} + z^{n-6} + \cdots + z^3 + z\right)\left(z^2 - 1\right) + z.$$

Thus

$$h(z) = \begin{cases} z^{n-2} + z^{n-4} + z^{n-6} + \cdots + z^2 + 1, & \text{if } n \text{ is even,} \\ z^{n-2} + z^{n-4} + z^{n-6} + \cdots + z^3 + z, & \text{if } n \text{ is odd.} \end{cases},$$

and

$$r(z) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ z, & \text{if } n \text{ is odd.} \end{cases}.$$

I.2.1
Express all values of the following expressions in both polar and Cartesian coordinates, and plot them.

(a)	$\sqrt{i}$	**(c)**	$\sqrt[4]{-1}$	**(e)**	$(-8)^{1/3}$	**(g)**	$(1+i)^8$
(b)	$\sqrt{i-1}$	**(d)**	$\sqrt[4]{i}$	**(f)**	$(3-4i)^{1/8}$	**(h)**	$\left(\frac{1+i}{\sqrt{2}}\right)^{25}$

Solution
(a)

$$\sqrt{i} = \left\{\left(e^{i(\pi/2+2k\pi)}\right)^{1/2} = e^{i(\pi/4+k\pi)},\ k=0,1\right\} = \left\{e^{i\pi/4}, e^{i5\pi/4}\right\} = \left\{\pm\left(1+i\right)/\sqrt{2}\right\}.$$

(b)

$$\begin{aligned}\sqrt{i-1} &= \left\{\left(\sqrt{2}e^{i(3\pi/4+2k\pi)}\right)^{1/2} = 2^{1/4}e^{i(3\pi/8+k\pi)},\ k=0,1\right\} = \\ &= \left\{2^{1/4}e^{i3\pi/8}, 2^{1/4}e^{i11\pi/8}\right\} = \left\{\pm 2^{1/4}\left(\cos\left(3\pi/8\right)+i\sin\left(3\pi/8\right)\right)\right\}.\end{aligned}$$

(c)

$$\begin{aligned}\sqrt[4]{-1} &= \left\{\left(e^{i(\pi+2k\pi)}\right)^{1/4} = e^{i(\pi/4+k\pi/2)},\ k=0,1,2,3\right\} = \\ &= \left\{e^{i\pi/4}, e^{i3\pi i/4}, e^{i5\pi/4}, e^{i7\pi/4}\right\} = \left\{(1\pm i)/\sqrt{2}, \left(-1\pm i\right)/\sqrt{2}\right\}.\end{aligned}$$

(d)

$$\begin{aligned}\sqrt[4]{i} = \left\{\left(e^{i(\pi/2+2k\pi)}\right)^{1/4} e^{i(\pi/8+k\pi/2)},\ k=0,1,2,3 = \left\{e^{i\pi/8}, e^{i5\pi/8}, e^{i9\pi/8}, e^{i13\pi/8}\right\}\right\} = \\ = \left\{\pm\left(\cos\left(\pi/8\right)+i\sin\left(\pi/8\right)\right), \pm\left(\cos\left(5\pi/8\right)+i\sin\left(5\pi/8\right)\right)\right\}.\end{aligned}$$

(e)

$$\begin{aligned}(-8)^{1/3} = \left\{\left(2^3e^{i(\pi+2k\pi)}\right)^{1/3} = 2e^{i(\pi/3+2k\pi/3)},\ k=0,1,2\right\} = \\ = \left\{2e^{i\pi/3}, 2e^{i\pi}, 2e^{i5\pi/3}\right\} = \left\{1+i\sqrt{3}, -2, 1-i\sqrt{3}\right\}.\end{aligned}$$

(f)

Draw figure and get $\tan\theta_0 = -4/3 \Rightarrow \theta_0 = \tan^{-1}\left(-\frac{4}{3}\right)$.

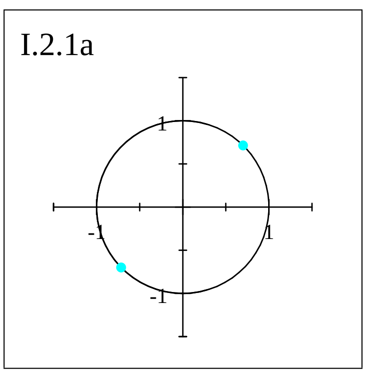
$$(3-4i)^{1/8} = \left\{\left(5e^{i(\theta_0+2k\pi)}\right)^{1/8} = 5^{1/8}e^{i(\theta_0/8+k\pi/4)},\ k=0,1,\ldots,7\right\} =$$
$$= \left\{\pm 5^{1/8}e^{i(\theta_0/8)}, \pm 5^{1/8}e^{i(\theta_0/8+\pi/4)}, \pm 5^{1/8}e^{i(\theta_0/8+\pi/2)}, \pm 5^{1/8}e^{i(\theta_0/8+3\pi/4)}\right\} =$$
$$\left\{\pm 5^{1/8}\left(\cos\left(\theta_0/8\right) + i\sin\left(\theta_0/8\right)\right), \pm 5^{1/8}\left(\cos\left(\theta_0/8+\pi/4\right) + i\sin\left(\theta_0/8+\pi/4\right)\right),\right.$$
$$\left.\pm 5^{1/8}\left(\cos\left(\theta_0/8+\pi/2\right) + i\sin\left(\theta_0/8+\pi/2\right)\right), \pm 5^{1/8}\left(\cos\left(\theta_0/8+3\pi/4\right) + i\sin\left(\theta_0/8+3\pi/4\right)\right)\right\}.$$

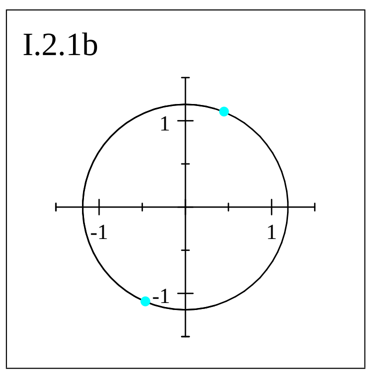
(g)

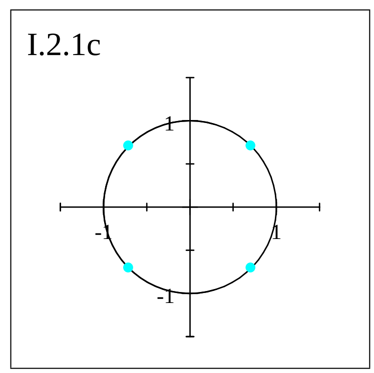
$$(1+i)^8 = \left(\sqrt{2}e^{i(\pi/4+2k\pi)}\right)^8 = 16e^{i(2\pi+16k\pi)} = 16e^{i2\pi} = 16.$$

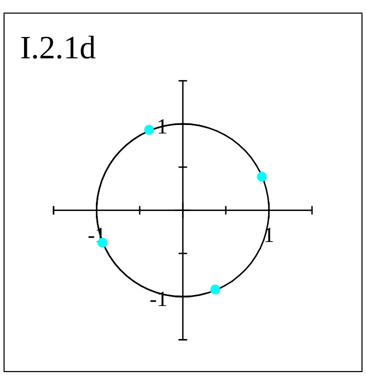
(h)

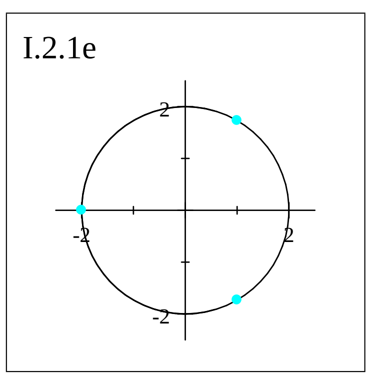
$$\left(\frac{1+i}{\sqrt{2}}\right)^{25} = \left(e^{i(\pi/4+2k\pi)}\right)^{25} = e^{i(25\pi/4+50k\pi)} = e^{i(\pi/4+6\pi+50k\pi)} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}.$$

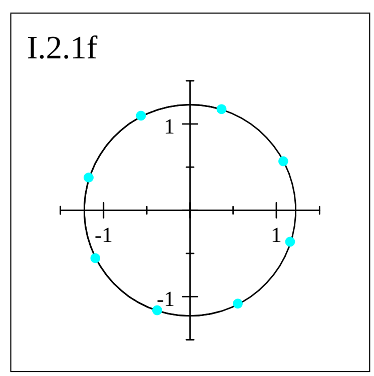


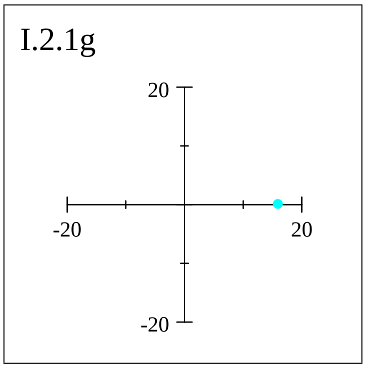


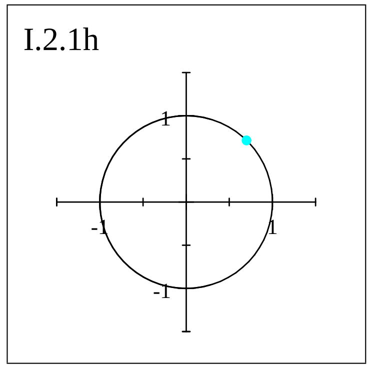








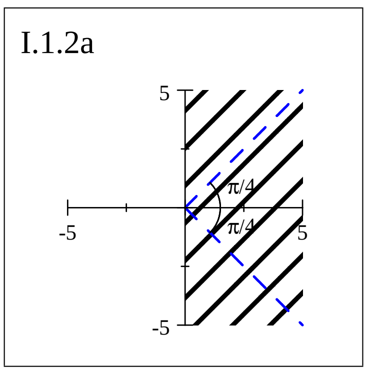

I.2.1g
20
-20
20
-20

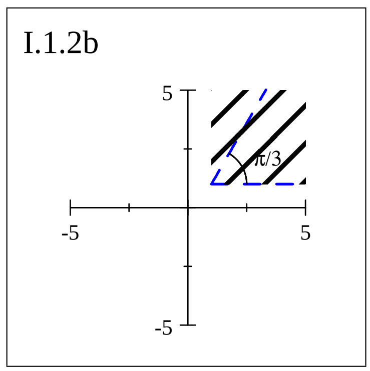

I.2.1h
1
-1
1
-1

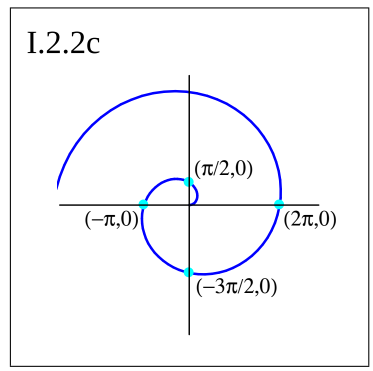
I.2.2
Sketch the following sets

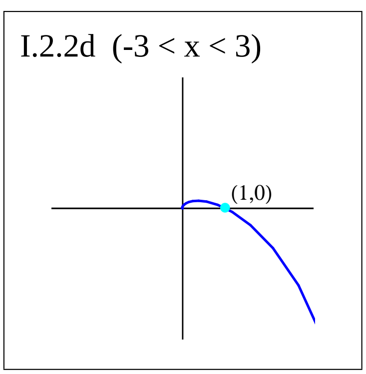
(a)	$\lvert \arg z \rvert < \pi/4$	**(c)**	$\lvert z \rvert = \arg z$
(b)	$0 < \arg(z - 1 - i) < \pi/3$	**(d)**	$\log \lvert z \rvert = -2 \arg z$

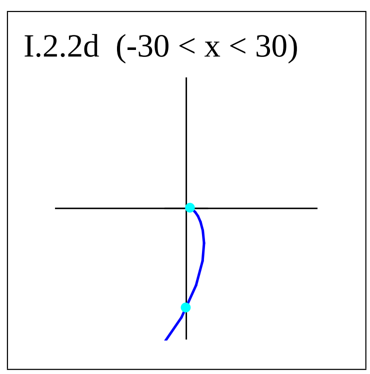
Solution
a) Sector. b) Sector. c) Is a spiral curve starting at 0, spiraling to ∞. (d) Is a spiral curve, spiraling to 0, and to ∞.

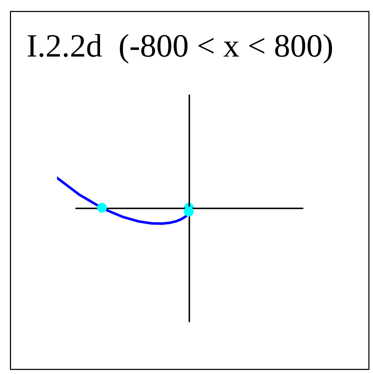






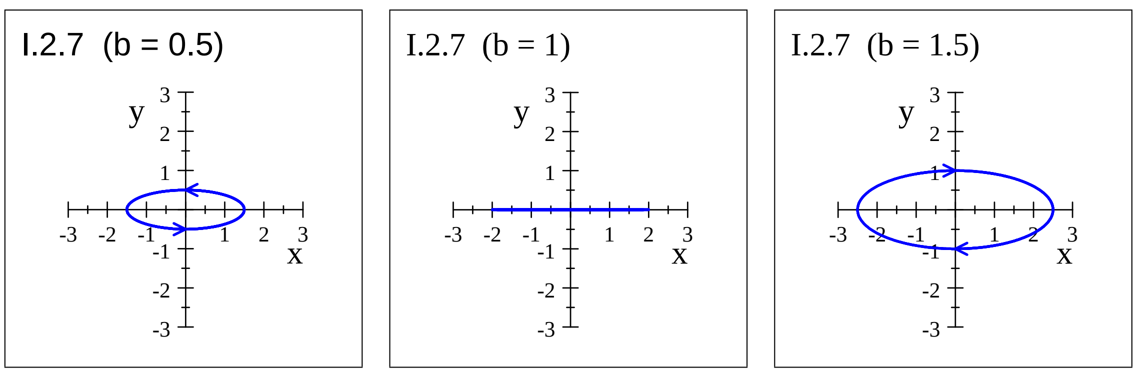






I.2.3
For a fixed complex number b, sketch the curve $\left\{e^{i\theta}+be^{-i\theta}:0\leq\theta\leq 2\pi\right\}$. Differentiate between the cases $|b|<1$, $|b|=1$ and $|b|<1$.
Hint. First consider the case $b>0$, and then reduce the general case to this case by a rotation.

Solution
For $0<b<1$, an ellipse $x^2/\left(1+b\right)^2+y^2/\left(1-b\right)^2=1$, traversed in positive direction with increasing θ. For $b=1$, an interval $[-2,2]$. For $1<b<+\infty$, an ellipse traversed in negative direction. For $b=\rho e^{i\varphi}$, express equation as $e^{i\varphi/2}\left(e^{i(\theta-\varphi/2)}+\rho e^{-i(\theta-\varphi/2)}\right)$ to see that curve is rotate of ellipse or interval by $\varphi/2$.



I.2.4
For which n is i an n-th root of unity?

Solution.
i is an n^{th} root of unity for $i^n = 1$, $n = 4k$, $k = 1, 2, 3, \ldots$

I.2.5
For $n \geq 1$, show that
(a) $1 + z + z^2 + \cdots + z^n = \left(1 - z^{n+1}\right) / \left(1 - z\right), \qquad z \neq 1$
(b) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\theta/2}$.

Solution
(a)
Set

$$s_n = 1 + z + z^2 + \cdots + z^n,$$

and multiply s_n with z and have

$$zs_n = z + z^2 + z^3 + \cdots + z^{n+1}.$$

Now subtract zs_n from the sum s_n, and have

$$s_n\left(1 - z\right) = 1 - z^{n+1}.$$

If $z \neq 1$ we have after division by $1 - z$,

$$s_n = \frac{1 - z^{n+1}}{1 - z}.$$

(b)
Apply (a) to $z = e^{i\theta}$ and to $z = e^{-i\theta}$,

$$\begin{aligned} 1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}, \\ 1 + e^{-i\theta} + e^{-i2\theta} + \cdots + e^{-in\theta} &= \frac{1 - e^{-i(n+1)\theta}}{1 - e^{-i\theta}}. \end{aligned}$$

Add the identities, and use the definitions of sine and cosine.

$$
\begin{aligned}
&2\left(1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta\right)= \\
&\qquad=\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}+\frac{1-e^{-i(n+1)\theta}}{1-e^{-i\theta}}= \\
&\qquad=\frac{\left(1-e^{i(n+1)\theta}\right)e^{-i\theta/2}}{e^{-i\theta/2}-e^{i\theta/2}}+\frac{\left(1-e^{-i(n+1)\theta}\right)e^{i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}}= \\
&\qquad=-\frac{1}{2i}\frac{e^{-i\theta/2}\left(1-e^{i(n+1)\theta}\right)}{\sin\left(\theta/2\right)}+\frac{1}{2i}\frac{e^{i\theta/2}\left(1-e^{-i(n+1)\theta}\right)}{\sin\left(\theta/2\right)}= \\
&=\frac{1}{2i}\left[\frac{e^{i\theta/2}-e^{-i\theta/2}+e^{i\left(n+\frac{1}{2}\right)\theta}-e^{-i\left(n+\frac{1}{2}\right)\theta}}{\sin\left(\theta/2\right)}\right]=\frac{\sin\left(\theta/2\right)+\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\theta/2\right)}= \\
&\qquad=1+\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\theta/2\right)}.
\end{aligned}
$$

Divide both sides with 2 and get

$$
1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\frac{1}{2}+\frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\theta/2}.
$$

I.2.6

Fix $n \geq 1$. Show that the $n-$ th roots of unity $w_0, \ldots, w_{n-1}$ satisfy

(a) $(z - w_0)(z - w_1)\ldots(z - w_{n-1}) = z^n - 1$.

(b) $w_0 + \cdots + w_{n-1} = 0$ if $n \geq 2$.

(c) $w_0 \cdots w_{n-1} = (-1)^{n-1}$.

(d) $\sum\limits_{j=0}^{n-1} w_j^k = \begin{cases} 0, & 1 \leq k \leq n-1, \\ n, & k = n. \end{cases}$

Solution

(a)

Let $w_0, \ldots, w_{n-1}$ be the $n-$ th roots of unity then $w_j = e^{2\pi ji/n}$, and w_j are a roots of $z^n - 1$ since

$$w_j^n - 1 = \left(e^{2\pi ji/n}\right)^n - 1 = e^{2\pi ji} - 1 = 0.$$

By the fundamental theorem of algebra, and since the root are simple and the coefficient for the z^n term is 1 it follows that

$$z^n - 1 = (z - w_0)(z - w_1)\cdots(z - w_{n-1}).$$

(b)

Using exercise (a) and multiply the factors in the product we get

$$z^n - 1 = z^n - (w_0 + w_1 + \cdots w_{n-1}) z^{n-1} + c_2 z^{n-2} + \cdots + (-1)^n w_0 w_1 \cdot \cdots \cdot w_{n-1}.$$

By identification of the coefficients we see that

$$w_0 + w_1 + \cdots + w_{n-1} = 0 \quad \text{if} \quad n \geq 2.$$

(c)

From (b) follows that

$$-1 = (-1)^n w_0 w_1 \cdots w_{n-1} \Leftrightarrow w_0 w_1 \cdots w_{n-1} = (-1)^{n-1}.$$

(d)

If $1 \leq k \leq n-1$

$$\sum_{j=0}^{n-1} w_j^k = \sum_{j=0}^{n-1} \left(e^{2\pi ji/n}\right)^k = \sum_{j=0}^{n-1} \left(e^{2\pi ki/n}\right)^j = \frac{1 - \left(e^{2\pi ki/n}\right)^n}{1 - e^{2\pi ki/n}} = \frac{1 - e^{2\pi ki}}{1 - e^{2\pi ki/n}} = 0.$$

If $k = n$

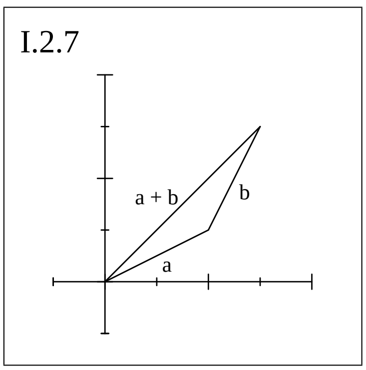
$$\sum_{j=0}^{n-1} w_j^k = \sum_{j=0}^{n-1} \left(e^{2\pi j i/n}\right)^n = \sum_{j=0}^{n-1} e^{2\pi j i} = \sum_{j=0}^{n-1} 1 = n.$$

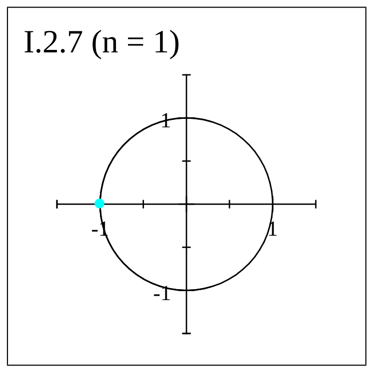
I.2.7
Fix $R > 1$ and $n \geq 1$, $m \geq 0$. Show that

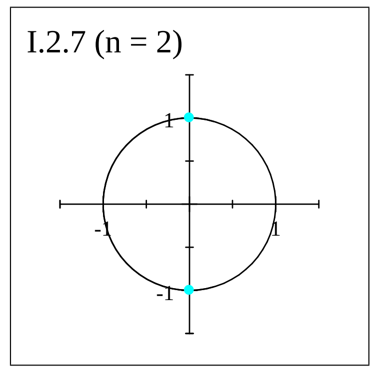
$$\left|\frac{z^m}{z^n+1}\right| \leq \frac{R^m}{R^n-1}, \qquad |z| = R.$$

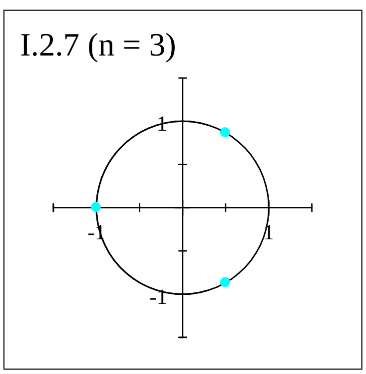
Sketch the set where equality holds. Hint. See (1.1) p.2.

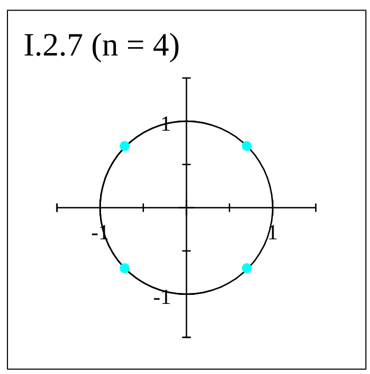
Solution

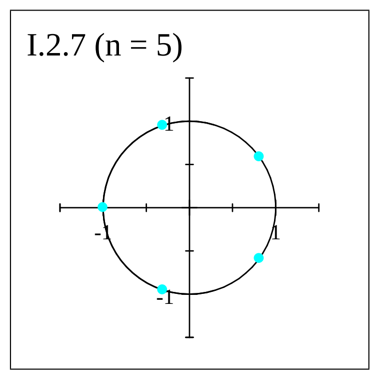












We use that $|z-w| \geq |z| - |w|$ see (1.1) on page 2 in CA, and have that

$$|z^n + 1| \geqslant |z^n| - 1 = R^n - 1,$$

where the last equaltity is due to that $|z| = R$.
Some rearrangement gives

$$\frac{1}{|z^n+1|} \leq \frac{1}{(R^n-1)},$$

and mulitplication by $|z^m| = R^m$ gives

$$\left|\frac{z^m}{z^n+1}\right| \leq \frac{R^m}{R^n-1}, \qquad |z| = R.$$

For equality, we must have

$$|z^n + 1| = R^n - 1,$$

because $|z|^m = R^m$.
We rearrarange this equality to

$$|-1| + |z^n + 1| = |z^n|$$

and we conclude that -1 and z^n+1 must lie on the same ray. (We have used the fact that $|a|+|b| = |a+b|$ implies that a, b lie on the same ray, see first figure I.2.7)
If

$$z = Re^{i\theta}$$

then

$$z^n = R^n e^{in\theta}.$$

Since $R > 1$, we require that $e^{in\theta} = -1$ thus

$$z = w_k R e^{\pi i/n},$$

where w_k is an $n-$ th root of unity.
If

$$z = w_k R e^{\pi i/n},$$

then

$$z^n + 1 = -R^n + 1,$$

and

$$\left|\frac{z^m}{z^n+1}\right| = \frac{R^m}{R^n-1}.$$

I.2.8

Show that $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\cos\theta\sin\theta$ using de Moivre's formulae. Find formulae for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Solution

Let $\theta \in \mathbb{R}$ be given. Then by de Moivre's formulae (on page 8 in CA) we have for all $n \in \mathbb{Z}$

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

Hence for $n = 2$ we get

$$\begin{aligned}
&\cos 2\theta + i\sin 2\theta = \\
&\qquad = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + i2\cos\theta\sin\theta - \sin^2\theta = \\
&\qquad\qquad = \left(\cos^2\theta - \sin^2\theta\right) + i\left(2\cos\theta\sin\theta\right),
\end{aligned}$$

then, by setting the real and imaginary parts equal to each other we obtain

$$\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\sin 2\theta &= 2\cos\theta\sin\theta.
\end{aligned}$$

Similarly, applying de Moivre's formulae for $n = 4$ we get (using the Binomial Theorem)

$$\begin{aligned}
\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 = \\
&= \cos^4\theta + i4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - i4\cos\theta\sin^3\theta + \sin^4\theta = \\
&= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right),
\end{aligned}$$

then, by setting the real and imaginary parts equal to each other we obtain

$$\begin{aligned}
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\
\sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.
\end{aligned}$$

as required.

I.3.1

Sketch the image under the spherical projection of the following sets on the sphere

(a) the lower hemisphere $Z \leq 0$

(b) the polar cap $\frac{3}{4} \leq Z \leq 1$

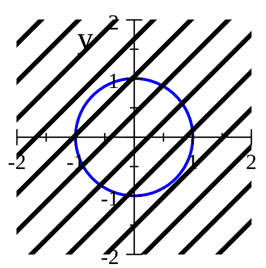
(c) lines of lattitude $X = \sqrt{1-Z^2}\cos\theta$, $Y = \sqrt{1-Z^2}\sin\theta$, for Z fixed and $0 \leq \theta \leq 2\pi$

(d) lines of longitude $X = \sqrt{1-Z^2}\cos\theta$, $Y = \sqrt{1-Z^2}\sin\theta$, for θ fixed and $-1 \leq Z \leq 1$

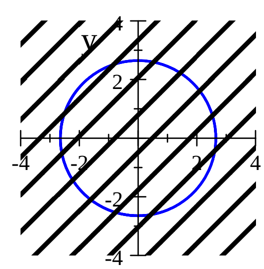
(e) the spherical cap $A \leq X \leq 1$, with center lying on the equator, for fixed A. Separate into cases, according to various ranges of A.

Solution

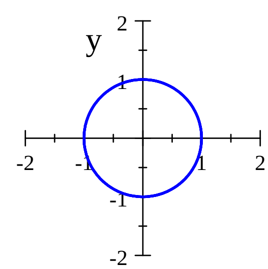
I.3.1a



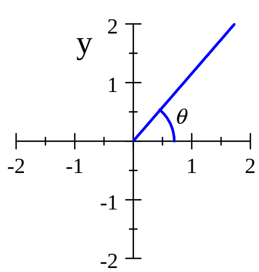
I.3.1b



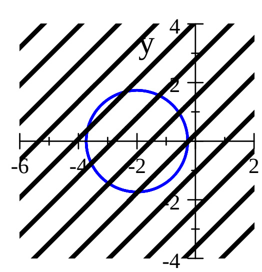
I.3.1c



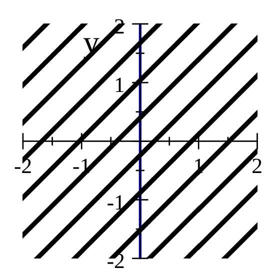
I.3.1d

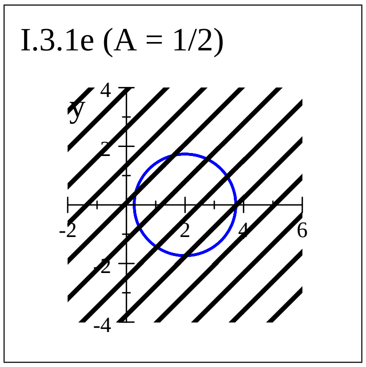


I.3.1e (A = -1/2)



I.3.1e (A = 0)





(a)
Image is the unit disk.
(b)
Set $Y = 0$ and $Z = 3/4$ in $X^2 + Y^2 + Z^2 = 1$, we have $X = \sqrt{7}/4$. We have the formula $x = X/(1-Z)$, thus $x = \sqrt{7}$. Image is the exterior of a disk, with centre 0 and radius $\sqrt{7}$.
(c) Image is the disk, with centre 0 and radius $\sqrt{1+z}/\sqrt{1-z}$. We have that $X = \sqrt{1-Z^2}\cos\theta$ and $Y = \sqrt{1-Z^2}\sin\theta$, we can take the radius to image for $\theta = 0$, thus $r = \frac{\sqrt{1-Z^2}}{1-Z} = \frac{\sqrt{1+Z}}{\sqrt{1-Z}}$.
(d) Image is lines of longitude θ issuing from 0.
(e)
Case 1: $-1 < A < 0$
Image is the exterior of the disk centered at $1/A$ with radius $\sqrt{1-A^2}/|A|$.
Case 2: $A = 0$
Image is the left half–plane.
Case 3: $0 < A < 1$
Image is a disk centered at $1/A$ with radius $\sqrt{1-A^2}/|A|$.
Set $Y = 0$ then Z goes from $-\sqrt{1-A^2}$ to $\sqrt{1-A^2}$. We have the formula $x = X/(1-Z)$ thus x goes from $A/\left(1+\sqrt{1-A^2}\right) = \left(1-\sqrt{1-A^2}\right)/A$ to $A/\left(1-\sqrt{1-A^2}\right) = \left(1+\sqrt{1-A^2}\right)/A$.

I.3.2
If the point P on the sphere corresponds to z under the stereographic projection, show that the antipodal point $-P$ on the sphere corresponds to $-1/\bar{z}$.

Solution
For $z = x + iy$ the corresponding point on the sphere under transformation given on p. 12 in CA is given by (X, Y, Z) where

$$\begin{aligned} X &= \frac{2x}{|z|^2+1}, \\ Y &= \frac{2y}{|z|^2+1}, \\ Z &= \frac{|z|^2-1}{|z|^2+1}. \end{aligned}$$

For $z \neq 0$, we have

$$-\frac{1}{\bar{z}} = -\frac{1}{x-iy} = -\frac{x+iy}{(x-iy)(x+iy)} = -\frac{x+iy}{x^2+y^2} = -\frac{x}{|z|^2} - i\frac{y}{|z|^2}$$

which corresponds to the point on the sphere given by (X', Y', Z') where

$$\begin{aligned} X' &= \frac{\frac{-2x}{|z|^2}}{\left|\frac{1}{\bar{z}}\right|^2+1} = \frac{-2x}{|z|^2+1} = -X, \\ Y' &= \frac{\frac{-2y}{|z|^2}}{\left|\frac{1}{\bar{z}}\right|^2+1} = \frac{-2y}{|z|^2+1} = -Y, \\ Z' &= \frac{\left|\frac{1}{\bar{z}}\right|^2-1}{\left|\frac{1}{\bar{z}}\right|^2+1} = -\frac{|z|^2-1}{|z|^2+1} = -Z. \end{aligned}$$

Hence, the map $(X, Y, Z) \longmapsto (-X, -Y, -Z)$ on the sphere corresponds to $z \longmapsto -1/\bar{z}$ of $\mathbb{C}$.

I.3.3

Show that as z travers a small circle in the complex plane in the positive (counterclockwise) direction, the corresponding point P on the sphere traverses a small circle in the negative (clockwise) direction with respect to someone standing at the center of the circle and with body outside the sphere. (Thus the stereographic projection is orientation reversing, as a map from the sphere with orientation determined by the unit outer normal vector to the complex plane with the usual orientation.)

Solution

Draw the picture. Or argue as follows. The orientation of the image circle is the same for all circles on the sphere orientated so that N is outside the circle. This can be seen by moving one circle continuously to the other, and seeing that the image circles moves continuously. Thus we need to shrink it only for the equator of the sphere oriented as indicated ($\circlearrowright$), if the South Pole is inside it. And the image circle is the unity and positive direction of the unit circle ($\circlearrowleft$) is the converse. So the orientation of the image is clockwise (negative).

I.3.4
Show that a rotation of the sphere of $180°$ about the X-axis corresponds under stereographic projection to the inversion $z \longmapsto 1/z$ of $\mathbb{C}$.

Solution
For $z = x + iy$ the corresponding point on the sphere under transformation given on p. 12 in CA is given by (X, Y, Z) where

$$\begin{aligned} X &= \frac{2x}{|z|^2 + 1}, \\ Y &= \frac{2y}{|z|^2 + 1}, \\ Z &= \frac{|z|^2 - 1}{|z|^2 + 1}. \end{aligned}$$

For $z \neq 0$, we have

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{x}{|z|^2} - i\frac{y}{|z|^2}$$

which corresponds to the point on the sphere given by (X', Y', Z') where

$$\begin{aligned} X' &= \frac{\frac{2x}{|z|^2}}{\left|\frac{1}{z}\right|^2 + 1} = \frac{2x}{|z|^2 + 1} = X, \\ Y' &= \frac{\frac{-2y}{|z|^2}}{\left|\frac{1}{z}\right|^2 + 1} = \frac{-2y}{|z|^2 + 1} = -Y, \\ Z' &= \frac{\frac{1}{|z|^2} - 1}{\frac{1}{|z|^2} + 1} = -\frac{|z|^2 - 1}{|z|^2 + 1} = -Z. \end{aligned}$$

Hence, the map $(X, Y, Z) \longmapsto (X, -Y, -Z)$ on the sphere corresponds to inversion $z \longmapsto 1/z$ of $\mathbb{C}$. Moreover, the map $(X, Y, Z) \longmapsto (X, -Y, -Z)$ is given by rotation of the sphere by $180°$ about the X-axis.

I.3.5

Suppose $(x, y, 0)$ is the spherical projection of (X, Y, Z). Show that the product of the distances from the north pole N to (X, Y, Z) and from N to $(x, y, 0)$ is 2. What is the situation when (X, Y, Z) lies on the equator on the sphere?

Solution

The distance from from the north pole $N = (0, 0, 1)$ to (X, Y, Z) is

$$\sqrt{X^2 + Y^2 + (Z-1)^2},$$

and the distance from the north pole $N = (0, 0, 1)$ to $(x, y, 0) = \left(\frac{X}{1-Z}, \frac{Y}{1-Z}, 0\right)$ is

$$\sqrt{\frac{X^2}{(1-Z)^2} + \frac{Y^2}{(1-Z)^2} + 1}.$$

The product of distances is

$$\left(X^2 + Y^2 + (Z-1)^2\right)^{1/2} \left(\frac{X^2}{(1-Z)^2} + \frac{Y^2}{(1-Z)^2} + 1\right)^{1/2} = \frac{X^2 + Y^2 + (Z-1)^2}{1-Z} = 2.$$

When (X, Y, Z) lie on the equator, the product is simply the square of the distance from N to a point on the equator. By the Pythagorean Law, this is $1 + 1 = 2$.

I.3.6
We define the chordal distance $d(z,w)$ between two points $z,w \in \mathbb{C}^*$ to be the length of the straight line segment joining to the points P and Q on the unit sphere whose stereographic projections are z and w, respectively.
(a) Show that the chordal distance is a metric, that is, it is symmetric, $d(z,w) = d(w,z)$; satisfies the triangle inequality $d(z,w) \leq d(z,\zeta) + d(\zeta,w)$; and $d(z,w) = 0$ if and only if $z = w$.
(b) Show that the chordal distance from z to w is given by

$$d(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, \qquad z,w \in \mathbb{C}.$$

(c) What is $d(z,\infty)$? Remark. The expression for $d(z,w)$ shows that infinitesimal arc length corresponding to the chordal metric is given by

$$d\sigma(z) = \frac{2ds}{1+|z|^2},$$

where $ds = |dz|$ is the usual Euclidean infinitesimal arc length. The infinitesimal arc length $d\sigma(z)$ determines another metric, the spherical metric $\sigma(z,w)$, on the extended complex plane. See Section IX.3.

Solution
(a)
Follows from fact that Euclidean distance in $\mathbb{R}^3$ is a metric on the sphere.
(b)
Set $z = x_1 + iy_1, w = x_2 + iy_2$, we have

$$\begin{aligned}
|z-w|^2 &= \\
&= (z-w)\overline{(z-w)} = (z-w)(\overline{z}-\overline{w}) = \\
&= z\overline{z} - z\overline{w} - \overline{z}w + w\overline{w} = |z|^2 + |w|^2 - z\overline{w} - \overline{z}w = \\
&= |z|^2 + |w|^2 - (x_1+iy_1)(x_2-iy_2) - (x_1-iy_1)(x_2+iy_2) = \\
&= |z|^2 + |w|^2 - 2x_1x_2 - 2y_1y_2 \ . \qquad (1)
\end{aligned}$$

We take square of distance between P and Q

$$\begin{aligned}
d(z,w)^2 &= (X-X')^2+(Y-Y')^2+(Z-Z')^2 = \\
&= X^2+Y^2+Z^2+X'^2+Y'^2+Z'^2-2(XX'+YY'+ZZ') = \\
&= 2-2\left(\frac{4x_1x_2+4y_1y_2+\left(|z|^2-1\right)\left(|w|^2-1\right)}{\left(|z|^2+1\right)\left(|w|^2+1\right)}\right) = \\
&= 2\left(\frac{\left(|z|^2+1\right)\left(|w|^2+1\right)-4x_1x_2-4y_1y_2-\left(|z|^2-1\right)\left(|w|^2-1\right)}{\left(|z|^2+1\right)\left(|w|^2+1\right)}\right) = \\
&= 2\left(\frac{2|z|^2+2|w|^2-4x_1x_2-4y_1y_2}{\left(|z|^2+1\right)\left(|w|^2+1\right)}\right) \overset{(1)}{=} \frac{4|z-w|^2}{\left(|z|^2+1\right)\left(|w|^2+1\right)}.
\end{aligned}$$

Taking the positive square root we have

$$d(z,w)=\frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, \qquad z,w\in\mathbb{C}.$$

(c)

$$d(z,\infty)=\lim_{w\to\infty}\frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}=\lim_{w\to\infty}\frac{2|z/w-1|}{\sqrt{1+|z|^2}\sqrt{1/|w|^2+1}}=\frac{2}{\sqrt{1+|z|^2}}.$$

I.3.7

Consider the sphere of radius $\frac{1}{2}$ in $(X, Y, Z)-$ space, resting on the $(X, Y, 0)-$ plane, with south pole at the origin $(0, 0, 0)$ and north pole at $(0, 0, 1)$. We define a stereographic projection of the sphere onto the complex plane as before, so that corresponding points (X, Y, Z) and $z \sim (x, y, 0)$ lie on the same line through the north pole. Find the equations for $z = x + iy$ in terms of X, Y, Z, and the equations for X, Y, Z in terms of z. What is the corresponding formula for the chordal distance? Note. In this case, the equation of the sphere is $X^2 + Y^2 + \left(Z - \frac{1}{2}\right)^2 = \frac{1}{4}$.

Solution

The line through $P = (X, Y, Z)$ and $N = (0, 0, 1)$ is on the form $N + t\,(P - N)$, and it meats the $xy-$ plane when

$$(x, y, 0) = (0, 0, 1) + t\left((X, Y, Z) - (0, 0, 1)\right) = \left(tX, tY, 1 + t\,(Z - 1)\right).$$

By simultaneous equations we have, and solve for X, Y and Z

$$\begin{cases} x = tX \\ y = tY \\ 1 + t\,(Z - 1) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} X = \frac{x}{t} \\ Y = \frac{y}{t} \\ Z = 1 - \frac{1}{t} \end{cases}$$

We solve for the t that is a point on the sphere $X^2 + Y^2 + \left(Z - \frac{1}{2}\right)^2 = \frac{1}{4}$ we have

$$\frac{x^2}{t^2} + \frac{y^2}{t^2} + \left(\frac{1}{2} - \frac{1}{t}\right)^2 = \frac{1}{4} \Leftrightarrow |z|^2 + \left(\frac{t}{2} - 1\right)^2 = \frac{t^2}{4} \Leftrightarrow$$
$$\Leftrightarrow |z|^2 - t + 1 = 0 \Leftrightarrow t = |z|^2 + 1$$

thus we have

$$\begin{cases} X = \frac{x}{1+|z|^2} \\ Y = \frac{y}{1+|z|^2} \\ Z = \frac{|z|^2}{1+|z|^2}. \end{cases}$$

Set $z = x_1 + iy_1, w = x_2 + iy_2$, we have

$$
\begin{aligned}
&|z-w|^2 = \\
&\qquad = (z-w)\overline{(z-w)} = (z-w)(\overline{z}-\overline{w}) = \\
&\qquad = z\overline{z} - z\overline{w} - \overline{z}w + w\overline{w} = |z|^2 + |w|^2 - z\overline{w} - \overline{z}w = \\
&\qquad = |z|^2 + |w|^2 - (x_1 + iy_1)(x_2 - iy_2) - (x_1 - iy_1)(x_2 + iy_2) = \\
&\qquad = |z|^2 + |w|^2 - 2(x_1x_2 + y_1y_2)\ . \qquad (1)
\end{aligned}
$$

And the coordical distance

$$d\left(z,w\right)^2=\left(X_1-X_2\right)^2+\left(Y_1-Y_2\right)^2+\left(Z_1-Z_2\right)^2 \;=$$

$$=\left(\frac{x_1}{1+\left|z\right|^2}-\frac{x_2}{1+\left|w\right|^2}\right)^2+\left(\frac{y_1}{1+\left|z\right|^2}-\frac{y_2}{1+\left|w\right|^2}\right)^2+\left(\frac{\left|z_1\right|^2}{1+\left|z\right|^2}-\frac{\left|z_2\right|^2}{1+\left|w\right|^2}\right)^2=$$

$$=\frac{\left(x_1\left(1+\left|w\right|^2\right)-x_2\left(1+\left|z\right|^2\right)\right)^2+\left(y_1\left(1+\left|w\right|^2\right)-y_2\left(1+\left|z\right|^2\right)\right)^2+\left(\left|z\right|^2-\left|w\right|^2\right)^2}{\left(1+\left|z\right|^2\right)^2\left(1+\left|w\right|^2\right)^2}=$$

$$=\frac{\left(x_1^2+y_1^2\right)\left(1+\left|w\right|^2\right)^2+\left(x_2^2+y_2^2\right)\left(1+\left|z\right|^2\right)^2+\left|z\right|^4-2\left|z\right|^2\left|w\right|^2+\left|w\right|^4}{\left(1+\left|z\right|^2\right)^2\left(1+\left|w\right|^2\right)^2}+$$

$$+\frac{-2x_1x_2\left(1+\left|w\right|^2\right)\left(1+\left|z\right|^2\right)-2y_1y_2\left(1+\left|w\right|^2\right)\left(1+\left|z\right|^2\right)}{\left(1+\left|z\right|^2\right)^2\left(1+\left|w\right|^2\right)^2}=$$

$$=\frac{\left|z\right|^2\left(1+\left|w\right|^2\right)^2+\left|w\right|^2\left(1+\left|z\right|^2\right)^2+\left|z\right|^4-2\left|z\right|^2\left|w\right|^2+\left|w\right|^4}{\left(1+\left|z\right|^2\right)^2\left(1+\left|w\right|^2\right)^2}+$$

$$+\frac{-2\left(x_1x_2+y_1y_2\right)\left(1+\left|w\right|^2\right)\left(1+\left|z\right|^2\right)}{\left(1+\left|z\right|^2\right)^2\left(1+\left|w\right|^2\right)^2}=$$

$$=\frac{\left|z\right|^2+2\left|z\right|^2\left|w\right|^2+\left|z\right|^2\left|w\right|^4+\left|w\right|^2+2\left|w\right|^2\left|z\right|^2+\left|w\right|^2\left|z\right|^4+\left|z\right|^4-2\left|z\right|^2\left|w\right|^2+\left|w\right|^4}{\left(1+\left|z\right|^2\right)^2\left(1+\left|w\right|^2\right)^2}+$$

$$+\frac{-2\left(x_1x_2+y_1y_2\right)\left(1+\left|w\right|^2\right)\left(1+\left|z\right|^2\right)}{\left(1+\left|z\right|^2\right)^2\left(1+\left|w\right|^2\right)^2}=$$

$$=\frac{\left(\left|z\right|^2+\left|w\right|^2\right)\left(1+\left|w\right|^2\right)\left(1+\left|z\right|^2\right)-2\left(x_1x_2+y_1y_2\right)\left(1+\left|w\right|^2\right)\left(1+\left|z\right|^2\right)}{\left(1+\left|z\right|^2\right)^2\left(1+\left|w\right|^2\right)^2}=$$

$$=\frac{\left(\left|z\right|^2+\left|w\right|^2-2\left(x_1x_2+y_1y_2\right)\right)\left(1+\left|w\right|^2\right)\left(1+\left|z\right|^2\right)}{\left(1+\left|z\right|^2\right)^2\left(1+\left|w\right|^2\right)^2}=$$

$$=\frac{\left(\left|z\right|^2+\left|w\right|^2-2\left(x_1x_2+y_1y_2\right)\right)}{\left(1+\left|z\right|^2\right)\left(1+\left|w\right|^2\right)}\overset{(1)}{=}$$

$$\overset{(1)}{=}\frac{\left|z-w\right|^2}{\left(\left|z\right|^2+1\right)\left(\left|w\right|^2+1\right)}.$$

This gives

$$d\left(z,w\right)=\frac{\left|z-w\right|}{\sqrt{\left(\left|z\right|^{2}+1\right)}\sqrt{\left(\left|w\right|^{2}+1\right)}}.$$

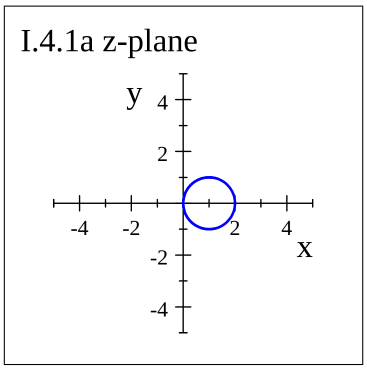
I.4.1

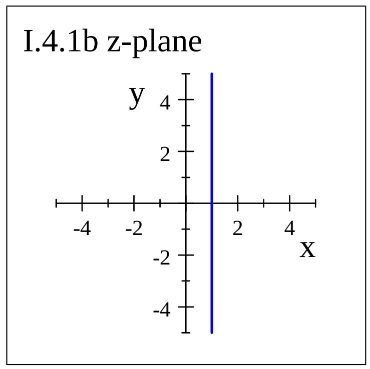
Sketch each curve and its image under $w = z^2$.

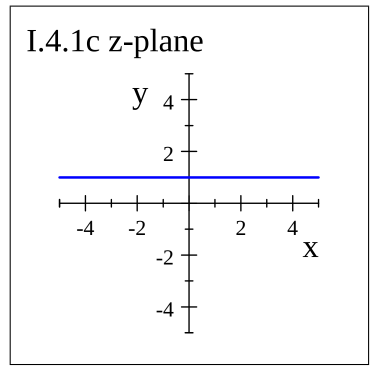
(a) $|z-1| = 1$ **(c)** $y = 1$ **(e)** $y^2 = x^2 - 1,\ x > 0$

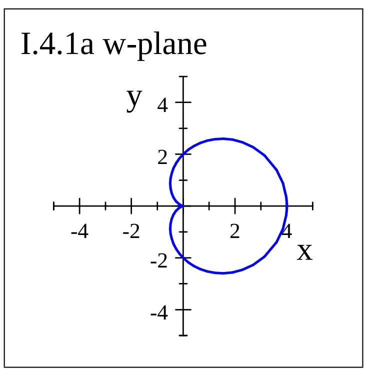
(b) $x = 1$ **(d)** $y = x + 1$ **(f)** $y = 1/x,\ x \neq 0$

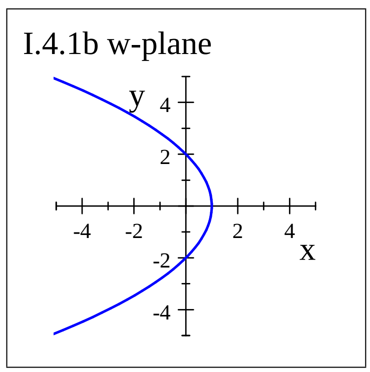
Solution

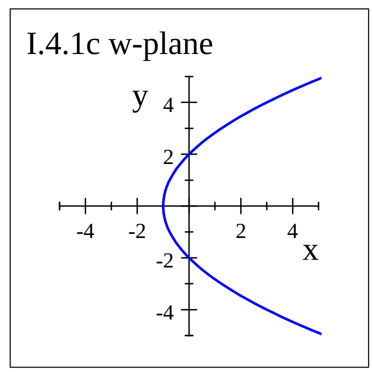


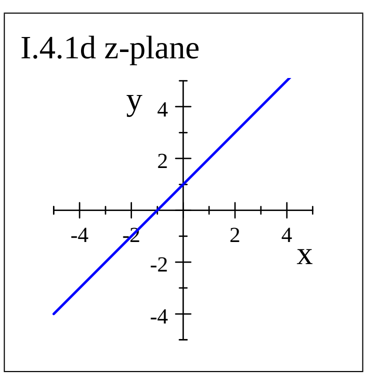


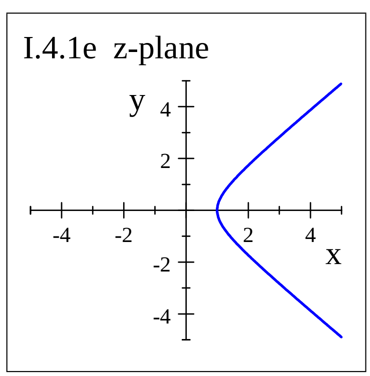


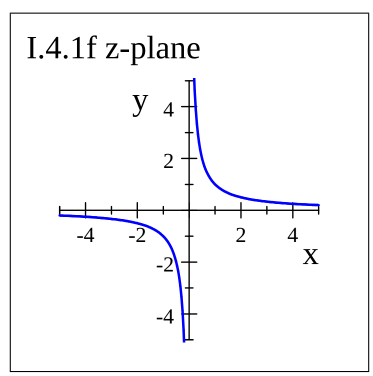


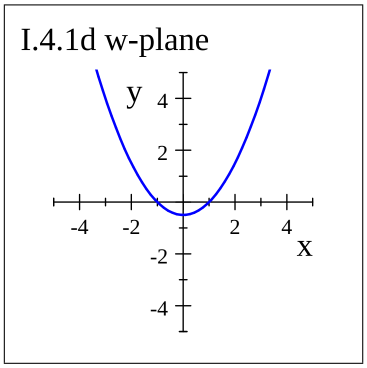


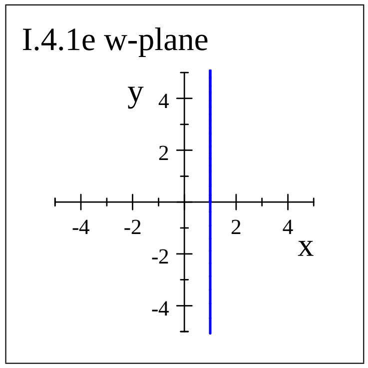


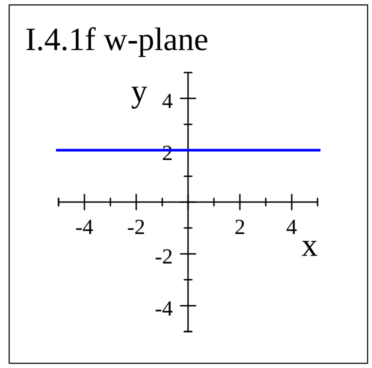







I.4.1d w-plane
y
4
2
-4
-2
2
4
x
-2
-4


I.4.1e w-plane
y
4
2
-4
-2
2
4
x
-2
-4

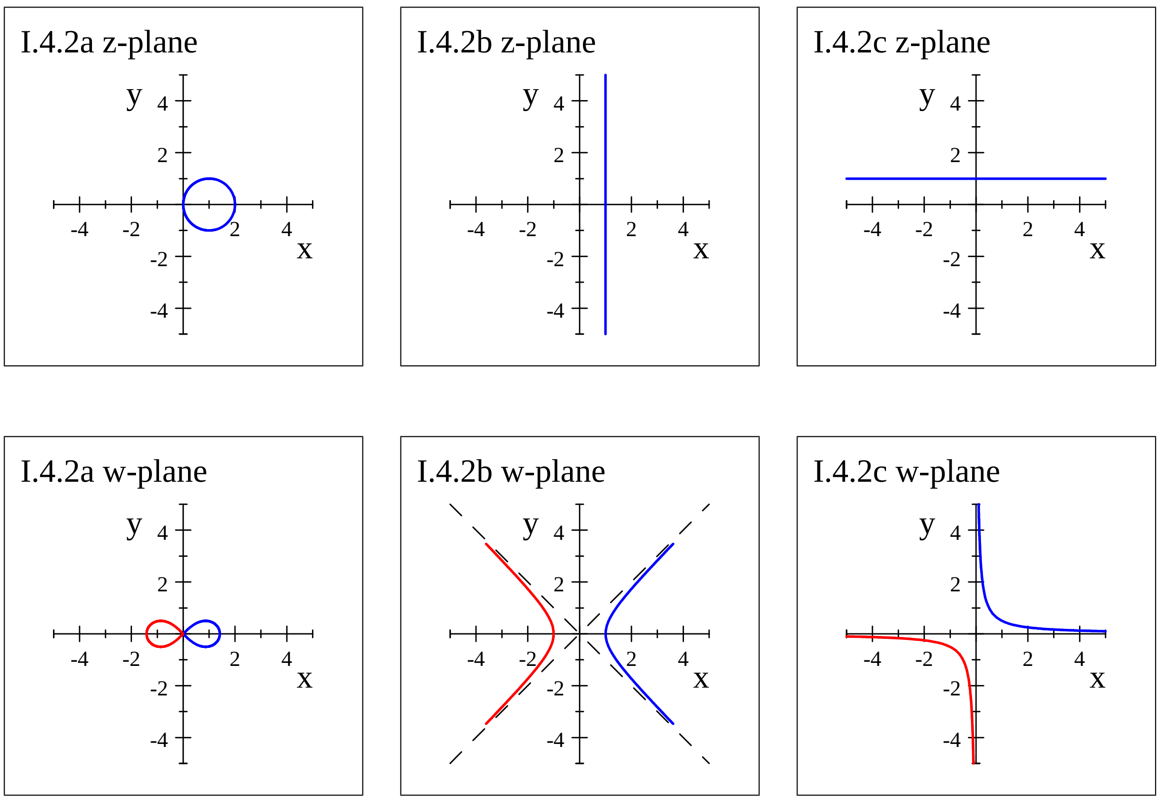

I.4.1f w-plane
y
4
2
-4
-2
2
4
x
-2
-4

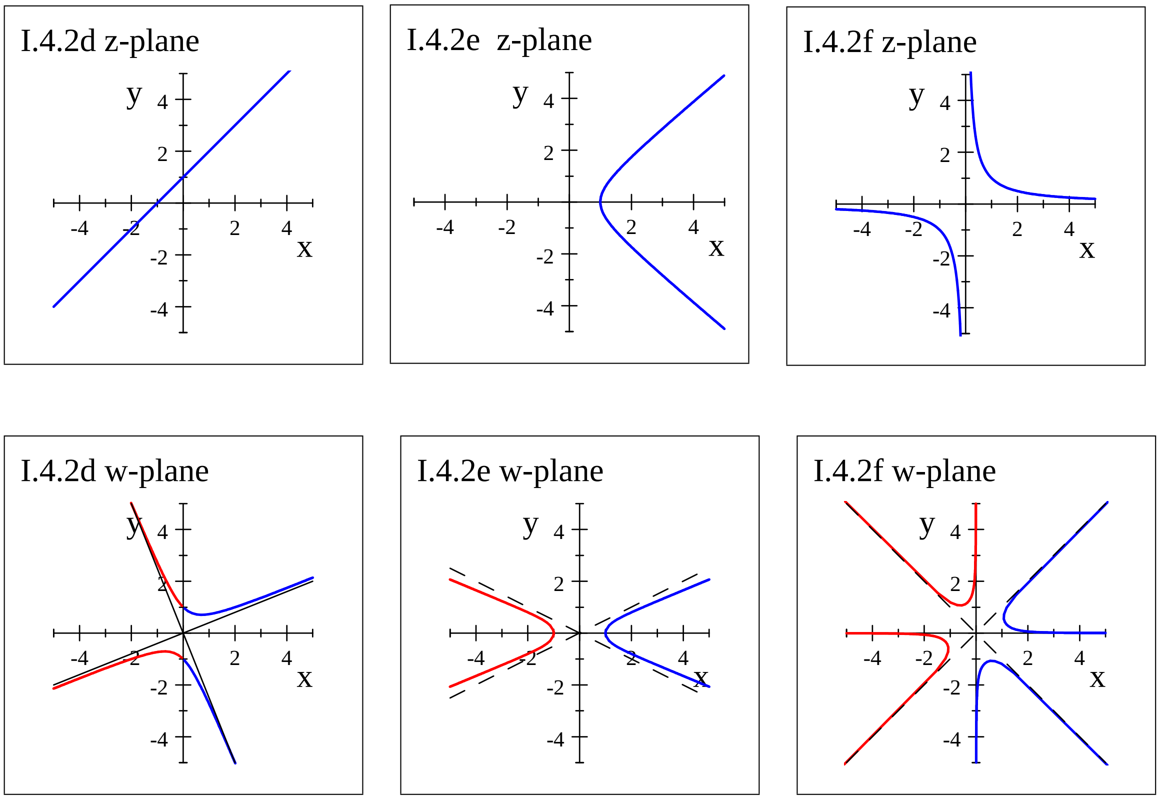
I.4.2

Sketch the image of each curve in the preceding problem under the principal branch of $w = \sqrt{z}$, and also sketch, on the same grid but in a different color, the image of each curve under the other branch of $\sqrt{z}$.

Solution

$$w = z^{1/2} = \left(|z|\, e^{i\,\arg|z|}\right)^{1/2} = \left(|z|\, e^{i\,\mathrm{Arg}|z| + i2\pi n}\right)^{1/2} = \pm\sqrt{|z|}e^{i\,\mathrm{Arg}\, z/2}$$




I.4.2d z-plane
I.4.2e z-plane
I.4.2f z-plane
I.4.2d w-plane
I.4.2e w-plane
I.4.2f w-plane
x
y
-4
-2
2
4

I.4.3

(a) **Give a brief description of the function $z \longmapsto w = z^3$, considered as a mapping from the $z-$ plane to the w- plane. (Describe what happens to w as z traverses a ray emanating from the origin, and as z traverses a ray a circle centered at the origin.) (b) Make branch cuts and define explicitly three branches of the inverse mapping. (c) Describe the construction of the Riemann surface of $z^{1/3}$.**

Solution

(a)

The function $w = f(z) = z^3$. For $z = re^{i\theta}$, we have $z^3 = r^3 e^{i3\theta}$. The radical rays at angle θ are mapped to rays at angle 3θ, that is, $\arg w = 3 \arg z$. The magnitude of a complex number is cubed, $|w| = |z|^3$. Circles, centered at the origin of radius r, are mapped to cocentric circles with radius r^3.

(b)

We make branch cuts at $(-\infty, 0]$,

$$z = w^{1/3} = \left(|w|\, e^{i \arg w}\right)^{1/3} = \left(|w|\, e^{i \operatorname{Arg} w + i2\pi n}\right)^{1/3} = e^{i2\pi n/3} \sqrt{|w|} e^{i \operatorname{Arg} w/3}$$

we choose

$g(z) = z^{1/3} = r^{1/3} e^{i\theta/3}$, $-\pi < \theta < \pi$.

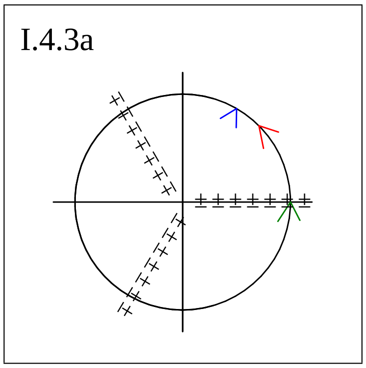
Sheet 1 : Take $f_1(z) = g(z)$,

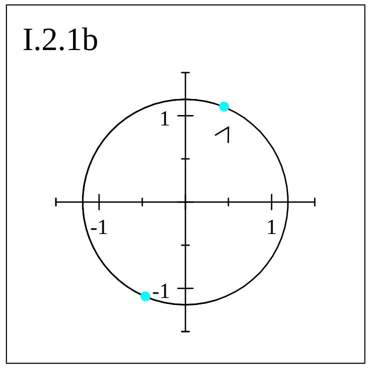
Sheet 2 : Take $f_2(z) = e^{2\pi i/3} g(z)$,

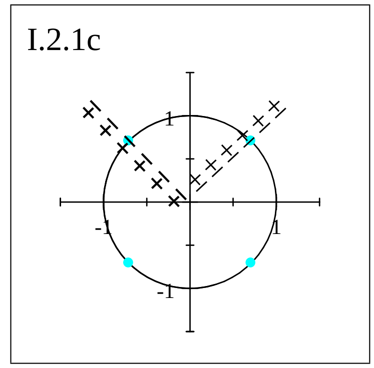
Sheet 3 : Take $f_3(z) = e^{4\pi i/3} g(z)$.

(c) Top edge of cut on sheet 1 to bottom edge of cut on sheet 2. Top edge of cut on sheet 2 to bottom edge of cut on sheet 3. Top edge of cut on sheet 3 to bottom edge of cut on sheet 1. The endpoints for $f(z)$ is continuous on surface

Spara







Spara

I.4.4

Describe how to construct the Riemann surfaces for the following functions
(a) $w = z^{1/4}$, (b) $w = \sqrt{z-i}$, (c) $w = (z-1)^{2/5}$.

Remark. To describe the Riemann surface of a multivalued function, begin with one sheet for each branch of the function, make branch cuts so that the branches are defined continuously on each sheet, and identify each edge of a cut on one sheet to another appropriate edge so that the function values match up continuously.

Solution

(a) Use four sheets, can make branch cuts along real axis from $-\infty$ to 0.
(b) Use two sheets, can make branch cuts along horizontal line from $-\infty + i$ to i.
(c) Use five sheets, can make branch cuts along real axis from $-\infty$ to 1.

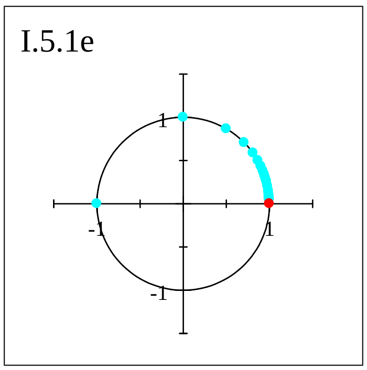
I.5.1

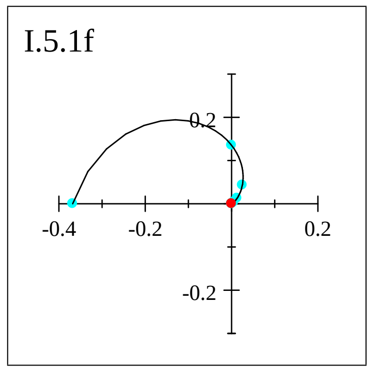
Calculate and plot for e^z for the following points z.

(a) 0 **(c)** $\pi(i-1)/3$ **(e)** $\pi i/m, \quad m = 1,2,3,\ldots$

(b) $\pi i + 1$ **(d)** $37\pi i$ **(f)** $m(i-1) \quad m = 1,2,3,\ldots$

Solution





(a)

$$e^0 = 1.$$

(b)

$$e^{\pi i+1} = e^{\pi i}e^1 = -e.$$

(c)

$$e^{\pi(i-1)3} = e^{-\pi/3}e^{\pi i/3} = e^{-\pi/3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 0.351 + 0.006i.$$

(d)

$$e^{37\pi i} = e^{36\pi i}e^{\pi i} = -1.$$

(e)

We take the limit for the sequence $e^{\pi i/m}$, $m = 1,2,3,\ldots$ as $m \to \infty$, and have

$$e^{\pi i/m} \to 1,$$

as $m \to \infty$. Because $\left|e^{\pi i/m}\right| = 1$ the sequence approaches its limit along a circle with radius 1.
(f)
We take the limit for the sequence $e^{m(i-1)} = e^{-m}e^{mi}$, $m = 1, 2, 3, \ldots$ as $m \to \infty$, and have

$$e^{-m}e^{mi} \to 0,$$

as $m \to \infty$. Because $|e^{-m}e^{mi}| = e^{-m}$ the sequence spiraling to its limit in origo.

I.5.2

Sketch each of the following figures and its image under the exponential map $w = e^z$. Indicate the images of horizontal and vertical lines in your sketch.

(a) the vertical strip $0 < \operatorname{Re} z < 1$,

(b) the horizontal strip $5\pi/3 < \operatorname{Im} z < 8\pi/3$,

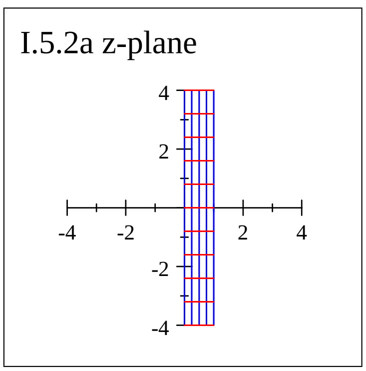
(c) the rectangle $0 < x < 1$, $0 < y < \pi/4$,

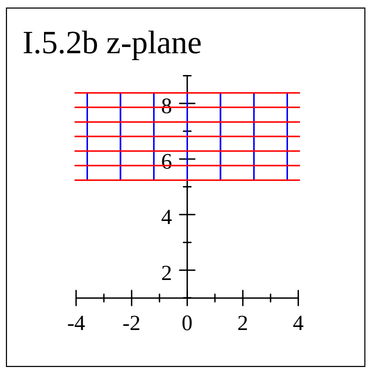
(d) the disk $|z| \leq \pi/2$,

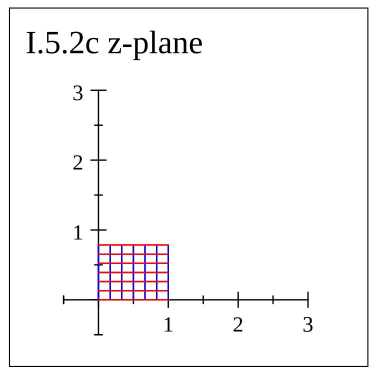
(e) the disk $|z| \leq \pi$,

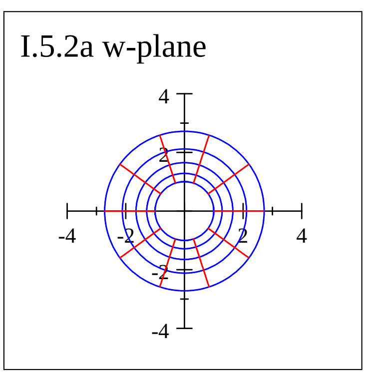
(f) the disk $|z| \leq 3\pi/2$.

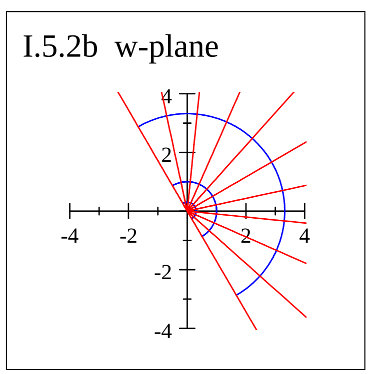
Solution

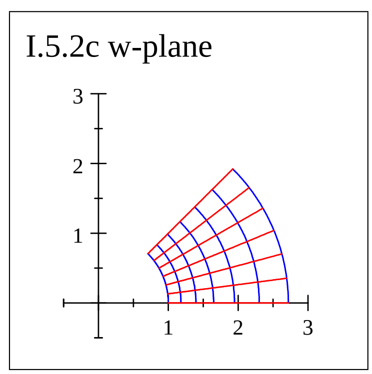


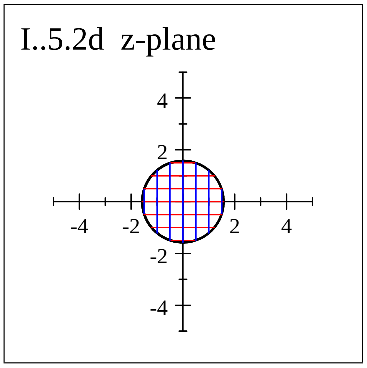


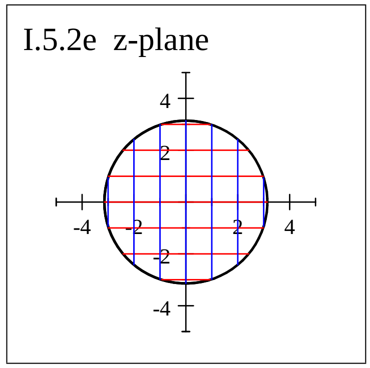


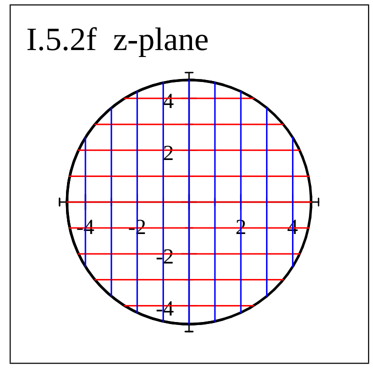


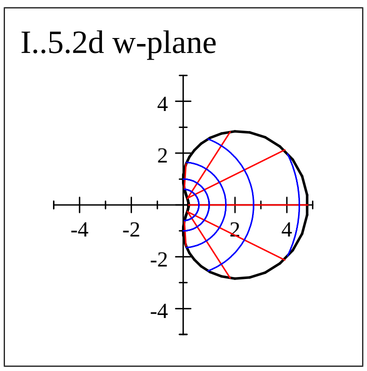


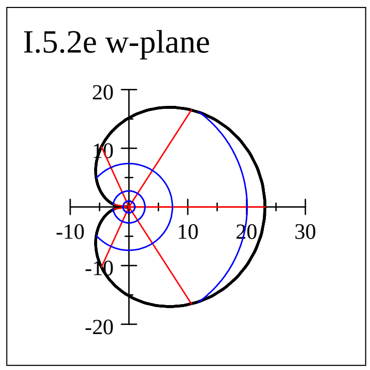


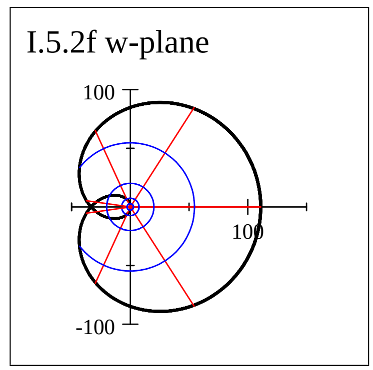

I..5.2d z-plane
4
2
-4
-2
2
4
-2
-4


I.5.2e z-plane
4
2
-4
-2
2
4
-2
-4


I.5.2f z-plane
4
2
-4
-2
2
4
-2
-4


I..5.2d w-plane
4
2
-4
-2
2
4
-2
-4


I.5.2e w-plane
20
10
-10
10
20
30
-10
-20


I.5.2f w-plane
100
100
-100

I.5.3
Show that $e^{\bar{z}} = \overline{e^z}$.

Solution

$$
\begin{aligned}
e^{\bar{z}} = e^{x-iy} = e^x e^{-iy} = e^x\left(\cos\left(-y\right) + i\sin\left(-y\right)\right) = e^x\left(\cos y - i\sin y\right) = \\
= \overline{e^x}\,\overline{(\cos y + i\sin y)} = \overline{e^x\left(\cos y + i\sin y\right)} = \overline{e^x e^{iy}} = \overline{e^{x+iy}} = \overline{e^z}.
\end{aligned}
$$

I.5.4
Show that the only periods of e^z are the integral multiples of $2\pi i$, that is, if $e^{z+\lambda} = e^z$ for all z, then λ is an integer times $2\pi i$.

Solution
Show that the only periods of e^z are the integral muliples of $2\pi i$ that is, if $e^{z+\lambda} = e^z$ for all z, then λ is an integer times $2\pi i$.

$$e^z = e^{z+\lambda} = e^z e^\lambda \Rightarrow e^\lambda = 1 \Rightarrow \lambda = 2\pi m i$$

I.6.1
Find and plot $\log z$ for the following complex numbers z. Specify the principal value.
(a) 2 **(b)** i **(c)** $1+i$ **(d)** $\left(1+i\sqrt{3}\right)/2$

Solution
(a)
Suppose that $n = 0, \pm 1, \pm 2, \ldots$

$$\log 2 = \log|2| + i\operatorname{Arg} 2 + 2\pi n i = \log 2 + i2\pi n.$$

(b)
Suppose that $n = 0, \pm 1, \pm 2, \ldots$

$$\log i = \log|i| + i\operatorname{Arg} i + 2\pi n i = i\pi/2 + i2\pi n.$$

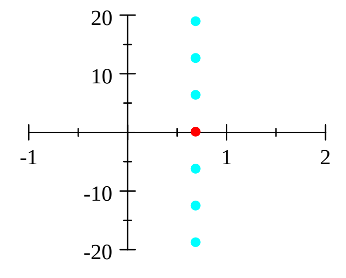
(c)
Suppose that $n = 0, \pm 1, \pm 2, \ldots$

$$\begin{aligned}&\log(1+i) = \\ &\quad = \log|1+i| + i\operatorname{Arg}(1+i) + 2\pi n i = \log\sqrt{2} + i\pi/4 + i2\pi n = \\ &\quad\quad = \log 2/2 + i\pi/4 + i2\pi n.\end{aligned}$$

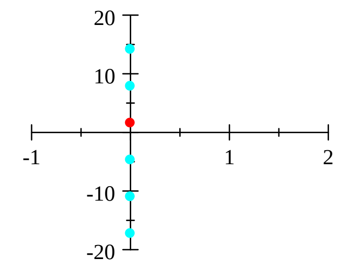
(d)
Suppose that $n = 0, \pm 1, \pm 2, \ldots$

$$\begin{aligned}&\log\left(\tfrac{1+i\sqrt{3}}{2}\right) = \\ &\quad = \log\left|\tfrac{1+i\sqrt{3}}{2}\right| + i\operatorname{Arg}\left(\tfrac{1+i\sqrt{3}}{2}\right) + 2\pi n i = \\ &\quad\quad = \pi i/3 + i2\pi n.\end{aligned}$$

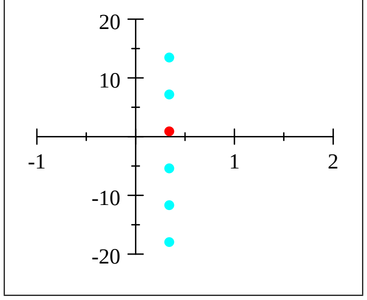
I.6.1a



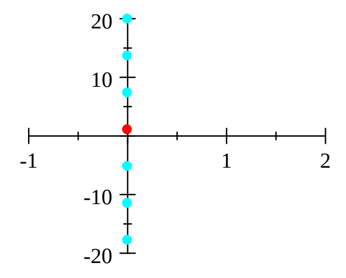
I.6.1b



I.6.1c



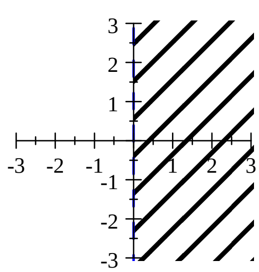
I.6.1d



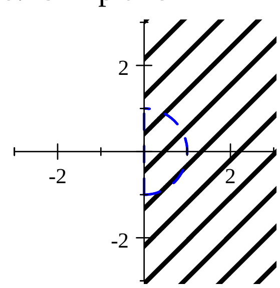
I.6.2
Sketch the image under the map $w = \operatorname{Log} z$ of each of the following figures.
(a) the right halv-plane $\operatorname{Re} z > 0$,
(b) the half-disk $|z| < 1$, $\operatorname{Re} z > 0$,
(c) the unit circle $|z| = 1$,
(d) the slit annulus $\sqrt{e} < |z| < e^2$, $z \notin (-e^2, -\sqrt{e})$,
(e) the horizontal line $y = e$,
(f) the vertical line $x = e$.

Solution
(b) We have a disk with radius less then 1, this means $|z| < 1$, thus $\log|z| < 0$ and is unbounded, therefore, it goes from 0 to $-\infty$. Since $\operatorname{Re}(z) > 0$, the polar angle is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
(d) Here, we have an annulus, the polar angle is from $-\pi$ to π, thus the argument is in this range. Since, the log function is the inverse of the exponential map, circles in $z-$ plane are stright lines in the $w-$ plane ($w = \log z$). Therefore, the image of this annulus under the log function is the rectangle, bounded by $x = \log|\sqrt{e}| = \frac{1}{2}$ and $x = \log|e^2| = 2$.

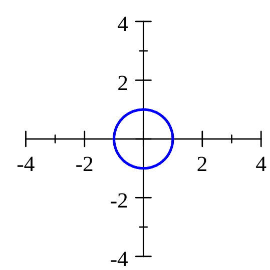
I.6.2a z-plane



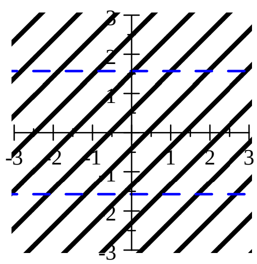
I.6.2b z-plane



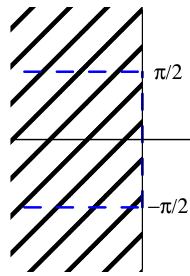
I.6.2c z-plane



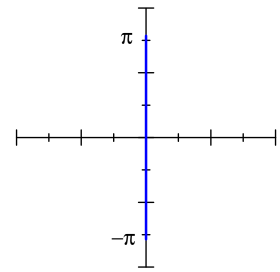
I.6.2a w-plane



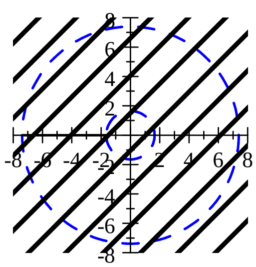
I.6.2b w-plane



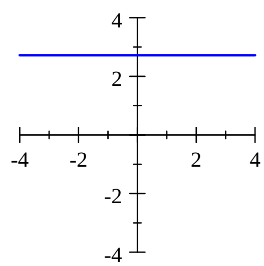
I.6.2c w-plane



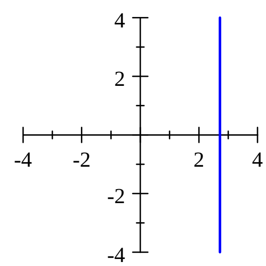
I.6.2d z-plane



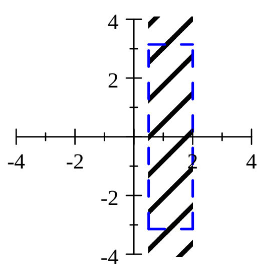
I.6.2e z-plane



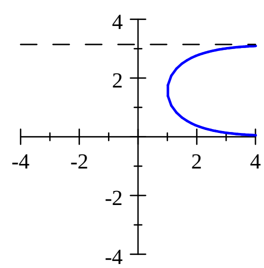
I.6.2f z-plane



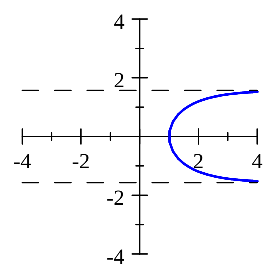
I.6.2d w-plane



I.6.2e w-plane

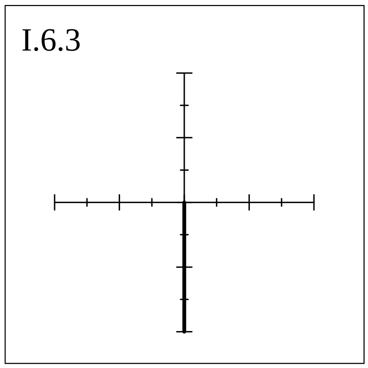


I.6.2f w-plane



I.6.3
Define explicitly a continuous branch of $\log z$ in the complex plane slit along the negative imaginary axis, $\mathbb{C} \setminus [0, -i\infty)$.

Solution



We have $\log z = \log re^{i\theta} = \log r + i\theta$. To avoid the negative imaginary axis we chose $-\pi/2 < \theta < 3\pi/2$. We use the branch

$$f\left(re^{i\theta}\right) = \log r + i\theta, \; -\pi/2 < \theta < 3\pi/2,$$

of $\log z$.

I.6.4
How would you make a branch cut to define a single-valued branch of the function $\log(z+i-1)$? How about $\log(z-z_0)$?

Solution
Any straight line cut from z_0 to ∞, in any direction, will do.

I.7.1
Find all values and plot them.
(a) $(1+i)^i$ **(b)** $(-i)^{1+i}$ **(c)** $2^{-1/2}$ **(d)** $\left(1+i\sqrt{3}\right)^{(1-i)}$

Solution
(a)
Suppose that $m, n = 0, \pm 1, \pm 2, \dots$

$$\begin{aligned}(1+i)^i &= \\ = e^{i\log(1+i)} = e^{i(\log|1+i| + i\operatorname{Arg}(1+i) + i2\pi m)} = e^{i\left(\log\sqrt{2} + i\pi/4 + i2\pi m\right)} &= e^{-2\pi m} e^{-\pi/4} e^{i\log\sqrt{2}} = [m=-n] = \\ &= e^{2\pi n} e^{-\pi/4} e^{i\log\sqrt{2}}.\end{aligned}$$

(b)
Suppose that $m, n = 0, \pm 1, \pm 2, \dots$

$$\begin{aligned}(-i)^{1+i} &= \\ &= e^{(1+i)\log(-i)} = e^{(1+i)(\log|-i| + i\operatorname{Arg}(-i) + i2\pi m)} = e^{(1+i)(-i\pi/2 + i2\pi m)} = \\ &= e^{-2\pi m} e^{\pi/2} e^{i(-\pi/2 + 2\pi m)} = [m=-n] = e^{2\pi n} e^{\pi/2} e^{i(-\pi/2 - 2\pi n)} = \\ &= -ie^{2\pi n} e^{\pi/2}.\end{aligned}$$

(c)
Suppose that $m = 0, \pm 1, \pm 2, \dots$

$$\begin{aligned}2^{-1/2} &= \\ &= e^{-\log 2/2} = e^{-(\log|2| + i\operatorname{Arg} 2 + i2\pi m)/2} = e^{-(\log 2 + i2\pi m)/2} = e^{-\log 2/2} e^{-i\pi m} = \\ &= \pm 1/\sqrt{2}.\end{aligned}$$

(d)
Suppose that $n = 0, \pm 1, \pm 2, \dots$

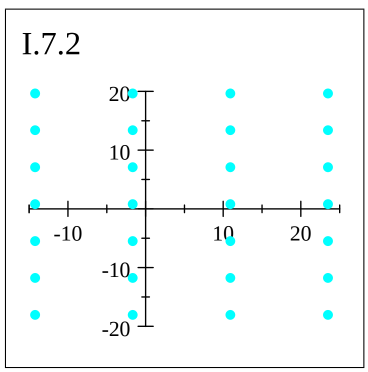
$$
\begin{aligned}
&\left(1+i\sqrt{3}\right)^{1-i} = \\
&\quad = e^{(1-i)\log\left(1+i\sqrt{3}\right)} = e^{(1-i)\left(\log\left|1+i\sqrt{3}\right|+i\,\mathrm{Arg}\left(1+i\sqrt{3}\right)+i2\pi n\right)} = \\
&\quad = e^{(1-i)(\log 2+i\pi/3+i2\pi n)} = e^{2\pi n}e^{\log 2+\pi/3}e^{i(\ -\log 2+\pi/3+2\pi n)} = \\
&\quad = e^{2\pi n}e^{\log 2+\pi/3}e^{i(\ -\log 2+\pi/3)}.
\end{aligned}
$$

I.7.2
Compute and plot $\log\left[(1+i)^{2i}\right]$**.**

Solution
We rewrite the expression, suppose that $k,m,n=0,\pm1,\pm2,\ldots$

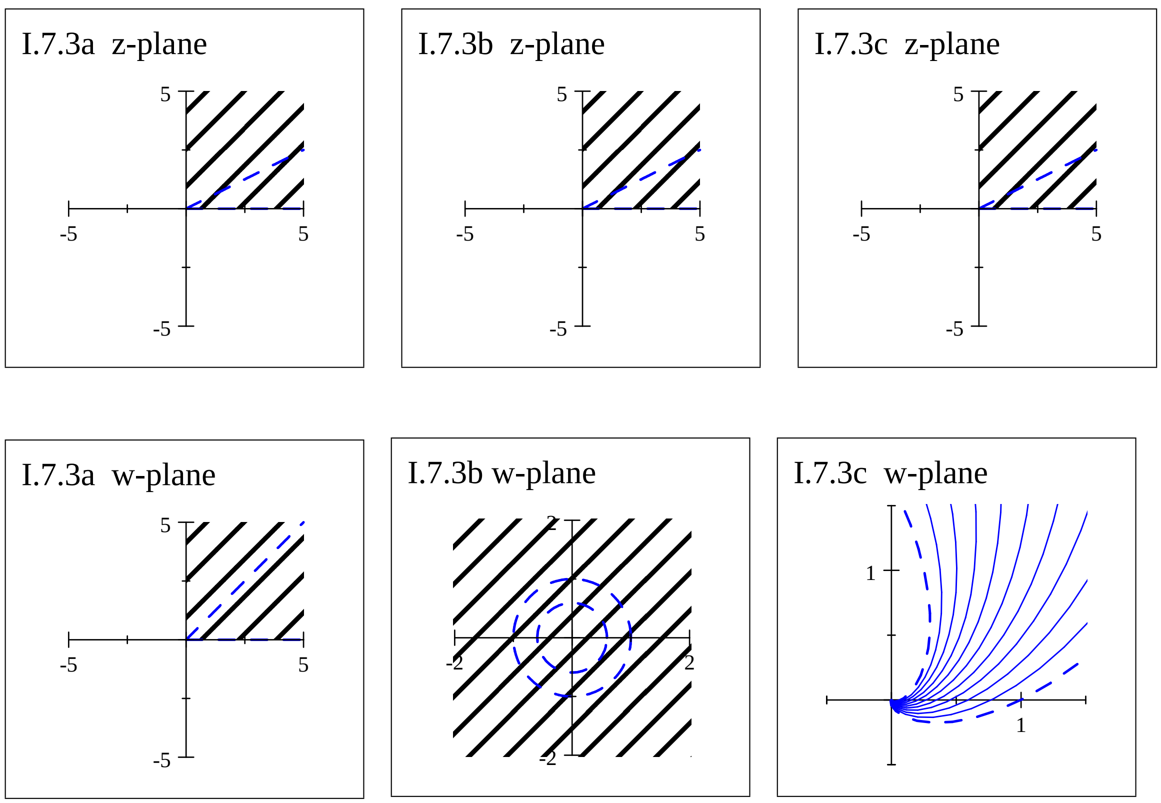
$$
\begin{aligned}
&\log\left[(1+i)^{2i}\right]= \\
&\quad=\log\left[\left(e^{\log(1+i)}\right)^{2i}\right]=\log\left[e^{2i\log(1+i)}\right]=\log\left[e^{2i(\log|1+i|+i\operatorname{Arg}(1+i)+i2\pi k)}\right]= \\
&=\log\left[e^{2i\left(\log\sqrt{2}+i\pi/4+i2\pi k\right)}\right]=\log\left[e^{-4\pi k}e^{-\pi/2}e^{i\log 2}\right]=[k=-m]=\log\left[e^{4\pi m}e^{-\pi/2}e^{i\log 2}\right]= \\
&=\log\left|e^{4\pi m}e^{-\pi/2}e^{i\log 2}\right|+i\operatorname{Arg}\left(e^{4\pi m}e^{-\pi/2}e^{i\log 2}\right)+i2\pi n=4\pi m-\pi/2+i\log 2+i2\pi n= \\
&\qquad\qquad\qquad\qquad=-\pi/2+i\log 2+4\pi m+2\pi i n.
\end{aligned}
$$

Thus the set is a square lattice.



I.7.3
Sketch the image of the sector $\{0 < \arg z < \pi/6\}$ under the map $w = z^a$ for
(a) $a = \frac{3}{2}$ **(b)** $a = i$ **(c)** $a = i + 2$
Use only the principal branch of z^a.

Solution



Some points on the y-axis is marked, and the point $(1, 0)$ too.

I.7.4

Show that $(zw)^a = z^a w^a$, where on the right we take all possible products.

Solution

Let $\lambda \in (zw)^a$, and $k, m, n = 0, \pm 1, \pm 2, \dots$

$$\lambda =$$
$$= e^{a\log(zw)} = e^{a(\log|zw|+i\operatorname{Arg}(zw)+i2\pi n)} = e^{a(\log|z|+\log|w|+i\operatorname{Arg}z+i\operatorname{Arg}w+i2\pi m+i2\pi n)} =$$
$$= e^{a(\log|z|+i\operatorname{Arg}z+i2\pi(m+n))}e^{a(\log|w|+i\operatorname{Arg}w)} \in z^a w^a.$$

Conversely, if $\lambda \in z^a w^a$, say

$$\lambda =$$
$$= e^{a\log z}e^{a\log w} = e^{a(\log|z|+i\operatorname{Arg}z+i2\pi m)}e^{a(\log|w|+i\operatorname{Arg}w+i2\pi n)} =$$
$$= e^{a(\log|z|+\log|w|+i\operatorname{Arg}z+i\operatorname{Arg}w+i2\pi(m+n))} = e^{a(\log|zw|+i\operatorname{Arg}(zw)+i2\pi k+i2\pi(m+n))} =$$
$$= e^{a(\log|zw|+i\operatorname{Arg}(zw)+i2\pi(k+m+n))} =$$
$$= e^{a\log(zw)} \in (zw)^a$$

We have that $(zw)^a = z^a w^a$ as sets

I.7.5
Find i^{i^i}. Show that is does not coincide with $i^{i\cdot i} = i^{-1}$.

Solution
We rewrite the expression i^i, suppose that $k = 0, \pm1, \pm2, \dots$

$$
\begin{multline}
i^i = \\
= e^{i\log i} = e^{i(\log|i| + i\,\mathrm{Arg}(i) + i2\pi k)} = e^{i(i\pi/2 + i2\pi k)} = \\
= e^{-2\pi k - \pi/2}.
\end{multline}
$$

Now we can find i^{i^i} we use that we have i^i, suppose that $k, m = 0, \pm1, \pm2, \dots$

$$
\begin{multline}
i^{i^i} = \\
= \left(i^i\right)^i = e^{i\log\left(i^i\right)} = e^{i\left(\log\left(e^{-2\pi k - \pi/2}\right)\right)} = e^{i\left(\log\left|e^{-2\pi k-\pi/2}\right| + i\,\mathrm{Arg}\left(e^{-2\pi k - \pi/2}\right) + i2\pi m\right)} = \\
= e^{i(-2\pi k - \pi/2 + i2\pi m)} = e^{-2\pi m} e^{i(-2\pi k - \pi/2)} = -ie^{-2\pi m} = \\
= -ie^{2\pi m}.
\end{multline}
$$

Which not coincide with $i^{i\cdot i} = i^{-1} = 1/i = -i$ which is smaller than $\{-ie^{2\pi m}, m = 0, \pm1, \pm2, \dots\}$.

I.7.6

Determine the phase factors of the function $z^a(1-z)^b$ at the branch points $z=0$ and $z=1$. What conditions on a and b guarantee that $z^a(1-z)^b$ can be defined as a (continuous) single-valued function on $\mathbb{C}\setminus[0,1]$?

Solution

Phase factors is $e^{2\pi ia}$ at 0 and $e^{2\pi ib}$ at 1. Require $e^{2\pi ia}e^{2\pi ib}=1$, or $a+b=n$, where $n\in\mathbb{Z}$, to have a continuous single–valued determination of $z^a(1-z)^b$ on $\mathbb{C}\setminus[0,1]$.

I.7.7

Let $x_1 < x_2 < \cdots < x_n$ be n consecutive points on the real axis. Describe the Riemann surface of $\sqrt{(z-x_1)\cdots(z-x_n)}$. Show that for $n=1$ and $n=2$ the surface is topologically a sphere with certain punctures corresponding to the branch points and ∞. What is it when $n=3$ or $n=4$? Can you say anything for general n? (Any compact Riemann surface is topologically a sphere with handles. Thus a torus is topologically a sphere with one handle. For a given n, how many handles are there, and where do they come from?)

Solution

For $n=1$ and $n=2$, the functions $f_1 = \sqrt{z-x_1}$ and $f_2 = \sqrt{(z-x_1)(z-x_2)}$, and for a general n we have $f_n(x) = \sqrt{(z-x_1)(z-x_2)\dots(z-x_n)}$. To constuct the Riemann Suface we use two sheets with slits $[x_1, x_2], [x_3, x_4], \dots$. If n is odd, we also need a slit $[x_n, +\infty)$. Identify top edge of slit on one sheet with bottom edge of slit on other sheet. If $n=1$ and $n=2$ the surface is topologically a sphere with certain punctures corresponding to the branch points and ∞. If $n=3$ or $n=4$, surface is a torus. For a general even n, $f_n(x)$ will be continuous on $\mathbb{C}\backslash([x_1,x_2]\cup[x_3,x_4]\cup\dots\cup[x_{n-1},x_n])$, and its Riemann surface will be a sphere with $\frac{n}{2}-1$ holes. For a general odd n though, $f_n(x)$ will be continuous on $\mathbb{C}\backslash([x_1,x_2]\cup[x_3,x_4]\cup\dots\cup[x_n,+\infty])$, its Riemann surface will be a sphere with $\frac{n-1}{2}$ holes.

I.7.8

Show that $\sqrt{z^2 - 1/z}$ can be defined as a (single-valued) continuous function outside the unit disk, that is, for $|z| > 1$. Draw branch cuts so that the function can be defined continuous off the branch cuts. Describe the Riemann surface of the function.

Solution

The function is $z\sqrt{1 - 1/z^3}$. If $|z| > 1$, can use the principal value of the square root to define a branch of the function. There are branch points at $z = 0$ and $z^3 = 1$, that is at $0, 1, e^{2\pi i/3}$ and $e^{-2\pi i/3}$. Make two branch cuts by connecting any two pairs of point by curves; for instance, connect 0 to 1 by a straight line, and the other cube roots of unity by a straight line or arc of unit circle. The resulting two-sheeted surface with identification of cuts and with points at infinity is a torus.

I.7.9

Consider the branch of the function $\sqrt{z\left(z^3-1\right)\left(z+1\right)^3}$ that is positive at $z=2$. Draw branch cuts so that this branch of the function can be defined continuously off the branch cuts. Describe the Riemann surface of the function. To what value at $z=2$ does this branch return if it is continued continuously once counterclockwise around the circle $\{|z|=2\}$?

Solution

We have that the function is

$$(z+1)\, z^2 \sqrt{z} \sqrt{\left(1-1/z^3\right)\left(1+1/z\right)}.$$

If $|z|>1$, the second square root can be defined to be single-valued for $|z|>1$. The value of the function at $z=2$ returns to the negative of their initial value then we travers the circle $|z|=2$, because $\sqrt{z}$ does. We can construct a Riemann Surface by making cuts at $-1, 0, 1, e^{2\pi i/3}, e^{-2\pi i/3}$ and ∞. The function for the Riemann Surface is

$$\pm (z+1) \sqrt{z\,(z-1)\,(z+1)\left(z-e^{2\pi i/3}\right)\left(z-e^{-2\pi i/3}\right)}.$$

I.7.10

Consider the branch of the function $\sqrt{z\left(z^{3}-1\right)(z+1)^{3}(z-1)}$ that is positive at $z=2$. Draw branch cut so that this branch of the function can be defined continuously off the branch cuts. Describe the Riemann surface of the function. To what value at $z=2$ does this branch return if it is continued continuously once counterclockwise around the circle $\{|z|=2\}$?

Solution

We have that the function is

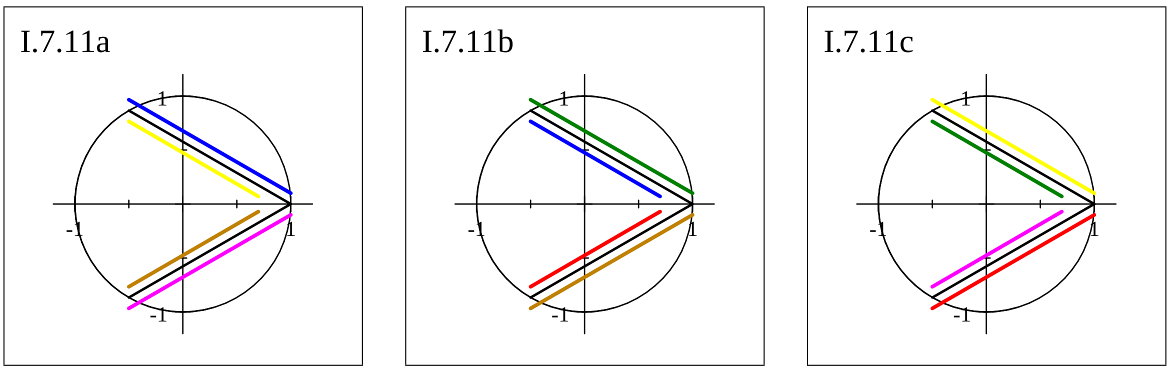
$$z^{4} \sqrt{\left(1-1/z^{3}\right)(1+1/z)^{3}(1-1/z)}$$

The branch that is positive for $z=2$ is, it is defined continuously for $|z|>1$. Branch return to original value around circle $|z|=2$. We can construct a Riemann Surface by making cuts at $0,-1, e^{2\pi i/3}$ and $e^{-2\pi i/3}$.
The function for the Riemann Surface is

$$\pm(z-1)(z+1) \sqrt{z(z+1)\left(z-e^{2\pi i/3}\right)\left(z-e^{-2\pi i/3}\right)}.$$

I.7.11
Find the branch points of $\sqrt[3]{z^3-1}$ and describe the Riemann surface of the function.

Solution



We rewrite as follows

$$\sqrt[3]{z^3-1} = \sqrt[3]{(z-1)\left(z-e^{2\pi i/3}\right)\left(z-e^{-2\pi i/3}\right)}.$$

This equation will have 3 branch points, which are the cube root of unity. So the phase factor is
$e^{2\pi i/3}$. The Riemann surface is obtained ty pasting three sheets with the corresponding branch cuts, we end up with a one hole torus.
Make two cuts, from 1 to $e^{\pm 2\pi i/3}$, on each sheet. In this case the cuts share common endpoint.
We nee three shets where $f_0(z)$, $f_1(z) = e^{2\pi i/3} f_0(z)$ and $f_2(z) = e^{-2\pi i/3} f_0(z)$
Note: Can use Riemann–formula to see that surface is a torus.
Cheek by going around each little tip what phase change to, which of for your sheet.

I.8.1
Establish the following addition formulae
(a) $\cos(z+w) = \cos z \cos w - \sin z \sin w$,
(b) $\sin(z+w) = \sin z \cos w + \cos z \sin w$,
(c) $\cosh(z+w) = \cosh z \sinh w + \sinh z \sinh w$,
(d) $\sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w$,

Solution
(a)
Using the definitions of sine and cosine functions given on page 29 in CA we have,

$$\begin{aligned}
&\cos(z+w) = \\
&= \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} = \frac{1}{4}\left[2e^{i(z+w)} + 2e^{-i(z+w)}\right] = \\
&= \frac{1}{4}\left[\left(e^{i(z+w)} + e^{i(z-w)} + e^{-i(z-w)} + e^{-i(z+w)}\right) + \right. \\
&\left. + \left(e^{i(z+w)} - e^{i(z-w)} - e^{-i(z-w)} + e^{-i(z+w)}\right)\right] = \\
&= \frac{e^{iz} + e^{-iz}}{2}\frac{e^{iw} + e^{-iw}}{2} - \frac{e^{iz} - e^{-iz}}{2i}\frac{e^{iw} - e^{-iw}}{2i} = \\
&= \cos z \cos w - \sin z \sin w.
\end{aligned}$$

(b)
Using the addition formula (a), and using that $\sin z = \cos(z - \pi/2)$ and $\cos z = -\sin(z - \pi/2)$ we have,

$$\begin{aligned}
\sin(z+w) &= \cos\left(z + w - \frac{\pi}{2}\right) = \\
&= \cos z \cos\left(w - \frac{\pi}{2}\right) - \sin z \sin\left(w - \frac{\pi}{2}\right) = \\
&= \cos z \sin w + \sin z \cos w = \sin z \cos w + \cos z \sin w.
\end{aligned}$$

(c)
Using the addition formula (a) and the formulas $\cos(iz) = \cosh z$ and $\sin(iz) = i \sinh z$ given on page 30 in CA we have,

$$\cosh(z+w) = \cos(i(z+w)) = \cos(iz)\cos(iw) - \sin(iz)\sin(iw) =$$
$$= \cosh z \cdot \cosh w - i\sinh z \cdot i\sinh w = \cosh z\cosh w + \sinh z\sinh w.$$

(d)
Using the addition formula (b) and the formulas $\cos(iz) = \cosh z$ and $\sin(iz) = i\sinh z$ given on page 30 in CA we have,

$$\sinh(z+w) = -i\sin(i(z+w)) = -i[\sin(iz)\cos(iw) + \cos(iz)\sin(iw)] =$$
$$= -i[i\sinh z \cdot \cosh w + \cosh z \cdot i\sinh w] = \sinh z\cosh w + \cosh z\sinh w.$$

I.8.2
Show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$, where $z = x + iy$. Find all zeros and periods of $\cos z$.

Solution.
Use trigonometric formulas from page 29 and 30 in CA we have,

$$\cos z = \cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy) = \cos x \cosh y - i \sin x \sinh y.$$

Now take the modulus squared, and use $\cosh^2 y = 1 + \sinh^2 y$,

$$\begin{aligned}|\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \\ &= \cos^2 x \left(1 + \sinh^2 y\right) + \sin^2 x \sinh^2 y = \\ &= \cos^2 x + \left(\cos^2 x + \sin^2 x\right) \sinh^2 y = \cos^2 x + \sinh^2 y.\end{aligned}$$

The identity for $|\cos z|^2$ shows that the only zeros of $\cos z$ are the zeros of $\cos z$ on the real axis, because

$$\cos z = 0 \Leftrightarrow \begin{cases} \cos x = 0 & \Leftrightarrow \quad x = \frac{\pi}{2} + m\pi, \quad m = 0, \pm 1, \pm 2, \dots, \\ \sinh y = 0 & \Leftrightarrow \quad y = 0. \end{cases}$$

Translation by any period λ of $\cos z$ sends zeros to zeros. Thus any period is an integral multiple of π, and since odd integral multiples are not periods, the only periods of $\cos z$ are $2\pi n$, $-\infty < n < \infty$.

I.8.3
Find all zeros and periods of $\cosh z$ and $\sinh z$.

Solution
We have that $\cosh z = \cos(iz)$, thus the zeros of $\cosh z$ are at $z = i\pi/2 + im\pi$, $m = 0, \pm 1, \pm 2, \ldots$, and the periods of $\cosh z$ are $2\pi mi$, $m = 0, \pm 1, \pm 2, \ldots$.

We have that $\sinh z = -i \sin(iz)$, thus the zeros of $\sinh z$ are at $z = m\pi i$, $m = 0, \pm 1, \pm 2, \ldots$, and the periods of $\cosh z$ are $2\pi mi$, $m = 0, \pm 1, \pm 2, \ldots$.

I.8.4
Show that

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right),$$

where both sides of the identity are to be interpreted as subsets of the complex plane. In other words, show that $\tan w = z$ if and only if $2iw$ is one of the values of the logarithm featured on the right.

Solution
Set $z = \tan w$, we have that

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{e^{iw} - e^{-iw}}{i\left(e^{iw} + e^{-iw}\right)} = \frac{e^{2iw} - 1}{i\left(e^{2iw} + 1\right)}.$$

Solve for e^{2iw}

$$e^{2iw} = \frac{1+iz}{1-iz},$$

and take logarithm and solve for w

$$w = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right).$$

Because $z = \tan w$ then $w \in \tan^{-1} z$ and we have the identity

$$\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right).$$

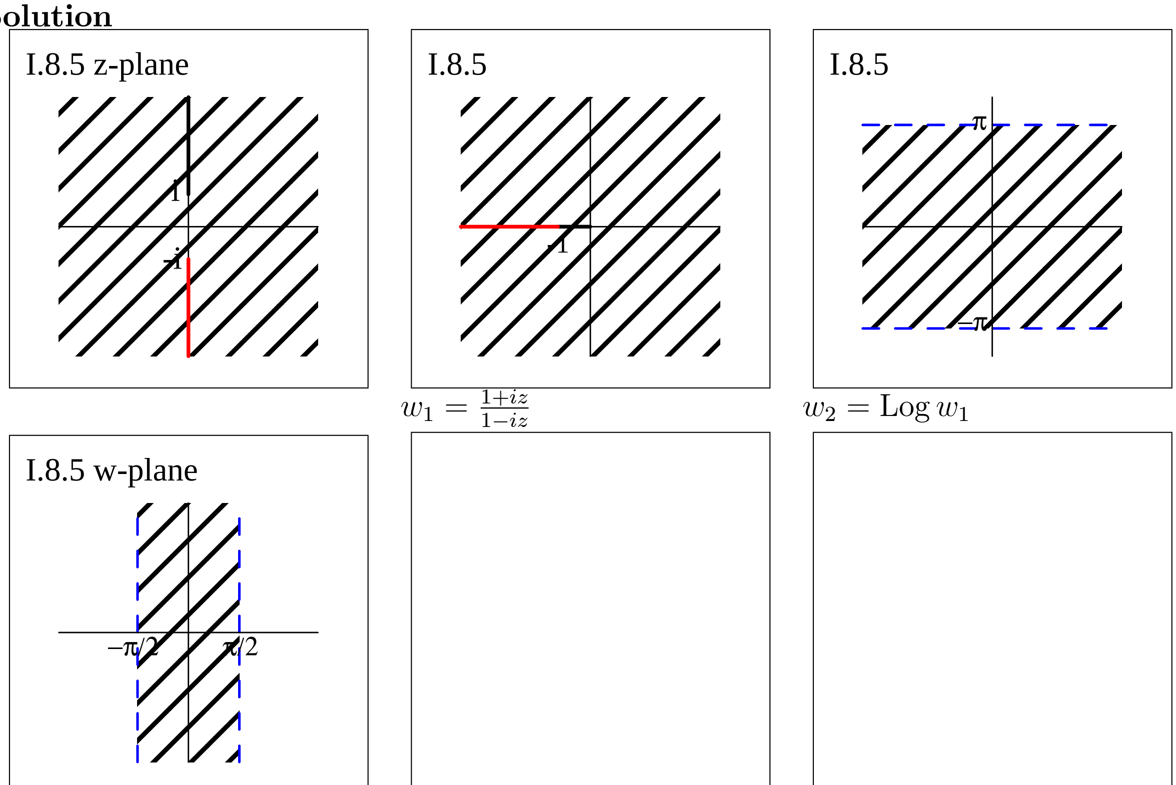
I.8.5

Let S denote the two slits along the imaginary axis in the complex plane, one running from i to $+i\infty$, the other from $-i$ to $-i\infty$. Show that $(1+iz)/(1-iz)$ lies on the negative real axis $(-\infty,0]$ if and only if $z \in S$. Show that the principal branch

$$\operatorname{Tan}^{-1} z = \frac{1}{2i}\operatorname{Log}\left(\frac{1+iz}{1-iz}\right)$$

maps the slit plane $\mathbb{C}\backslash S$ one-to-one onto the vertical strip $\{|\operatorname{Re} w| < \pi/2\}$.

Solution



$w_1 = \frac{1+iz}{1-iz}$ $\quad w_2 = \operatorname{Log} w_1$

$w = \frac{1}{2i} w_2 = \frac{1}{2i}\operatorname{Log}\left(\frac{1+iz}{1-iz}\right)$

We begin to show that $\frac{1+iz}{1-iz} \in (-\infty, 0]$ if and only if $z \in (-i\infty, -i] \cup [i, i\infty)$. Set

$$\frac{1+iz}{1-iz} = w_1$$

and solve for z, thus

$$z = \frac{1 - w_1}{1 + w_1} i$$

As w_1 goes from $-\infty$ to -1, $\frac{1-w_1}{1+w_1} i$ goes from $-i$ to $-i\infty$ along $i\mathbb{R}$, and when w_1 goes from -1 to 0, $\frac{1-w_1}{1+w_1} i$ goes from $i\infty$ to i along $i\mathbb{R}$.
Since we have that

$$\frac{1 + iz}{1 - iz} = w_1 \Rightarrow z = \frac{1 - w_1}{1 + w_1} i$$

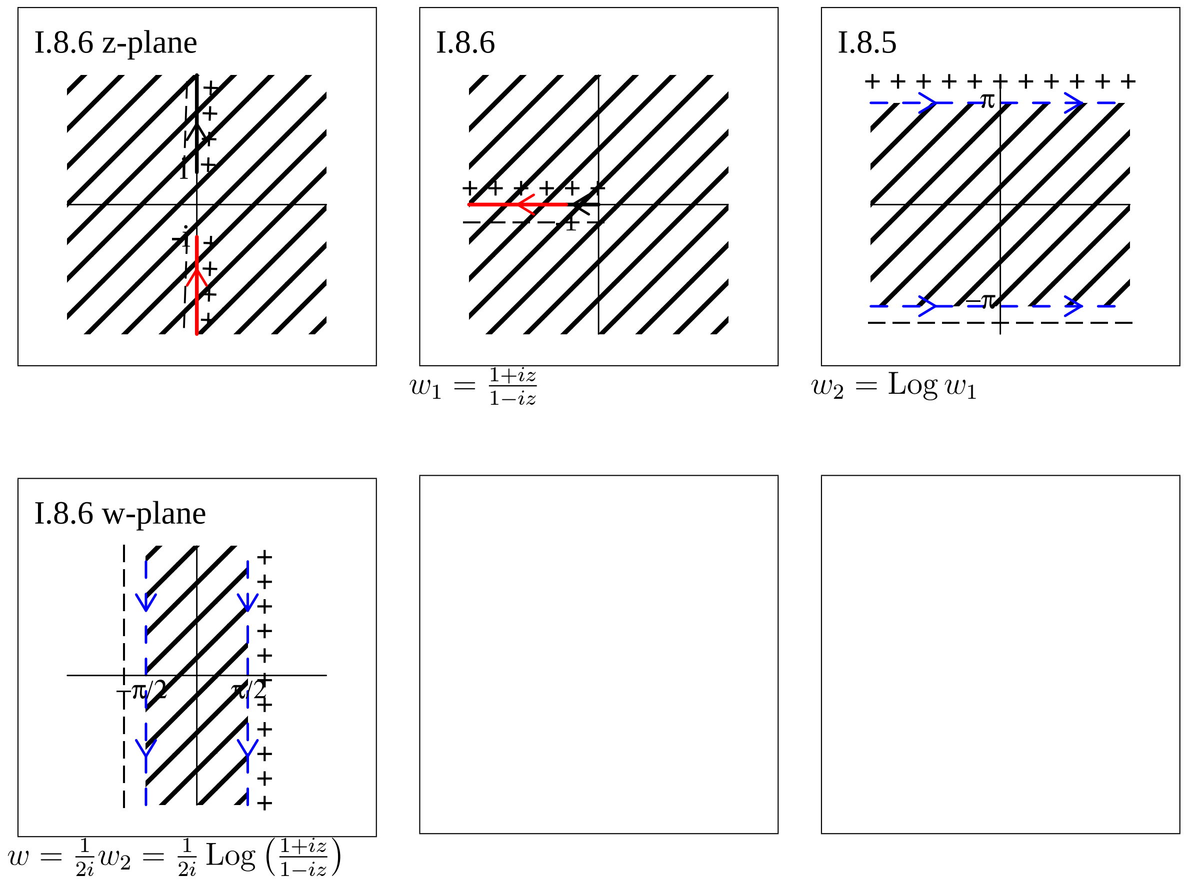
the map is one to one.
Remark that the map makes correspondence between the interval $[-i. - i\infty]$ in $z -$ plane and $[-\infty, -1]$ in $w_1 -$ plane, and between interval $[i, i\infty]$ in z $-$ plane and $[-1, 0]$ in $w_1 -$ plane.
We have that $w_2 = \operatorname{Log} w_1$ maps $C \backslash (-\infty, 0)$ onto $\{|\operatorname{Im} w| < \pi\}$, thus $w = \frac{w_2}{2i}$ maps $|\operatorname{Im} w_2| < \pi$ onto $\left\{|\operatorname{Re} w| < \frac{\pi}{2}\right\}$, se figures.
We have that the function

$$\operatorname{Tan}^{-1} z = \frac{1}{2i} \operatorname{Log}\left(\frac{1 + iz}{1 - iz}\right)$$

maps $C \backslash S$ onto $\left\{|\operatorname{Re} w| < \frac{\pi}{2}\right\}$, where $\operatorname{Tan}^{-1} z$ is the principal branch for $\tan^{-1} z$. The other branches are given by $f_n(z) = \operatorname{Tan}^{-1} z + n\pi$, $-\infty < n < \infty$.

I.8.6
Describe the Riemann surface for $\tan^{-1} z$.

Solution



We have that

$$f_0(z) = \operatorname{Tan}^{-1} z = \frac{1}{2i} \operatorname{Log}\left(\frac{1+iz}{1-iz}\right),$$

other branches of $\tan^{-1} z$ are $f_n(z) = f_0(z) + n\pi$, where $-\infty < n < \infty$. Use one copy of the double slit plane S for each integer n, and define $f_n(z) = \operatorname{Tan}^{-1} z + n\pi$ on the n^{th} sheet to the $(n+1)$ th sheet along one of the cuts, so that $f_n(z)$ and $f_{n+1}(z)$ have same value at the junction.

Make infinite many copies of S and call them S_n. Define $f_n(z) = \operatorname{Tan}^{-1} z + n\pi$ on S_n. Identify "+" side of cut on S_n to "$-$" side of cut on S_{n+1}. Then f_n on S_n continuous to f_{n+1} on S_{n+1}, and f_n maps S_n onto vertical strip $\left\{n - \frac{1}{2} < \operatorname{Re} z < n + \frac{1}{2}\right\}$. Note that composite function is not defined at $\pm i$ and $\pm\infty$ (endpoints of slits), and its image omits the sequence $\frac{1}{2} + n$, $-\infty < n < \infty$.

I.8.7

Set $w = \cos z$ and $\zeta = e^{iz}$. Show that $\zeta = w \pm \sqrt{w^2 - 1}$. Show that

$$\cos^{-1} w = -i \log \left[w \pm \sqrt{w^2 - 1} \right],$$

where both sides of the identity are to be interpreted as subsets of the complex plane.

Solution

Set $w = \cos z$ and $\zeta = e^{iz}$, we have that

$$w = \cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{\zeta + 1/\zeta}{2}.$$

Solve for ζ

$$\zeta = w \pm \sqrt{w^2 - 1},$$

and set $\zeta = e^{iz}$ and take the logarithm and solve for z

$$z = -i \log \left(w \pm \sqrt{w^2 - 1} \right).$$

Because $w = \cos z$ then $z \in \cos^{-1} w$ we have the identity

$$\cos^{-1} w = -i \log \left(w \pm \sqrt{w^2 - 1} \right).$$

I.8.8

Show that the vertical strip $|\mathrm{Re}\,(w)| < \pi/2$ is mapped by the function $z\,(w) = \sin w$ one-to-one onto the complex $z-$ plane with two slits $(-\infty, -1]$ and $[+1, +\infty)$ on the real axis. Show that the inverse function is the branch of $\sin^{-1} z = -i\,\mathrm{Log}\left(iz + \sqrt{1-z^2}\right)$ obtained by taking the principal value of the square root. Hint. First show that the function $1 - z^2$ on the slit plane omits the negative real axis, so that the principal value of the square root is defined and continuous on the slit plane, with argument in the open interval between $-\pi/2$ and $\pi/2$.

Solution

Set $w = x + iy$, by formulas on page 30 in CA we have that

$$\sin w = \sin\,(x+iy) = \sin\,(x)\cos\,(iy) + \cos\,(x)\sin\,(iy) = \\ = \sin x \cosh\,(y) + i\cos\,(x)\sinh y.$$

If $|\mathrm{Re}\, w| = |x| < \pi/2$ then $\sin w$ is mapped on $\mathbb{C}\backslash\,(-\infty, -1] \cup [+1, +\infty)$, because

$$\mathrm{Im}\,(\sin w) = 0 \Rightarrow y = 0 \Rightarrow \sin w = \sin\,(x)$$

and

$$-1 < \sin\,(x) < 1$$

for these x.

Let now $z = \sin w$. The value of w is referred to as $\sin^{-1} z$, that is, the complex number whose sine is z. Note that

$$z = \frac{e^{iw} - e^{-iw}}{2i}. \tag{1}$$

Now with $\zeta = e^{iw}$ and $1/\zeta = e^{-iw}$ in (1), we have

$$z = \frac{\zeta - 1/\zeta}{2i}.$$

Mulitplying the above by $2i\zeta$ and doing some rearranging, we find that

$$2iz\zeta = \zeta^2 - 1 \qquad \text{or} \qquad \zeta^2 - 2iz\zeta - 1 = 0.$$

With the quadratic formula, we solve this equation for ζ and find

$$\zeta = iz + \sqrt{1-z^2} \qquad \text{or} \qquad e^{iw} = iz + \sqrt{1-z^2}$$

We now take the logarithm of both sides of the last equation and divide the result by i to obtain

$$w = \frac{1}{i}\log\left(iz + \sqrt{1-z^2}\right)$$

and, since $w \in \sin^{-1} z$,

$$\sin^{-1} z = -i\log\left(iz \pm \sqrt{1-z^2}\right).$$

II	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1		X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		
2	X	X	X	X	X	X													
3	X	X	X		X			X											
4		X		X		X	X		X										
5	X	X	X	X	X	X	X	X											
6																			
7																			
8																			

II.1.1
Establish the following:

(a) $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$ **(c)** $\lim\limits_{n\to\infty} \frac{2n^p+5n+1}{n^p+3n+1} = 2, p > 1$
(b) $\lim\limits_{n\to\infty} \frac{n}{n^2+1} = 0$ **(d)** $\lim\limits_{n\to\infty} \frac{z^n}{n!} = 0, z \in \mathbb{C}.$

Solution
(a)

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$$

(b)

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n^2} = 0.$$

(c)

$$\lim_{n\to\infty} \frac{2n^p+5n+1}{n^p+3n+1} = \lim_{n\to\infty} \frac{2+5n^{1-p}+n^{-p}}{1+3n^{1-p}+n^{-p}} = 2.$$

(d)
Let m be a positive integer so that $m > 2\,|z|$ and let $n > m$, then

$$\begin{aligned} 0 \le \left|\frac{z^n}{n!}\right| &= \frac{|z|^m\,|z|^{n-m}}{m!\cdot(m+1)\cdot(m+2)\cdot\dots\cdot n} = \\ &= \frac{|z|^m}{m!}\cdot\frac{|z|}{m+1}\cdot\frac{|z|}{m+2}\cdot\dots\cdot\frac{|z|}{n} \le \frac{|z|^m}{m!}\cdot\frac{|z|}{m}\cdot\frac{|z|}{m}\cdot\dots\cdot\frac{|z|}{m} \le \\ &\le \frac{|z|^m}{m!}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\dots\cdot\frac{1}{2} = \frac{|z|^m}{m!}\cdot\frac{1}{2^{n-m}} \to 0 \end{aligned}$$

as $n \to \infty$, it follows by instängning that $\left|\frac{z^n}{n!}\right| \to 0$ as $n \to \infty$ as was to be shown.

II.1.2
For which values of z is the sequence $\{z^n\}_{n=1}^{\infty}$ bounded? For which values of z does the sequence converge to 0?

Solution
The sequence is bounded for $|z| \leq 1$, and the series converge to 0 for $|z| < 1$.

II.1.3
Show that $\{n^n z^n\}$ converges only for $z = 0$.

Solution
If we choose

$$n > \frac{2}{|z|}$$

then

$$2 < n \cdot |z|$$

and

$$|n^n z^n| = |n \cdot z|^n > 2^n \to \infty$$

when $n \to \infty$ for $z \neq 0$.

II.1.4
Show that $\lim\limits_{N\to\infty} \frac{N!}{N^k (N-k)!} = 1, \qquad k \geq 0$**.**

Solution
We have that

$$\frac{N!}{N^k (N-k)!} = \frac{N(N-1)\ldots(N-k+1)}{N \cdot N \cdot \ldots \cdot N} = \left(1 - \frac{1}{N}\right)\left(1 - \frac{2}{N}\right)\ldots\left(1 - \frac{k-1}{N}\right) \to 1$$

as $N \to \infty$ (k is fixed).

II.1.5
Show that the sequence

$$b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n, \qquad n \geq 1,$$

is decreasing, while the sequence $a_n = b_n - 1/n$ is increasing. Show that the sequences both converge to the same limit γ. Show that $\frac{1}{2} < \gamma < \frac{3}{5}$. Remark. The limit of the sequence is called Euler's constant. It is not known whether Euler's constant is a rational number or an irrational number.

Solution
We have that

$$\begin{aligned} b_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n} - \log n, \\ a_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \log n. \end{aligned}$$

We trivially have that

$$\log n = \int_1^n \frac{1}{t} dt \tag{(1)}$$

An elementary estimation gives

$$\frac{1}{k+1} < \int\limits_k^{k+1} \frac{1}{t} dt < \frac{1}{k} \tag{(2)}$$

We take the difference $a_{n+1} - a_n$ and by (1) and (2) we have that

$$a_{n+1} - a_n = \frac{1}{n} - \log(n+1) + \log n = \frac{1}{n} - \int\limits_n^{n+1} \frac{dt}{t} > 0,$$

thus $a_{n+1} > a_n$, so the sequence a_n is increasing.
We take the difference $b_{n+1} - b_n$ and by (1) and (2) we have that

$$b_{n+1} - b_n = \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \int\limits_n^{n+1} \frac{dt}{t} < 0,$$

thus $b_{n+1} < b_n$, so the sequence b_n is decreasing.
We have after some investigation on the increasing sequence a_n that

$$a_n \geq a_7 = \frac{49}{20} - \log 7 > \frac{1}{2}, \qquad n \geq 7,$$

thus $\frac{1}{2} < a_n$, if $n \geq 7$.
We have after some investigation on the decreasing sequence b_n that

$$b_n \leq b_{22} = \frac{19\,093\,197}{5173\,168} - \log 22 < \frac{3}{5}, \qquad n \geq 22$$

thus $b_n \leq 3/5$, if $n \geq 22$.
The both sequences converge to the same limit γ because $b_n - a_n = 1/n$ have the limit 0 as $n \to \infty$. And this limit γ is in the interval $\frac{1}{2} < \gamma < \frac{3}{5}$.

II.1.6
For a complex number α, we define the binomial coefficient "α choose n" by

$$\binom{\alpha}{0} = 1, \qquad \binom{\alpha}{n} = \frac{\alpha\,(\alpha-1)\cdots(\alpha-n+1)}{n!}, \qquad n \geq 1.$$

Show the following
(a) The sequence $\binom{\alpha}{n}$ is bounded if and only if $\operatorname{Re}\alpha \geq -1$.
(b) $\binom{\alpha}{n} \to 0$ if and only if $\operatorname{Re}\alpha > -1$.
(c) If $\alpha \neq 0, 1, 2, \ldots$, then $\binom{\alpha}{n+1} \Big/ \binom{\alpha}{n} \to -1$.
(d) If $\operatorname{Re}\alpha \leq -1$, $\alpha \neq -1$, then $\left|\binom{\alpha}{n+1}\right| > \left|\binom{\alpha}{n}\right|$ for all $n \geq 0$.
(e) If $\operatorname{Re} a > -1$ and α is not an integer, then $\left|\binom{\alpha}{n+1}\right| < \left|\binom{\alpha}{n}\right|$ for n large

Solution
(a)
We have that

$$\left|\binom{\alpha}{n}\right| = \prod_{k=1}^{n} \left|1 - \frac{\alpha+1}{k}\right| = \prod_{k=1}^{\infty} = \sqrt{1 - \frac{2\,(1+\operatorname{Re}\alpha)}{k} + \frac{(1+\operatorname{Re}\alpha)^2}{k^2} + \frac{(1+\operatorname{Im}\alpha)^2}{k^2}}.$$

Let

$$t_n = \frac{(1+\operatorname{Re}\alpha)^2}{k^2} + \frac{(1+\operatorname{Im}\alpha)^2}{k^2} - \frac{2\,(1+\operatorname{Re}\alpha)}{k}$$

If $\operatorname{Re}\alpha < -1$, we have that $t_k > 0$ and $\left|\binom{\alpha}{n}\right| \to \infty$ then $n \to \infty$ because $\sum t_k$ diverge is $\binom{\alpha}{n}$ unlimited.

If $\operatorname{Re}\alpha = -1$, we have that $t_k = \frac{(\operatorname{Im}\alpha)^2}{k^2}$ and $\left|\binom{\alpha}{n}\right| \to A$ then $n \to \infty$, and $A \neq 0$ becauxe for all k we have that $\left|1 - \frac{\alpha+1}{k}\right| > 1$ and $\sum t_k = \sum \frac{(\operatorname{Im}\alpha)^2}{k^2} < \infty$ so $\binom{\alpha}{n}$ is limited.

If $\operatorname{Re}\alpha > -1$ we have that it exists $k \in \mathbb{N}$ so that $-1 < t_k < 0$ for all $k > K$. This gives that $\left|\binom{\alpha}{n}\right|$ is a decreasing series with respect to n, and so limited.

The conclution is that $\binom{\alpha}{n}$ is limited if and only if $\operatorname{Re}\alpha \geq -1$.

b)
We have that $\binom{\alpha}{n} \to 0$ if and only if $\left|\binom{\alpha}{n}\right| \to 0$. From (a) we see that the only possibility for this is $\operatorname{Re}\alpha > -1$. For $\operatorname{Re}\alpha > -1$ we can see that $\left|\binom{\alpha}{n}\right| \geq 0$ is decreasing. Let $M = \lim\limits_{n\to\infty}\binom{\alpha}{n}$. Then we hav that $M = \lim\limits_{n\to\infty}\binom{\alpha}{n} = \lim\limits_{n\to\infty}\binom{\alpha}{n}\binom{\alpha-n}{n+1} = -M$ so $M = -M \iff M = 0$ so $\binom{\alpha}{n} \to 0$, then $n \to \infty$ if and only if $\operatorname{Re}\alpha > -1$.

c)
If $\alpha \neq 0, 1, 2, \ldots$, then $\binom{\alpha}{n} \neq 0$ and we can divide, to get

$$\frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} = \frac{\frac{\alpha(\alpha-1)\ldots(\alpha-(n+1)+1)}{(n+1)!}}{\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}} = \frac{\alpha-n}{n+1} = \frac{\frac{\alpha}{n}-1}{1+\frac{1}{n}} \to -1$$

as $n \to \infty$.

d)
$\frac{\left|\binom{\alpha}{n+1}\right|}{\left|\binom{\alpha}{n}\right|} = \frac{|\alpha-n|}{n+1} = \frac{\text{distance}(\alpha,n)}{\text{distance}(-1,n)} > 1$, if $\operatorname{Re}\alpha \leq -1$, $\alpha \neq -1$

e)
Let α be fixed and $\operatorname{Re}\alpha > -1$. Then

$$\frac{\left|\binom{\alpha}{n+1}\right|}{\left|\binom{\alpha}{n}\right|} = \sqrt{\frac{(\operatorname{Re}\alpha-n)^2+(\operatorname{Im}\alpha)^2}{(n+1)^2}} < 1 \qquad \text{for} \quad n > \frac{|\alpha|^2-1}{2(1+\operatorname{Re}\alpha)}.$$

Parts (a) and (b) seems to require some facts about products:

$\prod\limits_{k=1}^{n}\left(1+\frac{\beta}{k}\right) \to \infty$ as $n \to \infty$, $\beta > 0$
$\prod\limits_{k=1}^{n}\left(1-\frac{\beta}{k}\right) \to \infty$ as $n \to \infty$, $\beta > 0$
$\prod\limits_{k=1}^{n}\left(1-\frac{\beta}{k^2}\right) \to A \neq 0$ as $n \to \infty$

II.1.7

Define $x_0 = 0$, and define by induction $x_{n+1} = x_n^2 + \frac{1}{4}$ for $n \geq 0$. Show that $x_n \to \frac{1}{2}$.

Hint. Show that the sequence is bounded and monotone, and that any limit satisfies $x = x^2 + \frac{1}{4}$.

Solution

We first prove by induction that for every integer $n \geq 0$ we have

$$0 \leq x_n \leq \frac{1}{2} \qquad \text{and} \qquad x_n \leq x_{n+1}.$$

For $n = 0$, we have $x_0 = 0$ and $x_1 = x_0^2 + \frac{1}{4} = \frac{1}{4}$ and so

$$0 \leq x_0 \leq \frac{1}{2} \qquad \text{and} \qquad x_0 = 0 \leq \frac{1}{4} = x_1.$$

Hence, the assertion holds for $n = 0$. Now suppose that the assertion holds for $n = k$, that is,

$$0 \leq x_k \leq \frac{1}{2} \qquad \text{and} \qquad x_k \leq x_{k+1}.$$

For $0 \leq x \leq \frac{1}{2}$, it is straightforward to show that $x \leq x^2 + \frac{1}{4} \leq \frac{1}{2}$. Since $0 \leq x_k \leq \frac{1}{2}$ and $x_{k+1} = x_k^2 + \frac{1}{4}$ we have

$$0 \leq x_k \leq x_{k+1} = x_k^2 + \frac{1}{4} \leq \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2}$$

and hence $0 \leq x_{k+1} \leq \frac{1}{2}$, and

$$x_{k+1} \leq x_{k+1}^2 + \frac{1}{4} = x_{k+2}.$$

Hence, the inductive step holds and so the sequence $\{x_n\}_n$ is bounded and increasing. It follows by Bounded Monotone Sequence Theorem that it converges, say $x_n \to x$. Note that the real-valued function f given by $f(t) = t^2 + \frac{1}{4}$ for $t \in \mathbb{R}$ is continuous and therefore we have,

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x) = x^2 + \frac{1}{4}.$$

It follows that $x = x^2 + \frac{1}{4}$ and so $x = \frac{1}{2}$. Hence, $x_n \to \frac{1}{2}$ as required.

II.1.8
Show that if $s_n \to s$, then $|s_n - s_{n-1}| \to 0$.

Solution
We use standard $\varepsilon/2$ proof and let $s_n \to s$.
Let $\varepsilon > 0$, choose N such that

$$|s_n - s| < \varepsilon/2$$

for $n > N$.
If $n > N$, then we have

$$|s_{n+1} - s_n| = |s_{n+1} - s + s - s_n| \leq |s_{n+1} - s| + |s_n - s| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus

$$|s_{n+1} - s_n| \to 0$$

as $n \to \infty$.

II.1.9

Plot each sequence and determine its $\liminf$ **and** $\limsup$**.**

(a) $s_n = 1 + \frac{1}{n} + (-1)^n$ **(c)** $s_n = \sin(\pi n/4)$

(b) $s_n = (-n)^n$ **(d)** $s_n = x^n$ ($x \in \mathbb{R}$ **fixed**)

Solution

(a)

$\limsup s_n = 2$ $\liminf s_n = 0$

(b)

$\limsup s_n = \infty$ $\liminf s_n = -\infty$

(c)

$\limsup s_n = 1$ $\liminf s_n = 1$

(d)

$$\limsup s_n = \begin{cases} +\infty, & |x| > 1, \\ 1, & |x| = 1, \\ 0, & |x| < 1. \end{cases}$$

and

$$\liminf s_n = \begin{cases} -\infty, & -\infty < x < -1, \\ -1, & x = -1, \\ 0, & |x| < 1, \\ 1, & x = 1, \\ +\infty, & x > 1. \end{cases}$$

II.1.10

At what points are the following function continuous? Justify your answer.

(a)	z	**(c)**	$z^2/\|z\|$
(b)	$z/\|z\|$	**(d)**	$z^2/\|z\|^3$

Solution

(a)

continuous everywhere

(b)

Continuous except at $z = 0$, where it is not defined

(c)

Continuous for $|z| \neq 0$. It is not defined at $z = 0$, but it has a limit 0 as $z \to 0$. If we define the function to be 0 at $z = 0$, it is contionuous there.

(d)

Contionuous for $|z| \neq 0$. It has no limit as $z \to 0$

II.1.11
At what points does the function $\operatorname{Arg} z$ have a limit? Where is $\operatorname{Arg} z$ continuous? Justify your answer.

Solution
$\operatorname{Arg} z$ have a limit at each point of $\mathbb{C}\setminus(-\infty, 0]$. It is continuous in $\mathbb{C}\setminus(-\infty, 0]$. It is discontinuous at each point of $(-\infty, 0]$. Values of function $\to \{\pm\pi\}$ at points of $(-\infty, 0]$, the function may take any value in intervall $[-\pi, \pi]$ as $z \to 0$. The value is depending in that direktion we approach $z = 0$.

II.1.12

Let $h(z)$ be the restriction of the function $\operatorname{Arg} z$ to the lower half-plane $\{\operatorname{Im} z < 0\}$. At what points does $h(z)$ have a limit? What is the limit?

Solution

No limit at 0, has limit at all other points of closed lower half-plane, limit $= -\pi$ at points of $(-\infty, 0)$.

II.1.13
For which complex values of α does the principal value of z^α have a limit as z tends to 0? Justify your answer.

Solution
Set $z = re^{i\theta} = e^{\operatorname{Log} r} e^{i\theta} = e^{\operatorname{Log} r + i\theta}$ and $\alpha = \operatorname{Re}\alpha + i \operatorname{Im}\alpha$, then

$$\begin{aligned} z^\alpha = \left(e^{\operatorname{Log} r + i\theta}\right)^{\operatorname{Re}\alpha + i\operatorname{Im}\alpha} &= e^{\operatorname{Log} r \operatorname{Re}\alpha + i \operatorname{Log} r \operatorname{Im}\alpha + i\theta \operatorname{Re}\alpha - \theta \operatorname{Im}\alpha} = \\ &= e^{\operatorname{Log} r \operatorname{Re}\alpha} e^{-\theta \operatorname{Im}\alpha} e^{i(\operatorname{Log} r \operatorname{Im}\alpha + \theta \operatorname{Re}\alpha)} = r^{\operatorname{Re}\alpha} e^{-\theta\operatorname{Im}\alpha} e^{i(\operatorname{Log} r \operatorname{Im}\alpha + \theta \operatorname{Re}\alpha)}, \end{aligned}$$

thus

$$|z^\alpha| = r^{\operatorname{Re}\alpha} e^{-\theta \operatorname{Im}\alpha}.$$

Because $e^{-\theta\operatorname{Im}\alpha}$ is bounded we look at $r^{\operatorname{Re}\alpha}$ in three intervals
If $\operatorname{Re}\alpha < 0$, $r^{\operatorname{Re}\alpha} \to \infty$ as $r \to 0$ thus $|z^\alpha|$ have no limit as $z \to 0$, and not z^α either.
If $\operatorname{Re}\alpha = 0$, we get $z^\alpha = e^{-\theta\operatorname{Im}\alpha}$, have no limit unless $\operatorname{Im}\alpha = 0$, thus we have limit if $\alpha = 0$.
If $\operatorname{Re}\alpha > 0$, $r^{\operatorname{Re}\alpha} \to 0$ as $r \to 0$ thus $|z^\alpha|$ have limit 0 as $z \to 0$, and $z^\alpha \to 0$ as $z \to 0$.
The conclution is $z^\alpha \to 0$ if $\operatorname{Re}\alpha > 0$, and $z^\alpha \to 1$ if $\alpha = 0$, otherwise we have no limit at 0.

II.1.14
Let $h(t)$ be a continuous complex-valued function on the unit interval $[0,1]$, and consider

$$H(z) = \int_0^1 \frac{h(t)}{t-z}dt.$$

Where is $H(z)$ defined? Where is $H(z)$ continuous? Justify your answer. Hint. Use the fact if $|f(t) - g(t)| < \varepsilon$ for $0 \le t \le 1$, then $\int_0^1 |f(t) - g(t)|\, dt < \varepsilon$.

Solution
We have that $H(z)$ is defined for $z \in \mathbb{C}\backslash [0,1]$. If $h(z_1) = 0$ for some $z_1 \in [0,1]$. Then $H(z)$ is also defined for $z = z_1$.
Then $H(z) = \int\limits_0^1 \frac{h(t)}{t-z} dz$ is continuous for $z \in \mathbb{C}\backslash [0,1]$. If $z_n \to z \in \mathbb{C}\backslash [0,1]$, then $\frac{h(t)}{t-z_n} \to \frac{h(t)}{t-z}$ uniformly for $0 \le t \le 1$, so the used proof shows that $H(z_n) \to H(z)$.

II.1.15

Which of the following sets are open subsets of $\mathbb{C}$? Which are close? Sketch the sets.

(a) **The punctured plane,** $\mathbb{C}\setminus\{0\}$.

(b) **The exterior of the open unitdisk in the plane,** $\{|z| \geq 1\}$.

(c) **The exterior of the closed unitdisk in the plane,** $\{|z| > 1\}$.

(d) **The plane with the open unitinterval removed,** $\mathbb{C}\setminus(0,1)$.

(e) **The plane with the closed unitinterval removed,** $\mathbb{C}\setminus[0,1]$.

(f) **The semidisk,** $\{|z| < 1, \operatorname{Im}(z) \geq 0\}$.

(g) **The complex plane,** $\mathbb{C}$.

Solution

(a) open	(c) open	(e) open	(g) both open and closed
(b) closed	(d) neither	(f) neither	

Note: Sketches are difficult to do reasonably.

II.1.16

Show that the slit plane $\mathbb{C}\setminus(-\infty,0]$ is star-shaped but not convex. Show that the slit plane $\mathbb{C}\setminus[-1,1]$ is not star-shaped. Show that a punctured disk is not star-shaped.

Solution

(a) $\mathbb{C}\setminus(-\infty,0]$ is star–shaped, because we can see every point from $z=1$. To show that $\mathbb{C}\setminus(-\infty,0]$ is not convex we may choose any two points such that the stright line segment joining them contains a point in $(-\infty,0]$. For example take $-1+i$ and $-1-i$. These two points are joined by a vertical line which contain -1. From this follows that $\mathbb{C}\setminus(-\infty,0]$ is not convex.

(b) $\mathbb{C}\setminus[-1,1]$ is not star–shaped. Given any z we have that $-z$ can not bee seen from a stright line between z and $-z$ that must contain 0. We also have that if z belong to the domain then $-z$ belongs also to the domain.

(c) Same argument as in (b) shows that $\mathbb{C}\setminus\{0\}$ is not star–shaped.

II.1.17
Show that a set is convex if and only if it is star-shaped with respect to each of its points.

Solution (A. Kumjian)
Let $D \subset \mathbb{C}$, since the empty set trivially satisfies both conditions we will assume that $D \neq \varnothing$.
Suppose first that D is convex. We must show that D is star-shaped with respect to each of its points. So fix $z_0 \in D$, to show that D is star-shaped with respect to z_0. We must show that for every point $z \in D$ the line segment connecting z_0 to z is contained in D. Let $z \in D$ be given, then since D is convex it contains the line segment joining z_0 and z. Hence, D is star-shaped with respect to z_0, then, since z_0 was chosen arbitrarily, D is star-shaped with respect to each of its points.

Conversely, suppose that D is star-shaped with respect to each of its points. To show that D is convex we must show that given two points in D, the line segment joining them is also contained in D. So let $z_0, z_1 \in D$ be given. Then since D is star-shaped with respect to z_0, D contains the line segment joining z_0 and z_1. Hence, D is convex.

II.1.18

Show that the following are equivalent for an open subset U of the complex plane.

(a)

Any two points of U can be joined by a path consisting of straight linesegments parallel to the coordinate axis.

(b)

Any continuously differentiable function $h(x, y)$ on U such that $\nabla h = 0$ is constant.

(c)

If V and W are disjoint open subsets of U such that $U = V \cup W$, then either $U = V$ or $U = W$. Remark. In the context of topological spaces, this latter property is taken as definition of connectedness.

Solution

Show that the following equivalent for an open subset U of the complex plane.
(a) $\Rightarrow$ (b) Suppose $\nabla h = 0$ in D. Fix $z_0 \in D$. If $z_1 \in D$, join by polygonal curve. $\nabla h = 0 \Rightarrow h$ is constant on each segment of curve $\Rightarrow h(z_0) = h(z_1)$. $\Rightarrow h$ is constant in D.
(b) $\Rightarrow$ (c) Suppose $\nabla h = 0$, h not constant, say $h(z_0) = 0$ for some z_0. Let $W = \{h = 0\}$, $V = \{h \neq 0\}$. Evidently V is open. Since $\nabla h = 0$, h is constant in a neighborhood of each point, so $\{h = 0\}$ is open and W is open. $W \neq \varnothing$, $V \neq \varnothing$, $W \cap V = \varnothing$, $W \cup V = U$
(c) $\Rightarrow$ (a) Suppose $z_0 \in U$. Let $D =$ points that can be joined to z_0 by a polygonal curves, intervals parallel to coordinate axis. Evidently D is open. Also if can connected points near $z_0 \in U$, then can connect to U, so $U \backslash D$ is open. By (c), $U \backslash D$ must be empty, some $D \neq \varnothing$.

II.1.19
Give a proof of the fundamental theorem of algebra along the following lines. Show that if $p(z)$ is a nonconstant polynomial, then $|p(z)|$ attains its minimum at some point $z_0 \in \mathbb{C}$. Assume that the minimum is attained at $z_0 = 0$, and that $p(z) = 1 + az^m + \cdots$, where $m \geq 1$ and $a \neq 0$. Contradict the minimality by showing that $\left|P\left(\varepsilon e^{i\theta_0}\right)\right| < 1$ for an appropriate choice of θ_0.

Solution (K. Seip)
Set

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where we assume $a_n \neq 0$. Since

$$\lim_{|z|\to\infty} \frac{|p(z)|}{|z|^n} = |a_n|,$$

$\exists\ R > 0$ such that $|p(z)| > |a_0|$ for all $|z| > R$. Thus $|p(z)|$ attains its minimum in $|z| \leq R$. We may assume it is attained at $z_0 = 0$. Suppose $a_0 \neq 0$, say $a_0 = 1$. Then $p(z) = 1 + az^m +$ higher order terms, $a \neq 0$. Choose $z = \varepsilon\left(-\frac{1}{a}\right)^{\frac{1}{m}}$ (any $m-$ th root does the job). Then

$$p(z) = 1 - \varepsilon^m + O\left(\varepsilon^{m+1}\right).$$

Thus for ε sufficiently small ε we have $|p(z)| < 1$, which is a contradiction.

Follow the suggestion.
Let

$$p(z) = a_0 + a_1 z + a_{m+1} z^{m+1} + ... + a_N z^N, a_N \neq 0.$$

Choose R so that

$$|a_N| R^N > |a_0| + |a_1| R + |a_{m+1}| R^{m+1} + ... + |a_{N-1}| R^{N-1}.$$

Then $p(z) \neq 0$ for $|z| = R$, the term $a_N z^N$ dominates and further $|p(z)| > |p(0)|$ for $|z| = R$.
Let z_0 be a point in the disk $|z| \leqslant R$ at which the continuous function for $|p(z)|$ attains a minimum. Then $|z_0| < R$, so that is a disk centred at z_0 so that $|p(z)| \geqslant |p(z_0)|$ on the disk. Suppose $p(z_0) \neq 0$. We can assume that $p(z_0) = 1$.
Write

$$p(z) = 1 + a_m P (z - z_0)^m + O\left((z - z_0)^{m+1}\right),$$

where $a_m \neq 0$. Suppose $a_m = r_0 e^{i\theta_0}$.
Conside

$$z_m = z_0 + \varepsilon e^{-i\theta_0/m} e^{-i\pi/m}, \varepsilon > 0.$$

We have

$$p(z_m) = 1 + r_0 e^{i\theta_0} e^m e^{-i\theta_0} e^{-i\pi} + O\left(\varepsilon^{m+1}\right) = 1 - r_0 \varepsilon^m + O\left(\varepsilon^{m+1}\right).$$

Have $p(z_m) \approx 1 - r_0 \varepsilon^m < 1$ for $\varepsilon > 0$ small. Contradiction! We conclude that $p(z_0) = 0$.

II.2.1

Find the derivatives of the following function.

(a)	$z^2 - 1$	**(c)**	$(z^2 - 1)^n$	**(e)**	$1/(z^2 + 3)$	**(g)**	$(az + b)/(cz + d)$
(b)	$z^n - 1$	**(d)**	$1/(1 - z)$	**(f)**	$z/(z^3 - 5)$	**(h)**	$1/(cz + d)^2$

Solution

(a)	$2z$	(c)	$n(z^2 - 1)^{n-1} 2z$	(e)	$-2z/(z^2 + 3)^2$	(g)	$(ad - bc)/(cz + d)^2$
(b)	nz^{n-1}	(d)	$1/(1 - z)^2$	(f)	$(-2z^3 - 5)/(z^3 - 5)^2$	(h)	$-2c/(cz + d)^3$

II.2.2
Show that

$$1+2z+3z^2+\cdots+nz^{n-1}=\frac{1-z^n}{(1-z)^2}-\frac{nz^n}{1-z}.$$

Solution
Use the geometric sum

$$1+z+z^2+z^3+\cdots+z^n=\frac{1-z^{n+1}}{1-z}, \qquad z\neq 1.$$

Differentiate both sides

$$\begin{aligned}
&1+2z+3z^2+\cdots+nz^{n-1}=\\
&\qquad=\frac{(n+1)(-1)z^n(1-z)+\left(1-z^{n+1}\right)}{(1-z)^2}=\\
&\qquad=\frac{nz^{n+1}-nz^n+z^{n+1}-z^n+1-z^{n+1}}{(1-z)^2}=\\
&\qquad\qquad=\frac{1-z^n}{(1-z)^2}+\frac{nz^{n+1}-nz^n}{(1-z)^2}=\frac{1-z^n}{(1-z)^2}-\frac{nz^n}{(1-z)}.
\end{aligned}$$

II.2.3
Show from the definition that the functions $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are not complex differentiable at any point.

Solution
Differentiation $f(z) = x = \operatorname{Re} z$ (from the definition)

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\operatorname{Re}(z + \Delta z) - \operatorname{Re}(z)}{\Delta z} =$$
$$= \lim_{\Delta z \to 0} \frac{\operatorname{Re} \Delta z}{\Delta z} = \begin{cases} 1, & \text{if } \Delta z = \Delta x \\ 0, & \text{if } \Delta z = i\Delta y \end{cases} .$$

Differentiation $g(z) = y = \operatorname{Im} z$ (from the definition)

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{\operatorname{Im}(z + \Delta z) - \operatorname{Im}(z)}{\Delta z} = \\ &= \lim_{\Delta z \to 0} \frac{\operatorname{Im} \Delta z}{\Delta z} = \begin{cases} 0, & \text{if } \Delta z = \Delta x \\ -i, & \text{if } \Delta z = i\Delta y \end{cases} . \end{aligned}$$

The functions $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are not complex differentiable at any point, because the functions have different limit as $\Delta z \to 0$ through real and imaginary axis.

II.2.4 (+ IV.8.2)

Suppose $f(z) = az^2 + bz\bar{z} + c\bar{z}^2$, where a, b, and c are fixed complex numbers. By differentiating $f(z)$ by hand, show that $f(z)$ is complex differentiable at z if and only if $bz + 2c\bar{z} = 0$. Where is $f(z)$ analytic?

Solution

We will find $\frac{d}{dz}f(z)$ using the limit definition

$$
\begin{aligned}
&\frac{f(z+\Delta z) - f(z)}{\Delta z} = \\
&= \frac{a(z+\Delta z)^2 + b(z+\Delta z)\overline{(z+\Delta z)} + c\overline{(z+\Delta z)}^2 - (az^2 + bz\bar{z} + c\bar{z}^2)}{\Delta z} = \\
&= \frac{a\left(z^2 + 2z\Delta z + \Delta z^2\right) + b\left(z\bar{z} + z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\overline{\Delta z}\right) + c\left(\bar{z}^2 + 2\bar{z}\overline{\Delta z} + \overline{\Delta z}^2\right) - (az^2 + bz\bar{z} + c\bar{z}^2)}{\Delta z} \\
&= \frac{2az\Delta z + a\Delta z^2 + bz\overline{\Delta z} + b\bar{z}\Delta z + b\Delta z\overline{\Delta z} + 2c\bar{z}\overline{\Delta z} + c\overline{\Delta z}^2}{\Delta z} = \\
&= 2az + a\Delta z + bz\frac{\overline{\Delta z}}{\Delta z} + b\bar{z} + b\overline{\Delta z} + 2c\bar{z}\frac{\overline{\Delta z}}{\Delta z} + c\frac{\overline{\Delta z}^2}{\Delta z} = \\
&= 2az + a\Delta z + b\bar{z} + b\overline{\Delta z} + c\overline{\Delta z}\frac{\overline{\Delta z}}{\Delta z} + (bz + 2c\bar{z})\frac{\overline{\Delta z}}{\Delta z}
\end{aligned}
$$

We know $\bar{z}$ is not analytic at any open set of $\mathbb{C}$ and $\lim\limits_{\triangle z \to 0} \frac{\Delta\bar{z}}{\Delta z}$ does not exist. Therefore, unless $bz + 2c\bar{z} = 0$, the derivative $\frac{d}{dz}f(z)$ would not exist. Therefore $f(z)$ is analytic on $\mathbb{C}$ if $b = c = 0$, otherwise is the function not analytic on any open set.

II.2.5
Show that if f is analytic on D, then $g(z) = \overline{f(\bar{z})}$ is analytic on the reflected domain $D^* = \{\bar{z} : z \in D\}$, and $g'(z) = \overline{f'(\bar{z})}$.

Solution (A. Kumjian)
Let f be analytic on the domain D and define g on D^* as above (note that D^* is also a domain). For a complex-valued function φ defined near a point z_0 it is easy to show that

$$\lim_{z \to z_0} \varphi(z) \text{ exists iff } \lim_{z \to z_0} \overline{\varphi(z)} \text{ exists,}$$

and if either exists, then the two limits are complex conjugates of each other. Let $z \in D^*$ be given. We first show that g is differentiable at z and that $g'(z) = \overline{f'(\bar{z})}$.

$$\lim_{h \to 0} \frac{1}{h} \left(g(z+h) - g(z)\right) = \lim_{h \to 0} \frac{1}{h} \left(\overline{f\left(\overline{z+h}\right)} - \overline{f(\bar{z})} \right) \stackrel{*}{=} \overline{\lim_{k \to 0} \frac{1}{k} \left(f(\bar{z}+k) - f(\bar{z})\right)} = \overline{f'(\bar{z})}$$

where equation $(*)$ follows using the substitution $k = \bar{h}$ and basic algebraic properties of conjugation, note that conjugation is a continuous function so $h \to 0$ iff $\bar{h} \to 0$. Since f is analytic on the domain D, its derivative f' is continuous on D. Further, since $g'(z) = \overline{f'(\bar{z})}$ for all $z \in D^*$, g' is the composite of continuous functions and therefore is itself continuous. Hence, g is analytic on D^*.

II.2.6
Let $h(t)$ be a continuous complex-valued function on the unit interval $[0,1]$, and define

$$H(z)=\int_0^1 \frac{h(t)}{t-z}dt, \qquad z\in\mathbb{C}\setminus[0,1].$$

Show that $H(z)$ is analytic and compute its derivative. Hint. Differentiate by hand, that is, use the defining identity (2.4) of complex derivative.

Solution
We use definition (2.4) of derivative to find $H'(z)$.

$$\begin{aligned}
&H'(z)= \\
&=\lim_{\Delta z\to 0}\frac{H(z+\Delta z)-H(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{1}{\Delta z}\left(\int_0^1 \frac{h(t)}{t-z-\Delta z}dt-\int_0^1\frac{h(t)}{t-z}dt\right)= \\
&\qquad=\lim_{\Delta z\to 0}\frac{1}{\Delta z}\left(\int_0^1 h(t)\left(\frac{1}{t-z-\Delta z}-\frac{1}{t-z}\right)\right)= \\
&\qquad=\lim_{\Delta z\to 0}\left(\int_0^1\frac{h(t)}{(t-z-\Delta z)(t-z)}\right)=\int_0^1\frac{h(t)}{(t-z)(t-z)}dt= \\
&\qquad=\int_0^1\frac{h(t)}{(t-z)^2}dt.
\end{aligned}$$

$H'(z)$ is continuous in z, also by uniform convergence of integrand $H(z)$ is analytic for $z\in\mathbb{C}\setminus[0,1]$.

II.3.1

Find the derivatives of the following functions.

(a) $\tan z = \frac{\sin z}{\cos z}$ **(b)** $\tanh z = \frac{\sinh z}{\cosh z}$ **(c)** $\sec z = 1/\cos z$

Solution

(a)

$$\frac{d}{dz}\tan z = \frac{d}{dz}\frac{\sin z}{\cos z} = \frac{\cos z \cdot \cos z - \sin z \cdot (-\sin z)}{\cos^2 z} = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z}.$$

(b)

$$\frac{d}{dz}\tanh z = \frac{d}{dz}\frac{\sinh z}{\cosh z} = \frac{\cosh z \cdot \cosh z - \sinh z \cdot \sinh z}{\cosh^2 z} = \frac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \frac{1}{\cosh^2 z}.$$

(c)

$$\frac{d}{dz}\sec z = \frac{d}{dz}\frac{1}{\cos z} = -\frac{1}{\cos^2 z}(-\sin z) = \frac{\sin z}{\cos^2 z} = \tan z \sec z.$$

II.3.2
Show that $u = \sin x \sinh y$ and $v = \cos x \cosh y$ satisfy the Cauchy-Riemann equations. Do you recognize the analytic function $f = u + iv$? (Determine its complex form)

Solution
We have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\sin x \sinh y = \cos x \sinh y = \frac{\partial}{\partial y}\cos x \cosh y = \frac{\partial v}{\partial y},\\
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\sin x \sinh y = \sin x \cosh y = -\frac{\partial}{\partial x}\cos x \cosh y = -\frac{\partial v}{dx}.
\end{aligned}$$

Hence, u and v satisfy Cauchy-Riemann equations, and now we calculate $f(z)$.

$$\begin{gathered}
f(z) = \\
= u + iv = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)\left(\frac{e^{y} - e^{-y}}{2}\right) + i\left(\frac{e^{ix} + e^{-ix}}{2}\right)\left(\frac{e^{y} + e^{-y}}{2}\right) = \\
= -i\left(\frac{e^{ix+y} - e^{ix-y} - e^{-ix+y} + e^{-ix-iy}}{4}\right) + i\left(\frac{e^{ix+y} + e^{ix-y} + e^{-ix+y} + e^{-ix-y}}{4}\right) = \\
= i\frac{e^{ix-y} + e^{-ix+y}}{2} = i\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = i\frac{e^{iz} + e^{-iz}}{2} = \\
= i\cos z.
\end{gathered}$$

II.3.3

1	2	3	P	L	K
				LLL	

Show that if f and $\bar{f}$ are both analytic on a domain D, then f is constant.

Solution (A. Kumjian)
Let D be a domain and let f be a complex valued function such that both f and $\bar{f}$ are analytic on D. Then both $\operatorname{Re} f = \frac{1}{2}\left(f + \bar{f}\right)$ and $\operatorname{Im} f = \frac{1}{2i}\left(f - \bar{f}\right)$ must also be analytic on D. Since $\operatorname{Re} f$ and $\operatorname{Im} f$ are also both real-valued, it follows by the theorem on page 50, that both are constant. Hence, $f = \operatorname{Re} f + i \operatorname{Im} f$ is constant.

II.3.4
Show that if f is analytic on a domain D, and if $|f|$ is constant, then f is constant.
Hint. Write $\bar{f} = |f|^2 / f$.

Solution (with use of hint)
If $f(z) = 0$ for some $z \in D$. Then $f \equiv 0$ on D, since $|f|$ is constant. Assume that $f(z) \neq 0$ for every $z \in D$. Since f is analytic in D we have that $1/f$ and $\bar{f} = |f|^2 / f$ (where $|f|$ is constant) are analytic. From this follows that $\operatorname{Re} f$ and $\operatorname{Im} f$ are analytic and real-valued and therefore constant. So $f = \operatorname{Re} f + i \operatorname{Im} f$ is constant.

II.3.4

Show that if f is analytic on a domain D, and if $|f|$ is constant, then f is constant.

Hint. Write $\bar{f} = |f|^2/f$.

Solution

Set $f = u + iv$, where u and v are realvalued functions. Now suppose that $|f| = k$ is constant. We have that $u^2 + v^2 = k^2$. Differentiate both sides with respect to x respectively y

$$\begin{cases} uu_x^{'} + vv_x^{'} = 0, \\ uu_y^{'} + vv_y^{'} = 0. \end{cases} \tag{1}$$

We use Cauchy-Riemanns equations $u_x^{'} = v_y^{'}$ and $u_y^{'} = -v_x^{'}$ in (1) and have

$$\begin{cases} uu_x^{'} - vu_y^{'} = 0, \\ uu_y^{'} + vu_x^{'} = 0. \end{cases} \tag{2}$$

In the simultaneusly systems of equations in (2) we first first multiply the first row with u and second row with v in (2) and add them together, and have $(u^2 + v^2)\, u_x^{'} = 0$. And if we in the same system multiply the first row with $-v$ and the second row with u and add them we have $(u^2 + v^2)\, u_y^{'} = 0$. Thus

$$\begin{cases} (u^2 + v^2)\, u_x^{'} = 0, \\ (u^2 + v^2)\, u_y^{'} = 0. \end{cases} \tag{3}$$

If in (3) $u^2 + v^2 = 0$, and we have that $u = v = 0$ and thus f is constant. Suppose that $u^2 + v^2 \neq 0$. We have that (3) gives that $u_x^{'} = u_y^{'} = 0$, with says that u is constant. Cauchy-Riemanns equations gives

$$\begin{cases} 0 = u_x^{'} = v_y^{'}, \\ 0 = u_y^{'} = -v_y^{'}, \end{cases} \tag{4}$$

thus by (4) f is constant.

II.3.5
If $f = u + iv$ is analytic, then $|\nabla u| = |\nabla v| = |f'|$.

Solution (K. Seip)
Because $f = u + iv$ is analytic. Then

$$\nabla u = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle = \left\langle \frac{\partial u}{\partial x}, -\frac{\partial v}{\partial x} \right\rangle = \left\langle \frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x} \right\rangle$$

from which it follows that

$$|\nabla u| = |f'(z)| = |\nabla v|.$$

II.3.6
If $f = u + iv$ is analytic on D, then ∇v is obtained by rotating ∇u by $90°$. In Particular, ∇u and ∇v are orthogonal.

Solution (A. Kumjian)
Let $\theta \in \mathbb{R}$ then rotation of a vector $(x, y) \in \mathbb{R}^2$ by the angle θ (about the origin) is given by matrix multiplication:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

Hence, rotation by $90°$ is given by the transformation $(x, y) \mapsto (-y, x)$. By the Cauchy-Riemann equations we have

$$\nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right) = A_{\pi/2}\nabla u.$$

Thus, ∇v is obtained by rotating ∇u by $90°$. It follows that ∇u and ∇v are orthogonal.

Solution (D. Jakobsson)
Given since f is analytic, then

$$(0.6) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ and $\nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$ all we need to check is whether $\nabla v \cdot \nabla v = 0$. To prove that we use (0.6).

$$\nabla u \cdot \nabla v = \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} = 0$$

We conclude that ∇u and ∇v are orthogonal.
Notice that the operation that sends $\nabla u \to \nabla v$ is given by the counter-clockwise rotation matrix with $\theta = \frac{\pi}{2}$. Keeping in mind (0.6), one can write the following.

$$\nabla v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_y \\ -v_x \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \begin{pmatrix} v_y \\ -v_x \end{pmatrix} = R\left(\frac{\pi}{2}\right)\nabla u$$

Remark.
We have that

$$v_x = \frac{\partial v}{\partial x}, \qquad v_y = \frac{\partial v}{\partial y}, \qquad u_x = \frac{\partial u}{\partial x}, \qquad u_y = \frac{\partial u}{\partial y},$$

and

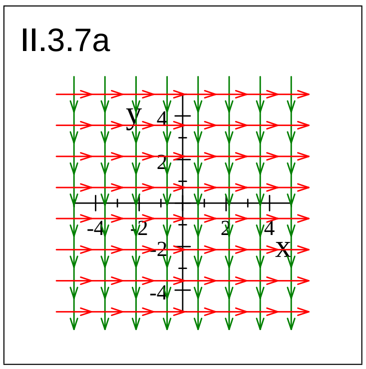
$$R\left(\pi/2\right) = A_{\pi/2}.$$

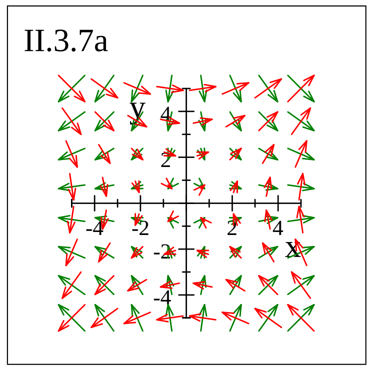
II.3.7

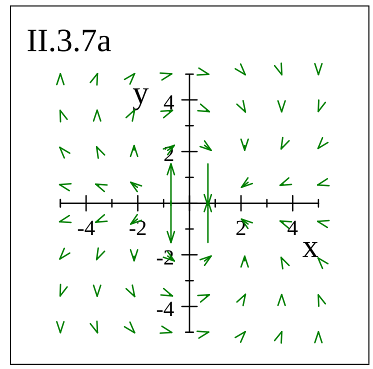
Sketch the vector fields ∇u and ∇v for the following functions $f = u + iv$.

(a) iz **(b)** z^2 **(c)** $1/z$

Solution







(a)

$$f(z) = iz = -y + ix \Rightarrow \begin{cases} u = -y \\ v = x \end{cases} \Rightarrow \begin{cases} \nabla u = (0, -1) \\ \nabla v = (1, 0) \end{cases}$$

(b)

$$f(z) = z^2 = x^2 - y^2 + 2ixy \Rightarrow \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases} \Rightarrow \begin{cases} \nabla u = (2x, -2y) \\ \nabla v = (2y, 2x) \end{cases}$$

(c)

$$f(z) = \frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} \Rightarrow \begin{cases} u = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r} \\ v = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r} \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\sin^2\theta - \cos^2\theta}{r^2} = \frac{-\cos 2\theta}{r^2} \\ \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} = \frac{-2\cos\theta\sin\theta}{r^2} = \frac{-\sin 2\theta}{r^2} \\ \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} = \frac{2\cos\theta\sin\theta}{r^2} = \frac{\sin 2\theta}{r^2} \\ \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\sin^2\theta - \cos^2\theta}{r^2} = \frac{-\cos 2\theta}{r^2} \end{cases}$$

$$\begin{cases} \nabla u = \frac{1}{r^2}(-\cos 2\theta, -\sin 2\theta) \\ \nabla v = \frac{1}{r^2}(-\sin 2\theta, -\cos 2\theta) \end{cases}$$

II.3.8 (see III.4.2) (36)
Derive the polar form of the Cauchy-Riemann equations for u and v,

$$\frac{\partial u}{\partial r} = \tfrac{1}{r}\tfrac{\partial v}{\partial \theta}, \qquad \tfrac{\partial u}{\partial \theta} = -r\tfrac{\partial v}{\partial r}.$$

Check that for any integer m, the functions $u\left(re^{i\theta}\right) = r^m \cos(m\theta)$ and $v\left(re^{i\theta}\right) = r^m \sin(m\theta)$ satisfy the Cauchy-Riemann equations.

Solution
Set $x = r\cos\theta$ and $y = r\sin\theta$, then take the partial derivative,

$$\begin{array}{ll} \frac{\partial x}{\partial r} = \cos\theta & \frac{\partial x}{\partial \theta} = -r\sin\theta \\ \frac{\partial y}{\partial r} = \sin\theta & \frac{\partial y}{\partial \theta} = r\cos\theta \end{array}$$

Use Cauchy-Riemanns equations and the derivative to get the first equation

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta = \\ &= \frac{1}{r}\left(-r\frac{\partial v}{\partial x}\sin\theta + r\frac{\partial v}{\partial y}\cos\theta\right) = \frac{1}{r}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}\right) = \frac{1}{r}\frac{\partial v}{\partial \theta}, \end{aligned}$$

Use Cauchy-Riemanns equations and the derivative to get the second equation

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\frac{\partial u}{\partial x}\sin\theta + r\frac{\partial u}{\partial y}\cos\theta = -r\frac{\partial v}{\partial y}\sin\theta - r\frac{\partial v}{\partial x}\cos\theta = \\ &= -r\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right) = -r\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right) = -r\frac{\partial v}{\partial r}. \end{aligned}$$

we get

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}. \end{cases}$$

Set $u\left(re^{i\theta}\right) = r^m \cos(m\theta)$ and $v\left(re^{i\theta}\right) = r^m \sin(m\theta)$ and use it in the Cauchy-Riemann's equations,

$$
\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial}{\partial r}\left(r^m \cos\left(m\theta\right)\right) = mr^{m-1}\cos\left(m\theta\right) = \frac{1}{r}r^m m\cos\left(m\theta\right) = \frac{1}{r}\frac{\partial}{\partial \theta}\left(r^m \sin m\theta\right) = \frac{1}{r}\frac{\partial v}{\partial \theta},\\
\frac{\partial u}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(r^m \cos\left(m\theta\right)\right) = -mr^m \sin\left(m\theta\right) = -rmr^{m-1}\sin\left(m\theta\right) = -r\frac{\partial}{\partial r}\left(r^m \sin\left(m\theta\right)\right) = -r\frac{\partial v}{\partial r}.
\end{aligned}
$$

II.4.1
Sketch the gradient vector fields ∇u and ∇v for
(a) $u + iv = e^z$ **(b)** $u + iv = \operatorname{Log} z$

Solution
a)
We have

$$u + iv = e^z = e^x \cos y + ie^x \sin y$$

and get

$$\nabla u = (e^x \cos y, -e^x \sin y) = e^x (\cos y, -\sin y)$$

(in black), and

$$\nabla v = (e^x \sin y, e^x \cos y) = e^x (\sin y, \cos y)$$

(in red).
b)
We have

$$u + iv = \operatorname{Log} z = \log r + i\theta$$

and get

$$\begin{aligned} \nabla u &= \frac{1}{r}\overrightarrow{u_r}, \\ \nabla v &= \frac{1}{r}\overrightarrow{u_\theta}. \end{aligned}$$

FIGURE II.4.1a NF FIGURE II.4.1b NF

II.4.2
Let a be a complex number $a \neq 0$, and let $f(z)$ be an analytic branch of z^a on $\mathbb{C} \setminus (-\infty, 0]$. Show that $f'(z) = af(z)/z$. (Thus $f'(z) = az^{a-1}$, where we pick the branch of z^{a-1} that corresponds to the original branch of z^a divided by z.)

Solution
We may write the branch as follows

$$f(z) = z^a = e^{a \log z} = e^{a(\operatorname{Log} z + 2\pi i m)} = e^{a \operatorname{Log} z + 2\pi i a m} = ce^{a \operatorname{Log} z},$$

where $c = e^{2\pi i a m}$ for some $m \in \mathbb{Z}$. It follows by the chain rule that

$$f'(z) = ce^{a \operatorname{Log} z} \frac{a}{z} = \frac{af(z)}{z}.$$

So, the desired result follows.

II.4.3
Consider the branch of $f(z) = \sqrt{z(1-z)}$ on $\mathbb{C}\setminus[0,1]$ that has positive imaginary part at $z = 2$. What is $f'(z)$? Be sure to specify the branch of the expression for $f'(z)$.

Solution
Set $f(z) = \sqrt{z(1-z)}$ (principal branch) and differentiate

$$f'(z) = \frac{1}{2}\frac{1}{\sqrt{z(1-z)}} \cdot \frac{d}{dz}(z(1-z)) = \frac{1}{2}\frac{1-2z}{\sqrt{z(1-z)}}.$$

Branch of $f(z)$ is $i\cdot$ increasing for $x > 0$ large, thus $f'(z)$ is $i\cdot$ positive for $x > 0$ large.
Use branch of $f'(z) = \sqrt{z(1-z)}$ that it is $i\cdot$ positive, i.e., same on $f(z)$.

$$f'(z) = \frac{1}{2}\frac{1-2z}{f(z)}.$$

Solution (D. Jakobsson)
Given $f(z) = \sqrt{z(1-z)}$, we can define $w = f(z)$ and get $w^2 = z(1-z)$.

$$\frac{d}{dz}(1-z)z = \frac{d}{dz}w^2$$

$$1 - 2z = 2w\frac{dw}{dz}$$

One gets

$$\frac{d}{dz}\sqrt{z(1-z)} = \frac{1-2z}{2\sqrt{z(1-z)}}$$

and is defined on $\mathbb{C}\setminus[0,1]$. Take the principal branch with the positive imaginary part.

II.4.4

Recall that the principal branch of the inverse tangent function was defined on the complex plane with two slits on the imaginary axis by

$$\operatorname{Tan}^{-1} = \tfrac{1}{2i} \operatorname{Log}\left(\tfrac{1+iz}{1-iz}\right), \qquad z \notin (-i\infty, -i] \cup [i, i\infty).$$

Find the derivative of $\operatorname{Tan}^{-1} z$**. Find the derivative of** $\tan^{-1} z$ **for any analytic branch of the function defined on a domain** D**.**

Solution

We have that

$$\frac{d}{dz} \operatorname{Tan}^{-1} z = \frac{1}{2i}\left[\frac{1}{1+iz} i - \frac{1}{1-iz}(-i)\right] = \frac{1}{2}\left[\frac{1-iz+1+iz}{(1+iz)(1-iz)}\right] = \frac{1}{1+z^2}.$$

Any two branches of $\tan^{-1} z$ differ by a constant, so derivatives are same.

II.4.5
Recall that $\cos^{-1}(z) = -i\log\left[z \pm \sqrt{z^2-1}\right]$**. Suppose** $g(z)$ **is an analytic branch of** $\cos^{-1}(z)$**, defined on a domain** D**. Find** $g'(z)$**. Do different branches of** $\cos^{-1}(z)$ **have the same derivative?**

Solution
We have that

$$\frac{d}{dz}\cos^{-1} z =$$

$$= \frac{-i}{z \pm \sqrt{z^2-1}}\left(1 \pm \frac{1}{2}\left(z^2-1\right)^{-1/2} \cdot 2z\right) =$$

$$= \frac{-i}{z \pm \sqrt{z^2-1}}\left(1 \pm \frac{z}{\sqrt{z^2-1}}\right) = \frac{\pm i}{\sqrt{z^2-1}} =$$

$$= \frac{\pm 1}{\sqrt{1-z^2}}.$$

Derivatives of branches of $\cos^{-1} z$ are not always the same.

II.4.6
Suppose $h(z)$ is an analytic branch of $\sin^{-1}(z)$, defined on a domain D. Find $h'(z)$. Do different branches of $\sin^{-1}(z)$ have the same derivative?

Solution (A. Kumjian)
Regard $h(z)$ as a local inverse of $f(z)$ where $f(z) = \sin z$. Then $f'(z) = \cos z$, so using the formula given in the statement of theorem on page 51 we have

$$h'(z) = \frac{1}{f'(h(z))} = \frac{1}{\cos h(z)}$$

if $\cos h(z) \neq 0$.
One local inverse of f satisfies $h_1(0) = 0$, while another local inverse of f satisfies $h_2(0) = \pi$. Using the formula above we see

$$h_1'(0) = \frac{1}{\cos 0} = 1 \qquad \text{and} \qquad h_2'(0) = \frac{1}{\cos \pi} = -1.$$

Hence, different branches of $\sin^{-1}(z)$ do not necessarily have the same derivative.

II.4.7

Let $f(z)$ be a bounded analytic function, defined on a bounded domain D in the complex plane, and suppose that $f(z)$ is one-one-one. Show that the area of $f(D)$ is given by

$$\operatorname{Area}(f(D)) = \iint_D |f'(z)|^2 \, dxdy.$$

Solution (K. Seip)

We have $f : D \to \mathbb{C}$, $|f(z)| \leq M$ for some $M < \infty$ when $z \in D$. Since f is assumed to be one-to-one, we may compute

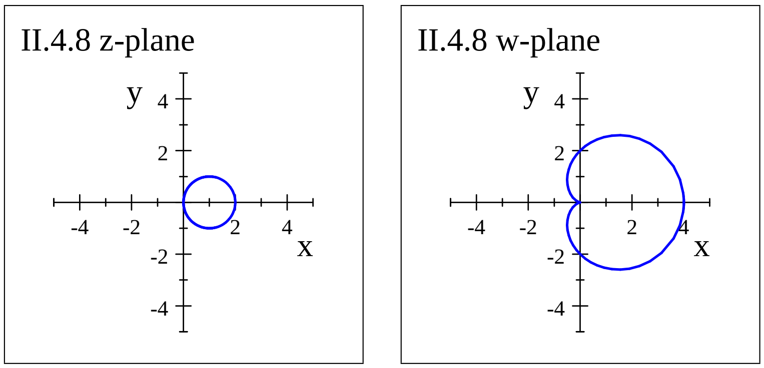
$$A(f(D)) = \iint\limits_{f(D)} dudv$$

by the change of variables $(u(x,y), v(x,y)) \to (x,y)$, and since $\det J_f = |f'(z)|^2$, we get

$$\operatorname{Area}(f(D)) = \iint\limits_{D} |f'(z)|^2 \, dxdy.$$

II.4.8
Sketch the image of the circle $\{|z-1| \leq 1\}$ under the map $w = z^2$. Compute the area of the image.

Solution



We have that $f(z) = z^2$, that gives $f'(z) = 2z$ and $J_f(z) = 4|z|^2$.
The area of the image is

$$\iint J_f dxdy = 4 \iint\limits_{(x-1)^2+y^2\leqslant 1} \left(x^2+y^2\right) dxdy.$$

Set $t = x-1$, get

$$4 \iint\limits_{t^2+y^2\leqslant 1} \left(t^2+2t+1+y^2\right) dtdy =$$

$$= 4\left[\iint\limits_{t^2+y^2\leqslant 1} \left(t^2+y^2\right) dtdy + 0 + \iint\limits_{t^2+y^2\leqslant 1} dtdy\right] =$$

$$= 4\left[\frac{\pi}{2}+\pi\right] = 6\pi.$$

Solution (D. Jakobsson)
Given $w = f(z) = z^2$, one can differentiate $f(z)$ and get $\frac{d}{dz}f(z) = 2z = 2(x+iy)$, we can use maple to integrate and get the following

$$\text{Area}\,(f\,(D)) =$$

$$= \iint_D |f'(z)|^2\, dxdy = \int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} 4\,|x+iy|^2\, dx\ dy =$$

$$= \int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} 4\left(x^2+y^2\right) dx\ dy = \int_0^2 8x^2\sqrt{2x-x^2} + 8/3\left(2x-x^2\right)^{3/2} dx =$$

$$= 6\pi$$

Change of variables

$$\begin{cases} x = 1+\sin t \\ dx = \cos t dt \end{cases}$$

$$\int_{-\pi/2}^{\pi/2} 8\,(1+\sin t)^2 \sqrt{1-\sin^2 t} + \frac{8}{3}\left(1-\sin^2\right)^{3/2} \cos t dt =$$

$$= \int_{-\pi/2}^{\pi/2} 8\left(1 + 2\underbrace{\sin t}_{\text{odd}} + \sin^2 t\right) \cos^2 + \frac{8}{3}\cos^4 t dt dt =$$

$$= 16\int_0^{\pi/2} 2\cos^2 - \frac{2}{3}\cos^4 t dt =$$

$$= 8\int_0^{\pi/2} 2\,(1+\cos 2t) - \frac{1}{3}\left(1+2\cos 2t+\cos^2 2t\right) dt =$$

$$= 8\int_0^{\pi/2} \frac{5}{3} + \frac{4}{3}\cos 2t - \frac{1}{3}\,(1+\cos 4t)\, dt =$$

$$= 8\cdot\frac{\pi}{2}\left(\frac{5}{3}-\frac{1}{6}\right) = 4\pi\frac{9}{6} = 6\pi.$$

II.4.9
Compute

$$\iint_D |f'(z)|^2 \, dxdy,$$

for $f(z) = z^2$ and D the open unit disk $\{|z| < 1\}$. Interpret your answer in terms of areas.

Solution
By the result above

$$\iint\limits_{\substack{|z|<1 \\ x>0}} |f'(z)|^2 \, dxdy = \iint\limits_{\substack{|z|<1 \\ x<0}} |f'(z)|^2 \, dx \; dy = \pi,$$

thus

$$\iint\limits_{|z|<1} |f'(z)|^2 \, dx \; dy = 2\pi.$$

Remark.
The function maps the top and bottom halves of the unit disk one-to-one onto the unit disk, so by Exersice II.4.7 the integrals on top and bottom halves are each π.

II.4.10

1	2	3	P	L	K

For smooth functions g and h defined on a bounded domain U, we define the Dirichlet form $D_U(g,h)$ by

$$D_U(g,h)=\iint_U\left[\frac{\partial g}{\partial x}\frac{\overline{\partial h}}{\partial x}+\frac{\partial g}{\partial y}\frac{\overline{\partial h}}{\partial y}\right]dx\,dy$$

Show that if $z=f(\zeta)$ is a one-to-one analytic function from the bounded domain V onto U, then

$$D_U(g,h)=D_V(g\circ f,h\circ f).$$

Remark. This shows that the Dirichlet form is a "conformal invariant".

Solution

$D_u(g,h)=\iint\limits_U \nabla g\overline{\nabla h}dxdy$ Assume h,g ????

$D_u(g,h)=\iint\limits_U \nabla(g\circ f)\overline{\nabla(h\circ f)}dudv=$

$\iint\limits_U g(u(x,y),v(x,y))\,h(u(x,y),v(x,y))\,dudv$

$\frac{\partial g(u(x,y),v(x,y))}{\partial x}\frac{\partial h(u(x,y),v(x,y))}{\partial x}+$

$\frac{\partial g(u(x,y),v(x,y))}{\partial y}\frac{\partial h(u(x,y),v(x,y))}{\partial y}=$

$\left(\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}\right)\left(\frac{\partial h}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial h}{\partial v}\frac{\partial v}{\partial x}\right)+\left(\frac{\partial g}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial y}\right)\left(\frac{\partial h}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial h}{\partial v}\frac{\partial v}{\partial y}\right)=$

$\frac{\partial g}{\partial u}\frac{\partial h}{\partial u}\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}\frac{\partial h}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}\frac{\partial h}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\frac{\partial h}{\partial v}\left(\frac{\partial v}{\partial x}\right)^2+$

$\frac{\partial g}{\partial u}\frac{\partial h}{\partial u}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial g}{\partial u}\frac{\partial u}{\partial y}\frac{\partial h}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial y}\frac{\partial h}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial g}{\partial v}\frac{\partial h}{\partial v}\left(\frac{\partial v}{\partial y}\right)^2=$

$\frac{\partial g}{\partial u}\frac{\partial h}{\partial u}\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial g}{\partial v}\frac{\partial h}{\partial v}\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial g}{\partial u}\frac{\partial h}{\partial u}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial g}{\partial v}\frac{\partial h}{\partial v}\left(\frac{\partial v}{\partial y}\right)^2=$

$\frac{\partial g}{\partial u}\frac{\partial h}{\partial u}\underbrace{\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)}_{|f'(z)|^2}+\frac{\partial g}{\partial v}\frac{\partial h}{\partial v}\underbrace{\left(\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right)}_{|f'(z)|^2}+$

$\underbrace{\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}\frac{\partial h}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}\frac{\partial h}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial g}{\partial u}\frac{\partial u}{\partial y}\frac{\partial h}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial y}\frac{\partial h}{\partial u}\frac{\partial u}{\partial y}}_{\text{Use C-R equations}}$

$\iint\limits_U \nabla g\nabla h\,|f'(z)|^2dxdy=\iint\limits_U \nabla g\nabla hdudv$

II.5.1

Show that the following functions are harmonic, and find its harmonic conjugates:

(a)	$x^2 - y^2$	**(c)**	$\sinh x \sin y$	**(e)**	$\tan^{-1}(y/x), x > 0$
(b)	$xy + 3x^2y - y^3$	**(d)**	$e^{x^2-y^2}\cos(2xy)$	**(f)**	$x/(x^2+y^2)$

Solution

(a)

Set

$$u(x,y) = x^2 - y^2 \Rightarrow \begin{cases} u_x^{'} = 2x \\ u_y^{'} = -2y \end{cases} \Rightarrow \begin{cases} u_{xx}^{''} = 2 \\ u_{yy}^{''} = -2 \end{cases} \Rightarrow \triangle u = 0,$$

thus is the function $u(x,y)$ is harmonic. Cauchy Riemann give us

$$\begin{cases} v_x^{'} = -u_y^{'} = 2y \Rightarrow v(x,y) = 2xy + g(y) \Rightarrow v_y^{'} = 2x + g'(y) \\ v_y^{'} = u_x^{'} = 2x \end{cases}$$

Thus we have that

$$g'(y) = 0 \Rightarrow g(y) = C,$$

thus

$$v(x,y) = 2xy + C.$$

(b)

Set

$$u(x,y) = xy + 3x^2y - y^3 \Rightarrow \begin{cases} u_x^{'} = y + 6xy \\ u_y^{'} = x + 3x^2 - 3y^2 \end{cases} \Rightarrow \begin{cases} u_{xx}^{''} = 6y \\ u_{yy}^{'} = -6y \end{cases} \Rightarrow \triangle u = 0,$$

thus is the function $u(x,y)$ is harmonic. Cauchy Riemann give us

$$\begin{cases} v_x^{'} = -u_y^{'} = -x - 3x^2 + 3y^2 \Rightarrow v(x,y) = -\frac{x^2}{2} - x^3 + 3xy^2 + g(y) \Rightarrow v_y^{'} = 6xy + g'(y) \\ v_y^{'} = u_x^{'} = y + 6xy \end{cases}$$

Thus we have that

$$g'(x) = y \Rightarrow g(y) = \frac{y^2}{2} + C,$$

thus

$$v(x,y) = \frac{y^2}{2} + 3xy^2 - \frac{x^2}{2} - x^3 + C.$$

In paranthesis we remark that the analytic function $f(z) = u + iv$ is given by

$$f(z) = xy + 3x^2y - y^3 + i\left(\frac{y^2}{2} + 3xy^2 - \frac{x^2}{2} - x^3 + C\right) = -iz^2(z + 1/2) + iC.$$

(c)
Set

$$u(x,y) = \sinh x \sin y \Rightarrow \begin{cases} u_x' = \cosh x \sin y \\ u_y' = \sinh x \cos y \end{cases} \Rightarrow \begin{cases} u_{xx}'' = \sinh x \sin y \\ u_{yy}' = -\sinh x \sin y \end{cases} \Rightarrow \triangle u = 0,$$

thus the function $u(x,y)$ is harmonic. Cauchy Riemann give us

$$\begin{cases} v_x' = -u_y' = -\sinh x \cos y \Rightarrow v(x,y) = -\cosh x \cos y + g(y) \Rightarrow v_y' = \cosh x \sin y + g'(y) \\ v_y' = u_x' = \cosh x \sin y \end{cases}$$

Thus we have that

$$g'(y) = 0 \Rightarrow g(y) = C,$$

thus

$$v(x,y) = -\cosh x \cos y + C$$

In paranthesis we remark that the analytic function $f(z) = u + iv$ is given by

$$f(z) = \sinh x \sin y - i(\cosh x \cos y + C) =$$
$$= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^{iy} - e^{-iy}}{2i}\right) - i\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^{iy} + e^{-iy}}{2}\right) + iC =$$
$$= -\frac{i}{4}\left(\left(e^x - e^{-x}\right)\left(e^{iy} - e^{-iy}\right) + \left(e^x + e^{-x}\right)\left(e^{iy} + e^{-iy}\right)\right) + iC =$$
$$= -i\left(\frac{e^{x+iy} + e^{-(x+iy)}}{2}\right) + iC = -i\cosh z + iC.$$

(d)
Set

$$u(x,y) = e^{x^2-y^2}\cos(2xy) \Rightarrow \begin{cases} u_x^{'} = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy) \\ u_y^{'} = -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy) \end{cases} \Rightarrow$$
$$\Rightarrow \begin{cases} u_{xx}^{''} = 2e^{x^2-y^2}\left(\cos 2xy + 2x^2\cos 2xy - 2y^2\cos 2xy - 4xy\sin 2xy\right) \\ u_{yy}^{'} = -2e^{x^2-y^2}\left(\cos 2xy + 2x^2\cos 2xy - 2y^2\cos 2xy - 4xy\sin 2xy\right) \end{cases} \Rightarrow \triangle u = 0,$$

thus the function $u(x,y)$ is harmonic. Cauchy Riemann give us

$$\begin{cases} v_x^{'} = -u_y^{'} = 2ye^{x^2-y^2}\cos(2xy) + 2xe^{x^2-y^2}\sin(2xy) \Rightarrow v(x,y) = e^{x^2-y^2}\sin(2xy) + g(y) \Rightarrow \\ \Rightarrow v_y^{'} = -2ye^{x^2-y^2}\sin(2xy) + 2xe^{x^2-y^2}\cos(2xy) + g'(y) \\ v_y^{'} = u_x^{'} = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy) \end{cases}$$

Thus we have that

$$g'(y) = 0 \Rightarrow g(y) = C,$$

thus

$$v(x,y) = e^{x^2-y^2}\sin(2xy) + C$$

In paranthesis we remark that the analytic function $f(z) = u + iv$ is given by

$$f(z) = e^{x^2-y^2}\cos(2xy) + i\left(e^{x^2-y^2}\sin(2xy) + C\right) = e^{z^2} + iC.$$

(e)

Set

$$u(x,y)=\tan^{-1}(y/x)\,,x>0\Rightarrow$$
$$\Longrightarrow\begin{cases}u_x^{'}=\dfrac{1}{1+(y/x)^2}\left(-\dfrac{y}{x^2}\right)=\dfrac{-y}{x^2+y^2}\\ u_y^{'}=\dfrac{x}{x^2+y^2}\end{cases}\Longrightarrow$$
$$\Rightarrow\begin{cases}u_{xx}^{''}=\dfrac{2xy}{(x^2+y^2)^2}\\ u_{yy}^{'}=\dfrac{-2xy}{(x^2+y^2)^2}\end{cases}\Rightarrow\triangle u=0,$$

thus the function $u(x,y)$ is harmonic. Cauchy Riemann give us

$$\begin{cases}v_x^{'}=-u_y^{'}=\dfrac{-x}{x^2+y^2}\Rightarrow v(x,y)=a-\frac{1}{2}\log(x^2+y^2)+g(y)\Rightarrow v_y^{'}=\dfrac{-y}{x^2+y^2}+g'(y)\\ v_y^{'}=u_x^{'}=\dfrac{-y}{x^2+y^2}\end{cases}$$

Thus we have that

$$g'(y)=0\Rightarrow g(y)=C,$$

thus

$$v(x,y)=-\frac{1}{2}\log\left(x^2+y^2\right)+C.$$

(f)
Set

$$u(x,y)=\frac{x}{x^2+y^2}\Rightarrow\begin{cases}u_x^{'}=-\dfrac{x^2-y^2}{(x^2+y^2)^2}\\ u_y^{'}=-\dfrac{2xy}{(x^2+y^2)^2}\end{cases}\Rightarrow\begin{cases}u_{xx}^{''}=-2x\dfrac{3y^2-x^2}{(x^2+y^2)^3}\\ u_{yy}^{'}=2x\dfrac{3y^2-x^2}{(x^2+y^2)^3}\end{cases}\Rightarrow\triangle u=0,$$

thus is the function $u(x,y)$ is harmonic. Cauchy Riemann give us

$$\begin{cases} v_x^{'} = -u_y^{'} = \dfrac{2xy}{\left(x^2+y^2\right)^2} \Rightarrow v\left(x,y\right) = \dfrac{-y}{\left(x^2+y^2\right)} + g\left(y\right) \Rightarrow v_y^{'} = -\dfrac{x^2-y^2}{\left(x^2+y^2\right)^2} + g'\left(y\right) \\ v_y^{'} = u_x^{'} = -\dfrac{x^2-y^2}{\left(x^2+y^2\right)^2} \end{cases}$$

Thus we have that

$$g'\left(y\right) = 0 \Rightarrow g\left(y\right) = C,$$

thus

$$v\left(x,y\right) = \frac{-y}{\left(x^2+y^2\right)} + C$$

In paranthesis we remark that the analytic funktion $f\left(z\right) = u + iv$ is given by

$$f\left(z\right) = \frac{x}{x^2+y^2} + i\left(\frac{-y}{\left(x^2+y^2\right)} + C\right) = \frac{\bar{z}}{\left|z\right|^2} + iC = \frac{\bar{z}}{z\bar{z}} + iC = \frac{1}{z} + iC.$$

II.5.2

Show that if v is a harmonic conjugate for u, then $-u$ is a harmonic conjugate for v.

Solution

If $f(z) = u(x,y) + iv(x,y)$ is analytic then $-if(z) = v(x,y) - iu(x,y)$ is analytic. Thus $-u$ is a harmonic conjugate of v.

II.5.3
Define $u(z) = \operatorname{Im}(1/z^2)$ for $z \neq 0$, and set $u(0) = 0$.
(a) Show that all partial derivatives of u with respect to x exist at all points of the plane $\mathbb{C}$, as do all partial derivative of u with respect to y.
(b) Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.
(c) Show that u is not harmonic on $\mathbb{C}$.
(d) Show that $\frac{\partial^2 u}{\partial x \partial y}$ does not exist at $(0,0)$.

Solution
(a)
We first define

$$u(z) = \begin{cases} \operatorname{Im} \dfrac{1}{z^2} = \operatorname{Im} \dfrac{\bar{z}^2}{|z|^4} = \operatorname{Im} \dfrac{(x-iy)^2}{(x^2+y^2)^2} = \dfrac{-2xy}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For all $(x,y) \neq (0,0)$, the $n-$ th derivative of $u(z)$ will be $\frac{f(x,y)}{(x^2+y^2)^{n+2}}$ with $f(x,y)$ a polynomial of degree $n+2$ in (x,y). At $(0,0)$, the function itself is 0, thus $u(x,y) \in \mathbb{C}^{\infty}(x)$. By symmetry $u(x,y) \in \mathbb{C}^{\infty}(y)$.
(b)
We have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x}\left(2y\frac{3x^2-y^2}{(x^2+y^2)^3}\right) + \frac{\partial u}{\partial y}\left(2x\frac{3y^2-x^2}{(x^2+y^2)^3}\right) = \\ = -24xy\frac{x^2-y^2}{(x^2+y^2)^4} + 24xy\frac{x^2-y^2}{(x^2+y^2)^4} = 0 \end{aligned}$$

(c)
Notice that the function $u(x,y)$ is not harmonic because the function itself, its first and second derivative are not continuous at the origin. The limit of $u(x,y)$ as $(x,y) \to (0,0)$ does not exist, for instance $u\left(\frac{1}{n}, \frac{1}{n}\right)$ goes to $-\infty$ as $n \to \infty$ whereas we defined the function u to be 0 at the origin.

$$\lim_{n\to\infty} u\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} -\frac{n^2}{2} = -\infty.$$

(d)

$$\frac{\partial u}{\partial y}=\frac{-2x\cdot\left(x^2+y^2\right)^2+2xy\cdot 2\left(x^2+y^2\right)\cdot 2y}{\left(x^2+y^2\right)^4}=\frac{-2x^3+6xy^2}{\left(x^2+y^2\right)^3}$$

thus

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\left(-6x^2+6y^2\right)\cdot\left(x^2+y^2\right)^3-\left(-2x^3+6xy^2\right)\cdot 3\left(x^2+y^2\right)^2\cdot 2x}{\left(x^2+y^2\right)^6}=\frac{6x^4-36x^2y^2+6y^4}{\left(x^2+y^2\right)^4}.$$

We have

$$\frac{\partial^2 u}{\partial x\partial y}=\begin{cases}\frac{6x^4-36x^2y^2+6y^4}{\left(x^2+y^2\right)^4} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$$\frac{\partial^2 u}{\partial x\partial y}(x,0)=\begin{cases}6/x^4, & x\neq 0\\ 0, & x=0\end{cases}.$$

This partial derivative is not continuous at $x=0$, so $\frac{\partial^2 u}{\partial x\partial y}$ does not exist at $(0,0)$.

.

II.5.4

Show that if $h(z)$ is a complex-valued harmonic function (solution of Laplace's equation) such that $zh(z)$ is also harmonic, then $h(z)$ is analytic.

Solution

First partial derivative for $zh(z)$ is,

$$\begin{aligned}\frac{\partial(zh)}{\partial x} &= \frac{\partial z}{\partial x}h + z\frac{\partial h}{\partial x},\\ \frac{\partial(zh)}{\partial y} &= \frac{\partial z}{\partial y}h + z\frac{\partial h}{\partial y}.\end{aligned}$$

Second derivative for $zh(z)$ is,

$$\begin{aligned}\frac{\partial^2(zh)}{\partial x^2} &= \frac{\partial^2 z}{\partial x^2}h + \frac{\partial z}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial h}{\partial x} + z\frac{\partial^2 h}{\partial x^2},\\ \frac{\partial^2(zh)}{\partial y^2} &= \frac{\partial^2 z}{\partial y^2}h + \frac{\partial z}{\partial y}\frac{\partial h}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial h}{\partial y} + z\frac{\partial^2 h}{\partial y^2}.\end{aligned}$$

We have $z = x+iy$, thus $\partial z/\partial x = 1$, $\partial z/\partial y = i$, and $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2 = 0$, therefore

$$\begin{aligned}\frac{\partial^2(zh)}{\partial x^2} &= 2\frac{\partial h}{\partial x} + z\frac{\partial^2 h}{\partial x^2},\\ \frac{\partial^2(zh)}{\partial y^2} &= i2\frac{\partial h}{\partial y} + z\frac{\partial^2 h}{\partial y^2}.\end{aligned}$$

Add the equations

$$\frac{\partial^2(zh)}{\partial x^2} + \frac{\partial^2(zh)}{\partial y^2} = 2\frac{\partial h}{\partial x} + 2i\frac{\partial h}{\partial y} + z\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right).$$

Because $h(z)$ and $zh(z)$ is harmonic, then both $\partial^2(zh)/\partial x^2 + \partial^2(zh)/\partial y^2 = 0$ and $\partial^2(h)/\partial x^2 + \partial^2(h)/\partial y^2 = 0$, we have

$$\frac{\partial h}{\partial x} = -i\frac{\partial h}{\partial y}.$$

Write $h = u + iv$, get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Now we take real and imaginary parts and get

$$\tfrac{\partial u}{\partial x} = \tfrac{\partial v}{\partial y} \qquad \text{and} \qquad \tfrac{\partial u}{\partial y} = -\tfrac{\partial v}{\partial x}.$$

Thus yields Cauchy-Riemann's equations for $u + v$, then $h\,(z)$ is analytic as was to be shown.

II.5.5
Show that Laplace's equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0.$$

Solution
Let $x=r\cos\theta$ and $y=r\sin\theta$, and get partial derivatives

$$\frac{\partial x}{\partial r}=\cos\theta,\quad \frac{\partial x}{\partial \theta}=-r\sin\theta,$$
$$\frac{\partial y}{\partial r}=\sin\theta,\quad \frac{\partial y}{\partial \theta}=r\cos\theta.$$

Taking the first derivative of u with respect to r

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta=\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}.$$

Taking the second derivative of u with respect to r

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2}&=\\ &=\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r}\cos\theta+\frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial r}\cos\theta+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial r}\sin\theta+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}\sin\theta=\\ &=\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r}\cos\theta+\frac{\partial^2 u}{\partial y\partial x}\sin\theta\cos\theta+\frac{\partial^2 u}{\partial x\partial y}\cos\theta\sin\theta+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}\sin\theta=\\ &=\cos^2\theta\frac{\partial^2 u}{\partial x^2}+2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y}+\sin^2\theta\frac{\partial^2 u}{\partial y^2}.\end{aligned}$$

Taking the first derivative of u with respect to θ

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}r\cos\theta.$$

Taking the second derivative of u with respect to θ

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial\theta^2} = \\
&\qquad = \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial\theta}\left(-r\sin\theta\right) + \frac{\partial^2 u}{\partial y\partial x}\frac{\partial y}{\partial\theta}\left(-r\sin\theta\right) + \frac{\partial u}{\partial x}\left(-r\cos\theta\right) + \\
&\qquad\quad + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial\theta}\left(r\cos\theta\right) + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial\theta}\left(r\cos\theta\right) + \frac{\partial u}{\partial y}\left(-r\sin\theta\right) = \\
&= \frac{\partial^2 u}{\partial x^2}\left(-r\sin\theta\right)\left(-r\sin\theta\right) + \frac{\partial^2 u}{\partial y\partial x}\left(r\cos\theta\right)\left(-r\sin\theta\right) + \frac{\partial u}{\partial x}\left(-r\cos\theta\right) + \\
&\quad + \frac{\partial^2 u}{\partial x\partial y}\left(-r\sin\theta\right)\left(r\cos\theta\right) + \frac{\partial^2 u}{\partial y^2}\left(r\cos\theta\right)\left(r\cos\theta\right) + \frac{\partial u}{\partial y}\left(-r\sin\theta\right) = \\
&= r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2} - r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y}.
\end{aligned}
$$

Combining these partial derivatives, one gets

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \\
&= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2} + \frac{1}{r}\left(\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right) + \\
&+\frac{1}{r^2}\left(r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2r^2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + r^2\cos^2\theta\frac{\partial^2 u}{\partial y^2} - r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y}\right) = \\
&= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2} + \frac{1}{r}\cos\theta\frac{\partial u}{\partial x} + \frac{1}{r}\sin\theta\frac{\partial u}{\partial y} + \\
&+ \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\cos\theta\frac{\partial u}{\partial x} - \frac{1}{r}\sin\theta\frac{\partial u}{\partial y} = \\
&\qquad = \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 u}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.
\end{aligned}
$$

Because u is a harmonic function we have $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ and we have,

$$
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
$$

II.5.6

Show using Laplace's equation in polar coordinates that $\log|z|$ is harmonic on the punctured plane $\mathbb{C}\setminus\{0\}$.

Solution

Set $z = re^{i\theta}$, we have $u(r,\theta) = \log|z| = \log r$. We can compute the Laplacian of $\log|z|$ without worrying about the origin, i.e. $r \neq 0$ because this case is taken out, thus we have

$$\Delta \log r = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial u^2}{\partial \theta^2} = -\frac{1}{r^2} + \frac{1}{r}\frac{1}{r} + 0 = 0.$$

II.5.7

Show that $\log|z|$ **has no conjugate harmonic function on the punctured plane** $\mathbb{C}\setminus\{0\}$**, though it does have a conjugate harmonic function on the slit plane** $C\setminus(-\infty,0)$**.**

Solution (A. Kumjamin)

Set $D := \mathbb{C}\setminus(-\infty,0]$ and let $u : D \to \mathbb{R}$ be given by $u(z) := \log|z|$, it is easily verified that $u(z) = \operatorname{Re}\operatorname{Log} z$ and this shows that u is harmonic on D by the main theorem in section II.5. Moreover, it follows by Cauchy-Riemann equations that a harmonic conjugate v is given by

$$v(z) := \operatorname{Im}\operatorname{Log} z = \operatorname{Arg} z \qquad \text{for} \qquad z \in D.$$

Any other harmonic conjugate differs from v by a constant. Now if we set $\tilde{D} := \mathbb{C}\setminus\{0\}$ and define $\tilde{u} : D \to \mathbb{R}$ by $\tilde{u}(z) := \log|z|$, it can easily be shown by an argument similar to the one above that $\tilde{u}$ is harmonic on $\tilde{D}$ even though it cannot be expressed as the real part of an analytic function on $\tilde{D}$ (the property of being harmonic is local). Suppose that $\tilde{u}$ does have a harmonic conjugate, say $\tilde{v}$, on $\tilde{D}$. Then its restriction to D would have to be a harmonic conjugate of u and so it would be of the form $z \longmapsto \operatorname{Arg} z + c$ for some $c \in \mathbb{R}$. But such a function has no continuous extension to $\tilde{D}$. This contradicts the fact that $\tilde{v}$ is harmonic on $\tilde{D}$ (since this implies $\tilde{v}$ is continuous on $\tilde{D}$). Therefore, $\log|z|$ has no conjugate harmonic function on the punctured plane $\mathbb{C}\setminus\{0\}$.

Solution (K. Seip)

In $\mathbb{C}\setminus(-\infty,0]$, we know that any harmonic conjugate of $\log|z|$ must have the form

$$v(x,y) = \operatorname{Arg} z + c.$$

Such a function can not be made continuous in $\mathbb{C}$ since for $x<0$, we have

$$\lim_{y\to 0^{0+}} v(x,y) = \pi + c,$$

and

$$\lim_{y\to 0^{0-}} v(x,y) = -\pi + c.$$

II.5.8

Show using Laplace's equation in polar coordinates that $u\left(re^{i\theta}\right) = \theta\log r$ is harmonic. Use the polar form of the Cauchy-Riemann equations. (Exercise 3.8) to find a harmonic conjugate v for u. What is the analytic function $u+iv$?

Solution

Set

$$u\left(re^{i\theta}\right) = \theta\log r \Rightarrow \begin{cases} u_r^{'} = \dfrac{\theta}{r} \\ u_\theta^{'} = \log r \end{cases} \Rightarrow \begin{cases} u_{rr}^{''} = -\dfrac{\theta}{r^2} \\ u_{\theta\theta}^{''} = 0 \end{cases}$$

The function $u\left(re^{i\theta}\right) = \theta\log r$ is harmonic because

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -\frac{\theta}{r^2} + \frac{\theta}{r^2} = 0.$$

Cauchy Riemann equations in polar form (page 50) gives us

$$\begin{cases} v_r^{'} = -\dfrac{1}{r}u_\theta^{'} = -\dfrac{\log r}{r} \Rightarrow v(r,\theta) = -\dfrac{(\log r)^2}{2} + g(\theta) \Rightarrow v_\theta^{'} = g'(\theta) \\ v_\theta^{'} = r u_r^{'} = \theta \end{cases}$$

Thus we have that

$$g'(\theta) = \theta \Rightarrow g(\theta) = \frac{\theta^2}{2} + C,$$

thus

$$v(r,\theta) = -\frac{(\log r)^2}{2} + \frac{\theta^2}{2} + C.$$

We have that the analytic function $f(z) = u + iv$ is given by

$$f(z) = \theta\log r + i\left(\frac{\theta^2}{2} - \frac{(\log r)^2}{2} + C\right) = -\frac{i}{2}\left(\log r + i\theta\right)^2 + C = -\frac{i}{2}\left(\log z\right)^2 + iC.$$

II.6.1
Sketch the families of level curves of u and v for the following functions $f = u + iv$.
(a) $f(z) = 1/z$ **(b)** $f(z) = 1/z^2$ **(c)** $f(z) = z^6$
Determine where $f(z)$ is conformal and where it is not conformal.

Solution
(a)
We have that

$$f(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} \Rightarrow \begin{cases} u = \dfrac{x}{x^2 + y^2} \\ v = -\dfrac{y}{x^2 + y^2} \end{cases}$$

Conformal except at $z = 0$, where it is not defined. (Conformal everywhere if we view it as a mapping of the sphere).
We have

$$u = \frac{1}{c}, \qquad \frac{1}{c}(x^2 + y^2) = x, \qquad \left(x - \frac{c}{2}\right)^2 + y^2 = \frac{c^2}{4},$$

which is circles centred on $x -$ axis and tangent to $y -$ axis.
We have

$$v = \frac{1}{a}, \qquad \frac{1}{a}(x^2 + y^2) = -y, \qquad x^2 + \left(y + \frac{a}{2}\right)^2 = \frac{a^2}{4},$$

which is circles centred on $y -$ axis and tangent to $x -$ axis.
(b)
We have

$$f(z) = \frac{1}{z^2} = \frac{\overline{z}^2}{|z|^4} = \frac{(x - iy)^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2 - 2ixy}{(x^2 + y^2)^2} \Rightarrow$$

$$\Longrightarrow \begin{cases} u = \frac{x^2 - y^2}{(x^2+y^2)^2} = \frac{\cos^2\theta - \sin^2\theta}{r^2} = \frac{\cos(2\theta)}{r^2} \\ v = \frac{-2xy}{(x^2+y^2)^2} = \frac{-2\cos\theta\sin\theta}{r^2} = \frac{-\sin(2\theta)}{r^2} \end{cases}$$

and

$$u=\frac{1}{c}, \qquad r^2=c\cos(2\theta), \qquad v=\frac{1}{c}, \qquad r^2=-c\sin(2\theta).$$

Conformal everywhere except at $z=0$.

(c)

We have that

$$f(z)=z^6=r^6(\cos(6\theta)+i\sin(6\theta)) \Rightarrow \begin{cases} u=r^6\cos(6\theta) \\ v=r^6\sin(6\theta) \end{cases}$$

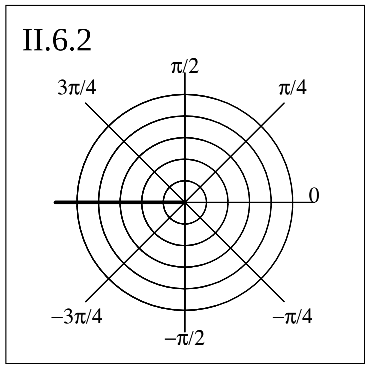
Figure repeats itself. Rotate of $\{u=0\}$ by $\pi/12$ is $\{v=0\}$. Each grind repeats itself every $\pi/6$.

Conformal everywhere except at $z=0$.

FIGURE II.6.1a NF FIGURE II.6.1b NF FIGURE II.6.1c NF

II.6.2
Sketch the families of level curves of u and v for $f(z) = \mathrm{Log}\, z = u+iv$. Relate your sketch to a figure in Section I.6.

Solution



We have

$$f(z) = \mathrm{Log}\, z = \log|z| + i\,\mathrm{Arg}\, z = \log r + i\theta \Rightarrow \begin{cases} u = \log|z| \\ v = \mathrm{Arg}\, z \end{cases} \Rightarrow \begin{cases} u = \log r \\ v = \theta \end{cases}.$$

If $u =$ constant and $r =$ constant we have circles centred at 0, and if $v =$ constanat and $\theta =$ constanat we have rays issuing from 0, $-\pi < \theta < \pi$. Refer to the Figure for the map $w = \mathrm{Log}\, z$ in section I.6. The sketch is of the inverse images of horizontal and vertical lines in that Figure.

II.6.3

Sketch the families of level curves of u and v for the function $f = u + iv$ given by (a) $f(z) = e^z$, (b) $f(z) = e^{\alpha z}$, where α is complex. Determine where $f(z)$ is conformal and where it is not conformal.

Solution

$f(z) = e^z = e^x(\cos y + i \sin y) \Rightarrow u = e^x \cos y v = e^x \sin y$
$u = \text{const}, \ x = -\log(c \cos y) = -\log c - \log(\cos y)$
$v = \text{const}, \ x = -\log(c \sin y) = -\log c - \log(\sin y)$
Level curves are invariant under ????? in x–direction. Level curves are invariant under ????? in y–direction by π. The function $f(z)$ is conformal everywhere.
$f(z) = e^{\alpha z} = e^{(a+ib)(x+iy)} = e^{ax-by}(\cos(ay+bx) + i \sin(ay+bx)) \Rightarrow$

$$\begin{cases} u = e^{ax-by}\cos(ay+bx) \\ v = e^{ax-by}\sin(ay+bx) \end{cases}$$

Figure repeat themselves
$u = 0$ at $ay + bx = n\pi + \frac{\pi}{2}$, $y = -\frac{b}{a}x + \frac{n\pi}{a} + \frac{\pi}{2a}$. $v = 0$ at $ay + bx = n\pi$, $y = -\frac{b}{a}x + \frac{n\pi}{a}$
FIGURE II.6.3a NF FIGURE II.6.3b NF

II.6.4
Find a conformal map of the horizontal strip $\{-A < \operatorname{Im} z < A\}$ onto the right half-plane $\{\operatorname{Re} w > 0\}$. Hint. Recall the discussion of the exponential function, or refer to the preceding problem.

Solution (A. Kumjian)

FIGURE II.6.4az FIGURE II.6.4aw

It is clearly intended that $A > 0$ so we will assume this. We define the map $h : \mathbb{C} \to \mathbb{C}$ by $h(z) = e^{\frac{\pi z}{2A}}$ for all $z \in \mathbb{C}$. Then

$$h'(z) = \frac{\pi}{2A} e^{\frac{\pi z}{2A}}$$

for all $z \in \mathbb{C}$. Hence, h is analytic with a nowhere vanishing derivative and so h is conformal. Note that the exponential functions maps the horizontal strip $\{z \in \mathbb{C} : -\pi/2 < \operatorname{Im} z < \pi/2\}$ onto the right half-plane

$$\{w \in \mathbb{C} : \operatorname{Re} w > 0\} = \{w \in \mathbb{C} : -\pi/2 < \operatorname{Arg} w < \pi/2\}.$$

Hence, h maps the horizontal strip $\{z \in \mathbb{C} : -A < \operatorname{Im} z < A\}$ onto the right half-plane $\{w \in \mathbb{C} : \operatorname{Re} w > 0\}$.

II.6.5
Find a conformal map of the wedge $\{-B < \arg z < B\}$ onto the right half-plane $\{\operatorname{Re} w > 0\}$. Assume $0 < B < \pi$.

Solution
Try $w = z^{\alpha}$, multiply angels by $\frac{\pi}{2B}$, so $e^{iB} \to e^{i\pi/2}$, $w = z^{\pi/(2B)}$ does the trick

FIGURE II.6.5az FIGURE II.6.5aw

II.6.6

Determine where the function $f(z) = z+1/z$ is conformal and where it is not conformal. Show that for each w, there are at most two values z for which $f(z) = w$. Show that if $r > 1$, $f(z)$ maps the circle $\{|z| = r\}$ onto an ellipse, and that $f(z)$ maps the circle $\{|z| = 1/r\}$ onto the same ellipse. Show that $f(z)$ is one-to-one on the exterior domain $D = \{|z| > 1\}$. Determine the image of D under $f(z)$. Sketch the images under $f(z)$ of the circles $\{|z| = r\}$ for $r > 1$, and sketch also the images of the parts of the rays $\{\arg z = \beta\}$ lying in D.

Solution

$f(z) = z + 1/z$, $f'(z) = 1 - 1/z^2$, $f'(z) = 0$ at $z^2 = 1$, $z = \pm 1$. Not defined at $z = 0$, not conformal at $z = \pm 1$. If $f(z) = \lambda$, then $z = \left(\lambda \pm \sqrt{\lambda^2 - 4}\right)/2$. If $z = \rho e^{i\theta}$, then $w = u + iv = \rho\cos\theta + i\rho\sin\theta + \cos\theta/\rho - i\sin\theta/\rho$, which can be written $u^2/(\rho + 1/\rho)^2 + v^2/(\rho - 1/\rho)^2 = 1$, i.e. image is an ellipse, replace ρ by $1/\rho$ and get same ellipse. Since $f(D) = f(\mathbf{D})$, and $f(z)$ is at most two–to–one, $f(z)$ is one–to–one on D. Since $f(z)$ maps $\partial\mathbf{D}$ onto the interval $[-2, 2]$, and $f(z)$ maps onto $\mathbb{C}$, the image of D is $\mathbb{C}\setminus[-2, 2]$.

II.6.7

For the function $f(z) = z + 1/z = u + iv$, sketch the families of level curves of u and v. Determine the images under $f(z)$ of the top half of the unit disk, the bottom half of the unit disk, the part of the upper half-plane outside the unit disk, and the part of the lower half-plane outside the unit disk. Hint. Start by locating the images of the curves where $u = 0$, where $v = 0$, and where $v = 1$. Note that the level curves are symmetric with respect to the real and imaginary axis, and they are invariant under the inversion $z \mapsto 1/z$ in the unit circle.

Solution

II.6.8 (From Hints and Solutions)

1	2	3	P	L	K

Consider $f(z) = z + e^{i\alpha}/z$, where $0 < \alpha < \pi$. Determine where $f(z)$ is conformal and where it is not conformal and where it is not conformal. Sketch the images under $f(z)$ of the unit circle $\{|z| = 1\}$ and the intervals $(-\infty, -1]$ and $[+1, +\infty)$ on the real axis. Show that $w = f(z)$ maps $\{|z| > 1\}$ conformally onto the complement of a slit plane in the $w-$ plane. Sketch roughly the images of the segments of rays outside the unit circle $\{\arg z = \beta, |z| \geq 1\}$ under $f(z)$. At what angels do they meet the slit, and at what angles do they approach ∞?

Solution

$f(z) = z + e^{i\alpha}/z$, $f'(z) = 1 - e^{i\alpha}/z^2$, $f'(z) = 0$ at $z^2 = e^{i\alpha}$. Not defined at $z = 0$, not conformal at $z = \pm e^{i\alpha/2}$. The expression $f(z) = e^{i\alpha/2}\left(e^{-i\alpha/2}z + 1/\left(e^{-i\alpha/2}z\right)\right)$, shows that $f(z)$ is a composition of the rotation by $-\alpha/2$, the function of Exercises 6 and 7, and a rotation by $\alpha/2$. Thus $f(z)$ maps $\{|z| > 1\}$ one–to–one onto the complement rotate of $[-2, 2]$ by $\alpha/2$.

II.6.9

Let $f = u+iv$ be a continuously differentiable complex-valued function on a domain D such that the Jacobian matrix of f does not vanish at any point of D. Show that if f maps orthogonal curves to orthogonal curves, then either f or $\bar{f}$ is analytic, with nonvanishing derivative.

Solution

Suppose $f(0) = 0$ and $u_x \neq 0$ at 0. The tangents in the orthogonal directions $(1,t)$ and $(-t,1)$ are mapped to the tangents in the directions $(u_x, v_x) + t(u_y, v_y)$ and $-t(u_x, v_x) + (u_y, v_y)$.
The orthogonality of these directions for all t gives

$$\langle (u_x, v_x) + t(u_y, v_y), -t(u_x, v_x) + (u_y, v_y)\rangle = 0,$$

can be rewritten

$$\langle (u_x, v_x), (u_y, v_y)\rangle + t\left(\langle (u_y, v_y), (u_y, v_y)\rangle - \langle (u_x, v_x), (u_x, v_x)\rangle\right) - t^2 \langle (u_y, v_y), (u_x, v_x)\rangle = 0.$$

The systems of equations

$$\begin{array}{c} (1) \\ (2) \end{array} \left\{ \begin{array}{c} u_x u_y + v_x v_y = 0 \\ u_y^2 + v_y^2 - (u_x^2 + v_x^2) = 0 \end{array} \right.$$

most hold to get orthogonality. Put (1) into (2) and get

$$\begin{aligned} u_x^2\left(u_y^2 + v_y^2 - u_x^2 - v_x^2\right) &= v_x^2 v_y^2 + u_x^2 v_y^2 - u_x^4 - u_x^2 v_x^2 = \\ &= v_x^2\left(v_y^2 - u_x^2\right) + u_x^2\left(v_y^2 - u_x^2\right) = \left(v_x^2 + u_x^2\right)\left(v_y^2 - u_x^2\right) = 0. \end{aligned}$$

If $u_x \neq 0$ we are lead to $u_x = \pm v_y$ and $u_y = \mp v_x$.
Tangent to curve $s \to (s, ts)$ is tangent to image curve $s + (u(s,ts), v(s,ts))$ $(1,0) \to (u_x, v_x)$, $(0,1) \to (u_y, v_y)$, orthogonal $\Rightarrow$ $(u_y, v_y) = C(-v_x, u_y)$ for some. A ????? gives $C = \pm 1$, same sign holds in ?????.
Either $u_x = v_y$, $u_y = -v_x \Rightarrow$ conformal in neighborhood of 0, or $u_x = -v_y$, $u_y = v_x \Rightarrow$ anti conformal in neighborhood of 0. $u_x = \pm v_y$ and $u_y = \mp v_x$.
So Cauchy Riemann is satisfied for either f or $\bar{f}$.

II.7.1

Compute explicitly the fractional linear transformations determined by the following correspondences of triples

(a) $(1+i,2,0) \longmapsto (0,\infty,i-1)$

(b) $(0,1,\infty) \longmapsto (1,1+i,2)$

(c) $(\infty,1+i,2) \longmapsto (0,1,\infty)$

(d) $(-2,i,2) \longmapsto (1-2i,0,1+2i)$

(e) $(1,2,\infty) \longmapsto (0,1,\infty)$

(f) $(0,\infty,i) \longmapsto (0,1,\infty)$

(g) $(0,1,\infty) \longmapsto (0,\infty,i)$

(h) $(1,i,-1) \longmapsto (1,0,-1)$

Solution

$$\frac{(w-w_0)(w_1-w_2)}{(w-w_2)(w_1-w_0)} = \frac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)}$$

(a)
We have the points

$z_0 = 1+i$	$w_0 = 0$
$z_1 = 2$	$w_1 = \infty$
$z_2 = 0$	$w_2 = i-1$

The mapping is

$$\frac{(w-w_0)(1-w_2/w_1)}{(w-w_2)(1-w_0/w_1)} = \frac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)},$$

take limit as $w_1 \to \infty$, to obtain

$$\frac{w-0}{w-(i-1)} = \frac{(z-(1+i))(2-0)}{(z-0)(2-(1+i))} \Rightarrow w = \frac{2iz+(2-2i)}{z-2}.$$

(b)
We have the points

$z_0 = 0$	$w_0 = 1$
$z_1 = 1$	$w_1 = 1+i$
$z_2 = \infty$	$w_2 = 2$

The mapping is

$$\frac{(w-w_0)(w_1-w_2)}{(w-w_2)(w_1-w_0)} = \frac{(z-z_0)(z_1/z_2-1)}{(z/z_2-1)(z_1-z_0)},$$

take limit as $z_2 \to \infty$, to obtain

$$\frac{(w-1)\,((1+i)-2)}{(w-2)\,((1+i)-1)} = \frac{z-0}{1-0} \Rightarrow w = \frac{2z-(1+i)}{z-(1+i)}.$$

(c)
We have the points

$z_0 = \infty$	$w_0 = 0$
$z_1 = 1+i$	$w_1 = 1$
$z_2 = 2$	$w_2 = \infty$

The mapping is

$$\frac{(w-w_0)\,(w_1/w_2-1)}{(w/w_2-1)\,(w_1-w_0)} = \frac{(z/z_0-1)\,(z_1-z_2)}{(z-z_2)\,(z_1/z_0-1)},$$

take limit as $z_0, w_2 \to \infty$, to obtain

$$\frac{w-0}{1-0} = \frac{(1+i)-2}{z-2} \Rightarrow w = \frac{i-1}{z-2}.$$

(d)
We have the points

$z_0 = -2$	$w_0 = 1-2i$
$z_1 = i$	$w_1 = 0$
$z_2 = 2$	$w_2 = 1+2i$

The mapping is

$$\frac{(w-w_0)\,(w_1-w_2)}{(w-w_2)\,(w_1-w_0)} = \frac{(z-z_0)\,(z_1-z_2)}{(z-z_2)\,(z_1-z_0)},$$

obtain

$$\frac{(w-(1-2i))\,(0-(1+2i))}{(w-(1+2i))\,(0-(1-2i))} = \frac{(z-(-2))\,(i-2)}{(z-2)\,(i-(-2))} \Rightarrow w = iz+1.$$

(e)
We have the points

$z_0 = 1$	$w_0 = 0$
$z_1 = 2$	$w_1 = 1$
$z_2 = \infty$	$w_2 = \infty$

The mapping is

$$\frac{(w-w_0)\,(w_1/w_2-1)}{(w/w_2-1)\,(w_1-w_0)}=\frac{(z-z_0)\,(z_1/z_2-1)}{(z/z_2-1)\,(z_1-z_0)},$$

take limit as $z_2, w_2 \to \infty$, to obtain

$$\frac{w-0}{1-0}=\frac{z-1}{2-1}\Rightarrow w=z-1.$$

(f)
We have the points

$z_0=0$	$w_0=0$
$z_1=\infty$	$w_1=1$
$z_2=i$	$w_2=\infty$

The mapping is

$$\frac{(w-w_0)\,(w_1/w_2-1)}{(w/w_2-1)\,(w_1-w_0)}=\frac{(z-z_0)\,(1-z_2/z_1)}{(z-z_2)\,(1-z_0/z_1)},$$

take limit as $z_1, w_2 \to \infty$, to obtain

$$\frac{w-0}{1-0}=\frac{z-0}{z-i}\Rightarrow w=\frac{z}{z-i}$$

(g)
We have the points

$z_0=0$	$w_0=0$
$z_1=1$	$w_1=\infty$
$z_2=\infty$	$w_2=i$

The mapping is

$$\frac{(w-w_0)\,(1-w_2/w_1)}{(w-w_2)\,(1-w_0/w_1)}=\frac{(z-z_0)\,(z_1/z_2-1)}{(z/z_2-1)\,(z_1-z_0)},$$

take limit as $z_2, w_1 \to \infty$, to obtain

$$\frac{w-0}{w-i}=\frac{z-0}{1-0}\Rightarrow w=\frac{iz}{z-1}.$$

(h)

We have the points

$z_0 = 1$	$w_0 = 1$
$z_1 = i$	$w_1 = 0$
$z_2 = -1$	$w_2 = -1$

The mapping is

$$\frac{(w - w_0)(w_1 - w_2)}{(w - w_2)(w_1 - w_0)} = \frac{(z - z_0)(z_1 - z_2)}{(z - z_2)(z_1 - z_0)},$$

obtain

$$\frac{(w - 1)(0 - (-1))}{(w - (-1))(0 - 1)} = \frac{(z - 1)(i - (-1))}{(z - (-1))(i - 1)} \Rightarrow w = \frac{iz + 1}{z + i}.$$

II.7.2

Consider the fractional linear transformation in Exercise 1a above, which maps $1+i$ to 0, 2 to ∞, and 0 to $i-1$. Without referring to an explicit formula, determine the image of the circle $\{|z-1|=1\}$, the image of the disk $\{|z-1|<1\}$, and the image of the real axis.

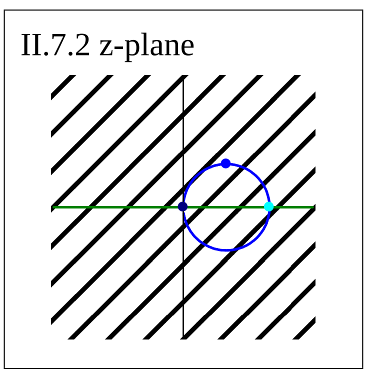
Solution

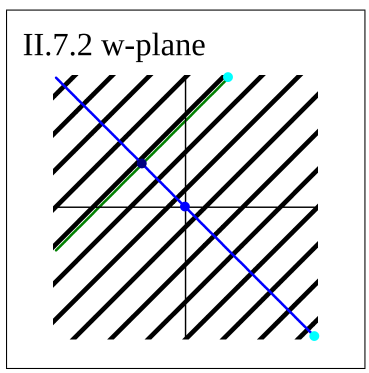
It is obvious that the tripple $(1+i, 2, 0)$ lies on the circle $|z-1| = 1$ in the $z-$ plane. From exercise II.7.1a we know that points in the $w-$ plane the tripple is mapped on, i.e,

$z_0 = 1+i$	$w_0 = 0$
$z_1 = 2$	$w_1 = \infty$
$z_2 = 0$	$w_2 = i-1$

The circle $|z-1| = 1$ is mapped to the straight line through 0 and $i-1$, i.e $\operatorname{Im} w = -\operatorname{Re} w$ since tree points determines the "circle". By preservation of orientation the disk goes to the half-plane to lower left of the straight line through 0 and $i-1$.

Image of the real axis is "circle" orthogonal to image of $|z-1| = 1$, that must be the straight line through $i-1$, with slope 1, i.e. the line $\operatorname{Im} w = \operatorname{Re} w + 2$.





II.7.3
Consider the fractional linear transformation that maps 1 to i, 0 to $1+i$, -1 to 1. Determine the image of the unit circle $\{|z|=1\}$, the image of the open unit disk $\{|z|<1\}$, and the image of the imaginary axis. Illustrate with a sketch.

Solution
We have the points

$z_0 = 1$	$w_0 = i$
$z_1 = 0$	$w_1 = 1+i$
$z_2 = -1$	$w_2 = 1$

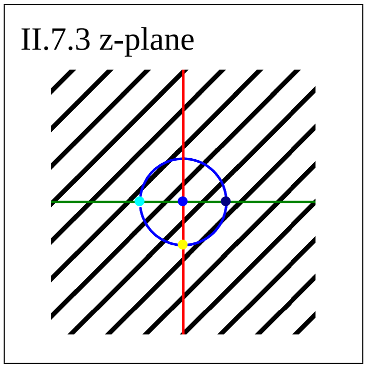
The mapping is

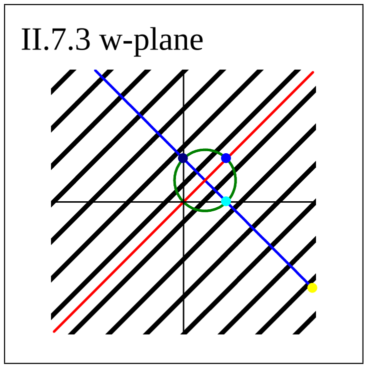
$$\frac{(w-w_0)(w_1-w_2)}{(w-w_2)(w_1-w_0)} = \frac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)},$$

obtain

$$\frac{(w-i)((1+i)-1)}{(w-1)((1+i)-i)} = \frac{(z-1)(0-(-1))}{(z-(-1))(0-1)} \Rightarrow w = \frac{i-1}{z+i}$$

By our mapping $w = (i-1)/(z+i)$, we have that $-i$ on the unitcircle in the $z-$ plane is mapped to ∞ in the $w-$ plane. The circle $|z|=1$ is mapped to the straight line through i and 1, i.e $\operatorname{Im} w = -\operatorname{Re} w + 1$ since tree points determines the "circle". By preservation of orientation the disk goes to the half-plane to upper left of the straight line through i and 1.
The tripple $(1, 0, -1)$ on the real axis is mapped to the points $(i, 1+i, 1)$ in the $w-$ plane, it must be so that the real axis in the $z-$ plane is mapped onto the circle $\left|w - \left(\frac{1}{2} + \frac{1}{2}i\right)\right| = \frac{\sqrt{2}}{2}$ in the $w-$ plane.
Imaginary axis must be mapped to a circle through $1+i$, that is orthogonal to image of real line, this is the straight line through $1+i$ and 0, i.e $\operatorname{Im} w = \operatorname{Re} w$.


II.7.3 z-plane


II.7.3 w-plane

II.7.4

Consider the fractional linear transformation that maps -1 to $-i$, 1 to $2i$, and i to 0. Determine the image of the unit circle $\{|z| = 1\}$, the image of the open unit disk $\{|z| < 1\}$, and the image of the interval $[-1, +1]$ on the real axis. Illustrate with a sketch.

Solution

We have the points

$z_0 = -1$	$w_0 = -i$
$z_1 = 1$	$w_1 = 2i$
$z_2 = i$	$w_2 = 0$

The mapping is

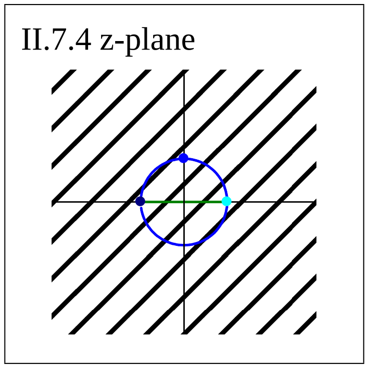
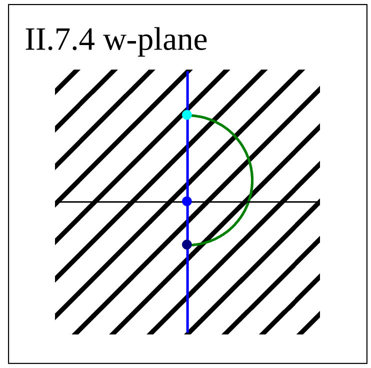
$$\frac{(w - w_0)(w_1 - w_2)}{(w - w_2)(w_1 - w_0)} = \frac{(z - z_0)(z_1 - z_2)}{(z - z_2)(z_1 - z_0)},$$

obtain

$$\frac{(w - (-i))(2i - 0)}{(w - 0)(2i - (-i))} = \frac{(z - (-1))(1 - i)}{(z - i)(1 - (-1))} \Rightarrow w = \frac{(-6 - 2i)z - 2 + 6i}{5z - 3 + 4i}$$

It is obvious that the tripple $(-1, 1, i)$ lies on the circle $|z| = 1$ in the $z-$plane. And we can see that thiese three points are mapped to three points on the imaginary axis in the $w-$ plane, i.e $\operatorname{Re} w = 0$, since tree points determines the "circle". By preservation of orientation the disk goes to the right half-plane.

The interval $[-1, 1]$ must be mapped to a circle through $-i$ and $2i$ that is orthogonal to image of the unit circle, this is an arc of the circle $\left|w - \frac{1}{2}i\right| = \frac{3}{2}$. We have that $w(0) = \frac{6}{5} - \frac{2}{5}i$ must be an point on the arc, thus the arc is the circle $\left|w - \frac{1}{2}i\right| = \frac{3}{2}$ in the right half-plane from $-i$ to $2i$.


II.7.4 z-plane

II.7.4 w-plane

II.7.5
What is the image of the horizontal line through i under the fractional linear transformation that interchanges 0 and 1 and maps -1 to $1+i$? Illustrate with a sketch.

Solution
We have the points

$z_0 = 0$	$w_0 = 1$
$z_1 = 1$	$w_1 = 0$
$z_2 = -1$	$w_2 = 1+i$

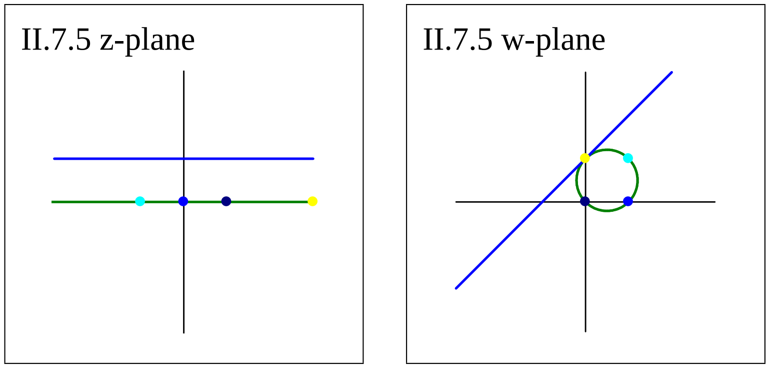
The mapping is

$$\frac{(w-w_0)(w_1-w_2)}{(w-w_2)(w_1-w_0)} = \frac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)},$$

obtain

$$\frac{(w-1)(0-(1+i))}{(w-(1+i))(0-1)} = \frac{(z-0)(1-(-1))}{(z-(-1))(1-0)} \Rightarrow w = \frac{iz-i}{z-i}$$

The tripple $(0,1,-1)$ on the real axis is mapped to the points $(1,0,1+i)$ in the $w-$ plane, it must be so that the real axis in the $z-$ plane is mapped onto the circle $\left|w-\left(\frac{1}{2}+\frac{1}{2}i\right)\right| = \frac{\sqrt{2}}{2}$ in the $w-$ plane, since tree points determines the "circle". The real axis and the line through i have one common point at infinity. The common point at infinity for both real axis and the line through i must be mapped to the same point in the $w-$ plane, we have that the point is $w(\infty) = i$. The two lines in the $z-$ plane is parallell, thus the images of the lines must go through i in the $w-$ plane and be parallell in that point, thus the mapping of the horizontal line through i must be the line through i, which is tangent line to circle at i, i.e. $\operatorname{Im} z = \operatorname{Re} z + 1$. Remark that the image can not be a circle because $w(i) = \infty$, so the image of the horizontal line must contain the point at infinity.


II.7.5 z-plane
II.7.5 w-plane

II.7.6
Show that the image of a straight line under the inversion $z \mapsto 1/z$ is a straight line or circle, depending on whether the line passes through the origin.

Solution
The image of the straight line $ax+by=c$ is (kontrollera $x-iy$ i lösningshäftet)

$$w=\frac{1}{z}=\frac{1}{z\bar{z}}=\frac{\bar{z}}{|z|^2}=(x-iy)\,\frac{1}{r^2}$$

where $u=x/r^2$ and $v=-y/r^2$. Get

$$au-bv=\frac{ax+by}{r^2}=\frac{c}{r^2}=\frac{c}{x^2+y^2}=c\left(u^2+v^2\right)$$

The image is the solution of $c\,(u^2+v^2)-au+bv=0$.
If $c=0$ we have that

$$-au+bv=0,$$

this is a straight line through 0.
If $c\neq 0$ we have

$$c\left(u^2+v^2\right)-au+bv=0,$$

which can be rewritten as

$$\left(u-\frac{a}{2c}\right)^2+\left(v+\frac{b}{2c}\right)^2=\left(\sqrt{\left(\frac{a}{2c}\right)^2+\left(\frac{b}{2c}\right)^2}\right)^2,$$

it is a circle through 0.

Solution (K. Seip)
Set $f(z)=\frac{1}{z}$. Then $\infty\in f(S)\Leftrightarrow 0\in S$, which proves the claim.

II.7.7
Show that the fractional linear transformation $f(z) = (az+b)/(cz+d)$ is the identity mapping z if and only if $b = c = 0$ and $a = d \neq 0$.

Solution
We have that

$$\frac{az+b}{cz+d} = z \Leftrightarrow az + b = cz^2 + dz \Leftrightarrow cz^2 + (d-a)z - b = 0.$$

Put in values on z for intance 0 and ± 1 the expession becomes

$$\begin{array}{lll} (1) & c - (d-a) = 0, & z = -1. \\ (2) & -b = 0 & z = 0 \\ (3) & c + (d-a) = 0, & z = 1. \end{array}$$

From (2) we have $b = 0$ and if we add (1) and (3) we have $c = 0$.
Thus the fractional linear transformation $f(z)$ is the identity mapping if and only if $b = c = 0$, $a = d \neq 0$.

Solution
Set $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. Suppose $f(z) = \frac{az+b}{cz+d}$ maps $\mathbb{R}^*$ to $\mathbb{R}^*$. Then $0 \mapsto \frac{b}{d} \in \mathbb{R}^*$, $\infty \mapsto \frac{a}{c} \in \mathbb{R}^*$, $0 \hookleftarrow -\frac{b}{a} \in \mathbb{R}^*$, $\infty \hookleftarrow -\frac{d}{c} \in \mathbb{R}^*$. At least one of theese rations is different from 0 and ∞, and since one of a, b, c, d can be chosen freely, the result follows.

II.7.8
Show that any fractional linear transformation can be represented in the form $f(z) = (az+b)(cz+d)$, where $ad - bc = 1$. Is this representation unique?

Solution
If

$$w = (\alpha z + \beta)(\gamma z + \delta)$$

divide each coefficient by the square root of

$$\alpha\delta - \beta\gamma$$

to obtain representation with

$$ad - bc = 1.$$

Representation is not unique, as can multiply all coefficients by -1.

II.7.9

Show that the fractional linear transformations that are real on the real axis are precisely those that can be expressed in the form $(az+b)/(cz+d)$, where $a, b, c,$and d are real.

Solution (K. Seip)

Set $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. Suppose $f(z) = \frac{az+b}{cz+d}$ maps $\mathbb{R}^*$ to $\mathbb{R}^*$. Then $0 \longmapsto \frac{b}{d} \in \mathbb{R}^*$, $\infty \mapsto \frac{a}{c} \in \mathbb{R}^*$, $0 \hookleftarrow -\frac{b}{a} \in \mathbb{R}^*$, $\infty \hookleftarrow -\frac{d}{c} \in \mathbb{R}^*$. At least one of these is different from 0 and ∞, and since one of a, b, c, d can be chosen freely, the result follows.

II.7.9
Show that the fractional linear transformations that are real on the real axis are precisely those that can be expressed in the form $(az+b)/(cz+d)$, where $a, b, c,$and d are real.

Solution (A. Kumjian)
Set $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$, note that a fractional linear transformation $f : \mathbb{C}^* \to \mathbb{C}^*$ is real on the real axis iff $f(\mathbb{R}^*) \subset \mathbb{R}^*$ (since f is continuous). For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\det A \neq 0$ we define the fractional linear transformation $f_A : \mathbb{C}^* \to \mathbb{C}^*$ by the formula $f_A(z) = (az+b)/(cz+d)$. It remains to prove that given a fractional linear transformation f, we have $f(\mathbb{R}^*) \subset \mathbb{R}^*$ iff $f = f_A$ for some 2×2 matrix A with real entries.

Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a fractional linear transformation. Suppose that $f = f_A$ where A is a 2×2 matrix with real entries a, b, c and d as above. Then for $x \in \mathbb{R}^*$ we have

$$f(x) = f_A(x) = \frac{ax+b}{cx+d} \in \mathbb{R}^*.$$

This is clear for $x \in \mathbb{R}$ but requires a little work for $x = \infty$. We have

$$f(\infty) = \lim_{z\to\infty} \frac{az+b}{cz+d} = \begin{cases} a/c & \text{if} \quad c \neq 0, \\ \infty & \text{if} \quad c = 0. \end{cases}$$

Hence, $f(\mathbb{R}^*) \subset \mathbb{R}^*$.
Conversely, suppose that $f(\mathbb{R}^*) \subset \mathbb{R}^*$, we must show that $f = f_A$ for some 2×2 matrix with real entries. If A is a 2×2 matrix (with non-zero determinant), then $f_{A^{-1}} = (f_A)^{-1}$. Hence, $f = f_A$ for some 2×2 matrix A with real entries iff its inverse f^{-1} has the same property.
By the unlikeness part of the theorem on page 64, f is completely determined by the three extended real numbers $x_0 = f(0)$, $x_1 = f(1)$ and $x_\infty = f(\infty)$ (recall that we have assumed $f(\mathbb{R}^*) \subset \mathbb{R}^*$). So it suffices to show that the fractional linear transformation g for which $g(x_0) = 0$, $g(x_1) = 1$ and $g(x_\infty) = \infty$ has the requisite properties, because we must have $g = f^{-1}$ (again by the uniqueness part of the theorem).
There are four cases to consider. The first case is when $x_0, x_1, x_\infty \neq \infty$ and the remaining three cases are $x_0 = \infty$, $x_1 = \infty$ and $x_\infty = \infty$. The formula for $g(z)$ in the four cases is given as follows (in the first case $k = \frac{x_1 - x_\infty}{x_1 - x_0} \in \mathbb{R}$):

$$
\begin{array}{llll}
k\dfrac{x-x_0}{x-x_\infty} & \text{if} \quad x_0, x_1, x_\infty \neq \infty, & \dfrac{x_1-x_\infty}{x-x_\infty} & \text{if} \quad x_0 = \infty, \\
\dfrac{x-x_0}{x-x_\infty} & \text{if} \quad x_1 = \infty, & \dfrac{x-x_0}{x_1-x_0} & \text{if} \quad x_\infty = \infty.
\end{array}
$$

In each case $g(z)$ has the desired form. Hence, $f = f_A$ for some 2×2 matrix A with real entries as required.

II.7.10

Suppose the fractional linear transformation $(az+b)/(cz+d)$ maps $\mathbb{R}$ to $\mathbb{R}$, and $ad-bc=1$. Show that $a, b, c,$ and d are real or they are all pure imaginary.

Solution

Because $\mathbb{R}$ is mapped on $\mathbb{R}$ and the orientation is perserved, $f(x)$ will be either increasing or decreasing.

Case 1 : Suppose $f(x)$ is increasing, then $f'(x) = 1/(cx+d)^2 > 0$ for all $x \Rightarrow c, d$ are real. $f(z) = (az+b)/(cz+d)$, then $ax+b$ is real for all $x \in \mathbb{R}$, so a, b are also real.

Case 2 : Suppose $f(x)$ is decreasing, then $f'(z) = 1/(cz+d)^2 < 0$ for all $x \Rightarrow c, d$ are pure imaginary. $f(z) = (az+b)/(cz+d)$, then $ax+b$ are pure imaginary for all $x \in \mathbb{R}$, so a, b are also pure imaginary.

II.7.11

Two maps f and g are conjugate if there is h such that $g = h \circ f \circ h^{-1}$. Here the conjugating map h is assumed to be one-to-one, with appropriate domain and range. We can think of f and g as the "same" map, after the change of variable $w = h(z)$. A point z_0 is a fixed point of f if $f(z_0) = z_0$. Show the following. (a) If f is conjugate to g, then g is conjugate to f.
(b) If f_1 is conjugate to f_2 and f_2 to f_3, then f_1 is conjugate to f_3.
(c) If f is conjugate to g, then $f \circ f$ is conjugate to $g \circ g$, and more generally, the $m-$fold composition $f \circ \cdots \circ f$ (m times) is conjugate to $g \circ \cdots \circ g$ (m times).
(d) If f and g are conjugate, then the conjugating function h maps the fixed points of f to fixed points of g. In particular, f and g have the same number of fixed points.

Solution

(a)
If f is conjugate to g then

$$g = h \circ f \circ h^{-1} \Rightarrow f = h^{-1} \circ g \circ h$$

thus g is conjugate to f

(b)
If $f_1 = h \circ f_2 \circ h^{-1}$ and $f_2 = g \circ f_3 \circ g^{-1}$ we get

$$f_1 = h \circ g \circ f_3 \circ g^{-1} \circ h^{-1} = (h \circ g) \circ f_3 \circ (h \circ g)^{-1}.$$

(c)

$$\begin{aligned} g = h \circ f \circ h^{-1} then \overbrace{g \circ g \circ \ldots \circ g}^{m \text{ times}} = \\ = h \circ f \circ h^{-1} \circ h \circ f \circ h^{-1} \circ \ldots \circ h \circ f \circ h^{-1} = \\ = h \circ \overbrace{f \circ f \circ \ldots \circ f}^{m \text{ times}} \circ h^{-1} \end{aligned}$$

(d)
$g = h \circ f \circ h^{-1}$, $f(z_0) = z_0$, and $w_0 = h(z_0)$, then

$$g(w_0) = h\left(f\left(h^{-1}(w_0)\right)\right) = h(f(z_0)) = h(z_0) = w_0.$$

We have that f and g have the same number of fixed points since every point of g is mapped to a fixed point of f by the function h^{-1}.

II.7.12

Classify the conjugacy classes of fractional linear transformations by establishing the following

(a)

A fractional linear transformation that is not the identity has either 1 or 2 fixed points, that is, points satisfying $f(z_0) = z_0$.

(b)

If a fractional linear transformation $f(z)$ has two fixed points, then it is conjugate to the dilation $z \mapsto az$ with $a \neq 0$, $a \neq 1$, that is, there is a fractional linear transformation $h(z)$ such that $h(f(z)) = ah(z)$. Is a unique? Hint. Consider a fractional linear transformation that maps the fixed points to 0 and ∞.

(c)

If a fractional linear transformation $f(z)$ has exactly one fixed point, then it is conjugate to the translation $\zeta \mapsto \zeta + 1$. In other words, there is a fractional linear transformation $h(z)$ such that $h(f(h^{-1}(\zeta))) = \zeta + 1$, or equivalently, such that $h(f(z)) = h(z) + 1$. Hint. Consider a fractional linear transformation that maps the fixed point to ∞.

Solution

a) $\frac{az+b}{cz+d} = z \Leftrightarrow az + b = cz^2 + dz \Leftrightarrow cz^2 + (d-a)z - b = 0$. If this is $= 0$ for every z, we get $w(z) = z$. It is not identically equall to zero, so it has either 1 or 2 finite solutions. If $c \neq 0$, it has 2 solutions possibly one with multiplicity and ∞ is not a fixed point. If $c = 0$, it has one finite solution $z_1 = b/(d-a)$, and $z_2 = \infty$ is a fixed point, so they are two.

b) Make a change of variables $h(z) = (\alpha z + \beta)/(\gamma z + \delta)$ that maps fixed points to 0 and ∞ . Then $h \circ f \circ h^{-1}$ is FLT with fixed points at 0 and ∞. $(h \circ f \circ h^{-1})(w) = aw$ for some $a \neq 0$, $a \neq 1$. f is conjugate to a dilation Suppose aw is conjugate to Aw, i.e. $\exists h$, $(h(ah^{-1}))(w) = Aw$, $h(az) = Ah(z)$. h must map fixed points to fixed points, so either $h(z) = cz$ or $h(z) = c/z$. So either $h(z) = cz \Rightarrow h(az) = caz = Ah(z) = Acz \Rightarrow A = a$ or $h(z) = c/z \Rightarrow h(az) = c/(az) = Ah(z) = Ac/z \Rightarrow A = 1/a$. Thus a is not unique.

c) Suppose h maps the fixed points to ∞, then $h \circ f \circ h^{-1}$ has only one fixed point at ∞, so $(h \circ f \circ h^{-1})(w) = aw+b$. Since this has no finite fixed points, $aw+b = w$ has no solutions and $a = 1$, $b \neq 0$. Thus $(h \circ f \circ h^{-1})(w) = w+b$.

Now $w+b$ is conjugate to $w+1$, by a dilation $g(w)=Aw$. $w \to w/A \to w/A+b \to w+Ab$, take $A=1/b$. So

$$\begin{aligned}\left(g \circ \left(h \circ f \circ h^{-1}\right) \circ g^{-1}\right)(w) = w+1 \iff \\ \iff \left((g \circ h) \circ f \circ (g \circ h)^{-1}\right)(w) = w+1 \iff \\ \iff ((g \circ h) \circ f)(z) = (g \circ h)(z) + 1.\end{aligned}$$

III	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1																			
2																			
3																			
4																			
5																			
6																			
7																			
8																			

III.1.1
Evaluate $\int_\gamma y^2dx + x^2dy$ along the following paths γ from $(0,0)$ to $(2,4)$,
(a) the arc of the parabola $y = x^2$,
(b) the horizontal interval from $(0,0)$ to $(2,0)$, followed by the vertical interval from $(2,0)$ to $(2,4)$, (c) the vertical interval from $(0,0)$ to $(0,4)$, followed by the horizontal interval from $(0,4)$ to $(2,4)$.

Solution
(a)

$$\int_\gamma y^2dx+x^2dy = \left[\begin{array}{lll} x=t & dx=dt, & 0\le t\le 2 \\ y=t^2 & dy=2t\,dt & \end{array}\right] = \int_0^2 \left(t^2\right)^2 dt + \int_0^2 t^2 2t\,dt = \frac{72}{5}.$$

(b)

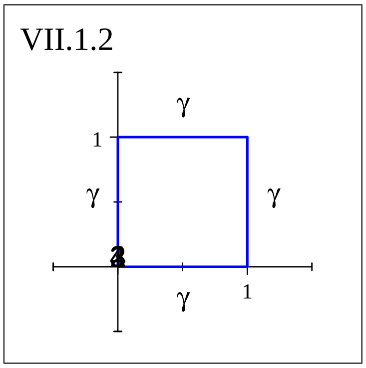
$$\int_{\gamma_1} y^2dx + x^2dy = \left[\begin{array}{lll} x=t & dx=dt & 0\le t\le 2 \\ y=0 & dy=0\,dt & \end{array}\right] = \int_0^2 0^2dt + \int_0^2 t^2\,0\,dt = 0$$
$$\int_{\gamma_2} y^2dx + x^2dy = \left[\begin{array}{lll} x=2 & dx=0\,dt & 0\le t\le 4 \\ y=t & dy=dt & \end{array}\right] = \int_0^4 t^2\,0\,dt + \int_0^4 2^2dt = 16$$
$$\int_\gamma y^2dx + x^2dy = \int_{\gamma_1} y^2dx + x^2dy + \int_{\gamma_2} y^2dx + x^2dy = 16.$$

(c)

$$\int_{\gamma_1} y^2dx + x^2dy = \left[\begin{array}{lll} x=0 & dx=0\,dx & 0\le t\le 4 \\ y=t & dy=dt & \end{array}\right] = \int_0^4 y^2\,0\,dt + \int_0^4 0^2dt = 0$$
$$\int_{\gamma_2} y^2dx + x^2dy = \left[\begin{array}{lll} x=t & dx=dt & 0\le t\le 2 \\ y=4 & dy=0dt & \end{array}\right] = \int_0^2 4^2dt + \int_0^2 t^2\,0\,dt = 32$$
$$\int_\gamma y^2dx + x^2dy = \int_{\gamma_1} y^2dx + x^2dy + \int_{\gamma_2} y^2dx + x^2dy = 32.$$

III.1.2
Evaluate $\int_\gamma xy\,dx$ both directly and using Green's theorem, where γ is the boundary of the square with vertices at $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

Solution (A. Kumjian)



Denote the square by D and note that $xy\,dx = P\,dx + Q\,dy$ where $P = xy$ and $Q = 0$. Then P and Q are continuously differentiable on $\bar{D}$ and $\gamma = \partial D$, hence by Green's Theorem we have,

$$\int_\gamma xy\,dx = \int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{dy}\right) dx\,dy = \int_0^1 \int_0^1 -x\,dx\,dy = -\frac{1}{2}.$$

Observe that $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where γ_1 and γ_3 are the bottom and top of the square while γ_2 and γ_4 are the last two sides taken in the order indicated by the order of the vertices in the statement of the problem (so the boundary is oriented counter clockwise). Note that the path integrals on γ_2 and γ_4 are zero because the edges are vertical.

$$\int_\gamma xy\,dx = \int_{\gamma_1} xy\,dx + \int_{\gamma_3} xy\,dx = \int_0^1 x \cdot 0\,dx - \int_0^1 x \cdot 1\,dx = 0 - \frac{1}{2} = -\frac{1}{2}.$$

III.1.3

Evaluate $\int_{\partial D} x^2 dy$ both directly and using Green's theorem, where D is the quarter-disk in the first quadrant bounded by the unit circle an the two coordinate axes.

Solution

Evaluate $\int_{\partial D} x^2 dy$ directly, set $\partial D = \gamma_1 + \gamma_2 + \gamma_3$.

$$\int_{\gamma_1} x^2 dy = \left[\begin{array}{lll} x = t & dx = dt & 0 \le t \le 1 \\ y = 0 & dy = 0\, dt & \end{array} \right] = \int_0^1 t^2\, 0\, dt = 0$$

$$\int_{\gamma_2} x^2 dy = \left[\begin{array}{lll} x = \cos t & dx = -\sin t\, dt & 0 \le t \le \pi/2 \\ y = \sin t & dy = \cos t\, dt & \end{array} \right] = \int_0^{\pi/2} \cos^2 t \cos t \; dt = \frac{2}{3}$$

$$\int_{\gamma_3} x^2 dy = \left[\begin{array}{lll} x = 0 & dx = 0\, dt & 0 \le t \le 1 \\ y = 1 - t & dy = -dt & \end{array} \right] = -\int_0^1 0^2 dt = 0$$

$$\int_{\partial D} x^2 dy = \int_{\gamma_1} x^2 dy + \int_{\gamma_2} x^2 dy + \int_{\gamma_3} x^2 dy = \frac{2}{3}$$

Now we evaluate $\int_{\partial D} x^2 dy$, this time using Green's theorem. In this case, $P(x, y) = 0$ and $Q(x, y) = x^2$.

$$\int_{\partial D} x^2 dy = \iint_D 2x\, dxdy =$$

$$= 2 \iint_D x\, dxdy = \left[\begin{array}{lll} x = r\cos\theta & dx\, dy = r\, dr\, d\theta & 0 \le r \le 1 \\ y = r\sin\theta & & 0 \le \theta \le \pi/2 \end{array} \right] = 2 \int_0^1 \int_0^{\pi/2} r\cos\theta\, r dr d\theta =$$

$$= 2 \int_0^1 r^2 dr \cdot \int_0^{\pi/2} \cos\theta d\theta = \frac{2}{3}.$$

III.1.4
Evaluate $\int_\gamma y\,dx$ both directly and using Green's theorem, where γ is the semicircle in the upper half-plane from R to $-R$.

Solution
Evaluate $\int_\gamma y\,dx$ directly, set $\gamma = \gamma_1 + \gamma_2$.

$$\int_{\gamma_1} y\,dx = \left[\begin{array}{lll} x = t & dx = dt & -R \le t \le R \\ y = 0 & dy = 0\,dt & \end{array}\right] = \int_{-R}^{-R} 0\ dt = 0$$

$$\int_{\gamma_2} y\,dx = \left[\begin{array}{lll} x = R\cos t & dx = -R\sin t\,dt & 0 \le t \le \pi \\ y = R\sin t & dy = R\cos t\,dt & \end{array}\right] = -\int_0^\pi R\sin t \cdot R\sin t\,dt = -\frac{\pi R^2}{2}$$

$$\int_{\partial D} x^2 dy = \int_{\gamma_1} x^2 dy + \int_{\gamma_2} x^2 dy = -\frac{\pi R^2}{2}$$

Now we evaluate $\int_\gamma y\,dx$, this time using Green's theorem. In this case, $P(x,y) = y$ and $Q(x,y) = 0$.

$$\int_{\partial D} y\,dx = -\iint_D dx\,dy =$$

$$= -\iint_D dx\,dy = \left[\begin{array}{lll} x = r\cos\theta & dx\,dy = r\,dr\,d\theta & 0 \le r \le R \\ y = r\sin\theta & & 0 \le \theta \le \pi \end{array}\right] = -\int_0^R \int_0^\pi r\,dr\,d\theta =$$

$$= -\int_0^R r\,dr \cdot \int_0^\pi d\theta = -\frac{\pi R^2}{2}.$$

III.1.5

Show that $\int_{\partial D} x\,dy$ is the area of D, while $\int_{\partial D} y\,dx$ is minus the area of D.

Solution

Using in the two cases

For $\int_{\partial D} x\,dy$, we have $P = 0$ and $Q = x$, using Green's theorem we have

$$\int_{\partial D} x\,dx = \int_D \int dx\,dy = \text{Area}\,D.$$

For $\int_{\partial D} y\,dx$, we have $P = y$ and $Q = 0$, using Green's theorem we have

$$\int_{\partial D} y\,dx = -\int_D \int dx\,dy = -\,\text{Area}\,D.$$

III.1.6

Show that if P and Q are continuous complex-valued functions on a curve γ, then

$$\int_\gamma \frac{Pdx}{z-w} + \int_\gamma \frac{Qdy}{z-w}, \qquad (z = x+iy)$$

is analytic for $w \in \mathbb{C}\backslash\gamma$. Express $F'(w)$ as a line integral over γ.

Solution

Differentiate by hand, use uniform convergence of

$$\frac{1}{\Delta w}\left[\frac{1}{z-(w+\Delta w)} - \frac{1}{z-w}\right] = \frac{1}{(z-w)(z-(w+\Delta w))} \to \frac{1}{(z-w)^2}$$

as $\Delta w \to 0$, uniformly for $z \in \gamma$. Get

$$F'(w) = \int_\gamma \frac{Pdx}{(z-w)^2} + \int_\gamma \frac{Qdx}{(z-w)^2}, \qquad z = x+iy$$

III.1.7

Show that the formula in Green's theorem is invariant under coordinate changes, in the sense that if the theorem holds for a bounded domain U with piecewise smooth boundary, and if $F(x,y)$ is a smooth function that maps U one-to-one onto another such domain V and that maps the boundary of U one-to-one smoothly onto the boundary of V, then Green's theorem holds for V. Hint. First note the change of variable formulae for line and area integrals, given by

$$\begin{aligned}\int_{\partial V} P\,d\xi &= \int_{\partial U} (P\circ F)\left(\frac{\partial\xi}{\partial x}dx + \frac{\partial\xi}{\partial y}dy\right),\\ \iint_V R\,d\xi\,d\eta &= \iint_U (R\circ F)\det\, J_F\,dx\,dy,\end{aligned}$$

where $F(x,y) = (\xi(x,y), \eta(x,y))$, and where J_F is the Jacobian matrix of F. Use these formulae, with $R = -\partial P/\partial\eta$. The summand $\int Q\,d\eta$ is treated similarly.

Solution

$$\iint\limits_V -\frac{\partial P}{\partial\eta}d\xi\,d\eta = \iint\limits_U -\left(\frac{\partial P}{\partial\eta}\right)(\xi(x,y), \eta(x,y))\det(J_F)\;dx\,dy =$$

$$F(x,y) = (\xi(x,y), \eta(x,y))$$

$$\begin{cases}\frac{\partial F}{\partial x} = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial x}\\ \frac{\partial F}{\partial y} = \frac{\partial\xi}{\partial y} + \frac{\partial\eta}{\partial y}\end{cases}$$

$$\begin{aligned}\det J_F(x,y) = \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x} &=\\ = -\iint\limits_U \frac{\partial P}{\partial\eta}(\xi(x,y), \eta(x,y))\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)dx\,dy\end{aligned}$$

Using Green's theorem

$$\int_{\partial V} P d\xi =$$

$$= \int_{\partial U} (P \circ F) \left(\frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy \right) = \int_{\partial U} P \left(\xi \left(x, y \right), \eta \left(x, y \right) \right) \left(\frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy \right) =$$

$$= \iint_U \left(-\frac{\partial}{\partial y} \left(P \frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial x} \left(P \frac{\partial \xi}{\partial y} \right) \right) dx\, dy =$$

$$= \iint\limits_U \left[-\frac{\partial}{\partial y} P \left(\xi \left(x, y \right), \eta \left(x, y \right) \right) \frac{\partial \xi}{\partial x} - P \frac{\partial^2 \xi}{\partial y \partial x} + \frac{\partial}{\partial x} P \left(\xi \left(x, y \right), \eta \left(x, y \right) \right) \frac{\partial \xi}{\partial y} + P \frac{\partial^2 \xi}{\partial x \partial y} \right] dx\, dy =$$

$$= \iint\limits_U \left[\left(-\frac{\partial P}{\partial \xi} \frac{\partial \xi}{\partial y} - \frac{\partial P}{\partial \eta} \frac{\partial \eta}{\partial y} \right) \frac{\partial \xi}{\partial x} + \left(\frac{\partial P}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial P}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \frac{\partial \xi}{\partial y} \right] dx\, dy =$$

$$= \iint\limits_U \left[-\frac{\partial P}{\partial \eta} \frac{\partial \eta}{\partial y} \frac{\partial \xi}{\partial x} + \frac{\partial P}{\partial \eta} \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial y} \right] dx\, dy =$$

$$= -\iint\limits_U \frac{\partial P}{\partial \eta} J_F \, dx\, dy.$$

Similar argument as above hold for $\int Q d\eta$
Here we use that

$$\int_{\partial V} Q d\eta = \int_{\partial U} (Q \circ F) \left(\frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy \right)$$

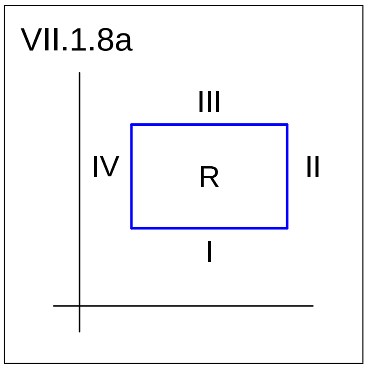
and replace $R = -\partial P / \partial \eta$ by R=$\partial Q / \partial \xi$ to conclude that

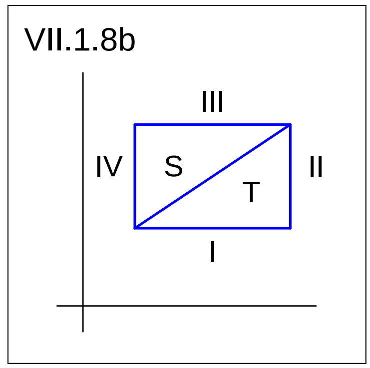
$$\int_{\partial V} Q d\eta = \iint_V \frac{\partial Q}{\partial \xi} d\xi d\eta.$$

III.1.8

Prove Green's theorem for the rectangle defined by $x_0 < x < x_1$ and $y_0 < y < y_1$ (a) directly, and (b) using the result for triangles.

Solution





(a)

Make figure and set the linesegment between (x_0, y_0) and (x_1, y_0) to γ_1 and proceed in this way in counterclockwise direction. We have

$$\int_{\gamma_1} Q\,dy = \int_{\gamma_2} P\,dx = \int_{\gamma_3} Q\,dy = \int_{\gamma_4} P\,dx = 0.$$

Integrate around the rectangle

$$
\begin{aligned}
&\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \\
&\quad = \int_{y_0}^{y_1} \left(\int_{x_0}^{x_1} \frac{\partial Q}{\partial x} dx\right) dy - \int_{x_0}^{x_1} \left(\int_{y_0}^{y_1} \frac{\partial P}{\partial y} dy\right) dx = \\
&\quad = \int_{y_0}^{y_1} \left[Q\left(x_1, y\right) - Q\left(x_0, y\right)\right]\, dy - \int_{x_0}^{x_1} \left[P\left(x, y_1\right) - P\left(x, y_0\right)\right]\, dx = \\
&= \int_{y_0}^{y_1} Q\left(x_1, y\right)\, dy - \int_{y_0}^{y_1} Q\left(x_0, y\right)\, dy - \int_{x_0}^{x_1} P\left(x, y_1\right)\, dx + \int_{x_0}^{x_1} P\left(x, y_0\right)\, dx = \\
&= \int_{x_0}^{x_1} P\left(x, y_0\right)\, dx + \int_{y_0}^{y_1} Q\left(x_1, y\right)\, dy - \int_{x_0}^{x_1} P\left(x, y_1\right)\, dx - \int_{y_0}^{y_1} Q\left(x_0, y\right)\, dy = \\
&\quad = \int_{\gamma_1} P\, dx + \int_{\gamma_2} Q\, dy + \int_{\gamma_3} P\, dx + \int_{\gamma_4} Q\, dy = \\
&\quad = \int_{\partial R} \left(P\, dx + Q\, dy\right).
\end{aligned}
$$

(b)
Write $R = S \cup T$. The integrals on the diagonal are in different directions and will cancel

$$
\begin{aligned}
&\int_{\partial R} \left(P\, dx + Q\, dy\right) = \\
&\quad = \int_{\partial S} \left(P\, dx + Q\, dy\right) + \int_{\partial T} \left(P\, dx + Q\, dy\right) = \\
&\quad = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy + \iint_T \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \\
&\quad = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.
\end{aligned}
$$

III.2.1

Determine whether each of the following line integrals is independent of path. If it is, find a function h such that $dh = P\,dx + Q\,dy$. If it is not, find a closed path γ around which the integral is not zero. (a) $x\,dx + y\,dy$, (b) $x^2dx + y^5dy$, (c) $y\,dx + x\,dy$, (d) $y\,dx - x\,dy$.

Solution

We compute the partial derivatve because if h exist we have that $dh = Pdx + Qdy$ where $\partial P/\partial y = \partial Q/\partial x$.

(a)

$$\begin{array}{ll} P = x & Q = y \\ \frac{\partial P}{\partial y} = 0 & \frac{\partial Q}{\partial x} = 0 \\ \int Pdx = \frac{x^2}{2} + g(y) & \int Qdy = \frac{y^2}{2} + f(x) \\ \Rightarrow h = \frac{x^2+y^2}{2} & \end{array}$$

(b)

$$\begin{array}{ll} P = x^2 & Q = y^5 \\ \frac{\partial P}{\partial y} = 0 & \frac{\partial Q}{\partial x} = 0 \\ \int Pdx = \frac{x^3}{3} + g(y) & \int Qdy = \frac{y^6}{6} + f(x) \\ \Rightarrow h = \frac{2x^3+y^6}{6} & \end{array}$$

(c)

$$\begin{array}{ll} P = y & Q = x \\ \frac{\partial P}{\partial y} = 1 & \frac{\partial Q}{\partial x} = 1 \\ \int Pdx = xy + g(y) & \int Qdy = xy + f(x) \\ \Rightarrow h = xy & \end{array}$$

(d)

$$\begin{array}{ll} P = y & Q = -x \\ \frac{\partial P}{\partial y} = 1 & \frac{\partial Q}{\partial x} = -1 \end{array}$$

The line integral is not independent of path, we use Greens' theorem

$$\int\limits_{|z|=1} (y\,dx - x\,dy) = \iint\limits_{|z|<1} -2\,dx\,dy = -2\pi.$$

III.2.2
Show that the differential

$$\frac{-ydx + xdy}{x^2+y^2}, \qquad (x,y) \neq (0,0)\,,$$

is closed. Show that it is not independent of path on any annulus centered at 0.

Solution

$$\begin{array}{ll} P = \frac{-y}{x^2+y^2} & Q = \frac{x}{x^2+y^2} \\ \frac{\partial P}{\partial y} = \frac{-\left(x^2+y^2\right)+y(2y)}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2} & \frac{\partial Q}{\partial x} = \frac{\left(x^2+y^2\right)-x(2x)}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2} \end{array}$$

Because $\partial P/\partial y = \partial Q/\partial x$,the differential is closed and we can use Green's Theorem

$$\oint\limits_{|z|=r} P\,dx + Q\,dy =$$

$$= \oint\limits_{|z|=r} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \frac{1}{r^2}\oint\limits_{|z|=r} -y\,dx + x\,dy =$$

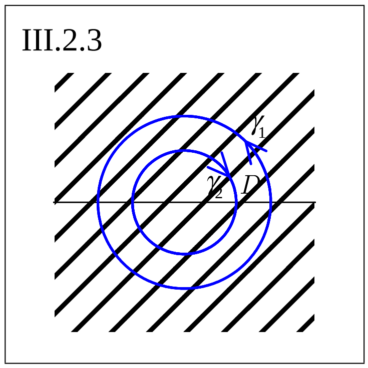
$$= \frac{1}{r^2}\iint\limits_{|z|<r} (1+1)\ dx\,dy = \frac{1}{r^2}\cdot 2\cdot r^2\cdot \pi = 2\pi \neq 0$$

Thus the line integral is not independent of path.

III.2.3

Suppose P and Q are smooth functions on the annulus $\{a < |z| < b\}$ that satisfy $\partial P/\partial y = \partial Q/dx$. Show directly using Green's theorem that $\oint_{|z|=r} Pdx + Qdy$ is independent of the radius r, for $a < r < b$.

Solution (A. Kumjian)



Let r_1, r_2 be given so that $a < r_1 < r_2 < b$. We must prove that

$$\oint_{|z|=r_1} Pdx + Qdy \stackrel{*}{=} \oint_{|z|=r_2} Pdx + Qdy.$$

Let D denote the annulus $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$. Observe that D is a bounded domain with piecewise smooth boundary $\partial D = \gamma_1 \cup \gamma_2$ where γ_1 denotes the circle $\{z \in \mathbb{C} : |z| = r_1\}$ with clockwise orientation while γ_2 denotes the circle $\{z \in \mathbb{C} : |z| = r_2\}$ with counter clockwise orientation. Moreover, both P and Q are continuously differentiable on $\bar{D} = D \cup \partial D$. Hence, Green's Theorem applies and we obtain

$$\int_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0,$$

and since, as noted above, ∂D consists of the two circles with opposite orientation,

$$\int_{\partial D} Pdx + Qdy = \oint_{|z|=r_2} Pdx + Qdy - \oint_{|z|=r_1} Pdx + Qdy.$$

Hence, equation $(*)$ holds and $\oint_{|z|=r} Pdx + Qdy$ is independent of the radius r, for $a < r < b$.

III.2.4

Let P and Q be smooth functions on D satisfying $\partial P/\partial y = \partial Q/\partial x$. Let γ_0 and γ_1 be two closed paths in D such that the straight line segment from $\gamma_0(t)$ to $\gamma_1(t)$ lies in D for every parameter value t. Then $\int_{\gamma_0} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy$. Use this to give another solution to the preceding exercise.

Solution

Use the theorem on page 81. Use the straight lines to deform γ_0 to γ_1. Define $\gamma_s(t) = s\gamma_1(t) + (1-s)\gamma_0(t)$, $0 \leqslant s \leqslant 1$, $a_1 \leqslant t \leqslant b_1$. The theorem on page 81 applies. Obtain the result for the annulus above by parameterizing the circles $|z| = r_0$ and $|z| = r_1$ by $\gamma_0(t) = r_0 e^{2\pi i t}$, $\gamma_1(t) = r_1 e^{2\pi i t}$, $0 \leqslant t \leqslant 1$. The straight line segments joining $\gamma_0(t)$ to $\gamma_1(t)$ are radical are in the annulus, so the first part applies.

III.2.5
Let $\gamma_0(t)$ and $\gamma_1(t)$, $0 \leq t \leq 1$, be paths in the slit annulus $\{a < |z| < b\} \setminus (-b, -a)$ from A to B. Write down explicitly a family of paths $\gamma_s(t)$ from A to B in the slit annulus that deforms continuously to γ_1.
Suggestion. Deform separately the modulus and the principal value of the argument.

Solution
Write $\gamma_0(t) = r_0 e^{i\theta_0(t)}$, $0 \leqslant t \leqslant 1$, when $-\pi < \theta_0(t) < \pi$, $r_0(t)$, $\theta_0(t)$ continuous. (Use the fact that $\theta(t) = \operatorname{Arg}\gamma_0(t)$ is continuous on the slit annulus) Also $\gamma_1(t) = r_1 e^{i\theta_1(t)}$.
Then consider

$$\gamma_s(t) = [sr_1(t) + (1-s)r_0(t)]\, e^{i[s\theta_1(t)+(1-s)\theta_0(t)]}, \qquad 0 \leq s \leq 1.$$

This does the trick, it deform γ_0 to γ_1 continuously in D, and each γ_s is a path in D from A to B.
We see that $\gamma_s(t)$ belongs to the slit annulus for $0 \leq s \leq 1$ and $0 \leq t \leq 1$ since for every $0 \leq 5 \leq 1$ we have

$$a < \min(r_0(t), r_1(t)) \leq sr_1(t) + (1-s)r_0(t) \leq \max(r_0(t), r_1(t)) < b$$

and

$$-\pi < \min(\theta_0(t), \theta_1(t)) \leq s\theta_1(t) + (1-s)\theta_0(t) \leq \max(\theta_0(t), \theta_1(t)) < \pi$$

III.2.6

Show that any closed path $\gamma(t)$, $0 \le t \le 1$, in the annulus $\{a < |z| < b\}$ can be deformed continuously to the circular path $\sigma(t) = \gamma(0)\, e^{2\pi i m t}$, $0 \le t \le 1$, for some integer m. Hint. Reduce to the case where $|\gamma(t)| \equiv |\gamma(0)|$ is constant. Then start by finding a subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\arg \gamma(t)$ has a continuous determination on each interval $t_{j-1} \le t \le t_j$.

Solution

We first note that there are two cases. For the closed path $\gamma(t)$ enclosing the origin we can deform $\gamma(t)$ to a curve with constant modulus. Suppose $|\gamma(t)| = r$ for $0 \leqslant t \leqslant 1$ and $\gamma(0) = \gamma(1) = 1$ where $a < r < b$. Follow hint, write $\gamma(t) = re^{i\theta_j(t)}$, $t_{j-1} \leqslant t \leqslant t_j$. Note $\theta_j(t_j) - \theta_{j+1}(t_j)$ is a integral multiple of 2π, since $\gamma(t)$ is continuous. Add multiples of 2π to the θ_j 's, obtain $\theta(t)$ continuous for $0 \leqslant t \leqslant 1$ such that $\theta(0) = 0$ and $\gamma(t) = re^{i\theta(t)}$. Note $\theta(1) = 2\pi m$ for some integer m. Deform by $\gamma_s(t) = re^{i[(1-s)\theta(t) + 2\pi s m]}$.
The second case is when $\gamma(t)$ does not enclose the origin. For this case $\gamma(t)$ the counter of a region that lies entirely inside the annulus. Therefrom the curve $\gamma(t)$ may be deformed to any point of this region. In particular it can be continuously deformed to the point $\gamma(0) = \sigma(t) = \gamma(0)\, e^{2\pi i m t}$ when $m = 0$.

III.2.7

Show that if 0 and ∞ lie in different connected components of the complement $\mathbb{C}^*\backslash D$ of D in the extended complex plane, then there is a closed path γ in D such that $\int_\gamma d\theta \neq 0$. Hint. The hypothesis means that there are $\delta > 0$and a bounded subset E of $\mathbb{C}\backslash D$ such that $0 \in E$, and every point of E has distance at least 5δ from every point of $\mathbb{C}\backslash D$ not in E. Lay down a grid of squares in the plane with side length δ,and let F be the union of the closed squares in the grid that meet E or that border on a square meeting E. Show that ∂F is a finite union of a closed paths in D, and that $\int_{\partial F} d\theta = 2\pi$.

Solution

Proceed as in the hint. Assume that 0 is the centre of one of the squares S_0 in the grid. Then$\int_{\partial S_0} d\theta = 2\pi$, while$\int_{\partial S} d\theta = 0$ for any other square in the grid. Thus $\sum \int_{\partial S_j} d\theta = 2\pi$, where we sum over squares. $S_0, S_1, \ldots, S_n$ in F. If two squares in F are ?????, the corresponding integrals along the edge cancel. Then this is the sum over the integrals over the edges of the S_j ´s that have F on one side and $\mathbb{C}\backslash F$ on the other, that is the edges that form ∂F. We have thus $\int_{\partial F} d\theta = 2\pi$. Note that by construction, $\partial F \subset D$. Now note how edges ∂F can meet. Either a vertex in ∂F have one edge coming in and one out, or it have two coming in and two out. By starting on following edges, and making a left turn at vertices where 4 edges meet, we must eventually end where we started, with a closed path. Three paths must be disjoint, except for vertices with four edges. Call the paths $\gamma_1, \ldots, \gamma_m$. Since $\sum \int_{\gamma_j} d\theta = 2\pi$, we must have $\int_\gamma d\theta \neq 0$ for one of the γ_j ‘s. (In fact, we will have $\int_{\gamma_j} d\theta = 2\pi$ on the γ_j ´s we get by moving from 0 to ∂F.)

III.3.1

For each of the following harmonic functions u, find du, find dv, and find v, the conjugate harmonic functions of u.

(a) $u(x,y) = x - y$ **(c)** $u(x,y) = \sinh x \cos y$

(b) $u(x,y) = x^3 - 3xy^2$ **(d)** $u(x,y) = \frac{y}{x^2+y^2}$

Solution

(a)

Use Cauchy-Riemanns equations page 83.

$$\begin{aligned}
&u(x,y) = x - y \\
&du = dx - dy \\
&dv = dx + dy \\
&\underline{v(x,y) = x + y}
\end{aligned}$$

(b)

Use Cauchy-Riemanns equations page 83.

$$\begin{aligned}
&u(x,y) = x^3 - 3xy^2 \\
&du = \left(3x^2 - 3y^2\right) dx - 6xy\, dy \\
&dv = 6xy\, dx + \left(3x^2 - 3y^2\right) dy \\
&\underline{v(x,y) = 3x^2 y - y^3}
\end{aligned}$$

(c)

Use Cauchy-Riemanns equations page 83.

$$\begin{aligned}
&u(x,y) = \sinh x \cos y \\
&du = \cosh x \cos y\, dx\ - \sinh x \sin y\, dy \\
&dv = \sinh x \sin y\, dx + \cosh x \cos y\, dy \\
&\underline{v(x,y) = \cosh x \sin y}
\end{aligned}$$

(d)

Use Cauchy-Riemanns equations page 83.

$$\begin{aligned}
&u(x,y) = \tfrac{y}{x^2+y^2} \\
&du = \tfrac{-2xy}{\left(x^2+y^2\right)^2} dx + \tfrac{x^2+y^2-y(2y)}{\left(x^2+y^2\right)^2} dy = \tfrac{-2xy}{\left(x^2+y^2\right)^2} dx + \tfrac{x^2-y^2}{\left(x^2+y^2\right)^2} dy \\
&dv = \tfrac{y^2-x^2}{\left(x^2+y^2\right)^2} dx - \tfrac{2xy}{\left(x^2+y^2\right)^2} dy \\
&\underline{v(x,y) = \tfrac{x}{x^2+y^2}}
\end{aligned}$$

III.3.2
Show that a complex-valued function $h(z)$ on a star-shaped domain D is harmonic if and only if $h(z) = f(z) + \overline{g(z)}$, where $f(z)$ and $g(z)$ are analytic on D.

Solution
Write $h = u + iw$ where u, w are harmonic. By the Theorem on page 83 both u and w have harmonic conjugate in D, (since D is star-shaped), so thus there are φ, ψ analytic such that $u = \operatorname{Re}\varphi$, $w = \operatorname{Re}\psi$.

$$u = (\varphi + \overline{\varphi})/2, \qquad w = \left(\psi + \overline{\psi}\right)/2,$$

gives

$$h = \tfrac{1}{2}\left[\varphi + \overline{\varphi} + i\psi + i\overline{\psi}\right] = \tfrac{1}{2}\left[\varphi + i\psi\right] + \tfrac{1}{2}\left[\overline{\varphi} + i\overline{\psi}\right].$$

Take

$$f = \tfrac{1}{2}\left[\varphi + i\psi\right], \qquad g = \tfrac{1}{2}\left[\varphi - i\psi\right],$$

then $h(z) = f(z) + \overline{g(z)}$.

To show the oposite direction. Assume that $h(z) = f(z) + \overline{g(z)}$ where $f(z)$ and $g(z)$ are analytic on D. Then $h = \operatorname{Re} f + \operatorname{Re} g + i(\operatorname{Im} f + \operatorname{Im} g)$ and from Cauchy-Riemanns equations follows that

$$\begin{aligned}
&\frac{\partial^2(\operatorname{Re} f)}{\partial x^2} + \frac{\partial^2(\operatorname{Re} g)}{\partial x^2} + i\left(\frac{\partial^2(\operatorname{Im} f)}{dx^2} - \frac{\partial^2(\operatorname{Im} g)}{\partial x^2}\right) + \\
&\qquad + \frac{\partial^2(\operatorname{Re} f)}{\partial y^2} + \frac{\partial^2(\operatorname{Re} g)}{\partial y^2} + i\left(\frac{\partial^2(\operatorname{Im} f)}{\partial y^2} - \frac{\partial^2(\operatorname{Im} g)}{\partial y^2}\right) = \\
&= \frac{\partial^2 \operatorname{Im} f}{\partial x \partial y} + \frac{\partial^2 \operatorname{Im} g}{\partial x \partial y} + i\left(-\frac{\partial^2 \operatorname{Re} f}{\partial x \partial y} + \frac{\partial^2 \operatorname{Re} g}{\partial x \partial y}\right) - \frac{\partial^2 \operatorname{Im} f}{\partial y \partial x} - \frac{\partial^2 \operatorname{Im} g}{\partial y \partial x} + \\
&\qquad + i\left(\frac{\partial^2 \operatorname{Re} f}{\partial y \partial x} - \frac{\partial^2 \operatorname{Re} g}{\partial y \partial x}\right) = 0
\end{aligned}$$

so $h(z)$ is harmonic.

III.3.3

Let $D = \{a < |z| < b\} \setminus (-b, -a)$, an annulus slit along the negative real axis. Show that any harmonic function on D has a harmonic conjugate on D.

Suggestion. **Fix c between a and b, and define $v(z)$ explicitly as a line integral along the path consisting of the straight line from c to $|z|$ followed by the circular arc from $|z|$ to z. Or map the slit annulus to a rectangle by $w = \operatorname{Log} z$.**

Solution

Integrate the C–R equations we have

$$v(z) = \int_{z_0}^{z} -\frac{1}{r}\frac{\partial u}{\partial \theta}dr + r\frac{\partial u}{\partial r}d\theta = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy$$

integrate along $r -$ interval, then $\theta -$ interval. Perception is same as getting from z_0 to z_1, then from z_1 to little disk. Integral is well-defined, continuous, harmonic and is harmonic conjugate for u.

III.3.4 (From Hints and Solutions)
Let $u(z)$ be harmonic on the annulus $\{a < |z| < b\}$. Show that there is a constant C such that $u(z) - C\log|z|$ has a harmonic conjugate on the annulus. Show that C is given by

$$C = \frac{r}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial r}\left(re^{i\theta}\right)\, d\theta,$$

where r is any fixed radius, $a < r < b$.

Solution
u has harmonic conjugate v_1 on the annulus slit along $(-b, -a)$, and also a harmonic conjugate v_2 on the annulus slit (a, b). Since $v_1 - v_2$ is constant above the slit $(-b, -a)$, and also constant below the slit, v_1 jumps by a constant across the slit. $\operatorname{Arg} z$ also jumps by a constant across the slit. By appropriate choice of C, $v_1 - C\operatorname{Arg} z$ is continuous across the slit $(-b, -a)$, and $u - C\log|z|$ has a harmonic conjugate $v_1 - C\log|z|$ on the annulus. For the identity, use the polar form of the Cauchy-Riemann equations to convert the r–derivative of u to a θ–derivative of v.

III.3.5
The flux of a function u across a curve γ is defined to be

$$\int_\gamma \frac{\partial u}{\partial n} ds = \int_\gamma \nabla u \cdot n \, ds,$$

where n is the unit normal vector to γ and ds is arc length. Show that if a harmonic function u on a domain D has a conjugate harmonic function v on D, then the integral giving the flux is independent of path in D. Further, the flux across a path γ in D from A to B is $v(B) - v(A)$.

Solution
$\overrightarrow{t} = \left(\frac{ds}{dx}, \frac{ds}{dy}\right)$, $\overrightarrow{n} = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$,
$\int_\gamma \nabla u \overrightarrow{n} \, ds = \int\limits_\gamma \left(\frac{\partial u}{\partial x}\frac{dy}{ds} - \frac{\partial u}{\partial y}\frac{dx}{ds}\right) ds = \int_\gamma -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \underbrace{=}_{*}$
$\int_\gamma \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = \int_\gamma dv$
$(*)$ C–R equations. If γ goes from P to Q, this is $v(Q) - v(P)$, which shows that the integral is independent of path in D.

III.4.1

Let $f(z)$ be a continuous function on a domain D. Show that if $f(z)$ has the mean value property with respect to circles, as defined above, then $f(z)$ has the mean value property with respect to disks, that is if $z_0 \in D$ and D_0 is a disk centered at z_0 with area A and contained in D, then $f(z_0) = \frac{1}{A}\iint_{D_0} f(z)\, dx\, dy$.

Note: This MVP for area is not exactly what is defined as the MVP for disks. What is then is that if $h(z_0)$ is the average of $h(z)$ for any circle $\{|z - z_0| = r\}$, $0 < r < 1$. Then $h(z)$ is the average of $h(z)$ over any disk $\{|z - z_0| = r_0\}$, $0 < r < r_0$. Express $dx\, dy$ in polar coordinates at z_0 and integrate first with respect to θ.

Solution

Suppose that f satisfies the mean value property,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)\, d\theta$$

for all $z_0 \in D$ and $r > 0$ such that $B_r(z_0) \subseteq D$. Let $D_0 = \{z \in \mathbb{C} : |z - z_0| \leq R\} \subseteq D$. We have, $A = \pi R^2$ and parametrizing in polar coordinates, and use the mean value property with respect to circles,

$$\begin{aligned}
\frac{1}{A}\iint_{D_0} f(z)\, dx\, dy &= \frac{1}{\pi R^2}\iint_{D_0} f(z)\, dx\, dy = \left[\begin{array}{lll} x = x_0 + r\cos\theta & dx\, dy = r\, dr\, d\theta & 0 \leq r \leq R \\ y = y_0 + r\sin\theta & & 0 \leq \theta \leq 2\pi \end{array}\right] \\
= \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R f\left(z_0 + re^{i\theta}\right) r\, dr\, d\theta &= \frac{1}{\pi R^2}\int_0^R \left[\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right)\, d\theta\right] r\, dr = \frac{1}{\pi R^2}\int_0^R 2\pi f(z_0)\, r\, dr \\
&= \frac{2f(z_0)}{R^2}\int_0^R r\, dr = \frac{1}{R^2}\left[\frac{r^2}{2}\right]_0^R f(z_0) = f(z_0).
\end{aligned}$$

Another note: It's better to use something other than $1/A$ as A was already used as the circle average.

III.4.2
Derive (4.2) from the polar form of the Cauchy-Riemann equations (Exercise II.3.8).

Solution
The polar form of Cauchy-Riemann's equations (from Exercise II.3.8)

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Now we have

$$r\int_0^{2\pi} \frac{\partial u}{\partial r}\left(z_0 + re^{i\theta}\right) d\theta = \int_0^{2\pi} \frac{\partial v}{\partial \theta}\left(z_0 + re^{i\theta}\right) d\theta = 0$$

III.4.3
A function $f(t)$ on an interval $I=(a,b)$ has the mean value property if

$$f\left(\frac{s+t}{2}\right)=\frac{f(s)+f(t)}{2},\qquad s,t\in I.$$

Show that any affine function $f(t)=At+B$ has the mean value property. Show that any continuous function on I with the mean value property is affine.

Solution (A. Kumjian)
Let $I:=(a,b)$ and let $f:I\to\mathbb{R}$ be given. Suppose first that f is an affine function, that is, there are $A,B\in\mathbb{R}$ such that $f(t)=At+B$ for all $t\in I$. Now let $s,t\in I$ be given, then

$$f\left(\frac{s+t}{2}\right)=A\left(\frac{s+t}{2}\right)+B=\frac{1}{2}(As+B)+\frac{1}{2}(At+B)=\frac{f(s)+f(t)}{2}.$$

Hence, f has the mean value property.
Next suppose that f is continuous and that it has the mean value property. We must prove that f is affine. We claim that for any $c,d\in I$, with $c<d$, and $t\in[0,1]$ we have

$$f(tc+(1-t)d)\overset{*}{=}tf(c)+(1-t)f(d).$$

So let $c,d\in I$ be given with $c<d$. We will first prove that equation $(*)$ holds for $t=k/2^m$ where k,m are nonnegative integers with $k\le 2^m$. We do this by induction on m. For $m=0$, we have $t=k=0,1$ and the assertion is obvious. Now suppose that the assertion holds for some integer $m\ge 0$ (that is, equation $(*)$ holds for $t=k/2^m$ for all $k=0,\ldots,2^m$). This inductive step entails showing that equation $(*)$ holds for $t=k/2^{m+1}$ for any nonnegative integer k with $k\le 2^{m+1}$. By the inductive hypothesis we may assume $k=2j+1$ for some integer $j=0,\ldots,2^m-1$ (for if k were even we could rewrite t as $j/2^m$ for some integer j with $0\le j\le 2^m$). Then setting $t_0=j/2^m$ and $t_1=(j+1)/2^m$, we have $t=(t_0+t_1)/2$ and $1-t=((1-t_0)+(1-t_1))/2$, thus,

$$tc + (1-t)\,d = \frac{1}{2}\left((t_0 c + (1-t_0)\,d + (t_1 c + (1-t_1)\,d)\right)$$

hence,

$$\begin{aligned}
&f\,(tc + (1-t)\,d) = \\
&\qquad = f\left(\frac{1}{2}((t_0 c + (1-t_0)\,d + (t_1 c + (1-t_1) d)\right) = \\
&= \frac{1}{2}\left(f\,(t_0 c + (1-t_0)\,d) + f\,(t_1 c + (1-t_1)\,d)\right) = \text{ by the mean value property} \\
&= \frac{1}{2}\left(t_0 f\,(c) + (1-t_0)\,f\,(d) + t_1 f\,(c) + (1-t_1)\,f\,(d)\right) = \text{ by inductive hypothesis} \\
&\qquad = tf\,(c) + (1-t)\,f\,(d)\,.
\end{aligned}$$

So we have proved that equation $(*)$ holds for $t = k/2^m$ where k, m are nonnegative integers with $k \leq 2^m$. Since such numbers are dense in $[0,1]$ the claim follows by the continuity of f. It is now straightforward to verify that for any $c, d \in I$, with $c < d$, there are unique constants A and B such that $f\,(x) = Ax + B$ for all $x \in [c,d]$ (take $A = (f\,(c) - f\,(d))\,/\,(c-d)$ and $B = f\,(c) - Ac$). The desired result now follows by observing that A and B do not depend on c, d.

III.4.4

Formulate the mean value property for a function on a domain in $\mathbb{R}^3$, and show that any harmonic function has the mean value property.

Hint. For $A \in \mathbb{R}^3$ and $r > 0$, let B_r be the ball of radius r centered at A, with volume element $d\tau$, and let ∂B_r be its boundary sphere, with area element $d\sigma$ and unit outward normal vector n. Apply the Gauss divergence theorem

$$\iint_{\partial B_r} F \cdot n \, d\sigma = \iiint_{B_r} \nabla \cdot F \, d\tau$$

to $F = \nabla u$.

Solution

Prove MV theorem for harmonic functions in $\mathbb{R}^n$, as follows $u\left(R\overrightarrow{x}\right) - u\left(\overrightarrow{0}\right) = \int_0^R \frac{\partial u}{\partial r}\left(r\overrightarrow{x}\right) dr$. Let $d\tau$ = area measure, $\overrightarrow{x}$ = unit vector, get $\int \frac{\partial u}{\partial r}\left(R\overrightarrow{x}\right) d\tau - \int \frac{\partial u}{\partial r}\left(r_0\overrightarrow{x}\right) d\tau = \int_0^R \int_S \frac{\partial^2 u}{\partial r^2}\left(r\overrightarrow{x}\right) dr d\tau\left(\overrightarrow{x}\right)$. Get directional derivative of Strokes formula. Better: Apply stroke's theorem to shell $\{\rho_0 < \rho < \rho_1\}$, get

$\int\limits_{\Sigma} \ldots \int \overrightarrow{V} \cdot \overrightarrow{n} d$ surface $= \int\limits_{\text{Volume}} \ldots \int \nabla \overrightarrow{V}$ volume. Apply to $\nabla u = \overrightarrow{v}$,

get$\int\limits_{\Sigma} \ldots \int \nabla u \overrightarrow{n} d$ surface $= \int \ldots \int \nabla^2 \overrightarrow{n} d$ volume $= 0$,

$\int\limits_{\left|\overrightarrow{x}-x_0\right|=\rho_1} \ldots \int \frac{\partial u}{\partial \rho} d\tau - \int\limits_{\left|\overrightarrow{x}-x_0\right|=\rho_0} \ldots \int \frac{\partial u}{\partial \rho} d\tau$

$\frac{\partial u}{\partial \rho} \int\limits_{\left|\overrightarrow{x}-x_0\right|=\rho} \ldots \int \dot{u} d\tau$ is constant.

III.5.1

Let D be a bounded domain, and let u be a real-valued harmonic function on D that extends continuously to the boundary ∂D. Show that if $a \leq u \leq b$, then $a \leq u \leq b$ on D.

Solution

Since $D \cup \partial$ is a compact set we have that u attains both maximum and minimum values. Assume that $u(z_0) = c > b$ fore some $z_0 \in D$. Then it follows by the maximum principle that $u \equiv c$ on D, but this is a contradiction to the asumption that u is continuously extended to ∂D. Thereby $u \leq b$ on D.

For $-u$ we have that $-b \leq -u \leq -a$ on ∂D. Assume that $-u(z_0) = c' > -a$ for seme $z_0 \in D$. Then by the maximum principle $-u \equiv c'$ on D, but this contradicts that $-u$ is continuously extended to ∂D with $-u \leq -a$ on ∂D. So $-u \leq -a$ on D, equivalently $a \leq u \leq b$ on D.

This show that $a \leq u \leq b$ on D.

III.5.2

Fix $n \geq 1$, $r > 0$ and $\lambda = \rho e^{i\varphi}$. What is the maximum modulus of $z^n + \lambda$ over the disk $\{|z| \leq r\}$? Where does $z^n + \lambda$ attain its maximum modulus over the disk?

Solution (A. Kumjian)

We claim that the maximum modulus of $z^n + \lambda$ over the disk $\{z \in \mathbb{C} : |z| \leq r\}$ is $r^n + \rho$ and this is attained at $\zeta = re^{i\theta}$ where $\theta = (2k\pi + \varphi)/n$ for $k = 0, 1, \ldots, n-1$. By the maximum modulus principle the maximum modulus occurs on the boundary $\{z \in \mathbb{C} : |z| = r\}$. For $z \in \mathbb{C}$ with $|z| = r$ we have by the triangle inequality

$$|z^n + \lambda| \leq |z^n| + |\lambda| = r^n + \rho = \left|(r^n + \rho)\, e^{i\varphi}\right| = |\zeta^n + \lambda|\,.$$

so the claim is proven.

III.5.3
Use the maximum principle to prove that the fundamental theorem of algebra, that any polynomial $p(z)$ of degree $n \geq 1$ has a zero, by applying the maximum principle to $1/p(z)$ on a disk of large radius.

Solution
It is sufficient to show that any $p(z)$ has one root, for by division we can then write $p(z) = (z - z_0)\, g(z)$, with $g(z)$ of lower degree.
Note that if

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

then as $|z| \to \infty$, $|p(z)| \to \infty$. This follows as

$$p(z) = z^n \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right|.$$

Assume $p(z)$ is non-zero everywhere. Then $\frac{1}{p(z)}$ is bounded when $|z| \geq R$. Also, $p(z) \neq 0$, so $\frac{1}{p(z)}$ is bounded for $|z| \leq R$ by continuity. Thus, $\frac{1}{p(z)}$ is a bounded entire function, which must be constant. Thus, $p(z)$ is constant, a contradiction which implies $p(z)$ must have a zero.

Solution (K. Seip)
If $p(z)$ has no zeros, then $1/p(z)$ is an entire function. Also, if we denote by $m(R)$ the maximum of $1/p(z)$ on the circle $|z| = R$, then $m(R) \to 0$ when $R \to \infty$, unless $p(z)$ is a constant. By the maximum principle $|1/p(z)| \leq m(R)$ when $|z| \leq R$, which means $1/p(z) = 0$. This is impossible, and so $1/p(z)$ is not an entire function.

III.5.4

Let $f(z)$ be an analytic function on a domain D that has no zeros on D. (a) Show that if $|f(z)|$ attains its minimum on D, then $f(z)$ is constant. (b) Show that if D is bounded, and if $f(z)$ extends continuously to the boundary ∂D of D, then $f(z)$ attains its minimum on ∂D.

Solution

(a)

If $|f(z)|$ attains its minimum on D, then $|1/f(z)|$ attains its maximum on D, which can only happens if $f(z)$ is a constant, by the maximum principle.

(b)

Since $D \cup \partial D$ is compact and $f(z)$ is continuous it follows that $|f(z)|$ attains both maximum and minimum value on $D \cup \partial D$. Assume that $|f(z)|$ does not attain its minimum on ∂D. Then $|f(z)|$ attains its minimum on D and it follows by (a) that f is constant so $|f(z)|$ attains its minimum on ∂D.

III.5.5
Let $f(z)$ be a bounded analytic function on the right half-plane. Suppose that $f(z)$ extends continuously to the imaginary axis and satisfies $|f(iy)| \leq M$ for all points iy on the imaginary axis. Show that $|f(z)| \leq M$ for all z in the right half-plane.
Hint. **For $\varepsilon > 0$ small, consider $(z+1)^{-\varepsilon} f(z)$ on a large semidisk.**

Solution
Now we consider the function $(z+1)^{-\varepsilon} f(z)$ in the right half-plane, because $f(z)$ is bounded here, we assume that $|f(z)| \leq C$ in this domain. On a semidisk of radius r in the right half-plane we have

$$\left|(z+1)^{-\varepsilon} f(z)\right| \leq \frac{C}{|r-1|^{\varepsilon}}$$

For $r = R$ sufficient large, we have

$$\left|(z+1)^{-\varepsilon} f(z)\right| \leq \frac{C}{(R-1)^{\varepsilon}} \leq M$$

for all $|z| \geq R$, when $\operatorname{Re} z \geq 0$.
On the imaginary axis we have that

$$\left|(z+1)^{-\varepsilon} f(z)\right| \leq M$$

because $|f(iy)| \leq M$ for all points iy on the imaginary axis.
By the maximum principle, we have

$$\left|(z+1)^{-\varepsilon} f(z)\right| \leq M$$

for $|z| \leq R$, when $\operatorname{Re} z \geq 0$.
Now let $\varepsilon \to 0$, we have

$$|f(z)| \leq M$$

for all z in the right half-plane.

Solution (K. Seip)
We assume that $|f(z)| \leq C$ for $\operatorname{Re} z > 0$ and $|f(iy)| \leq M$. For $\varepsilon > 0$ set

$$F_{\varepsilon}(z) = (1+z)^{-\varepsilon} f(z).$$

Then

$$|F_\varepsilon(iy)| \le M$$

and

$$\left|F_\varepsilon\left(Re^{i\theta}\right)\right| \le C \cdot R^{-\varepsilon} \le M$$

for sufficiently large R.
Thus for arbitary z, $\operatorname{Re} z > 0$ we have

$$|F_\varepsilon(z)| \le M,$$

or

$$|f(z)| \le M(1+|z|)^\varepsilon.$$

This holds for every $\varepsilon > 0$, thus $|f(z)| \le M$.

III.5.6
Let $f(z)$ be a bounded analytic function on the right half-plane. Suppose that $\limsup |f(z)| \leq M$ as z approaches any point of the imaginary axis. Show that $|f(z)| \leq M$ for all z in the half-plane.
Remark. **This is a technical improvement on the preceding exercise for students who can deal with $\limsup$ (see Section V.1).**

Solution
Set

$$g(z) = (z+1)^{-\varepsilon} f(z).$$

Then

$$|g(z)| \leqslant |f(z)|$$

becauce $|z+1| > 1$ for all z in the right half-plane.
Take R large so that $|g(z)| \leqslant M$ for $|z| > R$. The lim sup condition implies that there is $\delta > 0$ such that

$$|g(z)| \leqslant M + \varepsilon, \qquad |z| < R, \quad 0 < \operatorname{Re} z \leqslant \delta.$$

Apply the maximum principles to the domain $\{z_j|z| < R, \operatorname{Re} z > \delta\}$, to obtain $|g(z)| \leqslant M + \varepsilon$. Then $g(z) \to f(z)$ as $\varepsilon \to 0$.

III.5.7
Let $f(z)$ be a bounded analytic function on the open unit disk D. Suppose there are a finite number of points on the boundary such that $f(z)$ extends continuously to the arcs of ∂D separating the points and satisfies $\left|f\left(e^{i\theta}\right)\right| \leq M$ there. Show that $|f(z)| \leq M$ on D. Hint. In the case that there is only one exceptional point $z = 1$, consider the function $(1-z)^{\varepsilon} f(z)$.

Solution
Let the points be $a_1, \ldots, a_n$. Then $(z-a_j)^{\varepsilon}$, where $1 \leq j \leq n$ is analytic (take analytic branch). We have that $|z-a_j|^{\varepsilon} \to 0$ as $z \to a_j$ for $1 \leq j \leq n$. Then $\left[\prod (z-a_j)^{\varepsilon}\right] f(z) = f_{\varepsilon}(z)$ satisfies $f_{\varepsilon}(z)$ is continuous in $D \cup \partial D$. $|f_{\varepsilon}(z)| \leqslant M$ on ∂D. $|f_{\varepsilon}(z)| \leqslant M$ on D. Let $\varepsilon \to 0$, obtain $|f(z)| \leqslant M$ on D.

Solution
Let $C = \sup_{z \in D} |f(z)|$. By definition $|f(z)| \leq C$. Let $z_1, \ldots, z_n \in \partial D$ be all the points for which $f(z)$ does not extend continuously.
Now suppose that $|f(z)| \leq M$ for all $z \in \partial D$ except $\{z_1, \ldots, z_n\}$ and to the contrary $M < C$, then there is a $z_0 \in D$ such that $|f(z_0)| = M + \delta$ with $0 < \delta \leq C - M$.
Let

$$g(z) = (z-z_1)^{\varepsilon} \cdot \ldots \cdot (z-z_n)^{\varepsilon} f(z)$$

where ε is chosen so that $\forall z \in D \cup \partial D$.
We have that

$$\frac{M+\delta/2}{M+\delta} \leq |(z-z_1)^{\varepsilon} \cdot \ldots \cdot (z-z_n)^{\varepsilon}| \quad \text{and} \quad |(z-z_1)^{\varepsilon} \cdot \ldots \cdot (z-z_n)^{\varepsilon}| \leq \frac{M+\delta/3}{M}$$

Note that $g(z)$ extends continuously to ∂D and that $g(z)$ is analytic in D. By the maximum modulus principle, and since

$$|g(z_0)| = |(z_0-z_1)^{\varepsilon} \cdot \ldots \cdot (z_0-z_n)^{\varepsilon}| \, f(z_0) \geq (M+\delta) \frac{M+\delta/2}{M+\delta}$$

we have that

$$\frac{M+\delta/2}{M+\delta}\cdot(M+\delta) \leq |g(z_0)| \leq \sup_{z\in\partial D} |(z-z_1)^{\varepsilon}\cdot\ldots\cdot(z-z_n)^{\varepsilon}|\cdot|f(z)| \ \leq \frac{M+\delta/3}{M}\cdot M$$

which is a contradiction.

III.5.8

Let $f(z)$ be a bounded analytic function on a horizontal strip in the complex plane. Suppose that $f(z)$ extends continuously to the boundary lines of the strip and satisfies $|f(z)| \leq M$ there. Show that $|f(z)| \leq M$ for all z in the strip. Hint. Find a conformal map of the strip onto D and apply Exercise 7.

Solution

Assume strip is $\left[-\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}\right]$, so it is mapped to right half-plane by $w = -i \log z$, $z = e^{iw}$. Then $g(w) = f(e^{iw})$ hold on right half–plane, $\leqslant M$ at all point of $i\mathbb{R}$ except at $w = 0$. Modify proof idea of Exercise III.5.5 to get $|g| \leqslant M$ on the half–plane.

III.5.9
Let D be an unbounded domain, $D \neq \mathbb{C}$, and let $u(z)$ be a harmonic function on D that extends continuously to the boundary ∂D. Suppose that $u(z)$ is bounded below on D, and that $u(z) \geq 0$ on ∂D. Show that $u(z) \geq 0$ on D. Hint. Suppose $0 \in \partial D$, and consider functions of the form $u(z) + \rho \log|z|$ on $D \cap \{|z| > \varepsilon\}$.

Solution (From Hints and Solutions)
Let $\varepsilon > 0$. Take $\delta > 0$ such that $u(z) \geqslant -\varepsilon$ for $z \in D$, $0 < |z| < \delta$. Let $\rho > 0$. Take $R > 0$ so large that $u(z) + \rho \log|z| > 0$ for $|z| > R$. By the maximum principle, $u(z) + \rho \log|z| \geqslant -\varepsilon + \rho \log \delta$ for $z \in D$, $\delta < |z| < R$, hence for all $z \in D$ such that $|z| > \delta$. Let $\rho \to 0$, then let $\varepsilon \to 0$, to obtain $u(z) \geqslant 0$ on D.

Solution (K. Seip)
We may assume $0 \in \partial D$. By continuity of u at 0, we can find δ and ρ so that $(\delta, \rho > 0)$

$$u(z) + \rho \log|z| \geq -\varepsilon, \qquad z \in D \cap \{|z| = \delta\}$$

for arbitary $\varepsilon > 0$. Since u is bounded below, there is also $R_0 > 0$ such that for $R > R_0$

$$u(z) + \rho \log|z| \geq 0, \qquad z \in D \cap \{|z| = R\},$$

Applying the maximum principle to $-u(z) - \rho \log|z|$ in $D \cap \{\delta < |z| < R\}$, we get $u(z) \geq \varepsilon - \rho \log|z|$. Since ε and ρ can be chosen arbitarily small, the result follows.

III.5.10

Let D be a bounded domain, and let $z_0 \in \partial D$. Let $u(z)$ be a harmonic function on D that extends continuously to each boundary point of D except possibly z_0. Suppose $u(z)$ is bounded below on D, and that $u(z) \geq 0$ for all $z \in \partial D$, $z \neq z_0$. Show that $u(z) \geq 0$ on D.

Solution

D hold, $z_0 \in \partial D$, $u \geqslant -M$ on D, $u \geqslant 0$ $(\partial D) \setminus \{z_0\}$. The map $\varphi(z) = 1/(z - z_0)$ maps D onto an unbounded domain, to which the result of Exercise 9 applies. Or: consider $w(z) = u(z) + \rho \log(1/|z - z_0|)$. Assume $|z - z_0| \leqslant R$ on D, then take $\varepsilon > 0$ small, assume $R \geqslant 1$. Then $w(z) > \varepsilon \log(1/R)$ at all points of $(\partial D) \setminus \{z_0\}$, $w(z) \to +\infty$ as $z \to z_0$. By the maximum principle, $w(z) \geqslant \varepsilon \log(1/R)$ on D. Let $\varepsilon \to 0$, get $w(z) \geqslant 0$ on D.

III.5.11
Let E be a bounded set of integer lattice points in the complex plane. A point $m + ni$ of E is an interior point of E if its four immediate neighbors $m \pm 1 + ni$, $m + ni \pm i$ belong to E. Otherwise $m + ni$ is a boundary point of E. A function E is harmonic if its value at any interior point of E is the average of its values at the four immediate neighbors. Show that a harmonic function on a bounded set of lattice points attains its maximum modulus on the boundary of the set.

Solution
Suppose $u(z)$ attains its maximum at $m + ni$, call it M. Then $u(m + ni)$ is the average of its values at its four neighbors, so all their values must coincide with M. Consider $\{m + ni : u(m + ni) = M\}$. This is a finite set. It have a point with largest m. This point have a neighbor in the boundary of E. u attains value M at that point. $u = c$ at some point of boundary of E.
For complex–valued case, follow same approach as in book. Remark: Should just consider real-valued u, and show it attains its maximum and minimum at boundary points of E.

III.6.1

Consider the fluid flow with constant velocity $V = (2, 1)$. Find the velocity potential $\phi(z)$, the stream function $\psi(z)$, and the complex velocity potential $f(z)$ of flow. Sketch the streamlines of the flow. Determine the flux of the flow across the interval $[0, 1]$ on the real axis and across the interval $[0, i]$ on the imaginary axis.

Solution

$V = (2, 1)$, $V = \nabla\phi$, $\phi = 2x + y$, $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$, $\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \Rightarrow \psi = 2y - x$

$f(z) = \phi + i\psi = 2z - iz = (z - i)z$

Streamlines are straight line with slope $1/2$, flux across $[0, 1] = \psi(1) - \psi(0) = -1$, flux across $[0, i] = \psi(i) - \psi(0) = 2$.

III.6.2
Fix real numbers α and β, and consider the vector field given in polar coordinates by

$$V(r,\theta) = \frac{\alpha}{r}u_r + \frac{\beta}{r}u_\theta,$$

where u_r and u_θ are the unit vectors in r and θ directions, respectively.
(a)
Show that $V(r,\theta)$ is the velocity vector field of a fluid flow, and find the velocity potential $\phi(z)$ of the flow.
(b)
Find the stream function $\psi(z)$ and the complex velocity potential $f(z)$ of the flow.
(c)
Determine the flux of the flow emanating from the origin. When is 0 a source and when is 0 a sink?
(d)
Sketch the streamlines of the flow in the case $\alpha = -1$ and $\beta = 1$.

Solution
a)
$V(r,\theta) = \frac{\alpha}{r}u_r + \frac{\beta}{r}u_\theta$
$\nabla\phi = \frac{\partial\phi}{\partial r}u_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}u_\theta = \frac{\alpha}{r}u_r + \frac{\beta}{r}u_\theta$
Get $\frac{\partial\phi}{\partial r} = \frac{\alpha}{r}$, $\phi = \alpha\log r + h(\theta)$
$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = h'(\theta) = \frac{\beta}{r}$, $h'(\theta) = \beta$, $h(\theta) = \beta\theta \therefore V = \nabla\theta$
$\phi = \alpha\log r + \beta\theta$ is harmonic locally. is the velocity vector field of a flow
b)
$\phi = \mathrm{Re}(\alpha\log z + \beta\arg z)$ stream function $= \psi = \mathrm{Im}(\alpha\log z + \beta\arg z) = \alpha\arg z - \beta\log|z|$
complex velocity of a point $f(z) = (\alpha - i\beta)\log z$
c)
Flux $=$ of ψ around a circle centred at $2\pi\alpha$, $\sin u : \alpha > 0$, $\sin u : \alpha > 0$.
d)
$\alpha = -1$, $\beta = 1$. Note: ϕ and ψ are only locally defined.

III.6.3

Consider the fluid flow with velocity $V = \nabla\phi$, where $\phi(r,\theta) = (\cos\theta)/r$. Show that the streamlines of the flow are circles and sketch them. Determine the flux of the flow emanating from the origin.

Solution

$\phi = \frac{\cos\theta}{r} = \frac{r\cos\theta}{r^2} = \mathrm{Re}\left(\frac{1}{z}\right) = \mathrm{Re}\left(\frac{\overline{z}}{|z|^2}\right)$ $\psi = \mathrm{Im}\left(\frac{1}{z}\right) = \frac{-\sin\theta}{r}$

Since ψ is single-valued, flux $= \int_{|z|=\varepsilon} d\psi = 0$, and there is no source or sink at 0. Set $w = 1/z$, $z = 1/w$, these streamlines are the curves $\mathrm{Im}(1/z) =$ constant, which correspond to the curves $\mathrm{Im}\, w =$ constant get modified to tangent to real line at $0 = z(\infty)$, and real axis. $\nabla\phi = \frac{\partial\phi}{\partial r}u_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}u_\theta = -\frac{\cos\theta}{r^2}u_r - \frac{\sin\theta}{r^2}u_\theta$.

III.6.4
Consider the fluid flow with velocity $V = \nabla \phi$, where

$$\phi(z) = \log \left| \frac{z-1}{z+1} \right| .$$

Show that the streamlines of the flow are arcs of circles and sketch them. Determine the flux of the flow emanating from each of the singularities at ± 1.

Solution

ϕ is the composition of an analytic function $w = (z-1)/(z+1)$ and the harmonic function $\log |w|$, so ϕ is harmonic, and $\phi = \operatorname{Re} \log ((z-1)/(z+1))$. The stream function is $\psi = \operatorname{Im} (\log (z-1)/(z+1)) = \arg ((z-1)/(z+1))$, then ψ is not single-valued. $\oint_{|z-1|=\varepsilon} d\psi = \oint_{|z-1|=\varepsilon} d \arg (z-1) = 2\pi =$ flux emanating from $+1$. $\oint_{|z-1|=\varepsilon} d\psi = \oint_{|z-1|=\varepsilon} d \arg (z+1) = -2\pi =$ flux emanating from -1. $z = 1$ is a source, $z = -1$ is a sink.
Set $\zeta = (z-1)/(z+1)$, $w = \log \zeta$. The line $\operatorname{Im} w =$ constant in the $w-$ plane correspond to rays issuing from 0 in the $\zeta-$ plane. (since $\zeta = e^w$), and this correspond to arcs of circles from 1 to -1 in the z–plane, since FLT´s maps circles to circles. Hence streamlines are arcs of circle from 1 to -1, including the circle through ∞.

III.6.5
Consider the fluid flow in the horizontal strip $\{0 < \operatorname{Im} z < \pi\}$ with a sink at 0 and equal sources at $\pm\infty$. Find the stream function $\psi(z)$ and the velocity vector field $V(z)$ of the flow. Sketch the streamlines of the flow. Hint. Map the strip to a half-plane by $\zeta = e^z$ and solve a Dirichlet problem with constant boundary values on the three intervals in the boundary separating sinks and sources.

Solution
Consider fluid flow in a horizontal strip. $\{0 < \operatorname{Im} z < \pi\}$ with a sink 0, but no source at $\pm\infty$. Find the stream function $\psi(z)$. Sketch the flow lines. Find the complex vector field $V(z)$. Hint : Map the half-plane by.
Map to half-plane $\xi = e^z$. $A \arg \xi + B \arg(\xi - 1) + C$, $\arg \xi = \arg e^z = y$, $Ay + C + B \arg(e^z - 1)$, $c_3 = 0$, requires $\pi A + \pi B + C = 0 \Rightarrow C = -\pi(A + B)$, $A(y - \pi) + B[\arg(e^z - 1) - \pi]$. $C_1 = -C_2$ require $-A\pi + B\pi - B\pi = -[-A\pi - B\pi]$, $A[-(y - \pi) + 2[\arg(e^z - 1) - \pi]] = A[-(\pi + y) + 2\arg(e^z - 1)]$

III.6.6
For a fluid flow with velocity potential $\phi(z)$, we define the conjugate flow to be the flow whose velocity potential is the conjugate harmonic function $\psi(z)$ of $\phi(z)$. What is the stream function of the conjugate flow? What is the complex velocity potential of the conjugate flow?

Solution
Stream function of conjugate flow in direction $\nabla\psi$ is $-\phi$. Complex velocity potential is $g(z) = \psi - i\phi = -if(z) = -i(\phi + i\psi)$

III.6.7
Find the stream function and the complex velocity potential of the conjugate flow associated with the fluid flow with velocity vector u_r/r. Sketch the streamlines of the conjugate flow. Do the particles near the origin travel faster or slower than particles on the unit circle?

Solution
Flow u_r/r, $\phi = \log|z|$, $\psi = \arg z$. Complex velocity potential of conjugate flow is $-i\log z$. Streamlines of the conjugate flow are $-\phi = -\log|z| =$ constant $\Leftrightarrow r = |z| =$ constant, which are circles centered in origin. Speed $= |V(z)| = |f'(z)| = 1/|z|$. Particles travel faster near $z = 0$.

III.6.8
Find the stream function of the conjugate flow of

$$V(r,\theta) = \frac{1}{r}(-u_r + u_\theta).$$

Sketch the streamlines of both the flow and the conjugate flow on the same axis. (See Exercise 2d.)

Solution
This is the vector field from 2d. Hence $V = \nabla(-\log r + \theta)$. The velocity potential is $\phi = -\log r + \theta$. The conjugate flow has velocity potential $\psi = -\theta - \log r = \operatorname{Re}((i-1)\log z)$. The stream function of the conjugate flow is then $\operatorname{Im}((i-1)\log z) = -\theta + \log r = -\phi$. (The steam function of the conjugate is always the negative of the velocity potential of the flow. See Ex.6) Streamline of conjugate flow are orthogonal to the stream of the flow (Exercise 2d) and velocity vector field. Conjugate flow is the rotation by 90° of the velocity vector field of the flow, so the origin is a sink of both flows in this case.

III.7.1

Find the steady-state heat distribution $u(x, y)$ in a laminar plate corresponding to the half disk $\{x^2 + y^2 < 1, y > 0\}$, where the semi-circular top edge is held at temperature T_1 and the lower edge $(-1, 1)$ is held at temperature T_2. Find and sketch the isothermal curves for the heat distribution. Hint. Consider the steady-state heat distribution for the full unit disk with the top held at temperature T_1 and the bottom temperature T_3, where $T_2 = (T_1 + T_3)/2$.

Solution

$\phi(z) = \frac{2}{\pi}(\mathrm{Arg}(1 + z) - \mathrm{Arg}(1 - z))$ has temp $+1$ on top, -1 on bottom half and is 0 on the interval $(-1, 1)$. Take $u = A\phi + B$, $\left\{\begin{array}{l} T_1 = A + B \\ T_2 = B \end{array}\right. \Rightarrow$

$\left\{\begin{array}{l} A = T_1 - T_2 \\ B = T_2 \end{array}\right.$, get $u = (T_1 - T_2)\,\phi(z) + T_2$

$u = (T_1 - T_2)\frac{2}{\pi}[\mathrm{Arg}(1 + z) - \mathrm{Arg}(1 - z)] + T_2$

The isothermal curves are the curves where $u =$ konstant. these are the curves

$$\mathrm{Arg}\left(\frac{1 + z}{1 - z}\right) = \text{konsant}$$

then

$$(1 + z)(1 - \bar{z}) = 1 + 2i\,\mathrm{Im}\,z - |z|^2$$

and

$$\frac{\mathrm{Im}\,z}{1 - |z|^2} = \text{konstant.}$$

III.7.2

Find the potential function $\phi(x, y)$ for the electric field for a conducting laminar plate corresponding to the quater-disk $\{x^2 + y^2 < 1, x > 0, y > 0\}$, where the two edges on the coordinate axis are grounded (that is, $\phi = 0$ on the edges), and the semicircular edge is held at constant potential V_1. Find and sketch the equipotential lines and the lines of force for the electric field. Hint. Use the conformal map $\zeta = z^2$ and the solution to the preceding exercise.

Solution

$\phi(z) = \frac{2}{\pi}(\operatorname{Arg}(1 + w(z)) - \operatorname{Arg}(1 - w(z)))$, $\phi_1 = V_1 \frac{2}{\pi}(\operatorname{Arg}(1 + z^2) - \operatorname{Arg}(1 - z^2))$

III.7.3

Find the potential function $\phi(x,y)$ for the electric field for a conducting laminar plate corresponding to the unit disk where the boundary quater-circles in each quadrant are held at a constant voltages V_1, V_2, V_3 and V_4. Hint. Map the disk to the upper half-plane by $w = w(z)$ and consider potential functions of the form $\operatorname{Arg}(w-a)$.

Solution

Let $w = i(1-z)/(1+z)$, $w(1) = 0$, $w(-1) = \infty$, $w(i) = 1$, $w(-i) = -1$. $z \to w(z)$ maps unit disk to upper half–plane. Suffers to find a harmonic function $u(w)$ in upper half-plane with boundary values constant on each interval, as above. Any such function is a linear combination of $u_{-1}(w) = \frac{1}{\pi}\operatorname{Arg}(w+1)$, $u_0(w) = \frac{1}{\pi}\operatorname{Arg} w$, $u_1(w) = \frac{1}{\pi}\operatorname{Arg}(w-1)$, and if $\phi = A_{-1}u_{-1} + A_0u_0 + A_1u_1 + C$, we get a system we can back solve.

$$\left\{\begin{array}{r} A_{-1}+A_0+A_1+C=V_3 \\ A_0+A_1+C=V_3 \\ A_1+C=V_1 \\ C=V_2 \end{array}\right. \Rightarrow \left\{\begin{array}{l} C=V_2 \\ A_1=V_1-V_2 \\ A_0=V_4-V_1 \\ A_{-1}=V_3-V_4 \end{array}\right.$$

That does it-just plug in.

Example: $V_2 = 1, V_1 = V_3 = V_4 = 0$, this solution is $\phi_2(z) = 1 - \frac{1}{\pi}\operatorname{Arg}\left(i\frac{1-z}{1+z}\right)$, and we can get other elementary solutions, $\phi_1(z)$, $\phi_3(z)$, $\phi_4(z)$, and then $\phi(z) = \sum v_j\phi_j(z)$.

III.7.4

Find the steady-state heat distribution in a laminar plate corresponding to the vertical half-strip $\{|x| < \pi/2, y > 0\}$, where the vertical sides at $x = \pm\pi/2$ are held at temperature $T_0 = 0$ and the bottom edge $(-\pi/2, \pi/2)$ on the real axis is held at temperature $T_1 = 100$. Make a rough sketch of the isothermal curves and the lined of heat flow. Hint. Use $w = \sin z$ to map the strip to the upper half-plane, and make use of harmonic functions of the form $\operatorname{Arg}(w - a)$.

Solution

Try $\phi = a \operatorname{Arg}(w+1) + b \operatorname{Arg}(w-1) + c$, want $\begin{cases} a+b+c=0 \\ a-b+c=100 \\ -a-b+c=0 \end{cases} \Rightarrow$

$\begin{cases} a=50 \\ b=-50 \\ c=0 \end{cases}$

$\phi(z) = 50 \operatorname{Arg}(\sin z + 1) - 50 \operatorname{Arg}(\sin z - 1)$.

III.7.5

Find the steady-state heat distribution in a laminar plate corresponding to the vertical half-strip $\{|x| < \pi/2, y > 0\}$, where the side $x = -\pi/2$ is held at constant temperature T_0, the side $x = \pi/2$ is held at constant temperature T_1, and the bottom edge $(-\pi/2, \pi/2)$ on the real axis is insulated, that is, no heat passes through the bottom edge, so the gradient ∇u of the solution $u(x, y)$ is parallel there to the $x-$ axis. Hint. Try linear functions plus constants.

Solution

$u(x, y) = ax + b$ is harmonic. $\frac{\partial u}{\partial y} = 0$

$$\begin{cases} -\frac{\pi}{2}a + b = T_0 \\ \frac{\pi}{2}a + b = T_1 \end{cases} \Rightarrow \begin{cases} a = \frac{T_1 - T_0}{\pi} \\ b = \frac{T_0 + T_1}{2} \end{cases}$$

$u = \frac{1}{\pi}(T_1 - T_0)x + \frac{T_0 + T_1}{2}.$

III.7.6

Find the steady-state heat distribution in a laminar plate corresponding to the upper half-plane $\{y > 0\}$, where the interval $(-1, 1)$ is insulated, the interval $(-\infty, -1)$ is held at temperature T_0, and the interval $(1, \infty)$ is held at temperature T_1. Make a rough sketch of the isothermal curves and the lines of heat flow. Hint. Use the solution to Exercise 5 and the conformal map from Exercise 4.

Solution

Let $z = z(w)$ map the vertical half-strip to the upper half-plane with $z(-\pi/2) = -1$ and $z(\pi/2) = 1$, $w = w(z)$: upper half–plane $\to$ vertical semi strip. Take $z = \sin w$, this has desired mapping property, $w = \sin^{-1}(z)$. Then solution is composition of sine function with $\frac{2}{\pi}(T_1 - T_0)\operatorname{Re} w + \frac{T_0+T_1}{2}$. Get $u = \frac{2}{\pi}(T_1 - T_0)\operatorname{Re}\left(\sin^{-1}(z)\right) + \frac{T_0+T_1}{2}$, where we take branch $\sin^{-1} z$ with values in the vertical semi-strip. Note : This conformal map was used in Exercise 4.

III.7.7

The gravitational field near the surface of the earth is approximately constant, of the form $F = ck$, where k is the unit vector in the $z-$ direction in $(x, y, z)-$ space and the surface of the earth is represented by the plane where $z = 0$ (the flat earth theory). Show that F is conservative, and find a potential function ϕ for F.

Solution

$\phi(x, y, z) = cz$, $\nabla\phi = c\overrightarrow{k}$

Since

$$\nabla \times F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (0, 0, c) = (0, 0, 0),$$

we have that F is conservative.

III.7.8

Show that the inverse square force field $F = u_r/r^2$ on $\mathbb{R}^3$ is conservative. Find the potential function ϕ for F, and show that ϕ is harmonic.

Solution

$\overrightarrow{r} = \frac{\overrightarrow{u}_r}{r^2} = \nabla\phi$, $\phi\left(x, y, z\right) = -\frac{1}{r}$

III.7.9

For $n \geq 3$, show that the function $1/r^{n-2}$ is harmonic on $\mathbb{R}^n \backslash \{0\}$. Find the vector field F that has this function as its potential.

Solution

$\frac{1}{r^{n-2}} = \frac{1}{\left(x_1^2+\ldots+x_n^2\right)^{\frac{n-2}{2}}} = u = r^{2-n} = \left(x_1^2 + \ldots + x_n^2\right)^{1-\frac{n}{2}}$

$\frac{\partial u}{\partial x_j} = \left(1 - \frac{n}{2}\right)\left(x_1^2 + \ldots + x_n^2\right)^{-\frac{n}{2}} \cdot 2x_j = (2-n)\, x_j r^{-n}$

$\frac{\partial^2 u}{\partial x_j^2} = (2-n)\, r^{-n} + (2-n)\, x_j \left(-\frac{n}{2}\right)\left(x_1^2 + \ldots + x_n^2\right)^{-\frac{n}{2}-1} \cdot 2x_j =$

$\frac{2-n}{r^n} + \frac{(2-n)(\ -n)x_j^2}{r^{n+2}}$

$\Delta u = n\frac{2-n}{r^n} - \frac{(2-n)n}{r^{n+2}} \sum x_j^2 = 0$ this shows that u is harmonic on $\mathbb{R}^n \backslash \{0\}$

$F = \Delta u = \sum \frac{\partial u}{\partial x_j} e_j = \frac{(2-n)}{r^n} \sum x_j e_j = \frac{2-n}{r^{n-1}} \cdot \frac{\Sigma x_j}{r} e_j = \frac{(2-n)\vec{u}_r}{r^{n-1}}$

$F = (2r)\frac{r}{r^n} = \frac{c\overrightarrow{u}}{r^{n-1}}$

$\int F \cdot d\sum = \frac{c}{r^{n-1}} Area\,(s) = (n-2)\frac{Area(S)}{r^{n-1}}$

IV	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1																			
2																			
3																			
4																			
5																			
6																			
7																			
8																			

IV.1.1

1	2	3	P	L	K
					KKK

F

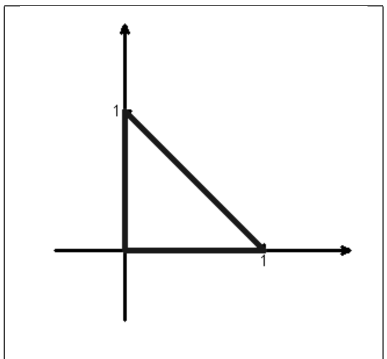
FFF

Let γ be the boundary of the triangle $\{0 < y < 1 - x,\ 0 < x < 1\}$, with the usual counterclockwise orientation. Evaluate the following integrals
(a) $\int_\gamma \operatorname{Re} z\, dz$ **(b)** $\int_\gamma \operatorname{Im} z\, dz$ **(c)** $\int_\gamma z\, dz$

Solution

a)

We begin do split the contur γ into three parts so that $\gamma = \gamma_1 + \gamma_2 + \gamma_3$. And integrate along each side in the triangle



$\gamma = \gamma_1 + \gamma_2 + \gamma_3$

We parametrizice $\gamma_1 = [0, 1]$, thus $\gamma_1(t) = t$, for $0 \le t \le 1$, then $\operatorname{Re}(t) = t$ and $dz = dt$

$$\int_{\gamma_1} \operatorname{Re} z\, dz = \int_{\gamma_1} \operatorname{Re}(t)\ dz = \int_0^1 t\, dt = \frac{1}{2}.$$

We parametrizice $\gamma_2 = [1, i]$, thus $\gamma_2(t) = 1 - t + it$, for $0 \le t \le 1$, then $\operatorname{Re}(1 - t + it) = 1 - t$ and $dz = (-1 + i)\ dt$, then

$$\int_{\gamma_2} \operatorname{Re} z\, dz = \int_0^1 (1 - t)(-1 + i)\, dt = -\frac{1}{2} + \frac{1}{2}i.$$

We parametrizice $\gamma_3 = [i, 0]$, thus $\gamma_3(t) = i(1 - t)$, for $0 \le t \le 1$, then, $\operatorname{Re}(i(1 - t)) = 0$ and $dz = -i\, dt$, then

$$\int_{\gamma_3} \operatorname{Re} z\, dz = \int_0^1 (0)(-i)\, dt = 0.$$

And now we take the parts in the integral together

$$\int_{\gamma} \operatorname{Re} z \, dz = \int_{\gamma_1} \operatorname{Re} z \, dz + \int_{\gamma_2} \operatorname{Re} z \, dz + \int_{\gamma_3} \operatorname{Re} z \, dz = \frac{i}{2}$$

b) For details see a)

$$\begin{aligned}
\int_{\gamma_1} \operatorname{Im} z \, dz &= \int_0^1 0 \, dt = 0, \\
\int_{\gamma_2} \operatorname{Im} z \, dz &= \int_0^1 t \, (-1 + i) \, dt = -\frac{1}{2} + \frac{1}{2} i, \\
\int_{\gamma_3} \operatorname{Im} z \, dz &= \int_0^1 (1 - t) \, (-i) \, dt = -\frac{1}{2} i.
\end{aligned}$$

thus

$$\int_{\gamma} \operatorname{Im} z \, dz = -\frac{1}{2}.$$

c) For details see a)

$$\begin{aligned}
\int_{\gamma_1} z \, dz &= \int_0^1 t \, dt = \frac{1}{2}, \\
\int_{\gamma_2} z \, dz &= \int_0^1 (1 - t + it) \, (-1 + i) \, dt = -1 \\
\int_{\gamma_3} z \, dz &= \int_0^1 (i \, (1 - t)) \, (-i) \, dt = \frac{1}{2}.
\end{aligned}$$

thus

$$\int_{\gamma} z \, dz = 0.$$

IV.1.2

1	2	3	P	L	K
111				LLL	

Absolutbeloppet i c) något är fel

Let γ be the unit circle $\{|z| = 1\}$, with the usual counterclockwise orientation. Evaluate the following integrals, for $m = 0, \pm 1, \pm 2, \ldots$.
(a) $\int_\gamma z^m \, dz$ **(b)** $\int_\gamma \bar{z}^m \, dz$ **(c)** $\int_\gamma z^m \, |dz|$

Solution
Set $z = e^{i\theta}$, where $0 \le \theta \le 2\pi$, and thus $dz = ie^{i\theta} d\theta$.
(a)

$$\int_\gamma z^m dz = \int_0^{2\pi} e^{im\theta} ie^{i\theta} d\theta = i \int_0^{2\pi} e^{i(m+1)\theta} d\theta = \left[\frac{e^{i(m+1)\theta}}{m+1}\right]_0^{2\pi} =$$
$$= \frac{1}{m+1}\left(e^{i2\pi(m+1)} - 1\right) = \begin{cases} 0, & m \neq -1 \\ 2\pi i & m = -1. \end{cases}$$

(b)

$$\int_\gamma \bar{z}^m dz = \int_0^{2\pi} e^{-im\theta} ie^{i\theta} d\theta = i \int_0^{2\pi} e^{i(1-m)\theta} d\theta =$$
$$= \left[\frac{e^{i(1-m)\theta}}{1-m}\right]_0^{2\pi} = \frac{1}{1-m}\left(e^{i2\pi(m-1)} - 1\right) = \begin{cases} 0, & m \neq 1 \\ 2\pi i & m = 1. \end{cases}$$

(c)

$$\int_\gamma z^m \, |dz| = \int_0^{2\pi} e^{im\theta} d\theta = \left[\frac{e^{im\theta}}{im}\right]_0^{2\pi} = \frac{1}{im}\left(e^{i2\pi m} - 1\right) = \begin{cases} 0, & m \neq 0 \\ 2\pi & m = -1. \end{cases}$$

IV.1.3

1	2	3	P	L	K
111					

Let γ be the circle $\{|z| = R\}$, with the usual counterclockwise orientation. Evaluate the following integrals, for $m = 0, \pm 1, \pm 2, \ldots$.
(a) $\int_\gamma |z^m|\, dz$ **(b)** $\int_\gamma |z^m|\, |dz|$ **(c)** $\int_\gamma \bar{z}^m\, dz$

Solution
Set $z = Re^{i\theta}$, where $0 \le \theta \le 2\pi$, and thus $dz = iRe^{i\theta}d\theta$ and $|z| = R$.
(a)

$$\int_\gamma |z^m|\, dz = \int_0^{2\pi} \left|R^m e^{im\theta}\right| iRe^{i\theta}d\theta = iR^{m+1}\int_0^{2\pi} e^{i\theta}d\theta = R^{m+1}\left[e^{i\theta}\right]_0^{2\pi} = R^{m+1}\left(e^{i2\pi} - 1\right) = 0.$$

(b)

$$\int_\gamma |z^m|\, |dz| = \int_0^{2\pi} \left|R^m e^{im\theta}\right| \left|iRe^{i\theta}\right| d\theta = R^{m+1}\int_0^{2\pi} d\theta = R^{m+1}\left[\theta\right]_0^{2\pi} = R^{m+1}\left(2\pi - 0\right) = 2\pi R^{m+1}.$$

(c)

$$\int_\gamma \bar{z}^m dz = \int_0^{2\pi} R^m e^{-im\theta} iRe^{i\theta} d\theta = iR^{m+1}\int_0^{2\pi} e^{i(1-m)\theta}d\theta = R^{m+1}\left[\frac{e^{i(1-m)\theta}}{1-m}\right]_0^{2\pi} =$$

$$= \frac{R^{m+1}}{1-m}\left(e^{i2\pi(1-m)} - 1\right) = \begin{cases} 0, & m \ne 1, \\ 2\pi i R^{m+1}, & m = 1, \end{cases} = \begin{cases} 0, & m \ne 1, \\ 2\pi i R^2, & m = 1. \end{cases}$$

IV.1.4

1	2	3	P	L	K
				LLL	

Show that if D is a bounded domain with smooth boundary, then

$$\int_{\partial D} \bar{z}\, dz = 2i \text{ Area}\,(D)\,.$$

Solution

Substitute $\bar{z} = x - iy$, $dz = dx + idy$, and apply Green's theorem.

IV.1.5

1	2	3	P	L	K
111					KKK

138

Show that

$$\left|\oint_{|z-1|=1}\frac{e^z}{z+1}dz\right|\leq 2\pi e^2.$$

Solution

We begin with parametrizize the circle $|z-1|=1$, as $z=1+e^{it}=1+\cos t+i\sin t$ where $0\leq t\leq 2\pi$.

For the nominator in the integrand $e^z/(z+1)$ we have

$$|e^z|=\left|e^{\operatorname{Re}z+i\operatorname{Im}z}\right|=\left|e^{\operatorname{Re}z}e^{i\operatorname{Im}z}\right|=\left|e^{\operatorname{Re}z}\right|\leq e^2,$$

since $\operatorname{Re}z\leq 2$ on the circle.

And for the denominator in the integrad we have

$$|1+z|=|2+\cos t+i\sin t|\geq \operatorname{Re}(2+\cos t+i\sin t)=2+\cos t\geq 1,$$

Thus

$$M=\left|\frac{e^z}{1+z}\right|=\frac{|e^z|}{|z+1|}\leq\frac{e^2}{1}=e^2.$$

Becauce the integrationcontour is a circle with radius 1 we have that $L=2\pi$. $ML-$ estimate gives

$$\left|\oint_{|z-1|=1}\frac{e^z}{z+1}dz\right|\leq 2\pi e^2.$$

IV.1.6

1	2	3	P	L	K
111					KKK

Show that there is a strict inequality

$$\left|\oint_{|z|=R} \frac{\operatorname{Log} z}{z^2} dz\right| \leq 2\sqrt{2}\pi \frac{\log R}{R}, \qquad R > e^{\pi}.$$

Solution

For the integrand $\operatorname{Log} z/z^2$, with the nominator $\operatorname{Log} z = \log R + i\theta$ with the principalargument $-\pi \leq \theta \leq \pi$ we have

$$\left|\frac{\operatorname{Log} R}{R^2}\right| = \frac{\sqrt{\log^2 R + \theta^2}}{R^2} \leqslant \frac{\sqrt{\log^2 R + \pi^2}}{R^2} \leqslant \frac{\sqrt{\log^2 R + \log^2 R}}{R^2} = \frac{\sqrt{2}\log R}{R^2}.$$

there the last inequalitys follows by given fact $R > e^{\pi}$ that gives $\pi^2 < \log^2 R$. Becauce the integrationcontour is a circle with radius R we have that $L = 2\pi R$.

$ML-$ estimate gives

$$\left|\oint_{|z|=R} \frac{\operatorname{Log} z}{z^2} dz\right| \leqslant \frac{\sqrt{2}\log R}{R^2} 2\pi R = 2\sqrt{2}\pi \frac{\log R}{R}, \qquad R > e^{\pi}.$$

IV.1.7

1	2	3	P	L	K

Show that there is a strict inequality

$$\left|\oint_{|z|=R} \frac{z^n}{z^m-1} dz\right| \leq \frac{2\pi R^{n+1}}{R^m-1}, \qquad R>1, m \geq 1, n \geq 1.$$

Solution

If $|f(z)| \leqslant m_0 < m$ on an arc γ_0 of the circle, $\gamma = \gamma_0 \cup \gamma_1$, then $\left|\int\limits_{\gamma} f(z)\, dz\right| =$ $\left|\int\limits_{\gamma_0} f(z)\, dz + \int\limits_{\gamma_1} f(z)\, dz\right| \leqslant m_0 \text{length}(\gamma_0) + m\text{length}(\gamma_1) = (m_0 - m)\,\text{length}(\gamma_0) +$ $m\text{length}(\gamma) < m\text{length}(\gamma)$

Thus if $\left|\int_\gamma f(z)\, dz\right| = \mathrm{ML}$, then $|f(z)| = M$ on γ. In this case, $|z^m| = R^n$, $\left|\frac{1}{z^m-1}\right| \leqslant \frac{1}{R^m-1}$, with strict inequality unless $z = \mathrm{R} \cdot \underbrace{e^{2\pi i k/m}}_{^{th}\text{ root of } w}$. $\therefore M = \frac{R^n}{R^m-1}$, $L = 2\pi R$, get $\left|\int_{|z|=R} \frac{z^n}{z^m-1} dz\right| < \frac{R^n}{R^m-1} \cdot 2\pi R = \frac{2\pi R^{n+1}}{R^m-1}$, $R>1$, $m \geqslant 1$, $n \geqslant 0$.

IV.1.8

1	2	3	P	L	K
				LLL	

Suppose the continuous function $f\left(e^{i\theta}\right)$ **on the unit circle satisfies** $\left|f\left(e^{i\theta}\right)\right| \leq M$ **and** $\left|\int_{|z|=1} f\left(z\right)\, dz\right| = 2\pi M$**. Show that** $f\left(z\right) = c\bar{z}$ **for some constant** c **with modulus** $|c| = M$**.**

Solution

Multiply f by a unimodular constant, assume $|f\left(z\right)| \leqslant M$, $\oint_{|z|=1} f\left(z\right) dz = \int_0^{2\pi} f\left(e^{i\theta}\right) d\theta = 2\pi M$. Take real parts, have $\int_0^{2\pi} \mathrm{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) d\theta = 2\pi M$, $\mathrm{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) d\theta \leqslant \left|f\left(e^{i\theta}\right)\right| \leqslant M$. If have strict inequality somewhere, then $\int_0^{2\pi} < 2\pi M$. We conclude that $\mathrm{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) d\theta \equiv M$. Since $\left|\left(f\left(e^{i\theta}\right) ie^{i\theta}\right)\right| \leqslant M$, we have $\mathrm{Im}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) d\theta \equiv 0$. $\therefore f\left(e^{i\theta}\right) ie^{i\theta} \equiv M$, $f\left(e^{i\theta}\right) \equiv -ie^{-i\theta} M$, $f\left(z\right) \equiv -iM\bar{z}$. $\therefore f\left(z\right) = c\bar{z}$ for a constant, $|c| = M$.

IV.1.9

1	2	3	P	L	K

Suppose $h(z)$ is a continuous function on a curve γ. Show that

$$H(w) = \int_\gamma \frac{h(z)}{z-w} dz, \qquad w \in \mathbb{C} \backslash \gamma,$$

is analytic on the complement of γ, and find $H'(w)$.

Solution

For the analyticity, differentiate by hand. (See Exercise III.1.6). The Derivative is $H'(w) = \lim_{w \to \infty} \frac{1}{\Delta w} \int_\gamma \left[\frac{h(z)}{z-(w+\Delta w)} - \frac{h(z)}{z-w} \right] dz =$
$\lim_{w \to \infty} \frac{1}{\Delta w} \int_\gamma \frac{h(z)\Delta w}{(z-(w+\Delta w))(z-w)} dz = \int_\gamma \frac{h(z)dz}{(z-w)^2}$

IV.2.1

1	2	3	P	L	K	K
111				LLL	KKK	246

Evaluate the following integrals, for a path γ that travels from $-\pi i$ to πi in the right half-plane, and also for a path γ from $-\pi i$ to πi in the left half-plane.

(a) $\int_\gamma z^4 dz$ **(b)** $\int_\gamma e^z dz$ **(c)** $\int_\gamma \cos z\, dz$ **(d)** $\int_\gamma \sinh z\, dz$

Solution

The integrals are all independent of path and can be evaluated by finding a primitive.

(a)

$$\int_\gamma z^4 dz = \left[\frac{z^5}{5}\right]_{-\pi i}^{\pi i} = \frac{(\pi i)^5}{5} - \frac{(-\pi i)^5}{5} = \frac{2\pi^5 i}{5}.$$

(b)

$$\int_\gamma e^z dz = [e^z]_{-\pi i}^{\pi i} = e^{\pi i} - e^{-\pi i} = -1 - (-1) = 0.$$

(c)

$$\int_\gamma \cos z\ dz = [\sin z]_{-\pi i}^{\pi i} =$$

$$= \sin(\pi i) - \sin(-\pi i) = \frac{e^{i(\pi i)} - e^{-i(\pi i)}}{2i} - \frac{e^{i(-\pi i)} - e^{-i(-\pi i)}}{2i} =$$

$$= \frac{2\left(e^{-\pi} - e^{\pi}\right)}{2i} = i\left(e^{\pi} - e^{-\pi}\right).$$

(d)

$$\int_\gamma \sinh z\ dz = [\cosh z]_{-\pi i}^{\pi i} = \cosh(\pi i) - \cosh(-\pi i) =$$

$$= \frac{e^{\pi i} + e^{-\pi i}}{2} - \frac{e^{-\pi i} + e^{-(-\pi i)}}{2} = 0.$$

IV.2.2

1	2	3	P	L	K
111				LLL	

K

246

Using an appropriate primitive, evaluate $\int_\gamma 1/z\, dz$ for a path γ that travels from $-\pi i$ to πi in the right half-plane, and also for a path γ from $-\pi i$ to πi in the left half-plane. For each path give a precise definition of the primitive used to evaluate the integral.

Solution

In right half–plane use $f(z) = \operatorname{Log} z = \log|z| + i\theta$, where $-\pi < \theta < \pi$ as a primitive, get

$$\int_\gamma \frac{1}{z} dz = [\operatorname{Log} z]_{-\pi i}^{\pi i} = \left(\log|\pi i| + i\frac{\pi}{2}\right) - \left(\log|-i\pi| - i\frac{\pi}{2}\right) = \pi i.$$

In left half–plane use $f(z) = \log_0 z = \log|z| + i\theta$, where $0 < \theta < 2\pi$ as a primitive, get

$$\int_\gamma \frac{1}{z} dz = [\log_0 z]_{-\pi i}^{\pi i} = \left(\log|\pi i| + i\frac{\pi}{2}\right) - \left(\log|-i\pi| + i\frac{3\pi}{2}\right) = -\pi i.$$

IV.2.3

1	2	3	P	L	K
111				LLL	KKK

Show that if $m \neq -1$, then z^m has a primitive on $\mathbb{C} \setminus \{0\}$.

Solution

The primitive is $\frac{z^{m+1}}{m+1}$, which is analytic for $z \neq 0$ and $m \in \mathbb{Z}$, $m \neq -1$.

IV.2.4

1	2	3	P	L	K

Let $D = \mathbb{C} \backslash (-\infty, 1]$, and consider the branch of $\sqrt{z^2-1}$ on D that is positive on the interval $(1, \infty)$.

(a)

Show that $z + \sqrt{z^2-1}$ omits the negative real axis, that is, the range of the function on D does not include any values in the interval $(-\infty, 0]$ on the real axis.

(b)

Show that $\operatorname{Log}\left(z + \sqrt{z^2-1}\right)$ is a primitive for $1/\sqrt{z^2-1}$ on D.

(c)

Evaluate

$$\int_\gamma \frac{dz}{\sqrt{z^2-1}},$$

where γ is the path from $-2i$ to $+2i$ in D counterclockwise around the circle $|z| = 2$.

(d)

Evaluate the integral above in the case γ is the entire circle $|z| = 2$, oriented counterclockwise. (Note that the primitive is discontinuous at $z = -2$.)

Solution

$D = \mathbb{C} \backslash (-\infty, 1]$, $\sqrt{z^2-1}$

a)

$z + \sqrt{z^2-1} = -t$ $(t \geqslant 0) \Rightarrow \sqrt{z^2-1} = -t - z$, $z^2 - 1 = t^2 - 2z + z^2$, $2z = t^2 + 1$, $z = (t^2+1)/2$.

But values of the branch are $\geqslant 0$ on $(1, \infty) = D \cap \mathbb{R}$. $\therefore$ It assumes no negative values.

b)

$\operatorname{Log}\left(z + \sqrt{z^2-1}\right)$ is the composition of $z + \sqrt{z^2-1}$ and $\operatorname{Log} w$ on $\mathbb{C} \backslash (-\infty, 0]$, so it's analytic. By the chain role, $\frac{d}{dz} \operatorname{Log}\left(z + \sqrt{z^2-1}\right) = \frac{1}{z+\sqrt{z^2-1}} \left(1 + \frac{1}{2}\left(z^2-1\right)^{-1/2}\right) 2z = \frac{1}{z+\sqrt{z^2-1}} \left(1 + \frac{z}{\sqrt{z^2-1}}\right) = \frac{1}{\sqrt{z^2-1}}$, where we use everywhere the constant branch of $\sqrt{z^2-1}$, by ????? it on $(1, \infty)$ or using $\frac{d}{dz} z^a = \frac{a z^a}{z}$

c)

$\int_\gamma \frac{dz}{\sqrt{z^2-1}} = \left[\operatorname{Log}\left(z+\sqrt{z^2-1}\right)\right]_{-2i}^{2i} = \operatorname{Log}\left(2i+i\sqrt{5}\right) - \operatorname{Log}\left(-2i-i\sqrt{5}\right) = \pi i$
Use fact that $\sqrt{z^2-1}$ is positive imaginary on positive imaginary axis, and also $\operatorname{Log}\left(2i+i\sqrt{5}\right) - \operatorname{Log}\left(-2i-i\sqrt{5}\right)$?????. d)
$\int_\gamma \frac{dz}{\sqrt{z^2-1}} = \left[\operatorname{Log}\left(z+\sqrt{z^2-1}\right)\right]_{-2+i0}^{-2+i0}$
At $-2+0t$, have$\operatorname{Log}\left(z+\sqrt{z^2-1}\right) \to$
$\log\left|-2+\sqrt{3}\right| + i\operatorname{Arg}\left(z+\sqrt{z^2-1}\right)$ at $z = -2$. As $z \to -2+i0$, have $\arg\sqrt{z^2-1} \to +\pi$, $|z^2-1| \to 3$. $\therefore z+\sqrt{z^2-1} \to -2+(-1)\sqrt{3} = -2-\sqrt{3}$, $\arg\left(z+\sqrt{z^2-1}\right) \to \pi$. As $z \to -2+i0$, have $\arg\sqrt{z^2-1} \to -\pi$, $|z^2-1| \to 3$. $\therefore z+\sqrt{z^2-1} \to -2+(-1)\sqrt{3} = -2-\sqrt{3}$, $\arg\left(z+\sqrt{z^2-1}\right) \to -\pi$. Values coincide $\therefore$
$\int_\gamma \frac{dz}{\sqrt{z^2-1}} = \left[\operatorname{Log}\left(z+\sqrt{z^2-1}\right)\right]_{-2+i0}^{-2+i0} = \log|| - \log|| + \pi i - (-\pi i) = 2\pi i$

IV.2.5

1	2	3	P	L	K

Show that an analytic function $f(z)$ has a primitive in D if and only if $\int_\gamma f(z)\,dx = 0$ for every closed path γ in D.

Solution (A. Kumjian)

Let f be an analytic function defined on a domain D. Suppose first that f has a primitive in D. Then by the Fundamental Theorem of Calculus for Analytic Functions (Part I) we have $\int_\gamma f(z)\,dx = 0$ for every closed path $\gamma \in D$ (since for any such path $A = B$).

Conversely, suppose $\int_\gamma f(z)\,dx = 0$ for every closed path $\gamma \in D$. By formula (1.1) on p. 102, we have

$$\int_\gamma f(z)\,dx + if(z)\,dy = \int_\gamma f(z)\,dx = 0$$

for every close path $\gamma \in D$. It follows that the integral on the left is path independent for paths which are not necessarily closed and hence, by the Lemma on page 77, there is a continuously differentiable function F on D such that $dF = f dx + if dy$, that is, $\frac{\partial}{\partial x}F = f$ and $\frac{\partial}{\partial y}F = if$. It follows that the real and imaginary parts of F satisfy the Cauchy-Riemann equations:

$$\begin{aligned}
\frac{\partial}{\partial x}\operatorname{Re}F &= \operatorname{Re}\frac{\partial F}{\partial x} = \operatorname{Re}f = \operatorname{Im}if = \operatorname{Im}\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\operatorname{Im}F, \\
\frac{\partial}{\partial y}\operatorname{Re}F &= \operatorname{Re}\frac{\partial F}{\partial y} = \operatorname{Re}if = -\operatorname{Im}f = -\operatorname{Im}\frac{\partial F}{\partial x} = -\frac{\partial}{\partial x}\operatorname{Im}F.
\end{aligned}$$

Moreover, $F'(z) = f(z)$ for all $z \in D$. Hence, f has a primitive in D, namely F.

IV.3.1

1	2	3	P	L	K
				LLL	

By integrating $e^{-z^2/2}$ around a rectangle with vertices $\pm R$, $it \pm R$, and sending R to ∞, show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-itx} dx = e^{-t^2/2}, \qquad -\infty < t < \infty.$$

Use the known value of the integral for $t = 0$. *Remark.* This shows that $e^{-x^2/2}$ is an eigenfunction of the Fourier transform with eigenvalue 1. For more see the next exercise.

Solution

$\int_{\partial D} e^{-z^2/2} dz = \int_{-R}^{R} e^{-x^2/2} dx - \int_0^t e^{-(-R+iy)^2/2} i dy + \int_0^t e^{-(R+iy)^2/2} i dy + \int_{-R}^{R} e^{-(x+it)^2/2} dx = 0$

$\left| e^{-(\pm R+iy)^2/2} \right| \Leftrightarrow 0$, uniformly for $0 \leqslant y \leqslant 1$ as $R \to \infty$. $\sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-x^2/2} dx =$
$\int_{-\infty}^{\infty} e^{-(x+it)^2/2} dx = \int_{-\infty}^{\infty} e^{-\left(x^2+2ixt-t^2\right)/2} dx =$
$e^{t^2/2} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ixt} dx$
$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = e^{-t^2/2}$

IV.3.2

1	2	3	P	L	K

We define the Hermite polynomials $H_n(x)$ and Hermite orthogonal functions $\phi_n(x)$ for $n \geq 0$ by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right), \qquad \phi_n(x) = e^{-x^2/2} H_n(x).$$

(a)
Show that $H_n(x) = 2^n x^n + \cdots$ is a polynomial of degree n such that is even when n is even and odd when n is odd.
(b)
By integrating the function

$$e^{(z-it)^2/2} \frac{d^n}{dz^n}\left(e^{-z^2}\right)$$

around a rectangle with vertices $\pm R, it \pm R$ and sending R to ∞, show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_n(x) e^{-itx} dx = (-i)^n \phi_n(t), \qquad -\infty < t < \infty.$$

Hint. Use the identity from Exercise 1, and also justify and use the identity

$$\frac{d^n}{dx^n} e^{-(x+it)^2} = \frac{1}{i^n} \frac{d^n}{dt^n} e^{-(x+it)^2}.$$

(c)
Show that $\phi_n'' - x\phi_n + (2n+1)\phi_n = 0$.
(d)
Using $\int \phi_n'' \phi_m \, dx = \int \phi_n \phi_n'' \, dx$ and (c), show that

$$\int_{-\infty}^{\infty} \phi_n(x) \phi_m(x) \; dx = 0, \qquad n \neq m.$$

Remark. This shows that the ϕ_n's form an orthogonal system of eigenfuctions for the (normalized) Fourier transform operator $\mathcal{F}$ with eigenvalues ± 1 and $\pm i$. Thus $\mathcal{F}$ extends to a unitary operator on square-integrable functions. Further, $\mathcal{F}^4$is the identity operator, and the inverse Fourier transform is given by $(\mathcal{F}^{-1} f)(x) = (\mathcal{F} f)(-x)$.

Solution

Part A

$0=\int_{\partial D_R} e^{(z-it)^2/2}\frac{d^n}{dz^n}e^{-z^2}dz$

$\to \int_{-\infty}^{\infty} e^{(x-it)^2/2}\frac{d^n}{dx^n}e^{-x^2}dx=\int_{-\infty}^{\infty} e^{-t^2/2}e^{-itx}e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2}dx=$

$e^{-t^2/2}\int_{-\infty}^{\infty} e^{-itx}\left(-1\right)^n\phi_n\left(x\right)dx=e^{-t^2/2}\left(-1\right)^n\hat{\phi}_n\left(t\right)$

$\int_{(x+it)\cap\partial D_R}\to\int_{-\infty}^{\infty} e^{x^2/2}\frac{d^n}{dx^n}e^{-(x+it)^2}dx=\int_{-\infty}^{\infty} e^{x^2/2}\frac{d^n}{d(it)^n}e^{-(x+it)^2}dx=$

$\left(-i\right)\frac{d^n}{dt^n}\int_{-\infty}^{\infty} e^{x^2/2}e^{-(x+it)^2}dx=\left(-i\right)\frac{d^n}{dt^n}\int_{-\infty}^{\infty} e^{x^2/2}e^{-2itx}e^{t^2}dx=$

$\left(-i\right)\frac{d^n}{dt^n}e^{-t^2}\underbrace{\int_{-\infty}^{\infty} e^{-x^2/2}e^{-2itx}dx}_{\sqrt{2\pi}e^{-(2t)^2/2}}=\sqrt{2\pi}\left(-i\right)^n\frac{d^n}{dt^n}e^{-t^2}=\sqrt{2\pi}e^{-t^2/2}i^n\phi_n\left(t\right)$

If we set these equal, we get

$\left(-i\right)^n e^{-t^2/2}\hat{\phi}_n\left(t\right)=\sqrt{2\pi}e^{-t^2/2}i^n\phi_n\left(t\right),\ \frac{1}{\sqrt{2\pi}}\hat{\phi}_n\left(t\right)=\left(-i\right)^n\phi_n\left(t\right)$

Part B

$0=\int_{\partial D_R} e^{z^2/2}\frac{d^n}{dz^n}e^{-z^2}dz$

$\int_{-\infty}^{\infty} e^{x^2/2}\frac{d}{dx}e^{-z^2}dx=\int_{-\infty}^{\infty} e^{x^2/2}\frac{d^n}{dx^n}e^{-(x+it)^2}dx=\left(-i\right)^n\int_{-\infty}^{\infty} e^{z^2/2}\frac{d^n}{dt^n}e^{-(x+it)^2}dx$

$\int_{-\infty}^{\infty} e^{(x-it)^2/2}\frac{d}{dz^n}e^{-x^2}dx=\int_{-\infty}^{\infty} e^{x^2/2}\frac{d}{dz^n}e^{-(x+it)^2}dx=$

$e^{-t^2/2}\int_{-\infty}^{\infty} e^{-itx}\underbrace{e^{x^2/2}\frac{d}{dx^n}e^{-x^2}}_{(-1)^n\phi_n(x)}dx+\left(-i\right)^n\int_{-\infty}^{\infty} e^{x^2/2}\frac{d}{dx^n}e^{-(x+it)^2}dx$

$e^{-t^2/2}\hat{\phi}_n\left(t\right)=i^n\frac{d^n}{dt^n}e^{-t^2}+\left(-i\right)^n\frac{d}{dx^n}\left(e^{t^2}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-2ixt}dx\right)$

$\hat{\phi}_n\left(t\right)=\left(-i\right)^n\phi\left(t\right)+\left(-i\right)^n\sqrt{2\pi}\frac{d}{dx^n}\left(e^{t^2}e^{-2t^2}\right)=\left(-i\right)^n\sqrt{2\pi}\frac{d}{dx^n}e^{t^2}$

$\phi_n\left(x\right)=e^{-x^2/2}H_n\left(x\right),\ H_n\left(x\right)=\left(-1\right)^n e^{x^2}\left(\frac{d^n}{dx^n}\right)e^{-x^2}=$

$H_0\left(x\right)=1,\ H_1\left(x\right)=e^{-x^2}\frac{d}{dx}\left(e^{-x^2}\right)=-e^{-x^2}\left(-2x\right)e^{-x^2}=2x$

$H_2\left(x\right)=e^{-x^2}\frac{d^2}{dx^2}\left(e^{-x^2}\right)=e^{x^2}\left(4x^2e^{-x^2}-2e^{-x^2}\right)=4x^2-2$

$H_n\left(x\right)=2^nx^n+$ lower order polynomial. $\int_{-\infty}^{\infty}\phi_n\left(x\right)e^{-itx}dx=\int_{-\infty}^{\infty} e^{-x^2/2}\left(-1\right)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)e^{-ixt}d$

$\left(-1\right)^n\int_{-\infty}^{\infty} D_x^n\left(e^{x^2}\right)e^{x^2/2-ixt}dx=\int_{-\infty}^{\infty} e^{-x^2/2}D_x^n\left(e^{x^2/2-ixt}\right)dx=$

$\int\limits_{-\infty}^{\infty} e^{-x^2}D_x^n\left(e^{(x-it)^2/2}\right)e^{t^2/2}dx=\left(i\right)^n\int_{-\infty}^{\infty} e^{-x^2}D_t^n\left(e^{(x-it)^2/2}\right)e^{t^2/2}dx=$

$e^{t^2/2}i^nD_t^n\int_{-\infty}^{\infty} e^{-x^2}e^{(x-it)^2/2}dx=e^{t^2/2}i^nD_t^n\int_{-\infty}^{\infty} e^{-x^2/2}e^{-ixt}e^{-t^2/2}dx=$

$\sqrt{2\pi}e^{t^2/2}i^nD_t^ne^{-t^2/2}e^{-t^2/2}$

IV.3.3

1	2	3	P	L	K

Let $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a polynomial.
(a)
If the c_k's are real, show that

$$\int_{-1}^{1} f(x)^2\, dx \leq \pi \int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right| \frac{d\theta}{2\pi} = \pi \sum_{k=0}^{n} c_k^2.$$

Hint. For the first inequality, apply Cauchy's theorem th the function $f(z)^2$ separately on the top half an the bottom half of the unit disk.
(b)
If the c_k's are complex, show that

$$\int_{-1}^{1} |f(x)|^2\, dx \leq \pi \int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right| \frac{d\theta}{2\pi} = \pi \sum_{k=0}^{n} |c_k|^2.$$

(c)
Establish the following variant of Hilbert's inequality, that

$$\left|\sum_{j,k=0}^{n} \frac{c_j c_k}{j+k+1}\right| \leq \pi \sum_{k=0}^{n} |c_k|^2,$$

with strict inequality unless the complex numbers $c_0, \ldots, c_n$ are all zero. Hint. Start by evaluating $\int_0^1 f(x)^2\, dx$.

Solution
(a)
$\int\limits_{-1}^{1} f(x)^2\, dx = -\int\limits_{-\gamma_1} f(z)^2\, dz \leqslant \int\limits_{0}^{\pi} \left|f\left(e^{i\theta}\right)\right|^2 d\theta$
$\int_{-1}^{1} f(x)^2\, dx = -\int_{-\gamma_1} f(z)^2\, dz \leqslant \int_{\pi}^{2\pi} \left|f\left(e^{i\theta}\right)\right|^2 d\theta$
Add, get $2\int_{-1}^{1} f(x)^2\, dx = \leqslant \int_{\pi}^{2\pi} \left|f\left(e^{i\theta}\right)\right|^2 d\theta$
$\int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^2 d\theta = 2\pi \sum\limits_{k=0}^{n} c_k^2$, by from $\left|f\left(e^{i\theta}\right)\right|^2 = \left(\sum\limits_{k=0}^{n} c_k e^{ik\theta}\right)\left(\sum\limits_{k=0}^{n} c_m e^{-im\theta}\right)$
(b) $\left|f(x)^2\right| = \left|\sum a_k x^k + i \sum b_k x^k\right|^2 = \left(\sum a_k x^k\right)^2 + \left(\sum b_k x^k\right)^2$
$\int_{-1}^{1} \left|f(x)^2\right| dx = \int_{-1}^{1} \left(\sum a_k x^k\right)^2 dx + \int_{-1}^{1} \left(\sum b_k x^k\right)^2 dx = \pi\left(\sum a_k x^k + \sum b_k x^k\right) = \pi \sum c_k x^k$

$\pi\left[\int_{-1}^{1}\left|\sum a_k e^{ik\theta}\right|\frac{d\theta}{2\pi}+\int_{-1}^{1}\left|\sum b_k e^{ik\theta}\right|\frac{d\theta}{2\pi}\right]$

(c)$\int_0^1\left(\sum c_j x^j\right)^2 dx=\int_0^1\sum\limits_{j,k=0}^{n} c_j c_k x^{j+k}dx=\left[\sum\limits_{j,k=0}^{n} c_j c_k \frac{x^{j+k+1}}{j+k+1}\right]_0^1=\sum\limits_{j,k=0}^{n}\frac{c_j c_k}{j+k+1}$

$||\leqslant\int_0^1|f(x)|^2dx\leqslant\int_0^1|f(x)|^2dx\leqslant\pi\sum|c_k|^2$, for equality, must have $\int_0^1|f(x)|^2=\int_0^1|f(x)|^2$, so $f(x)=0$, for $-1\leqslant x\leqslant 0$, and then $f(x)\equiv 0$

Suppose $P(z)=a_0+a_1z+...+a_nz^n$ is a polynomial. By integrating $P(z)^2\log z$, for an appropriate branch of $\log z$, around a keyhole contour, show that $\left|\sum\limits_{j,k=0}^{n}\frac{a_j a_k}{j+k+1}\right|=\left|\int\limits_0^1 P(x)^2dx\right|\leqslant\pi\sum\limits_{k=0}^{n}|a_k|^2$. $\int\limits_{\partial D}P(z)^2\log z\,dz=0$, gives $\int_0^1 P(z)^2\log x\,dx-\int_0^1 P(z)^2(\log x+2\pi i)\,dx+\int_0^{2\pi}P\left(e^{i\theta}\right)^2\theta ie^{i\theta}\,d\theta=0$. Get $2\pi i\int_0^1 P(x)^2dx=i\int_0^{2\pi}P\left(e^{i\theta}\right)e^{i\theta}\theta d\theta$. Note that $\int P\left(e^{i\theta}\right)d\theta$, so latter integral is $i\int_0^{2\pi}P\left(e^{i\theta}\right)^2e^{i\theta}(\theta-\pi)\,d\theta$. Thus $\int_0^1 P(x)^2dx=\int_0^{2\pi}P\left(e^{i\theta}\right)e^{i\theta}(\theta-\pi)\frac{d\theta}{2\pi}$

$\therefore\left|\int_0^1 P(x)^2dx\right|\leqslant\pi\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^2\frac{d\theta}{2\pi}=\pi\sum\limits_{k=0}^{n}|a_k|^2$

IV.3.4

1	2	3	P	L	K

Prove that a polynomial in z without zeros is constant (the fundamental theorem of algebra) using Cauchy's theorem, along the following lines. If $P(z)$ is a polynomial that is not a constant, write $P(z) = P(0) + zQ(z)$, divide by $zP(z)$, and integrate around a large circle. this will lead to a contradiction if $P(z)$ has no zeros.

Solution

$P(z)$ is a polynomial with no zeros. Assume $P(0) = 1$. Write $p(z) = 1 + zQ(z)$.

$\oint_{|z|=R} \frac{1}{z} dz = \oint\limits_{|z|=R} \frac{P(z)}{zP(z)} dz = \oint\limits_{|z|=R} \left(\frac{1}{zP(z)} + \frac{Q(z)}{P(z)} \right) dz = \oint\limits_{|z|=R} \frac{1}{zP(z)} dz 2 = \pi i$

$\to 0$ by ML-estimate if $\deg P \geqslant 1$.

$M = \max\limits_{z=R} \left| \frac{1}{zP(z)} \right| \sim \frac{1}{R^{n+1}}$, $L \sim R$, when $n = \det P$

$ML \sim \frac{1}{R^n} \to 0$, if $R \to \infty$, if $n \geqslant 1$.

IV.3.5

1	2	3	P	L	K

Suppose that D is a bounded domain with piecewise smooth boundary, and that $f(z)$ is analytic on $D \cup \partial D$. Show that

$$\sup_{z\in\partial D} |\bar{z} - f(z)| \geq 2\frac{\text{Area}(D)}{\text{Length}(D)}.$$

Show that this estimate is sharp, and that in fact there exist D and $f(z)$ for which equality holds. Hint. Consider $\int_{\partial D} |\bar{z} - f(z)|\, dz$, and use Exercise 4 in section 1.

Solution

$\int\limits_{\partial D} [\bar{z} - f(z)]\, dz = \int\limits_{\partial D} \bar{z} dz = 2i\text{Area}(D)$

$\left|\int_{\partial D} [\bar{z} - f(z)]\, dz\right| \leqslant \sup\limits_{z\in\partial D} |\bar{z} - f(z)| \,\text{Length}(\partial D)$

$\therefore \sup\limits_{z\in\partial D} |\bar{z} - f(z)| \geqslant 2\frac{\text{Area}(D)}{\text{Length}(\partial D)}$

To see its sharp, take $D = D =$ unit disk, get $\sup\limits_{z\in\partial D} |\bar{z} - f(z)| = 1$

$2\frac{\text{Area}(D)}{\text{Length}(\partial D)} = 2 \cdot \frac{\pi}{2\pi} = 1$

IV.3.6

1	2	3	P	L	K

Suppose $f(z)$ is continuous in the closed disk $\{|z| \le R\}$ and analytic on the open disk $\{|z| < R\}$. Show that $\oint_{|z|=R} f(z)\,dz = 0$. *Hint.* Approximate $f(z)$ uniformly by $f_r(z) = f(rz)$.

Solution (A. Kumjian)

To prove $\oint_{|z|=R} f(z)\,dz = 0$ it suffices to show that $\left|\oint_{|z|=R} f(z)\,dz\right| \le \varepsilon$, for every $\varepsilon > 0$. So let $\varepsilon > 0$ be given. Since f is continuous in the closed disk $\{|z| \le R\}$, it must be uniformly continuous there (since the close disk is both closed and bounded). Hence, there is a $\delta > 0$ such that for all z_1, z_2 in the closed disk, we have

$$|f(z_1) - f(z_2)| < \frac{\varepsilon}{2\pi R} \qquad \text{if} \qquad |z_1 - z_2| < \delta.$$

For $r \in (0,1)$, define f_r on the disk $\{z : |z| < R/r\}$ by $f_r(z) = f(rz)$ for all z such that $|z| < R/r$. Then since f_r is analytic on a domain containing the closed disk $\{|z| \le R\}$, we have $\oint_{|z|=R} f(z)\,dz = 0$ by Cauchy's theorem. Now let r be chosen such that $1 - \delta/R < r < 1$. Then for all z with $|z| = R$ we have $|z - zr| = R(1-r) < \delta$ and so by the choice of δ we have

$$|f(z) - f_r(z)| = |f(z) - f(rz)| < \frac{\varepsilon}{2\pi R}.$$

Hence we have

$$\left|\oint_{|z|=R} f(z)\,dz\right| = \left|\oint_{|z|=R} (f(z) - f_r(z))\,dz\right| \le \oint_{|z|=R} |(f(z) - f_r(z))|\,|dz| \le \frac{\varepsilon}{2\pi R} 2\pi R = \varepsilon.$$

by the ML-estimate theorem on p. 105.

IV.4.1

1	2	3	P	L	K
111				LLL	

Evaluate the following integrals, using the Cauchy integral formula:

(a) $\oint_{|z|=2} \frac{z^n}{z-1} dz, \quad n \geq 0$ **(e)** $\oint_{|z|=1} \frac{e^z}{z^m} dz, \quad -\infty < m < \infty$

(b) $\oint_{|z|=1} \frac{z^n}{z-2} dz, \quad n \geq 0$ **(f)** $\oint_{|z-1-i|=5/4} \frac{\operatorname{Log} z}{(z-1)^2} dz$

(c) $\oint_{|z|=1} \frac{\sin z}{z} dz$ **(g)** $\oint_{|z|=1} \frac{dz}{z^2(z^2-4)e^z}$

(d) $\oint_{|z|=1} \frac{\cosh z}{z^3} dz$ **(h)** $\oint_{|z-1|=2} \frac{dz}{z^2(z^2-4)e^z}$

Solution

We rearragnde the Cauchy integral formula. Let D be bounded domain with piecewise smooth boundary ∂D and let f be an analytic function defined in a domain which contains $\bar{D}$. Then for any $z_0 \in D$ and positive integer m we have,

$$\int_{\partial D} \frac{f(z)}{(z-z_0)^m} dz = \frac{2\pi i}{(m-1)!} \left[\frac{d^m}{dz^m} f(z) \right]_{z=z_0}.$$

(a)

The nominator has one zero at $z = 1$ inside $|z| = 2$, thus by by Cauchy´s theorem,

$$\oint_{|z|=2} \frac{z^n}{z-1} dz = \frac{2\pi i}{0!} [z^n]_{z=1} = 2\pi i.$$

(b)

The nominator has no zeros inside $|z| = 1$, thus by by Cauchy´s theorem,

$$\oint_{|z|=1} \frac{z^n}{z-2} dz = 0.$$

(c)

The nominator has one zero at $z = 0$ inside $|z| = 1$, thus by by Cauchy´s theorem,

$$\oint_{|z|=1} \frac{\sin z}{z} dz = \frac{2\pi i}{0!} [\sin z]_{z=0} = 0.$$

(d)

The nominator has one tripel zero at $z = 0$ inside $|z| = 1$, thus by by Cauchy´s theorem,

$$\oint_{|z|=1} \frac{\cosh z}{z^3} dz = \frac{2\pi i}{2!} \left[\frac{d^2}{dz^2} \cosh z \right]_{z=0} = \pi i.$$

(e)
The nominator has one tripel zero at $z = 0$ inside $|z| = 1$, thus by by Cauchy´s theorem.
If $m \leq 0$, then the integrand $z^{-m}e^z$ defines an analytic function on $\mathbb{C}$, hence, by Cauchy's Theorem the integral must be zero.
If $m \geq 1$, the nominator has a zero of order m at $z = 0$ inside $|z| = 1$, hence by by Cauchy´s theorem.

$$\oint_{|z|=1} \frac{e^z}{z^m} dz = \frac{2\pi i}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} e^z \right]_{z=0} = \frac{2\pi i}{(m-1)!}.$$

Thus

$$\oint_{|z|=1} \frac{e^z}{z^m} dz = \begin{cases} 0 & \text{if } m \leq 0, \\ \frac{2\pi i}{(m-1)!} & \text{if } m \geq 1. \end{cases}$$

(f)
Let f be the principal branch of the logarithm defined on the slit plane, that is, $f(z) = \operatorname{Log} z$ for all $z \neq 0$ with $\operatorname{Arg} z \neq -\pi$. Then f is analytic on a domain $|z - 1 - i| \leq 5/4$. The nominator has one double zero at $z = 1$ inside $|z - 1 - i| < 5/4$, thus by by Cauchy´s theorem.

$$\oint_{|z-1-i|=5/4} \frac{\operatorname{Log} z}{(z-1)^2} dz = \frac{2\pi i}{1!} \left[\frac{d}{dz} (\operatorname{Log} z) \right]_{z=1} = 2\pi i.$$

(g)
The nominator has one double zero at $z = 0$ inside $|z| = 1$, thus by by Cauchy´s theorem,

$$\oint_{|z|=1} \frac{dz}{z^2 (z^2 - 4) e^z} = \frac{2\pi i}{1!} \left[\frac{d}{dz} \left(\frac{e^{-z}}{z^2 - 4} \right) \right]_{z=0} = \frac{\pi i}{2}.$$

(h)
The nominator has one zero at $z = 0$, and one zero at $z = 2$ inside $|z - 1| = 2$, thus by by Cauchy´s theorem,

$$\oint_{|z-1|=2} \frac{dz}{z(z^2-4)e^z} = \frac{2\pi i}{0!} \left[\frac{1}{(z^2-4)e^z} \right]_{z=0} + \frac{2\pi i}{0!} \left[\frac{1}{z(z+2)e^z} \right]_{z=2} = -\frac{\pi i}{2} + \frac{\pi i}{4e^2}.$$

IV.4.2

1	2	3	P	L	K
				LLL	

Show that a harmonic function is C^∞, that is, a harmonic function has partial derivatives of all orders.

Solution

Write $u = \operatorname{Re} f$, and note that $\frac{\partial}{\partial x} u = \operatorname{Re} f'$, $\frac{\partial}{\partial y} u = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \operatorname{Re}(f')$. $v = \operatorname{Im} f = \operatorname{Re}(-if)$, $\frac{\partial v}{\partial x} = \operatorname{Re}(-if)$. Now take successive derivatives, get $\frac{\partial^m u}{\partial x^p \partial y^q}$ a multiple of $\operatorname{Re} f'$ is $\operatorname{Re}(if')$.

IV.4.3

1	2	3	P	L	K
				LLL	

Use the Cauchy integral formula to derive the mean value property of harmonic functions, that

$$u(z_0) = \int_0^{2\pi} u\left(z_0 + \rho e^{i\theta}\right) \frac{d\theta}{2\pi}, \qquad z_0 \in D,$$

whenever $u(z)$ **is harmonic in a domain** D **and the closed disk** $|z - z_0| \le \rho$ **is contained in** D**.**

Solution

Write $f(z_0) = \oint_{|z-z_0|=\rho} \frac{f(z)}{z-z_0} dz = \oint_0^{2\pi} f\left(z_0 + \rho e^{i\theta}\right) \frac{d\theta}{2\pi}$, which is the MVP.

IV.4.4

1	2	3	P	L	K

Let D be a bounded domain with smooth boundary ∂D, and let $z_0 \in D$. Using the Cauchy integral formula, show that there is a constant C such that

$$|f(z_0)| \leq D \sup\{|f(z)| : z \in \partial D\}$$

for any function $f(z)$ analytic on $D \cup \partial D$. By applying this estimate to $f(z)^n$, taking nth roots, and letting $n \to \infty$, show that the estimate holds with $C = 1$. Remark. This provides an alternative proof of the maximum principle for analytic function.

Solution

Write $f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-z_0} dz$. Let$\|f\| = \max\limits_{z \in \partial D} |f(z)|$, $\frac{1}{|z-z_0|} \leqslant \frac{1}{dist(z_0, \partial D)} = \frac{1}{d}$. By ML–estimate, $L = \text{length}\, \partial D$, $|f(z_0)| \leqslant \frac{1}{2\pi d} \|f\| L$. Apply to f^n, get $|f(z_0)|^n \leqslant \frac{1}{2\pi d} \|f^n\| L$. Use $\|f^n\| = \|f\|^n$, take nth roots, get $|f(z_0)| \leqslant \|f\| \left(\frac{L}{2\pi d}\right)^{1/n}$. Let $n \to \infty$, get $|f(z_0)| \leqslant \|f\|$.

IV.5.1

1	2	3	P	L	K
111				LLL	

Denna behöver Nog fixas till Lite

Show that if u is a harmonic function on $\mathbb{C}$ that is bounded above, then u is constant.
Hint. **Express u as the real part of an analytic function, and exponentiate.**

Solution
Let u be the real part of the analytic function $f = u + iv$, where we know that $u \leq C$, for some constant C. Then set $g = e^{u+iv}$ where g is an entire function. Now we have that

$$|g| = \left|e^{u+iv}\right| = \left|e^{u}e^{iv}\right| = |e^{u}|\left|e^{iv}\right| = e^{u}.$$

We can se that becauce u is bounded above, so is g bounded above. Now we now that g is both analytic and bounded, by Liouville's theorem g is constant. Thus f is constant.

IV.5.2

1	2	3	P	L	K
111				LLL	

Show that if $f(z)$ is an entire function, and there is a nonempty disk such that $f(z)$ does not attain any values in the disk, then $f(z)$ is constant.

Solution

If $f(z)$ does not attain values in the disk $|w - c| < \varepsilon$, then $1/(f(z) - c)$ is a bounded entire function, hence constant by Liouville's theorem, and $f(z)$ is constant.

Let w_0 be the center of the disk that f omits. Then $1/(f(z) - w_0)$

IV.5.2

1	2	3	P	L	K
				LLL	

Show that if $f(z)$ is an entire function, and there is a nonempty disk such that $f(z)$ does not attain any values in the disk, then $f(z)$ is constant.

Solution (A. Kumjian)

Let f be such an entire function. Then there is a disc $D = D_r(z_0)$ for some $z_0 \in \mathbb{C}$ and $r > 0$ such that $f(z) \notin D$ for all $z \in \mathbb{C}$. That is, $|f(z) - z_0| \geq r$ for all $z \in \mathbb{C}$. We define an entire function g by the formula

$$g(z) = \frac{1}{f(z) - z_0} \text{ for all } z \in \mathbb{C}.$$

It follows that $|g(z)| \leq 1/r$ for all $z \in \mathbb{C}$. Hence, g is a bounded entire function an so by Liouville's Theorem g must be a constant function. The desired result now follows immediately.

If f does not attain values in the disk $|w-c|<\varepsilon$, then $1/(f-c)$ is bounded, hence constant by Liouville's theorem, and f is constant.

IV.5.3

1	2	3	P	L	K
				LLL	

A function $f(z)$ on the complex plane is double periodic if there are two periods w_0 and w_1 of $f(z)$ that do not lie on the same line thought the origin (that is, w_0 and w_1 are linearly independent over the reals, and $f(z+w_0) = f(z+w_1) = f(z)$ for all complex numbers z). Prove that the only entire functions that are doubly periodic are the constants.

Solution

The "lattice" of points mw_0+nw_1, $m, n \in \mathbb{Z}$ are the corners of parallelograms that cover $\mathbb{C}$, each parallelogram being a translate $F + mw_0 + nw_1$ of "the fundamental region" F as shown in the figure:

Figure IV.5.3

Thus each $\zeta \in \mathbb{C}$ can be written $\zeta = z + mw_0 + nw_1$ with $z \in F$. Since we assume $f(\zeta) = f(z)$, it follows that f i bounded, thus a constant.

If $|f(z)| \in M$ for $|z| \leqslant R$. Suppose $|w_0|$, $|w_0| \leqslant R$. Let $P =$ parallelogram with vertices $0,\ w_0,\ w_1,\ w_0 + w_1$, and suppose for $z \in P$. Any $z \in \mathbb{C}$ have form $\xi + mw_0 + nw_1$ where $\xi \in P$. Thus $|f(z)| = |f(\xi)| \leqslant M$, so $f(z)$ is bounded and entire.

IV.5.4

1	2	3	P	L	K

Suppose that $f(z)$ is an entire function such that $f(z)/z^n$ is bounded for $|z| \geq R$. Show that $f(z)$ is a polynomial of degree at most n. What can be said if $f(z)/z^n$ is bounded on the entire complex plane?

Solution

Apply the Cauchy estimates for $f^{(m+1)}(z)$ to disk $|z - z_0| < R$, and let $R \to \infty$, to obtain $f^{(m+1)}(z_0) = 0$. If $f(z)$ is a polynomial of degree $\leqslant m$ such that $f(z)/z^m$ is bounded near 0, then $f(z) = cz^m$.

IV.5.5

1	2	3	P	L	K

Show that if $V(z)$ is the velocity vector field for a fluid flow in the entire complex plane, and if the speed $|V(z)|$ is bounded, then $V(z)$ is a constant flow.

Solution

Use $V(z) = \overline{f'(z)}$ (section III.6 p92), when f is entire. Get f' constant, by Liouville's theorem, i.e. $V(z)$ is constant.

IV.6.1

1	2	3	P	L	K
				LLL	

Let L be a line in the complex plane. Suppose $f(z)$ is a continuous complex-valued function on a domain D that is analytic on $D \backslash L$. Show that $f(z)$ is analytic on D.

Solution

Rotate and apply result in text.

IV.6.2

1	2	3	P	L	K

Let $h(t)$ be a continuous function on the interval $[a,b]$. Show that the Fourier transform

$$H(z) = \int_a^b h(t)\, e^{-itz} dt$$

is an entire function that satisfies

$$|H(z)| \leq Ce^{A|y|}, \qquad z = x + iy \in \mathbb{C},$$

for some constants $A, C > 0$. Remark. An entire function satisfying such a growth restriction is called an entire function of finite type.

Solution

The analyticity of $H(z)$ follows from the theorem in the text. If $|h| \leqslant M$, take $C = M(b-a)$ and $A = \max(|a|, |b|)$. We have $|e^{-itz}| = e^{ty}$. $|H(z)| \leqslant M(b-a) \max\limits_{a \leq t \leq b} e^{ty} \leqslant M(b-a)\, e^{A|y|}$

IV.6.3

1	2	3	P	L	K

Let $h(t)$ be a continuous function on a subinterval $[a, b]$ of $[0, \infty)$. Show that the Fourier transform $H(z)$, defined as above, is bounded in the lower half-plane.

Solution

IV.6.4

1	2	3	P	L	K

Let γ be a smooth curve in the plane $\mathbb{R}^2$, let D be a domain in the complex plane, and let $P(x, y, \zeta)$ and $Q(x, y, \zeta)$ be continuous complex-valued functions defined for (x, y) on γ and $\zeta \in D$. Suppose that the functions depend analytically on ζ for each fixed (x, y) on γ.
Show that

$$G(\zeta) = \int_\gamma P(x, y, \zeta)\, dz + Q(x, y, \zeta)\, dy$$

is analytic on D.

Solution
Reduce to Riemann integral and apply theorem in the text.

IV.7.1

1	2	3	P	L	K

Find an application for Goursat's theorem in which it is not patently clear by other means that the function in question is analytic.

Solution

Answer to this is not known (by me).

IV.8.1

1	2	3	P	L	K
				LLL	

Show from the definition that
$\frac{\partial}{\partial z} z = 1, \qquad \frac{\partial}{\partial \bar{z}} z = 0, \qquad \frac{\partial}{\partial z} \bar{z} = 0, \qquad \frac{\partial}{\partial \bar{z}} \bar{z} = 1.$

Solution
We use the first-order differential operators (page 124)

$$\frac{\partial}{\partial z} = \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right] \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right].$$

We have that

$$\begin{aligned}
\frac{\partial}{\partial z} z &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy) = \frac{1}{2}(1+1) = 1 \\
\frac{\partial}{\partial \bar{z}} z &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x+iy) = \frac{1}{2}(1-1) = 0 \\
\frac{\partial}{\partial z} \bar{z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x-iy) = \frac{1}{2}(1-1) = 0 \\
\frac{\partial}{\partial \bar{z}} \bar{z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x-iy) = \frac{1}{2}(1+1) = 1
\end{aligned}$$

IV.8.2

1	2	3	P	L	K
				LLL	

Compute $\frac{\partial}{\partial \bar{z}}\left(az^2 + bz\bar{z} + c\bar{z}^2\right)$**. Use the result to determine where** $az^2 + bz\bar{z} + c\bar{z}^2$ **is complex-differentiable and where it is analytic. (See Problem II.2.3)**

Solution

By the Leibnitz rule and the results of Exercise 1, the $\bar{z}-$derivative is $bz+2c\bar{z}$. The function is complex differentiable at z if and only if $bz + 2c\bar{z} = 0$. If $b = c = 0$, the function is entire. Otherwise this locus is either a point $\{0\}$ or a straight line through the origin, and there is no open set on which the function is analytic.

IV.8.3

1	2	3	P	L	K
				LLL	

Show that the Jacobian of a smooth function f is given by

$$\det\, J_f = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2.$$

Solution

$J_f = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

$\left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2 = \frac{1}{4}\left|\frac{\partial(u+iv)}{\partial x} - i\frac{\partial(u+iv)}{\partial y}\right|^2 - \frac{1}{4}\left|\frac{\partial(u+iv)}{\partial x} + i\frac{\partial(u+iv)}{\partial y}\right|^2$

$= \frac{1}{4}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right] - \frac{1}{4}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right]$

$= \frac{1}{4}\left[2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right] = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = \det J_f$

IV.8.4

1	2	3	P	L	K
				LLL	

Show that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

Deduce the following, for a smooth complex-valued function h.
(a) h is harmonic if and only if $\partial^2 h/\partial z \partial \bar{z} = 0$.
(b) h is harmonic if and only if $\partial h/\partial z$ is conjugate-analytic.
(c) h is harmonic if and only if $\partial h/\partial \bar{z} = 0$.
(d) If h is harmonic, then any mth order partial derivative of h is a linear combination of $\partial^m h/\partial z^m$ and $\partial^m h/\partial \bar{z}^m$.

Solution
$4\frac{\partial^2}{\partial z \partial \bar{z}} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} - i\frac{\partial^2}{\partial x \partial y} - i\frac{\partial^2}{\partial y \partial x} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
(a) clear
(b) h harmonic $\Leftrightarrow \frac{\partial^2 h}{\partial \bar{z} \partial z} = 0 \Leftrightarrow \frac{\partial h}{\partial z}$ is analytic
(c) h harmonic $\Leftrightarrow \frac{\partial^2 h}{\partial z \partial \bar{z}} = 0 \Leftrightarrow \frac{\partial}{\partial z}\left(\frac{\partial h}{\partial \bar{z}}\right) = 0$. Since $\frac{\partial h}{\partial z} = \overline{\frac{\partial \bar{h}}{\partial \bar{z}}}$, this ????? $\frac{\partial \bar{h}}{\partial \bar{z}}$ is analytic $\Leftrightarrow \frac{\partial h}{\partial \bar{z}}$ is conjugate analytic.
(d) If h is analytic, then $\frac{\partial^{k+l} h}{\partial z^k \partial \bar{z}^l} = 0$ unless $l = 0$. Only non-vanishing derivative is $\frac{\partial^m h}{\partial z^m} = h^m(z)$. If h is conjugate analytic, ????? $\frac{\partial^{k+l} h}{\partial \bar{z}^k \partial z^l} = 0$ unless $l = 0$. If h is harmonic, then $h =$ analytic + conjugate analytic, so all derivatives of h vanish except for $\frac{\partial^k h}{\partial z^k} = \frac{\partial^l h}{\partial \bar{z}^k}$.

(f) $\int_{|z-1-i|=5/4} \frac{\operatorname{Log} z}{(z-1)^2} dz = \frac{2\pi i}{1!} \frac{d}{dz} \operatorname{Log} z\Big|_{z=1} = 2\pi i \left.\frac{1}{z}\right|_{z=1} = 2\pi i$

(g) $\int_{|z|=1} \frac{dz}{z^2(z^2-4)e^z} = 2\pi i \frac{d}{dz} \left.\frac{e^{-z}}{z^2-4}\right|_{z=0} = 2\pi i \left.\frac{\left(-e^{-z}\right)\left(z^2-4\right)-e^{-z}(2z)}{\left(z^2-4\right)^2}\right|_{z=0}$

$= \frac{2\pi i \cdot 4}{16} = \frac{\pi i}{2}$

(h) $\int_{|z-1|=2} \frac{dz}{z(z^2-4)e^z} = 2\pi i \left.\frac{1}{(z^2-4)e^z}\right|_{z=0} + 2\pi i \left.\frac{1}{(z+2)e^z}\right|_{z=2} =$

$2\pi i \left[-\frac{1}{4} + \frac{1}{8e^2}\right] = \pi i \left[-\frac{1}{2} + \frac{1}{4e^2}\right]$

IV.8.5

1	2	3	P	L	K
				LLL	

With $d\bar{z} = dx - idy$, show that for a smooth function $f(z)$ that

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}.$$

Solution

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)(dx + idy) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)(dx - idy) =$

$\frac{1}{2}\frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial f}{\partial y}dy + \frac{1}{2}\frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial f}{\partial y}dy + \frac{i}{2}\left[\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx - \frac{\partial f}{\partial x}dy + \frac{\partial f}{\partial y}dx\right] = df$

IV.8.6

1	2	3	P	L	K

Show that if D is a domain with smooth boundary, and if $f(z)$ and $g(z)$ are analytic on $D \cup \partial D$, then

$$\int_{\partial D} f(z)\,\overline{g(z)}\,dz = 2i \iint_D f(z)\,\overline{g'(z)}\,dx\,dy.$$

Compare this formula with Exercise 1.4.

Solution

By (8.4), $\int_{\partial D} f(z)\,\overline{g(z)}\,dz = 2i \iint_D \frac{\partial}{\partial E}(f\,\bar{g})\,dz = 2i \iint_D f g dz$

Using the Leibnitz rule, obtain $\frac{\partial}{\partial \bar{z}}(f\,\bar{g}) = f\frac{\partial \bar{g}}{\partial \bar{z}} + \bar{g}\frac{\partial f}{\partial \bar{z}} = f\overline{\frac{\partial g}{\partial z}} = f\overline{g'}$

To get Ex1.4, take $f = 1$, $g(z) = z$, get $\int_{\partial D} \bar{z}\,dz = 2i \iint_D dx dt = 2i\text{Area}$

IV.8.7

1	2	3	P	L	K
				LLL	

Show that the Taylor series expansion at $z_0 = 0$ of a smooth function $f(z)$, through the quadratic terms, is given by

$$f(z) = f(0)+\frac{\partial f}{\partial z}(0)\ z+\frac{\partial f}{\partial \bar{z}}(0)\ \bar{z}+\frac{1}{2}\left[\frac{\partial^2 f}{\partial z^2}(0)\, z^2 + 2\frac{\partial^2 f}{\partial z \partial \bar{z}}(0)\, |z|^2\right]+o\left(|z|^3\right).$$

Solution

Since a smooth function f is a linear combination of the functions $1, z, \bar{z}, z^2, \bar{z}^2, |z|^2$ and a reminder term $O\left(|z|^3\right)$, it suffices to check the formula for these six functions. The formula holds fro each of these. For instance, for $|z|^2 = z\bar{z}$ are harmonic. $\frac{\partial f}{\partial z}(0) = 0$, $\frac{\partial f}{\partial \bar{z}}(0) = 0$, $\frac{\partial^2 f}{\partial \bar{z}^2}(0) = 0$, $f(0) = 0$, $\frac{\partial^2 f}{\partial z \partial \bar{z}}(0) = 1$. The term $O\left(|z|^3\right)$ has Taylor series $0 + O\left(|z|^3\right)$. That ????? it.

IV.8.8

1	2	3	P	L	K
				LLL	

Establish the following version of the chain rule for smooth complex-valued functions $w = w(z)$ and $h = h(w)$.

$$\begin{aligned}
\frac{\partial}{\partial \bar z}(h \circ w) &= \frac{\partial h}{\partial w}\frac{\partial w}{\partial \bar z} + \frac{\partial h}{\partial \bar w}\frac{\partial \bar w}{\partial \bar z}, \\
\frac{\partial}{\partial z}(h \circ w) &= \frac{\partial h}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial h}{\partial \bar w}\frac{\partial \bar w}{\partial z}.
\end{aligned}$$

Solution

Suppose $w(0) = 0$, $h(0) = 0$, write $h(w) = aw + b\bar w + O\left(|w|^2\right)$, $a = \frac{\partial h}{\partial w}(0)$, $b = \frac{\partial h}{\partial \bar w}(0)$ and $w(z) = \alpha w + \beta \bar w + O\left(|z|^2\right)$, $\alpha = \frac{\partial w}{\partial z}(0)$, $b = \frac{\partial w}{\partial \bar z}(0)$ then $h(w(z)) = a\left(\alpha z + \beta \bar z + O\left(|z|^2\right)\right) + b\left(\bar\alpha \bar z + \bar\beta z + O\left(|z|^2\right)\right) + O\left(|w|^2\right) =$ $a\alpha z + a\beta \bar z + b\bar\alpha \bar z + b\bar\beta z + O\left(|z|^2\right) = \left(a\alpha + b\bar\beta\right) z + \left(a\beta + b\bar\alpha\right)\bar z + O\left(|z|^2\right)$

This gives,

$\frac{\partial h\circ w}{\partial \bar z}(0) = a\beta + b\bar\alpha = \frac{\partial h}{\partial w}\frac{\partial w}{\partial \bar z} + \frac{\partial h}{\partial \bar w}\overline{\frac{\partial w}{\partial z}}_{\partial \bar w/\partial \bar z} = \frac{\partial h}{\partial w}\frac{\partial w}{\partial \bar z} + \frac{\partial h}{\partial \bar w}\frac{\partial \bar w}{\partial \bar z}$

$\frac{\partial h\circ w}{\partial z}(0) = a\alpha + b\bar\beta = \frac{\partial h}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial h}{\partial \bar w}\underbrace{\overline{\frac{\partial w}{\partial \bar z}}}_{\partial \bar w/\partial z} = \frac{\partial h}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial h}{\partial \bar w}\frac{\partial \bar w}{\partial z}$

IV.8.9

1	2	3	P	L	K
				LLL	

Show with the aid of the preceding exercise that if both $h(w)$ and $w(z)$ are analytic, then $(h \circ w)(z)$ and $(h \circ w)'(z) = h'(w(z))\, w'(z)$.

Solution

If the functions h and w are analytic, then $\frac{\partial h \circ w}{\partial \bar{z}} = 0 + 0 = 0$ so $h \circ w$ is analytic. $(h \circ w)'(z) = \frac{\partial}{\partial z} h \circ w = \frac{\partial h}{\partial w} \frac{\partial w}{\partial z} + 0 = h'(w(z))\, w'(z)$

IV.8.10

1	2	3	P	L	K

Let $g(z)$ be a continuously differentiable function on the complex plane that is zero outside of som compact set. Show that

$$g(w) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{\partial g}{\partial \bar{z}} \frac{1}{z-w} dx\, dy, \qquad w \in \mathbb{C}.$$

Remark. If we integrate this formally by parts, we obtain

$$g(w) = \frac{1}{\pi}\iint_{\mathbb{C}} g(z) \frac{\partial}{\partial \bar{z}} \left(\frac{1}{z-w}\right) dx\, dy.$$

Thus the "distribution derivative" of $1/(\pi(z-w))$ with respect to z is the point mass w ("Dirac delta-function"), in the sense that it is equal to 0 away from w, and it is infinite at w in such a way that its integral (total mass) is equal to 1.

Solution

Apply Pompeeiu‘s formula to a large disk $\{|z| < R\}$.

V	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1																			
2																			
3																			
4																			
5																			
6																			
7																			
8																			

V.1.1

1	2	3	P	L	K

(Harmonic Series) Show that

$$\sum_{k=1}^{n} \frac{1}{k} \geq \log n.$$

Deduce that the series $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. *Hint.* Use the estimate

$$\frac{1}{k} \geq \int_{k}^{k+1} \frac{1}{x} dx.$$

Solution

Let k be a nonnegative integer. Then since the function $x \mapsto 1/x$ is decreasing on the positive reals, we have for all $x \in [k, k+1]$ that $1/x \leq 1/k$. Hence locking at the oversum in the interval $[k, k+1]$

$$\frac{1}{k}\left((k+1)-k\right) = \frac{1}{k} \geq \int_{k}^{k+1} \frac{1}{x} dx.$$

Therefore, for $n \geq 1$

$$\sum_{k=1}^{n} \frac{1}{k} \geq \sum_{k=1}^{n} \int_{k}^{k+1} \frac{1}{x} dx = \int_{1}^{n+1} \frac{1}{x} dx = \log(n+1) \geq \log n.$$

Where the last inequality follows from the fact that $x \longmapsto \log x$ is an increasing function on the positive reals. And thus is the sum $\sum_{k=1}^{n} \frac{1}{k}$ have not limit as $n \to \infty$ and does not converge.

V.1.2

1	2	3	P	L	K

Show that if $p < 1$, then the series $\sum_{k=1}^{\infty} 1/k^p$ diverges. Hint. Use Exercise 1 and the comparison test.

Solution

If $k > 1$ then, $k^p < k$ for $0 < p < 1$, then

$$\frac{1}{k^p} > \frac{1}{k}.$$

Thus we have

$$\sum_{k=1}^{\infty} \frac{1}{k^p} > \sum_{k=1}^{\infty} \frac{1}{k}.$$

By the comparison test, we have that the since the sum $\sum_{k=1}^{n} 1/k$ diverge as $n \to \infty$ by Exercise 1, also the sum $\sum_{k=1}^{n} 1/k^p$ diverge as $n \to \infty$, which was to be shown.

V.1.3

1	2	3	P	L	K

Show that if $p > 1$, then the series $\sum_{k=1}^{\infty} 1/k^p$ converges to S, where

$$\left| S - \sum_{k=1}^{n} \frac{1}{k^p} \right| < \frac{1}{(p-1)\, n^{p-1}}.$$

Hint. Use the estimate $\frac{1}{k^p} < \int_{k-1}^{k} \frac{dx}{x^p}$.

Solution

Let k be a nonnegative integer. Then since the function $x \mapsto 1/x$ is decreasing on the positive reals, we have for all $x \in [k-1, k]$ that $1/k \leq 1/x$. Becauce $p > 1$, it follows that $1/k^p \leq 1/x^p$. Hence locking att the undersum in the interval $[k-1, k]$

$$\frac{1}{k^p}(k-(k-1)) = \frac{1}{k^p} \leq \int_{k-1}^{k} \frac{1}{x^p} dx$$

Therefore

$$\sum_{k=1}^{n} \frac{1}{k^p} \leq \sum_{k=1}^{n} \int_{k-1}^{k} \frac{dx}{x^p} = 1 + \int_{1}^{n} \frac{dx}{x^p} = 1 + \left[\frac{x^{-p+1}}{-p+1} \right]_1^n = 1 + \frac{1}{p-1} - \frac{n^{-p+1}}{p-1} \leq 1 + \frac{1}{p-1}.$$

Hence the serie converge to S, where

$$S \leq 1 + \frac{1}{p-1} = \frac{p}{p-1},$$

since its terms are > 0.

Now we have that

$$\sum_{k=n+1}^{N} \frac{1}{k^p} \leqslant \int_{n}^{N} \frac{dx}{x^p} = 1 + \left[\frac{x^{-p+1}}{-p+1} \right]_n^N = \frac{n^{-p+1} - N^{-p+1}}{p-1} \to \frac{n^{-p+1}}{p-1}$$

as $N \to \infty$.

Thus we have

$$\sum_{k=n+1}^{N} \frac{1}{k^p} = S - \sum_{k=1}^{n} \frac{1}{k^p} \leqslant \frac{n^{-p+1}}{p-1} = \frac{1}{(p-1)\, n^{p-1}}$$

thus

$$\left| S - \sum_{k=n+1}^{N} \frac{1}{k^p} \right| \leq \frac{1}{(p-1)\, n^{p-1}}.$$

V.1.4

1	2	3	P	L	K

Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges. Hint. Show that the partial sums of the series satisfy $S_2 < S_4 < S_6 < \cdots < S_5 < S_3 < S_1$**.**

Solution

We define the sum S_n by

$$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

This is the "alternating series test", which is standard signs alternate, and terms $\to 0$, $|\text{terms}| \downarrow$, so series converge.

V.1.5

1	2	3	P	L	K

Show that the series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots$$

converges to $3S/2$, where S is the sum of the series in Exercise 4. (It turns out that $S=\log 2$.) Hint. Organize the terms in the series in Exercise 4 in groups of four, and relate it to the groups of three in the above series.

Solution

We have the series S from exercise V.1.5 and arrange the terms in groups of four

$$S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\ldots$$

And now we arrange the terms in groups of four

$$S=\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right]+\left[\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\right]+\left[\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}\right]+\ldots$$

Now we rearrange the terms in groups of two, and divide the terms by 2

$$\frac{S}{2}=\left[\frac{1}{2}-\frac{1}{4}\right]+\left[\frac{1}{6}-\frac{1}{8}\right]+\left[\frac{1}{10}-\frac{1}{12}\right]+\ldots$$

Now add one group of four from the sum S with a group of two from the sum $S/2$, and so on

$$\begin{aligned}\frac{3}{2}S &= \left[1+\frac{1}{3}-2\cdot\frac{1}{4}\right]+\left[\frac{1}{5}+\frac{1}{7}-2\cdot\frac{1}{8}\right]+\left[\frac{1}{9}+\frac{1}{11}-2\cdot\frac{1}{12}\right]+\ldots+= \\ &= \left[1+\frac{1}{3}-\frac{1}{2}\right]+\left[\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right]+\left[\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\right]+\ldots\end{aligned}$$

Since series S in Exercise 4 converge, thus also $3S/2$ converge by ......

V.1.6

1	2	3	P	L	K

Show that

$$\int_e^R \frac{1}{k\log k} = [\log(\log k)]_e^R = \log(\log R) - \log e\,(\log e) \to \infty$$

diverges while $\sum_{k=1}^{\infty} \frac{1}{k(\log k)^2}$ **converges.**

Solution 1 (Comparison Test)
For the first sum

$$\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k\log k}$$

there are 2^n terms between $k = 2^n+1$ and $k = 2^{n+1}$, each of the terms greater than $\frac{1}{2^{n+1}\log 2^{n+1}}$, so these 2^n terms have sum greater than $\frac{1}{2(n+1)\log 2}$. Thus series diverges, by comparison with the harmonic series. The other series is treated similarly, using an upper estimate.

For the other sum

$$\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k\,(\log k)^2}$$

there are 2^n terms between $k = 2^n + 1$ and $k = 2^{n+1}$, each of the terms less than $\frac{1}{2^{n+1}\log 2^{n+1}}$, so these 2^n terms have sum less than $\frac{1}{2(n+1)\log 2}$. Thus series converges, by comparison with the harmonic series.

Solution 2 (Integral Test)
Note that $x\log x$ and $x\,(\log x)^2$ are increasing for $x > 1$, so the terms of both series are positive and decreasing. Vi can use Cauchy's integralkriterium and look att them corresponding integrals. Becauce of difficult with the integrand we split the integral into two parts

$$\int_e^R \frac{1}{x\log x} = [\log(\log x)]_e^R = \log(\log R) - \log(\log e) \to \infty,$$

as $R \to \infty$.

$$\int_e^R \frac{1}{x(\log x)^2} = \left[-\frac{1}{\log x}\right]_e^R = \frac{1}{\log e} - \frac{1}{\log R} \to 1,$$

as $R \to \infty$.
We have that the series

$$\sum_{k=1}^{\infty} \frac{1}{k \log k}$$

diverges, while the series

$$\sum_{k=1}^{\infty} \frac{1}{k(\log k)^2}$$

konverges.

V.1.7

1	2	3	P	L	K

Show that the series $\sum a_k$ converges if and only if $\sum_{k=m}^{k=n} \frac{1}{k(\log k)^2}$ converges.

Solution

If S_n is the partial-sum $\sum_{k=1}^{n} a_k$, then $S_n - S_{m-1} = \sum_{k=m}^{n} a_k$. This follows immediately from $\sum_{k=m}^{n} a_k = S_n - S_{m-1}$, when $S_n = \sum_{k=1}^{n} a_k$ is the n^{th} partial-sum of the series.

V.2.1 $\frac{d}{dx}\frac{x^k}{k+x^{2k}} = \frac{kx^{k-1}\left(k+x^{2k}\right)-x^k(2k)x^{2k-1}}{\left(1+x^{2k}\right)^2} = 0$ at $k^2x^{k-1} + kx^{3k-1} = 2kx^{3k-1}$, $k^2x^{k-1} = k^{3k-1}$, $k = x^{2k}$, $\sqrt{k} = x^k$. $\therefore$ Function $\uparrow$ for $0 \leqslant x \leqslant k^{1/2k}$, $\downarrow$ for $x \geqslant k^{1/2k}$, $\to 0$ as $x \to \infty$. $\frac{x^k}{k+x^{2k}} \leqslant \frac{\sqrt{k}}{k+k} = \frac{1}{2\sqrt{k}}$ = worst-case estimation. $\therefore$ Converge to 0 uniformly.

V.2.1

1	2	3	P	L	K

Show that $f_k(x) = x^k / \left(k + x^k\right)$ converges uniformly to 0 on $[0, \infty)$.
Hint. Determine the worst-case estimator ε_k by calculus.
Solution
Suppose $f_k(x)$ converge pointwise to a function $f(x)$ as $k \to \infty$.
We have

$$f_k(x) = \frac{x^k}{k + x^{2k}},$$

thus

$$f_k'(x) = \frac{kx^{k-1}\left(k + x^{2k}\right) - x^k (2k) x^{2k-1}}{\left(k + x^{2k}\right)^2},$$

there $f_k'(x) = 0$ for $x_1 = 0$ and $x_2 = k^{1/(2k)}$.
We have

$$\begin{array}{lccccc} x & 0 & & k^{1/(2k)} & & \\ f_k'(x) & 0 & + & 0 & & - \\ f_k(x) & & \nearrow & & & \searrow \end{array}$$

and the function is increasing for $0 \le x \le k^{1/(2k)}$, and decreasing for $x \ge$ for $x \ge k^{1/2k}$, and $f_k(x) \to 0$ as $x \to \infty$. We have that $f_k(x)$ attains its maximum for $x = k^{1(2k)}$.
We determine $f(x)$ applying the works case estimator ε_k to the function $f_k(x)$,

$$f(x) = \lim_{k\to\infty} \frac{x^k}{k + x^{2k}} \le \lim_{k\to\infty} \frac{\sqrt{k}}{k + k} = \lim_{k\to\infty} \frac{1}{2\sqrt{k}} = 0.$$

The series $f_k(x) = x^k / \left(k + x^k\right)$ converge uniformly to 0 on $[0, \infty)$.

V.2.2

1	2	3	P	L	K

Show that $g_k = x^k/\left(1+x^k\right)$ converges pointwise on $[0,\infty)$ but not uniformly. What is the limit function? On which subsets of $[0,\infty)$ does the sequence converge uniformly?

Solution

Suppose $g_k(x)$ converge pointwise to a function $g(x)$ as $k \to \infty$, we start to show that is is an increasing function

$$g_k^{'}(x) = \frac{kx^{k-1} - x^k kx^{k-1}}{\left(1+x^k\right)^2} = \frac{kx^{k-1}}{\left(1+x^k\right)^2} > 0$$

also is the function $g_k(x)$ increasing, now we take the limit in three intervals. If $0 \le x < 1$, in this interval we have that $x^k \to 0$ as $k \to \infty$, and we have

$$g(x) = \lim_{k\to\infty} \frac{x^k}{1+x^k} = \frac{0}{1+0} = 0.$$

If $x = 1$, then

$$g(1) = \lim_{k\to\infty} \frac{1^k}{1+1^k} = \frac{1}{1+1} = \frac{1}{2}.$$

If $x > 1$, in this interval we have that $x^k \to 0$ as $k \to \infty$, and we have

$$g(x) = \lim_{k\to\infty} \frac{x^k}{1+x^k} = \lim_{k\to\infty} \frac{1}{1/x^k+1} = \frac{1}{0+1} = 1.$$

We also have that $g_k(x) \to g(x)$, where

$$g(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < 1, \\ \frac{1}{2} & \text{for} \quad x = 1, \\ 1 & \text{for} \quad x > 1 \end{cases}$$

Convergence is uniform on $[0, 1-\varepsilon]$ for any $\varepsilon > 0$. Not uniform on $[0,1]$, because limit function is not continuous at 1. Convergence is uniform on $[1+\varepsilon, +\infty]$ for any $\varepsilon > 0$.

V.2.3

1	2	3	P	L	K

Show that $f_k(z) = z^k/k$ converges uniformly for $|z| < 1$. Show that $f_k'(z)$ does not converge uniformly for $|z| < 1$. What can be said about uniform convergence of $f_k'(z)$?

Solution

Suppose $f_k(z)$ converge pointwise to a function $f(z)$ as $k \to \infty$ in the interval $|z| \leq 1$, in this domain we have that $\left|z^k/k\right| \leq 1/k$ thus

$$f(z) = \lim_{k\to\infty} \left|\frac{z^k}{k}\right| \leq \lim_{k\to\infty} \frac{1}{k} = 0.$$

Thus $f_k(z)$ converges uniformly for $|z| < 1$.
Now suppose $f_k'(z)$ converge pointwise to a function $f'(z)$ as $k \to \infty$ in the interval $|z| \leq 1$, now we must split the interval when we take the limit

$$f'(z) = \lim_{k\to\infty} \left|z^{k-1}\right| = \begin{cases} 0 & \text{for} \quad |z| < 1, \\ 1 & \text{for} \quad |z| = 1. \end{cases}$$

If $|z| = 1$, then $\left|f_k'(z)\right| = 1$ so the series can not converge uniformly for $|z| < 1$. But for any $\varepsilon > 0$ the series $f_k'(z) = z^{k-1}$ converges uniformly for $|z| \leq 1 - \varepsilon$.

V.2.4

1	2	3	P	L	K

Show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \frac{x^k}{1+x^{2k}}$$

converges uniformly for $-\infty < x < +\infty$**.**

Solution

We apply the Weierstrass $M-$test (p. 135.) with $M_k = \frac{1}{k^2}$, where the sum $\sum_{k=0}^{\infty} M_k$ converges. First we show that $\left|x^k\right| \leq \left|1+x^{2k}\right|$ for all $-\infty < x < +\infty$.

We have

$$\begin{array}{ll} \left|x^k\right| \leq 1 \leq 1+x^{2k} = \left|1+x^{2k}\right| & \text{for} \quad |x| \leq 1 \\ \left|x^k\right| \leq x^{2k} < 1+x^{2k} = \left|1+x^{2k}\right| & \text{for} \quad |x| > 1 \end{array}$$

This establishes the fact that

$$\left|\frac{x^k}{1+x^{2k}}\right| \leq 1$$

for all $-\infty < x < +\infty$.

Hence, for each $k \geq 1$ and $-\infty < x < +\infty$,

$$\left|\frac{1}{k^2} \frac{x^k}{1+x^{2k}}\right| \leq \frac{1}{k^2} = M_k.$$

Thus convergence is uniform on $-\infty < x < +\infty$ by the Weierstrass $M-$test.

V.2.5

1	2	3	P	L	K

For which real numbers x does $\sum \frac{1}{k}\frac{x^k}{1+x^{2k}}$ converge?

Solution

Suppose $g_k(x)$ converge pointwise to a function $g(x)$ as $k \to \infty$, and take limit in three intervals

If $x < 1$, in this interval we have that $x^k \to 0$ as $k \to \infty$, and we have

$$\sum_{k=1}^{\infty} \frac{1}{k}\frac{x^k}{1+x^{2k}} < \sum_{k=1}^{\infty} \frac{1}{1/x^k + x^k} < \sum_{k=1}^{\infty} \frac{1}{1/x^k} = \sum_{k=1}^{\infty} x^k$$

which is a convergent series.

If $x = 1$, then $x^k = x^{2k} = 1$ then

$$\sum_{k=1}^{\infty} \frac{1}{k}\frac{x^k}{1+x^{2k}} = \sum_{k=1}^{\infty} \frac{1}{2k}$$

which is a divergent series.

If $x > 1$, in this interval we have that $x^k \to \infty$ as $k \to \infty$, and we have

$$\sum_{k=1}^{\infty} \frac{1}{k}\frac{x^k}{1+x^{2k}} = \sum_{k=1}^{\infty} \frac{1}{k}\frac{1}{1/x^k + x^k} < \sum_{k=1}^{\infty} \frac{1}{x^k}$$

which is a convergent series.

V.2.6

1	2	3	P	L	K

Show that for each $\varepsilon > 0$, the series $\sum \frac{1}{k} \frac{x^k}{1+x^{2k}}$ converges uniformly for $x \geq 1 + \varepsilon$.

Solution

We have

$$f_k(x) = \frac{x^k}{1 + x^{2k}},$$

thus

$$f_k^{'}(x) = \frac{kx^{k-1}\left(1 + x^{2k}\right) - x^k (2k)\, x^{2k-1}}{\left(1 + x^{2k}\right)^2} = \frac{x^{k-1} - x^{3k-1}}{\left(1 + x^{2k}\right)^2}$$

there $f_k^{'}(x) = 0$ for $x_1 = 0$ and $x_2 = 1$.

We have

$$\begin{array}{llllll} x & 0 & & 1 & \\ f_k^{'}(x) & 0 & + & 0 & - \\ f_k(x) & & \nearrow & & \searrow \end{array}$$

and the function is increasing for $0 \leq x \leq 1$, and decreasing for $x \geq 1$, and $f_k(x) \to 0$ as $x \to \infty$. We have that $f_k(x)$ attains its maximum for $x = 1$. If $x = 1$, then $x^k = x^{2k} = 1$ then

$$\sum_{k=1}^{\infty} \frac{1}{k} \frac{x^k}{1 + x^{2k}} = \sum_{k=1}^{\infty} \frac{1}{2k}$$

which is a divergent series.

If $x > 1$, in this interval we have that $x^k \to \infty$ as $k \to \infty$, and we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k} \frac{x^k}{1 + x^{2k}} &= \sum_{k=1}^{\infty} \frac{1}{k} \frac{(1+\varepsilon)^k}{1 + (1+\varepsilon)^{2k}} = \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{1/(1+\varepsilon)^k + (1+\varepsilon)^k} \leq \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{(1+\varepsilon)^k} \end{aligned}$$

V.2.7

1	2	3	P	L	K
				LLL	

Let a_n be a bounded sequence of complex numbers. Show that for each $\varepsilon > 0$, the series $\sum_{n=1}^{\infty} a_n n^{-z}$ converges uniformly for $\operatorname{Re} z \geq 1 + \varepsilon$. Here we choose the principal branch of n^{-z}.

Solution

$\sum\limits_{n=1}^{\infty} a_n n^{-z}$, $|a_n| \leqslant C$, $\operatorname{Re} z \geqslant 1 + \varepsilon$. Apply Weierstrass M-test, $|a_n n^{-z}| \leqslant Cn^{-\operatorname{Re} z} \leqslant \frac{C}{n^{1+\varepsilon}} = M_n$, $\sum M_n < \infty \Rightarrow \sum a_n n^{-z}$ converges uniformly for $\operatorname{Re} z \geqslant 1 + \varepsilon$.

V.2.8

1	2	3	P	L	K
				LLL	

Show that $\sum \frac{z^k}{k^2}$ converges uniformly for $|z| < 1$.

Solution

Apply the Weierstrass $M-$ test with $M = 1/k^2$ in $|z| \le 1$, we find that

$$\sum_{k=1}^{\infty} \left| \frac{z^k}{k^2} \right| = \sum_{k=1}^{\infty} \frac{|z|^k}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2},$$

which is a convergent series by Weierstass $M-$ test. Thus the series converges uniformly for $|z| < 1$.

V.2.9

1	2	3	P	L	K
				LLL	

Show that $\sum \frac{z^k}{k}$ does not converges uniformly for $|z| < 1$.

Solution

Suppose that we have uniform convergence for $|z| < 1$, then the series must converge for $|z| \leq 1$

Apply the Weierstrass $M -$ test with $M = 1/k$ in $|z| \leq 1$, we find that

$$\sum_{k=1}^{\infty} \left| \frac{z^k}{k} \right| = \sum_{k=1}^{\infty} \frac{|z|^k}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k},$$

which is a divergent series by Weierstass $M -$ test. Thus the series do not converge uniformly for $|z| < 1$.

V.2.10

1	2	3	P	L	K

Shoe that if a sequence of functions $\{f_k(x)\}$ converges uniformly on E_j for $1 \leq j \leq n$, then the sequence converges uniformly on the union $E = E_1 \cup E_2 \cup \cdots \cup E_n$.

Solution

Use Cauchy criterion. Let $\varepsilon > 0$. Choose N_j such that $|f_n(x) - f_k(x)| < k$ for $m, k \geqslant N_j$ and $x \in E_j$. Let $N = \max\{N_1, \ldots, N_n\}$. Then $|f_m(x) - f_k(x)| < \varepsilon$ for $m, k \geqslant N$ and $x \in E_1 \cup \ldots \cup E_n$. $\therefore$ $\{f_m\}$ converges uniformly.

V.2.11

1	2	3	P	L	K

Suppose that E is a bounded subset of a domain $D \subset \mathbb{C}$ at a positive distance from the boundary of D, that is $\delta > 0$ such that $|z - w| \geq \delta$ for all $z \in E$ and $w \in \mathbb{C} \backslash D$. Show that E can be covered by a finite number of closed disks contained in D. Hint. Consider all closed disks with centers at points $(m + ni)\,\delta/10$ and radius $\delta/10$ that meet E.

Solution

Follow the hint. There are only finitely many of the disks in ????? (because E is bounded); and they cover E, because the collection of all disks centred at $(m + ni)\,\delta/10$ of radius $\delta/10$ covers the complex plan, and they are contained in D, because $dist\,(\varepsilon, \partial D) \geqslant \delta$.

V.2.12

1	2	3	P	L	K

Let $f(z)$ be analytic on a domain D, and suppose $|f(z)| \leq M$ for all $z \in D$. Show that for each $\delta > 0$ and $m \geq 1$, $\left|f^{(m)}(z)\right| \leq m!M/\delta^m$ for all $z \in D$ whose distance from ∂D is at least δ. Use this to show that if $\{f_k(z)\}$ is a sequence of analytic functions on D that converges to $f(z)$ on D, then for each m the derivatives $f_k^{(m)}(z)$ converge uniformly to $f^{(m)}(z)$ on each subset of D at a positive distance from δD.

Solution

Use the Cauchy estimates for $f^{(m)}(w)$ on a disk centred at w of radius $\delta < d(w, \partial D)$, get $\left|f^{(m)}(w)\right| \leqslant \frac{m!m}{\delta^m}$.

V.3.1

1	2	3	P	L	K
				LLL	

Find the radius of convergence of the following power series:

(a) $\sum_{k=0}^{\infty} 2^k z^k$ **(d)** $\sum_{k=0}^{\infty} \frac{3^k z^k}{4^k+5^k}$ **(g)** $\sum_{k=1}^{\infty} \frac{k^k}{1+2^k k^k} z^k$

(b) $\sum_{k=0}^{\infty} \frac{k}{6^k} z^k$ **(e)** $\sum_{k=1}^{\infty} \frac{2^k z^{2k}}{k^2+k}$ **(h)** $\sum_{k=3}^{\infty} (\log k)^{k/2} z^k$

(c) $\sum_{k=1}^{\infty} k^2 z^k$ **(f)** $\sum_{k=1}^{\infty} \frac{z^{2k}}{4^k k^k}$ **(i)** $\sum_{k=1}^{\infty} \frac{k! z^k}{k^k}$

Solution

(a)

We apply the ratio test, thus

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{2^k}{2^{k+1}} \right| = \lim_{k\to\infty} \frac{1}{2} = \frac{1}{2}.$$

(b)

We apply the ratio test, thus

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{k}{6^k} \frac{6^{k+1}}{k+1} \right| = \lim_{k\to\infty} \left| \frac{6k}{k+1} \right| = \lim_{k\to\infty} \frac{6}{1+1/k} = 6.$$

(c)

We apply the ratio test, thus

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{k^2}{(k+1)^2} \right| = \lim_{k\to\infty} \frac{k^2}{k^2+2k+1} = \lim_{k\to\infty} \frac{1}{1+2/k+1/k^2} = 1.$$

(d)

We apply the ratio test, thus

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{3^k}{4^k+5^k} \frac{4^{k+1}+5^{k+1}}{3^{k+1}} \right| =$$

$$= \lim_{k\to\infty} \left| \frac{3^k \cdot 5^{k+1} \left(\left(\frac{4}{5}\right)^{k+1} + 1 \right)}{3^{k+1} \cdot 5^k \left(\left(\frac{4}{5}\right)^{k} + 1 \right)} \right| = \lim_{k\to\infty} \left| \frac{5 \left(\left(\frac{4}{5}\right)^{k+1} + 1 \right)}{3 \left(\left(\frac{4}{5}\right)^{k} + 1 \right)} \right| = \frac{5}{3}.$$

(e)
We apply the ratio test, thus

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{2^k z^{2k}}{k^2+k}\frac{(k+1)^2+k+1}{2^{k+1}z^{2(k+1)}}\right| = \lim_{k\to\infty}\left(\frac{k+2}{k}\right)\frac{1}{2|z|^2} =$$
$$= \lim_{k\to\infty}\left(\frac{1+2/k}{1}\right)\frac{1}{2|z|^2} = \begin{cases} 0, & 2|z|^2<1 \\ \infty, & 2|z|^2>1 \end{cases}.$$

The convergence radius $R = 1/\sqrt{2}$.

(f)
We apply the ratio test, thus

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{z^{2k}}{4^k k^k}\frac{4^{k+1}(k+1)^{k+1}}{z^{2(k+1)}}\right| = \lim_{k\to\infty}\left|\frac{4(k+1)}{z^2}\frac{(k+1)^k}{k^k}\right| =$$
$$= \lim_{k\to\infty}\frac{4(k+1)}{|z|^2}\left(\frac{1+1/k}{1}\right)^k = \infty.$$

(g)
We apply the ratio test, thus

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\left|\frac{k^k}{1+2^k k^k}\frac{1+2^{k+1}(k+1)^{k+1}}{(k+1)^{k+1}}\right| = \lim_{k\to\infty}\left|\frac{k^k}{(k+1)^k}\frac{1+2^{k+1}(k+1)^{k+1}}{(1+2^k k^k)(k+1)}\right| =$$
$$= \lim_{k\to\infty}\left|\left(\frac{1/k}{1+1/k}\right)^k\left(\frac{1}{(1+2^k k^k)(k+1)} + \frac{2^{k+1}(k+1)^k}{1+2^k k^k}\right)\right| =$$
$$= \lim_{k\to\infty}\left|\left(\frac{1/k}{1+1/k}\right)^k\left(\frac{1}{(1+2^k k^k)(k+1)} + 2\left(\frac{k+1}{k}\right)^k\frac{1}{1/(2^k k^k)+1}\right)\right| = 2.$$

(h)
We apply the root test, thus

$$R = \frac{1}{\lim\limits_{k\to\infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim\limits_{k\to\infty} \sqrt[k]{(\log k)^{k/2}}} = \frac{1}{\lim\limits_{k\to\infty} \log k} = 0.$$

(i)
We apply the ratio test, thus

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{k!}{k^k} \frac{(k+1)^{k+1}}{(k+1)!} \right| = \lim_{k\to\infty} \left(\frac{k+1}{k} \right)^k = \lim_{k\to\infty} \left(1 + \frac{1}{k} \right)^k = e.$$

V.3.2

1	2	3	P	L	K
				LLL	

Determine for which z the following series converge.

(a) $\sum\limits_{k=1}^{\infty} (z-1)^k$ **(c)** $\sum\limits_{m=0}^{\infty} 2^m (z-2)^m$ **(e)** $\sum\limits_{n=1}^{\infty} n^n (z-3)^n$

(b) $\sum\limits_{k=0}^{\infty} \frac{(z-i)^k}{k!}$ **(d)** $\sum\limits_{m=0}^{\infty} \frac{(z+1)^m}{m^2}$ **(f)** $\sum\limits_{n=3}^{\infty} \frac{2^n}{n^2} (z-2-i)^n$

Solution

(a)
We apply the ratio test for $a_k = 1$, thus

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{1}{1} = 1.$$

hence the radius of convergence is $R = 1$. Therefore, the series converges for all z satisfying $|z-1| < 1$ and diverges for all z satisfying $|z-1| > 1$. Now consider z such that $|z-1| = 1$ then

$$|z-1| = 1.$$

Since $\sum_{k=1}^{\infty} 1$ diverges, the power series diverges for such z also. Hence, the given series converges for all z satisfying $|z-1| < 1$.

(b)
We apply the ratio test for $a_k = 1/k!$, thus

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{1}{k!} \frac{(k+1)!}{1} = \lim_{k\to\infty} (k+1) = \infty.$$

Therefore, the series converges for all z in $\mathbb{C}$.

(c)
We apply the ratio test for $a_m = 2^m$, thus

$$R = \lim_{m\to\infty} \left| \frac{a_m}{a_{m+1}} \right| = \lim_{m\to\infty} \frac{2^m}{2^{m+1}} = \lim_{m\to\infty} \frac{1}{2} = \frac{1}{2}.$$

hence the radius of convergence is $R = 1/2$. Therefore, the series converges for all z satisfying $|z-2| < 1/2$ and diverges for all z satisfying $|z-2| > 1/2$. Now consider z such that $|z-2| = 1/2$ then

$$\left|2^m (z-2)^m\right| = 2^m \left(\frac{1}{2}\right)^m = 1.$$

Since $\sum_{m=1}^{\infty} 1$ converges, the power series diverges for such z also. Hence, the given series converges for all z satisfying $|z-2| < 1/2$.

(d)
We apply the ratio test for $a_m = 1/m^2$, thus

$$R = \lim_{m\to\infty} \left|\frac{a_m}{a_{m+1}}\right| = \lim_{m\to\infty} \frac{1}{m^2} \frac{(m+1)^2}{1} = \lim_{m\to\infty} \frac{m^2+2m+1}{m^2} = \lim_{m\to\infty} \frac{1+2/m+1/m^2}{1} = 1.$$

hence the radius of convergence is $R = 1$. Therefore, the series converges for all z satisfying $|z+i| < 1$ and diverges for all z satisfying $|z+i| > 1$. Now consider z such that $|z+i| = 1$ then

$$\left|\frac{(z+i)^m}{m^2}\right| = \frac{1^m}{m^2} = \frac{1}{m^2}.$$

Since $\sum_{m=1}^{\infty} 1/m^2$ converges, the power series converges for such z also. Hence, the given series converges for all z satisfying $|z+i| \le 1$.

(e)
We apply the ratio test for $a_n = n^n$, thus

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n (n+1)} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} = 0.$$

Therefore, the series diverges for all z satisfying $|z-3| > 0$. Now consider z such that $|z-3| = 0$ then

$$\left|n^n (z-3)^n\right| = n^n \cdot 0^n = 0.$$

Since $\sum_{n=1}^{\infty} 0$ converges, the power series converges for $z = 3$. Hence, the given series only converges for $z = 3$.

(f)
We apply the ratio test for $a_k = 2^k/k^2$, thus

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{2^k}{k^2}\frac{(k+1)^2}{2^{k+1}} = \lim_{k\to\infty}\frac{k^2+2k+1}{2k^2} = \lim_{k\to\infty}\frac{1+2/k+1/k^2}{2} = \frac{1}{2}.$$

hence the radius of convergence is $R = 1/2$. Therefore, the series converges for all z satisfying $|z-2-i| < 1/2$ and diverges for all z satisfying $|z-2-i| > 1/2$. Now consider z such that $|z-2-i| = 1/2$ then

$$\left|\frac{2^n}{n^2}(z-2-i)^n\right| = \frac{2^n}{n^2}\left(\frac{1}{2}\right)^n = \frac{1}{n^2}.$$

Since $\sum_{k=1}^{\infty} 1/n^2$ converges, the power series converges for such z also. Hence, the given series converges for all z satisfying $|z-2-i| \leq 1/2$.

V.3.3

1	2	3	P	L	K
				LLL	

Find the radius of convergence of the following series.

(a) $\sum_{n=0}^{\infty} z^{3^n} = z + z^3 + z^9 + z^{27} + z^{81} + \cdots$

(b) $\sum_{p \text{ prime}} z^p = z^2 + z^3 + z^5 + z^7 + z^{11} + \cdots$

Solutions

(a)

(b)

$$R = \frac{1}{\limsup \sqrt[k]{|a_k|}} = \frac{1}{1} = 1.$$

Neither series converges at $z = 1$, so $R = 1$.

V.3.4

1	2	3	P	L	K

Show that the function defined by $f(z) = \sum z^{n!}$ is analytic on the open unit disk $\{|z| < 1\}$. Show that $|f(r\lambda)| \to +\infty$ as $r \to 1$ whenever λ is a root of unity. Remark. Thus $f(z)$ does not extend analytically to any larger open set than the open unit disk.

Solution

Cauchy-Hadamard formula gives $R = 1$ (radius of convergence) $\therefore$ $f(z)$ is analytic for $|z| < 1$. Suppose λ is an N^{th} root of unity, $\lambda^N = 1$. Then $(r\lambda)^{n!} = r^{n!}\left(\lambda^N\right)^{n!/N} = r^{n!}$ for $n \geqslant N$. Now $\sum^{\infty} r^{n!} \uparrow 1$ as $r \uparrow 1$, because it is increasing, and the finite partial sums $\to 1$ as $r \to 1$. $\therefore$ $|f(r\lambda)| \to +\infty$ as $r \to 1$.

V.3.5

1	2	3	P	L	K
				LLL	

What functions are represented by the following power series?

(a) $\sum\limits_{k=1}^{\infty} kz^k$ (b) $\sum\limits_{k=1}^{\infty} k^2 z^k$

Solution

(a)

Differentiate the geometric series $\sum_{k=0}^{\infty} z^k$, obtain

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \Rightarrow \sum_{k=1}^{\infty} kz^{k-1} = \frac{1}{(1-z)^2}.$$

Multiply by z obtain

$$\sum_{k=1}^{\infty} kz^k = \frac{z}{(1-z)^2}.$$

(b)

Differentiate the geometric series $\sum_{k=0}^{\infty} z^k$ twice, obtain

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \Rightarrow \sum_{k=1}^{\infty} kz^{k-1} = \frac{1}{(1-z)^2} \Rightarrow \sum_{k=2}^{\infty} k(k-1) z^{k-2} = \frac{2}{(1-z)^3}.$$

Multiply by z^2 obtain

$$\sum_{k=2}^{\infty} \left(k^2 - k\right) z^2 = \frac{2z^2}{(1-z)^3}.$$

We get

$$\sum_{k=2}^{\infty} k^2 z^k = \sum_{k=2}^{\infty} kz^k + \frac{2z^2}{(1-z)^3} = \frac{z}{(1-z)^2} - z + \frac{2z^2}{(1-z)^3}$$

V.3.6

1	2	3	P	L	K

Show that the series $\sum a_k z^k$, the differentiated series $\sum k a_k z^{k-1}$, and the integrated series $\sum \frac{a_k}{k+1} z^{k+1}$ all have the same radius of convergence.

Solution

Can use Cauchy-Hadamard formula, and $\sqrt[k]{k} = 1$. Then $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \limsup_{k\to\infty} \sqrt[k]{k\,|a_k|} = \limsup_{k\to\infty} \sqrt[k]{\frac{|a_k|}{k+1}}$, so all series have the same radius of convergence. Can also use the characterization of R as the radius of the largest disk to which the function extends and ?????. This largest disk is clearly the same for f , f', and ????.

V.3.7

1	2	3	P	L	K

Consider the series

$$\sum_{k=0}^{\infty}\left(2+(-1)^k\right)^k z^k.$$

Use the Cauchy-Hadamard formula to find the radius of convergence of the series. What happens when the ratio test is applied? Evaluate explicitly the sum of the series.

Solution

We apply the Cauchy-Hadamard formula where $a_k = \left(2+(-1)^k\right)^k$. First observe that

$$\sqrt[k]{|a_k|} = \sqrt[k]{\left(2+(-1)^k\right)^k} = 2+(-1)^k$$

Note that

$$2+(-1)^k = \begin{cases} 3 & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

Hence

$$\sqrt[k]{|a_k|} = \begin{cases} 3 & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

Hence, since the lim sup of this sequence is 3 we have by the Cauchy-Hadamard formula

$$R = \frac{1}{\limsup \sqrt[k]{|a_k|}} = \frac{1}{3}.$$

Applying the ratio test we have

$$\left|\frac{a_k}{a_{k+1}}\right| = \frac{\left(2+(-1)^k\right)^k}{\left(2+(-1)^{k+1}\right)^{k+1}} = \left\{ \begin{array}{ll} 3^k & \text{if } k \text{ is even,} \\ \frac{1}{3^{k+1}} & \text{if } k \text{ is odd.} \end{array} \right|$$

Since the sequence does not converge, the ratio test is inconclusive, in that $\lim_{k\to\infty} |a_k| / |a_{k+1}|$ does not exist. However since $|a_k| / |a_{k+1}| \leq 3$, it does converge for $|z| < 1/3$.
Then we have with $|z| < 1/3$

$$\sum_{k=0}^{\infty} \left(2 + (-1)^k\right)^k z^k =$$

$$= \sum_{k=0}^{\infty} \left(2 + (-1)^{2n}\right)^{2n} z^{2n} + \sum_{k=0}^{\infty} \left(2 + (-1)^{2n+1}\right)^{2n+1} z^{2n+1} =$$

$$= \sum_{n=0}^{\infty} 3^{2n} z^{2n} + \sum_{n=0}^{\infty} 1^{2n+1} z^{2n+1} = \sum_{n=0}^{\infty} \left(9z^2\right)^n + z \sum_{n=0}^{\infty} \left(z^2\right)^n =$$

$$= \frac{1}{1 - 9z^2} + \frac{z}{1 - z^2}.$$

Note: The series is convergent because $|9z^2|, |z^2| < 1$ since $|z| < 1/3$, and that the series have singularities at $z = \pm 1/3$ and $z = \pm 1$.

V.3.7

Let $L = \limsup \sqrt[k]{|a_k|}$. The definition of ${}^n \ln \sup^n$ implies that given $\varepsilon > 0$, $\exists > 0$, $\exists N 3$ for $k \geqslant N$ we have $\sqrt[k]{|a_k|} \leqslant L + \varepsilon$, but $\sqrt[k]{|a_k|} \geqslant L - \varepsilon$ for infinitely many $k's$. If $k \geqslant N$, then $|a_k| < (L+\varepsilon)^k$. If $|z| < 1/(L+\varepsilon)$, then $\sum a_k z^k$ converges. $(L+\varepsilon)|z| < 1$, $\left|\phi_k z^k\right| \leqslant (L+\varepsilon)^k |z|^k = ((L+\varepsilon)|z|)^k$ converges by comparison with geometric series. $\therefore R \geqslant 1/(L+\varepsilon) \Rightarrow |a_k| \geqslant (L-\varepsilon)^k$. If $|z| \geqslant L - \varepsilon$, then $\left|a_k z^k\right| \geqslant (L-\varepsilon)^k \cdot \frac{1}{(L-\varepsilon)^k}$ for infinitely many $k's$. $\therefore$, and $\sum a_k z^k$ diverge. $\therefore R \leqslant \frac{1}{L-\varepsilon}$. Let $\varepsilon \to 0$, get $R \leqslant \frac{1}{L}$.

V.3.8

1	2	3	P	L	K

Write out a proof of the Cauchy-Hadamard formula (3.4).

Solution

V.4.1

1	2	3	P	L	K
				LLL	

Find the radius of convergence of the power series for the following functions, expanded about the indicated point.

(a)	$\frac{1}{z-1}$ about $z = i$,	**(d)**	$\operatorname{Log} z$ about $z = 1 + 2i$,
(b)	$\frac{1}{\cos z}$ about $z = 0$,	**(e)**	$z^{3/2}$ about $z = 3$,
(c)	$\frac{1}{\cosh z}$ about $z = 0$,	**(f)**	$\frac{z-i}{z^3-z}$ about $z = 2i$.

Solution

(d)

Since the values of $\operatorname{Log} z$ are unbounded in any neighborhood of the origin it may not be extended to an analytic function on any open disk containing the origin. But $\operatorname{Log} z$ is defined and analytic on the disk of radius $|1 + 2i| = \sqrt{5}$ centered at $1 + 2i$. Hence, the desired radius of convergence is $\sqrt{5}$ by the second Corollary on p. 146.

(a) $R = \sqrt{2}$ (b) $R = \pi/2$ (c) $R = \pi/2$ (d) $R = \sqrt{5}$ (e) $R = 3$ (f) $R = 2$

V.4.2

1	2	3	P	L	K
				LLL	

Show that the radius of convergence of the power series expansion of $(z^2-1)/(z^3-1)$ about $z=2$ is $\sqrt{7}$.

Solution

Rewrite $f(z)$ as $\frac{z+1}{\left(z-e^{2\pi i/3}\right)\left(z+e^{2\pi i/3}\right)}$. Singularities of $f(z)$ are at $\pm e^{2\pi i/3}$, and distance from 2 to nearest singularity is $\sqrt{7}$.

V.4.3

1	2	3	P	L	K
				LLL	

Find the power series expansion of $\operatorname{Log} z$ about the point $z = i - 2$. Show that the radius of convergence of the series is $R = \sqrt{5}$. Explain why this does not contradict the discontinuity of $\operatorname{Log} z$ at $z = -2$.

Solution

(From Hints and Solutions)
$\operatorname{Log} z$ extends to be analytic for $|z - (i - 2)| < \sqrt{5}$, through the extension does not coincide with $\operatorname{Log} z$ in the part of the disk in the lower half–plane.

V.4.4

1	2	3	P	L	K
				LLL	

Suppose $f(z)$ is analytic at $z = 0$ and satisfies $f(z) = z + f(z)^2$. What is the radius of convergence of the power series expansion of $f(z)$ about $z = 0$?

Solution

(From Hints and Solutions)

Near 0 the function coincides with one of the branches of $\left(1 \pm \sqrt{1-4z}\right)/2$. The radius of convergence of the power series of either branch is $1/4$, which is the distance to the singularity at $1/4$.

V.4.5

1	2	3	P	L	K

Deduce the identity $e^{iz} = \cos z + i \sin z$ from the power series expansions.

Solution

$e^{iz} = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!} = \sum_{n \text{ even}} + \sum_{n \text{ odd}} = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{(2m+1)!}$, where $n = 2m$ in the first and $n = 2m + 1$ in the second series.

V.4.6

1	2	3	P	L	K
				LLL	

Find the power series expansions of $\cosh z$ and $\sinh z$ about $z = 0$. What are the radii of convergence of the series?

Solution

(a) $\cosh z = \cos iz = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots$

(b) $\sinh z = -i \sin(iz) = (-i)\left((iz) - \frac{(iz)^3}{3!} + \frac{(iz)^5}{5!} - \ldots\right) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots$
$= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

V.4.7

1	2	3	P	L	K

Find the power series expansion of the principal branch $\mathrm{Tan}^{-1}(z)$ of the inverse tangent function about $z = 0$. What is the radius of convergence of the series? Hint. Find it by integrating its derivative (a geometric series) term by term.

Solution

We have

$$\frac{d}{dz}\,\mathrm{Tan}^{-1} z = \frac{1}{1+z^2} = \sum_{k=0}^{\infty} \left(-z^2\right)^k,$$

thus

$$\mathrm{Tan}^{-1} z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{2k+1}.$$

And we have $R = 1$.

V.4.8

1	2	3	P	L	K

Expand $\mathrm{Log}\,(1+iz)$ **and** $\mathrm{Log}\,(1-iz)$ **in power series about** $z=0$**. By comparing power series expansions (see the preceding exercise), establish the identity**

$$\mathrm{Tan}^{-1} z = \frac{1}{2i}\,\mathrm{Log}\left(\frac{1+iz}{1-iz}\right).$$

(See Exercise 5 in section I.8)

Solution

$$\mathrm{Log}\,(1+iz) = -\sum_{k=0}^{\infty} (-i)^{\,k+1}\frac{z^{k+1}}{k+1}, \qquad \mathrm{Log}\,(1-iz) = -\sum_{k=0}^{\infty} i^{k+1}\frac{z^{k+1}}{k+1}.$$

Thus

$$\frac{1}{2i}\,\mathrm{Log}\,\frac{1+iz}{1-iz} = \frac{2i}{2i}\sum_{j=0}^{\infty} i^{2j}\,\frac{z^{2j+1}}{2j+1} = \sum_{j=0}^{\infty} (-1)^{j}\,\frac{z^{2j+1}}{2j+1}.$$

V.4.9

1	2	3	P	L	K

Let a be real, and consider the branch of z^a that is real and positive on $(0, \infty)$. Expand z^a in a power series about $z = 1$. What is the radius of convergence of the series? Write down the series explicitly.

Solution

$f(z) = z^a = e^{a \operatorname{Log} z}$, $\operatorname{Re} z > 0$. Assume $a \neq 0, 1, 2, \ldots$. $f^{(m)}(z) = a(a-1)\ldots(a-m+1) z^{a-m}$, $f^{(m)}(1) = a(a-1)\ldots(a-m+1)$

$f(z) = \sum_{m=0}^{\infty} a_m (z-1)^m$, $a_m = \frac{f^{(m)}(1)}{m!} = \frac{a(a-1)\ldots(a-m+1)}{m!} = \binom{a}{m}$, $m \geqslant 1$, and $a_0 = a_1 = \binom{a}{0}$. $f(z) = \sum_{m=0}^{\infty} \binom{a}{m} (z-1)^m$. $R = 1$ since $\left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{a-m}{m+1} \right| \to 1$ as $m \to \infty$. If a is an integer $\geqslant 0$ get $R = \infty$.

V.4.10 (From Hints and Solutions)

1	2	3	P	L	K

Recall that for a complex number α, the binomial coefficient "αchoose n" is defined by

$$\binom{\alpha}{0} = 1, \qquad \text{and} \qquad \binom{\alpha}{n} = \frac{\alpha\,(\alpha-1)\cdots(\alpha-n+1)}{n!}, \qquad n \geq 1.$$

Find the radius of convergence of the binomial series

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n.$$

Show that the binomial series represents the principal branch of the function $(1+z)^\alpha$. For which α does the binomial series reduce to a polynomial?

Solution

Use $f^{(n)} = \alpha\,(\alpha-1)\ldots(\alpha-n+1)\,(1+z)^{\alpha-n}$ and the formula for the coefficient of z^n. The series reduces to a polynomial for $\alpha = 0, 1, 2, \ldots$. Otherwise radius of convergence is 1, which is distance to the singularity at -1. We can obtain the radius of convergence for the series also from the ratio test, $\frac{a_n}{a_{n+1}} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{\alpha(\alpha-1)\ldots(\alpha-n+1)(\alpha-n)} \frac{(n+1)!}{n!} = \frac{n+1}{\alpha-n} \to -1$ as $n \to \infty$.

V.4.11

1	2	3	P	L	K

For fixed $n \geq 0$, define the function $J_n(z)$ by the power series

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{n+2k}}{k!\,(n+k)!2^{n+2k}}.$$

Show that $J_n(z)$ is an entire function. Show that $w = J_n(z)$ satisfies the differential equation

$$w'' + \frac{1}{z}w' + \left(1 - \frac{n^2}{z^2}\right) w = 0.$$

Remark. This is Bessel's differential equation, and $J_n(z)$ is Bessel's function of order n.

Solution

$J_n(z) = \sum\limits_{k=0}^{\infty} \frac{(-1)^k z^{n+2k}}{k!(n+k)!2^{n+2k}} = \frac{z^n}{n!2^n} + O(z^{n+2})$

$J_n'(z) = \sum\limits_{k=0}^{\infty} \frac{(-1)^k (n+2k) z^{n+2k-1}}{k!(n+k)!2^{n+2k}} = \frac{z^{n-1}}{(n-1)!2^n} + O(z^{n+2})$

$J_n''(z) = \sum\limits_{k=0}^{\infty} \frac{(-1)^k (n+2k)(n+2k-1) z^{n+2k-2}}{k!(n+k)!2^{n+2k}} = \frac{z^{n-2}}{(n-2)!2^n} + O(z^n)$

Term multiplying z^{n+2k} in $z^2 J_n'' + zJ_n' + (z^2 - n^2) J_n$ is $\frac{(-1)^k (n+2k)(n+2k-1)}{k!(n+k)!2^{n+2k}} + \frac{(-1)^k (n+2k)}{k!(n+k)!2^{n+2k}} - \frac{(-1)^k n^2}{k!(n+k)!2^{n+2k}} - \frac{(-1)^k}{(k-1)!(n+k-1)!2^{n+2k-2}} =$

$\frac{(-1)^k}{k!(n+k)!2^{n+2k}} \left[(n+2k)(n+2k-1) + \underset{=0}{(n+2k)} - n^2 - 4k(n+k) \right]$

Should cheek separately to the sum terms of $J_0 + J_1$, when we divide by 0 above, ???? work out. (constant term ????? J_0, z term for J_1). It works, because the case $k = 0$ we replace $1/(k-1)!$ by $k/k! = 0$. Ratio test gives radius of convergence $= \infty$. (It's not clear).

1

V.4.12

1	2	3	P	L	K
				LLL	

Suppose that the analytic function $f(z)$ has power series expansion $\sum a_n z^n$. Show that if $f(z)$ is an even function, then $a_n = 0$ for n odd. Show that if $f(z)$ is an odd function, then $a_n = 0$ for n even.

Solution

If $f(z)$ is analytic in $D_r(0)$ then $f(-z)$ is also analytic in the same region and $f(-z) = \sum_{n=0}^{\infty} (-1)^n a_n z^n$. If f is even then $f(z) - f(-z) = 0$, and so $\sum_{n=0}^{\infty} 2a_{2n+1} z^{2n+1} = 0$. A power series is 0 iff all of its contents are 0 (since $a_n = f^{(n)}(0)/n!$ which shows $a_n = 0$ for n odd. The result for $f(z)$ odd is analogous.

$f(z)$ even, $f(-z) = f(z)$. Chain rule: $f'(-z) = -f'(z)$, $f''(-z) = f''(z)$, etc. $f^{(k)}(-z) = (-1)^k f^{(k)}(z)$. $f^{(k)}(0) = (-1)^k f^{(k)}(0) \Rightarrow f^{(k)}(0) = 0$ for k odd. $\therefore a_k = f^{(k)}(0)/k!$ for k odd. If $f(z)$ is odd, $f(-z) = -f(z)$, same argument gives $a_k = 0$ for k even. Alternatively, apply ever result to $zf(z)$ which is even.

V.4.13

1	2	3	P	L	K

Prove the following version of L'Hospitals's rule. If $f(z)$ and $g(z)$ are analytic, $f(z_0) = g(z_0) = 0$, and $g(z)$ is not identically zero, then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)},$$

in the sense that either both limits are finite and equal, or both limits are infinite.

Solution

Assume $z_0 = 0$, $f(z) = \sum\limits_{k=1}^{\infty} a_k z^k = f'(0)\,z + O(z^2)$, $g(z) = g'(0)\,z + O(z^2) = g'(0)\,z + a_N z^n + O\left(z^{N+1}\right)$. $\frac{f(z)}{g(z)} \to \frac{f'(0)}{g'(0)}$ if $g'(0) \neq 0 \to \begin{cases} \infty & \text{if } g(0) = 0 \text{ and } f(0) \neq 0 \text{ or } k < N \\ \frac{a_N}{b_N} & \text{if } g(0) = 0\ ,\ f(0) \neq 0,\ k = N \\ 0 & \text{if } g(0) = 0\ ,\ f(0) \neq 0,\ k > N \end{cases}$

$\frac{f'(z)}{g'(z)} = \frac{f'(0)+ka_k z^{k-1}+O\left(z^k\right)}{g'(0)+Nb_N z^{N-1}+O(z^N)}$, $\frac{f'(z)}{g'(z)} \to \frac{f'(0)}{g'(0)}$ if $g'(0) \neq 0$

$\to \begin{cases} \infty & g(0) = 0 \text{ and either } f(0) \neq 0 \text{ or } k < N \\ \frac{Na_N}{Nb_N} = \frac{a_N}{b_N} & g(0) = 0\ ,\ f(0) \neq 0,\ k = N \\ 0 & g(0) = 0\ ,\ f(0) \neq 0,\ k > N \end{cases}$

Then the limits are equal in all cases.

V.4.14

1	2	3	P	L	K

Let f be continuous function on the unit circle $T = \{|z| = 1\}$. Show that f can be approximated uniformly on T by a sequence of polynomials in z if and only if f has an extension F that is continuous on the closed disk $\{|z| \leq 1\}$ and analytic on the interior $\{|z| < 1\}$. Hint. To approximate such an F, consider dilates $F_r(z) = F(rz)$.

Solution

If f has an analytic extension F , then $F_r(z) = F(rz)$ has power series $F(z) = \sum a_n z^k$, $|z| < 1$. $F_r(z) = \sum\limits_{n=j}^{\infty} a_n r^n z^k$, $|z| < \frac{1}{r}$. F_r converge uniformly for $|z| \leqslant 1$, to $F(rz)$ and $F(rz) \to f(z)$ uniformly for $|z| = 1$ as $v \uparrow 1$. For $\varepsilon > 0$, take $r < 1$ with $|F_r(z) - F(z)| < \varepsilon$ for $|z| = 1$, then take N such that $\left|\sum\limits_{n=0}^{N} a_n r^n z^n - F_v(z)\right| \leqslant \frac{\varepsilon}{2}$ for $|z| = 1$. Then $\left|\sum\limits_{n=0}^{N} a_n r^n z^n - F(z)\right| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $|z| = 1$. Conversely if p_n are polynomials, $p_n \to f$ uniformly for $|z| = 1$, then $|p_n - p_m| \to 0$ uniformly for $|z| = 1$ as $n, m \to \infty$, so by the maximum principle, $|p_n - p_m| \to 0$ uniformly for $|z| \leqslant 1$. $\therefore$ $\{p_n\}$ converges uniformly to same function $F(z)$ for $|z| \leqslant 1$. Since $p_n(z) \to f(z)$ for $|z| = 1$, $F(z) = f(z)$ for $|z| = 1$. Since p_n is analytic for $|z| < 1$, also $F(z)$ is analytic for $|z| < 1$.

V.5.1

1	2	3	P	L	K
				LLL	

Expand the following functions in power series about ∞

(a) $\frac{1}{z^2+1}$ (b) $\frac{z^2}{z^3-1}$ (c) e^{1/z^2} (d) $z \sinh(1/z)$

Solution

(b)

Denote the above function by f and define g by

$$g(w) = f(1/w) = \frac{(1/w)^2}{(1/w)^3 - 1} = \frac{w}{1-w^3} \qquad \text{for } w \neq 0$$

and $g(0) = 0$. Note that g is analytic on the disk $D_1(0)$ with power series representation

$$g(w) = \frac{w}{1-w^3} = w \sum_{k=0}^{\infty} \left(w^3\right)^n = \sum_{k=0}^{\infty} w^{3n+1}$$

for all $w \in D_1(0)$ (note that $|w^3| \leq 1$ if $|w| \leq 1$). Hence, we have

$$f(z) = g(1/z) = \sum_{k=0}^{\infty} (1/z)^{3n+1} = \sum_{k=0}^{\infty} \frac{1}{z^{3n+1}}$$

for all z with $|z| > 1$.

(a) $\frac{1}{z^2+1} = 1 - z^2 + z^4 - \ldots = \sum\limits_{n=0}^{\infty} (-1)^n z^{2n} \frac{1}{z^2} \frac{1}{1+1/z^2} = \frac{1}{z^2} \frac{(-1)^n}{z^{2n}} =$
$\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}}$

(b) $\frac{z^2}{z^3-1} = \frac{1}{z} \frac{1}{1-1/z^3} = \frac{1}{z} \sum\limits_{n=0}^{\infty} \frac{1}{z^{3n}} = \sum\limits_{n=0}^{\infty} \frac{1}{z^{3n+1}}$

(c) $e^{1/z^2} = \sum\limits_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^{2n}}$

(d) $z \sinh \frac{1}{z} = z \left(\frac{1}{z} + \frac{1}{3! z^3} + \frac{1}{5! z^5} + \ldots \right) = \sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{1}{z^{2n}}$

V.5.2

1	2	3	P	L	K

Suppose $f(z)$ is analytic at ∞, with series expansion (5.1). With the notation $f(\infty) = b_0$ and $f'(\infty) = b_1$, show that

$$f'(\infty) = \lim_{z\to\infty} z\,|f(z) - f(\infty)|\,.$$

Solution

$f(z) = \sum\limits_{k=0}^{\infty} \frac{b_k}{z^k} = f(\infty) + \frac{f'(\infty)}{z} + \sum\limits_{k=2}^{\infty} \frac{b_k}{z^k}$. $z\,[f(z) - f(\infty)] = f'(\infty) + \sum\limits_{k=1}^{\infty} \frac{b_{k+1}}{z^k} \to f'(\infty)$ as $z \to \infty$, since $\sum\limits_{k=1}^{\infty} b_{k+1} w^k \to 0$ as $w \to 0$.

V.5.3

1	2	3	P	L	K

Suppose $f(z)$ is analytic at ∞, with series expansion (5.1). Let $\sigma \geq 0$ be the smallest number such that $f(z)$ extends to be analytic for $|z| > \sigma$. Show that the series (5.1) converges absolutely for $|z| > \sigma$ and diverges for $|z| < \sigma$.

Solution (A. Kumjian)

Recall that f is analytic at ∞ iff there is a function at 0 such that $f(z) = g(w)$ with $w = 1/z$ wherever $f(z)$ makes sense. We may suppose that $f(z)$ is defined for $|z| > \sigma$ and that f is analytic at these points. Then setting $R = 1/\sigma$ if $\sigma > 0$ and $R = \infty$ if $\sigma = 0$, we have that g is analytic on $D_R(z_0)$ and cannot be extended to an analytic function on any larger open disk centered at z_0. Hence, g has a power series representation

$$g(w) = \sum_{k=0}^{\infty} b_k w^k \qquad \text{for } |w| < R,$$

with radius of convergence R. Thus, the series converges absolutely for $|w| < R$ and diverges for $|w| > R$. It follows that by substituting $w = 1/z$, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{b_k}{z^k} \qquad \text{for } |z| > \sigma,$$

where the series converges absolutely for $|z| > \sigma$ and diverges for $|z| < \sigma$.

V.5.4 (From Hints and Solutions)

1	2	3	P	L	K

Let E be a bounded subset of the complex plane $\mathbb{C}$ over which area integrals can be defined, and set

$$f(w) = \iint_E \frac{dx\,dy}{w-z}, \qquad w \in \mathbb{C}\backslash E,$$

where $z = x + iy$. Show that $f(w)$ is analytic at ∞, and find a formula for the coefficients of the power series of $f(w)$ at ∞ in descending powers of w. Hint. Use a geometric series expansion.

Solution

If $|z| \leqslant M$ for $z \in E$, and $R > M$, then $1/(w-z) = \sum z^n/w^{n+1}$ converges uniformly for $z \in E$ and $|w| > R$. Integrate term by term, obtain $f(w) = \sum\limits_{N=0}^{\infty} \frac{b_n}{w^{n+1}}$, $|w| > R$, where $b_n = \iint\limits_E z^n dxdy$.

V.5.5 (From Hints and Solutions)

1	2	3	P	L	K

Determine explicitly the function $f(w)$ defined in Exercise 4, in the case that $E = \{|w| \leq 1\}$ is the unit disk. Hint. There are two formulae for $f(w)$, one valid for $|w| \geq 1$ and the other for $|w| \leq 1$. Be sure thy agree for $|w| = 1$.

Solution

To find the formula for $|w| < 1$, break the integral into two pieces corresponding to $|z| > |w|$ and to $|z| < |w|$, and use geometric series. $f(w) = \iint\limits_D \frac{dxdy}{w-z} = \underbrace{\iint\limits_{|z|<|w|} \frac{dxdy}{w-z}}_{(1)} + \underbrace{\iint\limits_{|z|>|w|} \frac{dxdy}{w-z}}_{(2)} = (1)+(2)$. If $|w| < 1$, get $\iint z^{n-1}dxdy = \iint r^n drd\theta e^{i(n-1)\theta} = 0$, $f(w) = \iint \frac{1}{w-z}dxdy = \frac{\pi}{w}$, $|w| > 1$.

If $|w| < 1$, get (1) $\frac{1}{w} \iint\limits_{|z|<|w|} \frac{dxdy}{1-z/w} = \frac{1}{w} \sum \iint\limits_{|z|\leqslant|w|} \frac{z^n}{w^n} dxdy = \frac{1}{w} \iint\limits_{|z|\leqslant|w|} dxdy = \frac{1}{w}\pi |w|^2 = \pi\bar{w}$

(2) $\iint\limits_{1>|z|>|w|} \frac{dxdy}{w-z} = -\iint \frac{1}{z} \frac{dxdy}{1-w/z} = -\sum\limits_{n=0}^{\infty} w^n \iint \frac{dxdy}{z^{n+1}} = 0$

$$f(w) = \begin{cases} \pi/w, & |w| > 1 \\ \pi\bar{w}, & |w| < 1 \end{cases}$$

V.6.1

1	2	3	P	L	K
				LLL	

Calculate the terms through order seven of the power series expansion about $z = 0$ of the function $1/\cos z$.

Solution

$\frac{1}{\cos z} = 1 + \left(\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} + O\left(z^8\right)\right) + (\ldots)^2 + (\ldots)^3 + O\left(z\right)$

$1 + \frac{z^2}{2!} + a_4 z^4 + a_6 z^6 + O\left(z^8\right)$

$z^4 \ : \ -\frac{1}{4!} + \left(\frac{1}{2!}\right)^2 = \frac{1}{4} - \frac{1}{24} = \frac{5}{24}$, $z^6 \ : \ -\frac{2}{2!4!} + \left(\frac{1}{2!}\right)^3 = \frac{1}{8} - \frac{1}{24} = \frac{1}{12}$

$\frac{\sin z}{\cos z} = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right)\left(1 + \frac{z^2}{2} + \frac{5}{24}z^4 - \frac{1}{12}z^6 + \ldots\right)$

$z : 1$

$z^3 : \frac{1}{2} - \frac{1}{3!} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

$z^5 : \frac{1}{5!} - \frac{1}{2} \cdot \frac{1}{3!} + \frac{5}{24} = \frac{1}{120} - \frac{1}{12} + \frac{5}{24} = \frac{16}{120} = \frac{4}{30} = \frac{2}{15}$

$z^7 : \frac{1}{12} - \frac{5}{24 \cdot 3!} + \frac{1}{2 \cdot 5!} - \frac{1}{7!} = \frac{7 \cdot 6 \cdot 5 \cdot 2 - 5^2 \cdot 7 + 3 \cdot 7 - 1}{7!} =$
$\frac{420 - 175 + 21 - 1}{7!} = \frac{265}{7!} = \frac{53}{7 \cdot 6 \cdot 4 \cdot 3 \cdot 2}$

V.6.2

1	2	3	P	L	K
				LLL	

Calculate the terms through order five of the power series expansion about $z=0$ of the function $z/\sin z$.

Solution

$\frac{z}{\sin z} = \frac{z}{z-z^3/3!+z^5/5!-\ldots} = \frac{1}{1-z^2/3!+z^4/5!-\ldots}$

$= 1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \ldots\right) + (\ldots)^2 + (\ldots)^3 + O\left(z^2\right) =$

$1 + z^2/6 + a_4 z^4 + a_6 z^6 + O\left(z^7\right)$

$a_4 = -\frac{1}{5!} + \frac{1}{(3!)^2} = \frac{1}{36} - \frac{1}{120} = \frac{10-3}{360} = \frac{7}{360}$

$a_6 = \frac{1}{7!} = -\frac{2}{3!5!} + \frac{1}{(3!)^3} = \frac{6(1-14)+4\cdot5\cdot7}{3!7!} = \frac{140-78}{3!7!} = \frac{62}{3!7!} = \frac{3}{2}$

$\frac{z}{\sin z} = 1 + \frac{z^2}{6} + \frac{7z^4}{360} + O\left(z^6\right)$

V.6.3

1	2	3	P	L	K
				LLL	

Show that

$$\frac{e^z}{1+z} = 1 + \frac{1}{2}z^2 - \frac{1}{3}z^3 + \frac{3}{8}z^4 - \frac{11}{30}z^5 + \cdots$$

Show that the general term of the power series is given by

$$a_n = (-1)^n \left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right], \qquad n \geq 2.$$

What is the radius of convergence of the series?

Solution

We have

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k \quad \text{for all } |z| < 1 \qquad \text{and} \qquad \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \text{for all } z \in \mathbb{C}.$$

Since the given function can be expressed as product of these two we have

$$\frac{e^z}{1+z} = \frac{1}{1+z}e^z = \sum_{n=0}^{\infty} c_n z^n \qquad \text{for all } |z| < 1$$

where

$$a_n = \frac{(-1)^n}{0!} + \frac{(-1)^{n-1}}{1!} + \frac{(-1)^{n-2}}{2!} + \cdots + \frac{(-1)^0}{n!}$$

by formula (6.1). Hence, $c_0 = 1$, $c_1 = 0$ and for $n \geq 2$ we have, since $0! = 1! = 1$,

$$a_n = (-1)^{n-1}(-1+1) + \frac{(-1)^{n-2}}{2!} + \cdots + \frac{(-1)^0}{n!} = (-1)^n \left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right].$$

The radius of convergence is 1.

Solve $e^z/\left(1+z\right) = a_0 + a_1 z + a_2 z^2 + \ldots$ by multiplying by $(1+z)$ and comparing with the coefficients of e^z.

$$\left.\begin{array}{l} a_0 = 1 \\ a_0 + a_1 = 1 \\ a_1 + a_2 = 1/2 \\ a_{n-1} + a_n = 1/n! \end{array}\right\} \Rightarrow \begin{array}{l} \text{Recursively} \\ a_0 = 1 \\ a_n = 1/n! - a_{-n} \end{array}$$

So $a_n = \frac{1}{n!} - \frac{1}{(n-1)!} + \frac{1}{(n-2)!} - \ldots + \left(-1\right)^n \frac{1}{0!}$ (the two last terms cancel.) The radius of convergence is 1 because the function has a pole at -1 . (In fact, $\left(-1\right)^n a_n \to e^{-1}$ when $n \to \infty$.)

V.6.4

1	2	3	P	L	K

Define the Bernoulli numbers B_n by

$$\frac{z}{2}\cot(z/2) = 1 - B_1\frac{z^2}{2!} - B_2\frac{z^4}{4!} - B_3\frac{z^6}{6!} - \cdots .$$

Explain why there are no odd terms in this series. What is the radius of convergence of the series? Find the first three Bernoulli numbers.

Solution

$\frac{z}{2}\cot(z/2) = 1 - B_1\frac{z^2}{2!} - B_2\frac{z^4}{4!} - B_3\frac{z^6}{6!} - \ldots,$

$\cot(z/2) = \frac{\cos(z/2)}{\sin(z/2)}$ is odd, so $\frac{z}{2}\cot(z/2)$ is even, so only even terms appear in the series. The function has singularities at $z/2 = \pm\pi$, i.e., at $\pm 2\pi$. Distance from 0 to nearest singularity $= 2\pi =$ radius of convergence.

Find the two Bernoullinumbers $B_1{+}B_2$. $w\cot w = w\frac{\cos w}{\sin w} = \frac{1-\frac{w^2}{2!}+\frac{w^4}{4!}-\frac{w^6}{6!}+\ldots}{1-\frac{w^2}{3!}+\frac{w^4}{5!}-\frac{w^6}{7!}+\ldots} =$

$\left[1 - \frac{w^2}{2!} + \frac{w^4}{4!} + O\left(w^6\right)\right]\left[1 + \frac{w^2}{6} - \frac{w^4}{120} + \left(\frac{w^2}{6}\right)^2 + O\left(w^6\right)\right]$

$\left(1 - \frac{w^2}{2} + \frac{w^4}{24}\right)\left(1 + \frac{w^2}{6} + \left(\frac{1}{36} - \frac{1}{120}\right)w^4\right) + O\left(w^6\right) =$

$1 + \left(\frac{1}{6} - \frac{1}{2}\right)w^2 + \left(\frac{7}{360} - \frac{1}{12} + \frac{1}{24}\right)w^4 + O\left(w^6\right) =$

$1 - \frac{1}{3}w^2 - \frac{1}{45}w^4 + O\left(w^6\right)$

$\cot w = 1 - B_1\frac{w^2}{2!} - B_2\frac{2^4 w^4}{4!} - \ldots$

$\therefore \frac{2^2 B_1}{2!} = \frac{1}{3}$, $B_1 = \frac{1}{6}$, $\frac{2^4}{4!}B_2 = \frac{1}{45}$, $B_2 = \frac{4!}{2^4 \cdot 45}$, $B_2 = \frac{1}{30}$

For B_3 , must keep all powers ????? to and ????? w^8. Better to just ????? for B_1 and B_2.

V.6.5

1	2	3	P	L	K

Define the Euler numbers E_n by

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n.$$

What is the radius of convergence of the series? Show that $E_n = 0$ for n odd. Find the first four nonzero Euler numbers.

Solution

$\frac{1}{\cosh z} = \sum\limits_{n=0}^{\infty} \frac{E_n}{n!} z^n$, $\cosh z$ has zeros at $\pm\frac{\pi}{2}i, \pm\frac{3\pi}{2}i, \ldots$. Distance from 0 to nearest singularity is $\pi/2 = R$. Since $\cosh z$ is even, $E_n = 0$ for n odd.

$\frac{1}{\cosh z} = \frac{1}{(1+z^2/2!+z^4/4!+\ldots)} = 1 + E_2\frac{z^2}{2!} + E_4\frac{z^4}{4!} + E_6\frac{z^6}{6!} =$

$1 - \left(\frac{z^2}{2!} + \frac{z^4}{4!} + \ldots\right) + \left(\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} \ldots\right)^2 - (\ldots)^3$, $E_0 = 1$

$\frac{E_2}{2!} = -\frac{1}{2!}$, $E_2 = -1$, $\frac{E_4}{4!} = -\frac{1}{4!} + \left(\frac{1}{2!}\right)^2 = \frac{1}{4} - \frac{1}{24} = \frac{5}{24}$, $E_4 = 5$

$\frac{E_6}{6!} = -\frac{1}{6!} + 2\frac{1}{2!4!} - \frac{1}{(2!)^3}$, $E_6 = -61$

$E_6 = -1 + 6 \cdot 5 - \frac{6!}{3^3} = 29 - 6 \cdot 5 \cdot 3 = 29 - 90 = -61$

V.6.6

1	2	3	P	L	K

Show that the coefficients of a power series "depend continuously" on the function they represent, in the following sense. If $\{f_m(z)\}$ is a sequence of analytic functions that converges uniformly to $f(z)$ for $|z| > \rho$, and

$$f_m(z) = \sum_{k=0}^{\infty} a_{k,m}\, z^k, \qquad f(z) = \sum_{k=0}^{m} a_k\, z^k,$$

then for each $k \geq 0$, we have $a_{k,m} \to a_k$ as $m \to \infty$.

V.7.1

1	2	3	P	L	K
				LLL	

Find the zeros and orders of zeros of the following functions

(a)	$\frac{z^2+1}{z^2-1}$	**(d)**	$\cos z - 1$	**(g)**	$e^z - 1$
(b)	$\frac{1}{z} + \frac{1}{z^5}$	**(e)**	$\frac{\cos z - 1}{z}$	**(h)**	$\sinh^2 z + \cosh^2 z$
(c)	$z^2 \sin z$	**(f)**	$\frac{\cos z - 1}{z^2}$	**(i)**	$\frac{\operatorname{Log} z}{z}$ (principal value)

Solution

(c)

Let $f(z) = z^2 \sin z$. Then f has zeros at all integer multiples of π i.e. $z_0 = k\pi$ for $k \in \mathbb{Z}$. Since $\sin z$ has a zero of order 1 at 0, so $\sin z = zh(z)$ for some analytic function h with $h(0) \neq 0$. Thus, $f(z) = z^3 h(z)$ and so f has a zero of order three at 0. Note that $f'(z) = 2z \sin z + z^2 \cos z$. Hence, for any other zero $z_0 = k\pi$ where $k \in \mathbb{Z} \setminus \{0\}$, we have

$$f'(k\pi) = 2k\pi \sin k\pi + (k\pi)^2 \cos k\pi = (-1)^k k\pi \neq 0.$$

Thus, f has a simple zero at $z_0 = k\pi$ when $k \neq 0$.

(d)

Note that $g(z) = \cos z - 1 = 0$ iff $z = 2k\pi$ for $k \in \mathbb{Z}$. We will show that all the zeros are double zeros. First observe that $g'(z) = -\sin z$ and so $g'(2k\pi) = -\sin 2k\pi = 0$ for all $k \in \mathbb{Z}$. But $g''(z) = -\cos z$ and so $g'(2k\pi) = -\cos 2k\pi = -1$ for all $k \in \mathbb{Z}$. Hence, all zeros are double zeros.

(a) simple zeros at $\pm i$, (b) simple zeros at $\pm e^{\pi i/4}$, $\pm e^{3\pi i/4}$, (c) triple zero at 0, simple zeros at $n\pi$, $n = \pm 1, \pm 2, \ldots$, (d) double zeros at $n\pi$, $n = \pm 1, \pm 2 \ldots$, (h) simple zeros at $-\pi i/4 + n\pi i/2$, $n = \pm 1, \pm 2, \ldots$, (i) no zeros.
(b) $\frac{1}{z} + \frac{1}{z^5} = (z^4 + 1)\frac{1}{z^5} = (z - z_1)(z - z_2)(z - z_3)(z - z_4)\frac{1}{z^5}$, where $z_j = e^{i(\pi/4 + j/4 \cdot 2\pi)}$, these four points are simple zeros.
(d) $\cos z = \cos(x + iy) = \cos x \cos(iy) - \sin x \sin(iy)$, $\cos(iy) = \frac{e^{-y}+e^{y}}{2} = \cosh y$, $\sin(iy) = \frac{e^{-y}-e^{y}}{2i} = i\left(\frac{e^{y}-e^{-y}}{2}\right) = i \sinh y$, $\cos z = 1 \Rightarrow \sin x \sinh y = 0 \Rightarrow y = 0$ or $x = n\pi$. If $x = n\pi$, then $\cos(n\pi)\cosh(n\pi) = (-1)^n \cosh(n\pi)$. This is $= 1$ only if $n = 0$, since $\cos(n\pi) > 1$ for $n \neq 0$. Get $x = 0$, $y = 0$. If $y = 0$, then $\cos x = 1$ only at $x = 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. Solutions are $z = 0, \pm 2\pi, \pm 4\pi, \ldots$. Since $\frac{d}{dz}(\cos z - 1) = -\sin z = 0$ at these points, $\frac{d^2}{dz^2}(\cos z - 1) = -\cos z \neq 0$, at these points. The zeros are double zeros.
(e) $\frac{\cos z - 1}{z}$, has only zeros at $z = 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. Simple poles at $z = 0$. At the points $\frac{d}{dz} = \frac{\cos z - 1}{z} = \frac{z(-\sin z) - (\cos(z-1))}{z^2} = -\frac{\cos(2n\pi) - 1}{(2n\pi)^2}$ for $n \neq 0$. Other zeros are double. Double zeros at $z = 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$.
(g) $e^z - 1 = z + \frac{z^2}{2!} + \ldots$, thus a simple zero at 0. Since $e^z - 1$ is periodic, we have simple zeros at $i2\pi k$, $k \in \mathbb{Z}$.
(i) $\operatorname{Log} z = 0$ when $|z| = 1$ and $\operatorname{Arg} z = 0$, thus at $z = 1$. We write $\operatorname{Log} z = \operatorname{Log}(1 - (1 - z)) = 1 - z - \frac{(1-z)^2}{2} + \frac{(1-z)^3}{3} + \ldots$, so we have a simple zero at 1.

V.7.2 (From Hints and Solutions)

1	2	3	P	L	K

Determine which of the functions in the preceding exercise are analytic at ∞, and determine the orders of any zeros at ∞.

Solution

(a) analytic at ∞ , (b) analytic at ∞, simple zero, (c) – (i) not analytic at ∞.

V.7.3

1	2	3	P	L	K
				LLL	

Show that the zeros of $\sin z$ and $\tan z$ are all simple.

Solution

Zeros of $\sin z$ are at $n\pi$, $-\infty < n < \infty$, since $\cos(n\pi) = \frac{d}{dz}\sin z|_{z=n\pi} = \pm 1 \neq 0$, the zeros are simple.

V.7.4

1	2	3	P	L	K
				LLL	

Show that $\cos(z+w) = \cos z \cos w - \sin z \sin w$, assuming the corresponding identity for z and w real.

Solution

$\cos(z+w) = \cos z \cos w + \sin z \sin w = F(z,w)$ is entire in z for each fixed w and entire in w for each fixed z. $F(z,w) = 0$ for z, w real. Apply theorem, with $D = \mathbb{C}$, $E = \mathbb{R}$. Get $F(z,w) = 0$, so $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

V.7.5

1	2	3	P	L	K

Show that

$$\int_{-\infty}^{\infty} e^{-zt^2+2wt}dt = \sqrt{\frac{\pi}{z}}e^{w^2/z}, \qquad z, w \in \mathbb{C},\ \operatorname{Re} z > 0,$$

where we take the principal branch of the square root. Compare the result to Exercise IV.3.1. Hint. Show that the integral is analytic in z and w, and evaluate it for $z = x > 0$ and w real by making a change of variable and using the known value $\sqrt{\pi}$ for $z = 1$ and $w = 0$.

Solution

$\int\limits_{-\infty}^{\infty} e^{at^2-2bt}dt$, note the integral is improper, so a limit process may be used for analytic. $\operatorname{Re} a < 0$. $x > 0$, $\int\limits_{-\infty}^{\infty} e^{at^2-2bt}dt = \frac{1}{\sqrt{x}} \int\limits_{-\infty}^{\infty} e^{-s^2-\frac{2b}{\sqrt{x}}s}ds$. $F(\lambda) = \int\limits_{-\infty}^{\infty} e^{-s^2}e^{\lambda s}ds = \int\limits_{-\infty}^{\infty} e^{-(s-\lambda/2)}e^{\lambda^2/4}ds = e^{\lambda^2/4} \underbrace{\int\limits_{-\infty}^{\infty} e^{-(s-\lambda/2)}ds}$

$\int\limits_{-\infty}^{\infty} e^{-s^2}e^{\lambda s}ds = \sqrt{\pi}e^{\lambda^2/4}$, $\int\limits_{-\infty}^{\infty} e^{-xt^2}e^{-2bt}dt = \frac{1}{\sqrt{x}}\sqrt{\pi}e^{b^2/x}$, x real

$\int\limits_{-\infty}^{\infty} e^{-zt^2}e^{-zbt}dt = \sqrt{\frac{\pi}{z}}e^{b^2/z}$

V.7.6

1	2	3	P	L	K
				LLL	

Suppose $f(z)$ is analytic on a domain D and $z_0 \in D$. Show that if $f^{(m)}(z_0) = 0$ for $m \geq 1$, then $f(z)$ is constant on D.

Solution

Suppose that $f^{(m)}(z_0) = 0$ for $m \geq$ and define g on D by $g(z) = f(z) - f(z_0)$. Then it suffices to show that g is identically zero. Suppose not, then since $g^{(m)}(z_0)$ for $m \geq 0$, the power series expansion for g is trivial and therefore g is identically zero on a disk on nonzero radius centered at z_0 but this violates the first theorem on page 156 that asserts that an analytic function which is not identically zero only has isolated zeros. This result may also be proved using the Uniqueness Principle (the second theorem on page 156).

V.7.7

1	2	3	P	L	K
				LLL	

Show that if $u(x, y)$ is a harmonic function on a domain D such that all the partial derivatives of $u(x, y)$ vanish at the same point of D, then $u(x, y)$ is constant on D.

Solution

V.7.8

1	2	3	P	L	K

With the convention that the function that is identically zero has a zero of infinite order at each point, show that if $f(z)$ and $g(z)$ have zeros of order n and m respectively at z_0, then $f(z) + g(z)$ has a zero of order $k \geq \min(n, m)$. Show that strict inequality can occur here, but that equality holds whenever $m \neq n$.

Solution

V.7.9

1	2	3	P	L	K
				LLL	

Show that the analytic function $f(z)$ has a zero of order N at z_0, then $f(z) = g(z)^N$ for some function $g(z)$ analytic near z_0 and satisfying $g'(z) \neq 0$.

Solution
(From Hints and Solutions)
Write $f(z) = (z - z_0)^N h(z)$, where $h(z)$ has a convergent power series and $h(z_0) \neq 0$. Take $g(z) = (z - z_0) e^{(\log(h(z)))/N}$ for an appropriate branch of the logarithm.

V.7.10

1	2	3	P	L	K

Show that if $f(z)$ is a continuous function on a domain D such that $f(z)^N$ is analytic on D for some integer N, then $f(z)$ is analytic on D.

Solution

$f(z)^N$ analytic. Zeros of $f(z)$ are isolated, for this need maximum principle. $f(z)$ is analytic, except possibly at the isolated points where $f(z)=0$. (By the Riemann's theorem, $f(z)$ is analytic ????? there but don't have this yet) Write $f(z)^N = (z-z_0)^m h(z)$, $h(z)$ analytic, $h(z) \neq 0$. Then $h(z)$ has N^{th} root near z_0, by preceding exercise. $f(z)^N = g(z)^N (z-z_0)^m$, $(f(z)/g(z))^N = (z-z_0)^m$. $(z-z_0)^{m/N}$ is continuous in a neighborhood of z_0. $f(z)/g(z)$ returns to original value, doing circle around $z-z_0$. m is an integral multiple of N.

V.7.11

1	2	3	P	L	K

Show that if $f(z)$ is a nonconstant analytic function on a domain D, then the image under $f(z)$ of any open set is open. Remark. This is the open mapping theorem for analytic functions. The proof is easy when $f'(z) \neq 0$, since the Jacobian of $f(z)$ coincides with $|f'(z)|^2$. Use Exercise 9 to deal with the points where $f'(z)$ is zero.

Solution

If $f'(z_0) \neq 0$, then f maps open disks centred at z_0 onto open sets, so $f(D)$ contains a disk centred at $f(z_0)$.

If $f'(z_0) = 0$, assume $f(z_0) = 0$, write $f(z) = (z - z_0)^N h(z)$, $h(z_0) \neq 0$. Then f has a N^{th} root near z_0, $f(z) = g(z)^N$, $g'(z_0) \neq 0$. g covers a ????? disk, so f cover disk centred at 0.

V.7.12

1	2	3	P	L	K

Show that the open mapping theorem for analytic functions implies the maximum principle for analytic functions.

Solutions

Clearly open mapping theorem $\Rightarrow$ strict maximum principle for analytic functions, since a non constant analytic function can't attain maximum at z_0 and cover a disk centred at z_0 . Then strict maximum principle $\Rightarrow$ maximum principle.

V.7.13 (From Hints and Solutions)

1	2	3	P	L	K

Let $f_n(z)$ be a sequence of analytic functions on a domain D such that $f_n(D) \subset D$, and suppose that $f_n(z)$ converges normally to $f(z)$ on D. Show that either $f(D) \subset D$, or else $f(D)$ consists of a single point on ∂D.

Solution

$f(D) \subset D \cup \partial D$. If $f(z)$ is not constant, then $f(D)$ is open, and $f(D)$ cannot contain any point of ∂D.

V.7.14

1	2	3	P	L	K

A set E is discrete if every point of E is isolated. Show that a closed discrete subset of a domain D either is finite or can be arranged in a sequence $\{z_k\}$ that accumulates only on $\{\infty\} \cup \partial D$.

Solution

Let $K_n = \{z \in D : d(z, \partial D) \geqslant 1/n,\ |z| \leqslant n\}$. E_n is compact, only finitely many points of E belongs to K_n ????? points, shading with those in $E \cap K_1$, the $E \cap (K_2 \backslash K_1)$,?????

V8.1

1	2	3	P	L	K

Suppose that the principal branch of $\sqrt{z^2-1}$ is continued analytically from $z=2$ around the figure eight path indicated above. What is the analytic continuation of the function at the end of the path? Answer the same question form the functions $\left(z^3-1\right)^{1/3}$ and $\left(z^6-1\right)^{1/3}$.

Solution

$\sqrt{z^2-1}$, phase change i at $+1$, -1 at -1, i at $+1$, returns to initial value. $\sqrt[3]{z^3-1}$, phase change $e^{i\pi/3}$ at $+1$, $e^{i\pi/3}$ at $+1$, returns to $e^{2\pi i/3}$ times initial value. $\sqrt[3]{z^6-1}$, phase change $e^{i\pi/3}$ at $+1$, $e^{-2\pi i/3}$ at -1, $e^{i\pi/3}$ at $+1$, returns to initial value.

V8.2

1	2	3	P	L	K

Show that $f(z) = \operatorname{Log} z = (z-1) - \frac{1}{2}(z-1)^2 + \cdots$ has an analytic continuation around the unit circle $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. Determine explicitly the power series f_t for each t. How is $f_{2\pi}$ related to f_0?

Solution

$\frac{df}{dz} = \frac{1}{z} = \sum$, $f^{(m)}(z) = \frac{(-1)^{m-1}}{z^m}(m-1)!$

Series is $\sum\limits_{m=0}^{\infty} \frac{f^{(m)}(e^{it})}{m!}(z - e^{it})^m = f(e^{it}) + \sum\limits_{m=1}^{\infty} (-1)^{m-1} \frac{e^{-itm}}{m}(z - e^{it})^m$

$f_t(z) = it + \sum\limits_{m=1}^{\infty} (-1)^{m-1} \frac{e^{itm}}{m}(z - e^{it})^m$, get $f_{2\pi}(z) = f_0(z) + 2\pi i$.

V.8.3

1	2	3	P	L	K

Show that each branch of $\sqrt{z}$ can be continued analytically along any path γ in $\mathbb{C}\backslash\{0\}$, and show that the radius of convergence of the power series $f_t(z)$ representing the continuation is $|\gamma(t)|$. Show that $\sqrt{z}$ cannot be analytically along path containing 0.

Solution

V.8.4

1	2	3	P	L	K

Let $f(z)$ be analytic on a domain D, fix $z_0 \in D$, and let $f(z) = \sum a_n (z - z_0)^n$ be the expansion of $f(z)$ about z_0. Let

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

be the indefinite integral of $f(z)$ for z near z_0. Show that $F(z)$ can be continued analytically along any path in D starting at z_0. What happens in the case $D = \mathbb{C} \setminus \{0\}$, $z_0 = 1$, and $f(z) = 1/z$? What happens in the case that D is star-shaped?

Solution

V.8.5

1	2	3	P	L	K

Show that the function defined by

$$f(z) = \sum z^{2^n} = z + z^2 + z^4 + z^8 + \cdots$$

is analytic on the open unit disk $\{|z| < 1\}$, and that it cannot be extended analytically to any larger open set. Hint. Observe that $f(z) = z + f(z^2)$, and that $f(r) \to +\infty$ as $r \to 1$.

Solution

V.8.6

1	2	3	P	L	K

Suppose $f(z) = \sum a_n z^n$, where $a_n = 0$ except for n in a sequence n_k that satisfies $n_{k+1}/n_k \geq 1+\delta$ for some $\delta > 0$. Suppose further that the series has radius of convergence $R = 1$. Show that $f(z)$ does not extend analytically to any point of the unit circle. Remark. Such a sequence with large gaps between successive nonzero terms is called a lacunary sequence. This result is the Hadamard gap theorem. There is a slick proof. If $f(z)$ extends analytically across $z = 1$, consider $g(w) = f(w^m(1+w)/2)$, where m is a large integer.

Show that the power series for $g(w)$ has radius of convergence $r > 1$, and that this implies that the power series of $f(z)$ converges for $|z-1| < \varepsilon$.

Solution

V.8.7

1	2	3	P	L	K

Suppose $f(z) = \sum a_n z^n$, where the series has radius of convergence $R < \infty$. Show that there is an angle α such that $f(z)$ does not have an analytic continuation along the path $\gamma(t) = te^{i\alpha}$, $0 \le t \le R$. Determine the radius of convergence of the power series expansion of $f(z)$ about $te^{i\alpha}$.

Solution

V.8.8

1	2	3	P	L	K

Let $f(z)$ be analytic at z_0, and let $\gamma(t)$, $a \leq t \leq b$, be a path such that $\gamma(a) = z_0$. If $f(z)$ cannot be continued analytically along γ, show that there is a parameter value t_1 such that there is an analytic continuation $f_t(z)$ for $a \leq t < t_1$, and the radius of convergence of the power series $f_t(z)$ tends to 0 as $t \to t_1$.

Solution

V.8.9

1	2	3	P	L	K

Let $P(z,w)$ be a polynomial in z and w, of degree n in w. Suppose that $f(z)$ is analytic at z_0 and satisfies $P(z, f(z)) = 0$. Show that if $f_t(z)$ is any analytic continuation of $f(z)$ along any path starting at z_0, then $P(z, f_t(z)) = 0$ for all t. Remark. An analytic function $f(z)$ satisfies a polynomial equation $P(z, f(z)) = 0$ is called an algebraic function. For instance, the branches of $\sqrt[n]{z}$ are algebraic functions, since the satisfy $z - w^n = 0$.

Solution

V.8.10

1	2	3	P	L	K

Let D be the punctured disk $\{0 < |z| < \varepsilon\}$, suppose $f(z)$ is analytic at $z_0 \in D$, and $e^{w_0} = z_0$. Show that $f(z)$ has an analytic continuation along any path in D starting at z_0 if and only if there is an analytic function $g(w)$ in the half-plane $\{\operatorname{Re} w < \log \varepsilon\}$ such that $f(e^w) = g(w)$ for w near w_0. Remark. If $f(z)$ does not extend analytically to D but has an analytic continuation along any path in D, we say that $f(z)$ has a branch point at $z = 0$. For the proof, use the fact that any path in D starting at z_0 is the composition of a unique path in the half-plane starting at w_0 and the exponential function e^w.

Solution

VI	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1																			
2																			
3																			
4																			
5																			
6																			
7																			
8																			

VI.1.1

1	2	3	P	L	K
111				LLL	

Find all possible Laurent expansions centered at 0 of the following functions.

(a) $\frac{1}{z^2-z}$ **(b)** $\frac{z-1}{z+1}$ **(c)** $\frac{1}{(z^2-1)(z^2-4)}$

Solution

(a)

In the region $0<|z|<1$, we have

$$\frac{1}{z^2-z}=-\frac{1}{z}\frac{1}{1-z}=-\frac{1}{z}\sum_{k=0}^{\infty}z^k=-\sum_{k=0}^{\infty}z^{k-1}=[k-1=n]=-\sum_{n=-1}^{\infty}z^n.$$

In the region$|z|>1$, we have

$$\frac{1}{z^2-z}=\frac{1}{z^2}\frac{1}{1-\frac{1}{z}}=\frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k=\frac{1}{z^2}\sum_{k=0}^{\infty}z^{-k-2}=[-k-2=n]=\sum_{n=-\infty}^{-2}z^n.$$

(b)

In the region $|z|<1$ we have

$$\frac{z-1}{z+1}=1-\frac{2}{z+1}=1-\frac{2}{1-(-z)}=1-2\sum_{n=0}^{\infty}(-1)^n z^n=-1-2\sum_{n=1}^{\infty}(-1)^n z^n.$$

In the region $|z|>1$ we have

$$\frac{z-1}{z+1}=1-\frac{2}{z+1}=1-\frac{2}{z}\frac{1}{1-\left(-\frac{1}{z}\right)}=1-\frac{2}{z}\sum_{k=0}^{\infty}\frac{(-1)^k}{z^k}=$$
$$=1-2\sum_{k=0}^{\infty}\frac{(-1)^k}{z^{k+1}}=[k+1=n]=1-2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^n}=1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{z^n}.$$

(c)

From computation, we have

$$\frac{1}{(z^2-1)(z^2-4)} = -\frac{1}{3}\frac{1}{z^2-1} + \frac{1}{3}\frac{1}{z^2-4}$$

In the region $|z| < 1$, we have

$$\frac{1}{(z^2-1)(z^2-4)} = -\frac{1}{3}\frac{1}{z^2-1} + \frac{1}{3}\frac{1}{z^2-4} = \frac{1}{3}\frac{1}{1-z^2} - \frac{1}{12}\frac{1}{1-\frac{z^2}{4}} =$$
$$= \frac{1}{3}\sum_{n=0}^{\infty} z^{2n} - \frac{1}{12}\sum_{n=0}^{\infty}\frac{z^{2n}}{4^n} = \frac{1}{12}\sum_{n=0}^{\infty}\left(4-4^{-n}\right)z^{2n}.$$

In the region $1 < |z| < 2$, we have

$$\frac{1}{(z^2-1)(z^2-4)} = -\frac{1}{3}\frac{1}{z^2-1} + \frac{1}{3}\frac{1}{z^2-4} = -\frac{1}{3}\frac{1}{z^2\left(1-\frac{1}{z^2}\right)} - \frac{1}{3}\frac{1}{4\left(1-\frac{z^2}{4}\right)} =$$
$$= -\frac{1}{3z^2}\sum_{k=0}^{\infty}\frac{1}{z^{2n}} - \frac{1}{12}\sum_{n=0}^{\infty}\frac{z^{2n}}{4^n} = -\frac{1}{3}\sum_{k=-\infty}^{-1} z^{2k} - \frac{1}{12}\sum_{n=0}^{\infty}\frac{z^{2n}}{4^n}$$

In the region $|z| > 2$, we have

$$\frac{1}{(z^2-1)(z^2-4)} = -\frac{1}{3}\frac{1}{z^2-1} + \frac{1}{3}\frac{1}{z^2-4} =$$
$$= -\frac{1}{3}\frac{1}{z^2\left(1-\frac{1}{z^2}\right)} + \frac{1}{3}\frac{1}{z^2\left(1-\frac{4}{z^2}\right)} =$$
$$= -\frac{1}{3z^2}\sum_{n=0}^{\infty}\frac{1}{z^{2n}} + \frac{1}{3z^2}\sum_{n=0}^{\infty}\frac{4^n}{z^{2n}} =$$
$$= -\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{z^{2(n+1)}} + \frac{1}{3}\sum_{n=0}^{\infty}\frac{4^n}{z^{2(n+1)}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(4^n-1\right)z^{-2(n+1)} = [n+1=-k] =$$
$$= \frac{1}{3}\sum_{k=-\infty}^{-1}\left(4^{-1-k}-1\right)z^{2k}.$$

VI.1.2

1	2	3	P	L	K
111				LLL	

For each of the functions in Exercise 1, find the Laurent expansion centered at $z=-1$ that converges at $z=\frac{1}{2}$. Determine the largest open set on which each series converges.

Solution

(a)

Partial fractions give us

$$\frac{1}{z^2-z}=-\frac{1}{z}+\frac{1}{z-1}$$

Laurent series for $-1/z$

$$-\frac{1}{z}=-\frac{1}{z+1-1}=-\frac{1}{z+1}\frac{1}{1-\frac{1}{z+1}}=-\frac{1}{z+1}\sum_{k=0}^{\infty}\frac{1}{(z+1)^k}=$$
$$=-\sum_{k=0}^{\infty}\frac{1}{(z+1)^{k+1}}=[k+1=-n]=-\sum_{n=-\infty}^{-1}(z+1)^n$$

Laurent series for $1/z-1$

$$\frac{1}{z-1}=\frac{1}{z+1-2}=-\frac{1}{2}\frac{1}{1-\frac{z+1}{2}}=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(z+1)^n}{2^n}=-\frac{1}{2^{n+1}}\sum_{n=0}^{\infty}(z+1)^n$$

Thus we have

$$\frac{1}{z^2-z}=\sum_{k=-\infty}^{\infty}a_n(z+1)^n\,,\ \text{where } a_n=\left\{\begin{array}{ll}-1 & \text{for } n\le -1\\ -\frac{1}{2^{n+1}} & \text{for } n\ge 0\end{array}\right|$$

The function $1/\left(z^2-z\right)$ converges on the set $1<|z+1|<2$.

(b)

We have

$$\frac{z-1}{z+1}=1-\frac{2}{z+1}$$

The function $(z-1)/(z+1)$ converges on the set $0<|z+1|<\infty$.

(c)
Partial fractions give us

$$\frac{1}{(z^2-1)(z^2-4)} = -\frac{1}{6}\frac{1}{z-1} + \frac{1}{6}\frac{1}{z+1} + \frac{1}{12}\frac{1}{z-2} - \frac{1}{12}\frac{1}{z+2} =$$
$$= -\frac{1}{6}\frac{1}{z+1-2} + \frac{1}{6}\frac{1}{z+1} + \frac{1}{12}\frac{1}{z+1-3} - \frac{1}{12}\frac{1}{z+1+1} =$$
$$= \frac{1}{12}\frac{1}{1-\frac{z+1}{2}} + \frac{1}{6}\frac{1}{z+1} - \frac{1}{36}\frac{1}{1-\frac{z+1}{3}} - \frac{1}{12(z+1)}\frac{1}{1+\frac{1}{z+1}} =$$
$$= \frac{1}{12}\sum_{n=0}^{\infty}\frac{(z+1)^n}{2^n} + \frac{1}{6}\frac{1}{z+1} - \frac{1}{36}\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^n} - \frac{1}{12}\sum_{k=0}^{\infty}\frac{(-1)^k}{(z+1)^{k+1}} =$$
$$= \frac{1}{12}\sum_{n=0}^{\infty}\frac{(z+1)^n}{2^n} + \frac{1}{6}\frac{1}{z+1} - \frac{1}{36}\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^n} + \frac{1}{12}\sum_{-\infty}^{n=-1}(-1)^n(z+1)^n =$$
$$\frac{1}{12}\sum_{n=0}^{\infty}(-1)^n(z+1)^n + \left(\frac{1}{6}-\frac{1}{12}\right)(z+1)^{-1} + \frac{1}{12}\sum_{n=0}^{\infty}\left(\frac{1}{12\cdot 2^n} - \frac{1}{36\cdot 3^n}\right)(z+1)^n,$$

The function $1/\left((z^2-1)(z^2-4)\right)$ converges on the set $1 < |z+1| < 2$.

VI.1.3

1	2	3	P	L	K
				LLL	

Recall the powe series for the Bessel function $J_n(z)$, $n \geq 0$, given in Exercise V.4.11, and define $J_{-n}(z) = (-1)^n J_n(z)$. For fixed $w \in C$, establish the Laurent series expansion

$$\exp\left[\frac{w}{2}(z - 1/z)\right] = \sum_{-\infty}^{\infty} J_n(w)\ z^n, \qquad 0 < |z| < \infty.$$

From the coefficient formula (1.4), deduce that

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(z\sin\theta - n\theta)}, \qquad z \in \mathbb{C}.$$

Remark. This Laurent expansion is called Schlömlich formula.

Solution

(a) $f(z) = \tan z$, $3 < |z| < 4$

$f(z) = \tan z = \frac{\sin z}{\cos z}$, singularities at $\frac{\pi}{2} + n\pi, n = 0, \pm 1, \pm 2, \ldots$

Simple poles at $\frac{\pi}{2} + n\pi$

$\lim\limits_{z\to\pi/2}(z - \pi/2)\frac{\sin z}{\cos z} = \sin\frac{\pi}{2}\lim\limits_{z\to\pi/2}\frac{z-\pi/2}{\cos z - \cos\pi/2} = \frac{\sin\pi/2}{-\sin\pi/2} = -1$

$\lim\limits_{z\to-\pi/2}(z + \pi/2)\frac{\sin z}{\cos z} = \sin\left(-\frac{\pi}{2}\right)\lim\limits_{z\to-\pi/2}\frac{z+\pi/2}{\cos z - \cos(-\pi/2)} = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1$

$\therefore$ Principal parts at $\pm\frac{\pi}{2}$ are $-\frac{1}{z\mp\pi/2}$.

$\therefore f_0(z) = f(z) + \frac{1}{z-\pi/2} + \frac{1}{z+\pi/2}$ is analytic for $|z| < \frac{3\pi}{2}$

$f_1(z) = \frac{-1}{z-\pi/2} - \frac{1}{z+\pi/2}$ is analytic for $|z| > \frac{\pi}{2}, \to 0$ at ?????

$\therefore f(z) = f_0(z) + \cancel{f}_1(z)$ is the Laurent decomposition for $3 < |z| < 4$, also for $\frac{\pi}{2} < |z| < \frac{3\pi}{2}$.

(b) $f_1(z) = -\left(\frac{z+\frac{\pi}{2}+z-\frac{\pi}{2}}{z^2-\pi^2/4}\right) = -\frac{2z}{z^2-\pi^2/4} = -\frac{2z}{z^2}\sum_{k=0}^{\infty}\left(\frac{\pi^2}{4}\cdot\frac{1}{z^2}\right)^n =$
$-2\sum_{n=0}^{\infty}\left(\frac{\pi^2}{4}\right)^n\cdot\frac{1}{z^{2n+1}}$, converges for $|z| > \pi/2$.

(c) $f(z)$ is odd, so $f_0(z)$ is odd, and $a_0 = a_2 = 0$.

$a_1 = \frac{1}{2\pi i}\oint_{|z|=3}\frac{\tan z}{z^2}dz = \frac{1}{2\pi i}\left(\oint_{|z|=\varepsilon} + \oint_{|z-\pi/2|=\varepsilon}\oint_{|z+\pi/2|=\varepsilon}\right)\frac{\tan z}{z^2}dz$

At $z \sim 0$, $\frac{\tan z}{z^2} \sim \frac{\sin z}{z\cos z}\cdot\frac{1}{z}$, $\oint_{|z|=\varepsilon} = 2\pi i$

At $z \sim \frac{\pi}{2}$, $\frac{\tan z}{z^2} \sim \frac{1}{z^2}\cdot\frac{-1}{z-\pi/2} \sim -\frac{4}{\pi^2}\frac{1}{z-\pi/2}$, $\oint_{|z-\pi/2|=\varepsilon} = \frac{-4}{\pi^2}\cdot 2\pi i$

At $z \sim -\frac{\pi}{2}$, $\frac{\tan z}{z^2} \sim \frac{1}{z^2}\cdot\frac{-1}{z+\pi/2} \sim -\frac{4}{\pi^2}\frac{1}{z+\pi/2}$, $\oint_{|z+\pi/2|=\varepsilon} = \frac{-4}{\pi^2}\cdot 2\pi i$

(d) Since $f_0(z) = f(z) - f_1(z)$ has poles at $\pm\frac{3\pi}{2}$, otherwise is analytic for $|z| < \frac{3\pi}{2}$, the series converges for $|z| < \frac{3\pi}{2}$.

VI.1.4

1	2	3	P	L	K
				LLL	

Suppose that $f(z) = f_0(z) + f_1(z)$ is the Laurent decomposition of an analytic function $f(z)$ on the annulus $\{A < |z| < B\}$. Show that if $f(z)$ is an even function, then $f_0(z)$ and $f_1(z)$ are even functions, and the Laurent series expansion of $f(z)$ has only even powers of z. Show that if $f(z)$ is an odd function, then $f_0(z)$ and $f_1(z)$ are odd functions, ant the Laurent series expansion of $f(z)$ has only odd powers of z.

Solution

Use $a_n = \oint\limits_{|z|=r} \frac{f(z)}{z^{n+1}} dz$, or definition of $f_0(z)+f_1(z)$, $f(z) = \sum\limits_{-\infty}^{\infty} a_n z^n$, $f(-z) = \sum (-1)^n a_n z^n$. f even $\Rightarrow a_n = -a_n$ for n odd $\Rightarrow a_n = 0$ for n odd. $\Rightarrow$ $f_0(z) + f_1(z)$ have only even ????? ?????.

VI.1.5

1	2	3	P	L	K
				LLL	

Suppose $f(z)$ is analytic on the punctured plane $D = \mathbb{C}\backslash\{0\}$. Show that there is a constant c such that $f(z) - c/z$ has a primitive in D. Give a formula for c in terms of an integral of $f(z)$.

Solution

By the Laurent Expansion Theorem we have for all $z \in D$ and $r > 0$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \qquad \text{where} \qquad a_k = \frac{1}{2\pi i}\int_{C_r(0)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta.$$

Setting $c = a_{-1} = \frac{1}{2\pi i}\int_{C_r(0)} f(\zeta)\, d\zeta$ we have

$$f(z) - c/z = \sum_{k\neq -1} a_k z^k.$$

Recall that an analytic function g has a primitive on a region D iff $\int_\gamma g(z)\, dz = 0$ for every piecewise smooth closed path γ in D. Let γ be such a path in D and observe the series $\sum_{k\neq -1} a_k z^k$ converges uniformly on any closed annulus $\{z : r \leq |z| \leq s\}$ with $0 < r < s$ and hence on γ. Hence, by the first theorem on page 126, we may integrate the series termwise we obtain:

$$\int_\gamma f(z) - c/z dz = \int_\gamma \sum_{k\neq -1} a_k z^k = \sum_{k\neq -1} a_k \int_\gamma z^k dz = \sum_{k\neq -1} a_k \cdot 0 = 0,$$

since z^k has a primitive if $k \neq -1$. Hence, $f(z) - c/z$ has a primitive in D.

VI.1.6

1	2	3	P	L	K

Fix an annulus $D = \{a < |z| < b\}$, and let $f(z)$ be a continuous function on its boundary ∂D. Show that $f(z)$ can be approximated uniformly on ∂D by polynomials in z and $1/z$ if and only if $f(z)$ has continuous extension to the closed annulus $D \cup \partial D$ that is analytic on D.

Solution

Use the conformity theorem for polynomial approximation, V5 14. If f is analytic, continuous ????? values, unit $f = f_0 + f_1$, its Laurent decomposing. Then $f_0 + f_1$ have continuous boundary values. Can approximate $f_0(z)$ uniformly by polynomials in z for $f_1(z)$ uniformly by polynomials in $1/z$. (?????? change of variable $w = 1/z$). That does it. If $f = \lim g_n(z)$ uniformly on ∂D, then by the maximum principle $|g_k - g_j| \to 0$ uniformly on $\bar{D}$, so $g_k \to f$ uniformly on $\bar{D}$, and clearly $F \backslash \partial D = f$.

VI.1.7

1	2	3	P	L	K

Show that a harmonic function u on an annulus $\{A < |z| < B\}$ has a unique expansion

$$u\left(re^{i\theta}\right) = \sum_{n=-\infty}^{\infty} a_n r^n \cos(n\theta) + \sum_{n\neq 0} b_n r^n \sin(n\theta) + c\log r,$$

which is uniformly convergent on each circle in the annulus. Show that for each r, $A < r < B$, the coefficients a_n, b_n and c satisfy

$$\begin{aligned} a_n r^n + a_{-n} r^{-n} &= \frac{1}{\pi}\int_{-\pi}^{\pi} u\left(re^{i\theta}\right)\cos(n\theta)\, d\theta, \qquad n \neq 0, \\ b_n r^n + b_{-n} r^{-n} &= \frac{1}{\pi}\int_{-\pi}^{\pi} u\left(re^{i\theta}\right)\sin(n\theta)\, d\theta, \qquad n \neq 0, \\ a_0 + c\log r &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(re^{i\theta}\right) d\theta. \end{aligned}$$

Hint. Use a decomposition of the form $u = \operatorname{Re} f + c\log|z|$, where f is analytic on the annulus. (See Exercise III.3.4.)

Solution

By III.3.4, we have $u = \operatorname{Re} f\ + c\log|z|$ for some e.

Write its Laurent series

$f(z) = \sum_{k=-a}^{\infty} c_k z^k$, $A < |z| < B$.

Then for $c_k = \alpha_k \beta_k$, we have

$\operatorname{Re} f = \sum_{k=-\infty}^{\infty} \alpha_k \operatorname{Re} z^k + \beta_k \operatorname{Im} z^k$

$u = \sum_{k=-\infty}^{\infty} \alpha_k r^k \cos(k\theta) + \beta_k r^k \sin(k\theta) + c\log$

The series converging uniformly on ????? The term w/β_0 ?????

Multiply by $\cos n\theta$, ????? use orthogonality, get for $n \neq 0$,

$\int_{-\pi}^{\pi} u(r,\theta)\cos n\theta d\theta = \alpha_n r^n \int_{-\pi}^{\pi} \cos^2 n\theta d\theta + \alpha_n r^{-n} \int_{-\pi}^{\pi} \cos^2 n\theta d\theta =$

$\pi\left(\alpha_n r^n + \alpha_{-n} r^{-n}\right)$

$\therefore \alpha_n r^n + \alpha_{-n} r^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} u(r,\theta)\cos n\theta d\theta$, $n \neq 0$.

The formula for $\beta_n r^n - \beta_{-n} r^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} u(r,\theta)\sin n\theta d\theta$, $n \neq 0$, is similar.

If we just integrate, we get $\frac{1}{\pi}\int_{-\pi}^{\pi} u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} c\log r d\theta + \frac{\alpha_0}{2\pi}\int d\theta = \alpha_0 + c\log r$

Note:

VI.1.8 (ej)
Multiply f by a unimodular constant, assume $|f(z)| \leqslant M$, $\oint\limits_{|z|=1} f(z)\,dz = \int\limits_0^{2\pi} f\left(e^{i\theta}\right) d\theta = 2\pi M$. Take real parts, have $\int\limits_0^{2\pi} \operatorname{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) d\theta = 2\pi M$, $\operatorname{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) d\theta \leqslant \left|f\left(e^{i\theta}\right)\right| \leqslant M$. If have strict inequality somewhere, then $\int\limits_0^{2\pi} < 2\pi M$. We conclude that $\operatorname{Re}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) d\theta \equiv M$. Since $\left|\left(f\left(e^{i\theta}\right) ie^{i\theta}\right)\right| \leqslant M$, we have $\operatorname{Im}\left(f\left(e^{i\theta}\right) ie^{i\theta}\right) d\theta \equiv 0$. $\therefore f\left(e^{i\theta}\right) ie^{i\theta} \equiv M$, $f\left(e^{i\theta}\right) \equiv -ie^{-i\theta}M$, $f(z) \equiv -iM\bar{z}$. $\therefore f(z) = c\bar{z}$ for a constant, $|c| = M$.

VI.1.9 (ej)
For the analyticity, differentiate by hand. (See Exercise III.1.6). The Derivative is $H'(w) = \lim\limits_{w\to\infty} \frac{1}{\Delta w} \int\limits_\gamma \left[\frac{h(z)}{z-(w+\Delta w)} - \frac{h(z)}{z-w}\right] dz$
$= \lim\limits_{w\to\infty} \frac{1}{\Delta w} \int\limits_\gamma \frac{h(z)\Delta w}{(z-(w+\Delta w))(z-w)} dz = \int\limits_\gamma \frac{h(z)dz}{(z-w)^2}$

VI.2.1

1	2	3	P	L	K
				LLL	

Find the isolated singularities of the following functions, and determine where they are removable, essential, or poles. Determine the order of any pole, and find the principal part at each pole.

(a)	$z/\left(z^2-1\right)^2$	**(d)**	$\tan z = \frac{\sin z}{\cos z}$	**(g)**	$\operatorname{Log}\left(1-\frac{1}{z}\right)$
(b)	$\frac{ze^z}{z^2-1}$	**(e)**	$z^2 \sin\left(\frac{1}{z}\right)$	**(h)**	$\frac{\operatorname{Log} z}{(z-1)^3}$
(c)	$\frac{e^{2z}-1}{z}$	**(f)**	$\frac{\cos z}{z^2-\pi^2/4}$	**(i)**	$e^{1/\left(z^2+1\right)}$

Solution

(a) (J.R.J Groves)

We have that $z=1$ is a pole of order 2 since $\frac{z}{(z^2-1)^2} = \frac{g(z)}{(z-1)^2}$ with $g(z) = \frac{z}{(z+1)^2}$ and $g(z)$ is analytic near $z=1$. Similarly we have, $z=-1$ is a pole of order 2 since $\frac{z}{(z-1)^2} = \frac{h(z)}{(z+1)^2}$ with $h(z) = \frac{z}{(z-1)^2}$ and $h(z)$ is analytic near $z=-1$.

(b) (J.R.J Groves)

We have that $z=1$ is a pole of order 1 since $\frac{ze^z}{z^2-1} = \frac{g(z)}{z-1}$ with $g(z) = \frac{ze^z}{z+1}$, and $g(z)$ is analytic near $z=1$. Similarly, $z=-1$ is a pole of order 1 since $\frac{ze^z}{z^2-1} = \frac{h(z)}{z+1}$ with $h(z) = \frac{ze^z}{z-1}$, and $h(z)$ is analytic near $z=-1$.

(c) (A. Kumjian)

Note that the given function is analytic on the punctured plane $D = \mathbb{C}\setminus\{0\}$. It has an isolated singularity at $z=0$. For $z \neq 0$ we have

$$\frac{e^{2z}-1}{z} = \frac{1}{z}\left(-1+\sum_{k=0}^{\infty}\frac{(2z)^k}{k!}\right) = \frac{1}{z}\sum_{k=1}^{\infty}\frac{2^k z^k}{k!} = \frac{1}{z}\sum_{j=0}^{\infty}\frac{2^{j+1}}{(j+1)!}z^j.$$

Since the function has a power series expansion for all $z \neq 0$, it extends to an analytic function at $z=0$. Hence, the function has a removable singularity at 0.

(d) (J.R.J Groves)

We have that $\cos z$ has simple zeros at $z = \pi/2 + k\pi$ for $k \in \mathbb{Z}$. So $\frac{1}{\cos z}$ has simple poles at $z = \pi/2 + k\pi$ for $k \in \mathbb{Z}$. Since $\sin z$ is analytic and different to 0 at these points, $\tan z = \frac{\sin z}{\cos z}$ has simple poles at $z = \pi/2 + k\pi$ for $k \in \mathbb{Z}$.

(e) (A. Kumjian)

The function is analytic on the punctured plane $D = \mathbb{C}\backslash\{0\}$. It has an isolated singularity at $z = 0$. For $z \neq 0$ we have

$$z^2 \sin\left(\frac{1}{z}\right) = z^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{z}\right)^{2k+1} =$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\frac{1}{z^{2k-1}} = z + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}\frac{1}{z^{2k-1}}.$$

Hence $z = 0$ is an essential singularity for the function since there are an infinite number of nonzero terms in the Laurent series with negative exponents.
(e) (J.R.J Groves)
The function $z^2 \sin\left(\frac{1}{z}\right)$ has an isolated singularity at $z = 0$. Since

$$z^2 \sin\left(\frac{1}{z}\right) = z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{2n-1}},$$

$z = 0$ is an essential singularity.
(f) (J.R.J Groves)
The function $z^2 - (\pi/2)^2 = (z - \pi/2)(z + \pi/2)$ has simple zeros at $z = -\pi/2$ and $z = \pi/2$. The function $\cos z$ has a simple zero at $z = \pi/2$. So we suspect a removable singularity. Set $w = z - \pi/2$. Then

$$\frac{\cos(z)}{z^2 - (\pi/2)^2} = \frac{\cos(z)}{(z - \pi/2)(z + \pi/2)} = \frac{\cos(w + \pi/2)}{w(w + \pi)} = -\frac{1}{w + \pi} \cdot \frac{\sin w}{w}.$$

Since we know that $\sin(w)/w = 1 - w^2/3! + \ldots$, we deduce that our function has a removable singularity at $w = 0$ and so at $z = \pi/2$. In a similar fashion, it also has a removable singularity at $z = -\pi/2$.
(g) (J.R.J Groves)
The function $\mathrm{Log}(z)$ is analytic on the region $\mathbb{C}\backslash(-\infty, 0]$, it follows that $\mathrm{Log}\left(1 - \frac{1}{z}\right)$ is analytic on $\mathbb{C}\backslash[0, 1]$. There are no isolated singularities.
(h) (A. Kumjian)
First observe that for $z \in \mathbb{C}$ with $|z - 1| < 1$ we have

$$\frac{1}{z} = \frac{1}{1 + (z - 1)} = \sum_{k=0}^{\infty} (-1)^k (z - 1)^k.$$

Since $\operatorname{Log} z$ is a primitive of $1/z$, its power series expansion centered at $z_0 = 1$ my be obtained from that of $1/z$ by integrating termwise: we obtain (note $\operatorname{Log} 1 = 0$)

$$\operatorname{Log} z = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (z-1)^k,$$

for $|z-1| < 1$. Hence, we have for $0 < |z-1| < 1$

$$\begin{aligned} \frac{\operatorname{Log} z}{(z-1)^3} &= \frac{1}{(z-1)^3} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (z-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (z-1)^{k-3} = \\ &= \frac{1}{(z-1)^2} - \frac{1}{2(z-1)} + \sum_{j=0}^{\infty} \frac{(-1)^j}{k+3} (z-1)^j \end{aligned}$$

Hence, $z = 0$ is a pole of order 2. The principal part is given by $\frac{1}{(z-1)^2} - \frac{1}{2(z-1)}$.

(h) (J.R.J Groves)

We have that $z = 1$ is a pole of order 2. This follows from the fact that $\operatorname{Log} z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots$ for $|z-1| < 1$ so that

$$\frac{\operatorname{Log} z}{(z-1)^3} = \frac{1}{(z-1)^2} - \frac{1}{2} \cdot \frac{1}{z-1} + \frac{1}{3} - \cdots.$$

(i) (J.R.J Groves)

There are isolated singularities at $z = -i$ and $z = i$. Moreover,

$$\frac{1}{z^2+1} = \frac{i}{2}\left[\frac{1}{z+i} - \frac{1}{z-i}\right]$$

so that

$$e^{\frac{1}{z^2+1}} = e^{\frac{i}{2}\cdot\frac{1}{z+i}} e^{-\frac{i}{2}\cdot\frac{1}{z-i}} = e^{\frac{i}{2}\cdot\frac{1}{z+i}} \sum\nolimits_{n=0}^{\infty} \frac{(-i)^n}{2^n n!} \cdot \frac{1}{(z-i)^n}.$$

since $e^{\frac{i}{2}\cdot\frac{1}{z+i}}$ is analytic and non-zero at $z = i$, $e^{\frac{1}{z^2+1}}$ has an essential singularity at $z = i$. Similarly, it has an essential singularity at $z = -i$.

(a) Double poles at ± 1 , principal parts $\mp (1/4)(z \pm 1)^2$.
(b) Singularity at $z = \pm 1$, simple poles $\frac{ze^z}{z^2-1} = \frac{1}{2}\left[\frac{e^z}{z-1} + \frac{e^z}{z+1}\right]$. Principal part $\frac{1/2}{z-1}$ at $z = 1$, $\frac{1/2}{z+1}$ at $z = -1$.
(d) $\tan z = \sin z / \cos z$, poles at $z = \pi/2 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. $\cos z = -\sin(\pi/2 + n\pi)(z - \pi/2 - n\pi) + O\left((z - \pi/2 - n\pi)^3\right)$
$\tan z = \frac{\sin(\pi/2+n\pi)+O\left((z-\pi/2+n\pi)^2\right)}{-\sin(\pi/2+n\pi)(z-\pi/2-n\pi)+O\left((\ldots)^3\right)} = \frac{-1}{z-\pi/2-n\pi} + O(z - \pi/2 - n\pi)$ Poles are simple, principal part $-1/(z - \pi/2 - n\pi)$.
(f) $\cos z / (z^2 - \pi^2/4)$, singularities at $\pm\pi/2$ are both removable.
(g) Defined and analytic for $1 - 1/z \in \mathbb{C}\backslash[-\infty, 0)$, $1/z - 1 \in \mathbb{C}\backslash[0, \infty)$, $1/z \in \mathbb{C}\backslash[1, \infty)$, $z \in \mathbb{C}\backslash[0, 1]$. No isolated singularities.
(h) Isolated singularity at $z = 1$, $\operatorname{Log} z = \int^{z} \frac{dw}{w} = \int_{1}^{2} \frac{dw}{1+w-1} = \int \sum (-1)^n (w-1)^n \, dw = \sum_{n=0}^{\infty} \frac{(-1)^n (z-1)^{n+1}}{n+1}$ $\operatorname{Log} z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \ldots$
$\frac{\operatorname{Log} z}{(z-1)^3} = \frac{1}{(z-1)^2} - \frac{1}{z(z-1)} + \text{analytic}$
Double pole, principal part $\frac{1}{(z-1)^2} - \frac{1}{z(z-2)}$ at $z = 1$. (i) $e^{1/\left(z^2+1\right)}$, singularities at $z = \pm i$. Values of $1/(z^2 + 1)$ approach ∞ from all directions as $z \to \pm i$, so values of $e^{1/\left(z^2+1\right)}$ do not converge, and singularities are essential.????? check $e^{1\left(1-t^2\right)} = f(it) \to +\infty$ as $t \uparrow 1$ or $t \downarrow -1$, $e^{1\left(1-t^2\right)} = f(it) \to 0$ as $t \downarrow 1$ or $t \uparrow -1$, so no limit at $\pm i$.

VI.2.2

1	2	3	P	L	K
				LLL	

Find the radius of convergence of the power series for the following functions, expanded about the indicated point.

(a) $\frac{z-1}{z^4-1}$, about $z = 3+i$, (c) $\frac{z}{\sin z}$, about $z = \pi i$,
(b) $\frac{\cos z}{z^2-\pi^2/4}$, about $z = 0$, (d) $\frac{z^2}{\sin^3 z}$, about $z = \pi i$.

Solution
(a) Removable singularity at $z = 1$, poles at $\pm i, -1$. $R = 3 =$?????? to i (nearest singularity).
(b) singularities at $\pm\pi/2$ are removable.
(c) removable at $z = 0$, poles at $\pm\pi, \pm 2\pi$. $R = |\pi i \pm \pi| = \pi\sqrt{2}$.
(d) $z^2/\sin^2 z$, simple poles at $z = 0$, $R = \pi$.

VI.2.3

1	2	3	P	L	K

Consider the function $f(z) = \tan z$ in the annulus $\{3 < |z| < 4\}$. Let $f(z) = f_0(z) + f_1(z)$ be the Laurent decomposition of $f(z)$, so that $f_0(z)$ is analytic for $|z| < 4$, and $f_1(z)$ is analytic for $|z| > 3$ and vanishes at ∞. (a) Obtain an explicit expression for $f_1(z)$.

(b)

Write down the series expansion for $f_1(z)$, and determine the largest domain on which it converges.

(c)

Obtain the coefficients a_0, a_1 and a_2 of the power series expansion of $f_0(z)$.

(d)

What is the radius of convergence of the power series expansion for $f_0(z)$?

Solution

(a) $\tan z$ has two poles in the disk $|z| < 4$, simple poles at $\pm\pi/2$, principal parts $-1/(z \pm \pi/2)$. If $f_1(z) = -1/(z - \pi/2) - 1(z + \pi/2)$, then $f_0(z) = f(z) - f_1(z)$ is analytic for $|z| < 4$, and $f_1(z)$ is analytic for $|z| > 3$ and $\to 0$ as $z \to \infty$. By uniqueness, $f(z) = f_1(z) + f_2(z)$ is the Laurent decomposition.

(b) Use geometric series. Converges for $|z| > \pi/2$. $f_1(z) = -\left(\frac{z+\pi/2+z-\pi/2}{z^2-\pi^2/4}\right) = -\frac{2z}{z^2-\pi^2/4} = -\frac{-2z}{z^2}\sum_{k=0}^{\infty}\left(\frac{\pi^2}{4}\frac{1}{z^2}\right)^n =$, converges for $|z| > \pi/2$.

$-2\sum_{n=0}^{\infty}\left(\frac{\pi^2}{4}\right)^n \frac{1}{z^{2n+1}}$

(c) $a_0 = a_2 = 0, a_1 = 1 + 8/\pi^2$. (d) Since $f_0(z) = f(z) - f_1(z)$ has poles at $\pm 3\pi/2$, otherwise is analytic for $|z| < 3$, the series converges for $|z| < 3\pi/2$, and $R = 3\pi/2$.

VI.2.4

1	2	3	P	L	K

Suppose $f(z)$ is meromorphic on the disk $\{|z| < s\}$, with only a finite number of poles in the disk. Show that the Laurent decomposition of $f(z)$ with respect to the annulus $\{s - \varepsilon < |z| < s\}$ has the form $f(z) = f_0(z) + f_1(z)$, where $f_1(z)$ is the sum of the principal parts of $f(z)$ at its poles

Solution

Since $f_1(z)$ is a sum of principal parts, with poles inside $\{|z| \leqslant \delta - \varepsilon\}$, $f_1(z)$ is analytic for $|z| \geqslant \delta - \varepsilon$, and $f_1(z) \to 0$ as $z \to \infty$. Further $f_0(z) = f(z) - f_1(z)$ is analytic at ????? ??????, hence for $|z| < 5$. By the uniqueness of the Laurent decomposition, f is $f_0(z) + f_1(z) = f(z)$. Use the uniqueness of the Laurent decomposition.

VI.2.5

1	2	3	P	L	K

By estimating the coefficients of the Laurent series, prove that if z_0 is an isolated singularity of f, and if $(z - z_0) f(z) \to 0$ as $z \to z_0$, then z_0 is removable. Give a second proof based on Morea's theorem.

Solution
Suppose that z_0 is an isolated singularity of f and that $(z - z_0)^N f(z)$ is bounded near z_0. Then by Riemann's Theorem on removable singularities (see p. 172), $(z - z_0)^N f(z)$ has a removable singularity at z_0, the Laurent expansion is of the following form: there is a $r > 0$ so that

$$(z - z_0)^N f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $0 < |z - z_0| < r$. And hence, we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^{k-N} = \sum_{j=-N}^{\infty} a_{j+N} (z - z_0)^j .$$

It follows that z_0 is a pole of order at most N unless $a_k = 0$ for all $k = 0, 1, \ldots, N - 1$, in which case z_0 is a removable singularity.

VI.2.6

1	2	3	P	L	K
				LLL	

Show that if $f(z)$ is continuous on a domain D, and if $f(z)^8$ is analytic on D, then $f(z)$ is analytic on D.

Solution

VI.2.7

1	2	3	P	L	K
				LLL	

Show that if z_0 is an isolated singularity of $f(z)$, and if $(z - z_0)^N f(z)$ is bounded near z_0, then z_0 is either removable or a pole of order at most N.

Solution

Suppose that z_0 is an isolated singularity of f and that $(z - z_0)^N f(z)$ is bounded near z_0. Then by Riemann's Theorem on removable singularities (see p. 172), $(z - z_0)^N f(z)$ has a removable singularity at z_0, the Laurent expansion is of the following form: there is $r > 0$ so that

$$(z - z_0)^N f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

for all $0 < |z - z_0| < r$. And hence, we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^{k-N} = \sum_{j=-N}^{\infty} a_{j+N} (z - z_0)^j .$$

It follows that z_0 is a pole of order at most N unless $a_k = 0$ for all $k = 0, 1, \ldots, N-1$, in which case z_0 is a removable singularity.

VI.2.8

1	2	3	P	L	K

A meromorphic function f at z_0 is said to have order N at z_0 if $f(z) = (z - z_0)^N g(z)$ for some analytic function g at z_0 such that $g(z_0) \neq 0$. The order of the function 0 is defined to be $+\infty$. Show that
(a) $\operatorname{order}(fg, z_0) = \operatorname{order}(f, z_0) + \operatorname{order}(g, z_0)$,
(b) $\operatorname{order}(1/f, z_0) = -\operatorname{order}(f, z_0)$,
(c) $\operatorname{order}(f + g, z_0) \geq \min\{\operatorname{order}(f, z_0), \operatorname{order}(g, z_0)\}$.
Show that the inequality can occur in (c), but that equality hold in (c) whenever f and g have different orders at z_0.

Solution

VI.2.9

1	2	3	P	L	K

Recall that "$f(z) = O(h(z))$ as $z \to z_0$" means that there is a constant C such that $|f(z)| \leq C|h(z)|$ for z near z_0. Show that if z_0 is an isolated singularity of an analytic function $f(z)$, and if $f(z) = O((z-z_0)^m)$ as $z \to z_0$, then the Laurent coefficients of $f(z)$ are 0 for $k < m$, that is the Laurent series of $f(z)$ has the form

$$f(z) = a_m(z-z_0) + a_{m+1}(z-z_0)^{m+1} + \cdots .$$

Remark. This shows that the use of notation $O(z^m)$ in section V.6 is consistent.

Solution

VI.2.10

1	2	3	P	L	K
				LLL	

Show that if $f(z)$ and $g(z)$ are analytic functions that both have the same order $N \geq 0$ at z_0, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(N)}(z_0)}{g^{(N)}(z_0)}.$$

Solution

VI.2.11

1	2	3	P	L	K

Suppose $f(z) = \sum a_k z^k$ is analytic for $|z| < R$, and suppose that $f(z)$ extends to be meromorphic for $|z| < R + \varepsilon$, with only one pole z_0 on the circle $|z| = R$. Show that $a_k / a_{k+1} \to z_0$ as $k \to \infty$.

Solution

If $f(z) = \sum a_n z^n$ is analytic for $|z| < R$, and is meromorphic for $|z| < R+\varepsilon$, with poles of ?????$\leqslant N$ on the circle $|z| = R$, then $a_n = O\left(n^{N-1}\right)$. If $f(z) = \sum a_n z^n$ as only one pole on the circle $|z| = R$, of ????? N, with $f(z) = A/\left(z - z_0\right)^N +$ lower order, then $a_k = \frac{A(-1)^{N+1} k^N}{(N-1)! z_0^{N+k+1}} \left(1 + O\left(1/k\right)\right)$, so that $a_k / a_{k+1} \to z_0$ as $n \to \infty$. Proof: Write $f(z) = \sum\limits_{l=0}^{N} \frac{A_l}{\left(z-z_0\right)^l} + \sum b_k z^k$ analytic, $h(z) = \sum b_k z^k$ analytic. We have $|b_k| \leqslant 1/s^k$ for some $s > R$, $b_k = O\left(1/\left|z_0\right|^k\right)$. $\frac{1}{z-z_0} = -\frac{1}{z_0} \frac{1}{1-z/z_0} = -\frac{1}{z_0} \sum\limits_{j=0}^{\infty} \frac{z^j}{z_0^j}$, $(-1)^d \frac{(l-1)!}{\left(z-z_0\right)^l} = -\frac{1}{z_0} \sum\limits_{j=l}^{\infty} \frac{j(j-1)\ldots(j-l+1) z^{j-l}}{z_0^l}$

$\frac{1}{\left(z-z_0\right)^l} = \frac{(-1)^{l+1}}{(l-1)!} \sum\limits_{k=0}^{\infty} \frac{1}{z_0^{k+l+1}} (k+l) \ldots (k+1) z^k$

$\frac{1}{\left(z-z_0\right)^l} = \frac{(-1)^{l+1}}{(l-1)!} \sum\limits_{k=0}^{\infty} \frac{1}{z_0^{k+l+1}} (k+l) \ldots (k+1) z^k$

VI.2.12

1	2	3	P	L	K
				LLL	

Show that if z_0 is an isolated singularity of $f(z)$ that is not removable, then z_0 is an essential singularity for $e^{f(z)}$.

Solution

Apply Arzela'-Weierstrass theorem. Suppose $z_0 = 0$, an isolated singularity of $f(z)$, not removable. 0 essential $\Rightarrow$ values of $f(z)$???? on $\mathbb{C}$ as $z \to 0$, $\to$ values of $e^{f(z)}$????? on $\mathbb{C}$ as $z \to 0$. 0 a pole $\Rightarrow$ $f(z) \to \infty$, as $z \to 0$, monotone, values of $f(z)$?????? of 0 cover the extremum of some ?????? $\{|w| \geqslant R\}$. In any neighbourhood of 0 there are values of $e^{f(z)} = e^{w}$ that are near 0 (e^{-m}) and that are near to (e^{+m}), so $e^{f(z)}$ does not have a limit as $z \to 0$. 0 is essential singularity of $e^{f(z)}$.

VI.2.13 (From Hints and Solutions)

1	2	3	P	L	K

Let S be a sequence converging to a point $z_0 \in \mathbb{C}$, and let $f(z)$ be analytic on some disk centered at z_0 except possibly at the points of S and z_0. Show that either $f(z)$ extends to be meromorphic on some neighborhood of z_0, or else for any complex number L there is a sequence $\{w_j\}$ such that $w_j \to z_0$ and $f(w_j) \to L$.

Solution

Suppose values of $f(z)$ do not cluster at L as $z \to z_0$. Then $g(z) = 1/(f(z) - L)$ is bounded for $|z - z_0| < \varepsilon, z \neq z_j$. Apply Riemann's theorem first to the z_j's for j large, then to z_0, to see that $g(z)$ extends to be analytic for $|z - z_0| < \varepsilon$, and $f(z)$ is meromorphic there

VI.2.14

1	2	3	P	L	K

Suppose $u\left(re^{i\theta}\right)$ is harmonic on a punctured disk $\{0 < r < 1\}$, with Laurent series as in Exercise 7 of Section 1. Suppose $\alpha > 0$ is such that $r^{\alpha} u\left(re^{i\theta}\right) \to 0$ as $r \to 0$. Show that $a_n = 0 = b_n$ for $n \leq -\alpha$.

Solution

$u\left(re^{i\theta}\right)$ harmonic, $0 < r < 1$. $u = \sum\limits_{n=0}^{\infty} a_n r^n \cos n\theta + \sum\limits_{n \neq 0}^{\infty} b_n r^n \sin n\theta + c \log r$.
Let $m \geqslant \alpha$, $m - \alpha \geqslant 0$. Have $a_m r^m + a_{-m} r^{-m} = O\left(r^{-\alpha}\right)$, $a_m r^{2m} + a_{-m} = O\left(r^{-\alpha+m}\right)$ as $r \to 0$. Let $r \to 0$, get $a_{-m} = 0$ for $m \geqslant \alpha$. Similarly $b_{-m} = 0$ for $m \geqslant \alpha$.

VI.2.15

1	2	3	P	L	K

Suppose $u(z)$ is harmonic on the punctured disk $\{0 < |z| < \rho\}$.
Show that if

$$\frac{u(z)}{\log(1/|z|)} \to 0$$

as $z \to 0$, then $u(z)$ extends to be harmonic at 0. What can you say if you know only that $|u(z)| \leq C\log(1/|z|)$ for some fixed constant C and $0 < |z| < \rho$?

Solution

Assume $u(z) = O(\log(1/|z|))$ as $z \to 0$. Consider Laurent series as in 14. Have. For $m \geqslant 1$, get $\int u\left(re^{i\theta}\right) d\theta = O(\log(1/r))$ $a_m r^m + a_{-m} r^{-m} = O(\log(1/r))$, $a_m r^{2m} + a_{-m} = O(r^m \log(1/r))$, $a_{-m} = 0$ for all $m > 0$, and $b_{-m} = 0$ for $m > 0$. $u\left(re^{i\theta}\right) = \operatorname{Re} f\left(re^{i\theta}\right) + c\log r$, when f is analytic, $c =$ constant. $\frac{u(r)}{\log(1/r)} = \frac{\operatorname{Re} f\left(re^{i\theta}\right)}{\log(1/r)} + \frac{c\log r}{\log(1/r)}$, $c = 0$, and $u = \operatorname{Re} f$ is analytic at $z = 0$.

VI.3.1

1	2	3	P	L	K

Consider the functions in Exercise 1 of Section 2 above. Determine which have isolated singularities at ∞, and classify them.

Solution

VI.3.2

1	2	3	P	L	K

Suppose that $f(z)$ is an entire function that is not a polynomial. What kind of singularity can $f(z)$ have at ∞?

Solution

VI.3.3

1	2	3	P	L	K

Show that if $f(z)$ is nonconstant entire function, then $e^{f(z)}$ has an essential singularity at $z = \infty$.

Solution

VI.3.4

1	2	3	P	L	K

Show that each branch of the following functions is meromorphic at ∞, and obtain the series expansion for each branch at ∞.

(a) $(z^2-1)^{5/2}$ (b) $\sqrt[3]{(z^3-1)}$ (c) $\sqrt{z^2-1/z}$

Solution

VI.4.1

1	2	3	P	L	K
				LLL	

Find the partial fractions decompositions of the following functions.

(a) $\frac{1}{z^2-z}$ (c) $\frac{1}{(z+1)(z^2+2z+2)}$ (e) $\frac{z-1}{z+1}$

(b) $\frac{z^2+1}{z(z^2-1)}$ (d) $\frac{1}{(z^2+1)^2}$ (f) $\frac{z^2-4z+3}{z^2-z-6}$

(a)

$$\frac{1}{z^2-z} = \frac{A}{z} + \frac{B}{z-1}$$

$$\begin{aligned} A &= \lim_{z\to 0} z\, f(z) = \left.\frac{1}{z-1}\right|_{z=0} = -1 \\ B &= \lim_{z\to 1} (z-1)\ f(z) = \left.\frac{1}{z}\right|_{z=1} = 1 \end{aligned}$$

$$\frac{1}{z^2-z} = -\frac{1}{z} + \frac{1}{z-1}$$

(b)

$$\frac{z^2+1}{z(z^2-1)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+1}$$

$$\begin{aligned} A &= \lim_{z\to 0} z\, f(z) = \left.\frac{z^2+1}{z^2-1}\right|_{z=0} = -1 \\ B &= \lim_{z\to 1} (z-1)\ f(z) = \left.\frac{z^2+1}{z(z+1)}\right|_{z=1} = 1 \\ C &= \lim_{z\to -1} (z+1)\ f(z) = \left.\frac{z^2+1}{z(z-1)}\right|_{z=-1} = 1 \end{aligned}$$

$$\frac{z^2+1}{z(z^2-1)} = -\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z+1}$$

(c)

$$\frac{1}{(z+1)(z^2+2z+2)} = \frac{A}{z+1} + \frac{B}{z+1+i} + \frac{C}{z+1-i}$$

$$\begin{aligned}
A &= \lim_{z\to -1} z\,f(z) = \left.\frac{1}{(z^2+2z+2)}\right|_{z=-1} = 1 \\
B &= \lim_{z\to -1-i} (z+1+i)\; f(z) = \left.\frac{1}{(z+1)(z+1-i)}\right|_{z=-1-i} = -\frac{1}{2} \\
C &= \lim_{z\to -1+i} (z+1-i)\; f(z) = \left.\frac{1}{(z+1)(z+1+i)}\right|_{z=-1+i} = -\frac{1}{2}
\end{aligned}$$

$$\frac{1}{(z+1)(z^2+2z+2)} = \frac{1}{z+1} - \frac{1/2}{z+1+i} - \frac{1/2}{z+1-i}$$

(d) (/snider 106) man så?

$$\frac{1}{(z^2+1)^2} = \frac{A}{(z-i)^2} + \frac{B}{(z-i)} + \frac{C}{(z+i)^2} + \frac{D}{z-i}$$

$$\begin{aligned}
A &= \lim_{z\to i} (z-i)^2\; f(z) = \left.\frac{1}{(z+i)^2}\right|_{z=i} = -\frac{1}{4} \\
B &= \lim_{z\to i} (z-i)^2\; f(z) = \left.\frac{d}{dz}\;\frac{1}{(z+i)^2}\right|_{z=i} = -\frac{i}{4} \\
C &= \lim_{z\to -i} (z+i)^2\; f(z) = \left.\frac{1}{(z-i)^2}\right|_{z=-i} = -\frac{1}{4} \\
D &= \lim_{z\to -i} (z+i)^2\; f(z) = \left.\frac{d}{dz}\;\frac{1}{(z-i)^2}\right|_{z=-i} = \frac{i}{4}
\end{aligned}$$

$$\frac{1}{(z^2+1)^2} = -\frac{1/4}{(z-i)^2} - \frac{i/4}{(z-i)} - \frac{1/4}{(z+i)^2} + \frac{i/4}{z-i}$$

(e)

$$\frac{z-1}{z+1} = \frac{A}{z+1} + B$$

$$\begin{aligned}
A &= \lim_{z\to -1} (z+1)\, f(z) = \left. z-1\right|_{z=-1} = -2 \\
B &= \lim_{z\to\infty} f(z) = 1
\end{aligned}$$

$$\frac{z-1}{z+1} = 1 - \frac{2}{z+1}$$

(f)

$$\frac{z^2-4z+3}{z^2-z-6} = \frac{A}{z+2} + \frac{B}{z-3} + C$$

$$\begin{aligned}
A &= \lim_{z\to -2} (z+2)\ f(z) = \left.\frac{z^2-4z+3}{z-3}\right|_{z=-2} = -3 \\
B &= \lim_{z\to 3} (z-3)\ f(z) = \left.\frac{z^2-4z+3}{z(z+1)}\right|_{z=3} = 0 \\
C &= \lim_{z\to\infty} f(z) = 1
\end{aligned}$$

$$\frac{z^2-4z+3}{z^2-z-6} = \frac{-3}{z+2} + 1$$

VI.4.2

1	2	3	P	L	K
				LLL	

Use the division algorithm to obtain the partial fractions decomposition of the following functions
(a) $\frac{z^3+1}{z^2+1}$ (b) $\frac{z^9+1}{z^6-1}$ (c) $\frac{z^6}{(z^2+1)(z-1)^2}$

(a)

$$\frac{z^3+1}{z^2+1} = z + \frac{-z+1}{z^2+1} = z + \frac{A}{z+i} + \frac{B}{z-i}$$

$$\begin{aligned} A &= \lim_{z\to -i} (z+i)\ f(z) = \left.\frac{-z+1}{z-i}\right|_{z=-i} = -\frac{1}{2} + \frac{1}{2}i \\ B &= \lim_{z\to i} (z-i)\ f(z) = \left.\frac{-z+1}{z+i}\right|_{z=i} = -\frac{1}{2} - \frac{1}{2}i \end{aligned}$$

$$\frac{z^3+1}{z^2+1} = z + \frac{-z+1}{z^2+1} = z + \frac{(-1+i)/2}{z+i} - \frac{(1+i)/2}{z-i}$$

(b)

$$\frac{z^9+1}{z^6-1} = z^3 + \frac{z^3+1}{z^6-1} = z^3 + \frac{1}{z^3-1} = z^3 + \frac{1}{(z-1)(z^2+z+1)} = z^3 + \frac{A}{z-1} + \frac{B}{z-w} + \frac{C}{z-\bar{w}}$$

Because $z^3 - 1 = 0$ have tree roots $z_1 = 1$, $z_2 = e^{2\pi i/3}$, and $z_3 = e^{4\pi i/3}$. Set $z_2 = w$, and we see that $z_3 = \bar{w}$.

$$\begin{aligned} A &= \lim_{z\to 1} (z-1)\ f(z) = \left.\frac{1}{z^2+z+1}\right|_{z=1} = \frac{1}{3} \\ B &= \lim_{z\to w} (z-w)\ f(z) = \left.\frac{1}{(z-1)(z-e^{-2\pi i/3})}\right|_{z=e^{2\pi i/3}} = \frac{w}{3} \\ C &= \lim_{z\to \bar{w}} (z-\bar{w})^2\ f(z) = \left.\frac{1}{(z-1)(z-e^{2\pi i/3})}\right|_{z=e^{-2\pi i/3}} = \frac{\bar{w}}{3} \end{aligned}$$

$$\frac{z^9+1}{z^6-1} = z^3 + \frac{1/3}{z-1} + \frac{w/3}{z-w} + \frac{\bar{w}/3}{z-\bar{w}}, \qquad \text{where } w = e^{2\pi i/3}$$

Konstigt svar i boken?
$\left(\frac{1/3}{z-1}+\frac{w/3}{z-w}+\frac{\bar{w}/3}{z-\bar{w}}\right)-\left(-\frac{1/3}{z-1}+\frac{w/3}{z-w}-\frac{w/3}{z+w}\right)=$
(c)

$$\frac{z^6}{(z^2+1)(z^2-2z+1)}=\frac{z^6}{z^4-2z^3+2z^2-2z+1}$$

$$\frac{z^6}{z^4-2z^3+2z^2-2z+1}=z^2+2z+2+\frac{2z^3-z^2+2z-2}{(z^2+1)(z-1)^2}$$

$$\frac{2z^3-z^2+2z-2}{(z^2+1)(z-1)^2}=\frac{A}{z+i}+\frac{B}{z-i}+\frac{C}{(z-1)^2}+\frac{D}{z-1}$$

$$\begin{aligned}
A &= \lim_{z\to -i}(z+i)^2\ f(z)=\left.\frac{2z^3-z^2+2z-2}{(z-i)(z-1)^2}\right|_{z=-i}=-\frac{1}{4}\\
B &= \lim_{z\to i}(z-i)\ f(z)=\left.\frac{2z^3-z^2+2z-2}{(z+i)(z-1)^2}\right|_{z=i}=-\frac{1}{4}\\
C &= \lim_{z\to 1}(z-1)^2\ f(z)=\left.\frac{2z^3-z^2+2z-2}{(z^2+1)}\right|_{z=1}=\frac{1}{2}\\
D &= \lim_{z\to 1}(z-1)^2\ f(z)=\frac{d}{dz}\left.\frac{2z^3-z^2+2z-2}{(z^2+1)}\right|_{z=1}=\frac{5}{2}
\end{aligned}$$

VI.4.3

1	2	3	P	L	K
				LLL	

Let V be the complex vector space of functions that are analytic on the extended complex plane except possibly at the points 0 and i, where they have poles of order at most two. What is the dimension of V? Write down explicitly a vector space basis for V.

Solution

VI.5.1

1	2	3	P	L	K

Show that if $f(z)$ and $g(z)$ have period w, then so do $f(z) + g(z)$ and $f(z)\,g(z)$.

Solution

VI.5.2

1	2	3	P	L	K

Expand $1/\cos(2\pi z)$ in a series of powers of $e^{2\pi iz}$ that converges in the upper half-plane. Determine where the series converge absolutely and where it converges uniformly.

Solution

VI.5.3

1	2	3	P	L	K

Expand $\tan z$ in a series of powers of exponentials e^{ikz}, $-\infty < k < \infty$, that converges in the upper half-plane. Also find an expansion of $\tan z$ as in an exponential series that converges in the lower half-plane.

Solution

VI.5.4

1	2	3	P	L	K

Let $f(z)$ be an analytic function in the upper half-plane that is periodic, with real period $2\pi\lambda > 0$. Suppose that there are $A, C > 0$ such that $|f(x+iy)| \leq Ce^{Ay}$ for $y > 0$. Show that

$$f(z) = \sum_{n \geq -A\lambda} a_n e^{inz/\lambda},$$

where the series converges uniformly in each half-plane $\{y \geq \varepsilon\}$, for fixed $\varepsilon > 0$.

Solution

VI.5.5

1	2	3	P	L	K

Suppose that ± 1 are periods of a nonzero doubly periodic function $f(z)$, and suppose that there are no periods w of $f(z)$ satisfying $0 < |w| < 1$. How many periods of $f(z)$ lie on the unit circle? Describe the possibilities, and sketch the set of periods for each possibility.

Solution

VI.5.6

1	2	3	P	L	K

We say that w_1 and w_2 generate the periods of a double periodic function if the periods of the function are precisely the complex numbers of the form $mw_1 + nw_2$ where m and n are integers. Show that if w_1 and w_2 generate the periods of a doubly periodic function $f(z)$, and if λ_1 and λ_2 are complex numbers, then λ_1 and λ_2 generate the periods of $f(z)$ if and only if there is a 2×2 matrix A with integer entries and with determinant ± 1 such that $A(w_1, w_2) = (\lambda_1, \lambda_2)$.

Solution

VI.5.7

1	2	3	P	L	K

Let $w_1$and w_2 be two complex numbers that do lie on the same line through 0. Let $k \geq 3$. Show that the series

$$\sum_{m,n=-\infty}^{\infty} \frac{1}{(z-(mw_1+nw_2))^k}$$

converges uniformly on any bounded subset of the complex plane to double periodic meromorphic function $f(z)$, whose periods are generated by w_1 and w_2. Strategy. Show that the number of periods in any annulus $\{N \leq |z| \leq N+1\}$ is bounded by CN for some constant C.

Solution

VI.6.1

1	2	3	P	L	K

Consider the continuous function $f\left(e^{i\theta}\right) = |\theta|$, $-\pi \leq \theta \leq \pi$. Find the complex Fourier series $f\left(e^{i\theta}\right)$ and show that it can be expressed as a cosine series. Sketch the graphs of the first three partial sums of the cosine series. Discuss the convergence of the series. Does it converge uniformly? Partial answer. The cosine series is

$$|\theta| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos\theta + \frac{1}{3^2}\cos 3\theta + \frac{1}{5^2}\cos 5\theta + \cdots\right).$$

Solution

VI.6.2

1	2	3	P	L	K

Let $f\left(e^{i\theta}\right)$, $-\pi < \theta \le \pi$ (the principal value of the argument). Find the complex Fourier series of $f\left(e^{i\theta}\right)$ and the sine series of $f\left(e^{i\theta}\right)$. Show that the complex Fourier series diverges at $\theta = \pm\pi$, while the sine series converges at $\pm\pi$. Differentiate the complex Fourier series term by term and determine where the differentiated series converges.

Solution

VI.6.3

1	2	3	P	L	K

Consider the continuous function $f\left(e^{i\theta}\right) = \theta^2$, $-\pi \leq \theta \leq \pi$. Find the complex Fourier series of $f\left(e^{i\theta}\right)$ and show that it can be expressed as a cosine series. Discuss the convergence of the series. Does it converge uniformly? By substituting $\theta = 0$, show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots .$$

Solution

VI.6.4

1	2	3	P	L	K

Consider the continuous function $f\left(e^{i\theta}\right) = \theta^4 - 2\pi^2\theta^2$, $-\pi \leq \theta \leq \pi$. Find the complex Fourier series of $f\left(e^{i\theta}\right)$ and show that it can be expressed as a cosine series. Relate the Fourier series to the series of the function in Exercise 3.

Solution

VI.6.5

1	2	3	P	L	K

Show that if $\sum c_k$ converges absolutely, then $\sum c_k e^{ik\theta}$ converges absolutely for each θ, and the series converges uniformly for $-\pi \leq \theta \leq \pi$.

Solution

VI.6.6

1	2	3	P	L	K

Show that any function $f\left(e^{i\theta}\right)$ on the unit circle with absolutely convergent Fourier series has the form $f\left(e^{i\theta}\right) = g\left(e^{i\theta}\right) + \overline{h\left(e^{i\theta}\right)}$, where $g(z)$ and $h(z)$ are continuous functions on the unit circle that extend continuously to be analytic on the open unit disk.

Solution

VI.6.7

1	2	3	P	L	K

If $f\left(e^{i\theta}\right) \sim \sum c_k e^{ik\theta}$, and the series converges uniformly to $f\left(e^{i\theta}\right)$, then

$$\int_{-\pi}^{\pi} \left|f\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} = \sum_{k=-\infty}^{\infty} |c_k|^2 .$$

Remark. This is called Parseval's identity. Formula (6.6) show Parseval's identity holds for a function $f\left(e^{i\theta}\right)$ if and only if the partial sums of the Fourier series of $f\left(e^{i\theta}\right)$ converge to $f\left(e^{i\theta}\right)$ in the sense of "mean-square" or "L^2- approximation".

Solution

VI.6.8

1	2	3	P	L	K

By applying Parseval's identity to the picewise constant function with series (6.5), show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots .$$

Use this identity and som algebraic manipulation to show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

Solution

VI.6.9

1	2	3	P	L	K

By applying Parseval's identity to the function of Exercise 1, show that

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots .$$

Use this identity and som algebraic manipulation to show that

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots .$$

Solution

VI.6.10

1	2	3	P	L	K

If $f(z)$ is analytic in some annulus containing the unit circle $|z| = 1$, with Laurent expansion $\sum a_k z^k$, then

$$\frac{1}{2\pi} \oint_{|z|=1} |f(z)|^2 \, |dz| = \sum_{k=-\infty}^{\infty} |a_k|^2 .$$

Solution

VI.6.11

1	2	3	P	L	K

Let $f\left(e^{i\theta}\right)$ be a continuous function on the unit circle, with Fourier series $\sum c_k e^{ik\theta}$. Show that $f\left(e^{i\theta}\right)$ extends to be analytic on some annulus containing the unit circle if and only if there exist $r < 1$ and $C > 0$ such that $|c_k| \leq Cr^{|k|}$ for $-\infty < k < \infty$.

Solution

VI.6.12

1	2	3	P	L	K

Using the convergence theorem for Fourier series, prove that every contintinously function on the unit circle in the complex plane can be approximated uniformly there by trigonometric polynomials, that is, by finite linear combinations of exponentials $e^{ik\theta}$, $-\infty < k < \infty$. Strategy. First approximate $f\left(e^{ik\theta}\right)$ by a smooth function.

Solution

VI.6.13

1	2	3	P	L	K

Let D be a domain bounded by a smooth boundary curve of length 2π. We parametrize the boundary of D by arc length s, sot the boundary is given by a smooth periodic function $\gamma(s)$, $0 \leq s \leq 2\pi$. Let $\sum c_k e^{iks}$ be the Fourier series of $\gamma(s)$.

(a)

Show that $\sum k^2 |c_k|^2 = 1$. Hint. Apply Parseval's identity to $\gamma'(s)$ and use $|\gamma'(s)| = 1$ for a curve parameterized by arc length.

(b)

Show that the area of D is $\pi \sum k |c_k|^2$. Hint. Use Exercise IV.1.4.

(c)

Show that the area of D is $\leq \pi$, with equality if and only if D is a disk. Remark. This proves the isoperimetric theorem. Among all smooth closed curves of a given length, the curve that surrounds the largest area is a circle.

Solution

VI.1.14

1	2	3	P	L	K

Show that

$$\int_{-\pi}^{\pi}\left|f\left(e^{i\theta}\right)-\sum_{k=-m}^{n}b_k e^{ik\theta}\right|^2\frac{d\theta}{2\pi}=\int_{-\pi}^{\pi}\left|f\left(e^{i\theta}\right)-\sum_{k=-m}^{n}c_k e^{ik\theta}\right|^2\frac{d\theta}{2\pi}+\sum_{k=-m}^{n}\left|b_k-c_k\right|^2,$$

for any choice of complex numbers b_k, $-m\le k\le n$. Remark. This shows that the best mean-square approximate to $f\left(e^{i\theta}\right)$ by exponential sums $\sum_{-m}^{n}b_k e^{ik\theta}$, for fixed m and n, is the corresponding partial sum of the Fourier series.

Solution

VI.1.15

1	2	3	P	L	K

Show that a continously differentiable function on the unit circle has an absolutely convergent Fourier series. Strategy. Write the Fourier coefficients c_k of $f\left(e^{i\theta}\right)$ as $a_k b_k$, where $a_k = 1/ik$ and b_k is the Fourier coefficient of the derivative. Use Bessel's inequality and the Cauchy-Schwarz inequality $\left|\sum \alpha_k \overline{\beta_k}\right| \leq \sqrt{\sum |\alpha_k|^2}\sqrt{\sum |\beta_k|^2}$.

Solution

VI.6.16

1	2	3	P	L	K

Let $f\left(e^{i\theta}\right)$ be a continuous function on the unit circle. Suppose that $f\left(e^{i\theta}\right)$ is piecewise continuously differentiable, in the sense that it has a continuous derivative except at a finite number of points, at each of which the derivative has limits from the left and from the right. Show that the Fourier series $f\left(e^{i\theta}\right)$ is absolutely convergent. Strategy. Cancel the discontinuities of the derivative using translates of the function in Exercise 3, whose Fourier series is absolutely convergent.

Solution

VI.6.17

1	2	3	P	L	K

Let $f\left(e^{i\theta}\right)$ be a piecewise continuously differentiable, in the sense that it is continuously differentiable except at a finite number of points, at each of which both the function and its derivative have limits from the left and from the right. Show that the Fourier series of $f\left(e^{i\theta}\right)$ converges at each point, to $f\left(e^{i\theta}\right)$ if the function is continuous at $e^{i\theta}$, and otherwise to the average of the limits of $f\left(e^{i\theta}\right)$ from the left and from the right. Strategy. Show that $f\left(e^{i\theta}\right) = f_1\left(e^{i\theta}\right) + \sum b_j h_j\left(e^{i\theta}\right)$, where $f_1\left(e^{i\theta}\right)$ satisfies the hypotheses of Exercise 15, and each $h_j\left(e^{i\theta}\right)$ is obtained from the function of Exercise 2 by change of variable $\theta \longmapsto \theta - \theta_j$.

Solution

VII	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	X	X	X			X													
2	X	X	X	X	X	X	X	X	X	X	X	X							
3	X	X	X	X	X	X	X												
4	X	X	X	X	X	X	X	X	X	X									
5	X	X	X	X	X	X													
6	X	X	X	X	X	X	X												
7		X	X		X														
8				X															

VII.1.1
Evaluate the following residues.

(a)	$\operatorname{Res}\left[\frac{1}{z^2+4}, 2i\right]$	(d)	$\operatorname{Res}\left[\frac{\sin z}{z^2}, 0\right]$	(g)	$\operatorname{Res}\left[\frac{z}{\operatorname{Log} z}, 1\right]$
(b)	$\operatorname{Res}\left[\frac{1}{z^2+4}, -2i\right]$	(e)	$\operatorname{Res}\left[\frac{\cos z}{z^2}, 0\right]$	(h)	$\operatorname{Res}\left[\frac{e^z}{z^5}, 0\right]$
(c)	$\operatorname{Res}\left[\frac{1}{z^5-1}, 1\right]$	(f)	$\operatorname{Res}[\cot z, 0]$	(i)	$\operatorname{Res}\left[\frac{z^n+1}{z^n-1}, w_k\right]$

Solution
a)
By rule 4,

$$\operatorname{Res}\left[\frac{1}{z^2+4}, 2i\right] = \left.\frac{1}{2z}\right|_{z=2i} = \frac{1}{4i} = -\frac{i}{4}.$$

b)
By rule 4,

$$\operatorname{Res}\left[\frac{1}{z^2+4}, -2i\right] = \left.\frac{1}{2z}\right|_{z=-2i} = -\frac{1}{4i} = \frac{i}{4}.$$

c)
By rule 4,

$$\operatorname{Res}\left[\frac{1}{z^5-1}, 1\right] = \left.\frac{1}{5z^4}\right|_{z=1} = \frac{1}{5}.$$

d)
By rule 1,

$$\operatorname{Res}\left[\frac{\sin z}{z^2}, 0\right] = \lim_{z\to 0}\frac{\sin z}{z} = 1.$$

e)
By rule 2,

$$\operatorname{Res}\left[\frac{\cos z}{z^2}, 0\right] = \lim_{z\to 0}\frac{d}{dz}\cos z = \left.-\sin z\right|_{z=0} = 0.$$

f)
By rule 3,

$$\operatorname{Res}[\cot z, 0] = \operatorname{Res}\left[\frac{\cos z}{\sin z}, 0\right] = \left.\frac{\cos z}{\cos z}\right|_{z=0} = 1.$$

g)
By rule 3,

$$\operatorname{Res}\left[\frac{z}{\operatorname{Log} z}, 1\right] = \left.\frac{z}{1/z}\right|_{z=1} = 1.$$

h)
By Laurent expansion

$$\frac{e^z}{z^5} = \frac{1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + o\left(z^5\right)}{z^5} = \frac{1}{z^5} + \frac{1}{1!z^4} + \frac{1}{2!z^3} + \frac{1}{3!z^2} + \frac{1}{4!z} + o\left(1\right),$$

hence

$$\operatorname{Res}\left[\frac{e^z}{z^5}, 0\right] = \frac{1}{4!} = \frac{1}{24}.$$

i)
We have poles of the form $w_k = e^{2\pi ik/n}$, where $k = 0, 1, 2, \ldots n-1$, remark that $w_k^n = 1$, by rule 3,

$$\operatorname{Res}\left[\frac{z^n+1}{z^n-1}, w_k\right] = \left.\frac{z^n+1}{nz^{n-1}}\right|_{z=w_k} = \frac{w_k^n+1}{nw_k^{n-1}} = \frac{2}{nw_k^{n-1}} = \frac{2w_k}{n} = \frac{2e^{2\pi ik/n}}{n}.$$

VII.1.2
Calculate the residue at each singularity in the complex plane of the following functions.
(a) $e^{1/z}$ (b) $\tan z$ (c) $\frac{z}{(z^2+1)^2}$ (d) $\frac{1}{z^2+z}$

Solution
(a)
The function $e^{1/z}$ has a singularity at $z = 0$, by Laurent expansion

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots.$$

Thus by definition

$$\operatorname{Res}\left[e^{1/z}, 0\right] = 1.$$

(b)
The function $\tan z = \frac{\sin z}{\cos z}$ has isolated singularities at $z = \frac{\pi}{2} + \pi n$, $-\infty < n < \infty$, they are simple poles.
By rule 3,

$$\operatorname{Res}\left[\tan z, \pi/2 + n\pi\right] = \operatorname{Res}\left[\frac{\sin z}{\cos z}, \pi/2 + n\pi\right] = \left.\frac{\sin z}{-\sin z}\right|_{z=\pi/2+n\pi} = -1.$$

(c)
The function $\frac{z}{(z^2+1)^2}$ has isolated singularities at $z = \pm i$, they are double poles.
By rule 2,

$$\operatorname{Res}\left[\frac{z}{(z^2+1)^2}, i\right] = \frac{d}{dz}\left.\frac{z}{(z+i)^2}\right|_{z=i} = \left.\frac{(z+i)^2 - 2z(z+i)}{(z+i)^4}\right|_{z=i} = 0,$$

$$\operatorname{Res}\left[\frac{z}{(z^2+1)^2}, -i\right] = \frac{d}{dz}\left.\frac{z}{(z-i)^2}\right|_{z=-i} = \left.\frac{(z-i)^2 - 2z(z-i)}{(z-i)^4}\right|_{z=-i} = 0.$$

(d)

The function $\frac{1}{z(z+1)}$ has isolated singularities at $z = 0$ and $z = -1$ they are simple poles.
By rule 1,

$$\operatorname{Res}\left[\frac{1}{z\,(z+1)}, 0\right] = \lim_{z\to 0}\frac{1}{z+1} = \left.\frac{1}{z+1}\right|_{z=0} = 1,$$

$$\operatorname{Res}\left[\frac{1}{z\,(z+1)}, -1\right] = \lim_{z\to -1}\frac{1}{z} = \left.\frac{1}{z}\right|_{z=-1} = -1.$$

VII.1.3

Evaluate the following integrals using residue theorem.

(a) $\int_{|z|=1} \frac{\sin z}{z^2} dz$ (c) $\int_{|z|=2} \frac{z}{\cos z} dz$ (e) $\int_{|z-1|=1} \frac{1}{z^8-1} dz$

(b) $\int_{|z|=2} \frac{e^z}{z^2-1} dz$ (d) $\int_{|z|=1} \frac{z^4}{\sin z} dz$ (f) $\int_{|z-1/2|=3/2} \frac{\tan z}{z} dz$

Solution

(a)

Using the residue theorem and rule 1,

$$\int_{|z|=1} \frac{\sin z}{z^2} dz = 2\pi i \operatorname{Res}\left[\frac{\sin z}{z^2}, 0\right] = 2\pi i \left(\lim_{z\to 0} \frac{\sin z}{z}\right) = 2\pi i\,(1) = 2\pi i.$$

(b)

Using the residue theorem and rule 3,

$$\begin{aligned}\int_{|z|=2} \frac{e^z}{z^2-1} dz &= \\ &= 2\pi i \left(\operatorname{Res}\left[\frac{e^z}{z^2-1}, 1\right] + \operatorname{Res}\left[\frac{e^z}{z^2-1}, -1\right]\right) = \\ &= 2\pi i \left(\left.\frac{e^z}{2z}\right|_{z=1} + \left.\frac{e^z}{2z}\right|_{z=-1}\right) = 2\pi i \left(\frac{e^1 - e^{-1}}{2}\right) = \\ &= 2\pi i \sinh 1.\end{aligned}$$

(c)

Using the residue theorem and rule 3,

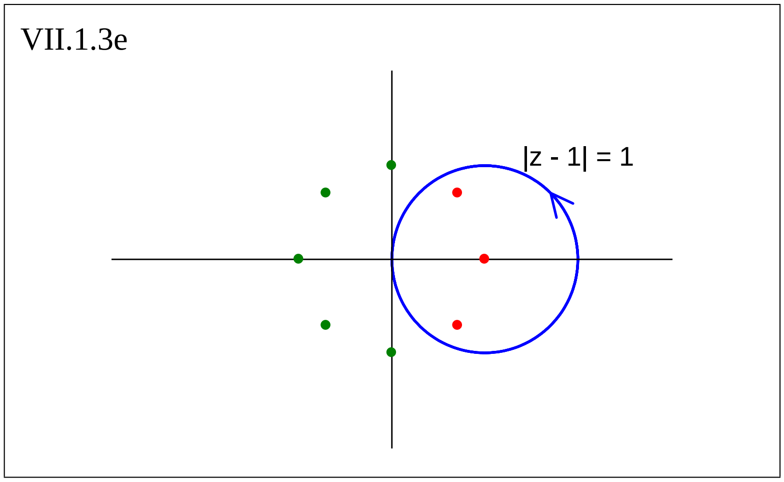
$$\begin{aligned}\int_{|z|=2} \frac{z}{\cos z} dz &= 2\pi i \left(\operatorname{Res}\left[\frac{z}{\cos z}, \frac{\pi}{2}\right] + \operatorname{Res}\left[\frac{z}{\cos z}, -\frac{\pi}{2}\right]\right) = \\ &= 2\pi i \left(\left.\frac{z}{-\sin z}\right|_{z=\pi/2} + \left.\frac{z}{-\sin z}\right|_{z=-\pi/2}\right) = 2\pi i \left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = \\ &= -2\pi^2 i.\end{aligned}$$

(d)

Using the residue theorem and rule 3,

$$\int_{|z|=1} \frac{z^4}{\sin z} dz = 2\pi i \operatorname{Res}\left[\frac{z^4}{\sin z}, 0\right] = 2\pi i \left(\left.\frac{z^4}{\cos z}\right|_{z=0} \right) = 0.$$

(e)



Using the residue theorem with the contour in Figure VII.1.3e and rule 4,

$$\int_{|z-1|=1} \frac{1}{z^8-1} dz = 2\pi i \left(\operatorname{Res}\left[\frac{1}{z^8-1}, 1\right] + \operatorname{Res}\left[\frac{1}{z^8-1}, e^{i\pi/4}\right] + \operatorname{Res}\left[\frac{1}{z^8-1}, e^{-i\pi/4}\right] \right) =$$

$$= 2\pi i \left(\left.\frac{1}{8z^7}\right|_{z=1} + \left.\frac{1}{8z^7}\right|_{z=e^{i\pi/4}} + \left.\frac{1}{8z^7}\right|_{z=e^{-i\pi/4}} \right) = \frac{\pi i}{4}\left[1 + e^{-\pi i/4} + e^{\pi i/4}\right] =$$

$$= \frac{\pi i}{4}\left[1 + \frac{1-i}{\sqrt{2}} + \frac{1+i}{\sqrt{2}}\right] = \frac{\pi i}{4}\left(1+\sqrt{2}\right).$$

(f)

The function $\frac{\tan z}{z} = \frac{\sin z/z}{\cos z}$ has removable singularity at $z=0$, and a isolated singularity at $z = \frac{\pi}{2}$ inside the domain $|z-1/2| = 3/2$. Using the residue theorem and rule 3,

$$
\int_{|z-1/2|=3/2} \frac{\tan z}{z} dz =
$$

$$
= 2\pi i \operatorname{Res}\left[\frac{\sin z/z}{\cos z}, \frac{\pi}{2}\right] = 2\pi i \left(\lim_{z\to\pi/2} \frac{\sin z/z}{-\sin z}\right) = 2\pi i \left(\left.\frac{\sin z/z}{-\sin z}\right|_{z=\pi/2}\right) = (2\pi i)\left(-\frac{2}{\pi}\right) =
$$

$$
= -4i.
$$

VII.1.4

1	2	3	P	L	K

Suppose $P(z)$ and $Q(z)$ are polynomials such that the zeros of $Q(z)$ are simple zeros at the points $z_1, \ldots, z_m$, and $\deg P(z) < \deg Q(z)$. Show that the partial fractions decomposition $P(z)/Q(z)$ is given by

$$\frac{P(z)}{Q(z)} = \sum_{j=1}^{m} \frac{P(z_j)}{Q'(z_j)} \frac{1}{z - z_j}.$$

Solution

Since the zeros of $Q(z)$ are simple, the polynomial quotient $P(z)/Q(z)$ has at most simple poles. By Rule 3, the residue at z_j is

$$P(z)/Q'(z).$$

Hence the principal part of $P(z)/Q(z)$ at z_j is

$$\frac{P(z_j)}{Q'(z_j)} \frac{1}{z - z_j}.$$

Since deg $P(z) < \deg Q(z)$, then

$$\frac{P(z)}{Q(z)} \to 0$$

as $z \to \infty$, and $P(z)/Q(z)$ has no principal part at ∞.

By partial fraction decomposition our polynomial quotient $P(z)/Q(z)$ or can be written as the sum of its principal parts at its pole,

$$\frac{P(z)}{Q(z)} = \sum_{j=1}^{m} \frac{P(z_j)}{Q'(z_j)} \frac{1}{z - z_j}.$$

VII.1.5

1	2	3	P	L	K

Let $f(z)$ be a meromorphic function on the complex plane that is doubly periodic, and suppose that none of the poles of $f(z)$ lie on the boundary of the period parallelogram P constructed in Section VI.5. By integrating $f(z)$ around of P, show that the sum of the residues at the poles of $f(z)$ in P is zero. Conclude that there is no doubly periodic meromorphic function with only one pole, a simple pole, in the period parallelogram.

Solution

Follow directions. $\int_{\partial P} = 0$, the curve integrals over opposite sides cancel. Thus the sum of the residues is zero. In particular, there can not be a single nonzero residue.

VII.1.6
Consider the integral

$$\int_{\partial D_R} \frac{e^{\pi i (z-1/2)^2}}{1-e^{-2\pi i z}} dz,$$

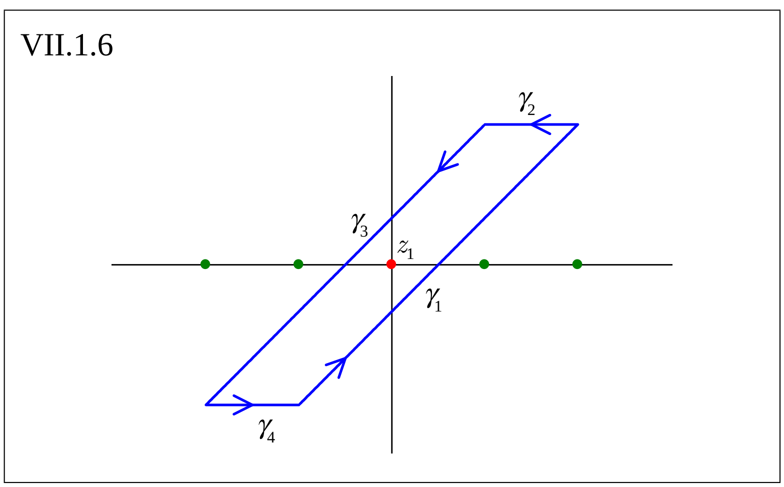
where D_R is the parallelogram with vertices $\pm\frac{1}{2} \pm (1+i)\,R$.
(a)
Use the residue theorem to show that the integral is $(1+i)/\sqrt{2}$.
(b)
By parameterizing the sides of the parallelogram, show that the integral tends to

$$(1+i)\int_{-\infty}^{\infty} e^{-2\pi t^2} dt$$

as $R \to \infty$.
(c)
Use (a) and (b) to show that

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}.$$

Solution
(a)



Set

$$I=\int_{\partial D_R}\frac{e^{\pi i(z-1/2)^2}}{1-e^{-2\pi i z}}dz,$$

i.e, we integrate

$$f\left(z\right)=\frac{e^{\pi i(z-1/2)^2}}{1-e^{-2\pi i z}}$$

along the parallelogram contour in Figure VII.1.6. The parallelogram has vertices at $\pm 1/2\pm\left(1+i\right)R$.
Residue at a simple pole at $z_1=0$, where by Rule 3,

$$\operatorname{Res}\left[\frac{e^{\pi i(z-1/2)^2}}{1-e^{-2\pi i z}},0\right]=\left.\frac{e^{\pi i(z-1/2)^2}}{2\pi i e^{-2\pi i z}}\right|_{z=0}=\frac{e^{\pi i/4}}{2\pi i}.$$

Using the *Residue Theorem*, we obtain that

$$\int_{\partial D_R}\frac{e^{\pi i(z-1/2)^2}}{1-e^{-2\pi i z}}dz=2\pi i\cdot\frac{e^{\pi i/4}}{2\pi i},$$

i.e.,

$$\int_{\partial D_R}\frac{e^{\pi i(z-1/2)^2}}{1-e^{-2\pi i z}}dz=\frac{1+i}{\sqrt{2}}.$$

(b)
Integrate along γ_1 parametrized by $z=\frac{1}{2}+\left(1+i\right)t$ with $-R\leq t\leq R$, and let $R\to\infty$. This gives

$$\int_{\gamma_1}f\left(z\right)dz=\int_{-R}^{R}\frac{e^{\pi i(z-1/2)^2}}{1-e^{-2\pi i z}}dz=\begin{bmatrix} z &=& \frac{1}{2}+\left(1+i\right)t \\ dz &=& \left(1+i\right)dt\end{bmatrix}=$$

$$=\int_{-R}^{R}\frac{e^{\pi i((1+i)t)^2}\left(1+i\right)}{1-e^{-2\pi i\left(\frac{1}{2}+(1+i)t\right)}}dt=\left(1+i\right)\int_{-R}^{R}\frac{e^{-2\pi t^2}}{1+e^{-2\pi i(1+i)t}}dt\to$$

$$\to\ \left(1+i\right)\int_{-\infty}^{\infty}\frac{e^{-2\pi t^2}}{1+e^{-2\pi i(1+i)t}}\ dt.$$

Now integrate along γ_2 parameterized by $z=-t+\frac{1}{2}+\left(1+i\right)R$ with $0\leq t\leq 1$, and let $R\to\infty$.

$$\left|\int_{\gamma_2} f(z)\,dz\right| =$$

$$= \left|\int_{\gamma_2} \frac{e^{\pi i(z-1/2)^2}}{1-e^{-2\pi i z}} dz\right| = \begin{bmatrix} z & = & -t+\frac{1}{2}+(1+i)R \\ dz & = & -dt \end{bmatrix} =$$

$$= \left|\int_0^1 \frac{e^{\pi i(-t+(1+i)R)^2}}{1-e^{-2\pi i(-t+1/2+(1+i)R)}} dt\right| = \left|\int_0^1 \frac{e^{\pi\left(t^2-2t\right)i} e^{-2\pi\left(R^2-Rt\right)}}{1-e^{\pi(2t-1-2R)i} e^{2\pi R}} dt\right| \leq$$

$$\leq \sup_{0\leq t\leq 1} \left|\frac{e^{\pi\left(t^2-2t\right)i} e^{-2\pi\left(R^2-Rt\right)}}{1-e^{\pi(2t-1-2R)i} e^{2\pi R}}\right| \cdot 1 \leq \sup_{0\leq t\leq 1} \frac{e^{-2\pi\left(R^2-Rt\right)}}{\left|1-\left|e^{\pi(2t-1-2R)i} e^{2\pi R}\right|\right|} =$$

$$= \sup_{0\leq t\leq 1} \frac{e^{-2\pi\left(R^2-Rt\right)}}{\left|1-e^{2\pi R}\right|} = \sup_{0\leq t\leq 1} \frac{e^{-2\pi\left(R^2-Rt\right)}}{e^{2\pi R}-1} \to 0.$$

Integrate along γ_3, parameterized by $z = -\frac{1}{2} + (1+i)\,t$ with $-R \leq t \leq R$, and let $R \to \infty$. This gives

$$\int_{\gamma_3} f(z)\,dz = \int_R^{-R} \frac{e^{\pi i(z-1/2)^2}}{1-e^{-2\pi i z}} dz dz = \begin{bmatrix} z & = & -\frac{1}{2}+(1+i)\,t \\ dz & = & (1+i)\,dt \end{bmatrix} =$$

$$= -\int_{-R}^{R} \frac{e^{\pi i(-1+(1+i)t)^2}(1+i)\,dt}{1-e^{-2\pi i\left(-\frac{1}{2}+(1+i)t\right)}} = (1+i)\int_{-R}^{R} \frac{e^{-2\pi i(1+i)t} e^{-2\pi t^2} dt}{1+e^{-2\pi i(1+i)t}} \to$$

$$\to (1+i)\int_{-\infty}^{\infty} \frac{e^{-2\pi i(1+i)t} e^{-2\pi t^2} dt}{1+e^{-2\pi i(1+i)t}} dt.$$

Now integrate along γ_4 parameterized by $z = t-\frac{1}{2}-(1+i)\,R$ with $0 \leq t \leq 1$, and let $R \to \infty$.

$$\left|\int_{\gamma_4} f(z)\,dz\right| =$$

$$= \left|\int_{\gamma_4} \frac{e^{\pi i(z-1/2)^2}}{1-e^{-2\pi i z}}dz\right| = \left[\begin{array}{rcl} z & = & t-\frac{1}{2}-(1+i)R \\ dz & = & dt \end{array}\right] =$$

$$= \left|\int_0^1 \frac{e^{\pi i(t-1-(1+i)R)^2}}{1-e^{-2\pi i(t-1/2-(1+i)R)}}dz\right| = \left|\int_0^1 \frac{e^{\pi\left(2R-2Rt+t^2-2t+1\right)i}e^{-2\pi\left(R^2+R-Rt\right)}}{1-e^{-\pi(2t-1-2R)i}e^{-2\pi R}}dz\right| =$$

$$\leq \sup_{0\leq t\leq 1}\left|\frac{e^{\pi\left(2R-2Rt+t^2-2t+1\right)i}e^{-2\pi\left(R^2+R-Rt\right)}}{1-e^{-\pi(2t-1-2R)i}e^{-2\pi R}}\right|\cdot 1 \leq \sup_{0\leq t\leq 1}\frac{e^{-2\pi\left(R^2+R-Rt\right)}}{|1-|e^{-\pi(2t-1-2R)i}e^{-2\pi R}||} \leq$$

$$\leq \sup_{0\leq t\leq 1}\frac{e^{-2\pi\left(R^2+R-Rt\right)}}{|1-e^{-2\pi R}|} \to 0.$$

Using the *Residue Theorem* and letting $R\to\infty$, we obtain that

$$\begin{aligned}(1+i)\int_{-\infty}^{\infty}\frac{e^{-2\pi t^2}}{1+e^{-2\pi i(1+i)t}}\ dt + 0 + (1+i)\int_{-\infty}^{\infty}\frac{e^{-2\pi i(1+i)t}e^{-2\pi t^2}dt}{1+e^{-2\pi i(1+i)t}}dt + 0 \ &= \\ &= \ 2\pi i\cdot\frac{e^{\pi i/4}}{2\pi i},\end{aligned}$$

and therefore

$$(1+i)\int_{-\infty}^{\infty}\frac{e^{-2\pi t^2}\left(1+e^{-2\pi i(1+i)t}\right)}{1+e^{-2\pi i(1+i)t}}\ dt = \frac{1+i}{\sqrt{2}},$$

i.e.,

$$(1+i)\int_{-\infty}^{\infty}e^{-2\pi t^2}\ dt = \frac{1+i}{\sqrt{2}}.$$

(c)
Comparing real parts thus yields,

$$\int_{-\infty}^{\infty}e^{-2\pi t^2}dt = \frac{1}{\sqrt{2}},$$

and changing variables as follows gives the desired result

$$\frac{1}{\sqrt{2}} = \int_{-\infty}^{\infty} e^{-2\pi t^2} dt = \begin{bmatrix} s & = & \sqrt{2\pi}t \\ ds & = & \sqrt{2\pi}dt \end{bmatrix} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds,$$

i.e.,

$$\int_{-\infty}^{\infty} e^{-s^2} dt = \sqrt{\pi}.$$

VII.2.1

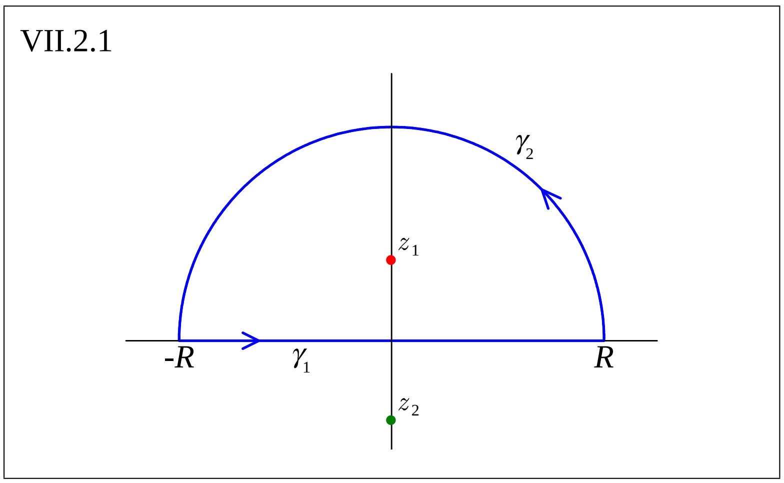
1	2	3	P	L	K

Show using residue theory that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}, \qquad a > 0.$$

Remark. **Check the result by evaluating the integral directly, using the arctangent function.**

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2},$$

and integrate

$$f(z) = \frac{1}{z^2+a^2} = \frac{1}{(z-ia)(z+ia)}$$

along the contour in Figure VII.2.1.
Residue at a simple pole at $z_1 = ia$, where by Rule 2,

$$\operatorname{Res}\left[\frac{1}{z^2+a^2}, ia\right] = \left.\frac{1}{2z}\right|_{z=ia} = -\frac{i}{2a}.$$

Integrate along γ_1, and let $R \to \infty$. This gives

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{dx}{x^2+a^2} \to \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = I.$$

Integrate along γ_2, and let $R \to \infty$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{1}{R^2-a^2} \cdot \pi R \sim \frac{\pi}{R} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$I + 0 = 2\pi i \cdot \left(-\frac{i}{2a}\right),$$

and therefore

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}, \qquad a > 0.$$

Remark.
Check of the result by evaluating the integral directly gives, because the integrand is positive the integral can be computed taking a single limit, namely,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \lim_{b\to\infty} \int_{-b}^{b} \frac{dx}{x^2+a^2} = \lim_{b\to\infty} \left[\frac{1}{a} \arctan \frac{x}{a}\right]_{-b}^{b} = \frac{\pi}{a}, \qquad a > 0.$$

VII.2.2

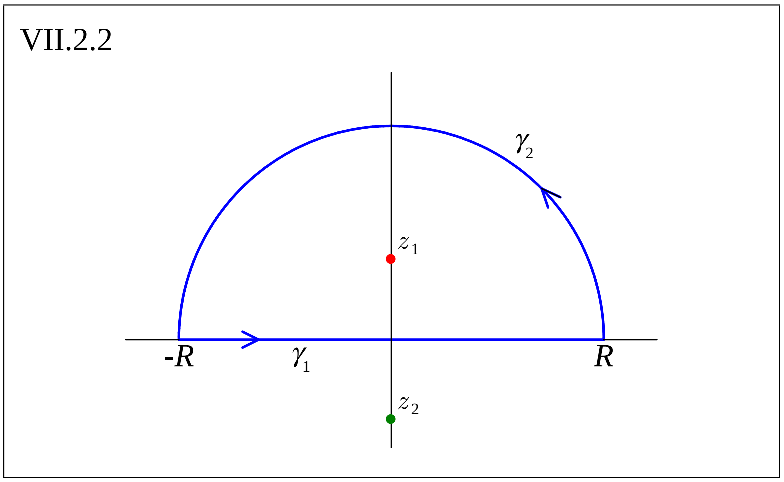
1	2	3	P	L	K

Show using residue theory that

$$\int_{-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^2} = \frac{\pi}{2a^3}.$$

Remark. **Check the result by differentiating the formula in the preceding exercise with respect to the parameter.**

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^2},$$

and integrate

$$f(z) = \frac{1}{\left(z^2+a^2\right)^2} = \frac{1}{\left(z-ia\right)^2\left(z+ia\right)^2}$$

along the contour in Figure VII.2.2.
Residue at a double pole at $z_1 = ia$, where by Rule 2,

$$\operatorname{Res}\left[\frac{1}{\left(z^2+a^2\right)^2}, ia\right] = \lim_{z\to ia}\frac{d}{dz}\frac{1}{\left(z+ia\right)^2} = \left.\frac{-2}{\left(z+ia\right)^3}\right|_{z=ia} = -\frac{i}{4a^3}.$$

Integrate along γ_1, and let $R \to \infty$. This gives

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{dx}{\left(x^2+a^2\right)^2} \to \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^2} = I.$$

Integrate along γ_2, and let $R \to \infty$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{1}{\left(R^2-a^2\right)^2} \cdot \pi R \sim \frac{\pi}{R^3} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that that

$$I + 0 = 2\pi i \cdot \left(-\frac{i}{4a^3}\right),$$

and therefore

$$\int_{-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^2} = \frac{\pi}{2a^3}.$$

Remark.
From Exercise VII.2.1 we have the formula

$$(1) \qquad \int_{-\infty}^{\infty} \frac{1}{x^2+a^2}dx = \frac{\pi}{a}, \qquad a > 0.$$

It is allowed to differentiate both sides in the formula, because for every compact interval with $a \neq 0$, both the integrand and its derivative are continuous, and the primitive of the derivative is uniformly bounded by a $a-$ independent function $h(x)$ such that $\int h(x) < \infty$ on every such like interval. We start by differentiate the left hand side in the formula with respect to the parameter a

$$(2) \qquad \frac{d}{da}\left(\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}dx\right) = \int_{-\infty}^{\infty} \frac{-2a}{\left(x^2+a^2\right)^2}dx = -2a \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^2},$$

and differentiate the right hand side in the formula in the same way

$$(3) \qquad \frac{d}{da}\frac{\pi}{a} = -\frac{\pi}{a^2}.$$

By (1) - (3), we have that

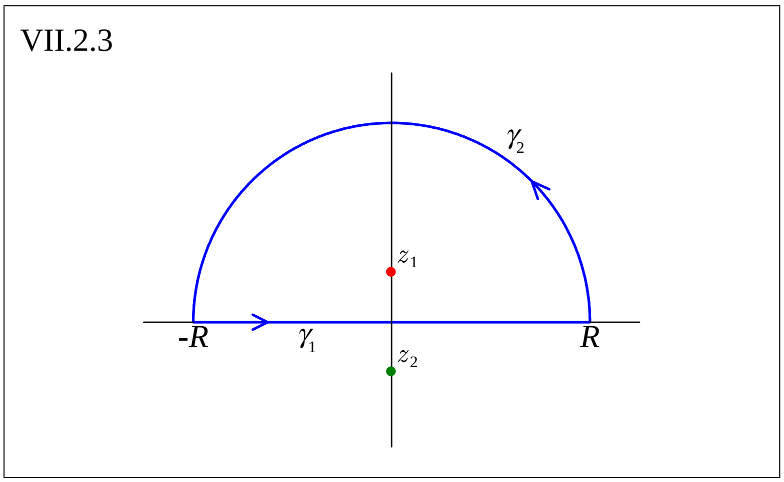
$$\int_{-\infty}^{\infty} \frac{dx}{\left(x^2 + a^2\right)^2} = \frac{\pi}{2a^3}.$$

VII.2.3
Show using residue theory that

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{\left(x^2+1\right)^2} = \frac{\pi}{2}.$$

Remark. **Check the result by combining the preceding two exercises.**

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{x^2 dx}{\left(x^2+1\right)^2},$$

and integrate

$$f\left(z\right) = \frac{z^2}{\left(z^2+1\right)^2} = \frac{z^2}{\left(z-i\right)^2\left(z+i\right)^2}$$

along the contour in Figure VII.2.3.
Residue at a double pole at $z_1 = i$, where by Rule 2,

$$\operatorname{Res}\left[\frac{z^2}{\left(z^2+1\right)^2}, i\right] = \lim_{z\to i}\frac{d}{dz}\frac{z^2}{\left(z^2+i\right)^2} = \left.\frac{2z\left(z+i\right)^2 - 2z^2\left(z+i\right)}{\left(z+i\right)^4}\right|_{z=i} = -\frac{i}{4}.$$

Integrate along γ_1, and let $R \to \infty$. This gives

$$\int_{\gamma_1} f(z)\, dz = \int_{-R}^{R} \frac{x^2 dx}{(x^2+1)^2} \to \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2} = I.$$

Integrate along γ_2, and let $R \to \infty$. This gives

$$\left| \int_{\gamma_2} f(z)\, dz \right| \leq \frac{R^2}{(R^2-1)^2} \cdot \pi R \sim \frac{\pi}{R} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$I + 0 = 2\pi i \cdot \left(-\frac{i}{4} \right),$$

and therefore

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2} = \frac{\pi}{2}.$$

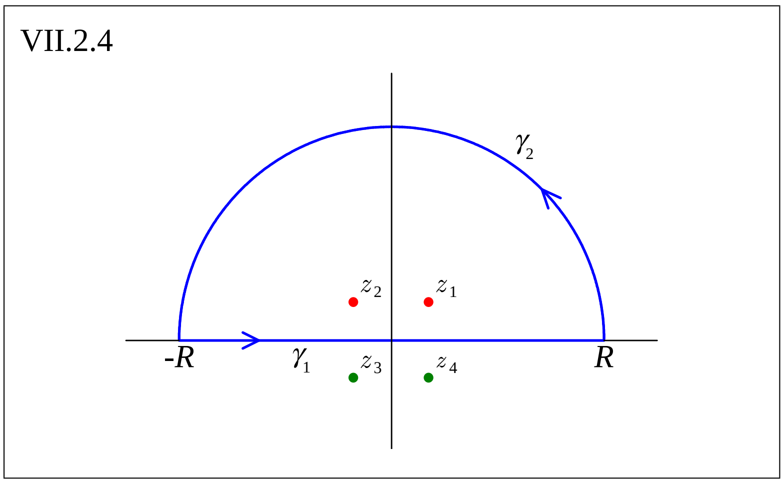
Remark.
We split the integrand using partial fraction, and put $a = 1$ in the formulas from the preceding two exercises

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2} = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} - \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

VII.2.4
Using residue theory, show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.$$

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^4+1},$$

and integrate

$$f(z) = \frac{1}{z^4+1} = \frac{1}{\left(z-\left(\frac{1+i}{\sqrt{2}}\right)\right)\left(z-\left(\frac{-1+i}{\sqrt{2}}\right)\right)\left(z-\left(\frac{-1-i}{\sqrt{2}}\right)\right)\left(z-\left(\frac{1-i}{\sqrt{2}}\right)\right)}$$

along the contour in Figure VII.2.4.
Residue at a simple pole at $z_1 = \frac{1+i}{\sqrt{2}}$, where by Rule 2,

$$\operatorname{Res}\left[\frac{1}{z^4+1}, \frac{1+i}{\sqrt{2}}\right] = \left.\frac{1}{4z^3}\right|_{z=(1+i)/\sqrt{2}} = \frac{\sqrt{2}}{8}(-1-i)$$

Residue at a simple pole at $z_2 = \frac{-1+i}{\sqrt{2}}$, where by Rule 2,

$$\operatorname{Res}\left[\frac{1}{z^4+1},\frac{-1+i}{\sqrt{2}}\right] = \left.\frac{1}{4z^3}\right|_{z=(-1+i)/\sqrt{2}} = \frac{\sqrt{2}}{8}(1-i)$$

Integrate along γ_1, and let $R \to \infty$. This gives

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{dx}{x^4+1} \to \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = I.$$

Integrate along γ_2, and let $R \to \infty$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \frac{1}{R^4-1}\cdot \pi R \sim \frac{\pi}{R^3} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$I + 0 = 2\pi i \cdot \left(\frac{\sqrt{2}}{8}(-1-i) + \frac{\sqrt{2}}{8}(1-i)\right),$$

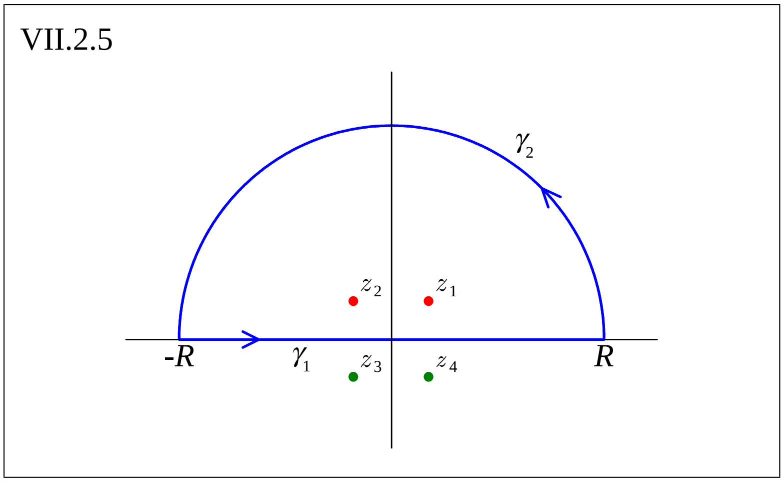
and therefore

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.$$

VII.2.5
Using residue theory, show that

$$\int_0^\infty \frac{x^2}{x^4+1}dx = \frac{\pi}{2\sqrt{2}}.$$

Solution



Set

$$I = \int_0^\infty \frac{x^2}{x^4+1}dx \Rightarrow 2I = \int_{-\infty}^\infty \frac{x^2}{x^4+1}dx$$

and integrate

$$f(z) = \frac{z^2}{z^4+1} = \frac{z^2}{\left(z-\left(\frac{1+i}{\sqrt{2}}\right)\right)\left(z-\left(\frac{-1+i}{\sqrt{2}}\right)\right)\left(z-\left(\frac{-1-i}{\sqrt{2}}\right)\right)\left(z-\left(\frac{1-i}{\sqrt{2}}\right)\right)}$$

along the contour in Figure VII.2.5.
Residue at a simple pole at $z_1 = \frac{1+i}{\sqrt{2}}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{z^2}{z^4+1}, \frac{1+i}{\sqrt{2}}\right] = \left.\frac{1}{4z}\right|_{z=(1+i)/\sqrt{2}} = \frac{\sqrt{2}}{8}(1-i).$$

Residue at a simple pole at $z_2 = \frac{-1+i}{\sqrt{2}}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{z^2}{z^4+1}, \frac{-1+i}{\sqrt{2}}\right] = \left.\frac{1}{4z}\right|_{z=(-1+i)/\sqrt{2}} = \frac{\sqrt{2}}{8}\left(-1-i\right).$$

Integrate along γ_1, and let $R \to \infty$. This gives

$$\int_{\gamma_1} f\left(z\right) dz = \int_{-R}^{R} \frac{x^2}{x^4+1} dx \to \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx = 2I.$$

Integrate along γ_2, and let $R \to \infty$. This gives

$$\left|\int_{\gamma_2} f\left(z\right) dz\right| \leq \frac{R^2}{R^4-1} \cdot \pi R \sim \frac{\pi}{R} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$2I + 0 = 2\pi i \cdot \left(\frac{\sqrt{2}}{8}\left(1-i\right) + \frac{\sqrt{2}}{8}\left(-1-i\right)\right),$$

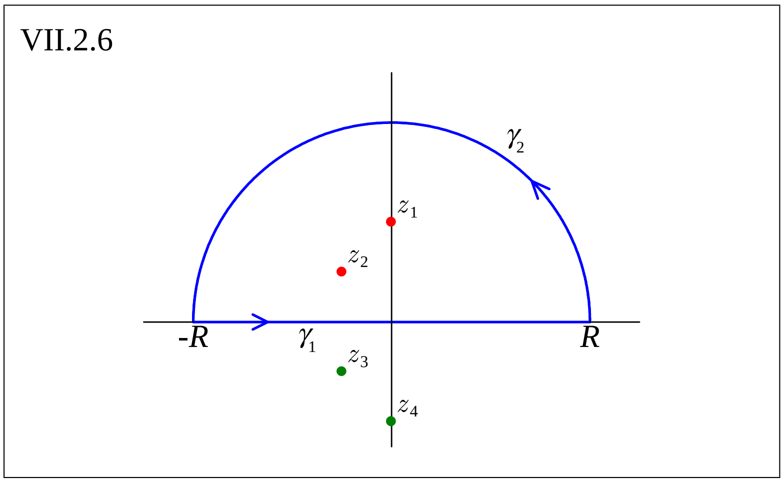
and therefore

$$\int_{0}^{\infty} \frac{x^2}{x^4+1} dx = \frac{\pi}{2\sqrt{2}}.$$

VII.2.6
Show that

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)} dx = -\frac{\pi}{10}.$$

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)} dx$$

and integrate

$$f(z) = \frac{z}{(z^2+2z+2)(z^2+4)} = \frac{z}{(z-(-1+i))(z-2i)(z-(-1-i))(z+2i)}$$

along the contour in Figure VII.2.6.
Residue at a simple pole at $z_1 = 2i$, where by Rule 3,

$$\text{Res}\left[\frac{z}{(z^2+2z+2)(z^2+4)}, 2i\right] = \frac{z}{(z^2+2z+2)\,2z}\bigg|_{z=2i} = -\frac{1}{20} - \frac{1}{10}i.$$

Residue at a simple pole at $z_2 = -1+i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{z}{\left(z^2+2z+2\right)\left(z^2+4\right)},-1+i\right]=\left.\frac{z}{\left(2z+2\right)\left(z^2+4\right)}\right|_{z=-1+i}=\frac{1}{20}+\frac{3}{20}i.$$

Integrate along γ_1, and let $R\to\infty$. This gives

$$\int_{\gamma_1} f\left(z\right)dz=\int_{-R}^{R}\frac{x}{\left(x^2+2x+2\right)\left(x^2+4\right)}dx\to$$
$$\to\int_{-\infty}^{\infty}\frac{x}{\left(x^2+2x+2\right)\left(x^2+4\right)}dx=I.$$

Integrate along γ_2, and let $R\to\infty$. This gives

$$\left|\int_{\gamma_2} f\left(z\right)dz\right|\leq\frac{R}{\left(R^2-2R-2\right)\left(R^2-4\right)}\cdot\pi R\sim\frac{\pi}{R^2}\to 0.$$

Using the *Residue Theorem* and letting $R\to\infty$, we obtain that

$$I+0=2\pi i\cdot\left(-\frac{1}{20}-\frac{1}{10}i+\frac{1}{20}+\frac{3}{20}i\right)$$

and therefore

$$\int_{-\infty}^{\infty}\frac{x}{\left(x^2+2x+2\right)\left(x^2+4\right)}dx=-\frac{\pi}{10}.$$

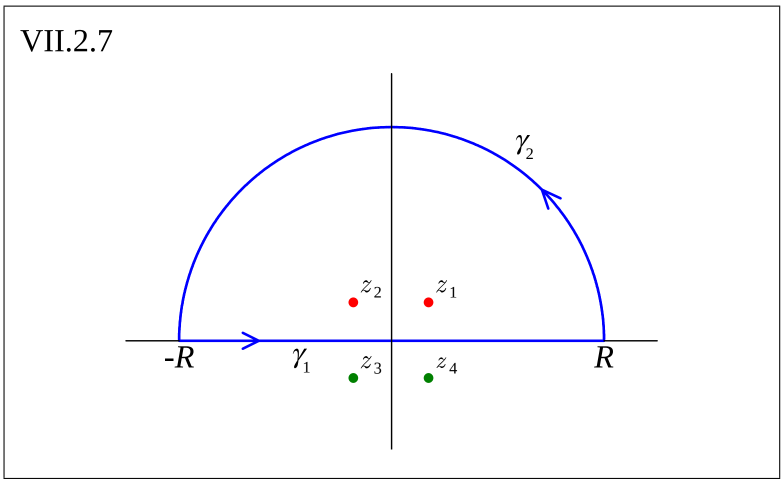
VII.2.7

1	2	3	P	L	K
				LLL	

Show that

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^4+1} dx = \frac{\pi}{\sqrt{2}} e^{-a/\sqrt{2}} \left(\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}} \right), \qquad a > 0.$$

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^4+1} dx \qquad a > 0,$$

and integrate

$$f(z) = \frac{e^{iaz}}{z^4+1} = \frac{e^{iaz}}{\left(z - \left(\frac{1+i}{\sqrt{2}}\right)\right)\left(z - \left(\frac{-1+i}{\sqrt{2}}\right)\right)\left(z - \left(\frac{-1-i}{\sqrt{2}}\right)\right)\left(z - \left(\frac{1-i}{\sqrt{2}}\right)\right)}$$

along the contour in Figure VII.2.7.
Residue at a simple pole at $z_1 = \frac{1+i}{\sqrt{2}}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{e^{iaz}}{z^4+1}, \frac{1+i}{\sqrt{2}}\right] = \left.\frac{e^{iaz}}{4z^3}\right|_{z=(1+i)/\sqrt{2}} =$$

$$= -\frac{\sqrt{2}e^{-a/\sqrt{2}}\cos\left(a/\sqrt{2}\right)}{8} - \frac{\sqrt{2}e^{-a/\sqrt{2}}\sin\left(a/\sqrt{2}\right)}{8}i - \frac{\sqrt{2}e^{-a/\sqrt{2}}\cos\left(a/\sqrt{2}\right)}{8}i + \frac{\sqrt{2}e^{-a/\sqrt{2}}\sin\left(a/\sqrt{2}\right)}{8}$$

Residue at a simple pole at $z_2 = \frac{-1+i}{\sqrt{2}}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{e^{iaz}}{z^4+1}, \frac{-1+i}{\sqrt{2}}\right] = \left.\frac{e^{iaz}}{4z^3}\right|_{z=(-1+i)/\sqrt{2}} =$$

$$= \frac{\sqrt{2}e^{-a/\sqrt{2}}\cos\left(a/\sqrt{2}\right)}{8} - \frac{\sqrt{2}e^{-a/\sqrt{2}}\sin\left(a/\sqrt{2}\right)}{8}i - \frac{\sqrt{2}e^{-a/\sqrt{2}}\cos\left(a/\sqrt{2}\right)}{8}i - \frac{\sqrt{2}e^{-a/\sqrt{2}}\sin\left(a/\sqrt{2}\right)}{8}$$

Integrate along γ_1, and let $R \to \infty$. This gives

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{e^{ax}}{x^4+1}dx \to \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^4+1}dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^4+1}dx = I + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^4+1}dx$$

Integrate along γ_2, and let $R \to \infty$. Because $|e^{iaz}| \le 1$ in the upper half plane if $a > 0$, this gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{1}{R^4-1} \cdot \pi R \sim \frac{\pi}{R^3} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$I + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^4+1}dx = 2\pi i \cdot \left(-\frac{\sqrt{2}e^{-a/\sqrt{2}}\cos\left(a/\sqrt{2}\right)}{4}i - \frac{\sqrt{2}e^{-a/\sqrt{2}}\sin\left(a/\sqrt{2}\right)}{4}i \right),$$

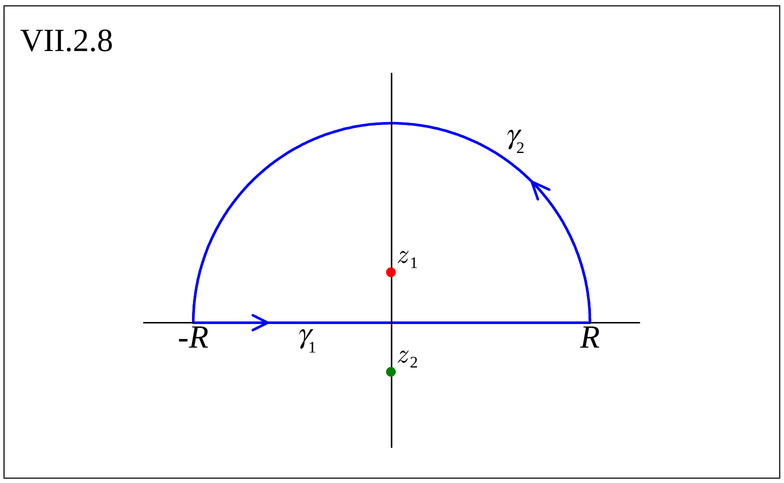
and therefore

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^4+1}dx = \frac{\pi}{\sqrt{2}}e^{-a/\sqrt{2}}\left(\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}}\right), \qquad a > 0.$$

VII.2.8
Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx = \frac{\pi}{e}.$$

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx$$

and integrate

$$f(z) = \frac{e^{iz}}{(z^2+1)^2} = \frac{e^{iz}}{(z-i)^2 (z+i)^2}$$

along the contour in Figure VII.2.8.
Residue at a double pole at $z_1 = i$, where by Rule 2,

$$\operatorname{Res}\left[\frac{e^{iz}}{(z^2+1)^2}, i\right] = \lim_{z\to i} \frac{d}{dz} \frac{e^{iz}}{(z+i)^2} == \left. \frac{ie^{iz}(z+i)^2 - 2e^{iz}(z+i)}{(z+i)^4} \right|_{z=i} = -\frac{e^{-1}}{2} i$$

Integrate along γ_1, and let $R \to \infty$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2} dx \to \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx + i \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)^2} dx =$$

$$= I + i \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)^2} dx.$$

Integrate along γ_2, and let $R \to \infty$. Because $|e^{iz}| \leq 1$ in the upper half plane, this gives

$$\left| \int_{\gamma_2} f(z)\,dz \right| \leq \frac{1}{(R^2-1)^2} \cdot \pi R \sim \frac{\pi}{R^3} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$I + i \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)^2} dx + 0 = 2\pi i \cdot \left(-\frac{e^{-1}}{2} i \right),$$

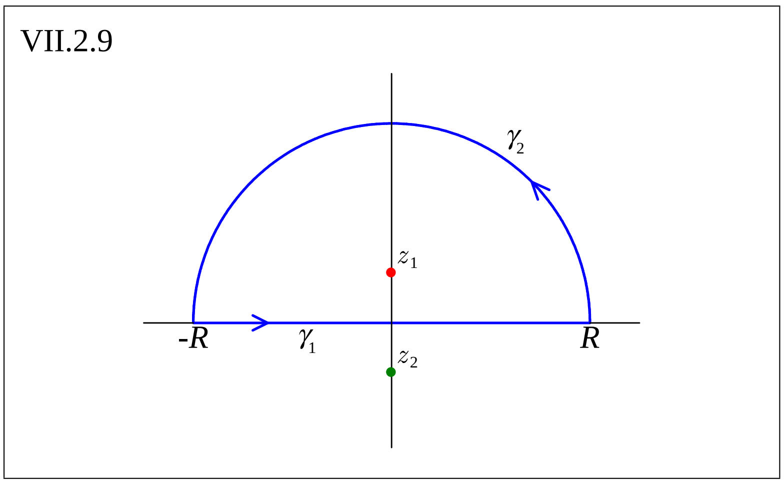
and therefore

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx = \frac{\pi}{e}.$$

VII.2.9
Show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1} dx = \frac{\pi}{2}\left[1-\frac{1}{e^2}\right].$$

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1} dx = \int_{-\infty}^{\infty} \frac{1-\cos 2x}{2\left(x^2+1\right)} dx$$

and integrate

$$f(z) = \frac{1-e^{i2z}}{2\left(z^2+1\right)} = \frac{1-e^{i2z}}{2(z-i)(z+i)}$$

along the contour in Figure VII.2.9.
Residue at a simple pole at $z_1 = i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{1-e^{i2z}}{2\left(z^2+1\right)}, i\right] = \left.\frac{1-e^{i2z}}{4z}\right|_{z=i} = \frac{1}{4} i e^{-2} - \frac{1}{4} i$$

Integrate along γ_1, and let $R \to \infty$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{-R}^{R} \frac{1-e^{i2x}}{2(x^2+1)}dx \to \int_{-\infty}^{\infty} \frac{1-\cos 2x}{2(x^2+1)}dx + i\int_{-\infty}^{\infty} \frac{\sin 2x}{2(x^2+1)}dx =$$

$$= I + i\int_{-\infty}^{\infty} \frac{\sin 2x}{2(x^2+1)}dx$$

Integrate along γ_2, and let $R \to \infty$. Because $|e^{iz}| \leq 1$ in the upper half plane, this gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \frac{2}{2(R^2-1)} \cdot \pi R \sim \frac{\pi}{R} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$I + i\int_{-\infty}^{\infty} \frac{\sin 2x}{2(x^2+1)}dx + 0 = 2\pi i \cdot \left(\frac{1}{4}ie^{-2} - \frac{1}{4}i\right),$$

and therefore

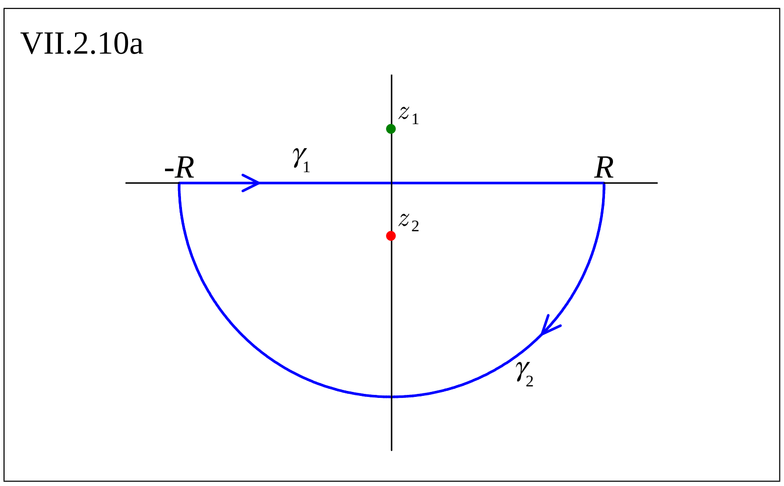
$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1}dx = \frac{\pi}{2}\left[1 - \frac{1}{e^2}\right].$$

VII.2.10
Show that

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}dx = \frac{\pi e^{-|a|b}}{b}, \qquad -\infty < a < \infty,\ b > 0.$$

For which complex values of the parameters a and b does the integral exist? Where does the integral depend analytically on the parameters?

Solution
The integral exists only for a real, and $\mathrm{Re}\, b \neq 0$ or $\cos(iab) = 0$. It depends analytically on b for $\mathrm{Re}\, b \neq 0$. It depends analytically on a in the intervals $-\infty < a < 0$ and $0 < a < \infty$.
Case 1: $a < 0$



Set

$$I = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2}dx, \qquad a < 0,\ b > 0$$

and integrate

$$f(z) = \frac{e^{iaz}}{z^2+b^2} = \frac{e^{iaz}}{(z-ib)(z+ib)}$$

along the contour in Figure 7.2.10a. (The lower half-plane)
Residue at a simple pole at $z_2 = -ib$, where by Rule 3,

$$\operatorname{Res}\left[\frac{e^{iaz}}{z^2+b^2}, -ib\right] = \left.\frac{e^{iaz}}{2z}\right|_{z=-ib} = \frac{e^{ab}}{2b}i$$

Integrate along γ_1 and let $R \to \infty$. This gives

$$\begin{aligned}
&\int_{\gamma_1} f(z)\,dz = \\
&\qquad = \int_{-R}^{R} \frac{e^{iax}}{x^2+b^2}dx \to \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+b^2}dx = \\
&\qquad\qquad\qquad\qquad\qquad\qquad = I + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+b^2}dx
\end{aligned}$$

Integrate along γ_2 and let $R \to \infty$. Because $|e^{iaz}| \le 1$ in the lower half plane if $a < 0$, this gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{1}{R^2-b^2}\cdot \pi R \sim \frac{\pi}{R} \to 0.$$

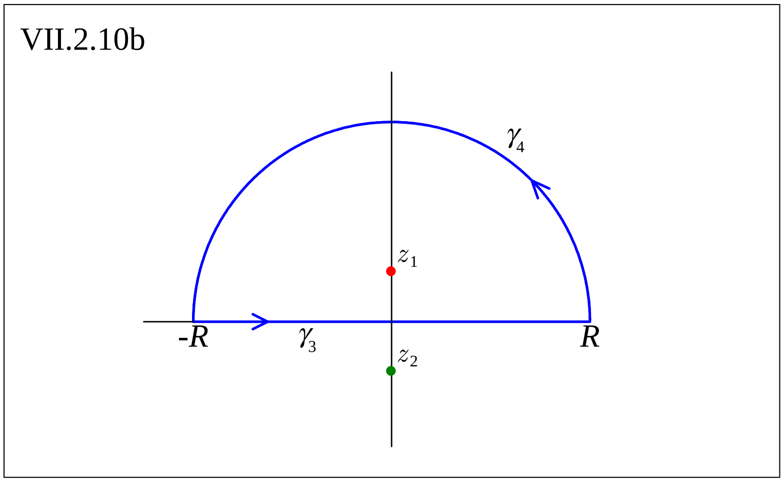
Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$I + 0 = -2\pi i \cdot \left(\frac{e^{ab}}{2b}i\right),$$

and hence

$$(1) \qquad I = \frac{\pi e^{ab}}{b},\ a < 0.$$

Case 2: $a > 0$



Set

$$I = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + b^2} dx, \qquad a > 0,\ b > 0,$$

and integrate

$$f(z) = \frac{e^{iaz}}{z^2 + b^2} = \frac{e^{iaz}}{(z - ib)(z + ib)}$$

along the contour in Figure VII.2.10b. (The upper half-plane)
Residue at a simple pole at $z_1 = ib$, where by Rule 3,

$$\operatorname{Res}\left[\frac{e^{iaz}}{z^2 + b^2}, ib\right] = \left.\frac{e^{iaz}}{2z}\right|_{z=ib} = -\frac{e^{-ab}}{2b} i$$

Integrate along γ_3 and let $R \to \infty$. This gives

$$\int_{\gamma_3} f(z)\, dz =$$

$$= \int_{-R}^{R} \frac{e^{iax}}{x^2 + b^2} dx \to \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2 + b^2} dx + i \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2 + b^2} dx =$$

$$= I + i \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2 + b^2} dx$$

Integrate along γ_4 and let $R \to \infty$. Because $|e^{iaz}| \le 1$ in the upper half plane if $a > 0$, this gives

$$\left| \int_{\gamma_4} f(z)\, dz \right| \le \frac{1}{R^2 - b^2} \cdot \pi R \sim \frac{\pi}{R} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$I + 0 = 2\pi i \cdot \left(-\frac{e^{-ab}}{2b} i \right),$$

and hence

$$(2) \qquad I = \frac{\pi e^{-ab}}{b}, \; a > 0.$$

By (1) and (2) we can concluede that

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2 + b^2} dx = \frac{\pi e^{-|a|b}}{b}, \qquad -\infty < a < \infty, \; b > 0.$$

VII.2.11

1	2	3	P	L	K

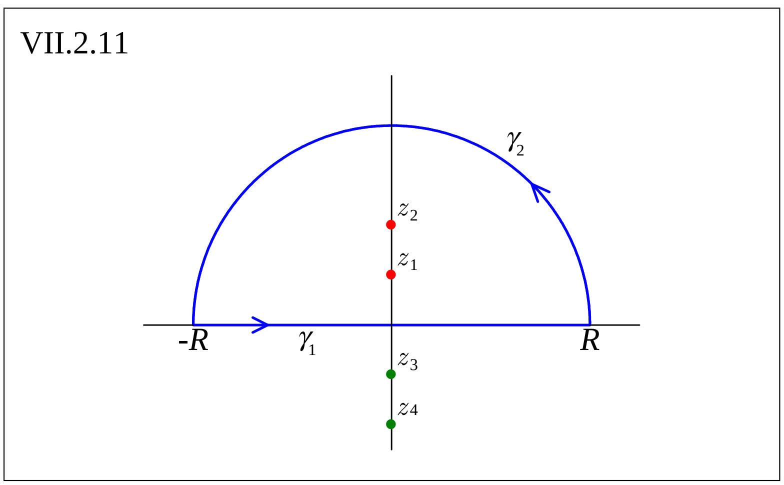
Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)\,(x^2+b^2)}\,dx.$$

Indicate the range of the parameters a and b.

Solution

The integral converges absolutely if $a^2, b^2 \notin (-\infty, 0]$, $a, b \notin i\mathbb{R}$. It also converges absolutely for certain points on the imaginary axis, so that $\pm ia$ are zeros of $\cos x$, $\pm ib$ are zeros of $\cos x$, but they are not the same zeros. The integrand, is an even function of a and b, so it suffices to evaluate it for a, b in the right half-plane. For this, we can assume that $a, b > 0$, and extend the formula by analyticity. We can also assume that $a \neq b$, and get other case in this limit. So we may as well assume that $0 < a < b$.



The integral $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$ converges absolutely if $a^2, b^2 \notin (-\infty, 0]$ i.e. $a, b \notin i\mathbb{R}$. It also converges absolutely for certain points on imaginary axis, so that $\pm ia$ are zeros of $\cos x$, $\pm ib$ are zeros of $\cos x$, but they are not same zeros. (Problem should may be simplified to avoid thus points.)

The integrand, is an even function of a and b, so it suffices to evaluate it for a, b in the right half-plane. For this, we can assume that $a, b > 0$, and extend the formula by analyticity. We can assume that $a \neq b$, and get other case in this limit. So we may as well assume $0 < a < b$.

Case 1: $(a, b > 0, a \neq b)$
Set

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx$$

and integrate

$$f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} = \frac{e^{iz}}{(z-ia)(z-ib)(z+ia)(z+ib)}$$

along the contour in Figure VII.2.11.
Residue at a simple pole at $z_1 = ai$, where by Rule 3,

$$\text{Res}\left[\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}, ia\right] = \left.\frac{e^{iz}}{2z(z^2+b^2)}\right|_{z=ai} = \frac{e^{-a}}{2a(b^2-a^2)i}$$

Residue at a simple pole at $z_1 = ib$, where by Rule 3,

$$\text{Res}\left[\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}, ib\right] = \left.\frac{e^{iz}}{2z(z^2+a^2)}\right|_{z=bi} = -\frac{e^{-b}}{2b(b^2-a^2)i}$$

Integrate along γ_1 and let $R \to \infty$. This gives

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)} dx \to$$
$$\to \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx + i\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+a^2)(x^2+b^2)} dx =$$
$$= I + i\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+a^2)(x^2+b^2)} dx$$

Integrate along γ_2 and let $R \to \infty$. Because $|e^{iz}| \leq 1$ in the upper half plane, this gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \frac{1}{(R^2-a^2)(R^2-b^2)} \cdot \pi R \sim \frac{\pi}{R^3} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$I + 0 = 2\pi i \cdot \left(\frac{e^{-a}}{2a(b^2-a^2)i} - \frac{e^{-b}}{2b(b^2-a^2)i}\right),$$

and therefore

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi \left(be^{-a} - ae^{-b}\right)}{ab\,(b^2-a^2)}.$$

Case 2: $(a, b > 0, a = b)$
From Case 1 we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi \left(be^{-a} - ae^{-b}\right)}{ab\,(b^2-a^2)} = \frac{\pi}{ab\,(b+a)} \frac{(b-a)\,e^{-a} + a\left(e^{-a} - e^{-b}\right)}{(b-a)}.$$

If we let $b \to a$, we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)^2} dx = \frac{\pi}{2a^3}\,(1+a)\,.$$

For $a = 1$, we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx = \frac{\pi}{e},$$

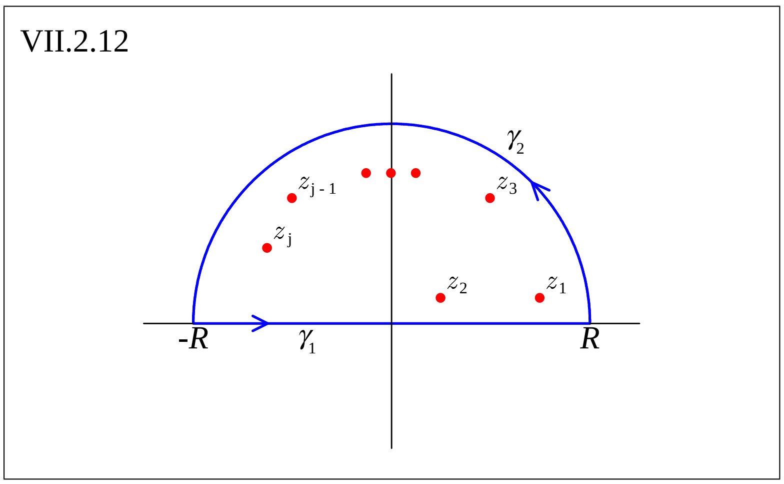
which thus yields the answer from Exercise 8.

VII.2.12
Let $Q(z)$ be a polynomial of degree m with no zeros on the real line, and let $f(z)$ be a function that is analytic in the upper half-plane and across the real line. Suppose there is $b < m-1$ such that $|f(z)| \leq |z|^b$ for z in the upper half-plane, $|z| > 1$. Show that

$$\int_{-\infty}^{\infty} \frac{f(x)}{Q(x)} dx = 2\pi i \sum \operatorname{Res}\left[\frac{f(z)}{Q(z)}, z_j\right],$$

summed over the zeros z_j of $Q(z)$ in the upper half-plane.

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{f(x)}{Q(x)} dx$$

and integrate

$$f(z) = \frac{f(z)}{Q(z)}$$

along the contour in Figure VII.2.12.
The sum of the residues at the poles $z_1, z_2, \ldots, z_j$ in the upper half plane,

$$\sum \operatorname{Res}\left[\frac{f(z)}{Q(z)}, z_j\right].$$

Integrate along γ_1, and let $R \to \infty$. This gives

$$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{f(x)}{Q(x)}dx \to \int_{-\infty}^{\infty} \frac{f(x)}{Q(x)}dx = I.$$

Integrate along γ_2, and let $R \to \infty$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| = \left[\begin{array}{c} z = Re^{i\theta} \\ 0 \leq \theta \leq \pi \end{array}\right] \leq \int_0^{\pi} \frac{\left|f\left(Re^{i\theta}\right)\right|}{\left|Q\left(Re^{i\theta}\right)\right|}d\theta \leq \frac{\left|f\left(Re^{i\theta}\right)\right|}{\left|Q\left(Re^{i\theta}\right)\right|} \cdot \pi R \leq$$

$$\leq \frac{R^b}{R^m\left(C + O\left(1/R\right)\right)} \leq \frac{\pi}{R\left(C + O\left(1/R\right)\right)} \sim \frac{\pi}{CR} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$I + 0 = 2\pi i \sum \operatorname{Res}\left[\frac{f(z)}{Q(z)}, z_j\right]$$

and therefore

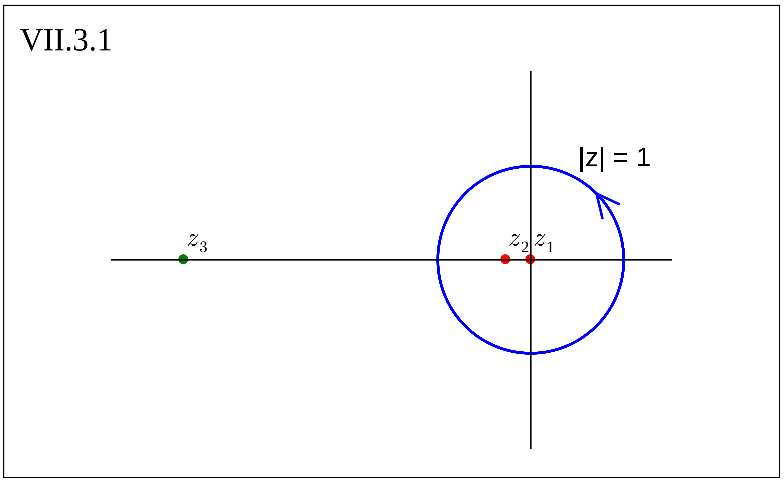
$$\int_{-\infty}^{\infty} \frac{f(x)}{Q(x)}dx = 2\pi i \sum \operatorname{Res}\left[\frac{f(z)}{Q(z)}, z_j\right]$$

Q.E.D.

VII.3.1
Show using residue theory that

$$\int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta} d\theta = 2\pi\left(1 - \frac{2}{\sqrt{3}}\right).$$

Solution



$$I = \int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta} d\theta.$$

A change of variables as in book page 203-204 gives

$$I = \int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta} d\theta = \oint_{|z|=1} \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{\left(2+\frac{1}{2}\left(z+\frac{1}{z}\right)\right)} \frac{dz}{iz} = \frac{1}{i} \oint_{|z|=1} \frac{z^2+1}{z^3+4z^2+z} dz.$$

Integrate

$$f(z) = \frac{z^2+1}{z^3+4z^2+z} = \frac{z^2+1}{z\left(z-\left(-2-\sqrt{3}\right)\right)\left(z-\left(-2+\sqrt{3}\right)\right)}$$

along the unit circle contour in Figure VII.3.1.
Residue at a simple pole at $z_1 = 0$, where by Rule 3,

$$\text{Res}\left[\frac{z^2+1}{z^3+4z^2+z}, 0\right] = \left.\frac{z^2+1}{3z^2+8z+1}\right|_{z=0} = 1.$$

Residue at a simple pole at $z_2 = -2+\sqrt{3}$, where by Rule 3,

$$\mathrm{Res}\left[\frac{z^2+1}{z^3+4z^2+z}, -2+\sqrt{3}\right] = \left.\frac{z^2+1}{3z^2+8z+1}\right|_{z=-2+\sqrt{3}} = -\frac{2\sqrt{3}}{3}.$$

Using the *Residue Theorem*, we thus obtain

$$I = 2\pi i \cdot \frac{1}{i}\left(1 - \frac{2\sqrt{3}}{3}\right),$$

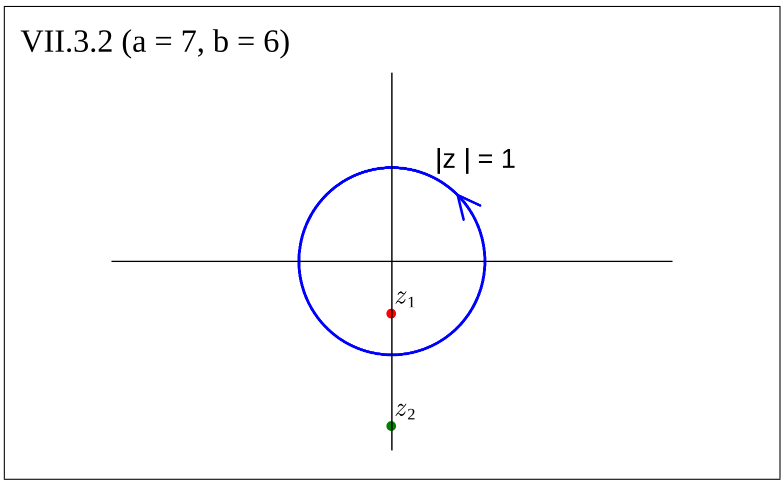
and therefore

$$\int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta}d\theta = 2\pi\left(1 - \frac{2}{\sqrt{3}}\right).$$

VII.3.2
Show using residue theory that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \qquad a > b > 0.$$

Solution



Set

$$I = \int_0^{2\pi} \frac{1}{a + b\sin\theta} d\theta.$$

A change of variables as in book page 203-204 gives

$$I = \int_0^{2\pi} \frac{1}{a + b\sin\theta} d\theta = \oint_{|z|=1} \frac{1}{\left(a + b \cdot \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)} \frac{dz}{iz} = \oint_{|z|=1} \frac{2}{bz^2 + 2azi - b} dz$$

Integrate

$$f(z) = \frac{2}{bz^2 + 2azi - b} = \frac{2}{b\left(z - \left(\frac{-a-\sqrt{a^2-b^2}}{b} i\right)\right)\left(z - \left(\frac{-a+\sqrt{a^2-b^2}}{b} i\right)\right)}$$

along the unit circle contour in Figure VII.3.2.
Residue at a simple pole at $z_1 = \frac{-a+\sqrt{a^2-b^2}}{b} i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{2}{bz^2+2azi-b}, \frac{-a+\sqrt{a^2-b^2}}{b}i\right] = \left.\frac{2}{2bz+2ai}\right|_{z=\frac{-a+\sqrt{a^2-b^2}}{b}i} = -\frac{i}{\sqrt{a^2-b^2}}.$$

Using the *Residue Theorem*, we thus obtain

$$I = 2\pi i \cdot \left(-\frac{1}{\sqrt{a^2-b^2}}i\right),$$

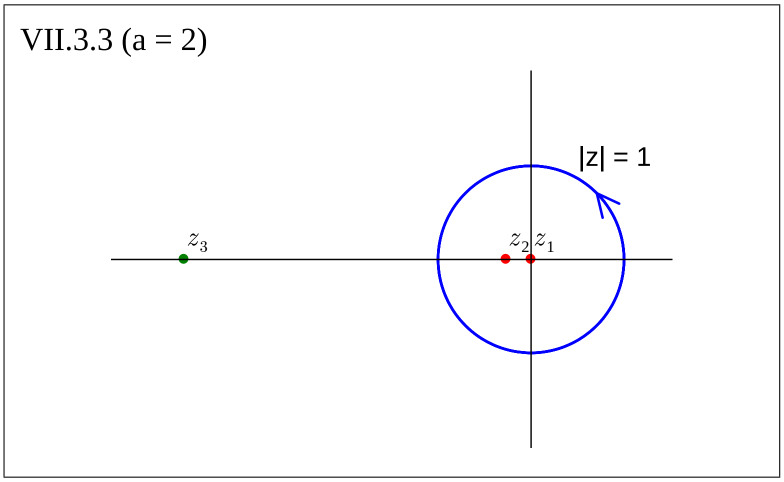
and therefore

$$\int_0^{2\pi} \frac{1}{a+b\sin\theta}d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}, \qquad a > b > 0.$$

VII.3.3
Show using residue theory that

$$\int_0^\pi \frac{\sin^2\theta}{a+\cos\theta}d\theta = \pi\left[a-\sqrt{a^2-1}\right], \qquad a>1.$$

Solution



Set

$$I = \int_0^\pi \frac{\sin^2\theta}{a+\cos\theta}d\theta = \left[\begin{array}{l}\theta = 2\pi - t\\ d\theta = -dt\end{array}\right] = -\int_{2\pi}^{\pi} \frac{\sin^2(2\pi-t)}{a+\cos(2\pi-t)}dt =$$
$$= \int_\pi^{2\pi} \frac{\sin^2(2\pi-t)}{a+\cos(2\pi-t)}dt = \int_\pi^{2\pi} \frac{\sin^2 t}{a+\cos t}dt.$$

We have

$$2I = \int_0^{2\pi} \frac{\sin^2\theta}{a+\cos\theta}d\theta.$$

A change of variables as in book page 203-204 gives

$$2I = \int_0^{2\pi} \frac{\sin^2\theta}{a+\cos\theta}d\theta = \oint_{|z|=1} \frac{\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^2}{\left(a+\frac{1}{2}\left(z+\frac{1}{z}\right)\right)}\frac{dz}{iz} = -\frac{1}{2i}\oint_{|z|=1} \frac{\left(z^2-1\right)^2}{z^2\left(z^2+2az+1\right)}dz.$$

Integrate

$$f(z)=\frac{\left(z^2-1\right)^2}{z^2\left(z^2+2az+1\right)}=\frac{\left(z^2-1\right)^2}{z^2\left(z-\left(-a+\sqrt{a^2-1}\right)\right)\left(z-\left(-a-\sqrt{a^2-1}\right)\right)}$$

along the unit circle contour in Figure VII.3.3.
Residue at a double pole at $z_1=0$, where by Rule 2,

$$\text{Res}\left[\frac{\left(z^2-1\right)^2}{z^2\left(z^2+2az+1\right)},0\right]=\lim_{z\to 0}\frac{d}{dz}\frac{\left(z^2-1\right)^2}{\left(z^2+2az+1\right)}=-2a.$$

Residue at a simple pole at $z_2=-a+\sqrt{a^2-1}$, where by Rule 3,

$$\text{Res}\left[\frac{\left(z^2-1\right)^2/z^2}{z^2+2az+1},-a+\sqrt{a^2-1}\right]=\left.\frac{\left(z^2-1\right)^2/z^2}{2z+2a}\right|_{z=-a+\sqrt{a^2-1}}=2\sqrt{a^2-1}.$$

Using the *Residue Theorem*, we thus obtain

$$2I=2\pi i\cdot\left(-\frac{1}{2i}\right)\left(-2a+2\sqrt{a^2-1}\right),$$

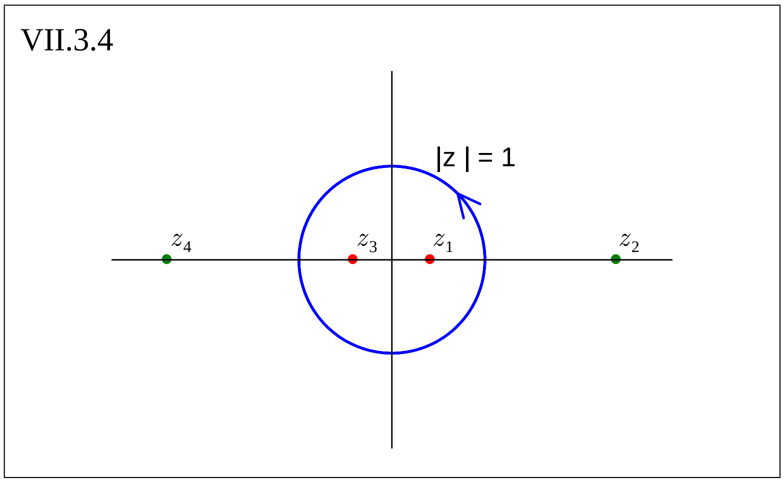
and therefore

$$\int_0^\pi\frac{\sin^2\theta}{a+\cos\theta}d\theta=\pi\left(a-\sqrt{a^2-1}\right),\qquad a>1.$$

VII.3.4
Show using residue theory that

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \pi\sqrt{2}.$$

Solution



Set

$$I = \int_{-\pi}^{\pi} \frac{1}{1+\sin^2\theta} d\theta = \begin{bmatrix} \theta = \pi - t \\ d\theta = -dt \end{bmatrix} =$$

$$= -\int_{2\pi}^{0} \frac{1}{1+\sin^2(\pi - t)} dt = \int_{0}^{2\pi} \frac{1}{1+\sin^2 t} dt.$$

We have

$$I = \int_0^{2\pi} \frac{1}{1+\sin^2\theta}\, d\theta.$$

A change of variables as in book page 203-204 gives

$$I = \int_0^{2\pi} \frac{1}{1+\sin^2\theta}\, d\theta = \oint_{|z|=1} \frac{1}{1+\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^2} \frac{dz}{iz} = -\frac{4}{i} \oint_{|z|=1} \frac{z}{z^4-6z^2+1}\, dz.$$

Integrate

$$f(z) = \frac{z}{z^4 - 6z^2 + 1} =$$
$$= \frac{z}{\left(z - \left(\sqrt{2}-1\right)\right)\left(z - \left(\sqrt{2}+1\right)\right)\left(z - \left(-\sqrt{2}+1\right)\right)\left(z - \left(-\sqrt{2}-1\right)\right)}$$

along the unit circle contour in Figure VII.3.4
Residues at a simple poles at $z_{1,2} = \pm\left(\sqrt{2}-1\right)$, where by Rule 3,

$$\operatorname{Res}\left[\frac{z}{z^4 - 6z^2 + 1}, \pm\left(\sqrt{2}-1\right)\right] = \left.\frac{z}{4z^3 - 12z}\right|_{z=\pm\left(\sqrt{2}-1\right)} = -\frac{1}{8\sqrt{2}}.$$

Using the *Residue Theorem*, we thus obtain

$$I = 2\pi i \cdot -\frac{4}{i}\left(-\frac{1}{8\sqrt{2}} - \frac{1}{8\sqrt{2}}\right),$$

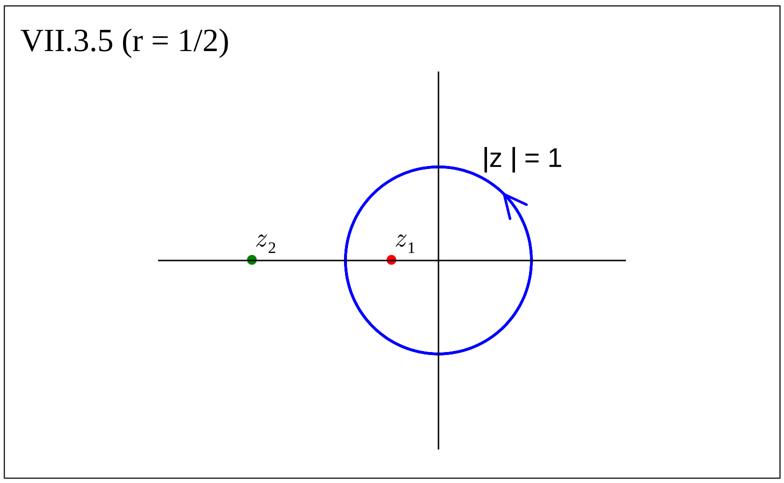
and therefore

$$\int_{-\pi}^{\pi} \frac{1}{1+\sin^2\theta}\, d\theta = \pi\sqrt{2}.$$

VII.3.5
Show using residue theory that

$$\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos\theta+r^2}\frac{d\theta}{2\pi} = 1, \qquad 0 \le r < 1.$$

Solution



Set

$$I = \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos\theta+r^2}\frac{d\theta}{2\pi} = \left[\begin{array}{c} \theta = \pi - t \\ d\theta = -dt \end{array}\right] =$$
$$= -\int_{2\pi}^{0} \frac{1-r^2}{1-2r\cos(\pi-t)+r^2}\frac{dt}{2\pi} =$$
$$= \int_{0}^{2\pi} \frac{1-r^2}{1+2r\cos t+r^2}\frac{dt}{2\pi}$$

A change of variables as in book page 203-204 gives

$$
\begin{aligned}
I = \int_0^{2\pi} \frac{1-r^2}{1+2r\cos t + r^2}\frac{dt}{2\pi} &= \\
&= \frac{1}{2\pi}\oint_{|z|=1} \frac{1-r^2}{1+2r\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)+r^2}\frac{dz}{iz} = \\
&= \frac{1}{2\pi i}\oint_{|z|=1} \frac{1-r^2}{(r+z)(rz+1)}\,dz.
\end{aligned}
$$

Integrate

$$
f(z) = \frac{1-r^2}{(r+z)(rz+1)} = \frac{1-r^2}{r(z+r)(z+1/r)}
$$

along the unit circle contour in Figure VII.3.4
Residue at a simple pole at $z_1 = -r$, where by Rule 3,

$$
\operatorname{Res}\left[\frac{1-r^2}{r(z+r)(z+1/r)}, -r\right] = \left.\frac{1-r^2}{\left(2rz+r^2+1\right)}\right|_{z=-r} = 1.
$$

Using the *Residue* Theorem, we thus obtain

$$
I = 2\pi i \cdot \frac{1}{2\pi i}(1),
$$

and therefore

$$
\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos\theta + r^2}\frac{d\theta}{2\pi} = 1, \qquad 0 \le r < 1.
$$

VII.3.6
By expanding both sides of the identity

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{w+\cos\theta}=\frac{1}{\sqrt{w^2-1}}$$

in a power series at ∞, show that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^{2k}\theta d\theta=\frac{(2k)!}{2^{2k}\left(k!\right)^2},\qquad k\geq 0.$$

Solution
Rewrite both sides in the given identity

$$(1)\qquad I=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{1+\frac{\cos\theta}{w}}=\left(1-\frac{1}{w^2}\right)^{-1/2},$$

where the integrand in the left side in the identity can be rewritten as

$$\frac{1}{1+\frac{\cos\theta}{w}}=\sum_{k=0}^{\infty}(-1)^k\frac{\cos^k\theta}{w^k}.$$

Now, we work on the left side of the identity

$$I=\frac{1}{2\pi}\int_0^{2\pi}\sum\nolimits_{k=0}^{\infty}(-1)^k\frac{\cos^k\theta}{w^k}d\theta=\frac{1}{2\pi}\sum\nolimits_{k=0}^{\infty}\frac{(-1)^k}{w^k}\int_0^{2\pi}\cos^k\theta d\theta.$$

Because $\int_0^{2\pi}\cos^k\theta d\theta=0$ if k odd, we have

$$(2)\qquad I=1+\sum_{k=1}^{\infty}\frac{1}{2\pi}\left(\int_0^{2\pi}\cos^{2k}\theta d\theta\right)\frac{1}{w^{2k}}.$$

Now, we use the power series expansion (MH p. 192.) for real α and $-1<x<1$,

$$(1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\cdots+\left(\begin{array}{c}\alpha\\ n\end{array}\right)x^n+\cdots,$$

we have

$$\left(1-\frac{1}{w^2}\right)^{-1/2}=$$

$$=1+\frac{\left(\frac{-1}{2}\right)}{1!}\left(\frac{-1}{w^2}\right)^1+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}\left(\frac{-1}{w^2}\right)^2+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}\left(\frac{-1}{w^2}\right)^3+$$

$$+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-7}{2}\right)}{4!}\left(\frac{-1}{w^2}\right)^4+\cdots+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-7}{2}\right)\cdots\left(\frac{1-2k}{2}\right)}{k!}\left(\frac{-1}{w^{2k}}\right)^k+\cdots=$$

$$=1+\sum\nolimits_{k=1}^{\infty}\frac{1\cdot 3\cdot\dots\cdot(2k-1)}{2^k k!}\frac{1}{w^{2k}}=$$

$$=1+\sum\nolimits_{k=1}^{\infty}\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot\ldots\cdot(2k-1)\cdot 2k}{2^k k!\cdot 2\cdot 4\cdot 6\cdot\ldots\cdot 2k}\frac{1}{w^{2k}}=$$

$$1+\sum\nolimits_{k=1}^{\infty}\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot\ldots\cdot(2k-1)\cdot 2k}{2^k k!\cdot 2^k\left(1\cdot 2\cdot 3\cdot\ldots\cdot k\right)}\frac{1}{w^{2k}}=$$

$$=1+\sum\nolimits_{k=1}^{\infty}\frac{(2k)!}{2^k k! 2^k k!}\frac{1}{w^{2k}}=1+\sum\nolimits_{k=1}^{\infty}\frac{(2k)!}{2^{2k}\left(k!\right)^2}\frac{1}{w^{2k}},$$

i.e.,

$$(2)\qquad \left(1-\frac{1}{w^2}\right)^{-1/2}=1+\sum\nolimits_{k=1}^{\infty}\frac{(2k)!}{2^{2k}\left(k!\right)^2}\frac{1}{w^{2k}}.$$

Comparing the two power series expressions (2) and (3) of the both sides in the identity (1) gives us together with the uniqueness theorem for the power series

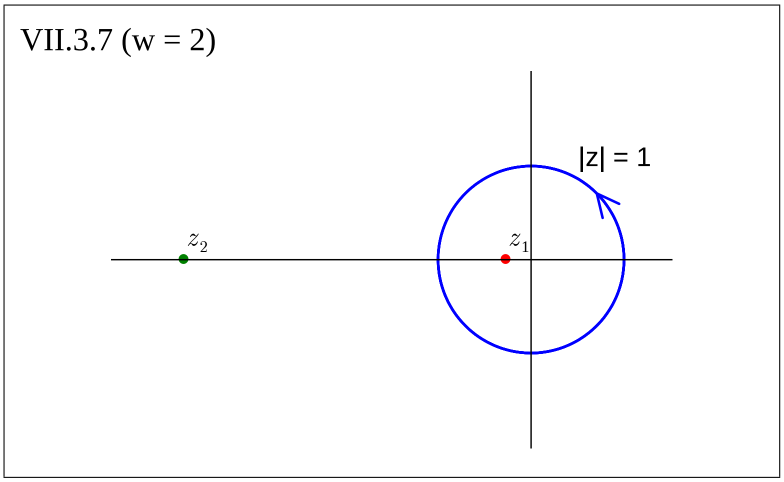
$$\frac{1}{2\pi}\int_0^{2\pi}\cos^{2k}\theta d\theta=\frac{(2k)!}{2^{2k}\left(k!\right)^2},\qquad k\geq 0.$$

VII.3.7
Show using residue theory that

$$\int_0^{2\pi} \frac{d\theta}{(w+\cos\theta)^2} = \frac{2\pi w}{(w^2-1)^{3/2}}, \qquad w \in \mathbb{C} \setminus [-1,1]\,.$$

Specify carefully the branch of the power function. Check you answer by differentiating the integral of $1/(w+\cos\theta)$ with respect to the parameter w.

Solution



Set

$$I = \int_0^{2\pi} \frac{d\theta}{(w+\cos\theta)^2}$$

A change of variables as in book page 203-204 gives

$$I = \int_0^{2\pi} \frac{d\theta}{(w+\cos\theta)^2} = \int_0^{2\pi} \frac{1}{\left(w+\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^2} \frac{dz}{iz} = \frac{4}{i} \oint_{|z|=1} \frac{z}{(z^2+2wz+1)^2} dz.$$

Integrate $f(z) = \frac{4z}{(z^2+2wz+1)^2}$ along the unit circle contour in Figure VII.3.7.
We choose branch of $\sqrt{w^2-1}$ on $\mathbb{C}\setminus[-1,1]$ that is positive for $w \in (1,\infty)$.
It suffices to check the identity for $w > 1$.
Residue at a double pole at $z_1 = -w + \sqrt{w^2-1}$, where by Rule 2,

$$\operatorname{Res}\left[\frac{z}{\left(z^2+2wz+1\right)^2},-w+\sqrt{w^2-1}\right]=$$

$$=\lim_{z\to -w+\sqrt{w^2-1}}\frac{d}{dz}\frac{z}{\left(z-\left(-w-\sqrt{w^2-1}\right)\right)^2}=$$

$$=\left.\frac{\left(z-\left(-w-\sqrt{w^2-1}\right)\right)^2-2z\left(z-\left(-w-\sqrt{w^2-1}\right)\right)}{\left(z-\left(-w-\sqrt{w^2-1}\right)\right)^4}\right|_{z=-w+\sqrt{w^2-1}}=$$

$$=\left.\frac{\left(z-\left(-w-\sqrt{w^2-1}\right)\right)-2z}{\left(z-\left(-w-\sqrt{w^2-1}\right)\right)^3}\right|_{z=-w+\sqrt{w^2-1}}=$$

$$=\frac{1}{4}\frac{w}{\left(w^2-1\right)^{3/2}}.$$

Using the *Residue Theorem*, we thus obtain

$$I=2\pi i\cdot\frac{4}{i}\left(\frac{1}{4}\frac{w}{\left(w^2-1\right)^{3/2}}\right),$$

and therefore

$$\int_0^{2\pi}\frac{d\theta}{\left(w+\cos\theta\right)^2}=\frac{2\pi w}{\left(w^2-1\right)^{3/2}},\qquad w\in\mathbb{C}\setminus[-1,1]\,.$$

Check.
From Exercise VII.3.6 we have the formula

$$(1)\qquad \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{w+\cos\theta}=\frac{1}{\sqrt{w^2-1}}$$

It is allowed to differentiate both sides in the formula, because for every compact interval with $w\in\mathbb{C}\setminus[-1,1]$, both the integrand and its derivative are continuous, and that the primitive of the derivative is uniformly bounded by an $w-$ independent funtion $h(\theta)$ such that $\int h(\theta)\,d\theta<\infty$ on every such like interval on w.
We start by differentiating the left hand side in the formula with respect to the parameter w

$$(2) \qquad \frac{d}{dw}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{w+\cos\theta}\right) = -\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{(w+\cos\theta)^2},$$

and differentiate the right hand side in the formula in the same way

$$(3) \qquad \frac{d}{dw}\frac{1}{\sqrt{w^2-1}} = -\frac{w}{(w^2-1)^{3/2}}.$$

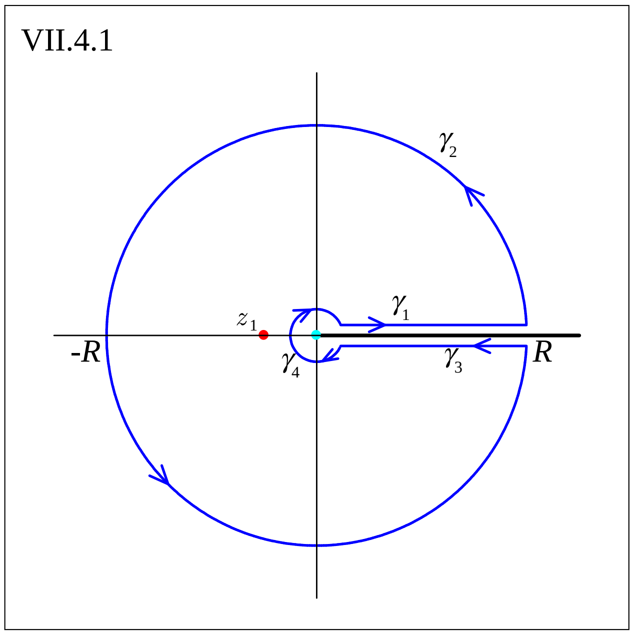
By (1) - (3), we thus have that

$$\int_0^{2\pi}\frac{d\theta}{(w+\cos\theta)^2} = \frac{2\pi w}{(w^2-1)^{3/2}}.$$

VII.4.1
By integrating around the keyhole contour, show that

$$\int_0^\infty \frac{x^{-a}}{1+x}dx = \frac{\pi}{\sin(\pi a)}, \qquad 0 < a < 1.$$

Solution



Set

$$I = \int_0^\infty \frac{x^{-a}}{1+x}dx$$

and integrate

$$f(z) = \frac{z^{-a}}{1+z} = \frac{|z|^{-a}e^{-ia\arg z}}{1+z}$$

along the keyhole contour in Figure VII.4.1. We make a branchcut for z^{-a} along the positive real axis, where $0 < \arg z < 2\pi$.
Residue at a simple pole at $z_1 = -1$, where by Rule 3,

$$\operatorname{Res}\left[\frac{|z|^{-a}e^{-ia\arg z}}{1+z}, -1\right] = |z|^{-a}e^{-ia\arg z}\Big|_{z=-1} = e^{-\pi ia}.$$

Integrate along γ_1, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$\int_{\gamma_1} \frac{|z|^{-a}\, e^{-ia\arg z}}{1+z} dz = \begin{bmatrix} z & = & xe^{0i} \\ dz & = & dx \end{bmatrix} = \int_{\varepsilon}^{R} \frac{x^{-a}}{1+x} dx \to$$

$$\to \int_0^{\infty} \frac{x^{-a}}{1+x} dx = I.$$

Integrate along γ_2 and let $R \to \infty$. This gives then $0 < a < 1$

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{R^{-a}}{R-1} \cdot 2\pi R \sim \frac{2\pi}{R^a} \to 0.$$

Integrate along γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{\gamma_3} \frac{|z|^{-a}\, e^{-ia\arg z}}{1+z} dz = \begin{bmatrix} z & = & xe^{2\pi i} \\ dz & = & dx \end{bmatrix} = \int_{R}^{\varepsilon} \frac{x^{-a}e^{-2\pi ia}}{1+x} dx \to$$

$$\to \int_{\infty}^{0} \frac{x^{-a}e^{-2\pi ia}}{1+x} dx = -e^{-2\pi ia} \int_0^{\infty} \frac{x^{-a}}{1+x} dx = -e^{-2\pi ia} I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives then $0 < a < 1$

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\varepsilon^{-a}}{1-\varepsilon} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{1-a} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 - e^{-2\pi ia} I + 0 = 2\pi i \cdot \left(e^{-\pi ia}\right).$$

Multiplying with $e^{\pi ia}$, this yields

$$\left(e^{\pi ia} - e^{-\pi ia}\right) I = 2\pi i,$$

and hence solving for I,

$$I = \frac{2\pi i}{e^{\pi i a} - e^{-\pi i a}} = \frac{\pi}{\sin(\pi a)},$$

i.e.,

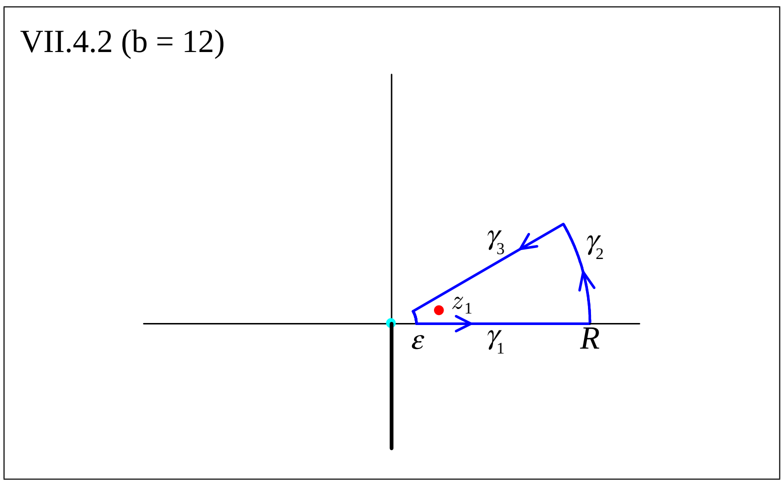
$$\int_0^\infty \frac{x^{-a}}{1+x} dx = \frac{\pi}{\sin(\pi a)}, \qquad 0 < a < 1.$$

VII.4.2
By integrating around the boundary of a pie-slice domain of aperture $2\pi/b$, show that

$$\int_0^\infty \frac{dx}{1+x^b} = \frac{\pi}{b\sin(\pi/b)}, \qquad b > 1.$$

Remark. Check the result by changing variable and comparing with Exercise 1.

Solution



Set

$$I = \int_0^\infty \frac{dx}{1+x^b},$$

and integrate

$$f(z) = \frac{1}{1+z^b} = \frac{1}{1+|z|^b e^{ib\arg z}}$$

along the pie-slice of aperture $2\pi/b$ contour in Figure VII.4.2. We make a branch cut for z^b along the negative imaginary axis, where $-\pi/2 < \arg z < 3\pi/2$.
Residue at a simple pole at $z_1 = e^{\pi i/b}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{1}{1+z^b}, e^{\pi i/b}\right] = \left.\frac{1}{bz^{b-1}}\right|_{z=e^{\pi i/b}} = \frac{-1}{b}e^{\pi i/b}.$$

Integrate along γ_1, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{\gamma_1} \frac{1}{1+|z|^b e^{ib\arg z}} dz = \begin{bmatrix} z & = & xe^{0i} \\ dz & = & dx \end{bmatrix} = \int_\varepsilon^R \frac{1}{1+x^b} dx \to$$

$$\to \int_0^\infty \frac{1}{1+x^b} dx = I.$$

Integrate along γ_2 and let $R \to \infty$. This gives then $b > 1$

$$\left| \int_{\gamma_2} f(z)\,dz \right| \leq \frac{1}{R^b - 1} \cdot \frac{2\pi R}{b} \sim \frac{2\pi}{R^{b-1}} \to 0.$$

Integrate along γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{\gamma_3} \frac{1}{1+|z|^b e^{ib\arg z}} dz = \begin{bmatrix} z & = & xe^{2\pi i/b} \\ dz & = & e^{2\pi i/b} dx \end{bmatrix} = \int_R^\varepsilon \frac{1}{1+x^b} e^{2\pi i/b} dx \to$$

$$\to \int_\infty^0 \frac{1}{1+x^b} e^{2\pi i/b} dx = -e^{2\pi i/b} \int_0^\infty \frac{1}{1+x^b} dx = -e^{2\pi i/b} I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives then $b > 1$

$$\left| \int_{\gamma_4} f(z)\,dz \right| \leq \frac{1}{1-\varepsilon^b} \cdot \frac{2\pi\varepsilon}{b} \sim \frac{2\pi\varepsilon}{b} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 - e^{2\pi i/b} I + 0 = 2\pi i \cdot \left(\frac{-1}{b} e^{\pi i/b} \right).$$

Multiplying with $e^{-\pi i/b}$, this yields

$$\left(e^{-\pi i/b} - e^{\pi i/b} \right) I = \frac{-2\pi i}{b},$$

and hence solving for I,

$$I = \frac{-2\pi i}{\left(e^{-\pi i/b} - e^{\pi i/b}\right)} = \frac{\pi}{b \sin\left(\pi/b\right)}$$

i.e.,

$$\int_0^\infty \frac{dx}{1+x^b} = \frac{\pi}{b\sin\left(\pi/b\right)}, \qquad b > 1.$$

Remark.
We begin with the first integral I, we make the substitution $x^b = t$ and use the integral from Exercise VII.4.1 where we choose $a = 1 - 1/b$. (Because $b > 1$, then $0 < 1 - 1/b < 1$ and the restriction that $0 < a < 1$ for the integral in Exercise VII.4.1 is satisfied.)

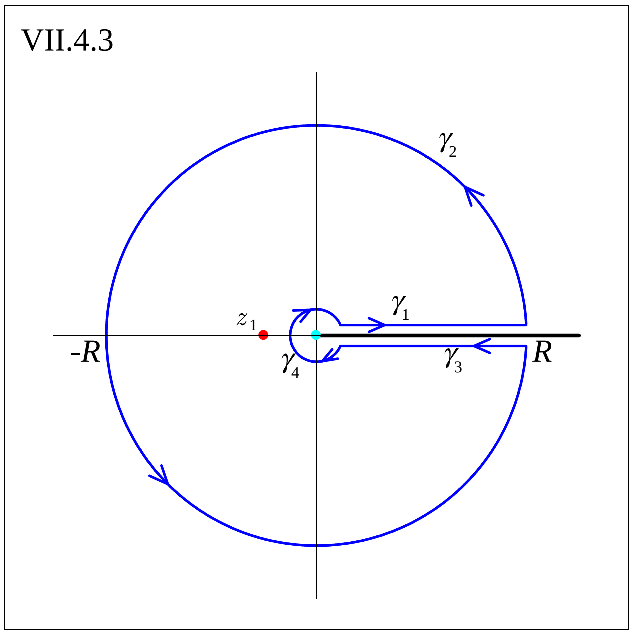
$$\begin{aligned} I = \int_0^\infty \frac{dx}{1+x^b} dx = \left[\begin{array}{ccc} x^b & = & t \\ x & = & t^{1/b} \\ dx & = & \frac{1}{b} t^{1/b-1} dt \end{array} \right] = \\ = \int_0^\infty \frac{1}{b} \frac{t^{1/b-1}}{1+t} dt = \frac{1}{b} \int_0^\infty \frac{t^{-(1-1/b)}}{1+t} dt = \\ = \frac{1}{b} \frac{\pi}{\sin\left(\pi\left(1/b-1\right)\right)} = \frac{\pi}{b\sin\left(\pi/b\right)}. \end{aligned}$$

VII.4.3
By integrating around the keyhole contour, show that

$$\int_0^\infty \frac{\log x}{x^a(x+1)}dx = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}, \qquad 0 < a < 1.$$

Remark. Check the result by differentiating the identity in Exercise 1.

Solution



Set

$$I = \int_0^\infty \frac{\log x}{x^a(x+1)},$$

and integrate

$$f(z) = \frac{\log x}{x^a(x+1)} = \frac{\log|z| + i\arg z}{|z|^a e^{ia\arg z}(z+1)}$$

along the keyhole contour in Figure VII.4.3, where the cimicircle γ_4 have radius ε. We make a branch cut for $\log z$ along the positive real axis, so that $0 < \arg z < 2\pi$.
Residue at a simple pole at $z_1 = -1$, where by Rule 3,

$$\operatorname{Res}\left[\frac{\log|z|+i\arg z}{|z|^a e^{ia\arg z}(z+1)},-1\right]=\left.\frac{\log|z|+i\arg z}{|z|^a e^{ia\arg z}}\right|_{z=-1}=\pi i e^{-\pi i a}.$$

Integrate along γ_1, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\begin{aligned}
&\int_{\gamma_1} f(z)\,dz= \\
&=\int_{\gamma_1}\frac{\log|z|+i\arg z}{|z|^a e^{ia\arg z}(z+1)}dz=\left[\begin{array}{rcl} z&=&xe^{0i}\\ dz&=&dx\end{array}\right]=\int_\varepsilon^R\frac{\log x}{x^a(x+1)}dx\to \\
&\qquad\to\int_0^\infty\frac{\log x}{x^a(x+1)}dx=I.
\end{aligned}$$

Integrate along γ_2 and let $R\to\infty$. This gives then $0<a<1$

$$\left|\int_{\gamma_2} f(z)\,dz\right|\le\frac{\sqrt{\log^2 R+(2\pi)^2}}{R^a(R-1)}\cdot 2\pi R\sim\frac{2\pi\log R}{R^\alpha}\to 0.$$

Integrate along γ_3, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\begin{aligned}
&\int_{\gamma_3} f(z)\,dz= \\
&=\int_{\gamma_3}\frac{\log|z|+i\arg z}{|z|^a e^{ia\arg z}(z+1)}dz=\left[\begin{array}{rcl} z&=&xe^{2\pi i}\\ dz&=&dx\end{array}\right]=\int_R^\varepsilon\frac{\log x+2\pi i}{x^a e^{2\pi i a}(x+1)}\to \\
&\qquad\to\int_\infty^0\frac{\log x+2\pi i}{x^a e^{2\pi i a}(x+1)}dx= \\
&=-e^{-2\pi i a}\int_0^\infty\frac{\log x}{x^a(x+1)}dx-2\pi i e^{-2\pi i a}\int_0^\infty\frac{dx}{x^a(x+1)}= \\
&\qquad=-e^{-2\pi i a}I-2\pi i e^{-2\pi i a}J
\end{aligned}$$

Integrate along γ_4, and let $\varepsilon\to 0^+$. This gives then $0<a<1$

$$\left|\int_{\gamma_4} f(z)\,dz\right|\le\frac{\sqrt{\log^2\varepsilon+2\pi}}{\varepsilon^a(1-\varepsilon)}\cdot 2\pi\varepsilon\sim 2\pi\varepsilon^{1-a}\left|\log\varepsilon\right|\to 0.$$

Using the *Residue Theorem* and letting $R\to\infty$ and $\varepsilon\to 0^+$, we obtain that

$$I + 0 - e^{-2\pi i a} I - 2\pi i e^{-2\pi i a} J + 0 = 2\pi i \cdot \left(\pi i e^{-\pi i a}\right).$$

Multiplying with $e^{\pi i a}$, this yields

$$\left(e^{\pi i a} - e^{-\pi i a}\right) I - 2\pi i e^{-\pi i a} J = -2\pi^2,$$

and thus

$$2i \sin(\pi a) I - 2\pi i e^{-\pi i a} J = -2\pi^2.$$

Separating real and imaginary parts, we get simultaneous equations

$$\begin{cases} & -2\pi \sin(\pi a) J &=& -2\pi^2 \\ 2\sin(\pi a) I & - 2\pi \cos(\pi a) J &=& 0, \end{cases}$$

and hence the solutions

$$I = \int_0^\infty \frac{\log x}{x^a (x+1)} dx = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}, \qquad J = \int_0^\infty \frac{dx}{x^a (x+1)} = \frac{\pi}{\sin(\pi a)},$$

i.e.

$$\int_0^\infty \frac{\log x}{x^a (x+1)} dx = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}, \qquad 0 < a < 1.$$

Remark.
From the Exercise VII.4.1 we have the formula

$$(1) \qquad \int_0^\infty \frac{x^{-a}}{1+x} dx = \frac{\pi}{\sin(\pi a)}, \qquad 0 < a < 1.$$

It is allowed to differentiate both sides in the formula, because for every compact interval with $0 < a < 1$, both the integrand and its derivative is continuous. And that the primitive of the derivative is uniformly limited by a constant on every compact interval on a, there $0 < a < 1$.
We begin to differentiate the left side in the formula with respect to the parameter a

$$(2) \qquad \frac{d}{da}\left(\int_0^\infty \frac{x^{-a}}{1+x} dx\right) = \int_{-\infty}^\infty \frac{-\log x \cdot x^{-a}}{1+x} dx = -\int_{-\infty}^\infty \frac{\log x}{x^a (x+1)} dx,$$

and differentiate the right hand side in the formula in the same way

$$\frac{d}{da}\frac{\pi}{\sin(\pi a)} = -\frac{\cos(\pi a)}{\sin^2(\pi a)}. \tag{3}$$

By (1) - (3), we have that

$$\int_0^\infty \frac{\log x}{x^a(x+1)}dx = \frac{\pi^2\cos(\pi a)}{\sin^2(\pi a)}.$$

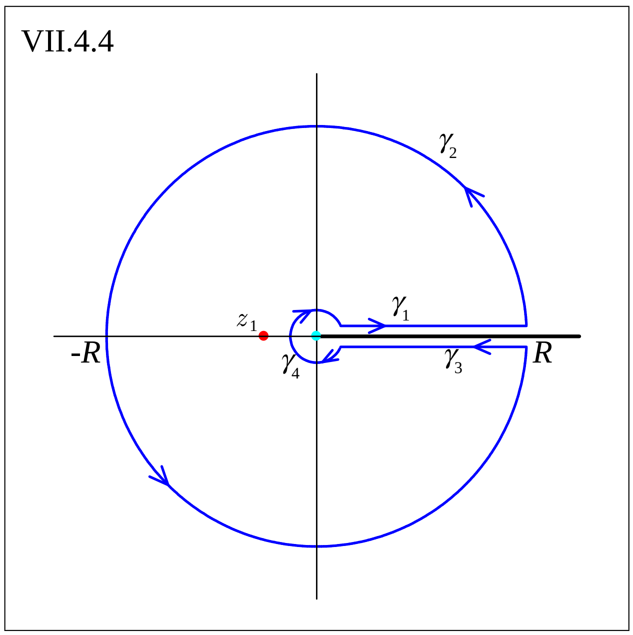
VII.4.4

For fixed $m \geq 2$, show that by integrating around the keyhole contour that

$$\int_0^\infty \frac{x^{-a}}{(1+x)^m}dx = \frac{\pi a\,(a+1)\cdots(a+m-2)}{(m-1)!\sin(\pi a)}, \qquad 1-m<a<1.$$

Remark. The result can be obtained also by integrating the formula in Exercise 1 by parts.

Solution



Set

$$I = \int_0^\infty \frac{x^{-a}}{(1+x)^m}dx$$

and integrate

$$f(z) = \frac{z^{-a}}{(1+z)^m} = \frac{|z|^{-a}\,e^{-ia\arg z}}{(1+z)^m}$$

along the keyhole contour in Figure VII.4.4. We make a branchcut for z^{-a} along the positive real axis, where $0 < \arg z < 2\pi$.

From *Saff/Snider* page 310 we have

$$\operatorname{Res}\left[f(z), z_0\right] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right].$$

Residue at a pole of order m at $z_1 = -1$

$$\begin{aligned}
\operatorname{Res}\left[\frac{z^{-a}}{(1+z)^m}, -1\right] &= \lim_{z \to -1} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left(z^{-a}\right) = \\
&= \lim_{z \to -1} \frac{1}{(m-1)!} (-a)(-a-1)\cdots(-a-m+2)\, z^{(-a-(m-1))} = \\
&= \lim_{z \to -1} \frac{1}{(m-1)!} (-1)^{m-1} a(a+1)\cdots(a+m-2)\, |z|^{(-a-(m-1))} e^{i(-a-(m-1))\arg z} = \\
&= \frac{1}{(m-1)!} (-1)^{m-1} a(a+1)\cdots(a+m-2)\, |-1|^{(-a-(m-1))} e^{i(-a-(m-1))\pi} = \\
&= \frac{1}{(m-1)!} (-1)^{m-1} a(a+1)\cdots(a+m-2)\, (-1)^{m-1} e^{-\pi i a} = \\
&= \frac{a(a+1)\cdots(a+m-2)}{(m-1)!} e^{-\pi i a}.
\end{aligned}$$

Integrate along γ_1, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\begin{aligned}
\int_{\gamma_1} f(z)\, dz = \int_{\gamma_1} \frac{|z|^{-a} e^{-ia \arg z}}{(1+z)^m} dz = \left[\begin{array}{ccc} z & = & xe^{0i} \\ dz & = & dx \end{array}\right] = \int_{\varepsilon}^{R} \frac{x^{-a}}{(1+x)^m} dx \to \\
\to \int_0^\infty \frac{x^{-a}}{(1+x)^m} dx = I.
\end{aligned}$$

Integrate along γ_2 and let $R \to \infty$. This gives then $1-m < a < 1$, where $m \geq 2$

$$\left|\int_{\gamma_2} f(z)\, dz\right| \leq \frac{R^{-a}}{(R-1)^m} \cdot 2\pi R \sim \frac{2\pi}{R^{m+a-1}} \to 0.$$

Integrate along γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$\int_{\gamma_3} \frac{|z|^{-a}\, e^{-ia\arg z}}{(1+z)^m} dz = \begin{bmatrix} z &=& xe^{2\pi i} \\ dz &=& dx \end{bmatrix} = \int_R^\varepsilon \frac{x^{-a} e^{-2\pi ia}}{(1+x)^m} dx \to$$

$$\to \int_\infty^0 \frac{x^{-a} e^{-2\pi ia}}{(1+x)^m} dx = -e^{-2\pi ia} \int_0^\infty \frac{x^{-a}}{(1+x)^m} dx = -e^{-2\pi ia} I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives then $1-m < a < 1$, where $m \geq 2$

$$\left| \int_{\gamma_4} f(z)\,dz \right| \leq \frac{\varepsilon^{-a}}{(1-\varepsilon)^m} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{1-a} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 - e^{-2\pi ia} I + 0 = 2\pi i \cdot \left(\frac{a(a+1)\cdots(a+m-2)}{(m-1)!} e^{-\pi ia} \right).$$

Multiplying with $e^{\pi ia}$, this yields

$$\left(e^{\pi ia} - e^{-\pi ia}\right) I = 2\pi i \frac{a(a+1)\cdots(a+m-2)}{(m-1)!},$$

and hence solving for I,

$$I = \frac{2i}{\left(e^{\pi ia} - e^{-\pi ia}\right)} \frac{\pi a(a+1)\cdots(a+m-2)}{(m-1)!} = \frac{\pi a(a+1)\cdots(a+m-2)}{(m-1)!\sin(\pi a)},$$

i.e.

$$\int_0^\infty \frac{x^{-a}}{(1+x)^m} dx = \frac{\pi a(a+1)\cdots(a+m-2)}{(m-1)!\sin(\pi a)}, \qquad 1-m < a < 1.$$

Remark.
Set (this is our induction hypothesis)

$$I_m(a) = \int_0^\infty \frac{x^{-a}}{(1+x)^m} dx = \frac{\pi a(a+1)\cdots(a+m-2)}{(m-1)!\sin(\pi a)}$$

From Exercise VII.4.1 we have

$$\int_0^\infty \frac{x^{-b}}{1+x}dx = \frac{\pi}{\sin(\pi b)}, \qquad 0 < b < 1.$$

Use Exercise VII.4.1 and integrate by part

$$\begin{aligned}\frac{\pi}{\sin(\pi b)} &= \int_0^\infty \frac{x^{-b}}{1+x}dx = \int_0^\infty x^{-b}\frac{1}{1+x}dx = \\ &= \left[\frac{x^{-b+1}}{1-b}\frac{1}{1+x}\right]_0^\infty - \int_0^\infty \frac{x^{-b+1}}{1-b}\frac{1}{(1+x)^2}dx = \\ &= \frac{1}{1-b}\int_0^\infty \frac{x^{-b+1}}{(1+x)^2}dx.\end{aligned}$$

Set $b = a+1$ and solve for the integral in the left side in the inequality above

$$\int_0^\infty \frac{x^{-a}}{(1+x)^2}dx = -a\frac{\pi}{\sin(\pi(a+1))} = \frac{\pi a}{\sin(\pi a)},$$

that is the same as $I_2(a)$, thus our induction hypothesis is valid for $m = 2$. Now suppose that our formula is valid for $m = p$, and integrate by parts,

$$\begin{aligned}&\frac{\pi b(b+1)\cdots(b+p-2)}{(p-1)!\sin(\pi b)} = \\ &= \int_0^\infty \frac{x^{-b}}{(1+x)^p}dx = \int_0^\infty x^{-b}\frac{1}{(1+x)^p}dx = \\ &= \left[\frac{x^{-b+1}}{1-b}\frac{1}{(1+x)^p}\right]_0^\infty - \int_0^\infty \frac{x^{-b+1}}{1-b}\frac{-p}{(1+x)^{p+1}}dx = \\ &= \frac{p}{1-b}\int_0^\infty \frac{x^{-b+1}}{(1+x)^{p+1}}dx.\end{aligned}$$

Set $b = a+1$ and solve for the integral in the left side in the inequality above

$$\int_0^\infty \frac{x^{-a}}{(1+x)^{p+1}}dx =$$

$$= \frac{-a}{p}\frac{\pi\,(a+1)\,(a+2)\cdots((a+1)+p-2)}{(p-1)!\sin\left(\pi\,(a+1)\right)} =$$

$$= \frac{\pi a\,(a+1)\,(a+2)\cdots(a+(p+1)-2)}{p!\sin\left(\pi a\right)},$$

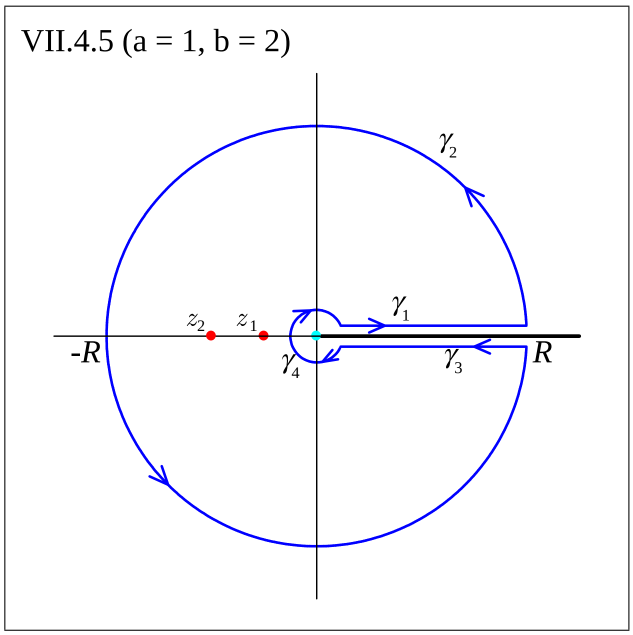
that is the same as $I_{p+1}\,(a)$, thus our induction hypothesis is valid for $m = p+1$.

We have showed that the formula $I_m\,(a)$ is valid for all $m > 2$, that was to be shown.

VII.4.5
By integrating a branch of $\frac{(\log z)^2}{(z+a)(z+b)}$ around the keyhole contour, show that

$$\int_0^\infty \frac{\log x}{(x+a)(x+b)}dx = \frac{(\log a)^2 - (\log b)^2}{2(a-b)}, \qquad a, b > 0,\ a \neq b.$$

Solution



Set

$$I = \int_0^\infty \frac{\log x}{(x+a)(x+b)}dx$$

and integrate

$$f(z) = \frac{(\log z)^2}{(z+a)(z+b)} = \frac{(\log|z| + i\arg z)^2}{(z+a)(z+b)}$$

around the keyhole contour in Figure VII.4.5. We make a branchcut for $\log z$ along the positive real axis, where $0 < \arg z < 2\pi$.
Residue at a simple pole at $z_1 = -a$, where by Rule 3,

$$\operatorname{Res}\left[\frac{(\log|z|+i\arg z)^2}{(z+a)(z+b)},-a\right]=\left.\frac{(\log|z|+i\arg z)^2}{(z+b)}\right|_{z=-a}=\frac{(\log a+i\pi)^2}{b-a}.$$

Residue at a simple pole at $z_1=-b$, where by Rule 3,

$$\operatorname{Res}\left[\frac{(\log|z|+i\arg z)^2}{(z+a)(z+b)},-b\right]=\left.\frac{(\log|z|+i\arg z)^2}{(z+a)}\right|_{z=-b}=\frac{(\log b+i\pi)^2}{a-b}.$$

Integrate along γ_1, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz=$$

$$=\int_{\gamma_1}\frac{(\log|z|+i\arg z)^2}{(z+a)(z+b)}dz=\left[\begin{array}{ccc} z & = & xe^{0i} \\ dz & = & dx \end{array}\right]=\int_{\varepsilon}^{R}\frac{(\log x)^2}{(x+a)(x+b)}dx\to$$

$$\to\int_0^\infty\frac{(\log x)^2}{(x+a)(x+b)}dx=J.$$

Integrate along γ_2 and let $R\to\infty$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right|\le\frac{\log^2 R+(2\pi)^2}{(R-a)(R-b)}\cdot 2\pi R\sim\frac{4\pi\log^2 R}{R}\to 0.$$

Integrate along γ_3, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz=$$

$$=\int_{\gamma_3}\frac{(\log|z|+i\arg z)^2}{(z+a)(z+b)}dz=\left[\begin{array}{ccc} z & = & xe^{2\pi i} \\ dz & = & dx \end{array}\right]=\int_R^{\varepsilon}\frac{(\log x+2\pi i)^2}{(x+a)(x+b)}dx\to$$

$$\to\int_\infty^0\frac{(\log x+2\pi i)^2}{(x+a)(x+b)}dx=-\int_0^\infty\frac{(\log x+2\pi i)^2}{(x+a)(x+b)}dx=$$

$$=-\int_0^\infty\frac{(\log x)^2}{(x+a)(x+b)}dx-4\pi i\int_0^\infty\frac{\log x}{(x+a)(x+b)}+\int_0^\infty\frac{4\pi^2}{(x+a)(x+b)}dx=$$

$$=-J-4\pi iI+K$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\log^2 \varepsilon + (2\pi)^2}{(a-\varepsilon)(b-\varepsilon)} \cdot 2\pi\varepsilon \sim \frac{2\pi\varepsilon \log^2 \varepsilon}{ab} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$J + 0 - J - 4\pi i I + K + 0 = 2\pi i \cdot \left(\frac{(\log|b| + i\pi)^2}{b-a} + \frac{(\log|a| + i\pi)^2}{a-b} \right),$$

and hence, setting imaginary parts equal,

$$I = \frac{(\log a)^2 - (\log b)^2}{2(a-b)},$$

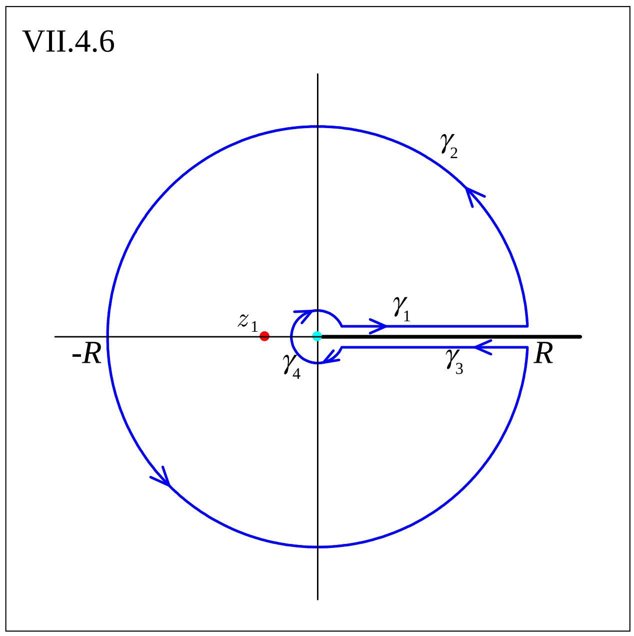
i.e.

$$\int_0^\infty \frac{\log x}{(x+a)(x+b)} dx = \frac{(\log a)^2 - (\log b)^2}{2(a-b)}, \qquad a, b > 0,\ a \ne b.$$

VII.4.6
Using residue theory, show that

$$\int_0^\infty \frac{x^a \log x}{(1+x)^2} dx = \frac{\pi \sin(\pi a) - a\pi^2 \cos(\pi a)}{\sin^2(\pi a)}, \qquad -1 < a < 1.$$

Solution



Set

$$I = \int_0^\infty \frac{x^a \log x}{(1+x)^2} dx$$

and integrate

$$f(z) = \frac{z^a \log z}{(1+z)^2} = \frac{|z|^a e^{ia \arg z} (\log |z| + i \arg z)}{(1+z)^2}$$

along the keyhole contour in Figure VII.4.6. We make a branchcut for z^a and $\log z$ along the real axis, where $0 < \arg z < 2\pi$.
Residue at a double pole at $z_1 = -1$, where by Rule 3,

$$\begin{aligned}
&\operatorname{Res}\left[\frac{|z|^a e^{ia\arg z}\left(\log|z|+i\arg z\right)}{(1+z)^2},-1\right]= \\
&\qquad = \frac{d}{dz}\left(z^a\log z\right)\Big|_{z=-1} = \frac{az^a\log z}{z}+\frac{z^a}{z}\Big|_{z=-1} = \\
&\qquad = \frac{z^a}{z}\left(a\log z+1\right)\Big|_{z=-1} = \\
&\qquad = \frac{|z|^a e^{ia\arg z}}{z}\left(a\left(\log|z|+i\arg z\right)+1\right)\Big|_{z=-1} = \\
&\qquad = e^{\pi ia}\left(-1-\pi ai\right).
\end{aligned}$$

Integrate along γ_1, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\begin{aligned}
&\int_{\gamma_1} f(z)\,dz = \\
&= \int_{\gamma_1}\frac{|z|^a e^{ia\arg z}\left(\log|z|+i\arg z\right)}{(1+z)^2}dz = \left[\begin{array}{ccc} z & = & xe^{0i} \\ dz & = & dx\end{array}\right] = \int_\varepsilon^R \frac{x^a\log x}{(1+x)^2}dx \to \\
&\qquad \to \int_0^\infty \frac{x^a\log x}{(1+x)^2}dx = I.
\end{aligned}$$

Integrate along γ_2 and let $R\to\infty$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{R^a\sqrt{\log^2 R+(2\pi)^2}}{(R-1)^2}\cdot 2\pi R \sim \frac{2\pi\log R}{R^{1-a}} \to 0.$$

Integrate along γ_3, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{\gamma_3} \frac{|z|^a e^{ia\arg z}\left(\log|z| + i\arg z\right)}{(1+z)^2}dz = \left[\begin{array}{ccc} z & = & xe^{2\pi i} \\ dz & = & dx \end{array}\right] = \int_R^{\varepsilon} \frac{x^a e^{2\pi ia}\left(\log|x| + 2\pi i\right)}{(1+x)^2}dx \to$$

$$\to \int_{\infty}^{0} \frac{x^a e^{2\pi ia}\left(\log|x| + 2\pi i\right)}{(1+x)^2}dx = -e^{2\pi ia}\int_0^{\infty} \frac{x^a\left(\log|x| + 2\pi i\right)}{(1+x)^2}dx =$$

$$= -e^{2\pi ia}\int_0^{\infty} \frac{x^a \log x}{(1+x)^2}dx - 2\pi i e^{2\pi ia}\int_0^{\infty}\frac{x^a dx}{(1+x)^2} =$$

$$= -e^{2\pi ia} I - 2\pi i e^{2\pi ia} J$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\varepsilon^a\sqrt{\log^2\varepsilon + (2\pi)^2}}{(1-\varepsilon)^2}\cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{1+a}\left|\log\varepsilon\right| \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 - e^{2\pi ia} I - 2\pi i e^{2\pi ia} J + 0 = 2\pi i \cdot e^{\pi ia}\left(-1 - \pi ai\right),$$

multiplying with $e^{-\pi ia}$, this yields

$$\left(e^{-\pi ia} - e^{\pi ia}\right) I - 2\pi i e^{\pi ia} J = 2\pi i\left(-1 - \pi ai\right),$$

and thus

$$-2i\sin(\pi a)\, I - 2\pi i e^{\pi ia} J = 2\pi^2 a - 2\pi i.$$

Separating real and imaginary parts, we get simultaneous equations

$$\left\{\begin{array}{rcl} 2\pi\sin(\pi a)\, J & = & 2\pi^2 a \\ -2\sin(\pi a)\, I - 2\pi\cos(\pi a)\, J & = & -2\pi, \end{array}\right.$$

and hence the solutions

$$I = \int_0^{\infty}\frac{x^a\log x}{(1+x)^2} = \frac{\pi\sin(\pi a) - a\pi^2\cos(\pi a)}{\sin^2(\pi a)}, \quad J = \int_0^{\infty}\frac{x^a dx}{(1+x)^2} = \frac{\pi a}{\sin(\pi a)},$$

i.e.

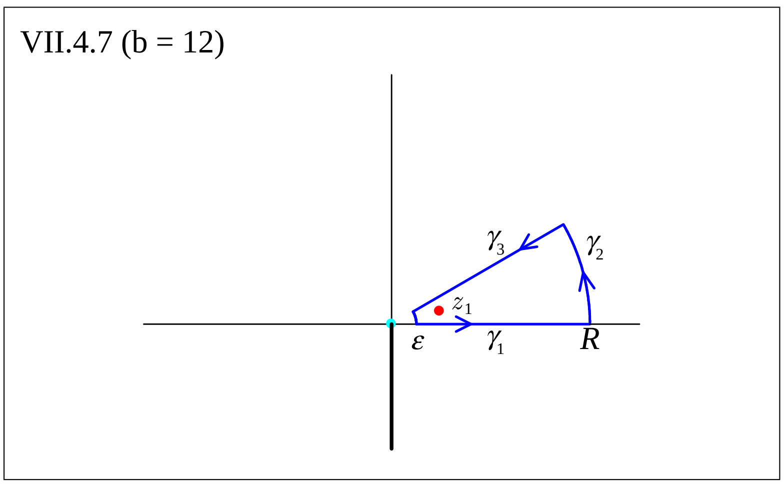
$$\int_0^\infty \frac{x^a \log x}{(1+x)^2} dx = \frac{\pi \sin(\pi a) - a\pi^2 \cos(\pi a)}{\sin^2(\pi a)}, \qquad -1 < a < 1.$$

VII.4.7
Show that

$$\int_0^\infty \frac{x^{a-1}}{1+x^b}dx = \frac{\pi}{b\sin\left(\pi a/b\right)}, \qquad 0 < a < b.$$

Determine for which complex values of the parameter a the integral exists (in the sense that the integral of the absolute value is finite), and evaluate it. Where does the integral depend analytically on the parameter a?

Solution



Set

$$I = \int_0^\infty \frac{x^{a-1}}{1+x^b},$$

and integrate

$$f\left(z\right) = \frac{z^{a-1}}{1+z^b} = \frac{\left|z\right|^{a-1} e^{i(a-1)\arg z}}{1+\left|z\right|^b e^{ib\arg z}}$$

along the pie-slice, of aperture $2\pi/b$, contour in Figure VII.4.7. We make a branch cut for z^{a-1} along the negative imaginary axis , where $-\pi/2 < \arg z < 3\pi/2$.
Residue at a simple pole at $z_1 = e^{\pi i/b}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{z^{a-1}}{1+z^b}, e^{\pi i/b}\right] = \left.\frac{z^{a-1}}{bz^{b-1}}\right|_{z=e^{\pi i/b}} = \frac{-1}{b}e^{\pi ia/b}.$$

Integrate along γ_1, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{\gamma_1} \frac{|z|^{a-1} e^{i(a-1)\arg z}}{1+|z|^b e^{ib\arg z}} dz = \left[\begin{array}{ccc} z & = & xe^{0i} \\ dz & = & dx \end{array}\right] = \int_\varepsilon^R \frac{x^{a-1}}{1+x^b} dx \to$$

$$\to \int_0^\infty \frac{x^{a-1}}{1+x^b} dx = I.$$

Integrate along γ_2 and let $R \to \infty$. This gives then $0 < a < b$

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \frac{R^{a-1}}{R^b - 1} \cdot \frac{2\pi R}{b} \sim \frac{2\pi}{bR^{b-a}} \to 0.$$

Integrate along γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{\gamma_3} \frac{|z|^{a-1} e^{i(a-1)\arg z}}{1+|z|^b e^{ib\arg z}} = \left[\begin{array}{ccc} z & = & xe^{2\pi i/b} \\ dz & = & e^{2\pi i/b}dx \end{array}\right] = \int_R^\varepsilon \frac{x^{a-1}e^{2\pi i(a-1)/b}}{1+x^b} e^{2\pi i/b} dx \to$$

$$\to \int_\infty^0 \frac{x^{a-1}e^{2\pi i(a-1)/b}}{1+x^b} e^{2\pi i/b} dx = -e^{2\pi ia/b} \int_0^\infty \frac{x^{a-1}}{1+x^b} dx = -e^{2\pi ia/b} I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives then $0 < a < b$

$$\left|\int_{\gamma_4} f(z)\,dz\right| \leq \frac{\varepsilon^{a-1}}{1 - \varepsilon^b} \cdot \frac{2\pi\varepsilon}{b} \sim \frac{2\pi\varepsilon^a}{b} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 - e^{2\pi ia/b} I + 0 = 2\pi i \cdot \left(\frac{-1}{b} e^{\pi ia/b}\right),$$

and multiplying with $e^{-\pi ia/b}$, this yields

$$\left(e^{-\pi i a/b} - e^{\pi i a/b}\right) I = -\frac{2\pi i}{b}$$

and hence solving for I,

$$I = \frac{2\pi i}{b\left(e^{\pi i a/b} - e^{-\pi i a/b}\right)} = \frac{\pi}{b \sin\left(\pi a/b\right)},$$

i.e.,

$$\int_0^\infty \frac{x^{a-1}}{1+x^b} dx = \frac{\pi}{b \sin\left(\pi a/b\right)}, \qquad 0 < a < b.$$

Now we suppose a and b complex and rewrite the nominator in the integrand

$$x^{a-1} = e^{(a-1)\log x} = e^{(\operatorname{Re} a - 1)\log x} e^{i \operatorname{Im} a \log x} = x^{\operatorname{Re} a - 1} e^{i \operatorname{Im} a \log x}.$$

The absolute value of the integrand is

$$\left|\frac{x^{a-1}}{1+x^b}\right| = \left|\frac{x^{\operatorname{Re} a - 1} e^{i \operatorname{Im} a \log x}}{1+x^b}\right| = \frac{x^{\operatorname{Re} a - 1}}{1+x^b},$$

thus $\operatorname{Re} a - 1 > -1$ and $\operatorname{Re} a - 1 - b < -1$. Thus the integrand depends analytically on the parameter a when $0 < \operatorname{Re} a < b$.

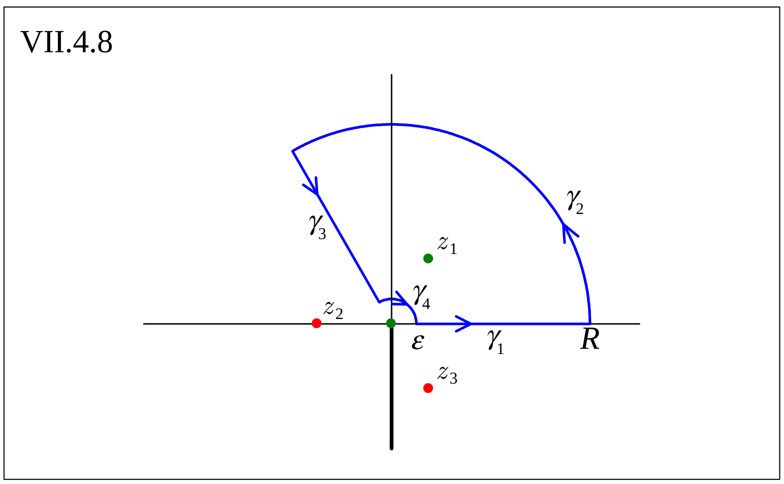
VII.4.8

By integrating a branch of $(\log z)/(z^3+1)$ around the boundary of an indented sector of aperture $2\pi/3$, show that

$$\int_0^\infty \frac{\log x}{x^3+1}dx = -\frac{2\pi^2}{27}, \qquad \int_0^\infty \frac{1}{x^3+1}dx = \frac{2\pi}{3\sqrt{3}}.$$

Remark. Compare the results with those of Exercise 3 (after changing variables) and Exercise 2.

Solution



Set

$$I = \int_0^\infty \frac{\log x}{x^3+1}dx, \qquad J = \int_0^\infty \frac{1}{x^3+1}dx$$

and integrate

$$f(z) = \frac{\log z}{z^3+1} = \frac{\log|z| + i\arg z}{\left(z-e^{\pi i/3}\right)\left(z-e^{\pi i}\right)\left(z-e^{5\pi i/3}\right)}$$

along the indented sector, of aperture $2\pi/3$, contour in Figure VII.4.8. We make a branchcut for $\log z$ along the negative imaginary axis, where $-\pi/2 < \arg z < 3\pi/2$.

Residue at a simple pole at $z_1 = e^{\pi i/3}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{\log|z|+i\arg z}{z^3+1},e^{\pi i/3}\right]=\left.\frac{\log|z|+i\arg z}{3z^2}\right|_{z=e^{\pi i/3}}=\frac{\pi i}{9}e^{-2\pi i/3}.$$

Integrate along γ_1, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\begin{aligned}&\int_{\gamma_1} f(z)\,dz=\\&\quad=\int_{\gamma_1}\frac{\log|z|+i\arg z}{z^3+1}dz=\left[\begin{array}{rcl} z&=&xe^{0i}\\ dz&=&dx\end{array}\right]=\int_\varepsilon^R\frac{\log x}{x^3+1}dx\to\\&\qquad\qquad\to\int_0^\infty\frac{\log x}{x^3+1}dx=I.\end{aligned}$$

Integrate along γ_2, and let $R\to\infty$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right|\le\frac{\sqrt{\log^2R+\left(\frac{2\pi}{3}\right)^2}}{R^3-1}\cdot\frac{2\pi R}{3}\sim\frac{2\pi\log R}{3R^2}\to 0.$$

Integrate along γ_3, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\begin{aligned}&\int_{\gamma_3} f(z)\,dz=\\&=\int_{\gamma_3}\frac{\log|z|+i\arg z}{z^3+1}dz=\left[\begin{array}{rcl} z&=&xe^{2\pi i/3}\\ dz&=&e^{2\pi i/3}dx\end{array}\right]=\int_R^\varepsilon\frac{\log x+2\pi i/3}{x^3+1}e^{2\pi i/3}dx\to\\&\to\int_\infty^0\frac{\log x+2\pi i/3}{x^3+1}e^{2\pi i/3}dx=-\int_0^\infty\frac{\log x+2\pi i/3}{x^3+1}e^{2\pi i/3}dx=\\&=-e^{2\pi i/3}\int_0^\infty\frac{\log x}{x^3+1}dx-\frac{2\pi i}{3}e^{2\pi i/3}\int_0^\infty\frac{1}{x^3+1}dx=-\,e^{2\pi i/3}I-\frac{2\pi i}{3}e^{2\pi i/3}J.\end{aligned}$$

Integrate along γ_4, and let $\varepsilon\to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right|\le\frac{\sqrt{\log^2\varepsilon+\left(\frac{2\pi}{3}\right)^2}}{1-\varepsilon^3}\cdot\frac{2\pi\varepsilon}{3}\sim\frac{2\pi\varepsilon\,|\log\varepsilon|}{3}\to 0.$$

Using the *Residue Theorem* and letting $R\to\infty$ and $\varepsilon\to 0^+$, we obtain that

$$I+0-e^{2\pi i/3}I-\frac{2\pi i}{3}e^{2\pi i/3}J+0=2\pi i\left(\frac{\pi i}{9}e^{-2\pi i/3}\right).$$

Multiplying with $e^{-\pi i/3}$, this yields

$$-2i\sin\left(\pi/3\right)I-\frac{2\pi}{3}ie^{\pi i/3}J=\frac{2\pi^2}{9},$$

and thus

$$-\sqrt{3}Ii+\frac{\pi\sqrt{3}}{3}J-\frac{\pi i}{3}J=\frac{2\pi^2}{9}.$$

Separating real and imaginary parts, we get the simultaneous equations

$$\left\{\begin{array}{rcl} \frac{\pi\sqrt{3}}{3}J & = & \frac{2\pi^2}{9} \\ -\sqrt{3}I-\frac{\pi}{3}J & = & 0, \end{array}\right.$$

and hence the solutions

$$I=\int_0^\infty \frac{\log x}{x^3+1}dx=-\frac{2\pi^2}{27},\qquad J=\int_0^\infty \frac{1}{x^3+1}dx=\frac{2\pi}{3\sqrt{3}}.$$

Remark.

We begin with the first integral I, we make the substitution $x^3=t$ and use the integral from Exercise VII.4.3, where we choose $a=2/3$.

$$I=\int_0^\infty \frac{\log x}{x^3+1}dx=\left[\begin{array}{rcl} x^3 & = & t \\ x & = & t^{1/3} \\ dx & = & \frac{1}{3}t^{-2/3}dt \end{array}\right]=$$

$$=\frac{1}{9}\int_0^\infty \frac{\log t}{t^{2/3}\left(t+1\right)}dt=\frac{1}{9}\frac{\pi^2\cos\left(\frac{2\pi}{3}\right)}{\sin^2\left(\frac{2\pi}{3}\right)}=-\frac{2\pi^2}{27}.$$

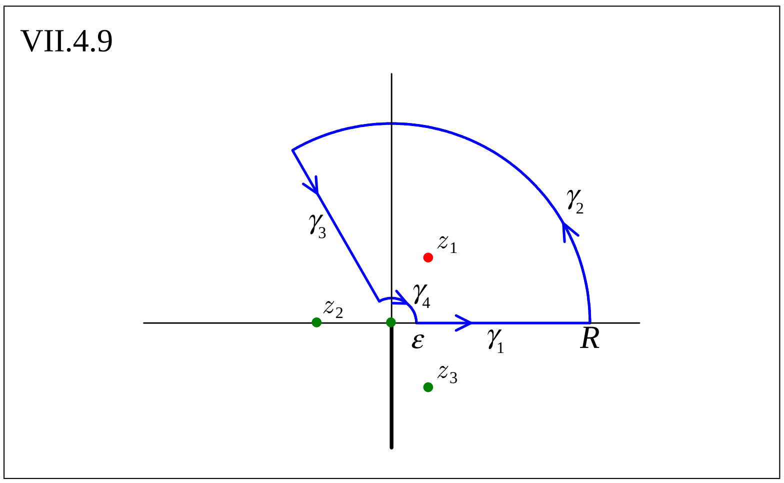
For the second integral J, we use the integral from Exercise VII.4.2, where we choose $b=3$.

$$J=\int_0^\infty \frac{dx}{1+x^b}=\frac{\pi}{3\sin\left(\pi/3\right)}=\frac{2\pi}{3\sqrt{3}}.$$

VII.4.9
By integrating around an appropriate contour and using the results of Exercise 8, show that

$$\int_0^\infty \frac{(\log x)^2}{x^3+1}dx = \frac{10\pi^3}{81\sqrt{3}}.$$

Solution



Set

$$I = \int_0^\infty \frac{(\log x)^2}{x^3+1}dx$$

Integrate

$$f(z) = \frac{(\log z)^2}{z^3+1} = \frac{(\log|z|+i\arg z)^2}{\left(z-e^{\pi i/3}\right)\left(z-e^{\pi i}\right)\left(z-e^{5\pi i/3}\right)}$$

along the pie-slice contour with $0 < \arg z < 2\pi/3$ in Figure VII.4.9. We make a branchcut for $\log z$ along the negative imaginary axis, where $-\pi/2 < \arg z < 3\pi/2$.
Residue at a simple pole at $z_1 = e^{\pi i/3}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{(\log|z|+i\arg z)^2}{z^3+1}, e^{\pi i/3}\right] = \left.\frac{(\log|z|+i\arg z)^2}{3z^2}\right|_{z=e^{\pi i/3}} = -\frac{\pi^2}{27}e^{-2\pi i/3}.$$

Integrate along γ_1, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$\int_{\gamma_1} \frac{(\log|z| + i\arg z)^2}{z^3+1} dz = \left[\begin{array}{ccc} z & = & xe^{0i} \\ dz & = & dx \end{array}\right] = \int_\varepsilon^R \frac{(\log x)^2}{x^3+1} dx \to$$

$$\to \int_0^\infty \frac{(\log x)^2}{x^3+1} dx = I.$$

Integrate along γ_2, and let $R \to \infty$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{\log^2 R + \left(\frac{2\pi}{3}\right)^2}{R^3-1} \cdot \frac{2\pi R}{3} \sim \frac{2\pi \log^2 R}{3R^2} \to 0.$$

Integrate along γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{\gamma_3} \frac{(\log|z| + i\arg z)^2}{z^3+1} dz = \left[\begin{array}{ccc} z & = & xe^{2\pi i/3} \\ dz & = & e^{2\pi i/3}dx \end{array}\right] = \int_R^\varepsilon \frac{(\log x + 2\pi i/3)^2}{x^3+1} e^{2\pi i/3} dx \to$$

$$\to \int_\infty^0 \frac{(\log x + 2\pi i/3)^2}{x^3+1} e^{2\pi i/3} dx = -\int_0^\infty \frac{(\log x + 2\pi i/3)^2}{x^3+1} e^{2\pi i/3} dx =$$

$$= -e^{2\pi i/3} \int_0^\infty \frac{(\log x)^2}{x^3+1} dx - \frac{4\pi i}{3} e^{2\pi i/3} \int_0^\infty \frac{\log x}{x^3+1} dx + \frac{4\pi^2}{9} e^{2\pi i/3} \int_0^\infty \frac{1}{x^3+1} dx =$$

$$= -e^{2\pi i/3} I - \frac{4\pi i}{3} e^{2\pi i/3} \int_0^\infty \frac{\log x}{x^3+1} dx + \frac{4\pi^2}{9} e^{2\pi i/3} \int_0^\infty \frac{1}{x^3+1}.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\log^2 \varepsilon + \left(\frac{2\pi}{3}\right)^2}{1-\varepsilon^3} \cdot \frac{2\pi\varepsilon}{3} \sim \frac{2\pi\varepsilon \log^2 \varepsilon}{3} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 - e^{2\pi i/3} I - \frac{4\pi i}{3} e^{2\pi i/3} \int_0^\infty \frac{\log x}{x^3+1} dx + \frac{4\pi^2}{9} e^{2\pi i/3} \int_0^\infty \frac{1}{x^3+1} + 0 = 2\pi i \left(-\frac{\pi^2}{27} e^{-2\pi i/3}\right).$$

multiplying with $e^{-\pi i/3}$, this yields

$$\left(e^{-\pi i/3}-e^{\pi i/3}\right) I-\frac{4\pi i}{3}\int_0^\infty \frac{\log x}{x^3+1}e^{\pi i/3}dx+\frac{4\pi^2}{9}\int_0^\infty \frac{1}{x^3+1}e^{\pi i/3}dx=\frac{2\pi^3}{27}i,$$

and thus, substituting the values for the integrals from Exercise 9,

$$-2i\sin\left(\pi/3\right) I-\frac{4\pi i}{3}e^{\pi i/3}\left(-\frac{2\pi^2}{27}\right)+\frac{4\pi^2}{9}e^{\pi i/3}\left(\frac{2\pi}{3\sqrt{3}}\right)=\frac{2\pi^3}{27}i.$$

And hence, setting imaginary parts equal,

$$-\sqrt{3}I+\frac{4\pi^3}{81}+\frac{12\pi^3}{81}=\frac{2\pi^3}{27},$$

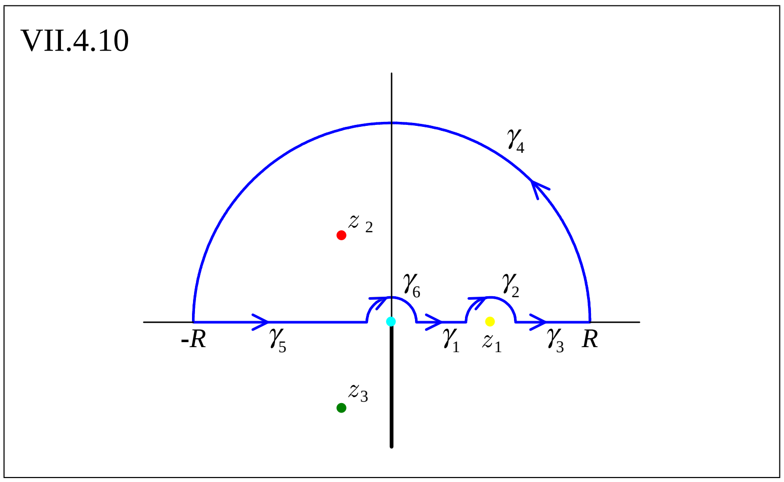
and therefore

$$\int_0^\infty \frac{(\log x)^2}{x^3+1}dx=\frac{10\pi^2}{81\sqrt{3}}.$$

VII.4.10
By integrating a branch of $(\log z)/(z^3-1)$ around the boundary of an indented half-disk and using the result of Exercise 8, show that

$$\int_0^\infty \frac{\log x}{x^3-1}dx = \frac{4\pi^2}{27}.$$

Solution



Set

$$I = \int_0^\infty \frac{\log x}{x^3-1}dx$$

Integrate

$$f(z) = \frac{\log z}{z^3-1} = \frac{\log|z| + i\arg z}{(z-1)\left(z-e^{2\pi i/3}\right)\left(z-e^{4\pi i/3}\right)}$$

along the indented half-disk contour in Figure VII.4.10. We make a branchcut for $\log z$ along the negative imaginary axis, where $-\pi/2 < \arg z < 3\pi/2$. Residue at a simple pole at $z_1 = 1$, where by Rule 3,

$$\text{Res}\left[\frac{\log|z| + i\arg z}{z^3-1}, 1\right] = \left.\frac{\log|z| + i\arg z}{3z^2}\right|_{z=1} = 0.$$

Residue at a simple pole at $z_2 = e^{2\pi i/3}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{\log|z|+i\arg z}{z^3-1}, e^{2\pi i/3}\right] = \left.\frac{\log|z|+i\arg z}{3z^2}\right|_{z=e^{2\pi i/3}} = \frac{2\pi i}{9}e^{-4\pi i/3}.$$

Integrate along γ_1 and γ_3, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\begin{aligned}
&\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = \\
&\qquad = \int_{\gamma_1} \frac{\log|z|+i\arg z}{z^3-1}dz + \int_{\gamma_3} \frac{\log|z|+i\arg z}{z^3-1}dz = \\
&\qquad = \left[\begin{array}{rcl} z & = & xe^{0i} \\ dz & = & dx \end{array}\right] = \\
&\qquad = \int_{\varepsilon}^{1-\varepsilon} \frac{\log x}{x^3-1}dx + \int_{1+\varepsilon}^{R} \frac{\log x}{x^3-1}dx \to \\
&\qquad\qquad \to \int_0^{\infty} \frac{\log x}{x^3-1}dx = I.
\end{aligned}$$

Integrate along γ_2, and let $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_2} f(z)\,dz \to -i(\pi-0)\cdot 0 = 0.$$

Integrate along γ_4, and let $R\to\infty$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\sqrt{\log^2 R+\pi^2}}{R^3-1}\cdot \pi R \sim \frac{\pi\log R}{R^2} \to 0.$$

Integrate along γ_5, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_5} f(z)\, dz =$$

$$= \int_{\gamma_5} \frac{\log|z| + i\arg z}{z^3 - 1} dz = \left[\begin{array}{ccc} z & = & xe^{\pi i} \\ dz & = & e^{\pi i} dx \end{array} \right] = \int_R^{\varepsilon} \frac{\log x + i\pi}{-x^3 - 1} e^{\pi i} d \to$$

$$\to \int_{\infty}^{0} \frac{\log x + i\pi}{-x^3 - 1} e^{\pi i} dx = -\int_0^{\infty} \frac{\log x + i\pi}{x^3 + 1} dx =$$

$$= -\int_0^{\infty} \frac{\log x}{x^3 + 1} dx - i\pi \int_0^{\infty} \frac{1}{x^3 + 1} dx =$$

$$= -\int_0^{\infty} \frac{\log x}{x^3 + 1} dx - i\pi \int_0^{\infty} \frac{1}{x^3 + 1} dx.$$

Integrate along γ_6, and let $\varepsilon \to 0^+$. This gives

$$\left| \int_{\gamma_6} f(z)\, dz \right| \leq \frac{\sqrt{\log^2 \varepsilon + \pi}}{1 - \varepsilon^3} \cdot \pi\varepsilon \sim \pi\varepsilon \left|\log \varepsilon\right| \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 + 0 - \int_0^{\infty} \frac{\log x}{x^3 + 1} dx - i\pi \int_0^{\infty} \frac{1}{x^3 + 1} dx + 0 = 2\pi i \left(\frac{2\pi i}{27} e^{-4\pi i/3} \right),$$

and thus, substituting the values for the integrals from Exercise 9,

$$I - \left(\frac{-2\pi^2}{27} \right) - i\pi \left(\frac{2\pi}{3\sqrt{3}} \right) = -\frac{4\pi^2}{9} e^{-4\pi i/3}.$$

And hence, setting imaginary parts equal,

$$I + \frac{2\pi^2}{27} = \frac{4\pi^2}{18},$$

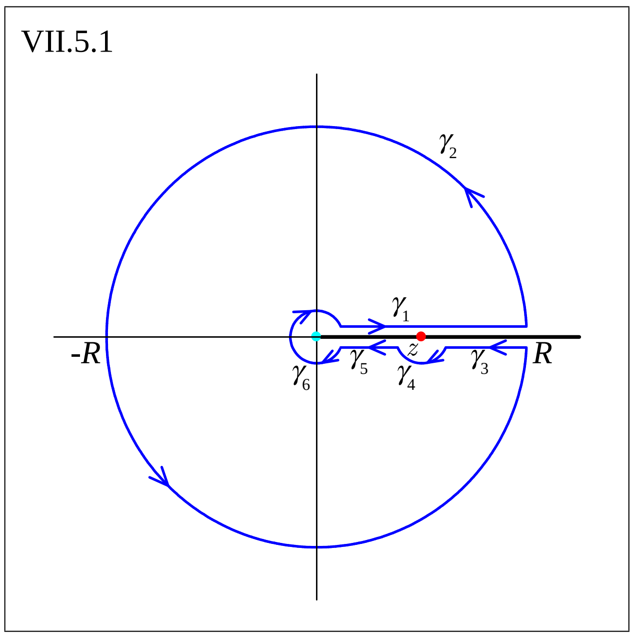
and therefore

$$\int_0^{\infty} \frac{\log x}{x^3 - 1} dx = \frac{4\pi^2}{27}.$$

VII.5.1
Use the keyhole contour indented on the lower edge of the axis at $x = 1$ to show that

$$\int_0^\infty \frac{\log x}{x^a (x-1)} dx = \frac{2\pi^2}{1-\cos(2\pi a)}, \qquad 0 < a < 1.$$

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{\log x}{x^a (x-1)} dx$$

and integrate

$$f(z) = \frac{\log z}{z^a (z-1)} = \frac{(\log|z| + i \arg z)}{e^{a(\log|z| + i \arg z)} (z-1)}$$

along the contour in Figure VII.5.1. We make a branchcut for $\log z$ along the positive imaginary axis, where $0 < \arg z < 2\pi$. Thus $f(z)$ is analytic on the keyhole domain. It extends analytically to $(0, \infty)$ from above, and the

apparent singularity at $z=1$ is removable. However, the extension to $(0,\infty)$ from below has a simple pole at $z=1$.
Residue at a simple pole at $z=1$, where by Rule 3,

$$\operatorname{Res}\left[\frac{(\log|z|+i\arg z)}{e^{a(\log|z|+i\arg z)}(z-1)},1\right]=\left.\frac{(\log|z|+i\arg z)}{e^{a(\log|z|+i\arg z)}}\right|_{z=1}=2\pi ie^{-2\pi ia}.$$

Integrate along γ_1, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\begin{aligned}&\int_{\gamma_1} f(z)\,dz=\\&=\int_{\gamma_1}\frac{(\log|z|+i\arg z)}{e^{a(\log|z|+i\arg z)}(z-1)}dz=\left[\begin{array}{ccc} z&=&xe^{0i}\\ dz&=&dx\end{array}\right]=\int_\varepsilon^R\frac{\log x}{x^a(x-1)}dx\to\\&\qquad\to\int_0^\infty\frac{\log x}{x^a(x-1)}dx=I.\end{aligned}$$

Integrate along γ_2, and let $\varepsilon\to 0^+$. This gives when $0<a<1$

$$\left|\int_{\gamma_2} f(z)\,dz\right|\le\frac{\sqrt{\log^2 R+(2\pi)^2}}{R^a(R-1)}\cdot 2\pi R\sim\frac{2\pi\log R}{R^a}\to 0.$$

Integrate along γ_3 and γ_5, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\begin{aligned}&\int_{\gamma_3} f(z)\,dz+\int_{\gamma_5} f(z)\,dz=\\&\quad=\int_{\gamma_3}\frac{(\log|z|+i\arg z)}{e^{a(\log|z|+i\arg z)}(z-1)}dz+\int_{\gamma_5}\frac{(\log|z|+i\arg z)}{e^{a(\log|z|+i\arg z)}(z-1)}dz=\\&=\left[\begin{array}{ccc} z&=&xe^{2\pi i}\\ dz&=&dx\end{array}\right]=e^{-2\pi ia}\int_R^{1+\varepsilon}\frac{(\log x+2\pi i)}{x^a(x-1)}dx+e^{-2\pi ia}\int_{1-\varepsilon}^{\varepsilon}\frac{(\log x+2\pi i)}{x^a(x-1)}dx\to\\&\quad\to e^{-2\pi ia}\int_\infty^0\frac{(\log x+2\pi i)}{x^a(x-1)}dx=-e^{-2\pi ia}\int_0^\infty\frac{(\log x+2\pi i)}{x^a(x-1)}dx=\\&=-e^{-2\pi ia}\int_0^\infty\frac{\log x}{x^a(x-1)}dx-2\pi ie^{-2\pi ia}\int_0^\infty\frac{1}{x^a(x-1)}dx=-e^{-2\pi ia}I-2\pi ie^{-2\pi ia}J.\end{aligned}$$

Integrate along γ_4, use fractional residue formula and let $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_4} f(z)\,dz \to -i(\pi - 0)\cdot 2\pi i e^{-2\pi i a} = 2\pi^2 e^{-2\pi i a}.$$

Integrate along γ_6, and let $\varepsilon \to 0^+$. This gives $0 < a < 1$

$$\left|\int_{\gamma_6} f(z)\,dz\right| \leq \frac{\sqrt{\log^2 \varepsilon + (2\pi)^2}}{\varepsilon^a (1-\varepsilon)} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{1-a}\left|\log\varepsilon\right| \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I - e^{-2\pi i a} I - 2\pi i e^{-2\pi i a} J + 2\pi^2 e^{-2\pi i a} + 0 = 0.$$

Multiplying with $e^{2\pi i a}$, this yields

$$I e^{2\pi i a} - I - 2\pi i J + 2\pi^2 = 0.$$

We equate real parts

$$(\cos(2\pi a) - 1)\, I + 2\pi^2 = 0,$$

and therefore

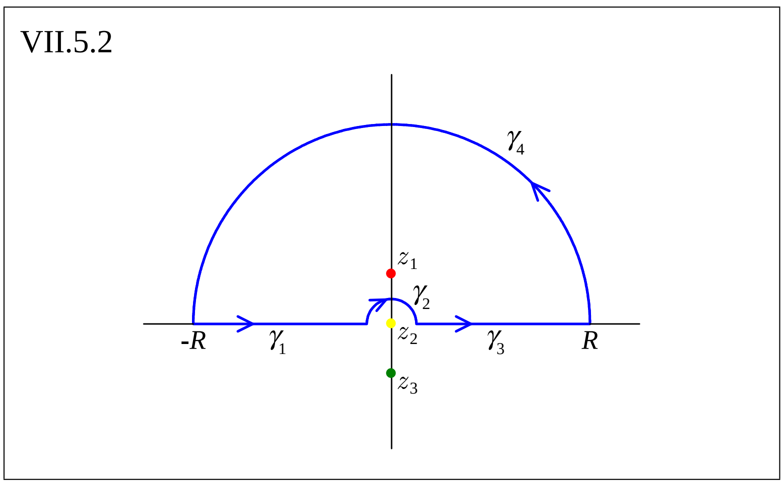
$$\int_0^\infty \frac{\log x}{x^a (x-1)} dx = \frac{2\pi^2}{1 - \cos(2\pi a)}, \qquad 0 < a < 1.$$

VII.5.2
Show using residue theory that

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+1)} dx = \pi\left(1 - e^{-a}\right), \qquad a > 0.$$

Hint. Replace $\sin(az)$ by e^{iaz}, and integrate around the boundary of a half-disk indented at $z = 0$.

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{\sin ax}{x(x^2+1)} dx$$

and integrate

$$f(z) = \frac{e^{iaz}}{z(z^2+1)} = \frac{e^{iaz}}{z(z-i)(z+i)}$$

along the contour in Figure VII.5.2.
Residue at a simple pole at $z_1 = 0$, where by Rule 3,

$$\text{Res}\left[\frac{e^{iaz}}{z(z^2+1)}, 0\right] = \left.\frac{e^{iaz}}{3z^2+1}\right|_{z=0} = 1.$$

Residue at a simple pole at $z_2 = i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{e^{iaz}}{z\left(z^2+1\right)}, i\right] = \left.\frac{e^{iaz}}{3z^2+1}\right|_{z=i} = -\frac{e^{-a}}{2}.$$

Integrate along γ_1 and γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\begin{aligned}
\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz &= \\
&= \int_{\gamma_1} \frac{e^{iaz}}{z\left(z^2+1\right)} dz + \int_{\gamma_3} \frac{e^{iaz}}{z\left(z^2+1\right)} dz = \left[\begin{array}{ccc} z & = & x \\ dz & = & dx \end{array}\right] = \\
&= \int_{-R}^{-\varepsilon} \frac{e^{iax}}{x\left(x^2+1\right)} dx + \int_{\varepsilon}^{R} \frac{e^{iax}}{x\left(x^2+1\right)} dx \to \\
\to \int_{-\infty}^{\infty} \frac{e^{iax}}{x\left(x^2+1\right)} dx &= \int_{-\infty}^{\infty} \frac{\cos ax}{x\left(x^2+1\right)} dx + i \int_{-\infty}^{\infty} \frac{\sin ax}{x\left(x^2+1\right)} dx = \\
&= \int_{-\infty}^{\infty} \frac{\cos ax}{x\left(x^2+1\right)} dx + iI.
\end{aligned}$$

Integrate along γ_2, use fractional residue formula and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_2} f(z)\,dz \to -i(\pi - 0) \cdot 1 = -\pi i.$$

Integrate along γ_4, and let $R \to \infty$. Because $|e^{iaz}| \leq 1$ in the upper half plane if $a > 0$, this gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \leq \frac{1}{R\left(R^2-1\right)} \cdot \pi R \sim \frac{\pi}{R^2} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x\left(x^2+1\right)} dx + iI - \pi i + 0 = 2\pi i \cdot \left(-\frac{e^{-a}}{2}\right),$$

and hence, setting imaginary parts equal,

$$I = \pi - \pi e^{-a},$$

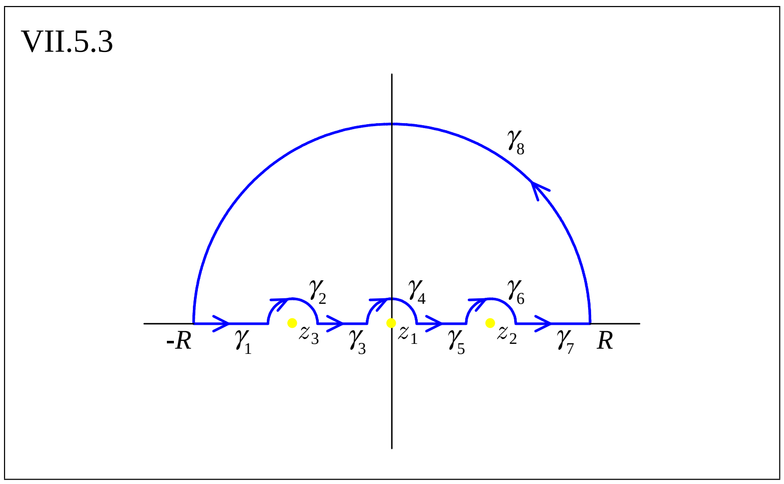
i.e.,

$$\int_{-\infty}^{\infty} \frac{\sin ax}{x\left(x^2+1\right)} dx = \pi\left(1-e^{-a}\right), \qquad a>0.$$

VII.5.3
Show using residue theory that

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(\pi^2 - a^2x^2)}dx = \frac{2}{\pi}, \qquad a > 0.$$

Solution



Set

$$I = \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(\pi^2 - a^2x^2)}dx$$

and integrate

$$f(z) = \frac{e^{iaz}}{z(\pi^2 - a^2z^2)} = \frac{e^{iaz}}{-a^2z(z - \pi/a)(z + \pi/a)}$$

along the contour in Figure VII.5.3.
Residue at a simple pole at $z_1 = 0$, where by Rule 3,

$$\text{Res}\left[\frac{e^{iaz}}{z(\pi^2 - a^2z^2)}, 0\right] = \left.\frac{e^{iaz}}{\pi^2 - 3a^2z^2}\right|_{z=0} = \frac{1}{\pi^2}.$$

Residue at a simple pole at $z_2 = \pi/a$, where by Rule 3,

$$\text{Res}\left[\frac{e^{iaz}}{z(\pi^2 - a^2z^2)}, \frac{\pi}{a}\right] = \left.\frac{e^{iaz}}{\pi^2 - 3a^2z^2}\right|_{z=\pi/a} = \frac{1}{2\pi^2}.$$

Residue at a simple pole at $z_3 = -\pi/a$, where by Rule 3,

$$\operatorname{Res}\left[\frac{e^{iaz}}{z\left(\pi^2 - a^2z^2\right)}, -\frac{\pi}{a}\right] = \left.\frac{e^{iaz}}{\pi^2 - 3a^2z^2}\right|_{z=-\pi/a} = \frac{1}{2\pi^2}.$$

Integrate along $\gamma_1, \gamma_3, \gamma_5$ and γ_7, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\begin{aligned}
&\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_5} f(z)\,dz + \int_{\gamma_7} f(z)\,dz = \\
&= \int_{\gamma_1} \frac{e^{iaz}}{z\left(\pi^2 - a^2z^2\right)}dz + \int_{\gamma_3} \frac{e^{iaz}}{z\left(\pi^2 - a^2z^2\right)}dz + \\
&+ \int_{\gamma_5} \frac{e^{iaz}}{z\left(\pi^2 - a^2z^2\right)}dz + \int_{\gamma_7} \frac{e^{iaz}}{z\left(\pi^2 - a^2z^2\right)}dz = \\
&= \left[\begin{array}{ccc} z & = & x \\ dz & = & dx \end{array}\right] = \\
&= \int_{-R}^{-\pi/a-\varepsilon} \frac{e^{iax}}{x\left(\pi^2 - a^2x^2\right)}dx + \int_{-\pi/a+\varepsilon}^{-\varepsilon} \frac{e^{iax}}{x\left(\pi^2 - a^2x^2\right)}dx + \\
&+ \int_{\varepsilon}^{\pi/a-\varepsilon} \frac{e^{iax}}{x\left(\pi^2 - a^2x^2\right)}dx + \int_{\pi/a+\varepsilon}^{R} \frac{e^{iax}}{x\left(\pi^2 - a^2x^2\right)}dx \to \\
&\to \int_{-\infty}^{\infty} \frac{e^{iax}}{x\left(\pi^2 - a^2x^2\right)}dx = \\
&= \int_{-\infty}^{\infty} \frac{\cos(ax)}{x\left(\pi^2 - a^2x^2\right)}dx + i\int_{-\infty}^{\infty} \frac{\sin(ax)}{x\left(\pi^2 - a^2x^2\right)}dx = \\
&= \int_{-\infty}^{\infty} \frac{\cos(ax)}{x\left(\pi^2 - a^2x^2\right)}dx + iI.
\end{aligned}$$

Integrate along γ_2, use fractional residue formula and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_2} f(z)\,dz \to -i(\pi - 0) \cdot \frac{1}{2\pi^2} = -\frac{1}{2\pi}i.$$

Integrate along γ_4, use fractional residue formula and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_4} f(z)\,dz \to -i(\pi - 0) \cdot \frac{1}{\pi^2} = -\frac{1}{\pi}i.$$

Integrate along γ_6, use fractional residue formula and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_6} f(z)\, dz \to -i(\pi - 0) \cdot \frac{1}{2\pi^2} = -\frac{1}{2\pi} i.$$

Integrate along γ_8, and let $R \to \infty$. Because $|e^{iaz}| \leq 1$ in the upper half plane if $a > 0$, this gives

$$\left| \int_{\gamma_2} f(z)\, dz \right| \leq \frac{1}{R(a^2R^2 - \pi^2)} \cdot \pi R \sim \frac{\pi}{a^2R^2} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x(x^2+1)} dx + iI - \frac{1}{2\pi} i - \frac{1}{\pi} i - \frac{1}{2\pi} i = 0,$$

and hence, setting imaginary parts equal

$$I - \frac{1}{2\pi} - \frac{1}{\pi} - \frac{1}{2\pi} = 0,$$

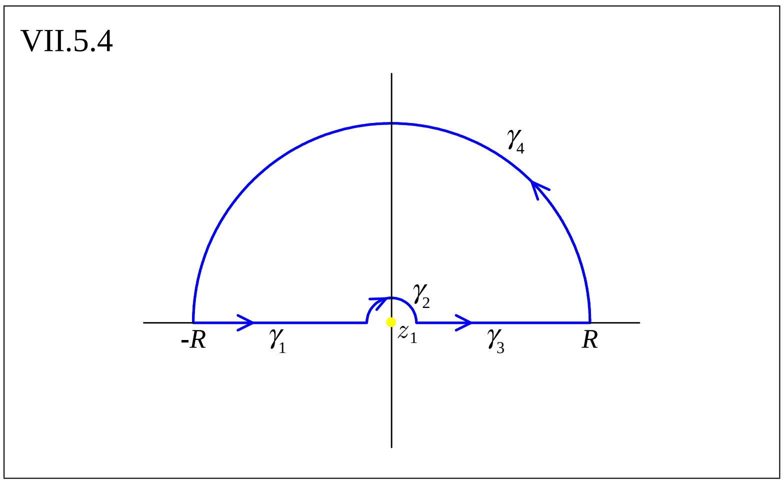
i.e.,

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(\pi^2 - a^2x^2)} dx = \frac{2}{\pi}, \qquad a > 0.$$

VII.5.4
Show using residue theory that

$$\int_0^\infty \frac{1-\cos x}{x^2}dx = \frac{\pi}{2}.$$

Solution



Set

$$I = \int_0^\infty \frac{1-\cos x}{x^2}dx \Rightarrow 2I = \int_{-\infty}^\infty \frac{1-\cos x}{x^2}dx$$

and integrate

$$f(z) = \frac{1-e^{iz}}{z^2}$$

along the contour in Figure VII.5.4.
Residue at a double pole at $z_1 = 0$, where by Rule 2,

$$\operatorname{Res}\left[\frac{1-e^{iz}}{z^2}, 0\right] = \lim_{z\to 0}\frac{d}{dz}\left(1-e^{iz}\right) = -ie^{iz}\big|_{z=0} = -i$$

Integrate along γ_1 and γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz =$$

$$= \int_{\gamma_1} \frac{1-e^{iz}}{z^2} dz + \int_{\gamma_3} \frac{1-e^{iz}}{z^2} dz =$$

$$= \left[\begin{array}{ccc} z & = & x \\ dz & = & dx \end{array} \right] =$$

$$\int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2} dx + \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2} dx \to \int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2} dx =$$

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2} dx - i \int_{-\infty}^{\infty} \frac{\sin x}{x^2} dx = 2I - i \int_{-\infty}^{\infty} \frac{\sin x}{x^2} dx.$$

Integrate along γ_2, use fractional residue formula and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_2} f(z)\,dz \to -i(\pi - 0) \cdot (-i) = -\pi.$$

Integrate along γ_4, and let $R \to \infty$. Because $|e^{iaz}| \le 1$ in the upper half plane if $a > 0$, this gives

$$\left| \int_{\gamma_4} f(z)\,dz \right| \le \frac{2}{R^2} \cdot \pi R \sim \frac{2\pi}{R} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$2I - i \int_{-\infty}^{\infty} \frac{\sin x}{x^2} dx - \pi + 0 = 0,$$

and hence, setting imaginary parts equal

$$2I - \pi = 0,$$

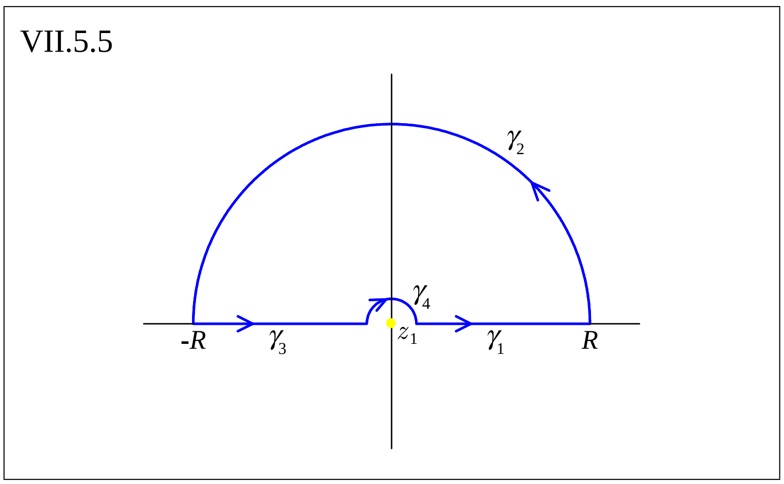
i.e.,

$$\int_{0}^{\infty} \frac{1-\cos x}{x^2} dx = \frac{\pi}{2}.$$

VII.5.5
By integrating $\left(e^{\pm 2iz}-1\right)/z^2$ over appropriate indented contours and using Cauchy´s theorem, show that

$$\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\pi.$$

Solution



Set

$$I=\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx=\int_{-\infty}^{\infty}\frac{1-\cos 2x}{2x^2}dx$$

and integrate

$$f(z)=\frac{e^{i2z}-1}{z^2}$$

along the contour in Figure VII.5.5.
Residue at a double pole at $z_1=0$, where by Rule 2,

$$\operatorname{Res}\left[\frac{e^{i2z}-1}{z^2},0\right]=\lim_{z\to 0}\frac{d}{dz}\left(e^{i2z}-1\right)=2ie^{i2z}\big|_{z=0}=2i$$

Integrate along γ_1 and γ_3, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz =$$

$$= \int_{\gamma_1} \frac{e^{i2z}-1}{z^2}dz + \int_{\gamma_3} \frac{e^{i2z}-1}{z^2}dz = \left[\begin{array}{ccc} z & = & x \\ dz & = & dx \end{array}\right] =$$

$$= \int_{-R}^{-\varepsilon} \frac{e^{i2x}-1}{x^2}dx + \int_{\varepsilon}^{R} \frac{e^{i2x}-1}{x^2}dx \to \int_{-\infty}^{\infty} \frac{e^{i2x}-1}{x^2}dx =$$

$$= \int_{-\infty}^{\infty} \frac{\cos 2x - 1}{x^2}dx + i\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2}dx = -2I + i\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2}dx.$$

Integrate along γ_2, use fractional residue formula and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_2} f(z)\,dz \to -i(\pi - 0)\cdot 2i = 2\pi.$$

Integrate along γ_4, and let $R \to \infty$. Because $|e^{iaz}| \le 1$ in the upper half plane if $a > 0$, this gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{2}{R^2}\cdot \pi R \sim \frac{2\pi}{R} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$-2I - i\int_{-\infty}^{\infty} \frac{\sin 2x}{2x^2}dx + 2\pi + 0 = 0,$$

and hence, setting real parts equal,

$$-2I + 2\pi = 0,$$

i.e.,

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}dx = \pi.$$

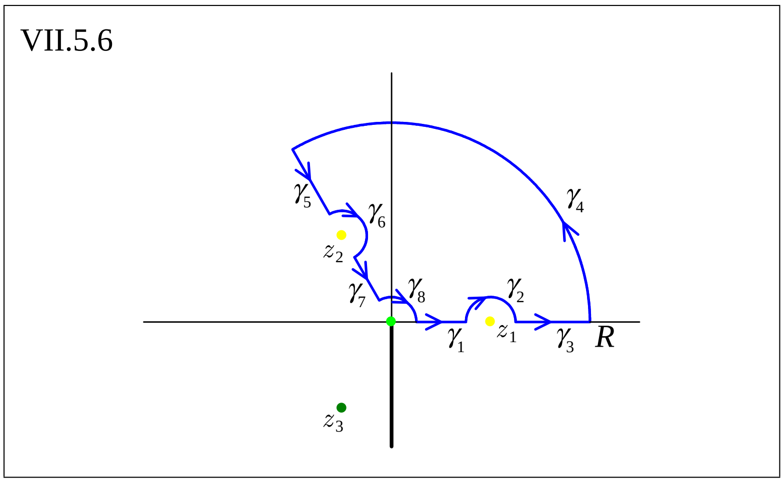
VII.5.6

By integrating a branch of $(\log z)/(z^3-1)$ around the boundary of an indented sector of aperture $2\pi/3$, show that

$$\int_0^\infty \frac{\log x}{x^3-1}dx = \frac{4\pi^2}{27}.$$

Remark. **See also Exercise 4.10.**

Solution



Set

$$I = \int_0^\infty \frac{\log x}{x^3-1}dx$$

and integrate

$$f(z) = \frac{\log z}{z^3-1} = \frac{\log|z| + i\arg z}{\left(z-e^{0i}\right)\left(z-e^{2\pi i/3}\right)\left(z-e^{4\pi i/3}\right)}$$

along the contour in Figure VII.5.6. We make a branchcut for $\log z$ along the negative imaginary axis, where $-\pi/2 < \arg z < 3\pi/2$.
Residue at a simple pole at $z_1 = 1$, where by Rule 3,

$$\text{Res}\left[\frac{\log|z| + i\arg z}{z^3-1}, 1\right] = \left.\frac{\log|z| + i\arg z}{3z^2}\right|_{z=1} = 0.$$

Residue at a simple pole at $z_2 = e^{2\pi i/3}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{\log|z|+i\arg z}{z^3-1},e^{2\pi i/3}\right]=\left.\frac{\log|z|+i\arg z}{3z^2}\right|_{z=e^{2\pi i/3}}=\frac{2\pi i}{9}e^{2\pi i/3}.$$

Integrate along γ_1 and γ_3, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz+\int_{\gamma_3} f(z)\,dz=$$
$$\int_{\gamma_1}\frac{\log|z|+i\arg z}{z^3-1}dz+\int_{\gamma_3}\frac{\log|z|+i\arg z}{z^3-1}dz=\left[\begin{array}{ccc} z & = & xe^{0i}\\ dz & = & dx\end{array}\right]=$$
$$=\int_{\varepsilon}^{1-\varepsilon}\frac{\log x}{x^3-1}dx+\int_{1+\varepsilon}^{R}\frac{\log x}{x^3-1}dx\to\int_0^\infty\frac{\log x}{x^3-1}dx=I.$$

Integrate along γ_2, use fractional residue formula and let $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_2} f(z)\,dz\to -i(\pi-0)\cdot 0=0.$$

Integrate along γ_4, and let $R\to\infty$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right|\le\frac{\sqrt{\log^2 R+\left(\frac{2\pi}{3}\right)^2}}{R^3-1}\cdot\frac{2\pi R}{3}\sim\frac{2\pi\log R}{R^2}\to 0.$$

Integrate along γ_5 and γ_7, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_5} f(z)\,dz+\int_{\gamma_7} f(z)\,dz=$$
$$=\int_{\gamma_5}\frac{\log|z|+i\arg z}{z^3-1}dz+\int_{\gamma_7}\frac{\log|z|+i\arg z}{z^3-1}dz=\left[\begin{array}{ccc} z & = & xe^{2\pi i/3}\\ dz & = & e^{2\pi i/3}dx\end{array}\right]=$$
$$=\int_R^{1+\varepsilon}\frac{\log x+2\pi i/3}{x^3-1}e^{2\pi i/3}dx+\int_{1-\varepsilon}^{\varepsilon}\frac{\log x+2\pi i/3}{x^3-1}e^{2\pi i/3}dx\to$$
$$\to e^{2\pi i/3}\int_\infty^0\frac{\log x+2\pi i/3}{x^3-1}dx=-e^{2\pi i/3}\int_0^\infty\frac{\log x}{x^3-1}dx-\frac{2\pi i}{3}e^{2\pi i/3}\int_0^\infty\frac{1}{x^3-1}dx=$$
$$=-e^{2\pi i/3}I-\frac{2\pi i}{3}e^{2\pi i/3}\int_0^\infty\frac{1}{x^3-1}dx.$$

Integrate along γ_6, use fractional residue formula and let $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_6} f(z)\, dz \to -i(\pi - 0) \cdot \left(\frac{2\pi i}{9} e^{2\pi i/3} \right) = \frac{2\pi^2}{9} e^{2\pi i/3}.$$

Integrate along γ_8, and let $\varepsilon \to 0^+$. This gives

$$\left| \int_{\gamma_8} f(z)\, dz \right| \leq \frac{\sqrt{\log^2 \varepsilon + \left(\frac{2\pi}{3}\right)^2}}{1 - \varepsilon^3} \cdot \frac{2\pi\varepsilon}{3} \sim \frac{2\pi\varepsilon\, |\log \varepsilon|}{3} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 - e^{2\pi i/3} I - \frac{2\pi i}{3} e^{2\pi i/3} \int_0^\infty \frac{1}{x^3 - 1} dx + \frac{2\pi^2}{9} e^{2\pi i/3} = 0.$$

Multiplying with $e^{-2\pi i/3}$, this yields

$$I \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right) - I - \frac{2\pi i}{3} \int_0^\infty \frac{1}{x^3 - 1} dx + \frac{2\pi^2}{9} = 0.$$

We equate real parts

$$-\frac{3}{2} I + \frac{2\pi^2}{9} = 0,$$

and hence

$$\int_0^\infty \frac{\log x}{x^3 - 1} dx = \frac{4\pi^2}{27}.$$

VII.6.1

Integrate $1/(1-x^2)$ directly, using partial fractions, and show that

$$\mathrm{PV}\int_0^\infty \frac{dx}{1-x^2} = 0.$$

Show that

$$\int_0^1 \frac{dx}{1-x^2} = +\infty, \qquad \int_1^\infty \frac{dx}{1-x^2} = -\infty.$$

Solution

Partial fractions gives

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)$$

$$\begin{aligned}
\mathrm{PV}\int_0^\infty \frac{dx}{1-x^2} &= \lim_{\varepsilon\to 0}\left(\int_0^{1-\varepsilon} \frac{dx}{1-x^2} + \int_{1+\varepsilon}^\infty \frac{dx}{1-x^2}\right) = \\
&= \lim_{\varepsilon\to 0}\left(\frac{1}{2}\left[\log(1+x) - \log(1-x)\right]_0^{1-\varepsilon} + \frac{1}{2}\left[\log|1+x| - \log|1-x|\right]_{1+\varepsilon}^\infty\right) = \\
&= \lim_{\varepsilon\to 0}\left(\frac{1}{2}\left[\log\frac{1+x}{1-x}\right]_0^{1-\varepsilon} + \frac{1}{2}\left[\log\left|\frac{1+x}{1-x}\right|\right]_{1+\varepsilon}^\infty\right) = \\
&= \lim_{\varepsilon\to 0}\frac{1}{2}\left(\log\left(\frac{2-\varepsilon}{\varepsilon}\right) - \log\left(\frac{2+\varepsilon}{\varepsilon}\right)\right) = \\
&= \lim_{\varepsilon\to 0}\frac{1}{2}\log\left(\frac{2-\varepsilon}{2+\varepsilon}\right) = 0.
\end{aligned}$$

Inspection gives

$$\begin{aligned}
\int_0^1 \frac{dx}{1-x^2} &= \lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon} \frac{dx}{1-x^2} = \lim_{\varepsilon\to 0^+}\frac{1}{2}\left[\log(1+x) - \log(1-x)\right]_0^{1-\varepsilon} = \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{2}\left[\log\frac{1+x}{1-x}\right]_0^{1-\varepsilon} = \lim_{\varepsilon\to 0^+}\frac{1}{2}\log\left(\frac{2-\varepsilon}{\varepsilon}\right) = +\infty\ .
\end{aligned}$$

and the second integral

$$\int_1^\infty \frac{dx}{1-x^2} = \lim_{\varepsilon\to 0^+} \int_{1+\varepsilon}^\infty \frac{dx}{1-x^2} = \lim_{\varepsilon\to 0^+} \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]_{1+\varepsilon}^\infty =$$
$$= \lim_{\varepsilon\to 0^+} \left(\frac{1}{2}\left[\log\left|\frac{1+x}{1-x}\right|\right]_{1+\varepsilon}^\infty\right) = \lim_{\varepsilon\to 0^+} \frac{-1}{2}\log\left(\frac{2+\varepsilon}{\varepsilon}\right) \to -\infty\ .$$

VII.6.2
Obtain the principal value in Exercise 1 by taking imaginary parts of the identity (5.4) p.211 in the preceding section and making a change of variable

Solution
Identity (5.4)

$$\int_0^\infty \frac{\log x}{x^2-1}dx+\int_{-\infty}^{-1-\varepsilon} \frac{\log|x|+\pi i}{x^2-1}dx+\int_{-1+\varepsilon}^{0} \frac{\log|x|+\pi i}{x^2-1}dx+\int_{C_\varepsilon} \frac{\log z}{z^2-1}dz=0.$$

Taking imaginary part of (5.4), get

$$\int_{-\infty}^{-1-\varepsilon} \frac{\pi}{x^2-1}dx+\int_{-1+\varepsilon}^{0} \frac{\pi}{x^2-1}dx+\operatorname{Im}\int_{C_\varepsilon} \frac{\log z}{z^2-1}dz=0.$$

Note

$$(*) \qquad \operatorname{Im}\int_{C_\varepsilon} \frac{\log z}{z^2-1}dz \to 0,$$

as $\varepsilon \to 0$, by estimation on p. 211.
Making the change of variables, $x \to -x$, and by using $(*)$ we get

$$\int_{-\infty}^{-1-\varepsilon} \frac{\pi}{x^2-1}dx+\int_{-1+\varepsilon}^{0} \frac{\pi}{x^2-1}dx=\begin{bmatrix} x & = & -x \\ dx & = & -dx \end{bmatrix}=$$
$$\int_{1+\varepsilon}^{\infty} \frac{\pi}{x^2-1}dx+\int_{0}^{1-\varepsilon} \frac{\pi}{x^2-1}dx \to 0 \text{ as } \varepsilon \to 0,$$

which gives that

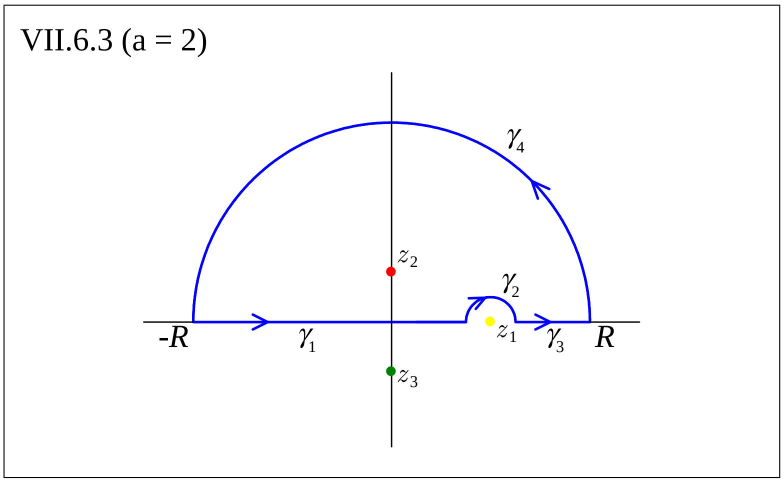
$$\mathrm{PV}\int_0^\infty \frac{dx}{1-x^2}=0.$$

Note. For a linear change of variables, we can take the limit either before or after change of variable. Otherwise we can not do this.

VII.6.3
By integrating around the boundary of an indented half-disk in the upper half-plane, show that

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(x-a)}dx=-\frac{\pi a}{a^2+1},\qquad -\infty<a<\infty.$$

Solution



Set

$$I=\mathrm{PV}\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(x-a)}dx$$

and integrate

$$f(z)=\frac{1}{(z^2+1)(z-a)}=\frac{1}{(z-i)(z+i)(z-a)}$$

along the contour in Figure VII.6.3.
Residue at a simple pole at $z_1=a$, where by Rule 1,

$$\mathrm{Res}\left[\frac{1}{(z^2+1)(z-a)},a\right]=\left.\frac{1}{z^2+1}\right|_{z=a}=\frac{1}{a^2+1}.$$

Residue at a simple pole at $z_1=i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{1}{(z^2+1)(z-a)}, i\right] = \left.\frac{1}{3z^2-2az+1}\right|_{z=i} = -\frac{1}{2(a^2+1)} + \frac{a}{2(a^2+1)}i.$$

Integrate along γ_1 and γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz =$$

$$= \int_{-R}^{a-\varepsilon} \frac{1}{(x^2+1)(x-a)}dx + \int_{a+\varepsilon}^{R} \frac{1}{(x^2+1)(x-a)}dx \to$$

$$\to \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x-a)}dx = I.$$

Integrate along γ_2, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_2} f(z)\,dz \to -i(\pi - 0) \cdot \left(\frac{1}{a^2+1}\right) = -\frac{\pi}{a^2+1}i.$$

Integrate along γ_4, and let $R \to \infty$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \leq \frac{1}{(R^2-1)(R-|a|)} \cdot \pi R \sim \frac{\pi}{R^2} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I - \frac{\pi}{a^2+1}i = 2\pi i\left(-\frac{1}{2(a^2+1)} + \frac{a}{2(a^2+1)}i\right),$$

i.e.,

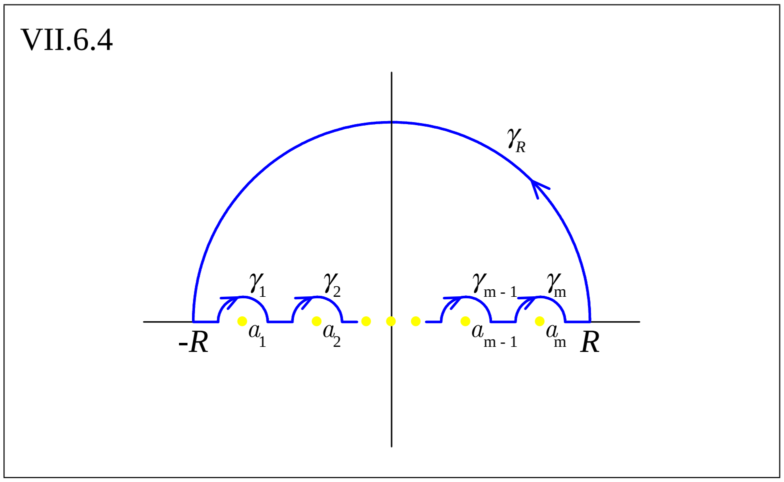
$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x-a)}dx = -\frac{\pi a}{a^2+1}.$$

VII.6.4

Suppose $m \geq 2$ and $a_1 < a_2 < \cdots < a_m$. By integrating around the boundary of an indented half-disk in the upper half-plane, show that

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{(x-a_1)(x-a_2)\cdots(x-a_m)}dx = 0.$$

Solution



Set

$$I = \mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{(x-a_1)(x-a_2)\cdots(x-a_m)}dx$$

and integrate

$$f(z) = \frac{1}{(z-a_1)(z-a_2)\cdots(z-a_m)} = \frac{1}{\prod_{k=1}^{m}(z-a_k)}$$

along the semicircular contour indented at $z = a_j$ with small semicircles of radii ε, see Figure VII.6.4.
Residue at a simple pole at $z_j = a_j$, $1 \leq j \leq m$, where by Rule 3,

$$\mathrm{Res}\left[\frac{1}{\prod_{k=1}^{m}(z-a_k)}, a_j\right] = \left.\frac{1}{\prod_{k=1,k\neq j}^{m}(z-a_k)}\right|_{z=a_j} = \frac{1}{\prod_{k=1,k\neq j}^{m}(a_j-a_k)}.$$

Integrate along the union of line segments on real axis denoted by Γ_L, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\Gamma_L} f(z)\,dz = \int_{-R}^{a_1-\varepsilon} f(z)\,dz + \sum_{k=1}^{m-1} \int_{a_k+\varepsilon}^{a_{k+1}-\varepsilon} f(z)\,dz + \int_{a_m+\varepsilon}^{R} f(z)\,dz =$$

$$= \int_{-R}^{a_1-\varepsilon} \frac{1}{\prod_{k=1}^{m}(z-a_k)}dz + \sum_{k=1}^{m-1} \int_{a_k+\varepsilon}^{a_{k+1}-\varepsilon} \frac{1}{\prod_{k=1}^{m}(z-a_k)}dz + \int_{a_m+\varepsilon}^{R} \frac{1}{\prod_{k=1}^{m}(z-a_k)}dz =$$

$$= \left[\begin{array}{ccc} z & = & x \\ dz & = & dx \end{array}\right] =$$

$$= \int_{-R}^{a_1-\varepsilon} \frac{1}{\prod_{k=1}^{m}(x-a_k)}dx + \sum_{k=1}^{m-1} \int_{a_k+\varepsilon}^{a_{k+1}-\varepsilon} \frac{1}{\prod_{k=1}^{m}(x-a_k)}dx + \int_{a_m+\varepsilon}^{R} \frac{1}{\prod_{k=1}^{m}(x-a_k)}dx \to$$

$$\to \int_{-\infty}^{\infty} \frac{1}{\prod_{k=1}^{m}(x-a_k)}dx = I.$$

Integrate along $\gamma_1, \gamma_2, \ldots \gamma_m$, and let $\varepsilon \to 0^+$. This gives

$$\sum_{j=1}^{m} \int_{\gamma_j} f(z)\,dz \to -i(\pi - 0) \cdot \left(\sum_{j=1}^{m} \frac{1}{\prod_{k=1,k\neq j}^{m}(a_j - a_k)} \right).$$

Integrate along γ_R, and let $R \to \infty$, this gives

$$\left| \int_{\gamma_R} f(z)\,dz \right| \leq \frac{1}{\prod_{k=1}^{m}(z - |a_k|)} \cdot \pi R \sim \frac{\pi}{R^{m-1}} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I - i\pi \sum_{j=1}^{m} \frac{1}{\prod_{k=1,k\neq j}^{m}(a_j - a_k)} = 0.$$

Now, note that the the sums of the residues at a simple pole at $z_j = a_j$, $1 \leq j \leq m$, are real, and identifying the real part, we have

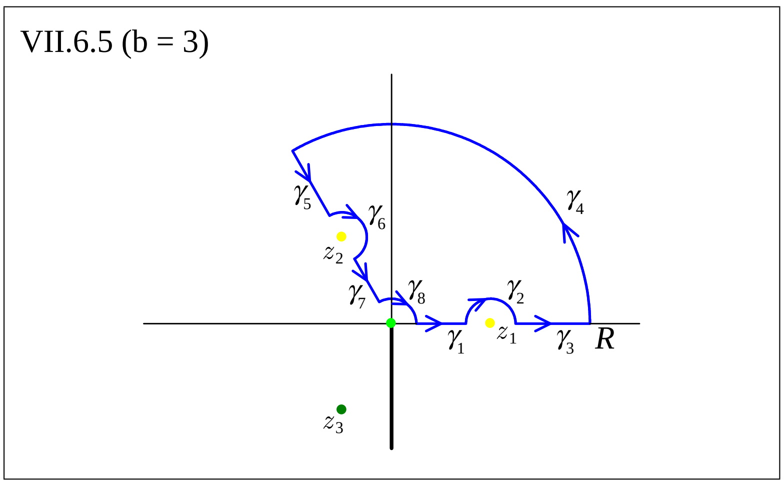
$$\mathrm{PV} \int_{-\infty}^{\infty} \frac{1}{(x-a_1)(x-a_2)\cdots(x-a_m)}dx = 0.$$

VII.6.5
Show that

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{x^{a-1}}{x^b-1}dx=-\frac{\pi}{b}\cot\left(\frac{\pi a}{b}\right),\qquad 0<a<b.$$

Hint. For $b>1$ one can integrate a branch of $z^{a-1}/\left(z^b-1\right)$ around a sector of aperture $2\pi/b$, indented at $z=1$ and $z=e^{2\pi i/b}$.

Solution



Set

$$I=\mathrm{PV}\int_{-\infty}^{\infty}\frac{x^{a-1}}{x^b-1}dx$$

and integrate

$$f\left(z\right)=\frac{z^{a-1}}{z^b-1}=\frac{|z|^{a-1}\,e^{i(a-1)\arg z}}{|z|^b\,e^{ib\arg z}-1}$$

along the contour in Figure VII.6.5. We make a branchcut for z^{a-1} along the negative imaginary axis, where $-\pi/2<\arg z<3\pi/2$.
Residue at a simple pole at $z_1=1$, where by Rule 3,

$$\mathrm{Res}\left[\frac{z^{a-1}}{z^b-1},1\right]=\left.\frac{|z|^{a-1}\,e^{i(a-1)\arg z}}{bz^{b-1}}\right|_{z=1}=\frac{1}{b}.$$

Residue at a simple pole at $z_2 = e^{2\pi i/b}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{z^{a-1}}{z^b-1}, e^{2\pi i/b}\right] = \left.\frac{|z|^{a-1}\, e^{i(a-1)\arg z}}{bz^{b-1}}\right|_{z=e^{2\pi i/b}} = \frac{e^{2\pi i/b}e^{2\pi i(a-1)/b}}{b}.$$

Integrate along γ_1 and γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\begin{aligned}
&\int_{\gamma_1} f(z)\, dz + \int_{\gamma_3} f(z)\, dz = \\
&= \int_{\gamma_1} \frac{|z|^{a-1}\, e^{i(a-1)\arg z}}{|z|^b\, e^{ib\arg z}-1} dz + \int_{\gamma_3} \frac{|z|^{a-1}\, e^{i(a-1)\arg z}}{|z|^b\, e^{ib\arg z}-1} dz = \begin{bmatrix} z & = & xe^{0i} \\ dz & = & dx \end{bmatrix} = \\
&\qquad = \int_{\varepsilon}^{1-\varepsilon} \frac{x^{a-1}}{x^b-1} dx + \int_{1+\varepsilon}^{R} \frac{x^{a-1}}{x^b-1} dx \to \int_0^\infty \frac{x^{a-1}}{x^b-1} dx = I.
\end{aligned}$$

Integrate along γ_2, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_2} f(z)\, dz \to -i(\pi - 0) \cdot \left(\frac{1}{b}\right) = -\frac{\pi i}{b}.$$

Integrate along γ_4, and let $R \to \infty$. This gives when $0 < a < b$

$$\left|\int_{\gamma_4} f(z)\, dz\right| \le \frac{R^{a-1}}{R^b-1} \cdot \frac{2\pi R}{b} \sim \frac{2\pi}{bR^{b-a}} \to 0.$$

Integrate along γ_5 and γ_7, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\begin{aligned}
&\int_{\gamma_5} f(z)\, dz + \int_{\gamma_7} f(z)\, dz = \\
&\int_{\gamma_5} \frac{|z|^{a-1}\, e^{i(a-1)\arg z}}{|z|^b\, e^{ib\arg z}-1} dz + \int_{\gamma_7} \frac{|z|^{a-1}\, e^{i(a-1)\arg z}}{|z|^b\, e^{ib\arg z}-1} dz = \begin{bmatrix} z & = & xe^{2\pi i/b} \\ dz & = & e^{2\pi i/b}dx \end{bmatrix} = \\
&\quad = \int_R^{1+\varepsilon} \frac{x^{a-1}e^{2\pi i(a-1)/b}}{x^b-1} e^{2\pi i/b} dx + \int_{1-\varepsilon}^{\varepsilon} \frac{x^{a-1}e^{2\pi i(a-1)/b}}{x^b-1} e^{2\pi i/b} dx \to \\
&\quad \to \int_\infty^0 \frac{x^{a-1}e^{2\pi i(a-1)/b}}{x^b-1} e^{2\pi i/b} dx = -e^{2\pi i/b}e^{2\pi i(a-1)/b} \int_0^\infty \frac{x^{a-1}}{x^b-1} dx = \\
&\qquad = -e^{2\pi i/b}e^{2\pi i(a-1)/b} I
\end{aligned}$$

Integrate along γ_6, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_6} f(z)\, dz \to -i(\pi - 0) \cdot \left(\frac{e^{2\pi i/b} e^{2\pi i(a-1)/b}}{b} \right) = -\pi i \frac{e^{2\pi i/b} e^{2\pi i(a-1)/b}}{b}.$$

Integrate along γ_8, and let $\varepsilon \to 0^+$. This gives when $0 < a < b$

$$\left| \int_{\gamma_8} f(z)\, dz \right| \leq \frac{\varepsilon^{a-1}}{1 - \varepsilon^b} \cdot \frac{2\pi\varepsilon}{b} \sim \frac{2\pi\varepsilon^a}{b} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I - \frac{\pi i}{b} + 0 - e^{2\pi i/b} e^{2\pi i(a-1)/b} I - \pi i \frac{e^{2\pi i/b} e^{2\pi i(a-1)/b}}{b} = 2\pi i \cdot 0.$$

Hence solving for I,

$$I = \frac{\pi i}{b} \frac{1 + e^{2\pi i/b} e^{2\pi i(a-1)/b}}{1 - e^{2\pi i/b} e^{2\pi i(a-1)/b}} = \frac{\pi i}{b} \frac{1 + e^{2\pi i a/b}}{1 - e^{2\pi i a/b}} = -\frac{\pi}{b} \frac{\cos\left(\frac{\pi a}{b}\right)}{\sin\left(\frac{\pi a}{b}\right)} = -\frac{\pi}{b} \cot\left(\frac{\pi a}{b}\right),$$

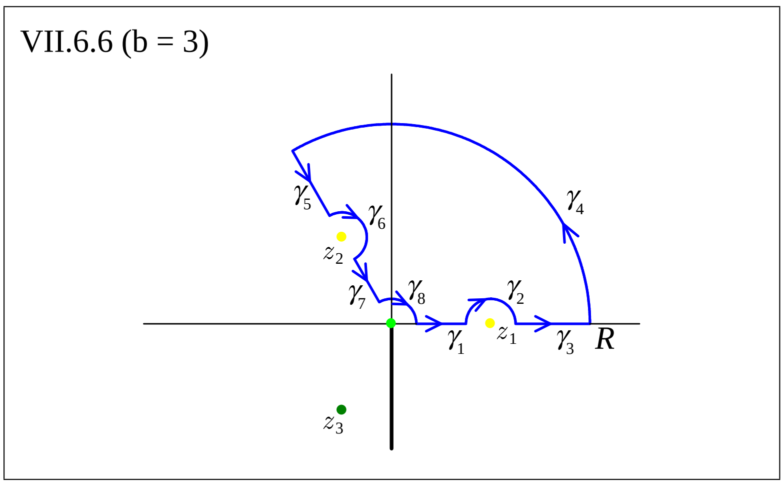
i.e.,

$$\mathrm{PV} \int_{-\infty}^{\infty} \frac{x^{a-1}}{x^b - 1} dx = -\frac{\pi}{b} \cot\left(\frac{\pi a}{b}\right), \qquad 0 < a < b.$$

VII.6.6
By integrating a branch of $(\log z)/\left(z^b-1\right)$ around an indented sector of aperture $2\pi/b$, show that for $b>1$,

$$\int_0^\infty \frac{\log x}{x^b-1}dx = \frac{\pi^2}{b^2\sin^2(\pi/b)}, \qquad \mathrm{PV}\int_0^\infty \frac{1}{x^b-1}dx = -\frac{\pi}{b}\cot(\pi/b).$$

Solution



Set

$$I=\int_0^\infty \frac{\log x}{x^b-1}dx \qquad J=\ \mathrm{PV}\int_0^\infty \frac{1}{x^b-1}dx$$

and integrate

$$f(z)=\frac{\log z}{z^b-1}=\frac{\log|z|+i\arg z}{|z|^b e^{ib\arg z}-1}$$

along the contour in Figure VII.6.5. We make a branchcut for $\log z$ along negative imaginary axis, where $-\pi/2<\arg z<3\pi/2$.
Residue at a simple pole at $z_1=1$, where by Rule 3,

$$\mathrm{Res}\left[\frac{\log z}{z^b-1},1\right]=\left.\frac{\log|z|+i\arg z}{bz^{b-1}}\right|_{z=1}=0.$$

Residue at a simple pole at $z_2=e^{2\pi i/b}$, where by Rule 3,

$$\operatorname{Res}\left[\frac{\log z}{z^b-1}, e^{2\pi i/b}\right] = \left.\frac{\log|z| + i\arg z}{bz^{b-1}}\right|_{z=e^{2\pi i/b}} = \frac{2\pi i e^{2\pi i/b}}{b^2}.$$

Integrate along γ_1 and γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_3} f(z)\,dz =$$
$$= \int_{\gamma_1} \frac{\log|z| + i\arg z}{|z|^b e^{ib\arg z} - 1}dz + \int_{\gamma_3} \frac{\log|z| + i\arg z}{|z|^b e^{ib\arg z} - 1}dz = \left[\begin{array}{ccc} z & = & xe^{0i} \\ dz & = & dx \end{array}\right] =$$
$$= \int_{\varepsilon}^{1-\varepsilon} \frac{\log x}{x^b-1}dx + \int_{1+\varepsilon}^{R} \frac{\log x}{x^b-1}dx \to \int_0^\infty \frac{\log x}{x^b-1}dx = I.$$

Integrate along γ_2, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_2} f(z)\,dz \to -i(\pi - 0)\cdot(0) = 0.$$

Integrate along γ_4, and let $R \to \infty$. This gives when $b > 1$

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\sqrt{\log^2 R + \left(\frac{2\pi}{b}\right)^2}}{R^b - 1}\cdot\frac{2\pi R}{b} \sim \frac{2\pi\log R}{bR^{b-1}} \to 0.$$

Integrate along γ_5 and γ_7, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_5} f(z)\,dz + \int_{\gamma_7} f(z)\,dz =$$
$$= \int_{\gamma_5} \frac{\log|z| + i\arg z}{|z|^b e^{ib\arg z} - 1}dz + \int_{\gamma_7} \frac{\log|z| + i\arg z}{|z|^b e^{ib\arg z} - 1}dz = \left[\begin{array}{ccc} z & = & xe^{2\pi i/b} \\ dz & = & e^{2\pi i/b}dx \end{array}\right] =$$
$$= \int_R^{1+\varepsilon} \frac{\log x + 2\pi i/b}{x^b-1}e^{2\pi i/b}dx + \int_{1-\varepsilon}^{\varepsilon} \frac{\log x + 2\pi i/b}{x^b-1}e^{2\pi i/b}dx \to$$
$$\to \int_\infty^0 \frac{\log x + 2\pi i/b}{x^b-1}e^{2\pi i/b}dx = -e^{2\pi i/b}\int_0^\infty \frac{\log x}{x^b-1}dx - \frac{2\pi i}{b}e^{2\pi i/b}\int_0^\infty \frac{1}{x^b-1}dx =$$
$$= -e^{2\pi i/b}I - \frac{2\pi i}{b}e^{2\pi i/b}J.$$

Integrate along γ_6, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_6} f(z)\,dz \to -i(\pi - 0) \cdot \left(\frac{2\pi i e^{2\pi i/b}}{b^2}\right) = \frac{2\pi^2 e^{2\pi i/b}}{b^2}.$$

Integrate along γ_8, and let $\varepsilon \to 0^+$. This gives when $b > 1$

$$\left|\int_{\gamma_8} f(z)\,dz\right| \leq \frac{\sqrt{\log^2 \varepsilon + \left(\frac{2\pi}{b}\right)^2}}{1 - \varepsilon^b} \cdot \frac{2\pi\varepsilon}{b} \sim \frac{2\pi\varepsilon\,|\log \varepsilon|}{b} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 - e^{2\pi i/b} I - \frac{2\pi i}{b} e^{2\pi i/b} J + \frac{2\pi^2 e^{2\pi i/b}}{b^2} = 2\pi i \cdot 0.$$

Multiplying with $e^{-2\pi i/b}$, this yields

$$\left(e^{-2\pi i/b} - 1\right) I - \frac{2\pi i}{b} J + \frac{2\pi^2}{b^2} = 0.$$

Separating real and imaginary parts, we get simultaneous equations

$$\begin{cases} \left(\cos\left(\frac{2\pi}{b}\right) - 1\right) I + \frac{2\pi^2}{b^2} & = & 0 \\ -\sin\left(\frac{2\pi}{b}\right) I - \frac{2\pi}{b} J & = & 0, \end{cases}$$

i.e.,

$$I = \int_0^\infty \frac{\log x}{x^b - 1} dx = \frac{\pi^2}{b^2 \sin^2(\pi/b)}, \qquad J = \mathrm{PV} \int_0^\infty \frac{1}{x^b - 1} dx = -\frac{\pi}{b} \cot(\pi/b).$$

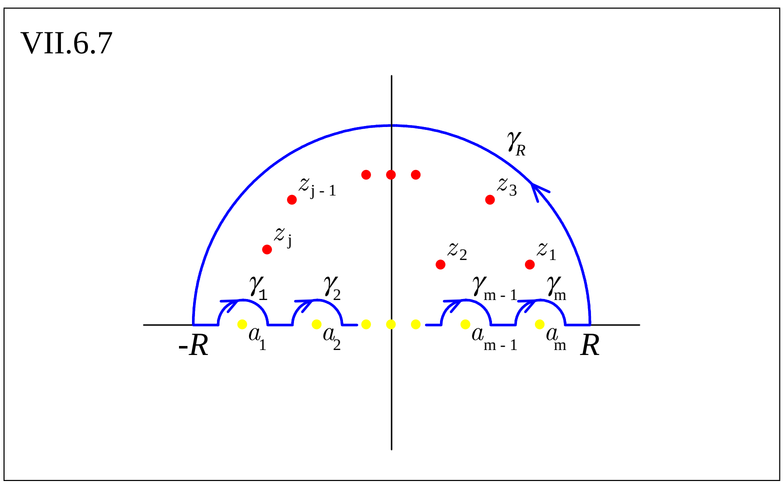
VII.6.7

Suppose that $P(z)$ and $Q(z)$ are polynomials, $\deg Q(z) \geq \deg P(z)+2$, and the zeros of $Q(z)$ on the real axis are all simple. Show that

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{P(z)}{Q(z)}dx = 2\pi i \sum \mathrm{Res}\left[\frac{P(z)}{Q(z)}, z_j\right] + \pi i \sum \mathrm{Res}\left[\frac{P(z)}{Q(z)}, x_k\right],$$

summed over the poles z_j of $P(z)/Q(z)$ in the open upper half-plane and the poles x_k of $P(z)/Q(z)$ on the real axis. Remark. In other words, the principal value of the integral is $2\pi i$ times the sum of the residues in the upper half-plane, where we count the poles on the real axis as being half in and half out of the upper half-plane.

Solution



Use a semicircular contour indented at $a_k = x_k$ with small semicircles of radii ε, see Figure VII.6.7.
Get

$$\int_{\gamma_k} \frac{P(z)}{Q(z)} \to -i(\pi - 0)\,\mathrm{Res}\left[\frac{P(z)}{Q(z)}, x_k\right] = -\pi i\,\mathrm{Res}\left[\frac{P(z)}{Q(z)}, x_k\right]$$

The condition $\deg Q \geq \deg P + 2$ guarantees that the integral converges at ∞. Otherwise we would have to use a residue at ∞ (in this case $\deg Q = \deg P + 1$).

Use the *Residue Theorem*, take limit, and the result follows

Set

$$I = \mathrm{PV}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx$$

and integrate

$$f(z) = \frac{P(z)}{Q(z)}$$

along the contour in Figure VII.6.7.
The sum of the residues at the simple poles $a_k = x_k$, $1 \le k \le m$ on the real axis is

$$\sum_{k=1}^{m}\mathrm{Res}\left[\frac{P(z)}{Q(z)}, x_k\right]$$

The sum of the residues at the simple poles z_k, $1 \le k \le j$ inside the integration contour is

$$\sum_{k=1}^{j}\mathrm{Res}\left[\frac{P(z)}{Q(z)}, z_j\right]$$

Integrate along segments on real axis, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int f(z)\,dz = \int_{-R}^{a_1-\varepsilon} f(z)\,dz + \sum_{k=1}^{m-1}\int_{a_k+\varepsilon}^{a_{k+1}-\varepsilon} f(z)\,dz + \int_{a_m+\varepsilon}^{R} f(z)\,dz =$$

$$= \int_{-R}^{a_1-\varepsilon}\frac{P(z)}{Q(z)}dz + \sum_{k=1}^{m-1}\int_{a_k+\varepsilon}^{a_{k+1}-\varepsilon}\frac{P(z)}{Q(z)}dz + \int_{a_m+\varepsilon}^{R}\frac{P(z)}{Q(z)}dz =$$

$$= \left[\begin{array}{ccc} z & = & x \\ dz & = & dx \end{array}\right] =$$

$$= \int_{-R}^{a_1-\varepsilon}\frac{P(x)}{Q(x)}dx + \sum_{k=1}^{m-1}\int_{a_k+\varepsilon}^{a_{k+1}-\varepsilon}\frac{P(x)}{Q(x)}dx + \int_{a_m+\varepsilon}^{R}\frac{P(x)}{Q(x)}dx \to$$

$$\to \int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx = I.$$

Integrate along γ_R, where we have $\deg Q(z) \geq \deg P(z) + 2$ and let $R \to \infty$. This gives

$$\left| \int_{\gamma_R} f(z)\, dz \right| \sim \frac{R^{\deg P}}{R^{\deg Q}} \cdot \pi R \leq \frac{\pi}{R} \to 0.$$

Integrate along $\gamma_1, \gamma_2, \ldots \gamma_m$, and let $\varepsilon \to 0^+$. This gives

$$\sum_{k=1}^{m} \int_{\gamma_k} f(z)\, dz \to -i(\pi - 0) \cdot \sum_{k=1}^{m} \left(\operatorname{Res}\left[\frac{P(z)}{Q(z)}, x_k \right] \right).$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I - i\pi \sum_{k=1}^{m} \left(\operatorname{Res}\left[\frac{P(z)}{Q(z)}, x_k \right] \right) = 2\pi i \left(\sum_{k=1}^{j} \operatorname{Res}\left[\frac{P(z)}{Q(z)}, z_k \right] \right),$$

i.e.,

$$\mathrm{PV} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{j=1}^{m} \operatorname{Res}\left[\frac{P(z)}{Q(z)}, z_j \right] + \pi i \operatorname{Res} \sum_{k=1}^{m} \left[\frac{P(z)}{Q(z)}, x_k \right].$$

VII.7.1

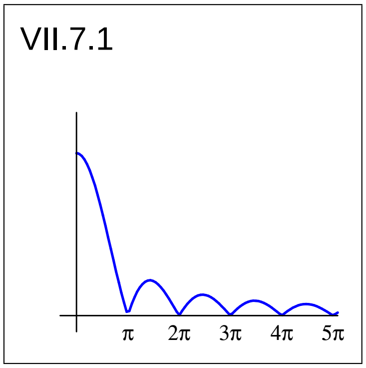
1	2	3	P	L	K
				LLL	

Show that

$$\int_0^\infty \frac{|\sin x|}{x} dx = +\infty.$$

Hint: Show that the area under the mth arc of $|\sin x|/x$ is $\sim 1/m$.

Solution



For x in the interval $(m-1)\pi + \frac{\pi}{4} \leq x \leq m\pi - \frac{\pi}{4}$, we have

$$|\sin x| \geq \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

and

$$\frac{1}{x} \geq \frac{1}{m\pi}.$$

We estimate the integrand

$$\frac{|\sin x|}{x} \geq \frac{1}{\sqrt{2}} \frac{1}{m\pi}.$$

We have

$$\int_{(m-1)\pi+\frac{\pi}{4}}^{m\pi-\pi/4} \frac{|\sin x|}{x} dx \geq \frac{1}{\sqrt{2}} \frac{1}{m\pi} \pi = \frac{C}{m}.$$

Hence

$$\int_0^{m\pi} \frac{|\sin x|}{x} dx \geq \sum_{k=1}^{m} \int_{(k-1)\pi+\frac{\pi}{4}}^{k\pi-\pi/4} \frac{|\sin x|}{x} dx = C\left(1+\frac{1}{2}+\ldots+\frac{1}{m}\right) \to +\infty$$

as $m \to \infty$.
We have used the fact that the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{m}$ tends to infinity as as $m \to \infty$, by taking the oversum for the integral $\int_1^{m+1} \frac{dx}{x}$,

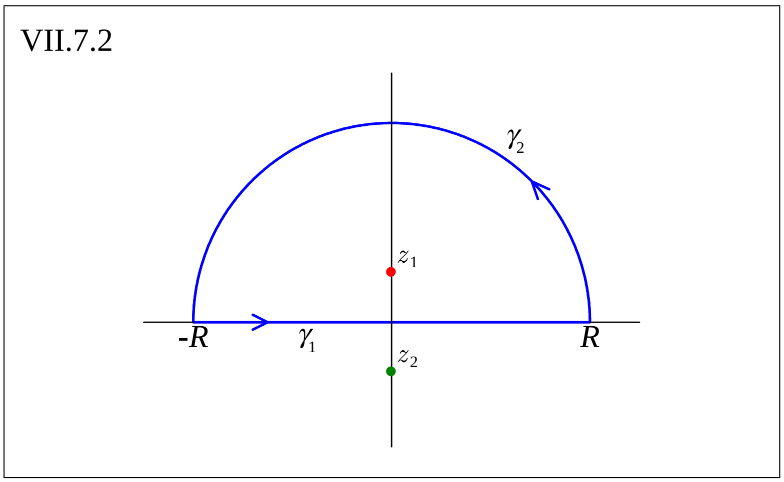
$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m} \geq \int_1^{m+1} \frac{dx}{x} = \log(m+1) \to +\infty$$

as $m \to \infty$.

VII.7.2
Show that

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^3\sin x}{\left(x^2+1\right)^2}dx=\frac{\pi}{2e}.$$

Solution



Set

$$I=\int_{-\infty}^{\infty}\frac{x^3\sin x}{\left(x^2+1\right)^2}dx$$

and integrate

$$f\left(z\right)=\frac{z^3e^{iz}}{\left(z^2+1\right)^2}=\frac{z^3e^{iz}}{\left(z-i\right)^2\left(z+i\right)^2}$$

along the contour in Figure VII.7.2.
Residue at a double pole at $z_1=i$, where by Rule 2,

$$\text{Res}\left[\frac{z^3e^{iz}}{\left(z^2+1\right)^2},i\right]=\lim_{z\to i}\frac{d}{dz}\frac{z^3e^{iz}}{\left(z+i\right)^2}=\frac{1}{4e}.$$

Integrate along γ_1 , and let $R\to\infty$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{\gamma_1} \frac{z^3 e^{iz}}{(z^2+1)^2} dz = \left[\begin{array}{ccc} z & = & x \\ dz & = & dx \end{array} \right] = \int_{-R}^{R} \frac{x^3 e^{ix}}{(x^2+1)^2} dx \rightarrow$$

$$\rightarrow \int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+1)^2} dx = \int_{-\infty}^{\infty} \frac{x^3 \cos x}{(x^2+1)^2} dx + i \int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2} dx =$$

$$= \int_{-\infty}^{\infty} \frac{x^3 \cos x}{(x^2+1)^2} dx + iI.$$

Integrate along γ_2, and let $R \rightarrow \infty$. Use Jordan's Lemma this gives

$$\left| \int_{\gamma_2} f(z)\,dz \right| \leq \frac{R^3}{(R^2-1)^2} \int_{\gamma_2} \left| e^{iz} \right| |dz| < \frac{\pi R^3}{(R^2-1)^2} \sim \frac{\pi}{R} \rightarrow 0.$$

Using the *Residue Theorem* and letting $R \rightarrow \infty$, we obtain that

$$\int_{-\infty}^{\infty} \frac{x^3 \cos x}{(x^2+1)^2} dx + iI = 2\pi i \cdot \left(\frac{1}{4e} \right),$$

and hence, setting imaginary parts equal,
i.e.,

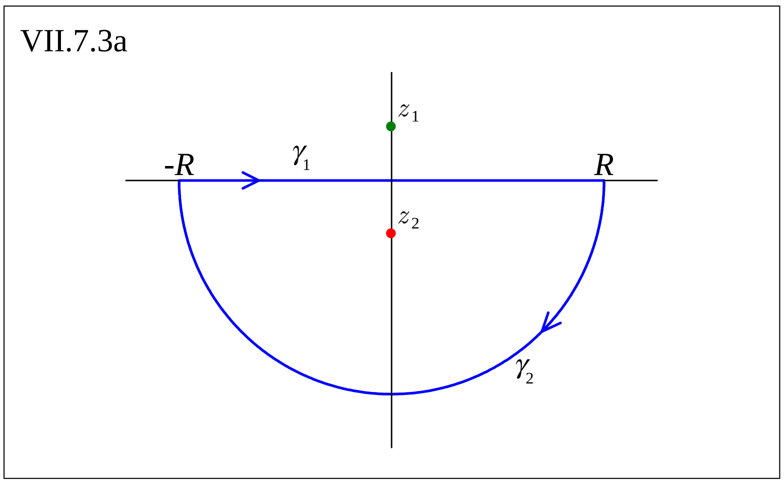
$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2} dx = \frac{\pi}{2e}.$$

VII.7.3
Evaluate the limits

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x\sin(ax)}{x^2+1}dx, \qquad -\infty < a < \infty.$$

Show that they do not depend continuously on the parameter a.

Solution



Case 1 $a < 0$.
Set

$$I = \int_{-\infty}^{\infty}\frac{x\sin(ax)}{x^2+1}dx$$

and integrate

$$f(z) = \frac{ze^{iaz}}{z^2+1} = \frac{ze^{iaz}}{(z-i)(z+i)}$$

along the contour in Figure VII.7.3a.
Residue at a simple pole at $z_2 = -i$, where by Rule 3,

$$\text{Res}\left[\frac{ze^{iaz}}{z^2+1}, -i\right] = \left.\frac{ze^{iaz}}{2z}\right|_{z=-i} = \frac{e^a}{2}.$$

Integrate along γ_1, and let $R\to\infty$. This gives

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_1} \frac{ze^{iaz}}{z^2+1}dz = \begin{bmatrix} z = xe^{0i} \\ dz = dx \end{bmatrix} = \int_{-R}^{R} \frac{xe^{iax}}{x^2+1}dx \to$$

$$\to \int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+1}dx = \int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^2+1}dx + i\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+1}dx =$$

$$= \int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^2+1}dx + iI.$$

Integrate along γ_2, and let $R \to \infty$. Use Jordan's Lemma this gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{R}{R^2-1}\int_{\gamma_2} \left|e^{iaz}\right| |dz| < \frac{\pi R}{R^2-1} \sim \frac{\pi}{R} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$\int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^2+1}dx + iI + 0 = -2\pi i \cdot \left(\frac{e^a}{2}\right),$$

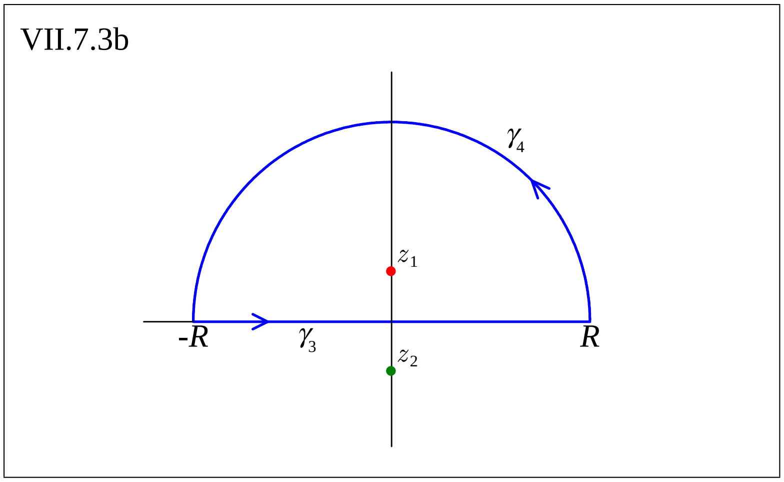
and hence, setting imaginary parts equal,

$$I = -\pi e^a.$$

Case 2 $a = 0$.

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(ax)}{x^2+1}dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{0}{x^2+1} = 0$$

Case 3 $a > 0$.



Set

$$J = \int_{-\infty}^{\infty} \frac{x \sin(ax)}{x^2+1} dx$$

and integrate

$$f(z) = \frac{ze^{iaz}}{z^2+1} = \frac{ze^{iaz}}{(z-i)(z+i)}$$

along the contour in Figure VII.7.3b
Residue at a simple pole at $z_1 = i$, where by Rule 3,

$$\text{Res}\left[\frac{ze^{iaz}}{z^2+1}, i\right] = \left.\frac{ze^{iaz}}{2z}\right|_{z=i} = \frac{e^{-a}}{2}.$$

Integrate along γ_3, and let $R \to \infty$. This gives

$$\int_{\gamma_3} f(z)\,dz = \int_{\gamma_3} \frac{ze^{iaz}}{z^2+1} dz = \begin{bmatrix} z = xe^{0i} \\ dz = dx \end{bmatrix} = \int_{-R}^{R} \frac{xe^{iax}}{x^2+1} dx \to$$

$$\to \int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+1} dx = \int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^2+1} dx + i\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+1} dx =$$

$$= \int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^2+1} dx + iJ.$$

Integrate along γ_4, and let $R \to \infty$. Use Jordan's Lemma this gives

$$\left|\int_{\gamma_4} f(z)\, dz\right| \leq \frac{R}{R^2-1}\int_{\gamma_4} \left|e^{iaz}\right| |dz| < \frac{\pi R}{R^2-1} \sim \frac{\pi}{R} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$\int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^2+1} dx + iI + 0 = 2\pi i \cdot \left(\frac{e^{-a}}{2}\right),$$

and hence, setting imaginary parts equal,

$$I = \pi e^{-a}.$$

and we have the solution

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2+1} dx = \begin{cases} -\pi e^{a}, & a < 0, \\ 0 & a = 0 \\ \pi e^{-a}, & a > 0. \end{cases}$$

Not continuous at $a = 0$ becauce $\sin 0 = 0$.

VII.7.4
By integrating $z^{a-1}e^{iz}$ around the boundary of a domain in the first quadrat bounded by the real and imaginary axes and a quater-circle, show that

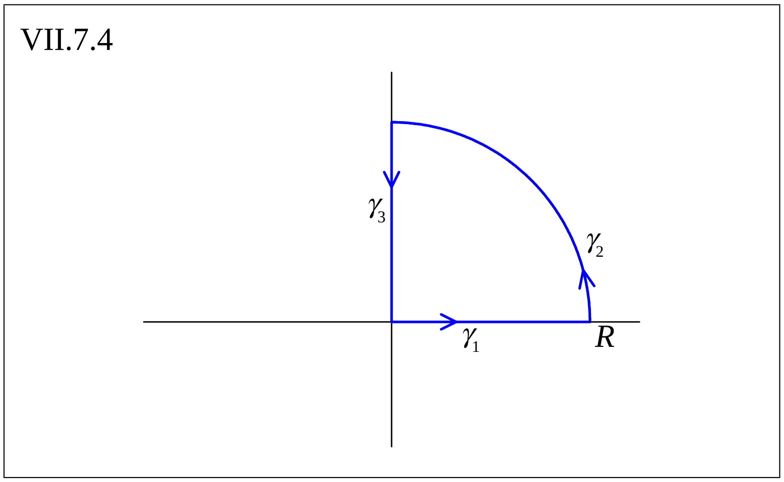
$$\begin{aligned}\lim_{R\to\infty}\int_0^R x^{a-1}\cos x dx &= \Gamma(a)\cos(\pi a/2), \qquad 0<a<1,\\ \lim_{R\to\infty}\int_0^R x^{a-1}\sin x dx &= \Gamma(a)\sin(\pi a/2), \qquad 0<a<1,\end{aligned}$$

where $\Gamma(a)$ is the gamma function defined by

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt.$$

Remark: The formula for the sine integral holds also for $-1<a<0$. To see this, integrate by parts.

Solution



Set

$$I = \lim_{R\to\infty}\int_0^R x^{a-1}\cos x dx \qquad J = \lim_{R\to\infty}\int_0^R x^{a-1}\sin x dx$$

where $0<a<1$, and integrate

$$f(z)=z^{a-1}e^{iz}=|z|^{a-1}\,e^{i(a-1)\arg z}e^{iz}$$

along the contour in Figure VII.7.4
Integrate along γ_1, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz=$$
$$=\int_{\gamma_1}|z|^{a-1}\,e^{i(a-1)\arg z}e^{iz}dz=\left[\begin{array}{ccc} z & = & xe^{0i}\\ dz & = & dx\end{array}\right]=\int_0^R x^{a-1}e^{ix}dx\to$$
$$\to\int_0^\infty x^{a-1}e^{ix}dx=\int_0^\infty x^{a-1}\cos x dx+i\int_0^\infty x^{a-1}\sin x dx=I+iJ.$$

Integrate along γ_2, and let $R\to\infty$. Use Jordan's Lemma this gives

$$\left|\int_{\gamma_2} f(z)\,dz\right|\le R^{a-1}\int_{\gamma_2}\left|e^{iz}\right||dz|<\pi R^{a-1}\sim\frac{\pi}{R^{1-a}}\to 0.$$

Integrate along γ_3, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz=$$
$$=\int_{\gamma_3}|z|^{a-1}\,e^{i(a-1)\arg z}e^{iz}dz=\left[\begin{array}{ccc} z & = & xe^{\pi i/2}\\ dz & = & e^{\pi i/2}dx\end{array}\right]=-\int_0^R x^{a-1}e^{\pi ia/2}e^{-x}dx\to$$
$$\to-\int_0^\infty x^{a-1}e^{\pi ia/2}e^{-x}dx=-e^{\pi ia/2}\int_0^\infty x^{a-1}e^{-x}dx=-e^{\pi ia/2}\Gamma(a)=$$
$$=-\Gamma(a)\cos(\pi a/2)-\Gamma(a)\sin(\pi a/2).$$

Using the *Residue Theorem* and letting $R\to\infty$, we obtain that

$$I+iJ-\Gamma(a)\cos(\pi a/2)-\Gamma(a)\sin(\pi a/2)=0.$$

We equate real and imaginary parts

$$I=\Gamma(a)\cos(\pi a/2)\qquad J=\Gamma(a)\sin(\pi a/2).$$

Formulas for the Gamma function, Mathematics Handbook page 278

$$\Gamma\left(z+1\right)=z\Gamma\left(z\right)$$

$$\int_0^R x^{a-1}\cos x dx=\left[x^{a-1}\sin x\right]_0^R-\int_0^R\left(a-1\right)x^{a-2}\sin x dx$$

$$\left(a-1\right)\int_0^R x^{a-2}\sin x dx=\left[x^{a-1}\sin x\right]_0^R-\int_0^R x^{a-1}\cos x dx$$

$$\left(a-1\right)\int_0^R x^{a-2}\sin x dx=R^{a-1}\sin R-\int_0^R x^{a-1}\cos x dx$$

$$\int_0^R x^{a-2}\sin x dx=\frac{R^{a-1}\sin R}{a-1}-\frac{1}{a-1}\int_0^R x^{a-1}\cos x dx$$

$$\int_0^R x^{a-2}\sin x dx=\frac{\sin R}{\left(1-a\right)R^{1-a}}-\frac{1}{a-1}\Gamma\left(a\right)\cos\left(\pi a/2\right)$$

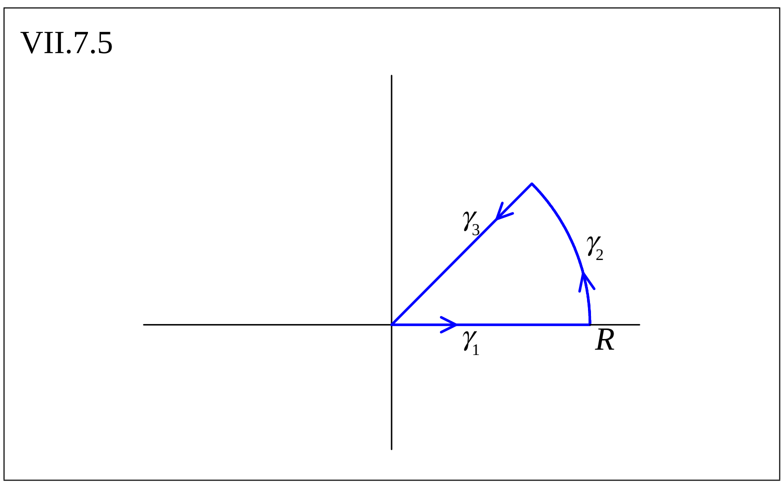
$$\begin{aligned}\lim_{R\to\infty}\int_0^R x^{a-1}\sin x dx&=\frac{-\Gamma\left(a+1\right)}{a}\cos\left(\frac{\pi}{2}\left(a+1\right)\right)=\\&=-\Gamma\left(a\right)\left[\cos\left(\pi a/2\right)\cos\left(\pi/2\right)-\sin\left(\pi a/2\right)\sin\left(\pi/2\right)\right]=\\&=\Gamma\left(a\right)\sin\left(\pi a/2\right).\end{aligned}$$

VII.7.5
Show that

$$\lim_{R\to\infty}\int_0^R \sin\left(x^2\right)dx = \lim_{R\to\infty}\int_0^R \cos\left(x^2\right)dx = \frac{\sqrt{\pi}}{2\sqrt{2}},$$

by integrating e^{iz} around the boundary of the pie-slice domain determined by $0 < \arg z < \pi/4$ and $|z| < R$. *Remark.* These improper integrals are called the **Fresnel integrals**. The identities can also be deduced from the preceding exercise by changing variable.

Solution



Set

$$I = \lim_{R\to\infty}\int_0^R \sin\left(x^2\right)dx \qquad J = \lim_{R\to\infty}\int_0^R \cos\left(x^2\right)dx$$

where $-1 < a < 0$, and integrate

$$f(z) = e^{iz^2}$$

along the contour in Figure VII.7.5
Integrate along γ_1, and let $R\to\infty$. This gives

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_1} e^{iz^2}dz = \begin{bmatrix} z &=& xe^{0i} \\ dz &=& dx \end{bmatrix} = \int_0^R e^{ix^2}dx \to$$
$$\to \int_0^\infty e^{ix^2}dx = \int_0^\infty \cos\left(x^2\right)dx + i\int_0^\infty \sin\left(x^2\right)dx = J + iI.$$

Integrate along γ_2 as we can parametrize as $z = Re^{it}$ where $0 \le t \le \pi/4$, and let $R \to \infty$. Use that $4x/\pi \le \sin 2x$ for $0 \le x \le \pi/4$, this gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| = \left|\int_{\gamma_2} e^{iz^2}dz\right| = \begin{bmatrix} z &=& Re^{it} \\ dz &=& iRe^{it}dt \end{bmatrix} =$$
$$= \left|\int_0^{\pi/4} e^{iR^2e^{2it}} iRe^{it}dt\right| \le R\int_0^{\pi/4} \left|e^{iR^2(\cos(2t)+i\sin(2t))}\right| dt =$$
$$= R\int_0^{\pi/4} e^{-R^2\sin 2t}dt \le R\int_0^{\pi/4} e^{-4R^2t/\pi}ds = -\frac{\pi}{4R}\left[e^{-4R^2t/\pi}\right]_0^{\pi/4} =$$
$$= \frac{\pi}{4R}\left(1 - e^{-R^2}\right) \sim \frac{\pi}{4R} \to 0.$$

Integrate along γ_3 as we can parametrize as $z = xe^{\pi i/4}$ where $0 \le x \le R$, and let $R \to \infty$. This gives

$$\int_{\gamma_3} f(z)\,dz = \int_{\gamma_3} e^{iz^2}dz = \begin{bmatrix} z &=& xe^{\pi i/4} \\ dz &=& e^{\pi i/4}dx \end{bmatrix} = -\int_0^R e^{-x^2}e^{\pi i/4}dx \to$$
$$\to -\int_0^\infty e^{-x^2}e^{\pi i/4}dx = -\frac{1+i}{\sqrt{2}}\int_0^\infty e^{-x^2}dx =$$
$$= -\frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}dx - i\frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}dx.$$

We will use integral (41) from Mathematics Handbook page 177 with $a = 1$,

$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

and hence

$$\int_{\gamma_3} f(z)\,dz = -\frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}dx - i\frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}dx = -\frac{\sqrt{\pi}}{2\sqrt{2}} - i\frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Using the *Residue Theorem* and letting $R \to \infty$, we obtain that

$$J + iI + 0 - \frac{\sqrt{\pi}}{2\sqrt{2}} - i\frac{\sqrt{\pi}}{2\sqrt{2}} = 0.$$

We equate real and imaginary parts

$$I = \frac{\sqrt{\pi}}{2\sqrt{2}} \qquad J = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

VII.8.1

1	2	3	P	L	K

Evaluate the residue at ∞ of the following functions.

(a) $\frac{z}{z^2-1}$ (c) $\frac{z^3+1}{z^2-1}$ (e) $z^n e^{1/z}$, $n=0,\pm 1,\dots$

(b) $\frac{1}{(z^2+1)^2}$ (d) $\frac{z^9+1}{z^6-1}$ (f) $\sqrt{\frac{z-a}{z-b}}$

Note. There are two possibilities for (f), one for each branch of the square root.

a)

$$\frac{z}{z^2-1}=\frac{1}{z}\frac{1}{1-\frac{1}{z^2}}=\frac{1}{z}\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\dots\right),$$

hence

$$\operatorname{Res}\left[\frac{z}{z^2-1},\infty\right]=-1.$$

b)

$$\frac{1}{(z^2+1)^2}=O\left(\frac{1}{z^4}\right)\text{ as } z\to\infty,$$

hence

$$\operatorname{Res}\left[\frac{1}{(z^2+1)^2},\infty\right]=0.$$

c)

$$\frac{z^3+1}{z^2-1}=\frac{z\left(z^2-1\right)+z+1}{z^2-1}=z+\frac{z}{z^2-1}+\frac{1}{z^2-1}=$$
$$=z+\frac{1}{z}\frac{1}{1-\frac{1}{z^2}}+\frac{1}{z^2-1}=z+\frac{1}{z}\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\dots\right)+\frac{1}{z^2-1},$$

hence

$$\operatorname{Res}\left[\frac{z^3+1}{z^2-1},\infty\right]=-1.$$

d)

$$\frac{z^9+1}{z^6-1}=\frac{z^3\left(z^6-1\right)+z^3+1}{z^6-1}=z^3+\frac{z^3+1}{z^6-1},$$

hence

$$\operatorname{Res}\left[\frac{z^9+1}{z^6-1},\infty\right]=0.$$

e)

$$\begin{aligned} z^n e^{1/z} = z^n\left(1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\frac{1}{3!z^3}+\frac{1}{4!z^4}+\ldots\right) &= \\ = z^n+z^{n-1}+\frac{1}{2!}z^{n-2}+\frac{1}{3!}z^{n-3}+\frac{1}{4!}z^{n-4}+ . \end{aligned}$$

We can see that the coefficient of $\frac{1}{z}$ is

$$\frac{1}{(n+1)!},$$

hence

$$\operatorname{Res}\left[\ z^n e^{1/z},\infty\right]=-\frac{1}{(n+1)!},\qquad n=0,\pm1,\ldots.$$

f)

$$\sqrt{\frac{z-a}{z-b}}=\sqrt{\frac{1-a/z}{1-b/z}}=\sqrt{\frac{1-aw}{1-bw}},$$

analytic at $w=0$.
coefficient of w is $g'(0)=\frac{1}{2}\left(\frac{1-aw}{1-bw}\right)^{-1/2}\left(\frac{-a(1-bw)+b(1-aw)}{(1-bw)^2}\right)\Big|_{w=0}=\frac{1}{2}\frac{1}{\sqrt{1}}\left(b-a\right)=$ $\pm\frac{b-a}{2}$.

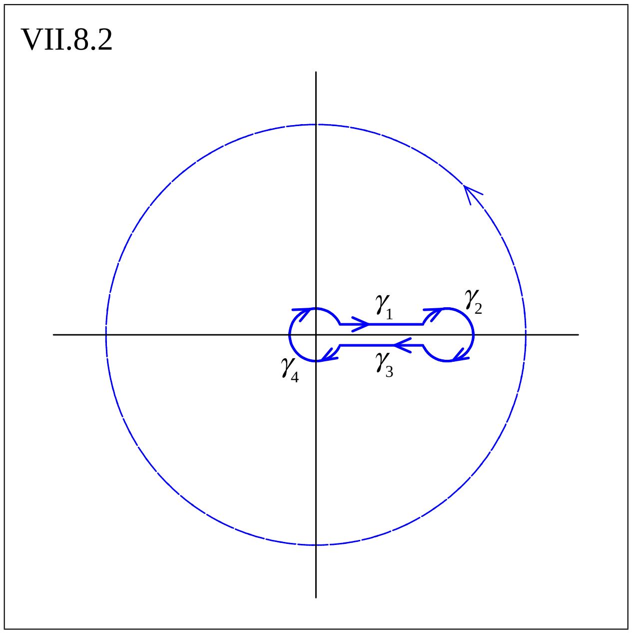
7.8.2

1	2	3	P	L	K

Show by integrating around the dogbone contour that

$$\int_0^1 \frac{x^4}{\sqrt{x(1-x)}}dx = \frac{35\pi}{128}.$$

Solution



Set

$$I = \int_0^1 \frac{x^4}{\sqrt{x(1-x)}}dx$$

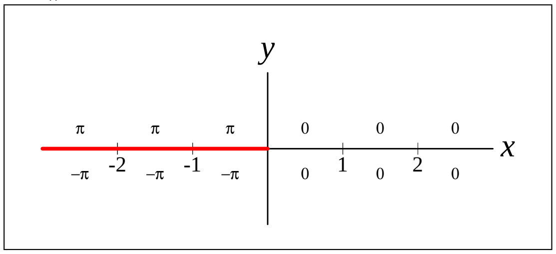
and integrate

$$f(z) = \frac{z^4}{\sqrt{z(1-z)}} = \frac{z^4}{\sqrt{|z|}e^{i\arg_\pi z/2}\sqrt{|1-z|}e^{i\arg_0(1-z)/2}}$$

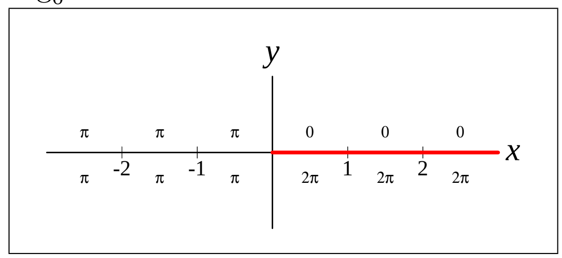
along the contour in Figure VII.8.2.

VIII.8.2

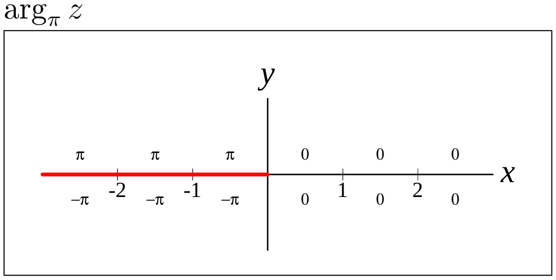
$\arg_{\pi} z$



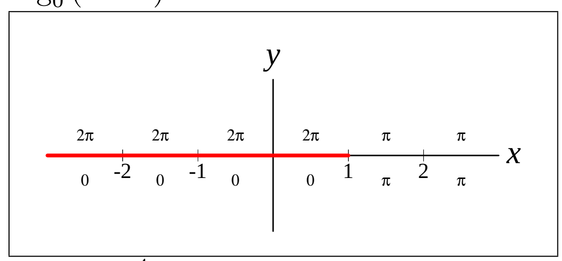
$\arg_{0} z$



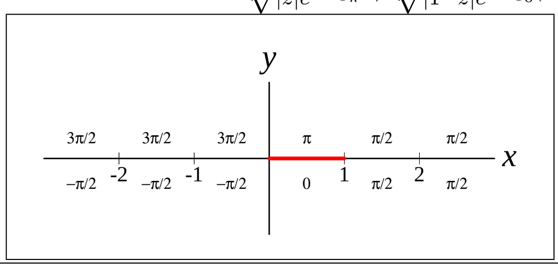
$\arg_{\pi} z$



$\arg_{0}(1-z)$



The argument for the function $f(z)=\frac{z^4}{\sqrt{|z|}e^{i\arg_{\pi} z/2}\sqrt{|1-z|}e^{i\arg_{0}(1-z)/2}}$



Residue at a simple pole at $z_1=\infty$, where by result from Exercise VII.8.13 and Rule 1,

$$\operatorname{Res}[f(z),\infty]=-\operatorname{Res}\left[\frac{1}{w^2}f\left(\frac{1}{w}\right),0\right]=-\operatorname{Res}\left[\frac{1}{w^2}\frac{\frac{1}{w^4}}{\sqrt{\frac{1}{w}\left(1-\frac{1}{w}\right)}},0\right]=$$

$$=-\operatorname{Res}\left[\frac{1}{w^5}\frac{1}{\sqrt{w-1}}\right]=-\lim_{w\to 0}\frac{1}{4!}\frac{d^4}{dz^4}\left[\frac{1}{\sqrt{w-1}}\right]=$$

$$=-\frac{35}{128\,(w-1)^{9/2}}=-\frac{35}{128\,(|w-1|)^{9/2}\,e^{i9\arg(w-1)/2}}=-\frac{35}{128}e^{-\pi i/2}=-\frac{35}{128}i$$

Integrate along γ_1, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{\varepsilon}^{1-\varepsilon} \frac{z^4}{\sqrt{|z|}e^{i\arg_\pi z/2}\sqrt{|1-z|}e^{i\arg_0(1-z)/2}}dz = \int_{\varepsilon}^{1-\varepsilon} \frac{x^4}{\sqrt{|x|}e^{i\cdot 0/2}\sqrt{|1-x|}e^{i\cdot 0/2}}dx =$$

$$= \int_{\varepsilon}^{1-\varepsilon} \frac{x^4}{\sqrt{x(1-x)}}dx \to \int_0^1 \frac{x^4}{\sqrt{x(1-x)}}dx = I.$$

Integrate along γ_2, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \frac{(1+\varepsilon)^4}{\sqrt{(1+\varepsilon)\,\varepsilon}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Integrate along γ_3, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{1-\varepsilon}^{\varepsilon} \frac{z^4}{\sqrt{|z|}e^{i\arg_\pi z/2}\sqrt{|1-z|}e^{i\arg_0(1-z)/2}}dz = \int_{1-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{|x|}e^{i\cdot 0/2}\sqrt{|1-x|}e^{i2\pi/2}}dx =$$

$$= \int_{\varepsilon}^{1-\varepsilon} \frac{x^4}{\sqrt{x(1-x)}}dx \to \int_0^1 \frac{x^4}{\sqrt{x(1-x)}}dx = I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \leq \frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 + I + 0 = 2\pi i \cdot \left(\frac{35}{128} i\right),$$

i.e.,

$$\int_0^1 \frac{x^4}{\sqrt{x(1-x)}}dx = \frac{35\pi}{128}.$$

7.8.3

1	2	3	P	L	K

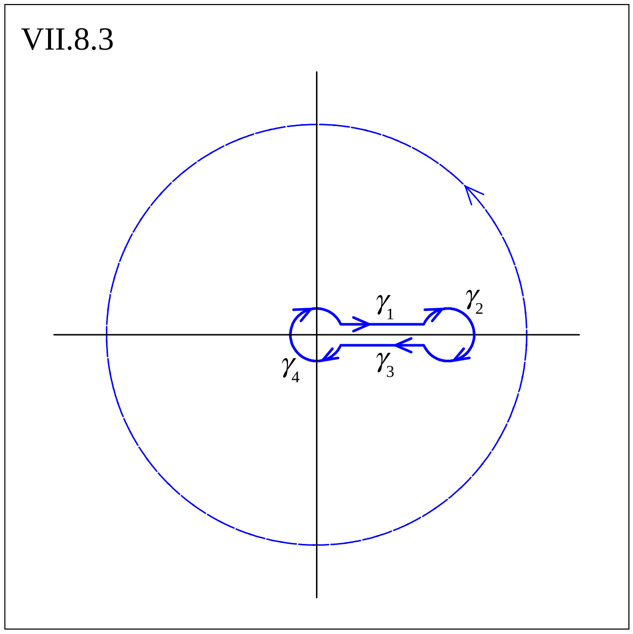
Fix an integer n, positive or negative. Determine for which complex values of the parameter a the integral

$$\int_0^1 \frac{x^n}{x^a(1-x)^{1-a}}dx$$

converges, and evaluate it.

Solution. (Obs fel i lösningen argumentet)
$x \in \mathbb{R}^+$
$1-x \in \mathbb{R}^+$
$a = \operatorname{Re} a + i \operatorname{Im} a$
$x^a = x^{\operatorname{Re} a + i \operatorname{Im} a} = x^{\operatorname{Re} a} x^{i \operatorname{Im} a}$
The integral is generalized in 0 and 1 $\varepsilon \in R$, $x = 0$
$g(x) = \frac{1}{x^{a-n}}$
$\frac{f(x)}{g(x)} = \frac{1}{(1-x)^{1-a}} \to 1$, as $x \to 0$
$\int_0^\varepsilon g(x)\,dx < \infty \Leftrightarrow a - n < 1 \Leftrightarrow \int_0^\varepsilon f(x)\,dx < \infty \Leftrightarrow a < n+1.$
The integral is generalized in 0 and 1 $\varepsilon \in R$, $x = 0$
$g(x) = \frac{1}{(1-x)^{1-a}}$
$\frac{f(x)}{g(x)} = \frac{x^n}{x^a} \to 1$, as $x \to 1$
$\int_\varepsilon^1 g(x)\,dx < \infty \Leftrightarrow 1 - a < 1 \Leftrightarrow \int_0^\varepsilon f(x)\,dx < \infty \Leftrightarrow a > 0.$
Thus
$\int_0^1 f(x)\,dx < \infty \Leftrightarrow 0 < a < n+1.$
Show that $x^{i \operatorname{Im} a}$ begränsad, set $\theta \in [\arg x]$
$\left|x^{i \operatorname{Im} a}\right| = \left|e^{i \operatorname{Im} a(\log x + i\theta)}\right| = \left|e^{i \operatorname{Im} a \log x}\right| = 1$
On principal arg 0

Determination for which complex values of the parameter a the integral converges



Set

$$I = \int_0^1 \frac{x^n}{x^a (1-x)^{1-a}} dx$$

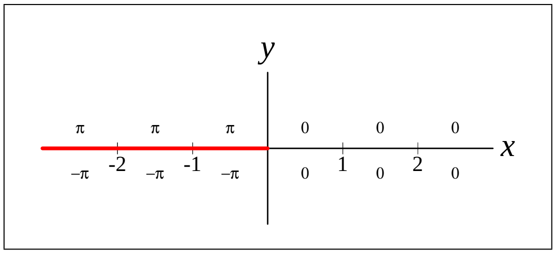
and integrate

$$f(z) = \frac{z^n}{z^a (1-z)^{1-a}} = \frac{z^n}{|z|^a e^{ai \arg_\pi z} |1-z|^{1-a} e^{(1-a)i \arg_0 (1-z)}}$$

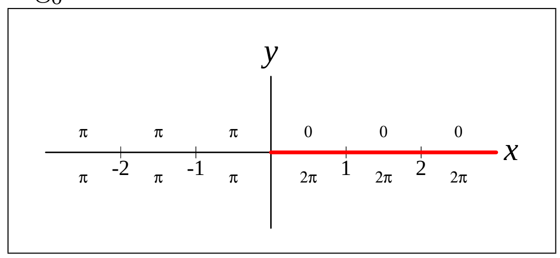
along the contour in Figure VII.8.3.

VIII.8.3

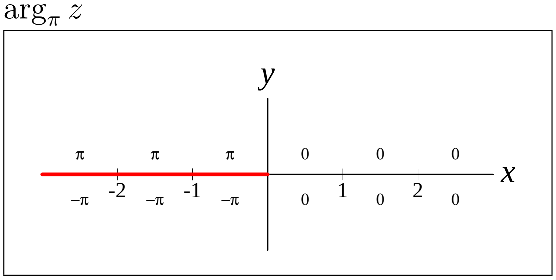
$\arg_{\pi} z$



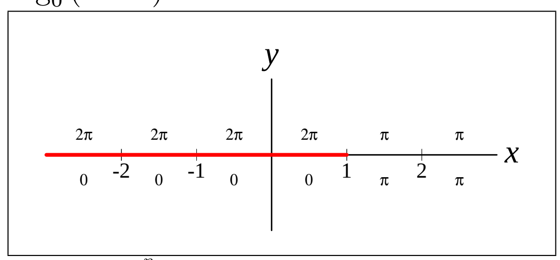
$\arg_0 z$



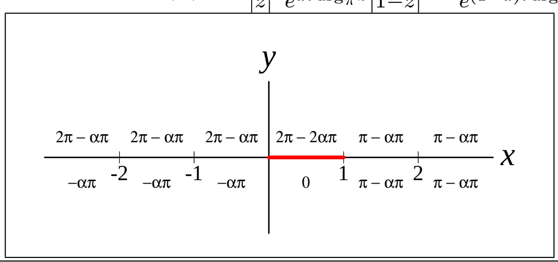
$\arg_{\pi} z$



$\arg_0 (1 - z)$



The argument for the function $f(z) = \frac{z^n}{|z|^a e^{ai\arg_{\pi} z} |1-z|^{1-a} e^{(1-a)i\arg_0(1-z)}}$



Residue at a simple pole at $z_3 = \infty$, where by result from Exercise VII.8.13 and Rule 1,

$$\operatorname{Res}[f(z), \infty] = -\operatorname{Res}\left[\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right] = -\operatorname{Res}\left[\frac{1}{w^2} \frac{\frac{1}{w^n}}{\frac{1}{w^a}\left(1 - \frac{1}{w}\right)^{1-a}}, 0\right] =$$

$$= -\operatorname{Res}\left[\frac{\sqrt{w^2-1}}{w(1+w^2)}, 0\right] = -\lim_{w \to 0} \frac{\sqrt{|w^2-1|} e^{i\arg\left(w^2-1\right)/2}}{w^2+1} = -e^{i\pi/2} = -i$$

Integrate along γ_1, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{\varepsilon}^{1-\varepsilon} \frac{z^n}{|z|^a e^{ai\arg_\pi z} |1-z|^{1-a} e^{(1-a)i\arg_0(1-z)}} = \int_{\varepsilon}^{1-\varepsilon} \frac{z^n}{|z|^a e^{ai\cdot 0} |1-z|^{1-a} e^{(1-a)i\cdot 0}} dx =$$

$$= \int_{\varepsilon}^{1-\varepsilon} \frac{z^n}{z^a (1-z)^{1-a}} dx \to$$

$$\to \int_0^1 \frac{z^n}{z^a (1-z)^{1-a}} dx = I.$$

Integrate along γ_2, and let $\varepsilon \to 0^+$. This gives

$$\left| \int_{\gamma_4} f(z)\,dz \right| \leq \frac{(1+\varepsilon)^4}{(1+\varepsilon)^a (1-(1+\varepsilon))^{1-a}} \cdot 2\pi\varepsilon \sim \frac{2\pi}{\varepsilon^{-a}} \to 0.$$

Integrate along γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{1-\varepsilon}^{\varepsilon} \frac{z^n}{|z|^a e^{ai\arg_\pi z} |1-z|^{1-a} e^{(1-a)i\arg_0(1-z)}} = \int_{1-\varepsilon}^{\varepsilon} \frac{z^n}{|z|^a e^{ai\cdot 0} |1-z|^{1-a} e^{(1-a)i\cdot 2\pi}} dx =$$

$$= -e^{2\pi i a} \int_{\varepsilon}^{1-\varepsilon} \frac{z^n}{z^a (1-z)^{1-a}} dx \to$$

$$\to -e^{2\pi i a} \int_0^1 \frac{z^n}{z^a (1-z)^{1-a}} dx = -e^{2\pi i a} I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\left| \int_{\gamma_4} f(z)\,dz \right| \leq \frac{\varepsilon^n}{\varepsilon^a (1-\varepsilon)^{1-a}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{n+1-a} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 - e^{2\pi i a} I + 0 = 2\pi i \cdot \left((-1)^n \frac{(a-1)\cdots(a-n)}{n!} \right).$$

Solve for I, we obtain that

$$I = \frac{(-1)^n \, 2\pi i \frac{(a-1)\cdots(a-n)}{n!}}{1 - e^{2\pi i a}} = \frac{(-1)^n \, 2\pi i \frac{(a-1)\cdots(a-n)}{n!}}{e^{\pi i a} - e^{-\pi i a}} = (-1)^n \, \frac{\pi}{\sin(\pi a)} \frac{(a-1)\cdots(a-n)}{n!},$$

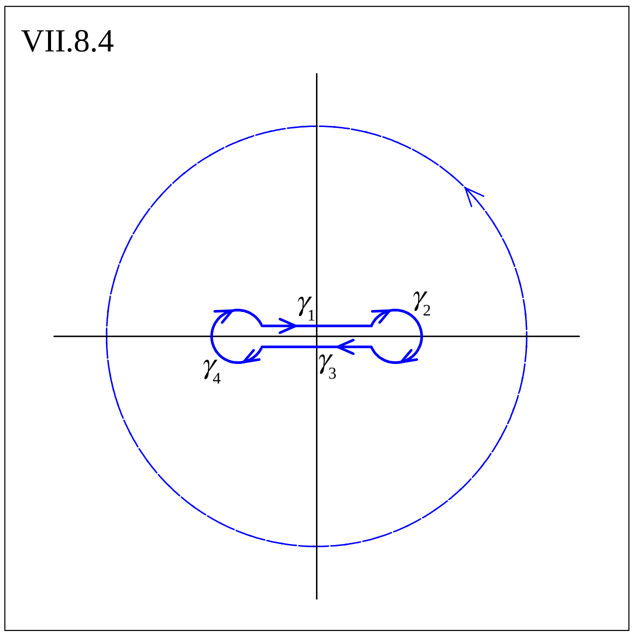
i.e.,

$$\int_0^1 \frac{x^n}{x^a (1-x)^{1-a}} dx = (-1)^n \, \frac{\pi}{\sin(\pi a)} \frac{(a-1)\cdots(a-n)}{n!}, \qquad \begin{array}{l} 0 < a < 1, \\ n = 0, 1, 2, \ldots \end{array}$$

VII.8.4
Show that

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx = \pi\left(\sqrt{2}-1\right).$$

Solution



Set

$$I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx$$

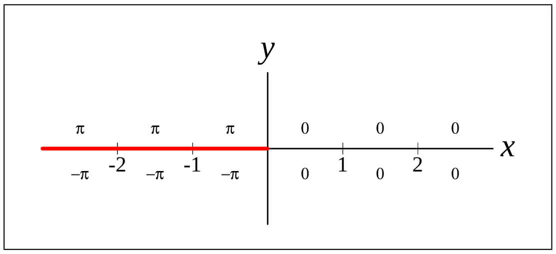
and integrate

$$f(z) = \frac{\sqrt{1-z^2}}{z^2+1} = \frac{\sqrt{|1+z|}e^{i\arg_\pi(1+z)/2}\sqrt{|1-z|}e^{i\arg_0(1-z)/2}}{z^2+1}$$

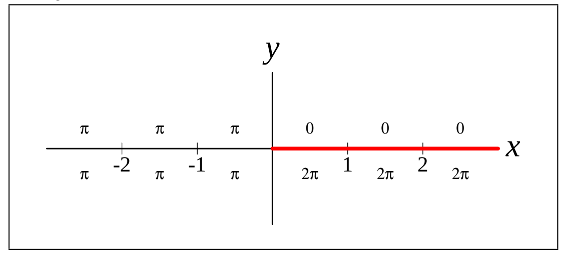
along the contour in Figure VII.8.4.

VIII.8.4

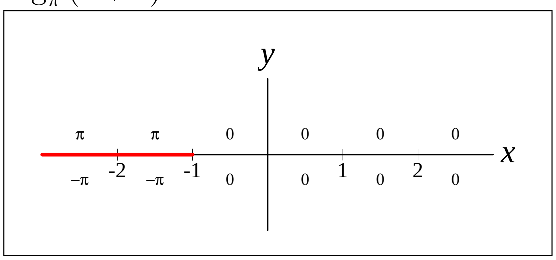
$\arg_\pi z$



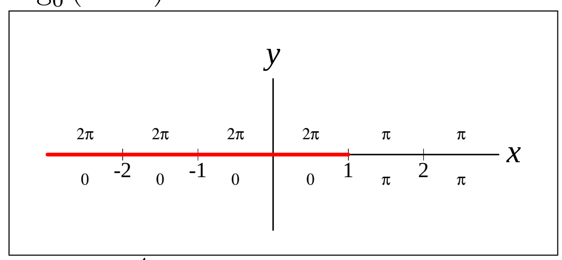
$\arg_0 z$



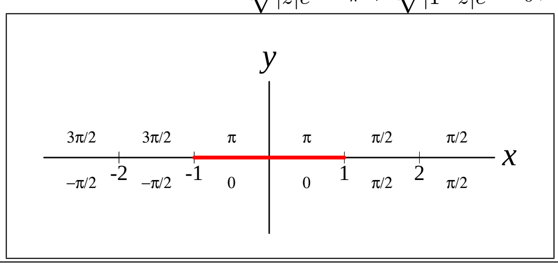
$\arg_\pi (1+z)$



$\arg_0 (1-z)$



The argument for the function $f(z) = \frac{z^4}{\sqrt{|z|e^{i\arg_\pi z/2}}\sqrt{|1-z|e^{i\arg_0(1-z)/2}}}$



Residue at a simple pole at $z_1 = i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{\sqrt{|1+z|}e^{i\arg(1+z)/2}\sqrt{|1-z|}e^{i\arg(1-z)/2}}{(z-i)(z+i)}, i\right] =$$

$$= \left.\frac{\sqrt{|1+z|}e^{i\arg(1+z)/2}\sqrt{|1-z|}e^{i\arg(1-z)/2}}{2z}\right|_{z=i} =$$

$$\frac{\sqrt{|1+i|}e^{i\arg(1+i)/2}\sqrt{|1-i|}e^{i\arg(1-i)/2}}{2i} = \frac{\sqrt{|1+i|}e^{(i\pi/4)/2}\sqrt{|1-i|}e^{(i7\pi/4)/2}}{2i}$$

$$= \frac{\sqrt{\sqrt{2}}\sqrt{\sqrt{2}}e^{\pi i}}{2i} = \frac{\sqrt{2}(-1)}{2i} = -\frac{\sqrt{2}}{2i} = \frac{i}{\sqrt{2}}$$

Residue at a simple pole at $z_2 = -i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{\sqrt{|1+z|}e^{i\arg(1+z)/2}\sqrt{|1-z|}e^{i\arg(1-z)/2}}{z^2+1}, -i\right] =$$

$$= \left.\frac{\sqrt{|1+z|}e^{i\arg(1+z)/2}\sqrt{|1-z|}e^{i\arg(1-z)/2}}{2z}\right|_{z=-i} =$$

$$= \frac{\sqrt{|1-i|}e^{i\arg(1-i)/2}\sqrt{|1+i|}e^{i\arg(1+i)/2}}{-2i} = \frac{\sqrt{|1-i|}e^{(-i\pi/4)/2}\sqrt{|1+i|}e^{(i\pi/4)/2}}{-2i} =$$

$$= \frac{\sqrt{\sqrt{2}}\sqrt{\sqrt{2}}}{-2i} = \frac{\sqrt{2}}{-2i} = \frac{1}{\sqrt{2}}i.$$

Residue at a simple pole at $z_3 = \infty$, where by result from Exercise VII.8.13 and Rule 1,

$$\operatorname{Res}\left[f(z), \infty\right] = -\operatorname{Res}\left[\frac{1}{w^2}f\left(\frac{1}{w}\right), 0\right] = -\operatorname{Res}\left[\frac{1}{w^2}\frac{\sqrt{1-\frac{1}{w^2}}}{\left(1+\frac{1}{w^2}\right)}, 0\right] =$$

$$= -\operatorname{Res}\left[\frac{\sqrt{w^2-1}}{w(1+w^2)}, 0\right] = -\lim_{w\to 0}\frac{\sqrt{|w^2-1|}e^{i\arg\left(w^2-1\right)/2}}{w^2+1} = -e^{i\pi/2} = -i$$

Integrate along γ_1, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{-1+\varepsilon}^{1-\varepsilon}\frac{\sqrt{|1+z|}e^{i\arg_\pi(1+z)/2}\sqrt{|1-z|}e^{i\arg_0(1-z)/2}}{z^2+1}dz = \int_{-1+\varepsilon}^{1-\varepsilon}\frac{\sqrt{|1+x|}e^{i\cdot 0/2}\sqrt{|1-x|}e^{i2\pi/2}}{1+x^2}dx \to$$

$$\to -\int_{-1+\varepsilon}^{1-\varepsilon}\frac{\sqrt{1-x^2}}{1+x^2}dx = -I.$$

Integrate along γ_2, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \frac{\sqrt{1-(1-\varepsilon)^2}}{1+(1-\varepsilon)^2}\cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{3/2} \to 0.$$

Integrate along γ_3, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{1-\varepsilon}^{-1+\varepsilon} \frac{\sqrt{|1+z|}e^{i\arg_\pi(1+z)/2}\sqrt{|1-z|}e^{i\arg_0(1-z)/2}}{z^2+1}dz = -\int_{-1+\varepsilon}^{1-\varepsilon} \frac{\sqrt{|1+x|}e^{i\cdot 0/2}\sqrt{|1-x|}e^{i\cdot 0/2}}{x^2+1}dx \to$$

$$\to -\int_{-1+\varepsilon}^{1-\varepsilon} \frac{\sqrt{1-x^2}}{1+x^2}dx = -I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\sqrt{1-(1-\varepsilon)^2}}{1+(1-\varepsilon)^2} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{3/2} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$-I + 0 - I + 0 = 2\pi i \cdot \left(\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}i - i\right)$$

i.e.,

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}dx = \pi\left(\sqrt{2}-1\right).$$

VII.8.5

1	2	3	P	L	K

Show that the sum of the residues of a rational function on the extended complex plane is equal to zero.

Solution

$$\oint_{|z|=R} f(z)\,dz = 2\pi i \sum \text{Residues in finite plane},$$

$$\oint_{|z|=R} f(z)\,dz = 2\pi i \operatorname{Res}[f,\infty]$$

Since $\oint_{\circlearrowleft} = -\oint_{\circlearrowright}$, we get $2\pi i \sum \operatorname{Res} = 0$.

VII.8.6

1	2	3	P	L	K

Find the residue of $z/(z^2+1)$ at each pole in the extended complex plane, and check that the sum of the residues is zero.

Solution

Residue at a simple pole at $z_1 = i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{z}{z^2+1}, i\right] = \left.\frac{z}{2z}\right|_{z=i} = \frac{1}{2}.$$

Residue at a simple pole at $z_1 = i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{z}{z^2+1}, -i\right] = \left.\frac{z}{2z}\right|_{z=-i} = \frac{1}{2}.$$

For residue at ∞, we have that

$$a_{-1} = \lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\frac{z^2}{z^2+1} = \lim_{z\to\infty}\frac{1}{1+1/z^2} = 1.$$

hence

$$\operatorname{Res}\left[\frac{z}{z^2+1}, \infty\right] = -a_{-1} = -1,$$

and the sum of the residues is zero in the extended complex plane.

(Since $\frac{z}{z^2+1} = \frac{a_{-1}}{z} + O\left(\frac{1}{z^2}\right)$ as $z \to \infty$).

VII.8.7

1	2	3	P	L	K

Find the sum of the residues of $(3z^4 + 2z + 1) / (8z^5 + 5z^2 + 2)$ **at its poles in the (finite) complex plane.**

Solution

For residue at ∞, we have that

$$a_{-1} = \lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{3z^5 + 2z^2 + z}{8z^5 + 5z^2 + z} = \lim_{z\to\infty} \frac{3 + 2/z^3 + 1/z^4}{8 + 5/z^3 + 1/z^4} = \frac{3}{8},$$

hence

$$\operatorname{Res}[f, \infty] = -\frac{3}{8}.$$

We have that the sum of the residues in the complex finite plane is $-\operatorname{Res}[f(z), \infty]$, thus the sum of residues in finite complex plane is 3/8.

$f(z) = \frac{a_{-1}}{z} + O\left(\frac{1}{z^2}\right)$ as $z \to \infty$, so again

VII.8.8

1	2	3	P	L	K

Fix $n \geq 1$ and $k \geq 0$. Find the residue of $z^k/(z^n-1)$ at ∞ by expanding $1/(z^n-1)$ in a Laurent series. Find the residue of $z^k/(z^n-1)$ at each finite pole, and verify that the sum of all the residues are zero.

Solution.
Let $f(z) = \frac{z^k}{z^n-1}$. At ∞, we have

$$f(z) = \frac{z^k}{z^n-1} = \frac{z^{k-n}}{1-\frac{1}{z^n}} = z^{k-n}\left(1 + z^{-n} + z^{-2n} + \dots\right) = z^{k-n} + z^{k-2n} + z^{k-3n} + \dots.$$

Then $\operatorname{Res}[f,\infty] = -1$ if $k - mn = -1$, thus $k = mn - 1$ for some integer $m \geq 1$, and otherwise $\operatorname{Res}[f,\infty] = 0$.

Let w_j be an nth root of unity, i.e $w_j = e^{2\pi ij/n}$ where $0 \leq j \leq n-1$. Then $f(z)$ has simple poles at $w_0, \dots, w_{n-1}$.
Residue at a simple pole at $z_j = w_j$, where by Rule 3 and that $w_j^n = 1$

$$\operatorname{Res}[f,w_j] = \left.\frac{z^k}{nz^{n-1}}\right|_{z=w_j} = \left.\frac{z^{k+1}}{nz^n}\right|_{z=w_j} = \frac{w_j^{k+1}}{nw_j^n} = \frac{w_j^{k+1}}{n}.$$

Let S be the sum of residues, then

$$S = \frac{1}{n}\sum_{j=0}^{n-1} w_j^{k+1} = \frac{1}{n}\left(1 + w + w^2 + \dots + w^{n-1}\right) = \frac{1}{n}\left(\frac{1-w^n}{1-w}\right) = 0,$$

unless $w = 1$.
If $w = 1$, then $w_j^{k+1} = w^j = 1$ for every j and $w = 1 \Leftrightarrow e^{\frac{2\pi i}{n}(k+1)} = 1 \Leftrightarrow \frac{k+1}{n} = m$, thus $k = mn - 1$ for some integer $m \geq 1$.
Thus we have that the sum of the residues in the complex plane are zero.

VII.8.9

1	2	3	P	L	K

Show that $f(z)$ is analytic at ∞, then

$$\operatorname{Res}\left[f(z), \infty\right] = -\lim_{z\to\infty} z\left(f(z) - f(\infty)\right).$$

Solution
Since $f(z)$ is analaytic at ∞, we have

$$f(z) = a_0 + \frac{a_{-1}}{z} + O\left(\frac{1}{z^2}\right).$$

Then

$$z\left(f(z) - f(\infty)\right) = z\left(a_0 + \frac{a_{-1}}{z} + O\left(\frac{1}{z^2}\right) - a_0\right) = a_{-1} + O\left(\frac{1}{z}\right) \to a_{-1}$$

as $z \to \infty$.
From this follows that

$$-\lim_{z\to\infty} z\left(f(z) - f(\infty)\right) = -a_{-1} = \operatorname{Res}\left[f(z), \infty\right].$$

VII.8.10

1	2	3	P	L	K

Let D be an exterior domain. Suppose that $f(z)$ is analytic on $D \cup \partial D$ and at ∞. Show that

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta = f(\zeta) - f(\infty), \qquad z \in D.$$

Solution.
Cauchy integral formula for exterior domain. If D is an exterior domaina, $f(z)$ is analytic in D and att ∞, then

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta = f(z) + \operatorname{Res}\left[\frac{f(\zeta)}{\zeta - z}, \infty\right] = f(z) - f(\infty)$$

IIII

$$\operatorname{Res}\left[\frac{f(\zeta)}{\zeta - z}, \infty\right] = -\lim_{\zeta\to\infty}\frac{\zeta f(\zeta)}{\zeta - z} = -f(\infty).$$

Or

$$f(\zeta) = a_0 + \frac{a_{-1}}{\zeta} + \frac{a_{-2}}{\zeta^2} + \ldots$$

$$\frac{f(\zeta)}{\zeta - z} = \frac{a_0}{\zeta - z} + O\left(\frac{1}{\zeta^2}\right)$$

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta = f(z) + \frac{1}{2\pi i}\int_{|z|=R}\frac{f(\zeta)}{\zeta - z}d\zeta =$$
$$= f(z) - \frac{1}{2\pi i}\int_{|z|=R}\frac{a_0}{\zeta - z}d\zeta = f(z) - a_0.$$

VII.8.11

1	2	3	P	L	K

If $f(z)$ is not integrable at ∞, we define the principal value

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \mathrm{PV}\int_{-R}^{R} f(x)\,dx.$$

Show that

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{x-a}dx = \begin{cases} i\pi, & \operatorname{Im} a > 0, \\ 0, & \operatorname{Im} a = 0, \\ -i\pi, & \operatorname{Im} a < 0. \end{cases}$$

Solution

Set $a = a_1 + a_2 i$, where a_1 and a_2 is fixed. We compute the principal value for the following 3 cases.

Case 1, $\operatorname{Im} a < 0 \Leftrightarrow a_2 < 0$. Integrate and let $R \to \infty$. This gives

$$\int_{-R}^{R} \frac{dx}{x-a} = \int_{-R}^{R} \frac{dx}{x-a_1-a_2 i} = [\log(x-a_1-a_2 i)]_{-R}^{R} = \log\left(\frac{R-a_1-a_2 i}{-R-a_1-a_2 i}\right) =$$
$$= \log\left|\frac{R-a_1-a_2 i}{-R-a_1-a_2 i}\right| + i\arg(R-a_1-a_2 i) - i\arg(-R-a_1-a_2 i) \to$$
$$\to 0 + i\cdot 0 - i\pi = -i\pi.$$

Case 2, $\operatorname{Im} a = 0 \Leftrightarrow a_2 = 0$. Integrate and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{-R}^{R} \frac{dx}{x-a} = \int_{-R}^{R} \frac{dx}{x-a_1} = \int_{-R}^{a_1-\varepsilon} \frac{dx}{x-a_1} + \int_{a_1+\varepsilon}^{R} \frac{dx}{x-a_1} = [\log|x-a_1|]_{-R}^{a_1-\varepsilon} + [\log|x-a_1|]_{a_1+\varepsilon}^{R} =$$
$$= \log\varepsilon - \log|R+a_1| + \log|R-a_1| - \log\varepsilon = \log\left|\frac{R-a_1}{R+a_1}\right| \to 0.$$

Case 3, $\operatorname{Im} a > 0 \Leftrightarrow a_2 > 0$. Integrate and let $R \to \infty$. This gives

$$\int_{-R}^{R} \frac{dx}{x-a} = \int_{-R}^{R} \frac{dx}{x-a_1-a_2 i} = [\log(x-a_1-a_2 i)]_{-R}^{R} = \log\left(\frac{R-a_1-a_2 i}{-R-a_1-a_2 i}\right) =$$
$$= \log\left|\frac{R-a_1-a_2 i}{-R-a_1-a_2 i}\right| + i\arg(R-a_1-a_2 i) - i\arg(-R-a_1-a_2 i) \to$$
$$\to 0 + i\cdot 2\pi - i\pi = i\pi.$$

We have that

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{x-a} dx = \begin{cases} i\pi, & \operatorname{Im} a > 0, \\ 0, & \operatorname{Im} a = 0, \\ -i\pi, & \operatorname{Im} a < 0. \end{cases}$$

VII.8.12

1	2	3	P	L	K

Suppose that $P(z)$ and $Q(z)$ are polynomials such that the degree of $Q(z)$ is strictly greater than the degree of $P(z)$. Suppose that the zeros $x_1, \ldots, x_m$ of $Q(z)$ on the real axis are all simple, and set $x_0 = \infty$. Show that

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum \operatorname{Res}\left[\frac{P(x)}{Q(x)}, z_j\right] + \pi i \sum \operatorname{Res}\left[\frac{P(x)}{Q(x)}, z_k\right],$$

summed over the poles z_j of $P(z)/Q(z)$ in the open upper half-plane and summed over the $x_k : s$ including ∞. *Hint.* Use the preceding exercise. See also Exercise 6.7

Solution.

By VII.8.11, formula holds for $\frac{1}{z-a}$ By Exercise b.7 (p. 215.) it holds if $\deg Q \geq \deg P + 2$,in initial case $\operatorname{Res}\left[\frac{P(x)}{Q(x)}, \infty\right] = 0$. By limiticity, it hods form sums, hence whenever $\deg Q \geq \deg P + 1$.

VII.8.13

1	2	3	P	L	K

Show that the analytic differential $f(z)\,dz$ transforms the change of variable $w = 1/z$ to $-f(1/w)\,dw/w^2$. Show that the residue of $f(z)$ at $z = \infty$ coincides with that of $-f(1/w)/w^2$ at $w = 0$.

Solution

We have that $\operatorname{Res}[f(z), \infty] = -a_1$ if $f(z) = \sum_{-\infty}^{\infty} a_j z^j$, $|z| > R$.
We transform $f(z)\ \ dz$ by the change of variable $z = \frac{1}{w}$, then $dz = -\frac{1}{w^2}dw$, and $f(z)\ \ dz = -f\left(\frac{1}{w}\right)\frac{1}{w^2}dw$.
Now we take the residue for $-f(1/w)\,dw/w^2$ at $w = 0$, we have that

$$-f\left(\frac{1}{w}\right)\frac{1}{w^2} = -\frac{1}{w^2}\sum_{-\infty}^{\infty}\frac{a_j}{w^j} = \sum_{-\infty}^{\infty}(-a_j)\,w^{-j-2} =$$
$$= \cdots - a_{-4}w^2 - a_{-3}w - a_{-2} - a_{-1}w^{-1} - a_0 w^{-2} - \cdots,$$

thus

$$\operatorname{Res}\left[-f\left(\frac{1}{w}\right)\frac{1}{w^2}, 0\right] = -a_{-1}$$

We have that

$$\operatorname{Res}[f(z), \infty] = -a_1 = \operatorname{Res}\left[-f\left(\frac{1}{w}\right)\frac{1}{w^2}, 0\right].$$

VII.9.1
Show using residue theory that

$$\int_0^1 \frac{1}{\sqrt[3]{x^2 - x^3}} dx = \frac{2\pi\sqrt{3}}{3}.$$

Solution
Set

$$I = \int_0^1 \frac{1}{\sqrt[3]{x^2 - x^3}} dx$$

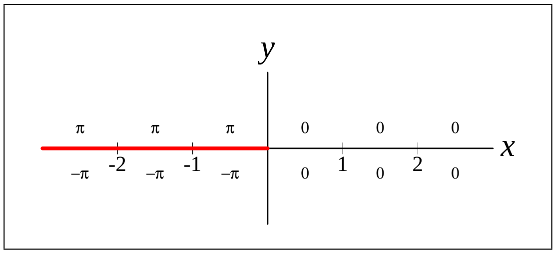
and integrate

$$f(z) = \int_0^1 \frac{1}{\sqrt[3]{z^2 - z^3}} dx = \frac{1}{\sqrt[3]{|z|^2} e^{2i \arg_\pi z/3} \sqrt[3]{|1-z|} e^{i \arg_0 (1-z)/3}}$$

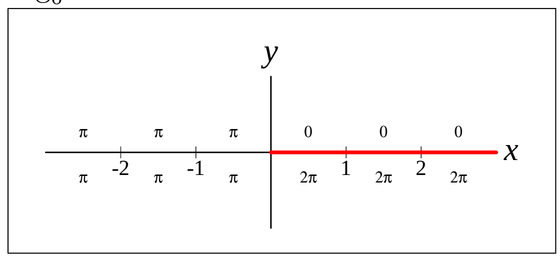
along the contour in Figure VII.9.1.

VIII.9.1

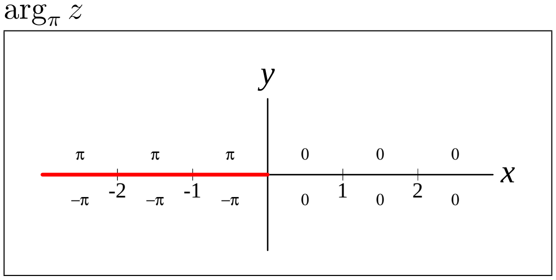
$\arg_\pi z$



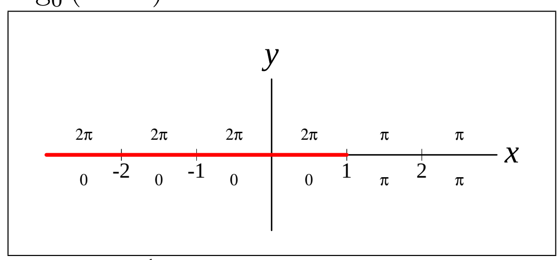
$\arg_0 z$



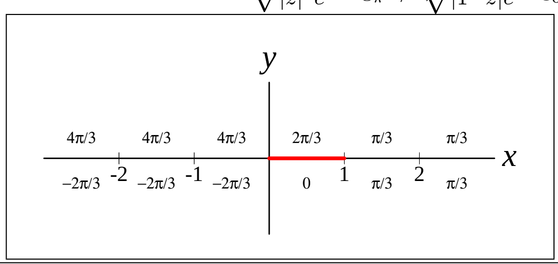
$\arg_\pi z$



$\arg_0 (1-z)$



The argument for the function $f(z) = \frac{1}{\sqrt[3]{|z|^2} e^{2i \arg_\pi z/3} \sqrt[3]{|1-z|} e^{i \arg_0 (1-z)/3}}$



Residue at a simple pole at $z_1 = \infty$, where by result from Exercise VII.8.13 and Rule 1,

$$\operatorname{Res}[f(z), \infty] = -\operatorname{Res}\left[\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right] = -\operatorname{Res}\left[\frac{1}{w^2} \frac{1}{\sqrt[3]{\frac{1}{w^2}\left(1-\frac{1}{w}\right)}}, 0\right] =$$

$$= -\operatorname{Res}\left[\frac{1}{w} \frac{1}{\sqrt[3]{w-1}}, 0\right] = -\lim_{w \to 0} \frac{1}{\sqrt[3]{|w-1|} e^{i \arg(w-1)/3}} = -e^{-\pi i/3}$$

Integrate along γ_1, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt[3]{|z|^2} e^{2i \arg_\pi z/3} \sqrt[3]{|1-z|} e^{i \arg_0 (1-z)/3}} dz =$$

$$= \int_{\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt[3]{|x|^2} e^{2i \cdot 0/3} \sqrt[3]{|1-x|} e^{i \cdot 2\pi/3}} dx = e^{-2\pi i/3} \int_{\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt[3]{x^2 - x^3}} dx \to$$

$$\to e^{-2\pi i/3} \int_0^1 \frac{1}{\sqrt[3]{x^2 - x^3}} dx = e^{-2\pi i/3} I.$$

Integrate along γ_2, and let $\varepsilon \to 0^+$. This gives

$$\left| \int_{\gamma_4} f(z)\,dz \right| \leq \frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Integrate along γ_3, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{1-\varepsilon}^{\varepsilon} \frac{1}{\sqrt[3]{|z|^2} e^{2i \arg_\pi z/3} \sqrt[3]{|1-z|} e^{i \arg_0 (1-z)/3}} dz =$$

$$= \int_{1-\varepsilon}^{\varepsilon} \frac{1}{\sqrt[3]{|x|^2} e^{2i \cdot 0/3} \sqrt[3]{|1-x|} e^{i \cdot 0/3}} dx = - \int_{\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt[3]{x^2 - x^3}} dx \to$$

$$\to - \int_0^1 \frac{1}{\sqrt[3]{x^2 - x^3}} dx = -I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\left| \int_{\gamma_4} f(z)\,dz \right| \leq \frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$e^{-2\pi i/3} I + 0 - I + 0 = 2\pi i \cdot \left(-e^{-\pi i/3}\right).$$

Solve for I, we obtain that

$$I = \frac{2\pi i e^{-\pi i/3}}{1 - e^{-2\pi i/3}} = \frac{2\pi i}{e^{\pi i/3} - e^{-\pi i/3}} = \frac{\pi}{\sin(\pi/3)} = \frac{2\pi\sqrt{3}}{3},$$

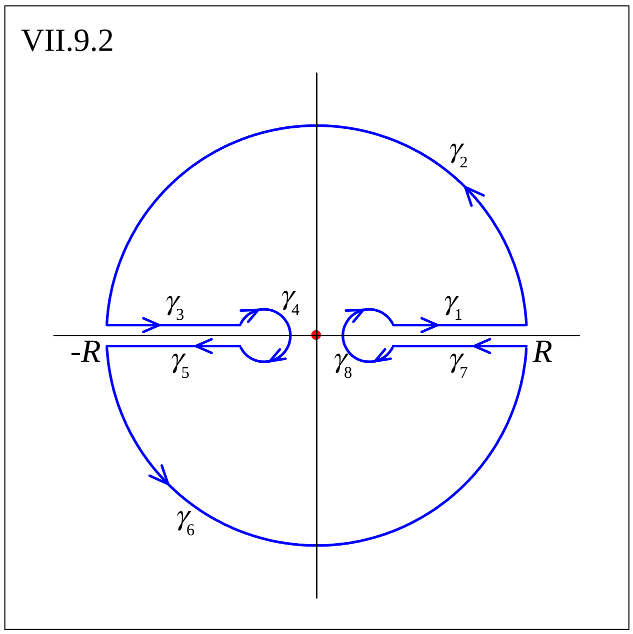
i.e.,

$$\int_0^1 \frac{1}{\sqrt[3]{x^2 - x^3}} dx = \frac{2\pi\sqrt{3}}{3}.$$

VII.9.2
Show using residue theory that

$$\int_0^\infty \frac{1}{x\sqrt{x^2-1}}dx = \frac{\pi}{2}.$$

Solution



Set

$$I = \int_0^\infty \frac{1}{x\sqrt{x^2-1}}dx$$

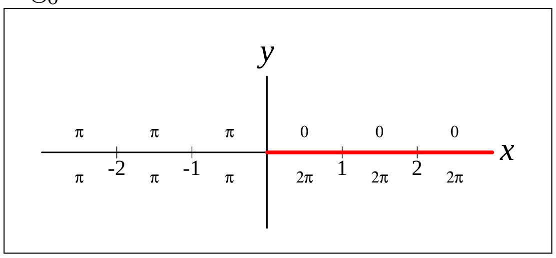
and integrate

$$f(z) = \frac{1}{z\sqrt{z^2-1}} = \frac{1}{z\sqrt{|z+1|e^{i\arg_0(z+1)/2}}\sqrt{|z-1|e^{i\arg_\pi(z-1)/2}}}$$

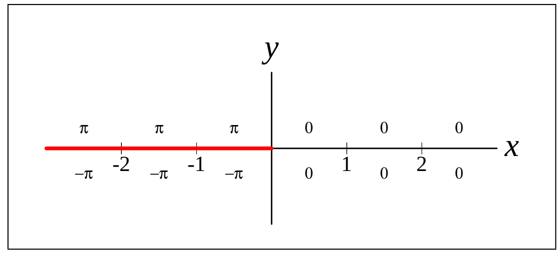
along the contour in Figure VII.9.2.

VIII.9.2

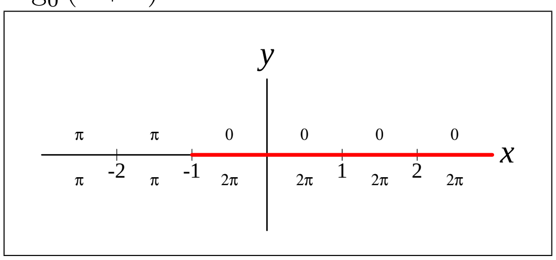
$\arg_0 z$



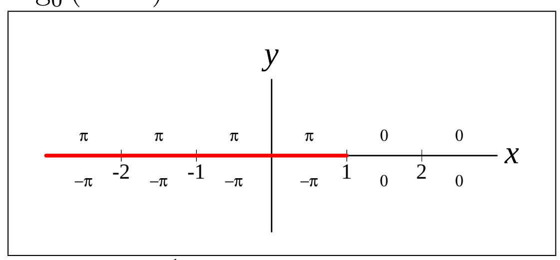
$\arg_\pi z$



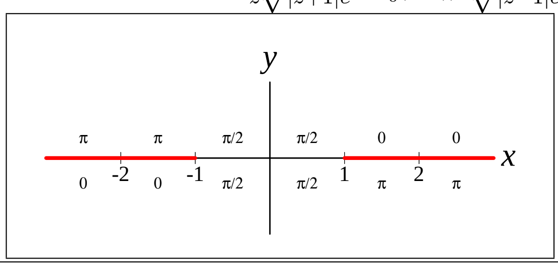
$\arg_0 (z+1)$



$\arg_0 (z-1)$



The argument for the function $f(z) = \frac{1}{z\sqrt{|z+1|}e^{i\arg_0(z+1)/2}\sqrt{|z-1|}e^{i\arg_\pi(z-1)/2}}$



Residue at a simple pole at $z_1 = i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{1}{z\sqrt{|z+1|}e^{i\arg(z+1)/2}\sqrt{|z-1|}e^{i\arg(z-1)/2}}, 0\right] =$$

$$\left.\frac{1}{\sqrt{|z+1|}e^{i\arg(z+1)/2}\sqrt{|z-1|}e^{i\arg(z-1)/2}}\right|_{z=0} = \frac{1}{e^{\pi i/2}} = -i$$

Integrate along γ_1, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{1+\varepsilon}^{R} \frac{1}{z\sqrt{|z+1|}e^{i\arg_0(z+1)/2}\sqrt{|z-1|}e^{i\arg_\pi(z-1)/2}}dz =$$

$$= \int_{1+\varepsilon}^{R} \frac{1}{x\sqrt{|x+1|}e^{i\cdot 0/2}\sqrt{|x-1|}e^{i\cdot 0/2}}dx =$$

$$= \int_{1+\varepsilon}^{R} \frac{1}{x\sqrt{x^2-1}}dx \to$$

$$\to \int_{1}^{\infty} \frac{1}{x\sqrt{x^2-1}}dx = I.$$

Integrate along γ_2, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_2} f(z)\,dz\right| \leq \frac{1}{R\sqrt{R^2-1}} \cdot \pi R \sim \frac{\pi}{R} \to 0.$$

Integrate along γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{-R}^{-1-\varepsilon} \frac{1}{z\sqrt{|z+1|}e^{i\arg_0(z+1)/2}\sqrt{|z-1|}e^{i\arg_\pi(z-1)/2}}dz =$$

$$= \int_{-R}^{-1-\varepsilon} \frac{1}{x\sqrt{|x+1|}e^{i\cdot\pi/2}\sqrt{|x-1|}e^{i\cdot\pi/2}}dx =$$

$$\int_{-1-\varepsilon}^{-R} \frac{1}{x\sqrt{x^2-1}}dx = \begin{bmatrix} x=-t \\ dx=-dt \end{bmatrix} = \int_{1}^{R} \frac{1}{t\sqrt{t^2-1}}dt = I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \leq \frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Integrate along γ_5, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_5} f(z)\,dz =$$

$$= \int_{-1-\varepsilon}^{-R} \frac{1}{z\sqrt{|z+1|}e^{i\arg_0(z+1)/2}\sqrt{|z-1|}e^{i\arg_\pi(z-1)/2}}dz =$$

$$= \int_{-1-\varepsilon}^{-R} \frac{1}{x\sqrt{|x+1|}e^{i\cdot\pi/2}\sqrt{|x-1|}e^{i\cdot(-\pi)/2}}dx =$$

$$\int_{-1-\varepsilon}^{-R} \frac{1}{x\sqrt{x^2-1}}dx = \begin{bmatrix} x=-t \\ dx=-dt \end{bmatrix} = \int_1^R \frac{1}{t\sqrt{t^2-1}}dt = I.$$

Integrate along γ_6, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_6} f(z)\,dz\right| \le \frac{1}{R\sqrt{R^2-1}} \cdot \pi R \sim \frac{\pi}{R} \to 0.$$

Integrate along γ_7, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_7} f(z)\,dz =$$

$$= \int_R^{1+\varepsilon} \frac{1}{z\sqrt{|z+1|}e^{i\arg_0(z+1)/2}\sqrt{|z-1|}e^{i\arg_\pi(z-1)/2}}dz =$$

$$= \int_R^{1+\varepsilon} \frac{1}{x\sqrt{|x+1|}e^{i\cdot 2\pi/2}\sqrt{|x-1|}e^{i\cdot 0/2}}dx =$$

$$= \int_{1+\varepsilon}^{R} \frac{1}{x\sqrt{x^2-1}}dx \to$$

$$\to \int_1^\infty \frac{1}{x\sqrt{x^2-1}}dx = I.$$

Integrate along γ_8, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$I + 0 + I + 0 + I + 0 + I + 0 = 2\pi i \cdot (-i)\,,$$

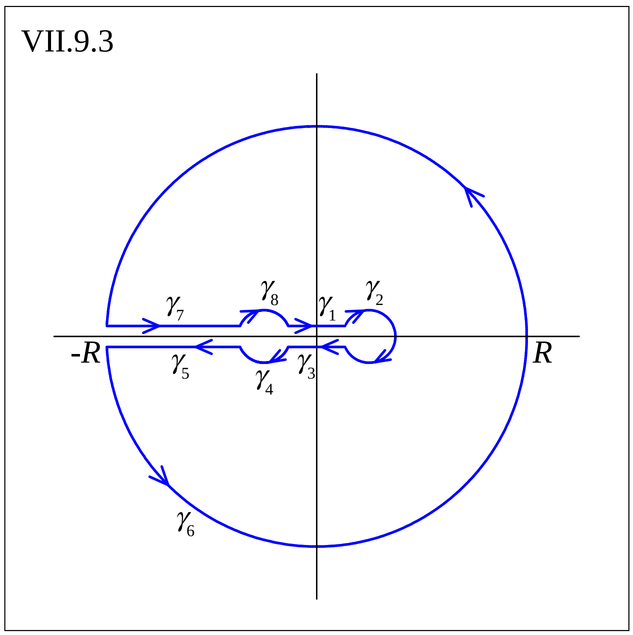
i.e.,

$$\int_0^\infty \frac{1}{x\sqrt{x^2-1}}dx = \frac{\pi}{2}.$$

VII.9.3
Show using residue theory that

$$\int_{-1}^{1} \frac{1}{(x^2+1)\sqrt{1-x^2}}dx = \frac{\pi}{\sqrt{2}}$$

Solution



Set

$$I = \int_{-1}^{1} \frac{1}{(x^2+1)\sqrt{1-x^2}}dx$$

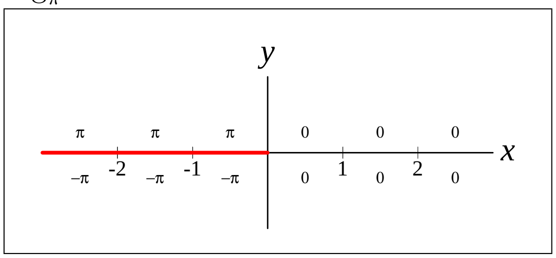
and integrate

$$f(z) = \frac{1}{(z^2+1)\sqrt{1-x^2}} = \frac{1}{(z-i)(z+i)\sqrt{|1+z|}e^{i\arg_\pi(1+z)/2}\sqrt{|1-z|}e^{i\arg_0(1-z)/2}}$$

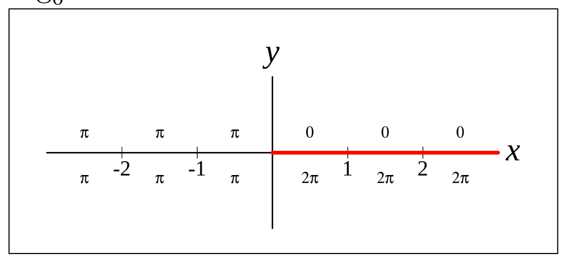
along the contour in Figure VII.9.3.

VIII.9.3

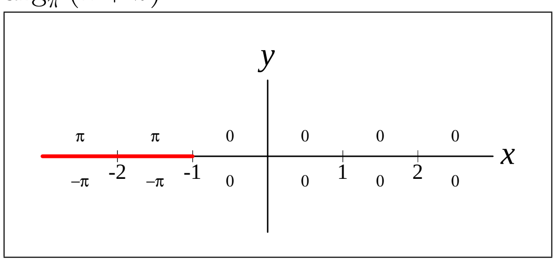
$\arg_\pi z$



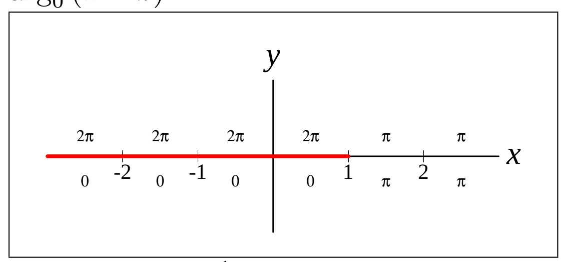
$\arg_0 z$



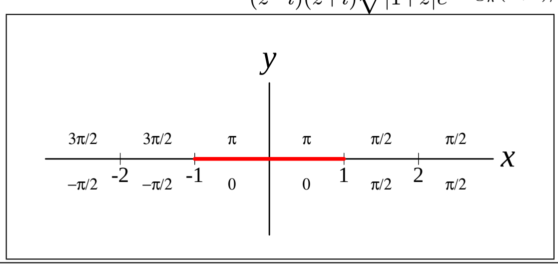
$\arg_\pi (1+z)$



$\arg_0 (1-z)$



The argument for the function $f(z) = \frac{1}{(z-i)(z+i)\sqrt{|1+z|}e^{i\arg_\pi(1+z)/2}\sqrt{|1-z|}e^{i\arg_0(1-z)/2}}$



Residue at a simple pole at $z_1 = i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{1}{(z-i)(z+i)\sqrt{|1+z|}e^{i\arg(1+z)/2}\sqrt{|1-z|}e^{i\arg(1-z)/2}}, i\right] =$$

$$\left.\frac{1}{(z+i)\sqrt{|1+z|}e^{i\arg(1+z)/2}\sqrt{|1-z|}e^{i\arg(1-z)/2}}\right|_{z=i} =$$

$$\frac{1}{(i+i)\sqrt{|1+z|}e^{i\arg(1+i)/2}\sqrt{|1-z|}e^{i\arg(1-i)/2}} = \frac{1}{2i\sqrt{|1+z|}e^{(i\pi/4)/2}\sqrt{|1-z|}e^{(i7\pi/4)/2}}$$

$$= \frac{1}{2i\sqrt{\sqrt{2}}\sqrt{\sqrt{2}}e^{\pi i}} = \frac{1}{2i\sqrt{2}(-1)} = -\frac{1}{2\sqrt{2}i} = \frac{1}{2\sqrt{2}}i$$

Residue at a simple pole at $z_2 = -i$, where by Rule 3,

$$\operatorname{Res}\left[\frac{1}{(z-i)\sqrt{|1+z|}e^{i\arg(1+z)/2}\sqrt{|1-z|}e^{i\arg(1-z)/2}}, -i\right] =$$

$$= \left.\frac{1}{(z-i)\sqrt{|1+z|}e^{i\arg(1+z)/2}\sqrt{|1-z|}e^{i\arg(1-z)/2}}\right|_{z=-i} =$$

$$\frac{1}{(-i-i)\sqrt{|1+z|}e^{i\arg(1-i)/2}\sqrt{|1-z|}e^{i\arg(1+i)/2}} = \frac{1}{-2i\sqrt{|1+z|}e^{(-i\pi/4)/2}\sqrt{|1-z|}e^{(i\pi/4)/2}} =$$

$$= \frac{1}{-2i\sqrt{\sqrt{2}}\sqrt{\sqrt{2}}e^{0i}} = \frac{1}{-2i\sqrt{2}} = -\frac{1}{2\sqrt{2}i} = \frac{1}{2\sqrt{2}}i$$

Integrate along γ_1, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{-1+\varepsilon}^{1-\varepsilon} \frac{1}{(z^2+1)\sqrt{|1+z|}e^{i\arg_\pi(1+z)/2}\sqrt{|1-z|}e^{i\arg_0(1-z)/2}}dz =$$

$$= \int_{-1+\varepsilon}^{1-\varepsilon} \frac{1}{(z^2+1)\sqrt{|1+x|}e^{i0/2}\sqrt{|1-x|}e^{i2\pi/2}}dx = -\int_{-1+\varepsilon}^{1-\varepsilon} \frac{1}{(x^2+1)\sqrt{1-x^2}}dx \to$$

$$-\int_{-1}^{1} \frac{1}{(x^2+1)\sqrt{1-x^2}}dx = -I.$$

Integrate along γ_2, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Integrate along γ_3, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz =$$

$$= \int_{1-\varepsilon}^{-1+\varepsilon} \frac{1}{(z^2+1)\sqrt{|1+z|}e^{i\arg_\pi(1+z)/2}\sqrt{|1-z|}e^{i\arg_0(1-z)/2}}dz =$$

$$= \int_{1-\varepsilon}^{-1+\varepsilon} \frac{1}{(z^2+1)\sqrt{|1+x|}e^{i\cdot 0/2}\sqrt{|1-x|}e^{i\cdot 0/2}}dx = -\int_{-1+\varepsilon}^{1-\varepsilon} \frac{1}{(x^2+1)\sqrt{1-x^2}}dx \to$$

$$\to -\int_{-1}^{1} \frac{1}{(x^2+1)\sqrt{1-x^2}}dx = -I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_4} f(z)\,dz \to 0.$$

Integrate along γ_5, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_5} f(z)\,dz =$$

$$= \int_{-1-\varepsilon}^{-R} \frac{1}{(z^2+1)\sqrt{|1+z|}e^{i\arg(1+z)/2}\sqrt{|1-z|}e^{i\arg(1-z)/2}}dz =$$

$$= \int_{-1-\varepsilon}^{-R} \frac{1}{(z^2+1)\sqrt{|1+x|}e^{i(-\pi)/2}\sqrt{|1-x|}e^{i0/2}}dx = -\int_{-1-\varepsilon}^{-R} \frac{1}{(x^2+1)\sqrt{1-x^2}\,(-i)}dx \to$$

$$\to -\int_{-1}^{-\infty} \frac{1}{(x^2+1)\sqrt{1-x^2}\,(-i)}dx = -J.$$

Integrate along γ_6, and let $R \to \infty$. This gives

$$\left|\int_{\gamma_6} f(z)\,dz\right| \le \frac{1}{(R^2-1)\sqrt{R^2-1}} \cdot 2\pi R \sim \frac{2\pi}{R^3} \to 0.$$

Integrate along γ_7, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_7} f(z)\,dz =$$

$$= \int_{-R}^{-1-\varepsilon} \frac{1}{(z^2+1)\sqrt{|1+z|}e^{i\arg(1+z)/2}\sqrt{|1-z|}e^{i\arg(1-z)/2}}dz =$$

$$= \int_{-R}^{-1-\varepsilon} \frac{1}{(z^2+1)\sqrt{|1+x|}e^{i\pi/2}\sqrt{|1-x|}e^{i2\pi/2}}dx = -\int_{-1-\varepsilon}^{-R} \frac{1}{(x^2+1)\sqrt{1-x^2}(-i)}dx \to$$

$$\to \int_{-1}^{-\infty} \frac{1}{(x^2+1)\sqrt{1-x^2}(-i)}dx = -J.$$

Integrate along γ_8, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \leq \frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$-I + 0 - I - J + 0 + J + 0 = 2\pi i \cdot \left(\frac{1}{2\sqrt{2}}i + \frac{1}{2\sqrt{2}}i\right)$$

i.e.,

$$\int_{-1}^{1} \frac{1}{(x^2+1)\sqrt{1-x^2}}dx = \frac{\pi}{\sqrt{2}}.$$

VII.9.4
Show using residue theory that

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{(1-x)(1+x)^2}} dx = \frac{2\pi}{\sqrt{3}}.$$

Solution
Set

$$I = \int_{-1}^{1} \frac{1}{\sqrt[3]{(1-x)(1+x)^2}} dx$$

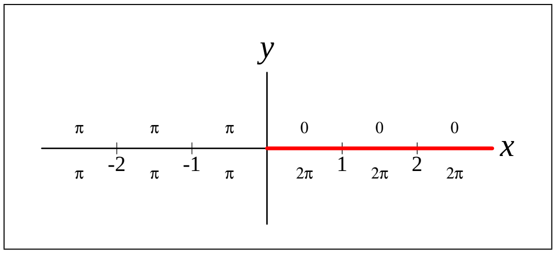
and integrate

$$f(z) = \frac{1}{\sqrt[3]{(1-z)(1+z)^2}} = \frac{1}{\sqrt[3]{|1-z|} e^{i \arg_0 (1-z)/3} \sqrt{|1+z|^2} e^{2i \arg_\pi (1+z)/3}}$$

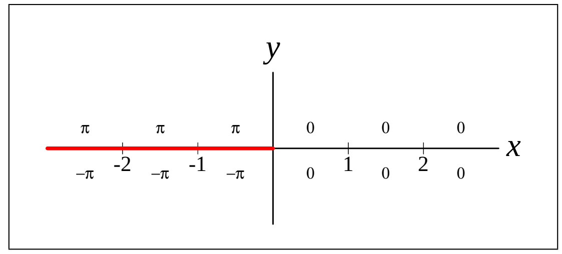
along the contour in Figure VII.9.4.

VIII.9.4

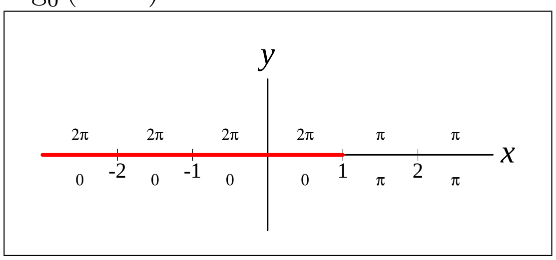
$\arg_0 z$



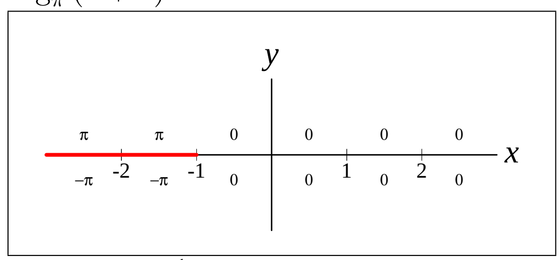
$\arg_\pi z$



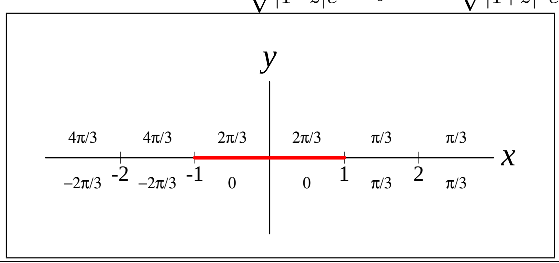
$\arg_0 (1-z)$



$\arg_\pi (1+z)$



The argument for the function $f(z) = \frac{1}{\sqrt[3]{|1-z|}e^{i\arg_0(1-z)/3}\sqrt{|1+z|^2}e^{2i\arg_\pi(1+z)/3}}$



Residue at a simple pole at $z_1 = \infty$, where by result from Exercise VII.8.13 and Rule 1,

$$\operatorname{Res}\left[f(z),\infty\right]=-\operatorname{Res}\left[\frac{1}{w^2}f\left(\frac{1}{w}\right),0\right]=-\operatorname{Res}\left[\frac{1}{w^2}\frac{1}{\sqrt[3]{\left(1-\frac{1}{w}\right)\left(1+\frac{1}{w}\right)^2}},0\right]=$$

$$=-\operatorname{Res}\left[\frac{1}{w}\frac{1}{\sqrt[3]{(w-1)(w+1)^2}},0\right]=$$

$$-\lim_{w\to 0}\left[\frac{1}{\sqrt[3]{|w-1|}e^{i\arg(w-1)/3}\sqrt{|w+1|^2}e^{2i\arg(w+1)/3}}\right]=-e^{-\pi i/3}$$

Integrate along γ_1, and let $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_1} f(z)\,dz=$$

$$=\int_{-1+\varepsilon}^{1-\varepsilon}\frac{1}{\sqrt[3]{|1-z|}e^{i\arg_0(1-z)/3}\sqrt{|1+z|^2}e^{2i\arg_\pi(1+z)/3}}dz=$$

$$=\int_{-1+\varepsilon}^{1-\varepsilon}\frac{1}{\sqrt[3]{|1-x|}e^{i2\pi/3}\sqrt{|1+x|^2}e^{2i\cdot 0/3}}dx=e^{-2\pi i/3}\int_{-1+\varepsilon}^{1-\varepsilon}\frac{1}{\sqrt[3]{(1-x)(1+x)^2}}dx\to$$

$$\to e^{-2\pi i/3}\int_{-1}^{1}\frac{1}{\sqrt[3]{(1-x)(1+x)^2}}dx=e^{-2\pi i/3}I.$$

Integrate along γ_2, and let $\varepsilon\to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right|\le\frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}}\cdot 2\pi\varepsilon\sim 2\pi\varepsilon^{9/2}\to 0.$$

Integrate along γ_3, and let $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{1-\varepsilon}^{-1+\varepsilon} \frac{1}{\sqrt[3]{|1-z|}e^{i\arg_0(1-z)/3}\sqrt{|1+z|^2}e^{2i\arg_\pi(1+z)/3}}dz =$$

$$= \int_{1-\varepsilon}^{-1+\varepsilon} \frac{1}{\sqrt[3]{|1-x|}e^{i\cdot 0/3}\sqrt{|1+x|^2}e^{2i\cdot 0/3}}dx = -\int_{-1+\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt[3]{(1-x)(1+x)^2}}dx \to$$

$$\to \int_{-1}^{1} \frac{1}{\sqrt[3]{(1-x)(1+x)^2}}dx = -I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\left| \int_{\gamma_4} f(z)\,dz \right| \leq \frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$e^{-2\pi i/3} I + 0 - I + 0 = 2\pi i \cdot \left(-e^{-\pi i/3}\right).$$

Solve for I, we obtain that

$$I = \frac{-2\pi i e^{-\pi i/3}}{e^{-2\pi i/3} - 1} = \frac{2\pi i}{e^{\pi i/3} - e^{-\pi i/3}} = \frac{\pi}{\sin(\pi/3)} = \frac{2\pi}{\sqrt{3}},$$

i.e.,

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{(1-x)(1+x)^2}}dx = \frac{2\pi}{\sqrt{3}}.$$

VII.9.5
Show using residue theory that

$$\int_0^1 \frac{1}{(x+1)\sqrt[4]{x(1-x)^3}}dx = \frac{\pi}{\sqrt[4]{2}}.$$

Solution
Set

$$I = \int_0^1 \frac{1}{(x+1)\sqrt[4]{x(1-x)^3}}dx$$

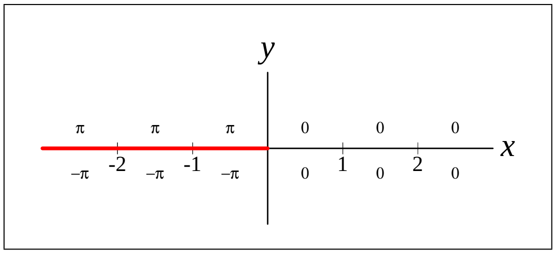
and integrate

$$f(z) = \frac{1}{(z+1)\sqrt[4]{z(1-z)^3}} = \frac{1}{(z+1)\sqrt[4]{|z|}e^{i\arg_\pi z/4}\sqrt[4]{|1-z|^3}e^{3i\arg_0(1-z)/4}}$$

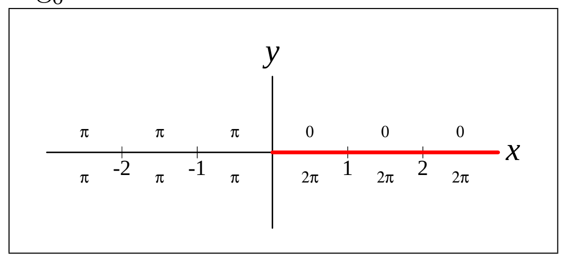
along the contour in Figure VII.9.5.

VIII.9.5

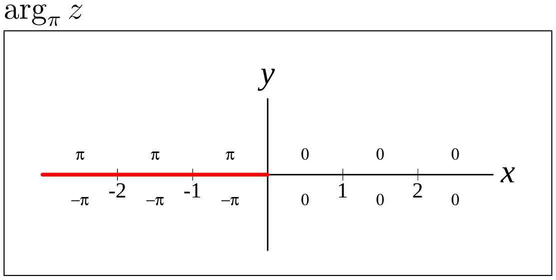
$\arg_\pi z$



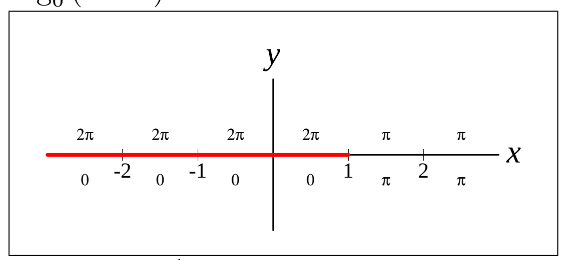
$\arg_0 z$



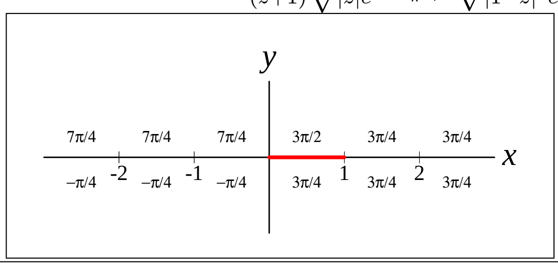
$\arg_\pi z$



$\arg_0 (1-z)$



The argument for the function $f(z) = \frac{1}{(z+1)\sqrt[4]{|z|}e^{i\arg_\pi z/4}\sqrt[4]{|1-z|^3}e^{3i\arg_0(1-z)/4}}$



Residue at a simple pole at $z_1 = -1$, where by Rule 3,

$$\operatorname{Res}\left[\frac{1}{(z+1)\sqrt[4]{|z|}e^{i\arg(z)/4}\sqrt[4]{|1-z|^3}e^{3i\arg(1-z)/4}},-1\right]=$$

$$\left.\frac{1}{\sqrt[4]{|z|}e^{i\arg(z)/4}\sqrt[4]{|1-z|^3}e^{3i\arg(1-z)/4}}\right|_{z=-1}=$$

$$=\frac{1}{\sqrt[4]{|z|}e^{i\arg(z)/4}\sqrt[4]{|1-z|^3}e^{3i\arg(1-z)/4}}=$$

$$=\frac{1}{\sqrt[4]{8}e^{-\pi i/4}}=\frac{1}{\sqrt[4]{8}}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=\frac{1}{2\sqrt[4]{2}}+i\frac{1}{2\sqrt[4]{2}}$$

Residue at a simple pole at $z_2=\infty$, where by result from Exercise VII.8.13 and Rule 1,

$$\operatorname{Res}[f(z),\infty]=-\operatorname{Res}\left[\frac{1}{w^2}f\left(\frac{1}{w}\right),0\right]=-\operatorname{Res}\left[\frac{1}{w^2}\frac{1}{\left(\frac{1}{w}+1\right)\sqrt[4]{\frac{1}{w}\left(1-\frac{1}{w}\right)^3}},0\right]=$$

$$=-\lim_{w\to 0}\left[\frac{1}{(1+w)\sqrt[4]{(w-1)^3}},0\right]=0$$

Residue at a simple pole at $z_2=-i$, where by Rule 3,
Integrate along γ_1, and let $R\to\infty$ and $\varepsilon\to 0^+$. This gives

$$\int_{\gamma_1}f(z)\,dz=$$

$$=\int_{\varepsilon}^{1-\varepsilon}\frac{1}{(z+1)\sqrt[4]{|z|}e^{i\arg_\pi z/4}\sqrt[4]{|1-z|^3}e^{3i\arg_0(1-z)/4}}dz=$$

$$=\int_{\varepsilon}^{1-\varepsilon}\frac{1}{(x+1)\sqrt[4]{|x|}e^{i\cdot 0/4}\sqrt[4]{|1-x|^3}e^{3i\cdot 2\pi/4}}dx=i\int_{\varepsilon}^{1-\varepsilon}\frac{1}{(x+1)\sqrt[4]{x(1-x)^3}}dx\to$$

$$\to i\int_0^1\frac{1}{(x+1)\sqrt[4]{x(1-x)^3}}dx=iI.$$

Integrate along γ_2, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Integrate along γ_3, and let $R \to \infty$ and $\varepsilon \to 0^+$. This gives

$$\int_{\gamma_3} f(z)\,dz =$$

$$= \int_{1-\varepsilon}^{\varepsilon} \frac{1}{(z+1)\sqrt[4]{|z|}e^{i\arg_\pi z/4}\sqrt[4]{|1-z|^3}e^{3i\arg_0(1-z)/4}}dz =$$

$$= \int_{1-\varepsilon}^{\varepsilon} \frac{1}{(x+1)\sqrt[4]{|x|}e^{i\cdot 0/4}\sqrt[4]{|1-x|^3}e^{3i\cdot 0/4}}dx = -\int_{\varepsilon}^{1-\varepsilon} \frac{1}{(x+1)\sqrt[4]{x(1-x)^3}}dx \to$$

$$\to -\int_0^1 \frac{1}{(x+1)\sqrt[4]{x(1-x)^3}}dx = -I.$$

Integrate along γ_4, and let $\varepsilon \to 0^+$. This gives

$$\left|\int_{\gamma_4} f(z)\,dz\right| \le \frac{\varepsilon^4}{\sqrt{\varepsilon(1-\varepsilon)}} \cdot 2\pi\varepsilon \sim 2\pi\varepsilon^{9/2} \to 0.$$

Using the *Residue Theorem* and letting $R \to \infty$ and $\varepsilon \to 0^+$, we obtain that

$$iI + 0 - I + 0 = 2\pi i \cdot \left(\frac{1}{2\sqrt[4]{2}} + i\frac{1}{2\sqrt[4]{2}} + 0\right),$$

i.e.,

$$\int_0^1 \frac{1}{(x+1)\sqrt[4]{x(1-x)^3}}dx = \frac{\pi}{\sqrt[4]{2}}.$$

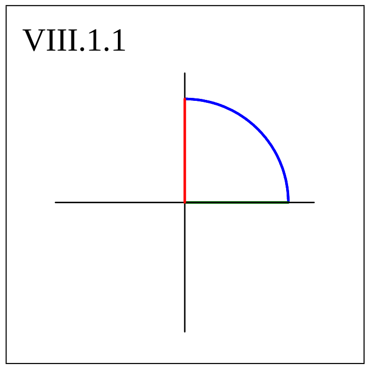
VIII	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1																			
2																			
3																			
4																			
5																			
6																			
7																			
8																			

VIII.1.1

1	2	3	P	L	K
				LLL	

Show that $z^4 + 2z^2 - z + 1$ has exactly one root in each quadrant.

Solution



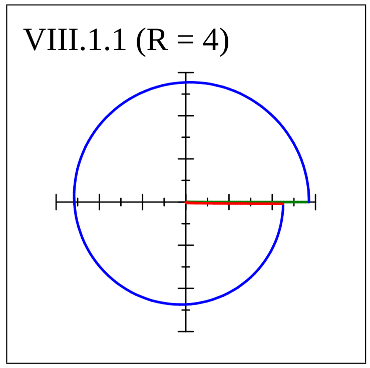
Set $p(z) = z^4 + 2z^2 - z + 1$, and compute $\triangle \arg p(z)$ along the three segments in the sector path in figure VIII.1.1. First of all $p(z)$ have no real zeros.

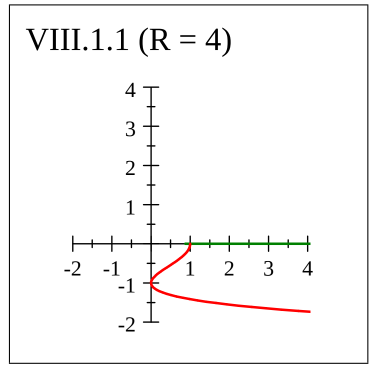
The positive real axis γ_1 from 0 to R we parametizize as $\gamma_x : z = t,\ 0 \le t \le R$, and our function $p(z)$ becomes $p(t) = t^4 + 2t^2 - t + 1 = (t^2+1)^2 - t > 0$. Since $p(x) > 0$ for $t \ge 0$, their is no change of argument on this line segment on the real axis, thus $\triangle \arg p(z) = 0$.

The arc γ_2 from R to iR we parametizize as $\gamma_2 : z = Re^{it},\ 0 \le t \le \pi/2$, and our function $p(z)$ becomes $p(t) = R^4e^{4it} + 2R^2e^{2it} - Re^{it} + 1$. Since the variation in argument is determined by the dominating term R^4e^{4it} for R large the change of argument on this arc is 4 times the variation in argument for the arc, and thus $\triangle \arg p(z) \approx 4 \cdot \frac{\pi}{2} = 2\pi$.

The positive imaginary axis γ_3 from iR to 0 we parametizize as $\gamma_3 : z = it,\ 0 \le t \le R$, and our function $p(z)$ becomes $f(t) = (it)^4 + 2(it)^2 - it + 1$ $= (t^2-1)^2 - it$. We find that the real part have a zero at $t = 1$, and the imaginary part have a zero for $t = 0$. We make the following table and do the sketches,

t	$\operatorname{Re} z$	$\operatorname{Im} z$	$\arg z$
∞	$+$	$-$	0
1	0	$-$	$-\frac{\pi}{2}$
0	1	0	0





Thus when we move from iR to 0, the argument for $p(z)$ remain in the 4-th quadrant except for touching the imaginary axis at $-i$ and terminating at 1, and their is no change of argument on this line segment on the imaginary axis, thus $\triangle \arg p(z) \approx 0$.

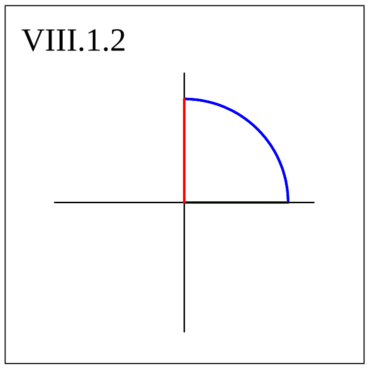
Now we have that the total change of argument for $p(z)$ is $\approx 2\pi$, so we have exactly one zero in the first quadrat. Because that the roots come in complex conjugate pairs it is plain that $p(z)$ have a root in the fourth quadrant too. Becauce that $p(x) = (x^2+1)^2 - x > 0$ it is clear that $p(z)$ have no real roots and $p(z)$ must have on has one zero in each quadrant.

VIII.1.2

1	2	3	P	L	K

Find the number of zeros of the polynomial $p(z) = z^4+z^3+4z^2+3z+2$ **in each quadrant.**

Solution



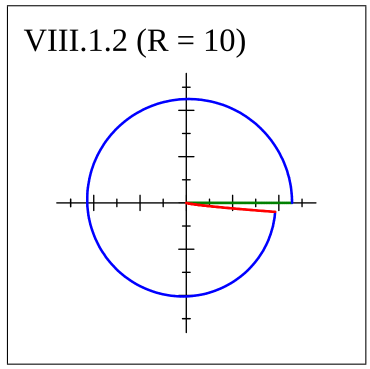
Set $p(z) = z^4 + z^3 + 4z^2 + 3z + 2$, and compute $\triangle \arg p(z)$ along the three segments in the sector path in figure VIII.1.2. First of all $p(z)$ have no real zeros.

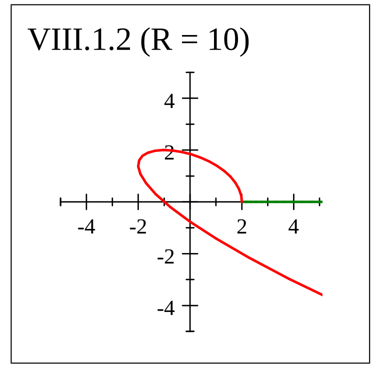
The positive real axis γ_1 from 0 to R we parametizize as $\gamma_x : z = t,\ 0 \leq t \leq R$, and our function $p(z)$ becomes $p(t) = t^4+t^3+4t^2+3t+2 > 0$. Since $p(x) > 0$ for $t \geq 0$, their is no change of argument on this line segment on the real axis, thus $\triangle \arg p(z) = 0$.

The arc γ_2 from R to iR we parametizize as $\gamma_2 : z = Re^{it}$, $0 \leq t \leq \pi/2$, and our function $p(z)$ becomes $p(t) = R^4e^{4it} + R^3e^{3it} + 4R^2e^{2it} + 3Re^{it} + 2$. Since the variation in argument is determined by the dominating term R^4e^{4it} for R large the change of argument on this arc is 4 times the variation in argument for the arc, and thus $\triangle \arg p(z) \approx 4 \cdot \frac{\pi}{2} = 2\pi$.

The positive imaginary axis γ_3 from iR to 0 we parametizize as $\gamma_3 : z = it,\ 0 \leq t \leq R$, and our function $p(z)$ becomes $f(t) = (it)^4 + (it)^3 + 4(it)^2 + 3it + 2 = t^4 - 4t^2 + 2 + i(-t^3 + 3t)$. We find that the real part have a zero at $t=\sqrt{2-\sqrt{2}}$ and $t=\sqrt{2+\sqrt{2}}$, and the imaginary part have a zero for $t=0$ and $t=\sqrt{3}$. We make the following table and do the sketches,

t	$\operatorname{Re} z$	$\operatorname{Im} z$	$\arg z$
∞	$+$	$-$	0
$\sqrt{2+\sqrt{2}}$	0	$-$	$-\frac{\pi}{2}$
$\sqrt{3}$	$-$	0	$-\pi$
$\sqrt{2-\sqrt{2}}$	0	$+$	$-\frac{3\pi}{2}$
0	$+$	0	-2π





Thus when we move from iR to 0, the argument for $p(z)$ on this line segment on the imaginary axis changes so that, $\triangle \arg p(z) \approx -2\pi$.

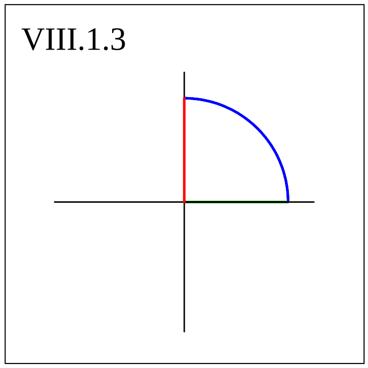
Now we have that the total change of argument for $p(z)$ is ≈ 0, so we have no zero in the 1st quadrant.
Since $p(z)$ has real coefficients, the roots come in conjugate pairs (since there are nor real roots), therefore there are no roots in the 4th quadras as well. By symmetry, there are 2 roots each in the 2nd and 3rd quadrants.

VIII.1.3

1	2	3	P	L	K

Find the number of zeros of the polynomial $p(z) = z^6 + 4z^4 + z^3 + 2z^2 + z + 5$ **in the first quadrant** $\{\operatorname{Re} z > 0, \operatorname{Im} z > 0\}$**.**

Solution



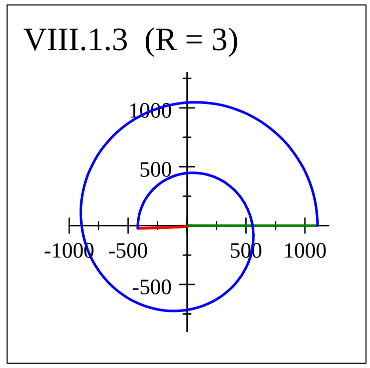
Set $p(z) = z^6 + 4z^4 + z^3 + 2z^2 + z + 5$, and compute $\triangle \arg p(z)$ along the three segments in the sector path in figure VIII.1.3. First of all $p(z)$ have no real zeros.

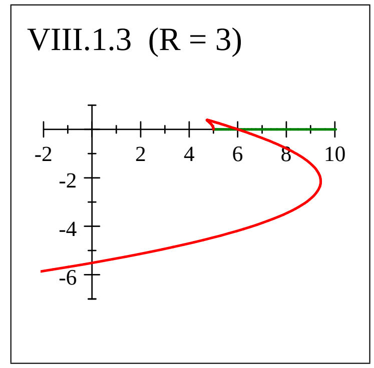
The positive real axis γ_1 from 0 to R we parametizize as $\gamma_x : z = t,\ 0 \leq t \leq R$, and our function $p(z)$ becomes $p(t) = t^6 + 4t^4 + t^3 + 2t^2 + t + 5 > 0$. Since $p(t) > 0$ for $t \geq 0$, their is no change of argument on this line segment on the real axis, thus $\triangle \arg p(z) = 0$.

The arc γ_2 from R to iR we parametizize as $\gamma_2 : z = Re^{it}$, $0 \leq t \leq \pi/2$, and our function $p(z)$ becomes $p(t) = R^6 e^{6it} + 4R^4 e^{4it} + 3R^3 e^{3it} + 2R^2 e^{2it} + Re^{it} + 5$. Since the variation in argument is determined by the dominating term $R^6 e^{6it}$ for R large the change of argument on this arc is 6 times the variation in argument for the arc, and thus $\triangle \arg p(z) \approx 6 \cdot \frac{\pi}{2} = 3\pi$.

The positive imaginary axis γ_3 from iR to 0 we parametizize as $\gamma_3 : z = it,\ 0 \leq t \leq R$, and our function $p(z)$ becomes $f(t) = (it)^6 + 4(it)^4 + (it)^3 + 2(it)^2 + it + 5 = -t^6 + 4t^4 - 2t^2 + 5 + i(-t^3 + t)$. We find that the real part have a zero at $t \approx 1,95$, and the imaginary part have a zero for $t = 0$ and $t = 1$. We make the following table and do the sketches,

t	$\operatorname{Re} z$	$\operatorname{Im} z$	$\arg z$
∞	$-$	$-$	$-\pi$
$1,95$	0	$-$	$-\frac{\pi}{2}$
1	$+$	0	0
0	$+$	0	0





Thus when we move from iR to 0, the argument for $p(z)$ on this line segment on the imaginary axis changes so that, $\triangle \arg p(z) \approx \pi$.

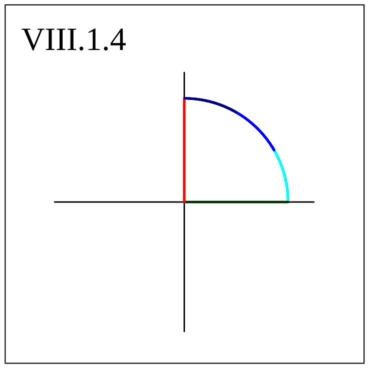
Now we have that the total change of argument for $p(z)$ is $\approx 4\pi$, so we have that $p(z)$ has exactly two zeros in the first quadrat. (None on real or imaginary axis)

VIII.1.4

1	2	3	P	L	K
				LLL	

Find the number of zeros of the polynomial $p(z) = z^9 + 2z^5 - 2z^4 + z + 3$ **in the right half-plane.**

Solution

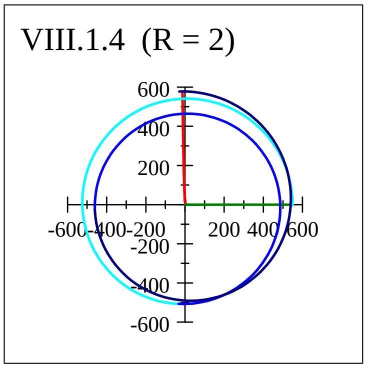


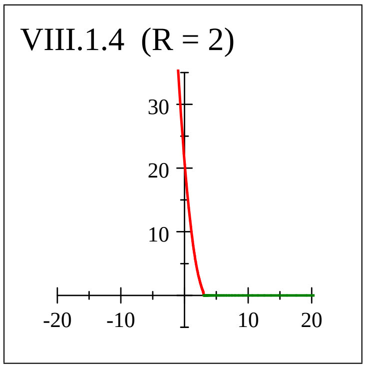
Set $p(z) = z^9 + 2z^5 - 2z^4 + z + 3$, and compute $\triangle \arg p(z)$ along the three segments in the sector path in figure VIII.1.4. First of all $p(z)$ have no real zeros.

The positive real axis γ_1 from 0 to R we parametizize as $\gamma_x : z = t,\ 0 \le t \le R$, and our function $p(z)$ becomes $p(t) = t^9 + 2t^5 - 2t^4 + t + 3$. Since $p(x) > 0$ for $t \ge 0$, their is no change of argument on this line segment on the real axis, thus $\triangle \arg p(z) = 0$.

The arc γ_2 from R to iR we parametizize as $\gamma_2 : z = Re^{it}$, $0 \le t \le \pi/2$, and our function $p(z)$ becomes $p(t) = R^9 e^{9it} + 2R^5 e^{5it} - 2R^4 e^{4it} + Re^{it} + 3$. Since the variation in argument is determined by the dominating term $R^9 e^{9it}$ for R large the change of argument on this arc is 9 times the variation in argument for the arc, and thus $\triangle \arg p(z) \approx 9 \cdot \frac{\pi}{2} = \frac{9\pi}{2}$.
The positive imaginary axis γ_3 from iR to 0 we parametizize as $\gamma_3 : z = it,\ 0 \le t \le R$, and our function $p(z)$ becomes $f(t) = (it)^9 + 2(it)^5 - 2(it)^4 + it + 3 = -2t^4 + 3 + iy(y^4 + 1)^2$. We find that the real part have a zero at $t = \sqrt[4]{\frac{3}{2}}$, and the imaginary part have only a zero for $t = 0$. We make the following table and do the sketches,

t	$\operatorname{Re} z$	$\operatorname{Im} z$	$\arg z$
∞	$-$	$+$	$\frac{\pi}{2}$
$\sqrt[4]{\frac{3}{2}}$	0	$+$	$\frac{\pi}{2}$
0	$+$	0	0





Thus when we move from iR to 0, the argument is changed from $\pi/2$ to 0, so the charge of argument on this on the imaginary axis is π, thus $\triangle \arg p(z) = -\pi/2$.

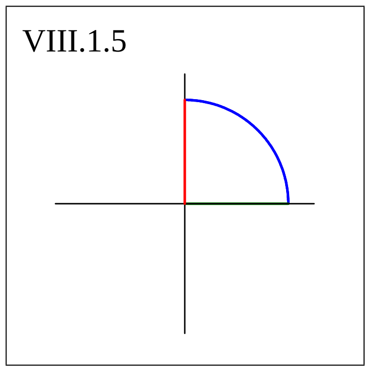
Now we have that the total change of argument for $p(z)$ is $\approx 4\pi$, so we have exactly two zeros in the first quadrat. Because that the roots come in complex conjugate pairs it is plain that $p(z)$ has four zeros in the right half-plane.

VIII.1.5

1	2	3	P	L	K

For a fixed real number α, find the number of zeros $z^4+z^3+4z^2+\alpha z+3$ satisfying $\operatorname{Re} z < 0$. (Your answer depends on α.)

Solution



Set $p(z) = z^4 + z^3 + 4z^2 + \alpha z + 3$, and compute $\triangle \arg p(z)$ along the three segments in the contuour in figure VIII.1.5.

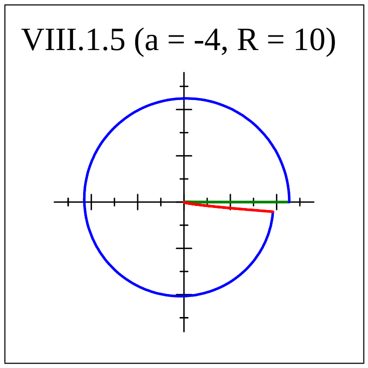
The positive real axis γ_1 from 0 to R we parametizize as $\gamma_x : z = t,\ 0 \leq t \leq R$, and our function $p(z)$ becomes $p(t) = t^4 + t^3 + 4t^2 + at + 3 = (t^2+1)^2 - t > 0$. Since $p(x) > 0$ for $t \geq 0$, their is no change of argument on this line segment on the real axis, thus $\triangle \arg p(z) = 0$.

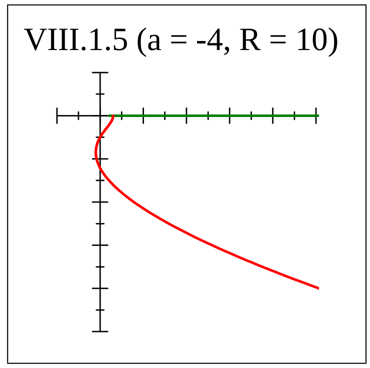
The arc γ_2 from R to iR we parametizize as $\gamma_2 : z = Re^{it}$, $0 \leq t \leq \pi/2$, and our function $p(z)$ becomes $p(t) = R^4e^{4it} + R^3e^{3it} + 4R^2e^{2it} + aRe^{it} + 3$. Since the variation in argument is determined by the dominating term R^4e^{4it} for R large (independent of the value of a) the change of argument on this arc is 4 times the variation in argument for the arc, and thus $\triangle \arg p(z) \approx 4 \cdot \frac{\pi}{2} = 2\pi$.

The positive imaginary axis γ_3 from iR to 0 we parametizize as $\gamma_3 : z = it,\ 0 \leq t \leq R$, and our function $p(z)$ becomes $f(t) = (it)^4 + (it)^3 + 4(it)^2 + ait + 3 = (t^2-1)(t^2-3) + i(-t^3+at)$. We find that the real part have a zero at $t = 1$ and $t = \sqrt{3}$ and the imaginary part have a zero for $t = 0$ and and a second root $t = \sqrt{a}$ if $a \geq 0$. We make the following table and do the sketches,

$a < 0$

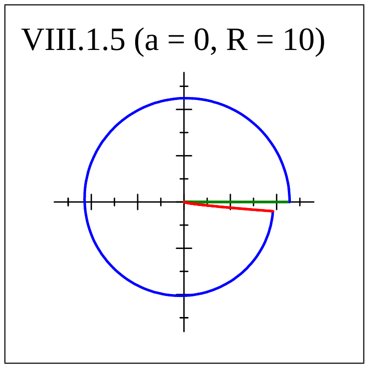
t	$\mathrm{Re}\, z$	$\mathrm{Im}\, z$	$\arg z$
∞	$+$	$-$	0
$\sqrt{3}$	0	$-$	$-\frac{\pi}{2}$
1	0	$-$	$-\frac{\pi}{2}$
0	$+$	0	0

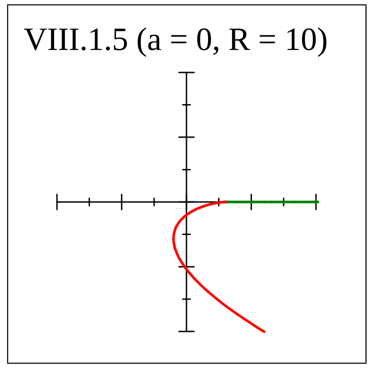




$a = 0$

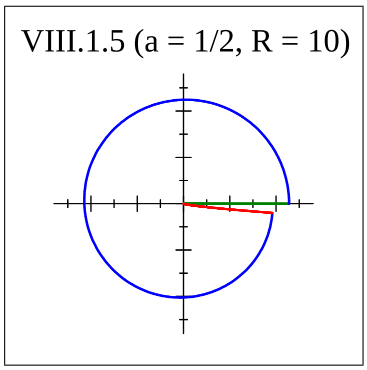
t	$\mathrm{Re}\, z$	$\mathrm{Im}\, z$	$\arg z$
∞	$+$	$-$	0
$\sqrt{3}$	0	$-$	$-\frac{\pi}{2}$
1	0	$-$	$-\frac{\pi}{2}$
0	$+$	0	0

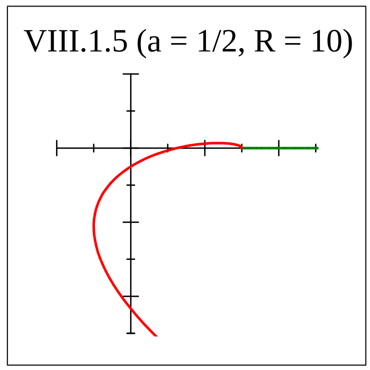




$0 < a < 1$

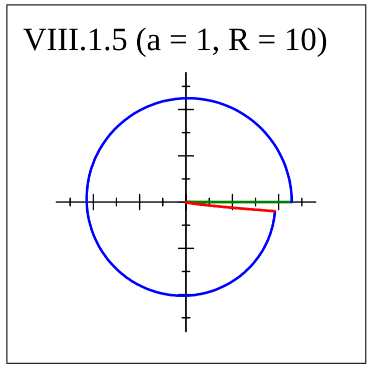
t	$\mathrm{Re}\, z$	$\mathrm{Im}\, z$	$\arg z$
∞	$+$	$-$	0
$\sqrt{3}$	0	$-$	$-\frac{\pi}{2}$
1	0	$-$	$-\frac{\pi}{2}$
$\sqrt{a}$	$+$	0	0
0	$+$	0	0

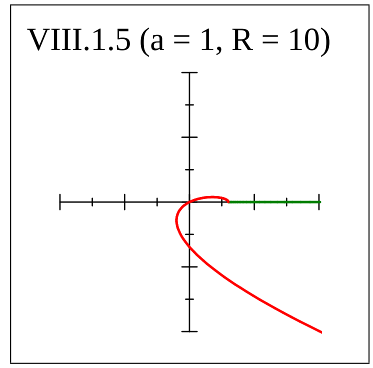




$a = 1$

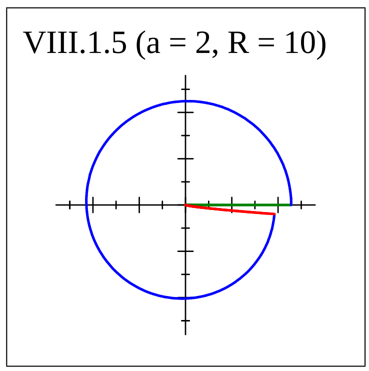
t	$\mathrm{Re}\, z$	$\mathrm{Im}\, z$	$\arg z$
∞	$+$	$-$	0
$\sqrt{3}$	0	$-$	$-\frac{\pi}{2}$
1	0	0	$*$
0	$+$	0	0

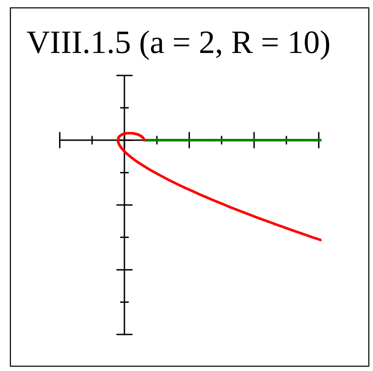




$1 < a < 3$

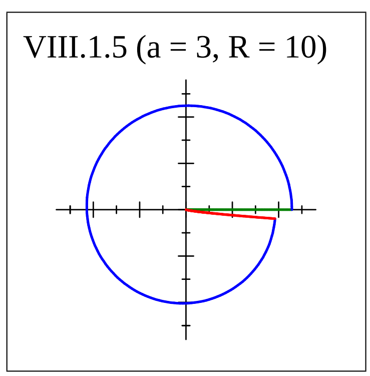
t	$\mathrm{Re}\, z$	$\mathrm{Im}\, z$	$\arg z$
∞	$+$	$-$	0
$\sqrt{3}$	0	$-$	$-\frac{\pi}{2}$
$\sqrt{a}$	$-$	0	$-\pi$
1	0	$+$	$-\frac{3\pi}{2}$
0	$+$	0	-2π

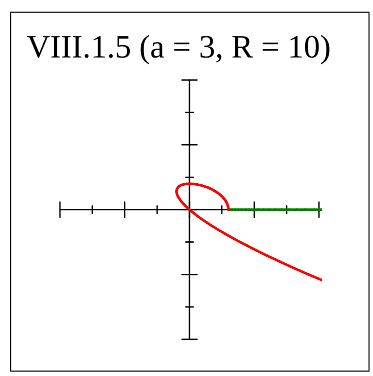




$a = 3$

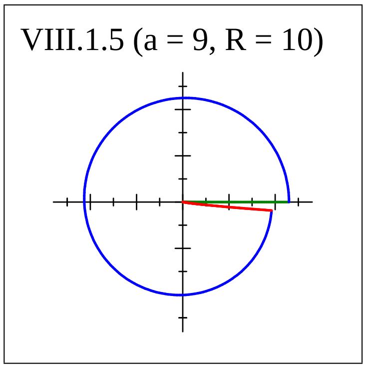
t	$\mathrm{Re}\, z$	$\mathrm{Im}\, z$	$\arg z$
∞	$+$	$-$	0
$\sqrt{3}$	0	0	$*$
1	0	$+$	$\frac{\pi}{2}$
0	$+$	0	0

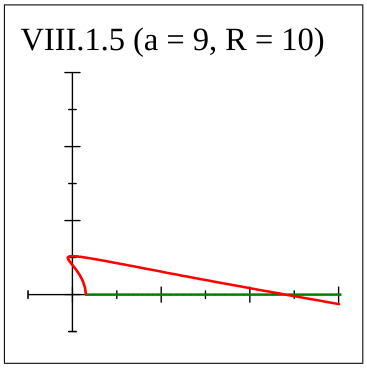




$a > 3$

t	$\operatorname{Re} z$	$\operatorname{Im} z$	$\arg z$
∞	$+$	$-$	0
$\sqrt{a}$	$+$	0	0
$\sqrt{3}$	0	$+$	$\frac{\pi}{2}$
1	0	$+$	$\frac{\pi}{2}$
0	$+$	0	0





Thus when we move from iR to 0, the argument for $p(z)$ only change if $1 < a < 3$ and we have that $\triangle \arg p(z) \approx -2\pi$, for all other value on a we have that $\triangle \arg p(z) \approx 0$. Remark that we can se that for $a = 1$ and $a = 3$ we have zeros on the imaginary axis.

Now we have that the total change of argument for $p(z)$ is $\approx 2\pi$, so we have exactly one zero in the first quadrat. Because that the roots come in complex conjugate pairs it is plain that $p(z)$ has one zero in each quadrant.

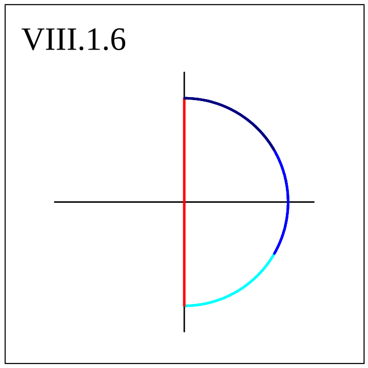
If $1 \leq \alpha \leq 3$, the total change of argument for $p(z)$ is ≈ 0 and no zeros on the imaginary axis, thus we must have four zeros in the open left half-plane. If $\alpha < 1$ and $\alpha > 3$, the total charge of argument for $p(z)$ is $\approx 2\pi$ and no zeros of the imaginary axis, thus we must have two zeros in the open left half-plane.
If $\alpha = 1$ and $\alpha = 3$, the total charge of argument for $p(z)$ is $\approx 2\pi$ and we have two zeros on the imaginary axis, thus we must have remaining two zeros in the open left half-plane.

VIII.1.6

1	2	3	P	L	K

For a fixed real number α, find the number of solutions of $z^5+2z^3-z^2+z=\alpha$ satisfying $\operatorname{Re} z>0$.

Solution

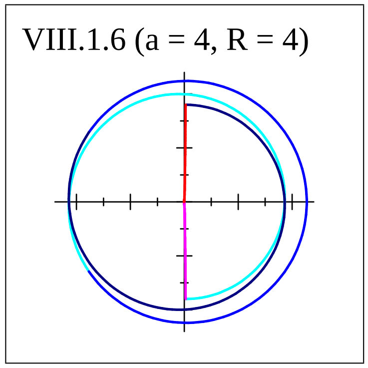


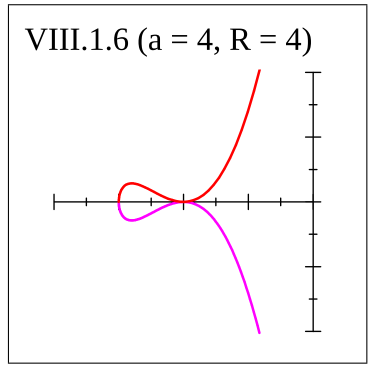
Set $p(z)=z^5+2z^3-z^2+z-\alpha$, and compute $\triangle \arg p(z)$ along the two segments in the contuour in figure VIII.1.6.

The arc γ_1 from $-iR$ to iR we parametizize as $\gamma_2 : z=Re^{it}$, $-\pi/2\leq t\leq\pi/2$, and our function $p(z)$ becomes $p(t)=R^5e^{5it}+2R^3e^{3it}-R^2e^{2it}+Re^{it}-\alpha$. Since the variation in argument is determined by the dominating term R^5e^{5it} for R large (independent of the value of α) the change of argument on this arc is 5 times the variation in argument for the arc, and thus $\triangle \arg p(z)\approx 5\cdot\frac{\pi}{2}=\frac{5\pi}{2}$.

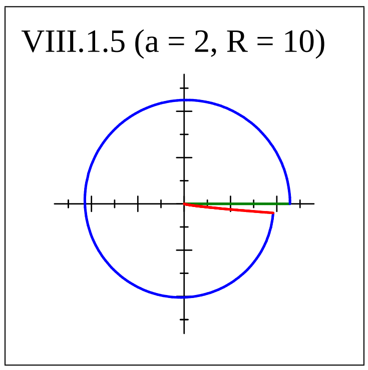
The positive imaginary axis γ_2 from iR to $-iR$ we parametizize as $\gamma_2 : z=it$, $-R\leq t\leq R$, and our function $p(z)$ becomes $f(t)=(it)^5+2(it)^3-(it)^2+it-\alpha=t^2-\alpha+it\left(t^2-1\right)^2$. We find that the real part have a zero at $t=-\sqrt{\alpha}$ and $t=\sqrt{\alpha}$ and the imaginary part have a zero for $t=-1$, $t=0$ and $t=1$. We make the following table and do the sketch,

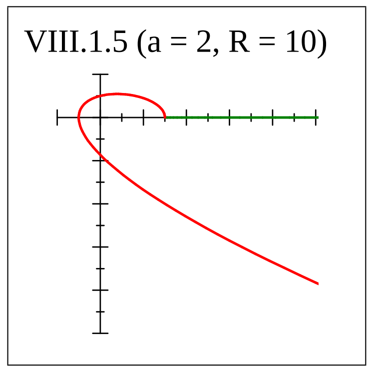
t	$\operatorname{Re} z$	$\operatorname{Im} z$	$\arg z$
∞			

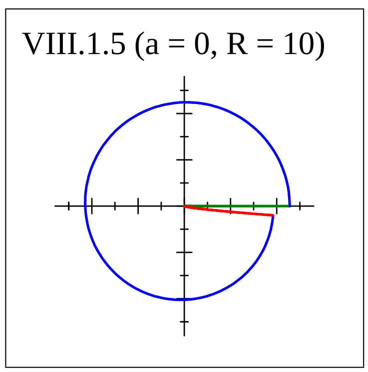


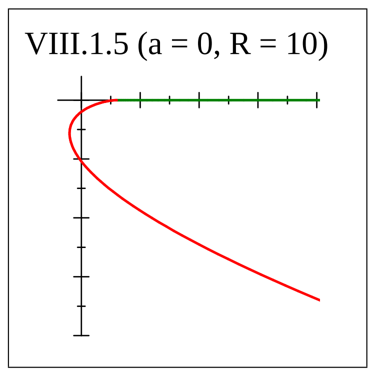


t	Re	Im	arg
∞	$+$	0	0
$\sqrt{3}$	0	$-$	$-\frac{\pi}{2}$
$\sqrt{a}$	$-$	0	$-\pi$
1	0	$+$	$-\frac{3\pi}{2}$
0	3	0	-2π









t	$\operatorname{Re} z$	$\operatorname{Im} z$	$\arg z$
∞			

Thus when we move from iR to 0, the argument for $p(z)$ remain in the 4-th quadrant except for touching the imaginary axis at $-i$ and terminating at 1, and their is no change of argument on this line segment on the imaginary axis, thus $\triangle \arg p(z) = 0$.

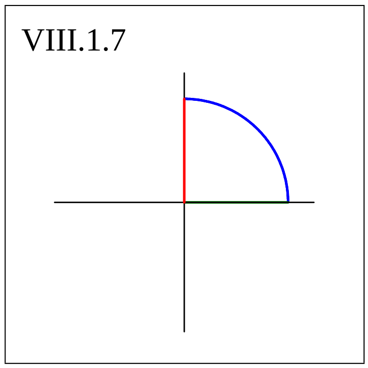
Now we have that the total change of argument for $p(z)$ is $\approx 2\pi$, so we have exactly one zero in the first quadrat. Because that the roots come in complex conjugate pairs it is plain that $p(z)$ has one zero in each quadrant.

VIII.1.7

1	2	3	P	L	K

For a fixed complex number λ, show that if m and n are large integers, then the equation $e^z = z + \lambda$ has exactly $m+n$ solutions in the horizontal strip $\{-2\pi i m < \operatorname{Im} z < 2\pi i n\}$.

Solution



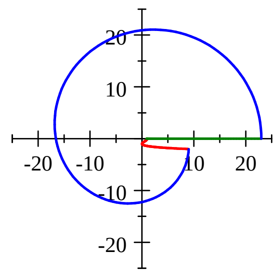
Set $p(z) = z^4 + 2z^2 - z + 1$, and compute $\triangle \arg p(z)$ along the three segments in the contuour in figure VIII.1.7.
The positive real axis γ_1 from 0 to R we parametizize as $\gamma_x : z = t,\ 0 \le t \le R$, and our function $p(z)$ becomes $p(t) = t^4 + 2t^2 - t + 1 = (t^2+1)^2 - t > 0$. Since $p(x) > 0$ for $t \ge 0$, their is no change of argument on this line segment on the real axis, thus $\triangle \arg p(z) = 0$.

The arc γ_2 from R to iR we parametizize as $\gamma_2 : z = Re^{it},\ 0 \le t \le \pi/2$, and our function $p(z)$ becomes $p(t) = R^4e^{4it} + 2R^2e^{2it} - Re^{it} + 1$. Since the variation in argument is determined by the dominating term R^4e^{4it} for R large the change of argument on this arc is 4 times the variation in argument for the arc, and thus $\triangle \arg p(z) \approx 4 \cdot \frac{\pi}{2} = 2\pi$.

The positive imaginary axis γ_3 from iR to 0 we parametizize as $\gamma_3 : z = it,\ 0 \le t \le R$, and our function $p(z)$ becomes $f(t) = (it)^4 + 2(it)^2 - it + 1 = (t^2-1)^2 - it$. We find that the real part have a zero at $t = 1$, and the imaginary part have a zero for $t = 0$ on the positive imaginary axis. We make the following table and do the sketch,

t	Re	Im	arg
∞	$+$	$-$	0
1	0	$-$	$-\frac{\pi}{2}$
0	$+$	0	0

VIII.1.7



Thus when we move from iR to 0, the argument for $p(z)$ remain in the 4-th quadrant except for touching the imaginary axis at $-i$ and terminating at 1, and their is no change of argument on this line segment on the imaginary axis, thus $\triangle \arg p(z) = 0$.

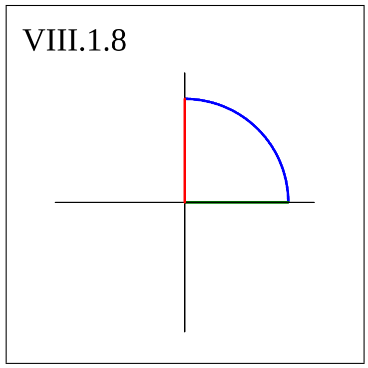
Now we have that the total change of argument for $p(z)$ is $\approx 2\pi$, so we have exactly one zero in the first quadrat. Because that the roots come in complex conjugate pairs it is plain that $p(z)$ has one zero in each quadrant.

VIII.1.8

1	2	3	P	L	K
				LLL	

Show that if $\operatorname{Re}\lambda > 1$, then the equation $e^z = z + \lambda$ has exactly one solution in the left half-plane.

Solution



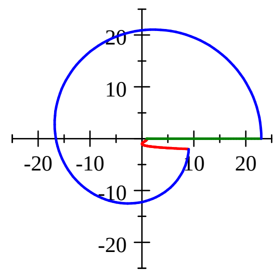
Set $p(z) = z^4 + 2z^2 - z + 1$, and compute $\triangle \arg p(z)$ along the three segments in the contuour in figure VIII.1.8.
The positive real axis γ_1 from 0 to R we parametizize as $\gamma_x : z = t,\ 0 \leq t \leq R$, and our function $p(z)$ becomes $p(t) = t^4 + 2t^2 - t + 1 = \left(t^2+1\right)^2 - t > 0$. Since $p(x) > 0$ for $t \geq 0$, their is no change of argument on this line segment on the real axis, thus $\triangle \arg p(z) = 0$.

The arc γ_2 from R to iR we parametizize as $\gamma_2 : z = Re^{it},\ 0 \leq t \leq \pi/2$, and our function $p(z)$ becomes $p(t) = R^4e^{4it} + 2R^2e^{2it} - Re^{it} + 1$. Since the variation in argument is determined by the dominating term R^4e^{4it} for R large the change of argument on this arc is 4 times the variation in argument for the arc, and thus $\triangle \arg p(z) \approx 4 \cdot \frac{\pi}{2} = 2\pi$.

The positive imaginary axis γ_3 from iR to 0 we parametizize as $\gamma_3 : z = it,\ 0 \leq t \leq R$, and our function $p(z)$ becomes $f(t) = (it)^4 + 2(it)^2 - it + 1$
$= \left(t^2-1\right)^2 - it$. We find that the real part have a zero at $t = 1$, and the imaginary part have a zero for $t = 0$ on the positive imaginary axis. We make the following table and do the sketch,

t	Re	Im	arg
∞	$+$	$-$	0
1	0	$-$	$-\frac{\pi}{2}$
0	$+$	0	0

VIII.1.8



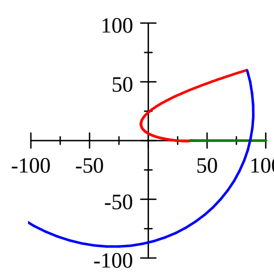
Thus when we move from iR to 0, the argument for $p(z)$ remain in the 4-th quadrant except for touching the imaginary axis at $-i$ and terminating at 1, and their is no change of argument on this line segment on the imaginary axis, thus $\triangle \arg p(z) = 0$.

Now we have that the total change of argument for $p(z)$ is $\approx 2\pi$, so we have exactly one zero in the first quadrat. Because that the roots come in complex conjugate pairs it is plain that $p(z)$ has one zero in each quadrant.

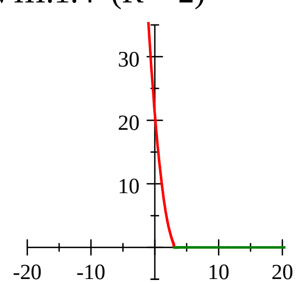
Set $f(z) = e^z - z - \lambda$, $\operatorname{Re}\lambda > 1$, and get $f(iy) = \cos y - \operatorname{Re}\lambda + i(\sin y - y) - i\operatorname{Im}\lambda$. Thus when we move from $f(-iR)$ to $f(iR)$, $f(iy)$ remains in the left half-plane and the change in $\arg f$ is $\approx \pi$.
When we move from $f(iR)$ to $f(-iR)$ along $f\left(Re^{i\theta}\right)$, $\frac{\pi}{2} < \theta < \frac{3\theta}{2}$, we cross the real axis from the 4-th to the 1 st quadrant, thus the change in $\arg f$ is again $\approx \pi$. (May also use Rouche's Theorem.)

t	Re	Im	arg
∞	$-$	$+$	$\frac{\pi}{2}$
$\sqrt[4]{\frac{3}{2}}$	0	$+$	$\frac{\pi}{2}$
0	$+$	$+$	0

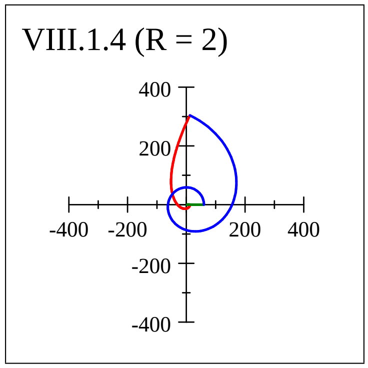
VIII.1.4 (R = 2)

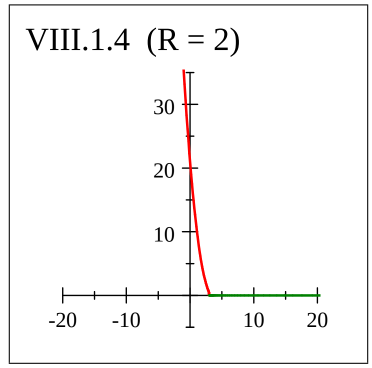


VIII.1.4 (R = 2)



t	Re	Im	arg
∞	$-$	$+$	$\frac{\pi}{2}$
$\sqrt[4]{\frac{3}{2}}$	0	$+$	$\frac{\pi}{2}$
0	$+$	$+$	0





$$(it)^4 - (it)^3 + 13\,(it)^2 - it\ +36 = t^4 + it^3 - 13t^2 - it + 36$$

VIII.1.9

1	2	3	P	L	K

Show that if $f(z)$ is analytic in a domain D, and if γ is a closed curve in D such that the values of $f(z)$ on γ lie in the slit plane $\mathbb{C}\backslash(-\infty, 0]$ then the increase in the argument of $f(z)$ around γ is zero.

Solution

VIII.2.1

1	2	3	P	L	K
				LLL	

Show that $2z^5 + 6z - 1$ has one root in the interval $0 < x < 1$ and four roots in the annulus $\{1 < |z| < 2\}$.

Solution

Set first $p(z) = f_1(z) + h_1(z)$, where $f_1(z) = 2z^5$ and $h_1(z) = 6z - 1$. On the circle $|z| = 2$ we have,
$|f_1(z)| = |2z^5| = 2 \cdot 2^5 = 64$ and $|h_1(z)| = |6z - 1| \leq 13$, then $|f_1(z)| > |h_1(z)|$ on $|z| = 2$ so by Rouche's Theorem $p(z)$ has all 5 roots in the disk $|z| = 2$.
Set now $p(z) = f_2(z) + h_2(z)$, where $f_2(z) = 6z$ and $h_2(z) = 2z^5 - 1$. On the circle $|z| = 1$ we have, $|f_2(z)| = |6z| = 6$ and $|h_2(z)| = |2z^5 - 1| \leq 3$, then $|f_2(z)| > |h_2(z)|$ on $|z| = 1$ so by Rouche's Theorem $p(z)$ has 1 roots in the disk $|z| = 1$.
It follows that $p(z) = 2z^5 + 6z - 1$ has $5 - 1 = 4$ roots in the annulus $1 < |z| < 2$. As any complex roots come in conjugate pairs that have the same magnitude, the root in the disk $|z| = 1$ must be real, we have that $p(0) = -1$ and $p(1) = 7$, thus this root must be in the interval $0 < x < 1$.

VIII.2.2

1	2	3	P	L	K
				LLL	

How many roots does $z^9 + z^5 - 8z^3 + 2z + 1$ **have between the circles** $\{|z| = 1\}$ **and** $\{|z| = 2\}$**?**

Solution

Set first $p(z) = f_1(z) + h_1(z)$, where $f_1(z) = z^9$ and $h_1(z) = z^5 - 8z^3 + 2z + 1$. On the circle $|z| = 2$ we have,
$|f_1(z)| = |z^9| = |z|^9 = 2^9 = 512$ and $|h_1(z)| = |z^5 - 8z^3 + 2z + 1| \leq 101$, then $|f_1(z)| > |h_1(z)|$ on $|z| = 2$ so by Rouche's Theorem $p(z)$ has 9 roots in the disk $|z| = 2$.
Set now $p(z) = f_2(z) + h_2(z)$, where $f_2(z) = z^9 + z^5 + 2z + 1$ and $h_2(z) = -8z^3$. On the circle $|z| = 1$ we have, $|f_2(z)| = |-8z^3| = 8$ and $|h_2(z)| = |z^9 + z^5 + 2z + 1| \leq 5$, then $|f_2(z)| > |h_2(z)|$ on $|z| = 2$ so by Rouche's Theorem $p(z)$ has 3 roots in the disk $|z| = 1$.
Thus it follows that $p(z) = z^9 + z^5 - 8z^3 + 2z + 1$ has $9 - 3 = 7$ roots in the annulus $1 < |z| < 2$.

VIII.2.3

1	2	3	P	L	K
				LLL	

Show that if m and n are positive integers, the the polynomial

$$p(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^m}{m!} + 3z^n$$

has exactly n zeros in the unit disk.

Solution

Set $p(z) = f(z) + h(z)$, where $f(z) = 3z^n$ and $h(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^m}{m!}$. On the circle $|z| = 1$ we have,
$|f(z)| = |3z^n| = 3|z|^n = 3 \cdot 1^n = 3$ and $|h(z)| = \left|1 + z + \frac{z^2}{2!} + \cdots + \frac{z^m}{m!}\right| \leq \sum_{k=0}^{\infty} \frac{1}{k!} = e$, then $|f(z)| > |h(z)|$ on $|z| = 1$ so by Rouche's Theorem $p(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^m}{m!} + 3z^n$ has exactly n roots in the unit disk $|z| = 1$.

VIII.2.4

1	2	3	P	L	K
				LLL	

Fix a complex number λ such that $|\lambda| < 1$. For $n \geq 1$, show that $(z-1)^n e^z - \lambda$ has n zeros satisfying $|z-1| < 1$ and no other zeros in the right half-plane. Determine the multiplicity of the zeros.

Solution

Set $p(z) = f(z) + h(z)$, where $f(z) = (z-1)^n e^z$ and $h(z) = -\lambda$. On the circle $|z-1| = 1$ which we parametrize by $z = 1 + e^{i\theta}$, $0 \leq \theta \leq 2\pi$ we have, $|f(z)| = |(z-1)^n e^z| = \left|\left(1 + e^{i\theta} - 1\right)\right|^n \left|e^{1+i\theta}\right| = e^{1+\cos\theta} \geq 1$, then $0 \leq \theta \leq 2\pi$ and $|h(z)| = |-\lambda| = |\lambda| < 1$ from the text, then $|f(z)| > |h(z)|$ on $|z-1| = 1$ so by Rouche's Theorem $p(z) = (z-1)^n e^z - \lambda$ has n roots in the disk $|z-1| = 1$. We have that $f'(z) = n(z-1)^{n-1} e^z + (z-1)^n e^z = (n + z - 1)(z-1)^{n-1} e^z$, thus the zeros of of $f'(z)$ are at $z = 1$ and at $z = 1 - n$, where the only zeros in the right half-plane are at $z = 1$. Since $f(1) = -\lambda$, the zeros of $f(z)$ must be simple unless $\lambda = 0$, in which case we ge a zero of order n at $z = 1$.

****

Här är lite konstigt den derivarande satsen, hur funkar den med nollställen
****

VIII.2.5

1	2	3	P	L	K

For a fixed λ satisfying $|\lambda| < 1$, show that $(z-1)^n e^z + \lambda (z+1)^n$ has n zeros in the right half-plane, which are all simple if $\lambda \neq 0$.

Solution

VIII.2.6

1	2	3	P	L	K

Let $p(z) = z^6 + 9z^4 + z^3 + 2z + 4$ be the polynomial treated in the example in this section.

(a)

Determine which quadrants contain the four zeros of $p(z)$ that lie inside the unit circle.

(b)

Determine which quadrants contain the two zeros of $p(z)$ that lie outside the unit circle.

(c)

Show that the two zeros of $p(z)$ that lie outside the unit circle satisfy $\{|z \pm 3i| < 1/10\}$.

Solution

VIII.2.7

1	2	3	P	L	K
				LLL	

Let $f(z)$ and $g(z)$ be analytic functions on the bounded domain D that extend continuously to ∂D and satisfy $|f(z)+g(z)| < |f(z)|+|g(z)|$ on ∂D. Show that $f(z)$ and $g(z)$ have the same number of zeros in D, counting multiplicity. Remark. This is a variant of Rouche's theorem, in which the hypotheses are symmetric in $f(z)$ and $g(z)$. Rouche's theorem is obtained by setting $h(z) = -f(z) - g(z)$. For the solution of the exercise, see Exercise 9 in the preceding section.

Solution

VIII.2.8

1	2	3	P	L	K

Let D be a bounded domain, and let $f(z)$ and $h(z)$ be meromorphic functions on D that extend to be analytic on ∂D. Suppose that $|h(z)| < |f(z)|$ on ∂D. Show by example that $f(z)$ and $f(z)+h(z)$ can have different numbers of zeros on D. What can be said about $f(z)$ and $f(z)+g(z)$? Prove your assertion.

Solution

VIII.2.9

1	2	3	P	L	K

Let $f(z)$ be a continuously differentiable function on a domain D. Suppose that for all complex constants a and b, the increase in the argument of $f(z)+az+b$ around any small circle in D on which $f(z)+az+b \neq 0$ is nonnegative. Show that $f(z)$ is analytic.

Solution

VIII.3.1

1	2	3	P	L	K
				LLL	

Let $\{f_k(z)\}$ be a sequence of analytic functions on D that converges normally to $f(z)$, and suppose that $f(z)$ has a zero of order N at $z_0 \in D$. Use Rouche's theorem to show that there exists $\rho > 0$ such that for k large, $f_k(z)$ has exactly N zeros counting multiplicity on the disk $\{|z - z_0| < \rho\}$.

Solution

VIII.3.2

1	2	3	P	L	K
				LLL	

Let S be the family of univalent functions $f(z)$ defined on the open unit disk $\{|z|<1\}$ that satisfy $f(0)=0$ and $f'(0)=1$. Show that S is closed under normal convergence, that is, if a sequence in S converges normally to $f(z)$, then $f \in S$. Remark. It is also true, but more difficult to prove, that S is a compact family of analytic functions, that is, every sequence in S has a normally convergent subsequence.

Solution

VIII.4.1

1	2	3	P	L	K
				LLL	

Suppose D is a bounded domain with piecewise smooth boundary. Let $f(z)$ be meromorphic and $g(z)$ analytic on D. Suppose that both $f(z)$ and $g(z)$ extend analytically across the boundary of D, and that $f(z) \neq 0$ on ∂D. Show that

$$\frac{1}{2\pi i} \oint_{\partial D} g(z) \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{n} m_j g(z_j),$$

where $z_1, \ldots, z_n$ are the zeros and poles of $f(z)$, and m_j is the order of $f(z)$ at z_j.

Solution

VIII.4.2

1	2	3	P	L	K

Let $f(z)$ be a meromorphic function on the complex plane that is doubly periodic. Suppose that the zeros and the poles of $f(z)$ are at the points $z_1, \ldots, z_n$ and at their translates by periods of $f(z)$, and suppose no z_j is a translate by a period of another z_k. Let m_j be the order of $f(z)$ at z_j. Show that $\sum m_j z_j$ is a period of $f(z)$. Hint. Integrate $zf'(z)/f(z)$ around the boundary of the fundamental parallelogram P constructed in Section VI.5.

Solution

VIII.4.3

1	2	3	P	L	K
				LLL	

Let $\{f_k(z)\}$ be a sequence of analytic functions on a domain D that converges normally to $f(z)$. Suppose that $f_k(z)$ attains each value w at most m times (counting multiplicity) in D. Show that either $f(z)$ is constant, or $f(z)$ attains each value w at most m times in D.

Solution

VIII.4.4

1	2	3	P	L	K

Let $f(z)$ be an analytic function on the open unit disk $D = \{|z| < 1\}$. Suppose there is an annulus $U = \{r < |z| < 1\}$ such that the restriction of $f(z)$ to U is one-to-one. Show that $f(z)$ is one-to-one on D.

Solution

VIII.4.5

1	2	3	P	L	K

Let $f(z) = p(z)/q(z)$ be a rational function, where $p(z)$ and $q(z)$ are polynomials that are relatively prime (no common zeros). We define the degree of $f(z)$ to be the larger of the degrees of $p(z)$ and $q(z)$. Denote the degree of $f(z)$ by d.

(a)

Show that each value $w \in \mathbb{C}$, $w \neq f(\infty)$, is assumed d times by $f(z)$ on $\mathbb{C}$.

(b)

Show that $f(z)$ attains each value $w \in \mathbb{C}^*$ d times on $\mathbb{C}^*$ (as always, counting multiplicity).

Solution

VIII.4.6

1	2	3	P	L	K

Let $f(z)$ be a meromorphic function on the complex plane, and suppose there is an integer m such that $f^{-1}(w)$ has at most m points for all $w \in \mathbb{C}$. Show that $f(z)$ is a rational function.

VIII.4.7

1	2	3	P	L	K

Let $F(z,w)$ be a continuous function of z and w that depends analytically on z for each fixed w, and let $F_1(z,w)$ denote the derivative of $F(z,w)$ with respect to z. Suppose $F(z_0,w_0)=0$, and $F_1(z_0,w_0)\neq 0$. Choose ρ such that $F(x,w_0)\neq 0$ for $0<|z-z_0|\leq\rho$.

(a)

Show that there exists $\delta>0$ such that if $|w-w_0|<\delta$, there is a unique $z=g(w)$ satisfying $|z-z_0|<\rho$ and $F(z,w)=0$.

(b)

Show that

$$g(w)=\frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho}\frac{\zeta F_1(\zeta,w)}{F(\zeta,w)}d\zeta,\qquad |w-w_0|<\delta.$$

(c)

Suppose further that $F(z,w)$ is analytic in w for each fixed z, and let $F_2(z,w)$ denote the derivative of $F(z,w)$ with respect to w. Show that $g(w)$ is analytic, and

$$g'(w)=-F_2(g(w),w)/F_1(g(w),w).$$

(d)

Derive the inverse function theorem given in this section, together with the formula for the derivative of the inverse function, as a corollary of (a), (b), and (c).

Remark. this is the implicit function theorem for analytic functions. Note that a specific formula is given for the function $g(w)$ defined implicitely by $F(g(w),w)=0$.

VIII.4.8

1	2	3	P	L	K

Let D be a bounded domain, and let $f(z)$ be a continuous function on $D \cup \partial D$ that is analytic on D. Show that $\partial(f(D)) \subseteq f(\partial D)$, that is, the boundary of the open set $f(D)$ is contained in the image under $f(z)$ of the boundary of D.

Solution

VIII.5.1

1	2	3	P	L	K

Find the critical points and critical values of $f(z) = z + 1/z$. Sketch the curves where $f(z)$ is real. Sketch the regions where $\operatorname{Im} f(z) > 0$ and where $\operatorname{Im} f(z) < 0$.

Solution.

VIII.5.2

1	2	3	P	L	K

Suppose $g(z)$ is analytic at $z = 0$, with power series $g(z) = 2 + iz^4 + O(z^5)$. Sketch and label the curves passing through $z = 0$ where $\operatorname{Re} g(z) = 2$ and $\operatorname{Im} g(z) = 0$.

Solution

VIII.5.3

1	2	3	P	L	K

Find the critical points and critical values of $f(z) = z^2 + 1$. Sketch the set of points z such that $|f(z)| \leq 1$, and locate the critical points of $f(z)$ on the sketch.

Solution

VIII.5.4

1	2	3	P	L	K

Suppose that $f(z)$ is analytic at z_0. Show that if the set of z such that $\operatorname{Re} f(z) = \operatorname{Re} f(z_0)$ consists of just one curve passing through z_0, then $f'(z_0) \neq 0$. Show also that if the set of z such that $|f(z)| = |f(z_0)|$ consists of just one curve passing through z_0, then $f'(z_0) \neq 0$.

Solution

VIII.5.5

1	2	3	P	L	K

How many critical points, counting multiplicity, does a polynomial of degree m have in the complex plane? Justify your answer.

Solution

VIII.5.6

1	2	3	P	L	K

Find and plot the critical points and critical values of $f(z) = z^2 + 1$ and of its iterates $f(f(z)) = (z^2 + 1)^2 + 1$ and $f(f(f(z)))$. Suggestion. Use the chain rule.

Solution

VIII.5.7

1	2	3	P	L	K

Let $f(z)$ be a polynomial of degree m. How many (finite) critical points does the $N-$ fold iterate $f \circ \cdots \circ f$ (N times) have? Describe them in terms of the critical points of $f(z)$.

Solution

VIII.5.8

1	2	3	P	L	K

We define a pole of $f(z)$ to be a critical point of $f(z)$ of order k if z_0 is a critical point of $1/f(z)$ of order k. We define $z = \infty$ to be a critical point of $f(z)$ of order k if $w = 0$ is a critical point of $g(w) = f(1/w)$ of order k. Show that with this definition, a point $z_0 \in \mathbb{C}^*$ is a critical point of order k for a meromorphic function $f(z)$ if and only if there are open sets U containing z_0 and V containing $w_0 = f(z_0)$ such that each $w \in v$, $w \neq w_0$, has exactly $k+1$ preimages in U. Remark. We say that $f(z)$ is a $(k+1) -$ sheeted covering of $f^{-1}(V \backslash \{w_0\}) \cap U$ over $V \backslash \{w_0\}$.

Solution

VIII.5.9

1	2	3	P	L	K

Show that a polynomial of degree m, regarded as a meromorphic function on C^*, has a critical point of order $m-1$ at $z_0 = \infty$.

Solution

VIII.5.10

1	2	3	P	L	K

Locate the critical points ana critical values in the extended complex plane of the polynomial $f(z) = z^4 - 2z^2$. Determine the order of each critical point. Sketch the set of points z such that $\operatorname{Im} z \geq 0$.

Solution

VIII.5.11

1	2	3	P	L	K

Show that if f is a rational function, and if g is a fractional linear transformation, then f and $g \circ f$ have the some critical points in the extended complex plane $\mathbb{C}^*$. What can be said about the critical values of $g \circ f$? What can be said about the critical points and critical values $f \circ g$?

Solution

VIII.5.12

1	2	3	P	L	K

Let $f(z) = p(z)/q(z)$ be a rational function of degree d, so that $p(z)$ and $q(z)$ are reactively prime, and d is the larger of the degrees of $p(z)$ and $q(z)$. (See Exercise 4.5.) Show that $f(z)$ has $2d-2$ critical points, counting multiplicity, in the extended complex plane $\mathbb{C}^*$. Hint if $\deg p \neq \deg q$, then the number of critical points of $f(z)$ in the finite plane $\mathbb{C}$ is $\deg(qp' - q'p) = \deg p + \deg q - 1$, while the order of the critical points at ∞ is $|\deg p - \deg q|$ -1.

Solution

VIII.5.13

1	2	3	P	L	K

Show that the set of solution points (w, z) of the equation $z^2 - 2(\cos w) + 1 = 0$ consists of the graphs of two entire functions $z_1(w)$ and $z_2(w)$ of w. Specify the entire functions, and determine where their graphs meet. Remark. The solutions set forms a reducible one-dimensional analytic variety in $\mathbb{C}^2$.

Solution

VIII.5.14

1	2	3	P	L	K

Let $a_0(w), \ldots, a_{m-1}(w)$ be analytic in a neighborhood of $w = 0$ and vanish at $w = 0$. Consider the monic polynomial in z whose coefficients are analytic functions in w,

$$P(z, w) = z^m + a_{m-1}(w) z^{m-1} + \cdots + a_0(w), \qquad |w| < \delta.$$

Suppose that for each fixed w, $0 < |w| < \delta$, there are m distinct solutions of $P(z, w) = 0$.

(a)

Show that the m roots of the equation $P(z, w) = 0$ determine analytic functions $z_1(w), \ldots, z_m(w)$ in the slit disk $\{|w| < \delta \backslash (-\delta, 0]\}$. Hint. Use the implicit function theorem (Exercise 4.7).

(b)

Glue together branch cuts to form an $m-$ sheeted (possibly disconnected) surface over the punctured disk $\{0 < |w| < \delta\}$ on which the branches $z_j(w)$ determine a continuous function.

(c)

Suppose that the indices are arranged so that for some fixed k, $1 \le k \le n$, the continuation of $z_j(w)$ once around $w = 0$ is $z_{j+1}(w)$ for $1 \le j \le k-1$, while the continuation of $z_k(w)$ once around $w = 0$ is $z_1(w)$. Show that $Q(z, w) = (z - z_1(w)) \cdots (z - z_k(w))$ determines a polynomial in z whose coefficients are analytic functions of w for $|w| < \delta$. Show further that the polynomial $Q(z, w)$ is an irreducible factor of $P(z, w)$, and that all irreducible factors of $P(z, w)$ arise from subsets of the $z_j(w)$'s in this way.

(d)

Show that if $f(z)$ is an analytic function that has a zero of order m at $z = 0$, and $z_1(w), \ldots, z_m(w)$ are solutions of the equation $w = f(z)$, then the polynomial $P(z, w) = (z - z_1(w)) \cdots (z - z_m(w))$ is irreducible.

Solution

VIII.5.15

1	2	3	P	L	K

Consider monic polynomial in z of the form

$$P(z,w) = z^m + a_{m-1}(w) + \cdots + a_0(w),$$

where the functions $a_0(w), \ldots, a_{m-1}(w)$ are defined and meromorphic in some disk centered at $w = 0$. Let $P_0(z,w)$ and $P_1(z,w)$ be two such polynomials, and consider the following algorithm. Using the division algorithm, find polynomials $A_2(z,w)$ and $P_2(z,w)$ such that $P_0(z,w) = A_2(z,w)\,P_1(z,w) + P_2(z,w)$ and the degree of $P_2(z,w)$ is less then the degree of $P_1(z,w)$. Continue in this fashion, finding polynomials $A_{j+1}(z,w)$ and $P_{j+1}(z,w)$ such that

$$P_{j-1}(z,w) = A_{j+1}(z,w)\,P_j(z,w) + P_{j+1}(z,w)$$

and $\deg P_{j+1}(z,w) < \deg P_j(z,w)$, until eventually we reach

$$P_l(z,w) = 0, \qquad P_{l-1}(z,w) = A_l(z,w)\,P_l(z,w).$$

Let $D(z,w)$ be the monic polynomials in z obtained by dividing $P_l(z,w)$ by the coefficient of the highest power of z.

(a)

Show that $D(z,w)$ is the greatest common divisor of $P_0(z,w)$ and $P_1(z,w)$, in the sense that $D(z,w)$ divides both $P_0(z,w)$ and $P_1(z,w)$, and each polynomial that divides both $P_0(z,w)$ and $P_1(z,w)$ also divides $D(z,w)$.

(b)

Show that there are polynomials $A(z,w)$ and $B(z,w)$ such that $D = AP_0 + BP_1$.

(c)

Show, that if $P_0(z,w)$ and $P_1(z,w)$ are relatively prime (that is $D(z,w) = 1$), then there is $\varepsilon > 0$ such that for each fixed w, $0 < |w| < \varepsilon$, the polynomials $P_0(z,w)$ and $P_1(z,w)$ have no common zeros.

(d)

Show that any polynomial $P(z,w)$ as above can be factored as a product of irreducible polynomials, and the factorization is unique up to the order of the factors and multiplication of a factor by a meromorphic function in w.

(e)

Show that if the coefficients of $P(z,w)$ are analytic at $w=0$, then the irreducible factors of $P(z,w)$ can be chosen so that their coefficients are analytic at $w=0$.

(f)

Show that if $P(z,w)$ is irreducible, then there is $\varepsilon>0$ such that for each fixed w, $0<|w|<\varepsilon$, the roots of $P(z,w)$ are distinct.

(g)

Show that the results of Exercise 14(a)-(c) hold without the supposition that the solutions of $P(z,w)=0$ are distinct.

Solution

VIII.6.1

1	2	3	P	L	K

Sketch the closed path $\gamma(t) = e^{it}\sin(2t)$, $0 \leq t \leq 2t$, and determine the winding number $W(\gamma, \zeta)$ for each point ζ not on the path.

Solution

VIII.6.2

1	2	3	P	L	K

Sketch the closed path $\gamma(t) = e^{-2it}\cos t$, $0 \le t \le 2\pi$, and determine the winding number $W(\gamma, \zeta)$ for each point ζ not on the path

Solution

VIII.6.3

1	2	3	P	L	K

Let $f(z)$ be analytic on an open set containing a closed path γ, and suppose $f(z) \neq 0$ on γ. Show that the increase in $\arg f(z)$ around γ is $2\pi W(f \circ \gamma, 0)$.

Solution

VIII.6.4

1	2	3	P	L	K

Let D be a domain, and suppose z_0 and z_1 lie in the same connected component of $\mathbb{C} \backslash D$.

(a)

Show that the increase in the argument of $f(z) = (z - z_0)(z - z_1)$ around any closed curve in D is an even multiple of 2π.

(b)

Show that $(z - z_0)(z - z_1)$ has an analytic square in D.

(c)

Show by example that $(z - z_0)(z - z_1)$ does not unnecessarily have an analytic cube in D.

Solution

VIII.6.5

1	2	3	P	L	K

Show that if γ is a piecewise smooth closed curve in the complex plane, with trace Γ, and if $z_0 \notin \Gamma$, then

$$\int_\gamma \frac{1}{(z-z_0)^n} dz = 0, \qquad n \geq 2.$$

Solution

VIII.6.6

1	2	3	P	L	K

Let γ be a closed path in a domain D such that $W(\gamma, \zeta) = 0$ for all $\zeta \notin D$. Suppose that $f(z)$ is analytic on D except possibly at a finite number of isolated singularities $z_1, \ldots, z_m \in D \backslash \Gamma$. Show that

$$\int_\gamma f(z)\, dz = 2\pi i \sum W(\gamma, z_k) \operatorname{Res}[f, z_k].$$

Hint. Consider the Laurent decomposition at each z_k, and use Exercise 5.

Solution

VIII.6.7

1	2	3	P	L	K

Evaluate

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z\,(z^2-1)},$$

where γ is the closed path indicated in the figure. Hint. Either use Exercise 6, or proceed directly with partial fractions.

VIII.6.8

1	2	3	P	L	K

Let $\gamma(t)$ and $\sigma(t)$, $a \leq t \leq b$, be closed paths.

(a)

Show that if $\zeta \in \mathbb{C}$ does not lie on the straight line segment between $\gamma(t)$ and $\sigma(t)$, for $a \leq t \leq b$, then $W(\sigma, \zeta) = W(\gamma, \zeta)$.

(b)

Show that if $|\sigma(t) - \gamma(t)| < |\zeta - \gamma(t)|$ for a $a \leq t \leq b$, then $W(\sigma, \zeta) = W(\gamma, \zeta)$.

Solution

VIII.6.9

1	2	3	P	L	K

Let $f(z)$ be a continuous complex-valued function on the complex plane such that $f(z)$ is analytic for $|z| < 1$, $f(z) \neq 0$ for $|z| \geq 1$, and $f(z) \to 1$ as $z \to \infty$. Show that $f(z) \neq 0$ for $|z| < 1$.

Solution

VIII.6.10

1	2	3	P	L	K

Let K be a nonempty closed bounded subset of the complex plane, and let $f(z)$ be a continuous complex-valued function on the complex plane that is analytic on $\mathbb{C}\backslash K$ and at ∞. Show that every value attained by $f(z)$ on $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is attained by $f(z)$ somewhere on K, that is $f(\mathbb{C}^*) = f(K)$.

Solution

VIII.7.1

1	2	3	P	L	K

Let $f(z)$ be an entire function, and suppose $g(\zeta)$ is analytic for ζ in the open upper hand lower half-planes and across the interval $(-1,1)$ on the real line. Suppose that

$$\int_{-1}^{1} \frac{f(x)}{z-\zeta} dx = g(\zeta)$$

for ζ in the upper half-plane. What is the value of the integral when ζ is in the lower half-plane? Justify your answer carefully.

Solution

VIII.7.2

1	2	3	P	L	K

Show that

$$\int_{-1}^{1} \frac{dx}{x-\zeta} = \operatorname{Log}\left(\frac{\zeta-1}{\zeta+1}\right), \qquad \zeta \in \mathbb{C}\setminus[-1,+1]\,.$$

(Note that we use the principal branch of the logarithm here.) Reconcile this result with your solution to Exercise 1.

Solution

VIII.7.3

1	2	3	P	L	K

Find the Cauchy integrals of the following functions around the unit circle $\Gamma = \{|z| =\}$, positively oriented.

(a) z (b) $\frac{1}{z}$ (c) $x = \operatorname{Re} z$ (d) $y = \operatorname{Im}(z)$

Solution

VIII.7.4

1	2	3	P	L	K

Suppose $f(z)$ is analytic on an annulus $\{\rho < |z| < \sigma\}$, and let $f(z) = f_0(z) + f_1(z)$ be the Laurent decomposition of $f(z)$. (See Section VI.1.) Fix r between ρ and σ, and let $F(\zeta)$ be the Cauchy integral of $f(z)$ around the circle $|z| = r$. Show that $f_0(\zeta) = F(\zeta)$ for $|\zeta| < r$, and $f_1(\zeta) = -F(\zeta)$ for $|\zeta| > r$. Show further that $f_0(\zeta) = F_-(\zeta)$ and $f_1(\zeta) = -F_+(\zeta)$. Remark The formula $f(z) = f_0(z) + f_1(z)$ reflects the jump theorem for the Cauchy integral of $f(z)$ around circles $|z| = r$.

Solution

VIII.7.5

1	2	3	P	L	K

Let γ be a piecewise smooth curve, and let $F(\zeta)$ be the Cauchy integral (7.1) of a continuous function $f(z)$ on γ. Show that if $g(z)$ is a smooth function on the complex plane that is zero off some bounded set, then

$$\int_\gamma g(z)\, f(z)\, dz = 2i \iint_{\mathbb{C}} \frac{\partial g}{\partial \bar{z}} F(z)\, dx\, dy.$$

Hint. Recall Pompeiu's formula (Section IV.8).

VIII.7.6

1	2	3	P	L	K

Determine whether the point z lie inside or outside. Explain.

Solution

VIII.7.7

1	2	3	P	L	K

A simple arc Γ in $\mathbb{C}$ is the image of a continuous one-to-one function $\gamma(t)$ from a closed interval $[a, b]$ to the complex plane. Show that a simple arc Γ in $\mathbb{C}$ has a connected complement, that is $\mathbb{C}\backslash\Gamma$ is connected. You may use the Tietze extension theorem, that a continuous real-valued function on a closed subset of the complex plane can be extended to a continuous real-valued function on the entire complex plane. Hint. Suppose z_0 belongs to a bounded complement of $\mathbb{C}\backslash\Gamma$. Find at continuous determination $h(z)$ of $\log(z - z_0)$ on Γ, extend $h(z)$ to a continuous function on $\mathbb{C}^*$, and define $f(z) = z - z_0$ on the component $\mathbb{C}\backslash\Gamma$ containing z_0, and $f(z) = e^{h(z)}$ on the remainder of $\mathbb{C}^*\backslash\Gamma$. Consider the increase in the argument of $f(z)$ around circles centered at z_0.

Prove the Jordan curve theorem for a simple closed curve γ by filling in the following proof outline.

(a)

Show that each component $\mathbb{C}\backslash\Gamma$ has boundary Γ. Hint. For $z_0 = \gamma(t_0) \in \Gamma$, apply the preceding exercise to the simple arc $\Gamma\backslash\gamma(I)$, where I is a small open parameter interval containing t_0.

(b)

Prove the Jordan curve theorem in the case where γ contains a straight line segment.

(c)

Show that for any $z_0 = \gamma(t_0) \in \Gamma$, any small disk D_0 containing z_0, and component U of $\mathbb{C}\backslash\Gamma$, there are points $z_1 = \gamma(t_1)$ and $z_2 = \gamma(t_2)$ such that the image of the parameter segment between t_1 and t_2 is contained in D_0 and such that z_1 and z_2 can be joined by a broken line segment in $U \cap D_0$.

(d)

With notation as in (b), let σbe the simple curve obtained by replacing the segment of γ in D_0 between z_0 and z_1 by the broken line segment in $U \cap D_0$ between them, and let τ be the simple closed curve in D_0 obtained by the following the segment of γ in D_0 from z_0 to z_1 and returning to z_0 along the broken line segment. Show that $W(\tau, \zeta) = 0$ and $W(\gamma, \zeta) = W(\sigma, \zeta)$ for $\zeta \in \mathbb{C}\backslash\Gamma$, $\zeta \neq D_0$.

(e)

Using (b) and (d), show that $\mathbb{C}\backslash\Gamma$ has at least two components and that $W(\gamma,\zeta) = \pm 1$ for ζ in each bounded component of $\mathbb{C}\backslash\Gamma$.
(f)
By taking U in (c) to be a bounded component of $\mathbb{C}\backslash\Gamma$, show that $W(\gamma,\zeta) = 0$ for ζ in any other component of $\mathbb{C}\backslash\Gamma$.

Solution

VIII.8.1

1	2	3	P	L	K

Which of the following domains in $\mathbb{C}$ are simply connected? Justify your answers. (a) $D = \{\operatorname{Im} z > 0\} \setminus [0, i]$, the upper half-plane with a vertical slit from 0 to i. (b) $D = \{\operatorname{Im} z > 0\} \setminus [i, 2i]$, the upper half-plane with a vertical slit from i to $2i$. (c) $D = \mathbb{C} \setminus [0, +\infty]$, the complex plane slit along the positive real axis. (d) $D = \mathbb{C} \setminus [-1, 1]$, the complex plane with an interval deleted.

Solution

VIII.8.2

1	2	3	P	L	K

Show that a domain D in the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is simply connected if and only if its complement $\mathbb{C}^* \backslash D$ is connected. Hint. If $D \neq \mathbb{C}^*$, move a point in the complement of D to ∞. If $D = \mathbb{C}^*$, first deform a given closed path to one that does not cover the sphere, then deform it to a point by pulling along arcs of great circles.

Solution

VIII.8.3

1	2	3	P	L	K

Which of the following domains in $\mathbb{C}^*$ are simply connected? Justify your answers. (a) $D = \mathbb{C}^* \setminus [-1, 1]$, the extended plane with an interval deleted, (b) $D = \mathbb{C}^* \setminus \{-1, 0, 1\}$, the thrice-punctured sphere.

Solution

VIII.8.4

1	2	3	P	L	K

Show that a domain D in the complex plane is simply connected if and only if any analytic function $f(z)$ on D does not vanish at any point of D has an analytic logarithm on D. Hint if $f(z) \neq 0$ on D, consider the function

$$G(z) = \int_{z_0}^{z} \frac{f'(z)}{f(w)} dw.$$

Solution

VIII.8.5

1	2	3	P	L	K

Show that a domain D is simply connected if and only if any analytic function $f(z)$ on D that does not vanish at any point of D has an analytic square root on D. Show that this occurs if and only if for any point $z_0 \notin D$ the function $z - z_0$ has an analytic square root on D.

Solution

VIII.8.6

1	2	3	P	L	K

Show that a domain D is simply connected if and only if each continuous function $f(z)$ on D that does not vanish at any point D has continuous logarithm on D.

Solution

VIII.8.7

1	2	3	P	L	K

Let E be a closed connected subset of the extended complex plane $\mathbb{C}^*$. Show that each connected component $\mathbb{C}^* \backslash E$ is simply connected.

Solution

VIII.8.8

1	2	3	P	L	K

Show that simple connectivity is a "topological property" that is, if U and V are domains, and φ is a continuous map of U onto V such that φ^{-1} is also continuous, then U is simply connected if and only if V is simply connected.

Solution

VIII.8.9

1	2	3	P	L	K

Suppose that $f(z)$ is analytic on a domain D, and $f'(z)$ has no zeros on D. Suppose also that $f(D)$ is simply connected, and that there is a branch $g(w)$ of f^{-1} that is analytic on $w_0 = f(z_0)$ and that can be continued analytically along any path in $f(D)$ starting at w_0. Show that $f(z)$ is one-to-one on D.

Solution

VIII.8.10

1	2	3	P	L	K

We define an integral $1-$ cycle in D to be an expression of the form $\sigma = \sum k_j \gamma_j$, where $\gamma_1, \ldots, \gamma_m$ are closed paths in D and $k_1, \ldots, k_m$ are integers. We define the winding number of σ about ζ to be $W(\sigma, \zeta) = \sum k_j W(\gamma_j, \zeta)$, $\zeta \in \mathbb{C} \backslash D$. Show that if $h(\zeta)$ is a continuous integer-valued function on $\mathbb{C}^* \backslash D$ such that $h(\infty) = 0$, then there is an integral 1-cycle σ on D such that $W(\sigma, \zeta) = h(\zeta)$ for all $\zeta \in \mathbb{C} \backslash D$.

Solution

VIII.8.11

1	2	3	P	L	K

An integral 1-cycle σ is homologous to zero i D if $W(\sigma, \zeta) = 0$ whenever $\zeta \notin D$. Let U be a bounded domain whose boundary consists of a finite number of piecewise smooth closed curves $\gamma_1, \ldots, \gamma_m$, oriented positively with respect to U, such that U together with its boundary is contained in D. Show that the 1-cycle $\partial U = \sum \gamma_j$ is homologous to zero in D.

Solution

VIII.8.12

1	2	3	P	L	K

Let D be a domain in $\mathbb{C}$ such that $\mathbb{C}^* \backslash D$ consists of $m+1$ disjoint closed connected sets. Show that there are m piecewise smooth closed curves $\gamma_1, \ldots, \gamma_m$ such that every integral 1-cycle σ can be expressed uniquely in the form $\sigma = \sigma_0 + \sum k_j \gamma_j$, where the k_j's are integers and σ_0 is homologous to zero in D. Remark. The γ_j's form a homology basis for D.

Solution