A First Course in Mathematics Concepts for Elementary School Teachers: Theory, Problems, and Solutions

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Contents

1 Pólya's Problem-Solving Process

Problem-solving is the cornerstone of school mathematics. The main reason of learning mathematics is to be able to solve problems. Mathematics is a powerful tool that can be used to solve a vast variety of problems in technology, science, business and finance, medecine, and daily life.
It is strongly believed that the most efficient way for learning mathematical concepts is through problem solving. This is why the National Council of Teachers of Mathematics NCTM advocates in *Principles and Standards for School Mathematics,* published in 2000, that mathematics instruction in American schools should emphasize on problem solving and quantitative reasoning. So, the conviction is that children need to learn to think about quantitative situations in insightful and imaginative ways, and that mere memorization of rules for computation is largely unproductive.
Of course, if children are to learn problem solving, their teachers must themselves be competent problem solvers and teachers of problem solving. The techniques discussed in this and the coming sections should help you to become a better problem solver and should show you how to help others develop their problem-solving skills.

Pólya's Four-Step Process
In his book *How to Solve It*, George Pólya identifies a four-step process that forms the basis of any serious attempt at problem solving. These steps are:

Step 1. Understand the Problem
Obviously if you don't understand a problem, you won't be able to solve it. So it is important to understand what the problem is asking. This requires that you read slowly the problem and carefully understand the information given in the problem. In some cases, drawing a picture or a diagram can help you understand the problem.

Step 2. Devise a Plan
There are many different types of plans for solving problems. In devising a plan, think about what information you know, what information you are looking for, and how to relate these pieces of information. The following are few common types of plans:
• Guess and test: make a guess and try it out. Use the results of your guess to guide you.

• Use a variable, such as x.
• Draw a diagram or a picture.
• Look for a pattern.
• Solve a simpler problem or problems first- this may help you see a pattern you can use.
• make a list or a table.

Step 3. Carry Out the Plan

This step is considered to be the hardest step. If you get stuck, modify your plan or try a new plan. Monitor your own progress: if you are stuck, is it because you haven't tried hard enough to make your plan work, or is it time to try a new plan? Don't give up too soon. Students sometimes think that they can only solve a problem if they've seen one just like it before, but this is not true. Your common sense and natural thinking abilities are powerful tools that will serve you well if you use them. So don't underestimate them!

Step 4. Look Back

This step helps in identifying mistakes, if any. Check see if your answer is plausible. For example, if the problem was to find the height of a telephone pole, then answers such as 2.3 feet or 513 yards are unlikely-it would be wise to look for a mistake somewhere. *Looking back* also gives you an opportunity to make connections: Have you seen this type of answer before? What did you learn from this problem? Could you use these ideas in some other way? Is there another way to solve the problem? Thus, when you *look back*, you have an opportunity to learn from your own work.

Solving Applied Problems

The term "word problem" has only negative connotations. It's better to think of them as "applied problems." These problems should be the most interesting ones to solve. Sometimes the "applied" problems don't appear very realistic, but that's usually because the corresponding real applied problems are too hard or complicated to solve at your current level. But at least you get an idea of how the math you are learning can help solve actual real-world problems.

Many problems in this book will be word problems. To solve such problems, one translates the words into an equivalent problem using mathematical symbols, solves this equivalent problem, and then interprets the answer. This process is summarized in Figure 1.1

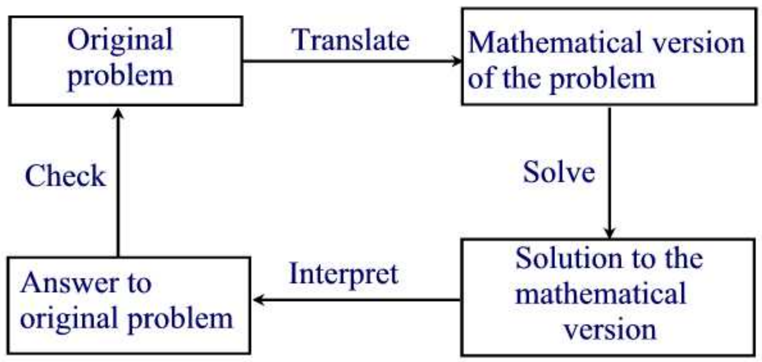


Figure 11

Example 1.1
In each of the following situations write the equation that describes the situation. Do not solve the equation:

(a) Herman's selling house is x dollars. The real estate agent received 7% of the selling price, and Herman's received \$84,532. What is the selling price of the house?
(b) The sum of three consecutive integers is 48. Find the integers.

Solution.
(a) The equation describing this situation is

$$x - 0.07x = 84,532.$$

(b) If x is the first integer then $x+1$ and $x+2$ are the remaining integers. Thus,

$$x + (x+1) + (x+2) = 48. \blacksquare$$

Practice Problems

In each of the following problems write the equation that describes each situation. Do not solve the equation.

Problem 1.1
Two numbers differ by 5 and have a product of 8. What are the two numbers?

Problem 1.2
Jeremy paid for his breakfest with 36 coins consisting of nickels and dimes. If the bill was \$3.50, then how many of each type of coin did he use?

Problem 1.3
The sum of three consecutive odd integers is 27. Find the three integers.

Problem 1.4
At an 8% sales tax rate, the sales tax Peter's new Ford Taurus was \$1,200. What was the price of the car?

Problem 1.5
After getting a 20% discount, Robert paid \$320 for a Pioneer CD player for his car. What was the original price of the CD?

Problem 1.6
The length of a rectangular piece of property is 1 foot less than twice the width. The perimeter of the property is 748 feet. Find the length and the width.

Problem 1.7
Sarah is selling her house through a real estate agent whose commission rate is 7%. What should the selling price be so that Sarah can get the \$83,700 she needs to pay off the mortgage?

Problem 1.8
Ralph got a 12% discount when he bought his new 1999 Corvette Coupe. If the amount of his discount was \$4,584, then what was the original price?

Problem 1.9
Julia framed an oil painting that her uncle gave her. The painting was 4 inches longer than it was wide, and it took 176 inches of frame molding. What were the dimensions of the picture?

Problem 1.10
If the perimeter of a tennis court is 228 feet and the length is 6 feet longer than twice the width, then what are the length and the width?

2 Problem-Solving Strategies

Strategies are tools that might be used to discover or construct the means to achieve a goal. They are essential parts of the "devising a plan step", the second step of Pólya's procedure which is considered the most difficult step. Elementary school children now learn strategies that they can use to solve a variety of problems. In this section, we discuss three strategies of problem solving: guessing and checking, using a variable, and drawing a picture or a diagram.

• **Problem-Solving Strategy 1: Guess and Check**
The guessing-and-checking strategy requires you to start by making a guess and then checking how close your answer is. Next, on the basis of this result, you revise your guess and try again. This strategy can be regarded as a form of trial and error, where the information about the error helps us choose what trial to make next.
This strategy may be appropriate when: there is a limited number of possible answers to test; you want to gain a better understanding of the problem; you can systematically try possible answers.
This strategy is often used when a student does not know how to solve a problem more efficiently of if the student does not have the tools to solve the problem in a facter way.

Example 2.1
In Figure 2.1 the numbers in the big circles are found by adding the numbers in the two adjacent smaller circles. Complete the second diagram so that the pattern holds.

Solution

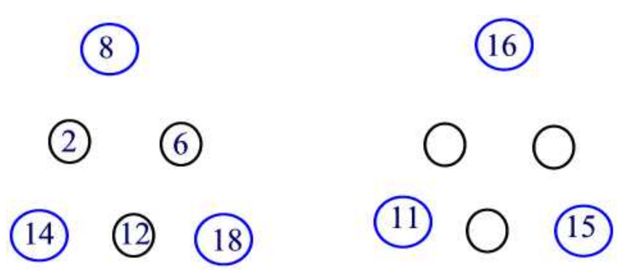


Figure 2.1

Understand the problem

In this example, we must find three numbers a, b, and c such that

$$\begin{aligned} a+b &= 16, \\ a+c &= 11, \\ b+c &= 15. \end{aligned}$$

See Figure 2.2.

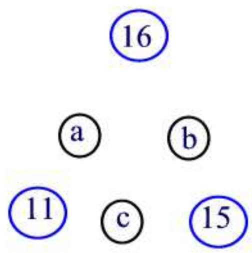


Figure 2.2

Devise a plan

We will try the *guess and check* strategy.

Carry out the plan

We start by guessing a value for a. Suppose $a = 10$. Since $a + b = 16$ then $b = 6$. Since $b + c = 15$ then $c = 9$. But then $a + c$ is 19 instead of 11 as it is supposed to be. This does not check.

Since the value of $a = 10$ yields a large $a + c$ then we will reduce our guess for a. Take $a = 5$. As above, we find $b = 11$ and $c = 4$. This gives $a + c = 9$ which is closer to 11 than 19. So our next guess is $a = 6$. This implies that $b = 10$ and $c = 5$. Now $a + c = 11$ as desired. See Figure 2.3.

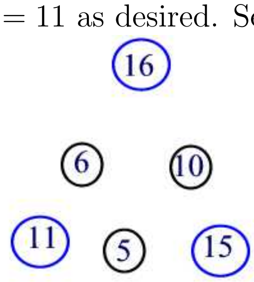


Figure 2.3

Look back

Is there an easier solution? Looking carefully at the initial example and the completed solution to the problem we notice that if we divide the sum of the numbers in the larger circles by 2 we obtain the sum of the numbers in the smaller circle. From this we can devise an easier solution. Looking at Figure 2.1, and the above discussion we have $a+b+c=21$ and $a+b=16$. This gives, $16+c=21$ or $c=5$. Since $a+c=11$ then $a=6$. Finally, since $b+c=15$ then $b=10$. ■

Example 2.2

Leah has \$4.05 in dimes and quarters. If she has 5 more quarters than dimes, how many of each does she have?

Solution.

Understand the problem

What are we asked to determine? We need to find how many dimes and how many quarters Leah has.
What is the total amount of money? \$4.05.
What else do we know? There are five more quarters than dimes.

Devise a plan

Pick a number, try it, and adjust the estimate.

Carry out the plan

Try 5 dimes. That would mean 10 quarters.

$$5 \times \$0.10 + 10 \times \$0.25 = \$3.00.$$

Increase the number of dimes to 7.

$$7 \times \$0.10 + 12 \times \$0.25 = \$3.70.$$

Try again. This time use 8 dimes.

$$8 \times \$0.10 + 13 \times \$0.25 = \$4.05$$

Leah has 8 dimes and 13 quarters.

Look back
Did we answer the question asked, and does our answer seem reasonable? Yes.■

Practice Problems

Problem 2.1
Susan made \$2.80 at her lemonade stand. She has 18 coins. What combination of coins does she have?

Problem 2.2
A rectangular garden is 4 feet longer than it is wide. Along the edge of the garden on all sides, there is a 2-foot gravel path. How wide is the garden if the perimeter of the garden is 28 feet? (Hint: Draw a diagram and use the guess and check strategy.)

Problem 2.3
There are two two-digit numbers that satisfy the following conditions:

(1) Each number has the same digits,
(2) the sum of digits in each number is 10,
(3) the difference between the two numbers is 54.
What are the two numbers?

Understanding the problem
The numbers 58 and 85 are two-digit numbers which have the same digits, and the sum of the digits is 13. Find two two-digit numbers such that the sum of the digits is 10 and both numbers have the same digits.
Devise a plan
Since there are only nine two-digit numbers whose digits have a sum of 10, the problem can be easily solved by guessing. What is the difference of your two two-digit numbers from part (a)? If this difference is not 54, it can provide information about your next guess.
Carry out the plan
Continue to guess and check. Which numbers has a difference of 54?
Looking back
This problem can be extended by changing the requirement that the sum of the two digits equal 10. Solve the problem for the case in which the digits have a sum of 12.

Problem 2.4
John is thinking of a number. If you divide it by 2 and add 16, you get 28. What number is John thinking of?

Problem 2.5
Place the digits 1, 2, 3, 4, 5, 6 in the circles in Figure 2.4 so that the sum of the three numbers on each side of the triangle is 12.

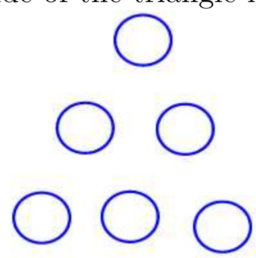


Figure 2.4

Problem 2.6
Carmela opened her piggy bank and found she had \$15.30. If she had only nickels, dimes, quarters, and half-dollars and an equal number of coins of each kind, how many coins in all did she have?

Problem 2.7
When two numbers are multiplied, their product is 759; but when one is subtracted from the other, their difference is 10. What are those two numbers?

Problem 2.8
Sandy bought 18 pieces of fruit (oranges and grapefruits), which cost \$4.62. If an orange costs \$0.19 and a grapefruit costs \$0.29, how many of each did she buy?

Problem 2.9
A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter," I count 24 heads and 80 feet. How many pigs and how many chickens are out there?"

Problem 2.10
At a benefit concert 600 tickets were sold and \$1,500 was raised. If there were \$2 and \$5 tickets, how many of each were sold?

Problem 2.11
At a bicycle store, there were a bunch of bicycles and tricycles. If there are 32 seats and 72 wheels, how many bicyles and how many tricycles were there?

Problem 2.12
If you have a bunch of 10 cents and 5 cents stamps, and you know that there are 20 stamps and their total value is $1.50, how many of each do you have?

• **Problem-Solving Strategy 2: Use a variable**
Often a problem requires that a number be determined. Represent the number by a variable, and use the conditions of the problem to set up an equation that can be solved to ascertain the desired number.
This strategy is most appropriate when: a problem suggests an equation; there is an unknown quantity related to known quantities; you are trying to develop a general formula.

Example 2.3
Find the sum of the whole numbers from 1 to 1000.

Solution.
Understand the problem
We understand that we are to find the sum of the first 1000 nonzero whole numbers.

Devise a plan
We will apply the use of variable strategy. Let s denote the sum, i.e.

$$s = 1 + 2 + 3 + \cdots + 1001 \tag{1}$$

Carry out the plan
Rewrite the sum in s in reverse order to obtain

$$s = 1000 + 999 + 998 + \cdots + 1 \tag{2}$$

Adding (1) - (2) to obtain

$$2s = 1001 + 1001 + 1001 + \cdots + 1001 = 1000 \times 1001.$$

Dividing both sides by 2 to obtain

$$s = \frac{1000 \times 1001}{2} = 500500.$$

Look back

Is it true that the process above apply to the sum of the first n whole integers? The answer is yes.■

Example 2.4

Lindsey has a total of \$82.00, consisting of an equal number of pennies, nickels, dimes and quarters. How many coins does she have in all?

Solution.

Understand the problem

We want to know how many coins Lindsey has.

How much money does she have total? \$82.00. How many of each coin does she have? We don't know exactly, but we know that she has an equal number of each coin.

Devise a plan

We know how much each coin is worth, and we know how much all of her coins are worth total, so we can write an equation that models the situation.

Carry out the plan

Let p be the number of pennies, n the number of nickels, d the number of dimes, and q the number of quarters. We then have the equation

$$p + 5n + 10d + 25q = 8200.$$

We know that she has an equal number of each coin, so $p = n = d = q$. Substituting p for the other variables gives an equation in just one variable. The equation above becomes $p + 5p + 10p + 25p = 41p = 8200$, so $p = 200$. Lindsey has 200 pennies. Since she has an equal number of each coin, she also has 200 nickels, 200 dimes and 200 quarters. Therefore, she has 800 coins.

Look back

Did we answer the question asked? Yes.

Does our answer seem reasonable? Yes, we know the answer must be less than 8200 (the number of coins if they were all pennies) and greater than

328 (the number of coins if they were all quarters).■

Practice Problems

Problem 2.13
A dog's weight is 10 kilograms plus half its weight. How much does the dog weigh?

Problem 2.14
The measure of the largest angle of a triangle is nine times the measure of the smallest angle. The measure of the third angle is equal to the difference of the largest and the smallest. What are the measures of the angles?(Recall that the sum of the measures of the angles in a triangle is $180°$)

Problem 2.15
The distance around a tennis court is 228 feet. If the length of the court is 6 feet more than twice the width, find the dimensions of the tennis court.

Problem 2.16
The floor of a square room is covered with square tiles. Walking diagonally across the room from corner to corner, Susan counted a total of 33 tiles on the two diagonals. What is the total number of tiles covering the floor of the room?

Problem 2.17
In three years, Chad will be three times my present age. I will then be half as old as he. How old am I now?

Problem 2.18
A fish is 30 inches long. The head is as long as the tail. If the head was twice as long and the tail was its present, the body would be 18 inches long. How long is each portion of the fish?

Problem 2.19
Two numbers differ by 5 and have a product of 8. What are the two numbers?

Problem 2.20
Jeremy paid for his breakfest with 36 coins consisting of nickels and dimes. If the bill was \$3.50, then how many of each type of coin did he use?

Problem 2.21
The sum of three consecutive odd integers is 27. Find the three integers.

Problem 2.22
At an 8% sales tax rate, the sales tax Peter's new Ford Taurus was \$1,200. What was the price of the car?

Problem 2.23
After getting a 20% discount, Robert paid \$320 for a Pioneer CD player for his car. What was the original price of the car?

Problem 2.24
The length of a rectangular piece of property is 1 foot less than twice the width. The perimeter of the property is 748 feet. Find the length and the width.

Problem 2.25
Sarah is selling her house through a real estate agent whose commission rate is 7%. What should the selling price be so that Sarah can get the \$83,700 she needs to pay off the mortgage?

Problem 2.26
Ralph got a 12% discount when he bought his new 1999 Corvette Coupe. If the amount of his discount was \$4,584, then what was the original price?

Problem 2.27
Julia framed an oil painting that her uncle gave her. The painting was 4 inches longer than it was wide, and it took 176 inches of frame molding. What were the dimensions of the picture?

Problem 2.28
If the perimeter of a tennis court is 228 feet and the length is 6 feet longer than twice the width, then what are the length and the width?

• **Problem-Solving Strategy 3: Draw a Picture**
It has been said that a picture worth a thousand words. This is particularly true in problem solving. Drawing a picture often provides the insight necessary to solve a problem.
This strategy may be appropriate when: a physical situation is involved; geometric figures or measurements are involved; a visual representation of the problem is possible.

Example 2.5
Can you cut a pizza into 11 pieces with four straight cuts?

Solution.
Understand the problem
The pieces need not have the same size and shape.

Devise a plan
If we try to cut the pizza with four cuts using the usual standard way we will end with a total of 8 equally shaped slices. See Figure 2.5.

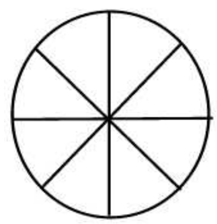


Figure 2.5

Carry out the plan
The cuts are made as shown in Figure 2.6.

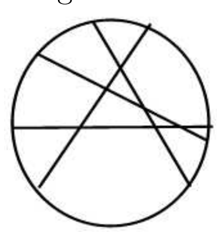


Figure 2.6

Look back
The above is not the only way to cut the pizza. There are many other ways.■

Example 2.6
In a stock car race the first five finishers in some order were a Ford, a Pontiac, a Chevrolet, a Buick, and a Dodge.

(a) The Ford finished seven seconds before the Chevrolet.
(b) The Pontiac finished six seconds after the Buick.

(c) The Dodge finished eight seconds after the Buick.
(d) The Chevrolet finished two seconds before the Pontiac.

In what order did the cars finish the race?

Solution.
Understand the problem
We are told to determine the order in which the five cars finished the race.

Devise a plan
We draw a line to represent the track at the finish of the race and place the cars on it according to the conditions of the problem. Mark the line off in time intervals of one second. We use the first letter of each car's name to represent the car. So the question is to order the letters B, C, D, F, and P on the line according to the given information.

Carry the plan
The finishing position of each of the five cars is given in Figure 2.7.

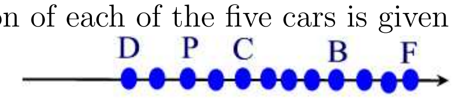


Figure 2.7

Look back
We see that pictures can help to solve problems.■

Practice Problems

Problem 2.29
Bob can cut through a log in one minute. How long will it take Bob to cut a 20-foot log into 2-foot sections?

Problem 2.30
How many posts does it take to support a straight fence 200 feet long if a post is placed every 20 feet?

Problem 2.31
Albright, Badgett, Chalmers, Dawkins, and Earl all entered the primary to seek election to the city council. Albright received 2000 more votes than

Badgett and 4000 fewer than Chalmers. Earl received 2000 votes fewer than Dawkins and 5000 votes more than Badgett. In what order did each person finish in the balloting?

Problem 2.32
A 9-meter by 12-meter rectangular lawn has a concrete walk 1 meter wide all around it outside lawn. What is the area of the walk?

Problem 2.33
An elevator stopped at the middle floor of a building. It then moved up 4 floors and stopped. It then moved down 6 floors, and then moved up 10 floors and stopped. The elevator was now 3 floors from the top floor. How many floor does the building have?

Problem 2.34
In the Falkland Islands, south of Argentina, Armado, a sheepherder's son, is helping his father build a rectangular pen to keep their sheep from getting lost. The pen will be 24 meters long, 20 meters wide, and have a fence posts 4 meters apart. How many fence posts do they need?

Problem 2.35
Five people enter a racquetball tournment in which each person must play every other person exactly once. Determine the total number of games that will be played.

Problem 2.36
When two pieces of ropes are placed end to end, their combined length is 130 feet. When the two pieces are placed side by side, one is 26 feet longer than the other. What are the lengths of the two pieces?

Problem 2.37
There are 560 third- and fourth-grade students in Russellville elementary school. If there are 80 more third graders than fourth graders, how many third graders are there in the school?

Problem 2.38
A well is 20 feet deep. A snail at the bottom climbs up 4 feet every day and slips back 2 feet each night. How many days will it take the snail to reach the top of the well?

Problem 2.39

Five friends were sitting on one side of a table. Gary set next to Bill. Mike sat next to Tom. Howard sat in the third seat from Bill. Gary sat in the third seat from Mike. Who sat on the other side of Tom?

3 More Problem-Solving Strategies

In this section we present three additional problem-solving strategies: look for a pattern, solve simpler problems, and make a list or a table.

• **Problem-Solving Strategy 3: Look for a pattern**
Lists several specific instances of a problem and then look to see whether a pattern emerges that suggests a solution to the entire problem.

Example 3.1
Use the pattern below to find the product 63×67.

$$\begin{array}{rcrcr} 3 & \times & 7 & = & 21 \\ 13 & \times & 17 & = & 221 \\ 23 & \times & 27 & = & 621 \\ 33 & \times & 37 & = & 1221 \end{array}$$

Solution.
Understand the problem
We need to find the value of the product 63×67 by observing the given products.

Devise a plan
Looking at the product above, observe the patterns in the factors. In each successive product, each factor is increased by 10. To find the product of 63 and 67, extend the pattern.

Carry out the plan
Now look at the products. Each product has 21 as the last two digits. The digits before 21 follow the pattern 0, 2, 6, 12. Take a close look at this pattern and extend it. See Figure 3.1.

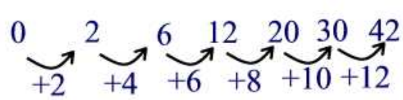


Figure 3.1

With this extension, you know the first two digits of each product. You already know the last two digits are 21. Extend the pattern.

$$\begin{array}{ccccc} 43 & \times & 47 & = & 2021 \\ 53 & \times & 57 & = & 3021 \\ 63 & \times & 67 & = & 4221 \end{array}$$

Look back

Is there another way to describe the pattern above? ■

Example 3.2

Laura was given an ant farm by her grandparents for her 13th birthday. The farm could hold a total of 100,000 ants. Laura's farm had 1500 ants when it was given to her. If the number of ants in the farm on the day after her birthday was 3000 and the number of ants the day after that was 6000, in how many days will the farm be full?

Solution.

Understand the problem

We need to know when the ant farm will be full.
How many ants will the farm hold? 100,000.
How many ants are in the farm the first day? 1500.
How many ants are in the farm the second day? 3000.
How many ants are in the farm the third day? 6000.

Devise a plan

Is a pattern developing? Yes, each day twice as many ants are in the farm as the day before. Make a table to count the ants systematically.

Carry out the plan

Draw a table with two lines for numbers.

The top line is the number of days after Laura's birthday, and the bottom line is the number of ants in the farm on that day.

#days	0	1	2	3	4	5	6	7
#ants	1500	3000	6000	12,000	24,000	48,000	96,000	192,000

The ant farm will be full seven days after her birthday.

Look back

Read the question again. Did we answer all of the questions? Yes. Does our answer seem reasonable? Yes. What assumption are we making? We are assuming that the pattern-the number of ants doubles each day-continues indefinitely.■

Practice Problems

Problem 3.1

Sequences like $2, 5, 8, 11, \cdots$, where each term is the previous term increased by a constant, are called **arithmetic sequences**. Compute the sum of the following arithmetic sequence

$$1 + 7 + 13 + \cdots + 73.$$

Problem 3.2

Sequences like $1, 2, 4, 8, 16, \cdots$, where each term is the previous term multiplied by a constant, are called **geometric sequences**. Compute the sum of the following geometric sequence

$$1 + 2 + 4 + 8 + \cdots + 2^{100}.$$

Problem 3.3

(a) Fill in the blanks to continue this sequence of equations.

$$\begin{array}{ccc} 1 & = & 1 \\ 1+2+1 & = & 4 \\ 1+2+3+2+1 & = & 9 \\ \rule{2.5cm}{0.4pt} & = & \_ \\ \rule{2.5cm}{0.4pt} & = & \_ \end{array}$$

(b) Compute the sum

$$\begin{array}{ccl} 1 & + & 2+3+\cdots+99+100+99 \\ & + & \cdots+3+2+1 = \rule{1cm}{0.4pt} \end{array}$$

(c) Compute the sum

$$\begin{array}{c} 1+2+3+\cdots+(n-1) \\ +n+(n-1)+\cdots+3+2+1 = \rule{1cm}{0.4pt} \end{array}$$

Problem 3.4
Find a pattern in the designs. How many squares will there be in the eighth design of your pattern?See Figure 3.2.

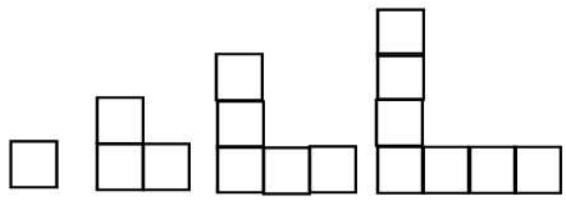


Figure 3.2

Problem 3.5
Find the sum of the first 100 even nonzero whole numbers.

Problem 3.6
James began writing a book. At the end of the first week, he'd written 10 pages. By the end of the second week, he'd written 6 more pages, for a total of 16 pages. At the end of the third week, he had a total of 23 pages and by the end of the fourth week he had 31 pages completed in his book. If he continues writing at this same rate, how many pages will his book have at the end of the seventh week?

Problem 3.7
Mary's five friends began an exercise group. They decided to walk along a trail each day. On the first day, they walked 2/3 of the trail. On the second day, they walked 3/5 of the trail. On the third day, they walked 4/7 and on the fourth day 5/9 of the trail. If this pattern continues, how far will Mary and her friends walk on the tenth day?

Problem 3.8
Patterns have been part of mathematics for a very long time. There are famous mathematicians who discovered patterns that are still used today. For example, Leonardo Fibonacci discovered the Fibonacci sequence. In this pattern, the first six numbers are: 1, 1, 2, 3, 5, 8. Work with a friend to find the next 5 numbers in this sequence. Write down the numbers that follow in the set and explain the pattern to your partner.

Problem 3.9
What is the units digit for 7^{3134}? (Hint: Work simpler problems to look for a pattern.)

Problem 3.10
Find these products: $7\times 9, 77\times 99, 777\times 999$. Predict the product for $77,777\times 99999$. What two numbers give a product of 77,762,223?

Problem 3.11
William is painting a design on a rug. He had time to paint a star, moon, sun, sun, moon, star, and moon before he had to quit. What shape will William paint next to finish the design?

Problem 3.12
Would you rather have \$100 a day for a month or \$1 on the first day and double it each day thereafter for a month?

• **Problem-Solving Strategy 5: Making a table or an organized list**
Making a table or a list is a way to organize data presented in a problem. This problem-solving strategy allows the problem solver to discover relationships and patterns among data.

Example 3.3
Customers at a particular yogurt shop may select one of three flavors of yogurt. They may choose one of four toppings. How many one-flavor, one-topping combinations are possible?

Solution.
Understand the problem
What question do we have to answer? How many flavor-topping combinations are possible?
How many flavors are available? Three.
How many toppings are available? Four.
Are you allowed to have more than one flavor or topping? No, the combinations must have only one flavor and one topping.

Devise a plan
How could we organize the possible combinations help? With letters and numbers in a list.

Carry out the plan

Make an organized list. Use F and T to denote either flavor or topping. Use the numbers 1-3 and 1-4 to mark different flavors and toppings.

$$\begin{array}{cccc} F1T1 & F1T2 & F1T3 & F1T4 \\ F2T1 & F2T2 & F2T3 & F2T4 \\ F3T1 & F3T2 & F3T3 & F3T4 \end{array}$$

Now count the number of combinations. There are 12 combinations possible.

Look back
Did we answer the question asked? Yes. ■

Example 3.4
Judy is taking pictures of Jim, Karen and Mike. She asks them, " How many different ways could you three children stand in a line?"

Solution.
Understand the problem
What do you need to know? You need to know that any of the students can be first, second or third.

Devise a plan
How can you solve the problem? You can make a list to help you find all the different ways. Choose one student to be first, and another to be second. The last one will be third.

Carry out the plan
When you make your list, you will notice that there are 2 ways for Jim to be first, 2 ways for Karen to be first and 2 ways for Mike to be first.

First	Second	Third
Jim	Karen	Mike
Jim	Mike	Karen
Karen	Jim	Mike
Karen	Mike	Jim
Mike	Karen	Jim
Mike	Jim	Karen

Look back
So, there are 6 ways that the children could stand in line. Does the answer make sense? Yes.■

Practice Problems

Problem 3.13
Take 25 marbles. Put them in 3 piles so an odd number is in each pile. How many ways can this be done?

Problem 3.14
A rectangle has an area of 120 square centimeters. Its length and width are whole numbers. What are the possibilities for the two numbers? Which possibility gives the smallest perimeter?

Problem 3.15
The product of two whole numbers is 96 and their sum is less than 30. What are possibilities for the two numbers?

Problem 3.16
Lonnie has a large supply of quarters , dimes, nickels, and pennies. In how many ways could she make change for 50 cents?

Problem 3.17
How many different four-digit numbers can be formed using the digits 1, 1, 9, and 9?

Problem 3.18
Which is greater : $5.00 or the total value of all combinations of three coins you can make using only pennies, nickels, dimes, and quarters?

Problem 3.19
The Coffee Hut sold 5 small cups of coffee at $.75 each, 7 medium cups of coffee at $1.25 each, and 12 large cups of coffee at $1.50 each. What were the total sales of The Coffee Hut?

Problem 3.20
Chris decided to use his birthday money to buy some candy at The Sweet Shop. He bought 7 pieces of bubble gum for $.35 each, 3 candy bars for $1.25 each, and 2 bags of jellybeans for $3.35 each. How much money did Chris spend at The Sweet Shop?

Problem 3.21
Sean and Brad were at the candy store. Together they had \$15 total. They saw Gummie Worms that were \$3 per pound, War Heads were \$2 per pound and Lollie Pops were \$1 per pound. How many different combinations of candy could they buy for \$15.00?

Problem 3.22
Doug has 2 pairs of pants: a black pair and a green pair. He has 4 shirts: a white shirt, a red shirt, a grey shirt, and a striped shirt. How many different outfits can he put together?

Problem 3.23
Ryan numbered his miniature race car collection according to the following rules:

1. It has to be a 3-digit number.
2. The digit in the hundreds place is less than 3.
3. The digit in the tens place is greater than 7.
4. The digit in the ones place is odd.

If Ryan used every possibility and each car had a different number, how many cars did Ryan have in his collection?

Problem 3.24
There will be 7 teams playing in the Maple Island Little League tournament. Each team is scheduled to play every other team once. How many games are scheduled for the tournament?

Problem 3.25
How many different total scores could you make if you hit the dartboard shown with three darts?See Figure 3.3.

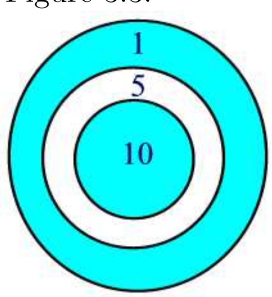


Figure 3.3

Problem 3.26
Sue and Ann earned the same amount of money, although one worked 6 days more than the other. If Sue earned \$36 per day and Ann earned \$60 per day, how many days did each work?

Problem 3.27
A bank has been charging a monthly service fee of \$2 per checking accounts plus 15 cents for each check announces that it will change its monthly fee to \$3 and that each check will cost 8 cents. The bank claims the new plan will save the customer money. How many checks must a customer write per month before the new plan is cheaper than the old plan?

Problem 3.28
Sasha and Francisco were selling lemonades for 25 cents per half cup and 50 cents per full cup. At the end of the day they had collected \$15 and had used 37 cups. How many full cups and how many half cups did they sell?

Problem 3.29
Harold wrote to 15 people, and the cost of postage was \$4.08. If it cost 20 cents to mail a postcard and 32 cents to mail a letter, how many postcards did he write?

Problem 3.30
I had some pennies, nickels, dimes, and quarters in my pocket. When I reached in and pulled out some change, I had less than 10 coins whose values was 42 cents. What are all the possibilities for the coins I had in my hand?

• **Problem-Solving Strategy 6: Solving a simpler problem**
Understanding a simple version of a problem often is the first step to understanding a whole lot more of it. So don't be scared of looking at a simple version of a problem, then gradually extending your investigation to the more complicated parts.

Example 3.5
A pie can be cut into seven pieces with three straight cuts. What is the largest number of pieces that can be made with eight straight cuts?

Solution.

Understand the problem

We want to find the largest number of pieces that can be made when using eight straight cuts.

Devise a plan

We will find the maximum number of pieces when using $1, 2, \cdots, 7$ cuts. This way we will see if there is a pattern and then use this pattern to figure out the number of pieces with eight straight cuts.

Carry out the plan

Figure 3.4 shows the maximum number of pieces when using 1, 2, 3, and 4 straight cuts.

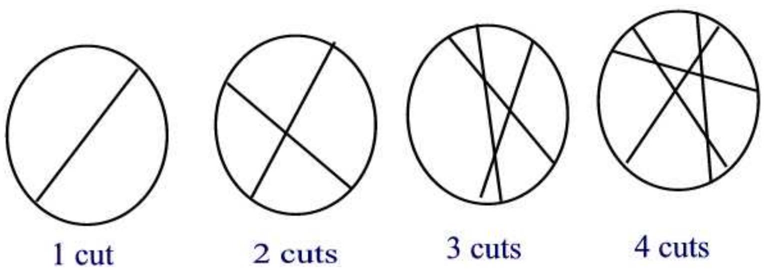


Figure 3.4

Thus, we can construct the following table

Cuts	1	2	3	4	5	6	7	8
# of pieces	2	4	7	11	16	22	29	37

Note that

$$\begin{aligned}
4 &= 2+2\\
7 &= 4+3\\
11 &= 7+4\\
16 &= 11+5\\
22 &= 16+6\\
29 &= 22+7\\
37 &= 29+8
\end{aligned}$$

Look back

The above pattern can be extended to any number of straight line cuts.■

Example 3.6
An exercise program requires you to do one sit-up the first day and double the number you do each day for seven days. How many sit-ups will you do on the sixth day?

Solution.
Understand the problem
We want to find out the number of sit ups you do on the sixth day according to the given information.

Devise a plan
We will find the number of sit ups for Days 1, 2, 3,4, and 5 and from that we will find the number of sit ups in the sixth day.

Carry out the plan

$$\begin{array}{ccccccc} Day & 1 & 2 & 3 & 4 & 5 & 6 \\ \#\ of\ sit-ups & 1 & 2 & 4 & 8 & 16 & 32 \end{array}$$

Look back
Does the answer make sense? Yes.■

Example 3.7
In a delicatessen, it costs \$2.49 for a half pound of sliced roast beef. The person behind the counter slices 0.53 pound. What should it cost?

Solution.
Try a simpler problem. How much would you pay if a half pound of sliced roast beef costs \$2 and the person slices 3 pounds? If a half pound costs \$2, then one pound would cost $2 \times \$2 = \$4$. Multiply by the number of pounds needed to get the total: $3 \times \$4 = \$12$.
Now try the original problem: If a half pound costs \$2.49, then one pound would cost $2 \times \$2.49$ or \$4.98. Multiply by the number of pounds needed to get the total: $0.53 \times \$4.98 = \$2.6394$ or \$2.64. ■

Practice Problems

Problem 3.31
There are 32 schools that participated in a statewide trivia tournament. In

each round, one school played one match against another school and the winner continued on until 1 school remained. How many total matches were played?

Problem 3.32
A.J. plays baseball. There are 7 teams in his league. For the baseball season, each team plays each of the other teams twice. How many games are in a season?

Problem 3.33
A total of 28 handshakes were exchanged at a party. Each person shook hands exactly once with each of the others. How many people were present at the party?

Problem 3.34
Mike is paid for writing numbers on pages of a book. Since different pages require different numbers of digits, Mike is paid for writing each digit. In his last book, he wrote 642 digits. How many pages were in the book?

Problem 3.35
A restaurant has 45 small square tables. Each table can seat only one person on each side. If the 45 tables are placed together to make one long table, how many people can sit there?

Problem 3.36
Drewby the goat loves green. Everything he has is green. He just built a brick wall and he's going to paint it green. The wall has 14 bricks across and is eleven bricks high. He is going to paint the front and back walls, and the sides that you can see. He is not going to paint the sides that touch one another. How many sides will Drewby paint?

Problem 3.37
Three shapes-a circle, a rectangle, and a square-have the same area. Which shape has the smallest perimeter?

Problem 3.38
How many palindromes are there between 0 and 1000? (A palindrome is a number like 525 that reads the same backward or forward.)

Problem 3.39
Tony's restaurant has 30 small tables to be used for a banquet. Each table can seat only one person on each side. If the tables are pushed together to make one long table, how many people can sit at the table?

4 Sets and Operations on Sets

The languages of set theory and basic set operations clarify and unify many mathematical concepts and are useful for teachers in understanding the mathematics covered in elementary school. Sets and relations between sets form a basis to teach children the concept of whole numbers. In this section, we introduce some of the basic concepts of sets and their operations.

Sets

In everyday life we often group objects to make things more managable. For example, files of the same type can be put in the same folder, all clothes in the same closet, etc. This idea has proved very convenient and fruitful in mathematics.

A **set** is a collection of objects called **members** or **elements**. For example, all letters of the English alphabet form a set whose elements are all letters of the English alphabet. We will use capital letters for sets and lower case letters for elements.

There are three ways to define a set:

•**verbal description:** $A = \{\text{all letters of the English alphabet}\}$

• **Roster notation or Listing in braces:**

$$A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

•**Set-builder notation:** $A = \{\text{x}|\text{x is a letter of the English alphabet}\}$.

In the last case a typical element of A is described. We read it as "A is the set of all x such that x is a letter of the English alphabet." The symbol "|" reads as "such that."

Example 4.1

(a) Write the set $\{2, 4, 6, \cdots\}$ using set-builder notation.

(b) Write the set $\{2n - 1|n \in \mathbb{N}\}$ by listing its elements. $\mathbb{N}$ is the set of **natural numbers** whose elements consists of the numbers $1, 2, 3, \cdots$.

Solution.

(a) $\{2, 4, 6, \cdots\} = \{2n|n \in \mathbb{N}\}$.

(b) $\{2n - 1|n \in \mathbb{N}\} = \{1, 3, 5, 7, \cdots\}$.■

Members of a set are listed without repetition and their order in the list is immaterial. Thus, the set $\{a, a, b\}$ would be written as $\{a, b\}$ and $\{a, b\} = \{b, a\}$.
Membership is symbolized by $\in$. If an element does not belong to a set then we use the symbol $\notin$. For example, if $\mathbb{N}$ is the set of natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, \cdots\}$ where the ellipsis "$\cdots$" indicates "and so on", then $15 \in \mathbb{N}$ whereas $-2 \notin \mathbb{N}$.
The set with no elements is called the **empty set** and is denoted by either $\{\}$ or the Danish letter $\emptyset$. For example, $\{x \in \mathbb{N} | x^2 = 2\} = \emptyset$.

Example 4.2
Indicate which symbol, $\in$ or $\notin$, makes each of the following statements true:
(a) 0_____$\emptyset$
(b) $\{1\}$_____$\{1, 2\}$
(c) $\emptyset$_____$\emptyset$
(d) $\{1, 2\}$_____$\{1, 2\}$
(e) 1024_____$\{2^n | n \in \mathbb{N}\}$
(f) 3002_____$\{3n - 1 | n \in \mathbb{N}\}$.

Solution.
(a) $0 \notin \emptyset$
(b) $\{1\} \notin \{1, 2\}$
(c) $\emptyset \notin \emptyset$
(d) $\{1, 2\} \notin \{1, 2\}$
(e) $1024 \in \{2^n | n \in \mathbb{N}\}$ since $1024 = 2^{10}$.
(f) $3002 \in \{3n - 1 | n \in \mathbb{N}\}$ since $3002 = 3 \times 1001 - 1$.$\blacksquare$

Two sets A and B are **equal** if they have the same elements. We write $A = B$. If A does not equal B we write $A \neq B$. This occurs, if there is an element in A not in B or an element in B not in A. For example, $\{x | x \in \mathbb{N}, 1 \leq x \leq 5\} = \{1, 2, 3, 4, 5\}$ whereas $\{1, 2\} \neq \{2, 4\}$.

Example 4.3
Which of the following represent equal sets?

$$\begin{array}{rclrcl} A & = & \{orange, apple\} & B & = & \{apple, orange\} \\ C & = & \{1,2\} & D & = & \{1,2,3\} \\ E & = & \{\} & F & = & \emptyset \\ G & = & \{a,b,c,d\} & & & \end{array}$$

Solution.
$A = B$ and $E = F$.■

If A and B are sets such that every element of A is also an element of B, then we say A is a **subset** of B and we write $A \subseteq B$. Every set A is a subset of itself. A subset of A which is not equal to B is called **proper subset.** We write $A \subset B$. For example, the set $\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$. Any set is a subset of itself, but not a proper subset.

Example 4.4
Given $A = \{1,2,3,4,5\}, B = \{1,3\}, C = \{2^n - 1 | n \in \mathbb{N}\}$.

(a) Which sets are subsets of each other?
(b) Which sets are proper subsets of each other?

Solution.
(a) $A \subseteq A, B \subseteq B, C \subseteq C, B \subseteq C$, and $B \subseteq A$.
(b) $B \subset A$ and $B \subset C$.■

Relationships between sets can be visualized using **Venn diagrams**. Sets are represented by circles included in a rectangle that represents the **universal set**, i.e., the set of all elements being considered in a particular discussion. For example, Figure 4.1 displays the Venn diagram of the relation $A \subseteq B$.

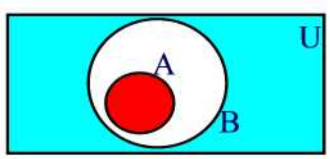


Figure 4.1

Example 4.5
Suppose M is the set of all students taking mathematics and E is the set of all students taking English. Identify the students described by each region in Figure 4.2

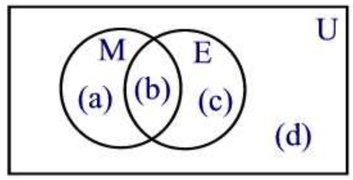


Figure 4.2

Solution.
Region (a) contains all students taking mathematics but not English.
Region (b) contains all students taking both mathematics and English.
Region (c) contains all students taking English but not mathematics.
Region (d) contains all students taking neither mathematics nor English.■

Practice Problems

Problem 4.1
Write a verbal description of each set.
(a) $\{4, 8, 12, 16, \cdots\}$
(b) $\{3, 13, 23, 33, \cdots\}$

Problem 4.2
Which of the following would be an empty set?
(a) The set of purple crows.
(b) The set of odd numbers that are divisible by 2.

Problem 4.3
What two symbols are used to represent an empty set?

Problem 4.4
Each set below is taken from the universe $\mathbb{N}$ of counting numbers, and has been described either in words, by listing in braces, or with set-builder notation. Provide the two remaining types of description for each set.
(a) The set of counting numbers greater than 12 and less than 17
(b) $\{x | x = 2n \textit{ and } n = 1, 2, 3, 4, 5\}$
(c) $\{3, 6, 9, 12, \cdots\}$

Problem 4.5
Rewrite the following using mathematical symbols:

(a) P is equal to the set whose elements are a, b, c, and d.
(b) The set consisting of the elements 1 and 2 is a proper subset of $\{1, 2, 3, 4\}$.
(c) The set consisting of the elements 0 and 1 is not a subset of $\{1, 2, 3, 4\}$.
(d) 0 is not an element of the empty set.
(e) The set whose only element is 0 is not equal to the empty set.

Problem 4.6
Which of the following represent equal sets?

$$\begin{array}{rcccrcr} A & = & \{a, b, c, d\} & & B & = & \{x, y, z, w\} \\ C & = & \{c, d, a, b\} & & D & = & \{x \in \mathbb{N} | 1 \leq x \leq 4\} \\ E & = & \emptyset & & F & = & \{\emptyset\} \\ G & = & \{0\} & & H & = & \{\} \\ I & = & \{2n + 1 | n \in W\} & where & W & = & \{0, 1, 2, 3, \cdots\} \\ J & = & \{2n - 1 | n \in \mathbb{N}\} & & & & \end{array}$$

Problem 4.7
In a survey of 110 college freshmen that investigated their high school backgrounds, the following information was gathered:
25 students took physics
45 took biology
48 took mathematics
10 took physics and mathematics
8 took biology and mathematics
6 took physics and biology
5 took all 3 subjects.

(a) How many students took biology but neither physics nor mathematics?
(b) How many students took biology, physics or mathematics?
(c) How many did not take any of the 3 subjects?

Problem 4.8
Twenty-four dogs are in a kennel. Twelve of the dogs are black, six of the dogs have short tails, and fifteen of the dogs have long hair. There is only one dog that is black with a short tail and long hair. Two of the dogs are black with short tails and do not have long hair. Two of the dogs have short

tails and long hair but are not black. If all of the dogs in the kennel have at least one of the mentioned characteristics, how many dogs are black with long hair but do not have short tails?Hint: Use Venn diagram.

Problem 4.9
True or false?
(a) $7 \in \{6, 7, 8, 9\}$ (b) $\frac{2}{3} \in \{1, 2, 3\}$
(c) $5 \notin \{2, 3, 4, 6\}$ (d) $\{1, 2, 3\} \subseteq \{1, 2, 3\}$
(e) $\{1, 2, 5\} \subset \{1, 2, 5\}$ (f) $\emptyset \subseteq \{\}$
(g) $\{2\} \not\subseteq \{1, 2\}$ (h) $\{1, 2\} \not\subseteq \{2\}$.

Problem 4.10
Which of the following sets are equal?

(a) $\{5, 6\}$ (b) $\{5, 4, 6\}$
(c) Whole numbers greater than 3
(d) Whole numbers less than 7
(e) Whole numbers greater than 3 or less than 7
(f) Whole numbers greater than 3 and less than 8
(g) $\{e, f, g\}$
(h) $\{4, 5, 6, 5\}$

Problem 4.11
Let $A = \{1, 2, 3, 4, 5\}, B = \{3, 4, 5\}$, and $C = \{4, 5, 6\}$. In the following insert $\in, \notin, \subseteq$, or $\not\subseteq$ to make a true statement.

(a) 2_____A (b) B_____A (c)C_____B (d) 6_____C.

Problem 4.12
Rewrite the following expressions using symbols.
(a) A is a subset of B.
(b) The number 2 is not an element of set T.

Set Operations

Sets can be combined in a number of different ways to produce another set. Here four basic operations are introduced and their properties are discussed.

The **union** of sets A and B, denoted by $A \cup B$, is the set consisting of all

elements belonging either to A or to B (or to both). The union of A and B is displayed in Figure 4.3(a). For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$ then $A \cup B = \{1, 2, 3, 4, 5\}$. Note that elements are not repeated in a set.
The **intersection** of sets A and B, denoted by $A \cap B$, is the set of all elements belonging to both A and B. The intersection of A and B is displayed in Figure 4.3 (b). For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$ then $A \cap B = \{2, 3\}$. If $A \cap B = \emptyset$ then we call the sets A and B **disjoint sets.** Figure 4.3(c) shows the two disjoint sets A and B. For example, $\{a, b\} \cap \{c, d\} = \emptyset$.

Example 4.6
Let $A = \{0, 2, 4, 6, \cdots\}$ and $B = \{1, 3, 5, 7, \cdots\}$. Find $A \cup B$ and $A \cap B$.

Solution.
$A \cup B = W$, where W is the set of whole numbers.
$A \cap B = \emptyset.\blacksquare$

The **difference** of sets A from B , denoted by $A - B$, is the set defined as

$$A - B = \{x | x \in A \ and \ x \notin B\}.$$

This set is displayed in Figure 4.3 (d). For example, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A - B = \{1\}$ and $B - A = \{4\}$. Note that in general $A - B \neq B - A$.

Example 4.7
If $U = \{a, b, c, d, e, f, g\}$, $A = \{d, e, f\}$, $B = \{a, b, c, d, e, f\}$, and $C = \{a, b, c\}$, find each of the following:
(a) $A - B$ (b) $B - A$ (c) $B - C$ (d) $C - B$.

Solution.
(a) $A - B = \emptyset$.
(b) $B - A = \{a, b, c\}$.
(c) $B - C = \{d, e, f\}$.
(d) $C - B = \emptyset.\blacksquare$

For a set A, the difference $U - A$, where U is the universe, is called the **complement** of A and it is denoted by $\overline{A}$. Thus, $\overline{A}$ is the set of everything that is not in A. Figure 4.3(e) displays the Venn diagram of $\overline{A}$.

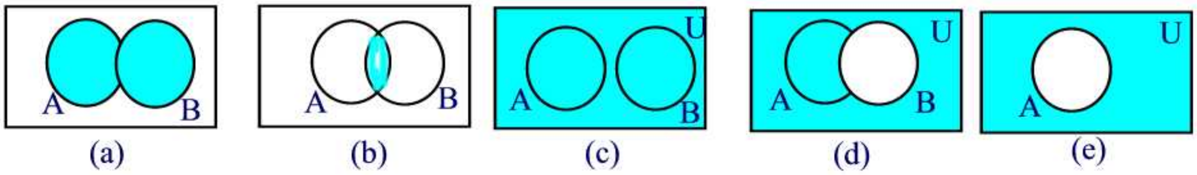


Figure 4.3

Example 4.8
(a) If $U = \{a, b, c, d\}$ and $A = \{c, d\}$, find $\overline{A}, \overline{U}, \overline{\emptyset}$.
(b) If $U = \mathbb{N}, A = \{2, 3, 6, 8, \cdots\}$, find $\overline{A}$.

Solution.
(a) $\overline{A} = \{a, b\}, \overline{U} = \emptyset, \overline{\emptyset} = U$.
(b) $\overline{A} = \{1, 3, 5, 7, \cdots\} = \{2n - 1 | n \in \mathbb{N}\}$.■

The fourth set operation is the Cartesian product. We first define an ordered pair and Cartesian product of two sets using it. By an **ordered pair** (a, b) we mean the set $\{\{a\}, \{a, b\}\}$. Note that $(a, b) = (c, d)$ if and only if $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ and this is equivalent to $\{a\} = \{c\}$ and $\{a, b\} = \{c, d\}$. Hence, $a = c$ and $b = d$.
The set of all ordered pairs (a, b), where a is an element of A and b is an element of B, is called the **Cartesian product** of A and B and is denoted by $A \times B$. For example, if $A = \{1, 2, 3\}$ and $B = \{a, b\}$ then

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

and

$$B \times A = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}.$$

Note that in general $A \times B \neq B \times A$.

Example 4.9
If $A = \{a, b, c\}, B = \{1, 2, 3\}$, find each of the following:
(a) $A \times B$ (b) $B \times A$ (c) $A \times A$.

Solution.
(a) $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$.
(b) $B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$.

(c) $A \times A = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}.\blacksquare$

Practice Problems

Problem 4.13
Draw Venn diagrams that represent sets A and B as described as follows:
(a) $A \subset B$ (b) $A \cap B = \emptyset$ (c) $A \cap B \neq \emptyset$.

Problem 4.14
Let $U = \{p,q,r,s,t,u,v,w,x,y\}$ be the universe, and let $A = \{p,q,r\}, B = \{q,r,s,t,u\}$, and $C = \{r,u,w,y\}$. Locate all 10 elements of U in a three-loop Venn diagram, and then find the following sets:
(a) $A \cup C$ (b) $A \cap C$ (c) $\overline{B}$ (d) $A \cup \overline{B}$ (e) $A \cap \overline{C}$.

Problem 4.15
If S is a subet of universe U, find each of the following:
(a) $S \cup \overline{S}$ (b) $\emptyset \cup S$ (c) $\overline{U}$ (d) $\overline{\emptyset}$ (e) $S \cap \overline{S}$.

Problem 4.16
Answer each of the following:
(a) If A has five elements and B has four elements, how many elements are in $A \times B$?
(b) If A has m elements and B has n elements, how many elements are in $A \times B$?

Problem 4.17
Find A and B given that

$$A \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}.$$

Problem 4.18
Let $A = \{x,y\}, B = \{a,b,c\}$, and $C = \{0\}$. Find each of the following:
(a) $A \times B$
(b) $B \times \emptyset$
(c) $(A \cup B) \times C$
(d) $A \cup (B \times C)$.

Problem 4.19
For each of the following conditions, find $A - B$:

(a) $A \cap B = \emptyset$ (b) $B = U$ (c) $A = B$ (d) $A \subseteq B$.

Problem 4.20
If $B \subseteq A$, find a simpler expression for each of the following:

(a) $A \cap B$ (b) $A \cup B$ (c) $B - A$ (d) $B \cap \overline{A}$.

Problem 4.21
Use a Venn diagram to decide whether the following pairs of sets are equal.

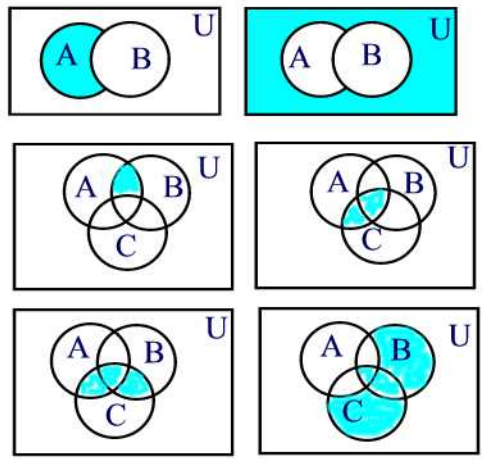
(a) $A \cap B$ and $B \cap A$
(b) $A \cup B$ and $B \cup A$
(c) $A \cap (B \cap C)$ and $(A \cap B) \cap C$
(d) $A \cup (B \cup C)$ and $(A \cup B) \cup C$
(e) $A \cup \emptyset$ and A
(f) $A \cup A$ and $A \cup \emptyset$.

Problem 4.22
In a survey of 6500 people, 5100 had a car, 2280 had a pet, 5420 had a television set, 4800 had a TV and a car, 1500 had a TV and a pet, 1250 had a car and a pet, and 1100 had a TV, a car, and a pet.

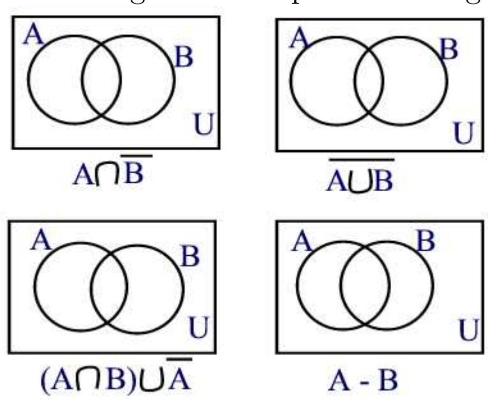
(a) How many people had a TV and a pet, but did not have a car?
(b) How many people did not have a pet or a TV or a car?

Problem 4.23
In a music club with 15 members, 7 people played piano, 6 people played guitar, and 4 people didn't play either of these instruments. How many people played both piano and guitar?

Problem 4.24
Use set notation to identify each of the following shaded region.



Problem 4.25
In the following, shade the region that represents the given sets:



Problem 4.26
Use Venn diagrams to show:

(a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$
(b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Problem 4.27
Let $G = \{n \in \mathbb{N} | n \textit{ divides } 90\}$ and $D = \{n \in \mathbb{N} | n \textit{ divides } 144\}$. Find $G \cap D$ and $G \cup D$.

Finite and Infinite Sets

The notion of one-to-one correspondence is so fundamental to counting that we don't even think about it. When we count out a deck of cards, we say, 1, 2, 3, ... , 52, and as we say each number we lay down a card. So we have a pairing of the cards with the numbers $1, 2, \cdots, 52$. This pairing defines a one-to-one correspondence. In general, we say that we have a **one-to-one correspondence** from a set A to a set B if every element of A is paired to exactly one element in B and vice versa every element in B is paired with exactly one element of A. In this case, the sets A and B are said to be **equivalent** and we write $A \sim B$. If A and B are not equivalent we write $A \not\sim B$. Figure 4.4 shows a one-to-one correspondence between two sets A and B.

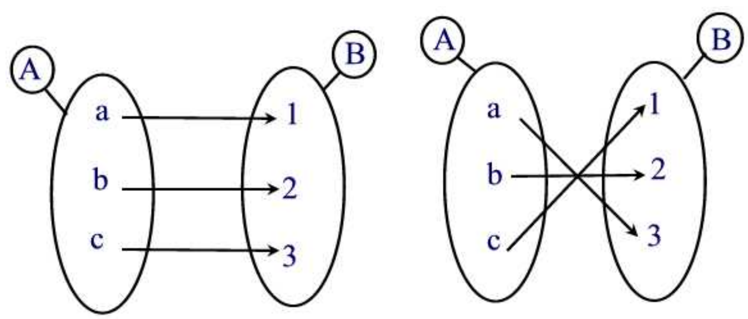


Figure 4.4

Example 4.10

Consider a set of three swimmers $\{A, B, C\}$ and a set of three swimming lanes.

(a) Exhibit all the one-to-one correspondence between the two sets.
(b) How many such one-to-one correspondence are there?

Solution.

(a) Figure 4.5 shows all the one-to-one correspondence between the two sets.
(b) There are six one-to-one correspondence.■

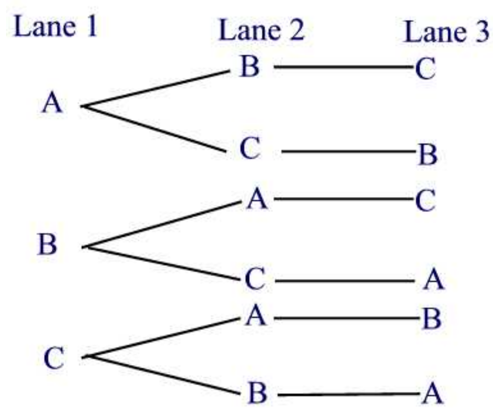


Figure 4.5

A set is **finite** if it is empty or can be put into a 1-1 correspondence with a set of the form $\{1, 2, 3, \cdots, n\}$ for some $n \in \mathbb{N}$. The number n represents the number of elements in A. A set that is not finite is said to be **infinite**. For example, the set $\{a, b, c, d\}$ is finite whereas the set of all even counting numbers is infinite.

Example 4.11
Decide whether each of the following sets is finite set or an infinite set.

(a) The set of whole numbers less than 6.
(b) The set of all the pencakes in Arizona right now.
(c) The set of counting numbers greater than 6.

Solution.
(a) $\{0, 1, 2, 3, 4, 5\}$ is finite.
(b) The set of all the pencakes in Arizona right now is a finite set.
(c) $\{7, 8, 9, \cdots\}$ is an infinite set.■

Practice Problems

Problem 4.28
Which of the following pairs of sets can be placed in one-to-one correspondence?

(a) $\{1, 2, 3, 4, 5\}$ and $\{m, n, o, p, q\}$.
(b) $\{m, a, t, h\}$ and $\{f, u, n\}$.
(c) $\{a, b, c, d, e, f, \cdots, m\}$ and $\{1, 2, 3, \cdots, 13\}$.
(d) $\{x|x \text{ is a letter in the word mathematics}\}$ and $\{1, 2, 3, \cdots, 11\}$.

Problem 4.29
How many one-to-one correspondence are there between the sets $\{x, y, z, u, v\}$ and $\{1, 2, 3, 4, 5\}$ if in each correspondence

(a) x must correspond to 5?
(b) x must correspond to 5 and y to 1?
(c) x, y, and z correspond to odd numbers?

Problem 4.30
True or false?

(a) The set $\{105, 110, 115, 120, \cdots\}$ is an infinite set.
(b) If A is infinite and $B \subseteq A$ then B is also infinite.
(c) For all finite sets A and B if $A \cap B = \emptyset$ then the number of elements in A plus the number of elements in B is equal to the number of elements in $A \cup B$.

Problem 4.31
Show three different one-to-one correspondence between the sets $\{1, 2, 3, 4\}$ and $\{x, y, z, w\}$.

Problem 4.32
Write a set that is equivalent but not equal to the set $\{a, b, c, d, e, f\}$.

Problem 4.33
Determine which of the following sets are finite. For those sets that are finite, how many elements are in the set?

(a) $\{\text{ears on a typical elephant}\}$
(b) $\{1, 2, 3, \cdots, 99\}$
(c) Set of points belonging to a line segment.
(d) A closed interval.

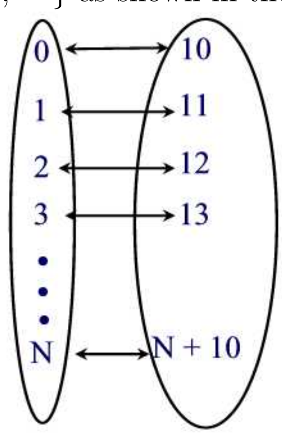
Problem 4.34
Decide whether each set is finite or infinite.

(a) the set of people named Lucky.
(b) the set of all perfect square numbers.

Problem 4.35
How many one-to-one correspondence are possible between each of the following pairs of sets?

(a) Two sets, each having two elements (b) Two sets, each having three elements (c) Two sets, each having four elements (d) Two sets, each having N elements.

Problem 4.36
A set A is infinite if it can be put into a one-to-one correspondence with a proper subset of itself. For example, the set $W = \{0, 1, 2, 3, \cdots\}$ of whole numbers is infinite since it can be put in a one-to-one correspondence with its proper subset $\{10, 11, 12, \cdots\}$ as shown in the figure below.



Show that the following sets are infinite:

(a) $\{0, 2, 4, 6, \cdots\}$
(b) $\{20, 21, 22, \cdots\}$.

Problem 4.37
Show that $\mathbb{N}$ and $S = \{1, 4, 9, 16, 25, \cdots\}$ are equivalent.

5 Numeration Systems

Numeration and the Whole Numbers
If you attend a student raffle, you might hear the following announcement when the entry forms are drawn

"The student with identification number 50768-973 has just won second prize-four tickets to the big game this Saturday."

This sentence contains three different types of numbers, each serving a different purpose.
• The number 50768-973 is an **identification** or **nominal number**. A nominal number is a sequence of digits used as a name or label. Telephone numbers, social security numbers, driver's license numbers are all examples of nominal numbers.
• The second type of numbers is called **ordinal numbers**. The words *first, second, third* and so on are used to describe the relative position of objects in an ordered sequence.
• The final use of number by the announcer is to tell how many tickets had been won. That is, the prize is a set of tickets and *four* tells us how many tickets are in the set. More generally, a **cardinal number** of a set is the number of objects in the set. If A is a finite set then we will denote the number of elements in A by $n(A)$. Some authors uses the notation $|A|$ for $n(A)$. So if $A = \{1, 2, \cdots, m\}$ then $n(A) = m$. We define $n(\emptyset) = 0$. The set of cardinal numbers of finite set is called the set of **whole numbers** and is denoted by W. Thus, $W = \{0, 1, 2, 3, \cdots\}$.
It should be noticed that numbers can be represented verbally (in a language) or symbolically (in a numeration system). For example, the winner in the above student raffle story wins four tickets to the game. The word four is represented by the symbol 4 in the Hindu-arabic numeration system.

Example 5.1
True or false?

(a) Two equivalent sets are equal.
(b) Two equivalent sets have the same number of elements.

Solution.
(a) This is false. For example, $\{a, b, c\} \sim \{1, 2, 3\}$ but $\{a, b, c\} \neq \{1, 2, 3\}$.
(b) This is always true. That is, if $A \sim B$ then $n(A) = n(B)$.■

Example 5.2
For each set give the whole number that gives the number of elements in the set.

(a) $A = \{x|x \text{ is a month of the year}\}$
(b) $B = \{n \in \mathbb{N}|n \text{ is square number between } 70 \text{ and } 80\}$.
(c) $C = \{0\}$.

Solution.
(a) $n(A) = 12$.
(b) $n(B) = 0$ since $B = \emptyset$.
(c) $n(C) = 1$.■

Ordering the Whole Numbers
We often wish to relate the number of elements of two given sets. For example, if each child in the class is given one cupcake, and there are some cupcakes left over, we would know that there are more cupcakes than children. Notice that children have been matched to a proper subset of the set of cupcakes.
The order of the whole numbers can be defined in the followign way: Let $n(A) = a$ and $n(B) = b$ be two whole numbers, where A and B are finite sets. If there is a one-to-one correspondence between A and a proper subset of B, we say that a **is less than** b and we write $a < b$. Equivalently, we can write $b > a$ which is read **"b is greater than a"**.
If $<$ or $>$ is combined with the equal sign we get the symbols $\leq$ and $\geq$.
There are three ways to compare whole numbers: (1) using sets, (2) counting chants, and (3) whole-number line as shown in the following example.

Example 5.3
Show that $4 < 7$ using the three methods introduced above.

Solution.
(1) Using sets: Figure 5.1(a) shows that a set with 4 elements is in a one-to-one correspondence with a proper subset of a set with 7 elements.

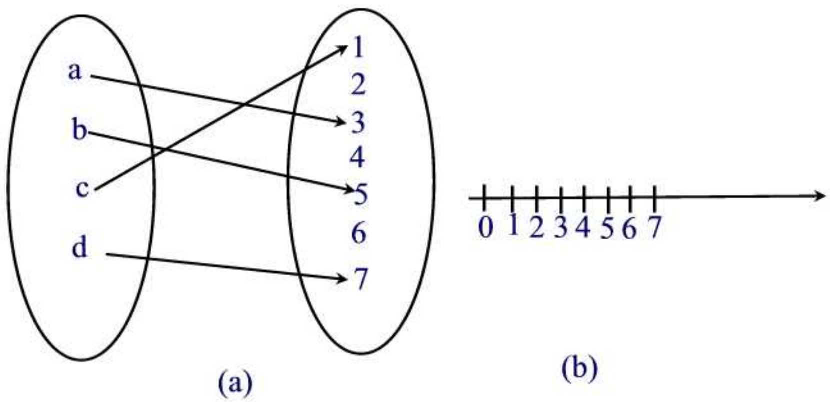


Figure 5.1

(2) Counting chant: one, two, three, four, five, six, seven. Since 4 precedes 7 then 4 is is less than 4.
(3) Whole-Number Line: Figure 5.1(b) shows that 4 is to the left of 7 on the number line, so 4 is less than 7 or 7 is greater than 4.■

Practice Problems

Problem 5.1
Let $A, B,$ and C be three sets such that $A \subset B \subseteq C$ and $n(B) = 5$.

(a) What are the possible values of $n(A)$?
(b) What are the possible values of $n(C)$?

Problem 5.2
Determine the cardinality of each of the following sets:

(a) $A = \{x \in \mathbb{N} | 20 \leq x < 35\}$
(b) $B = \{x \in \mathbb{N} | x + 1 = x\}$
(c) $C = \{x \in \mathbb{N} | (x - 3)(x - 8) = 0\}$.

Problem 5.3
Let A and B be finite sets.

(a) Explain why $n(A \cap B) \leq n(A)$. (b) Explain why $n(A) \leq n(A \cup B)$.

Problem 5.4
Suppose B is a proper subset of C.

(a) If $n(C) = 8$, what is the maximum number of elements in B?
(b) What is the least possible elements of B?

Problem 5.5
Suppose C is a subset of D and D is a subset of C.

(a) If $n(C) = 5$, find $n(D)$.
(b) What other relationships exist between C and D?

Problem 5.6
Use the definition of less than to show each of the following:

(a) $2 < 4$ (b) $3 < 100$ (c) $0 < 3$.

Problem 5.7
If $n(A) = 4, n(B) = 5$, and $n(C) = 6$, what is the greatest and least number of elements in

(a) $A \cup B \cup C$ (b) $A \cap B \cap C$?

Problem 5.8
True or false?If false give a counter example, i.e. an example that shows that the statement is false.

(a) If $n(A) = n(B)$ then $A = B$.
(b) If $n(A) < n(B)$ then $A \subset B$.

Problem 5.9
Suppose $n(A \cup B) = n(A \cap B)$. What can you say about A and B?

Problem 5.10
Let $U = \{1, 2, 3, \cdots, 1000\}$, F be a subset of U consisting of multiples of 5 and S the subset of U consisting of multiples of 6.

(a) Find $n(S)$ and $n(F)$.
(b) Find $n(F \cap S)$.
(c) Label the number of elements in each region of a two-loop Venn diagram with universe U and subsets S and F.

Problem 5.11
Finish labeling the number of elements in the regions in the Venn diagram shown, where the subsets A, B, and C of the universe U satisfy the conditions listed. See Figure 5.2.

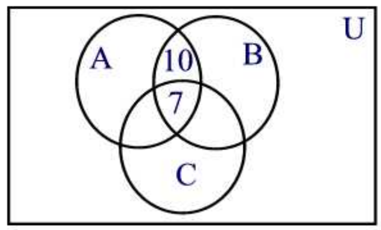


Figure 5.2

$$\begin{array}{rclrcl} n(U) & = & 100 & n(A) & = & 40 \\ n(B) & = & 50 & n(C) & = & 30 \\ n(A\cap B) & = & 17 & n(B\cap C) & = & 12 \\ n(A\cap C) & = & 15 & n(A\cap B\cap C) & = & 7 \end{array}$$

Problem 5.12
Let $S = \{s, e, t\}$ and $T = \{t, h, e, o, r, y\}$. Find $n(S), n(T), n(S\cup T), n(S\cap T), n(S\cap \overline{T})$, and $n(\overline{S}\cap T)$.

Problem 5.13
Suppose that $n(A) = m$ and $n(B) = n$. Find $n(A\times B)$.

Problem 5.14
Explain why $5 < 8$ using the definition of whole number inequality introduced in this section.

Problem 5.15
Let A and B be two sets in the universe $U = \{a, b, c, \cdots, z\}$. If $n(A) = 12, n(B) = 14$, and $n(A\cup B) = 21$, find $n(A\cap B)$ and $n(A\cap \overline{B})$.

Problem 5.16
Suppose that $n(A\times B) = 21$. What are all the possible values of $n(A)$?

Numeration Systems
A **numeration system** is a collection of properties and symbols agreed upon to represent numbers systematically. We will examine various numeration systems that have been used througout history.

Tally Numeration System
This is the earliest numeration system. Suppose you want to count a group of things (sheep or trees, etc). You could use a vertical line to each object you want to count as shwon is Figure 5.3.

Figure 5.3

One advantage of this system is its simplicity. However, a disadvantage of this system is its difficulty to read large numbers. For example, can you tell what number is represented by Figure 5.4?

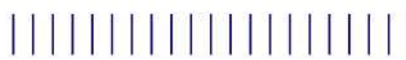


Figure 5.4

The tally system is improved by using **grouping.** The fifth tally mark was placed across every four to make a group of five. Thus, the number in Figure 5.4 is represented as shown in Figure 5.5.

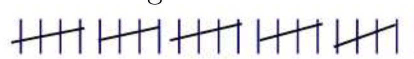


Figure 5.5

Example 5.4
Write the numerals 1 - 10 using the tally numeration system.

Solution.
The numerals 1 - 10 are as follows:

$$|,\ ||,\ |||,\ ||||,\ |||||,\ ||||||,\ |||||||,\ ||||||||,\ |||||||||,\ ||||||||||.\blacksquare$$

Egyptian Numeration System (3400 BC)
This system uses grouping by ten and special symbols representing powers of 10 as shown in Figure 5.6.

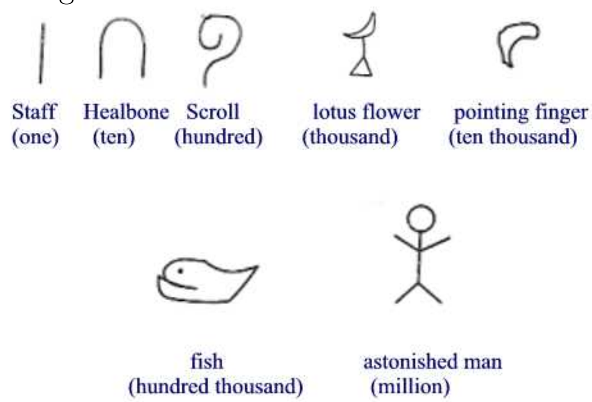


Figure 5.6

Thus, the number 10213 is represented by Figure 5.7.

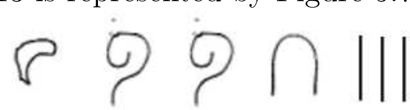


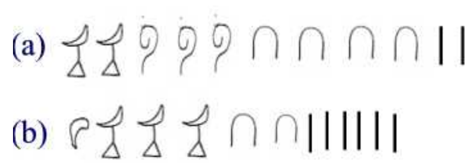
Figure 5.7

This system is called an **additive system** since the value of the symbols are added together to get the value of the number.
An advantage of this system is that fewer symbols are used than the tally system after you get to ten. The disadvantage is that this system is not easy when adding numbers.

Example 5.5
Write the following numbers, using Egyptian numerals:

(a) 2342 (b) 13,026.

Solution.



The Roman Numeration System (500 BC)
The Roman numeration system, an example of an additive system, was developed later than the Egyptian system. Roman numerals were used in most European countries until the eighteenth century. They are still commonly seen on buildings, on clocks, and in books. Roman numerals are selected letters of the Roman alphabet.
The basic Roman numerals are:

$$I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, M = 1000.$$

Roman numbers are made up of combinations of these basic numerals. For example, $CCLXXXI = 281$.
The Roman system has two advantages over the Egyptian system. The first is that it uses the **subtraction principle** as well as the **addition principle.** Starting from the left, we add each numeral unless its value is smaller than the numeral to its right. In that case, we subtract it from that numeral. For example, the symbol DC represents 500 + 100, or 600, and CD represents 500 - 100, or 400. Similarly, MC represents 1000 + 100, or 1100, and CM represents 1000 - 100, or 900. This is called a **positional system** since the same symbols can mean different values depending on position.

Example 5.6
Write in our numeration system.

(a) CLXII (b) DCXLVI .

Solution.
(a) Since each numeral is larger than the next one on its right, no subtraction is necessary.

$$CLXII = 100 + 50 + 10 + 1 + 1 = 162.$$

(b) Checking from left to right, we see that X has a smaller value than L. Therefore XL represents 50 - 10, or 40.

$$DCXLVI = 500 + 100 + (50 - 10) + 5 + 1. \blacksquare$$

In the roman numeral system, a symbol does not have to be repeated more than three consecutive times. For example, the number 646 would be written DCXLVI instead of DCXXXXVI.
The second advantage of the Roman numeration system over the Egyptian

system is that it makes use of the **multiplication principle** for numbers over 1000. A bar over the symbol or group of symbols indicates that the symbol or symbols are to be multiplied by 1000. So, the number 5000 would be written as $\overline{V}$ and the number 40,000 would be written as $\overline{XL}$.

Example 5.7
(a) Convert 9,389 into a Roman numeral.
(b) Convert the Roman number MMCCCLXXXIX into our numeration system.

Solution.
(a) $9,389 = 9000+300+80+9 = IX+CCC+LXXX+IX = \overline{IX}CCCLXXXIX.$
(b) $MMCCCLXXXIX = 2000 + 300 + 50 + 30 + (10 - 1) = 2,389.$■

The Babylonian Numeration System (3000 BC)
The Babylonian numeration system was developed about the same time as the Egyptian system. This system introduced the notion of **place value**, in which the position of a symbol in a numeral determined its value. This made it possible to write numerals for even very large numbers using very few symbols. Indeed, the system utilized only two symbols,  for 1 and  for 10 and combined these additively to form the digits 1 through 59. Thus

respectively, represented 21 and 34. Beyond 59, the system was positional to base sixty, where the positions from right to left represented multiples of successive powers of 60 and the multipliers were composite symbols for 1 through 59. Thus,

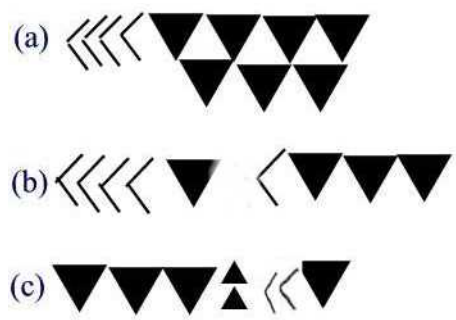
represented $2 \cdot 60^1 + 33 = 153$ and $1 \cdot 60^2 + 31 \cdot 60^1 + 22 = 5482$.
A difficulty with the babylonian system was the lack of a symbol for zero. In writing numerals, a space was left if a certain value was not to be used and since spacing was not uniform this always lead to confusion. To be more specific, the notation for 83 and 3623 were

and these could easily be confused if the spacing were not clear. Indeed, there was no way even to indicate missing position values of the extreme right of a numeral, so  could represent 1 or $1 \cdot 60$ or $1 \cdot 60^2$ and so on. Eventually, the Babylonians employed the symbol  as a **place holder** to indicate missing position values though they never developed the notion of zero as a number. Using this symbol,the numbers 83 and 3623 are now represented as

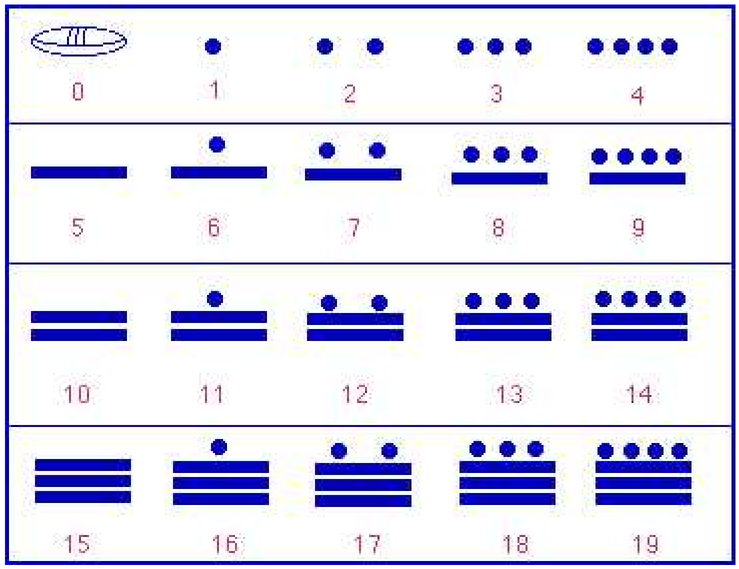
Example 5.8
Write the following numbers using Babylonian numeration:

(a) 47 (b) 2473 (c) 10,821.

Solution.



Mayan Numeration System (200 AD)
The Mayan number system was developed by the ancient Maya civilization in the region now known as the Yucatan Peninsula in Southeastern Mexico. The Maya seem to be the first people who used a place value system and a symbol for zero.
The Mayan numbers are based on three symbols:a dot, a bar, and a symbol for zero, or completion, usually a shell.
In the following table, you can see how the system of dots and bars works to create Mayan numerals and the modern equivalent numerals 0-19.



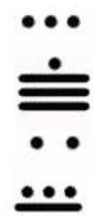
Like our numbering system, they used place values to expand this system to allow the expression of very large values. Their system has two significant differences from the system we use: 1) the place values are arranged vertically, and 2) they use a modified base 20 system. This means that, instead of the number in the second postion having a value 10 times that of the numeral (as in $21 = 2 \times 10 + 1$), in the Mayan system, the number in the second place has a value 20 times the value of the numeral. However, starting from the third place, the number in the third place has a value of $18 \cdot 20$ times the value of the numeral; and so on. The reason that $18 \cdot 20$ is used instead of $20 \cdot 20$ is that the main function of their number system was to keep track of time; their annual calendar consisted of 360 days. This above principle is illustrated in the example below.

Example 5.9
Write the following Mayan number in our numeration system.

Solution.
The number is: $6 \cdot 18 \cdot 20^2 + 0 \cdot 18 \cdot 20 + 14 \cdot 20 + 7 = 43,487.$■

Example 5.10
Write the number 27,408 in Mayan system.

Solution.
Since $21,600 = 3 \cdot 18 \cdot 20^2 + 5808, 5808 = 16 \cdot 18 \cdot 20 + 48, 48 = 2 \cdot 20 + 8$ then the Mayan representation of 27408 is



Practice Problems

Problem 5.17
Write the following in Egyptian system.

(a) 11 (b) 597 (c) 1949.

Problem 5.18
Write the following Roman notation using the subtraction principle as appropriate.

(a) 9 (b) 486 (c) 1945.

Problem 5.19
Write the following numbers in Babylonian system.

(a) 251 (b) 3022 (c) 18,741.

Problem 5.20
Write the following numbers in Mayan System.

(a) 12 (b) 584 (c) 12,473.

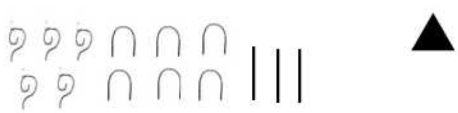
Problem 5.21
Write 2002, 2003, and 2004 in Roman numerals.

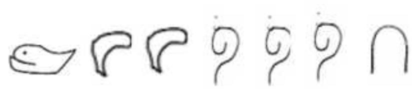
Problem 5.22
If the cornerstone represents when a building was built and it reads MCMXXII, when was this building built?

Problem 5.23
Write each of the following numbers in our numeration system, i.e. base 10.

(a) MDCCXXIX
(b) DCXCVII
(c) CMLXXXIV

(d) 

(e) 

(f) 

(g) 

(h) 

(i) 

(j) 

Problem 5.24
Convert the Roman numeral DCCCXXIV to Babylonian numeral.

Problem 5.25
Write the following numbers in the given system.

(a) Egyptian numeration:3275
(b) Roman numeration: 406
(c) Babylonian system: 8063
(d) Mayan numeration: 48

Problem 5.26
Represent the number 246 in the Mayan, Babylonian, and Egyptian numeration systems.

(a) In which system is the greatest number of symbols required?
(b) In which system is the smallest number of symbols required?

Problem 5.27
Some children go through reversal stage; that is they confuse 13 and 31, 27 and 72, 59 and 95. What numerals would give Roman children similar difficulties? How about Egyptian children?

Problem 5.28
A newspaper advertisement introduced a new car as follows: IV Cams, XXXII Valves, CCLXXX Horsepower, coming December XXVI- the new 1999 Lincoln Mark VII. Write the Roman numeral that represents the year of the advertisement.

Problem 5.29
After the credits of a film roll by, the Roman numeral MCMLXXXIX appears, representing the year in which the film was made. Express the year in our numeration system.

Problem 5.30
True of false?

(a) ||| is three in the tally system.
(b) $IV = VI$ in the Roman system.
(c) $\cap||| = |||\cap$

6 The Hindu-Arabic System (800 BC)

Today the most universally used system of numeration is the **Hindu-Arabic system**, also known as the **decimal system** or **base ten system.** The system was named for the Indian scholars who invented it at least as early as 800 BC and for the Arabs who transmitted it to the western world. Since the base of the system is ten, it requires special symbols for the numbers zero through nine. The following list the features of this system:

(1) **Digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9**. These symbols can be used in combination to represent all possible numbers.
(2) **Grouping by ten:** Grouping into sets of 10 is a basic principle of this system. Figure 6.1 shows how grouping is helpful when representing a collection of objects.

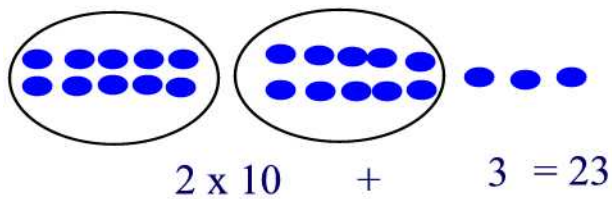


Figure 6.1

(3) **Place value:** The place value assigns a value of a digit depending on its placement in a numeral. To find the value of a digit in a whole number, we multiply the place value of the digit by its **face value,** where the face value is a digit. For example, in the numeral 5984, the 5 has place value "thousands", the 9 has place value "hundreds", the 8 has place value "tens", and 4 has place value "units".
(4) **Expanded form:** We can express the numeral 5984 as the sum of its digits times their respective place values, i.e. in **expanded form**

$$5984 = 5 \times 1000 + 9 \times 100 + 8 \times 10 + 4 = 5 \times 10^3 + 9 \times 10^2 + 8 \times 10^1 + 4.$$

Example 6.1
Express the number 45,362 in expanded form.

Solution.
We have: $45,362 = 4 \times 10^4 + 5 \times 10^3 + 3 \times 10^2 + 6 \times 10 + 2.$■

Word names for Hindu-Arabic numerals:
(1) unique names: $0(zero), 1(one), 2(two), 3(three), 4(four), 5(five), 6(six), 7(seven), 8(eight), 9(nine), 10(ten), 11(eleven), 12(twelve)$.
(2) $13, 14, \cdots, 19$ teen (for ten). For example, $14 = (4 + 10)$ four-teen.
(3) $20, 21, \cdots, 99$ $57 = 5 \times 10 + 7$ fifty seven.
(4) $100, 101, \cdots, 999$ is the combination of hundreds and previous names. For example, 538 is read "five hundreds thirty eight."
(5) In numerals containing more than three digits, groups of three digits are sets off by commas. For example, the number 864,456,221,653,127,851 is read: "eight hundred sixty four quadrillion four hundred fifty six trillion two hundred twenty one billion six hundred fifty three million one hundred twenty seven thousand eighthundred fifty one".

Nondecimal Numeration Systems
The decimal system discussed above is based on grouping by ten. Some other grouping are of interest such as grouping by two , three, four, etc. We apparently use base 10 because we (most of us) have ten fingers. Base 2 (also known as **binary**) is what computers use, internally. It has been joked that this is because they have two fingers (two electrical states, actually). In base 2, there are two different digits (0 and 1). And the first few numbers are $1, 10, 11, 100, 101, 110, 111, 1000, 1001, \cdots$.
It is important to label base two numbers (usually with a subscript 2) because they can be mistaken for base 10 numbers. For example $1010_{two} = 10_{ten}$.
Converting between binary and decimal numbers is fairly simple, as long as you remember that each digit in the binary number represents a power of two.

Example 6.2
Convert 101100101_{two} to the corresponding base-ten number.

Solution.
List the digits in order, and count them off from the RIGHT, starting with zero:

$$\begin{array}{rccccccccc} digits: & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ numbering: & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array}$$

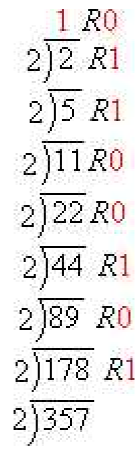
Use this listing to convert each digit to the power of two that it represents:
$1 \times 2^8 + 0 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$
$= 256 + 64 + 32 + 4 + 1 = 357$

Thus, $101100101_{two} = 357_{ten}$.■

Converting decimal numbers to binaries is nearly as simple: just divide by 2 as illustrated by the example below.

Example 6.3
Convert 357_{ten} to the corresponding binary number.

Solution.
To do this conversion, you need to divide repeatedly by 2, keeping track of the remainders as you go.



These remainders tell us what the binary number is. Read the numbers from around the outside of the division, starting on top and wrapping your way around the right-hand side. As you can see:

$$357_{ten} = 101100101_{two}.■$$

Conversions from any nondecimal system to base ten and vice versa, can be accomplished in a manner similar to that used for base two conversions.

Example 6.4
(a) Convert 11244_{five} to base ten.
(b) Convert 543 to base four.

Solution.
(a) Using the expanded notation we have

$$11244_{five} = 1 \times 5^4 + 1 \times 5^3 + 2 \times 5^2 + 4 \times 5 + 4 = 824.$$

(b) We use the process of repeated division by 4.

$$\begin{array}{rcl} 543 & = & 4 \times 135 + 3 \\ 135 & = & 4 \times 33 + 3 \\ 33 & = & 4 \times 8 + 1 \\ 8 & = & 4 \times 2 + 0 \\ 2 & = & 4 \times 0 + 2 \end{array}$$

Thus, $543 = 20133_{four}$.■

Practice Problems

Problem 6.1
Write each of the following numbers in expanded form.
(a) 70 (b) 746 (c) 840,001.

Problem 6.2
Write each of the following expressions in standard place-value form. That is, $1 \times 10^3 + 2 \times 10^2 + 7 = 1207$.
(a) $5 \times 10^5 + 3 \times 10^2$.
(b) $8 \times 10^6 + 7 \times 10^4 + 6 \times 10^2 + 5$.
(c) $6 \times 10^7 + 9 \times 10^5$.

Problem 6.3
Write the following numerals in words.
(a) $2,000,000,000$
(b) $87,000,000,000,000$
(c) $52,672,405,123,139$.

Problem 6.4
Write each of the following base seven numerals in expanded notation.
(a) 15_{seven} (b) 123_{seven} (c) 5046_{seven}.

Problem 6.5
Convert each base ten numeral into a numeral in the base requested.
(a) 395 in base eight
(b) 748 in base four
(c) 54 in base two.

Problem 6.6
The base twelve numeration system has the following twelve symbols:0,1,2,3,4,5,6,7,8,9,T,E. Change each of the following numerals to base ten numerals.
(a) 142_{twelve} (b) 503_{twelve} (c) $T9_{twelve}$ (d) $ETET_{twelve}$.

Problem 6.7
Write each of the numerals in base six and base twelve.
(a) 128 (b) 74 (c) 2438.

Problem 6.8
Convert the following base five numerals into base nine numerals.
(a) 12_{five} (b) 204_{five} (c) 1322_{five}.

Problem 6.9
(a) How many different symbols would be necessary for a base twenty system?
(b) What is wrong with the numerals 85_{eight} and 24_{three}?

Problem 6.10
The set of even whole numbers is the set $\{0, 2, 4, 6, \cdots\}$. What can be said about the ones digit of every even number in the following bases?
(a) Ten (b) Four (c) Two (d) Five

Problem 6.11
Translate the following numbers from one base to the other:
(a) $38_{ten} = \_\_\_\_\_\__{two}$.
(b) $63_{ten} = \_\_\_\_\_\__{two}$.

Problem 6.12
Translate the following numbers from one base to the other:
(a) $1101_{two} = \_\_\_\_\_\__{ten}$.
(b) $11111_{two} = \_\_\_\_\_\__{ten}$.

Problem 6.13
The sum of the digits in a two-digit number is 12. If the digits are reversed, the new number is 18 greater than the original number. What is the number?

Problem 6.14
State the place value of the digit 2 in each numeral.
(a) 6234 (b) 5142 (c) 2178

Problem 6.15
(a) Write out the first 20 base four numerals.
(b) How many base four numerals precede 2000_{four}?

Problem 6.16
True or false?
(a) $7_{eight} = 7_{ten}$ (b) $30_{four} = 30_{ten}$ (c) $8_{nine} = 8_{eleven}$ (d) $30_{five} = 30_{six}$

Problem 6.17
If all the letters of the alphabet were used as our single-digit numerals, what would be the name of our base system?

Problem 6.18
Find the base ten numerals for each of the following.
(a) 342_{five} (b) $TE0_{twelve}$ (c) 101101_{two}

Problem 6.19
The **hexadecimal** system is a base sixteen system used in computer programming. The system uses the symbols:0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F. Change each of the following hexadecimal numerals to base ten numerals.
(a) $213_{sixteen}$ (b) $1C2B_{sixteen}$ (c) $420E_{sixteen}$

Problem 6.20
Write each of the following base ten numerals in base sixteen numerals.
(a) 375 (b) 2941 (c) 9520 (d) 24,274

Problem 6.21
Rod used base twelve to write the equation:

$$g36_{twelve} = 1050_{ten}.$$

What is the value of g?

Problem 6.22
For each of the following decimal numerals, give the place value of the underlined digit:
(a) $827,\underline{3}67$ (b) $8,421,0\underline{0}0$ (c) $9\underline{7},998$

Problem 6.23

A certain three-digit whole number has the following properties: The hundreds digit is greater than 7; the tens digit is an odd number; and the sum of the digits is 10. What could the number be?

Problem 6.24

Find the number preceding and succeeding the number $EE0_{twelve}$.

7 Relations and Functions

In this section, we introduce the concept of relations and functions.

Relations
A **relation** R from a set A to a set B is a set of ordered pairs (a, b), where
• a is a member of A,
• b is a member of B,
• The set of all first elements (a) is the **domain** of the relation, and
• The set of all second elements is the **range** of the relation.
Often we use the notation $a\ R\ b$ to indicated that a and b are related, rather then the order pair notation (a, b). We refer to a as the **input** and b as the **output.**

Example 7.1
Find the domain and range of the relation $R = \{(2, 3), (2, 4), (3, 7), (5, 2)\}$.

Solution.
The domain is the set $\{2, 3, 5\}$ and the range is the set $\{2, 3, 4, 7\}$.■

Note that a relation R is just a subset of the Cartesian product $A \times B$.
We can also represent a relation as an **arrow diagram.** For example, the relation $\{(1, 2), (0, 1), (3, 4), (2, 1), (0, -2)\}$ can be represented by the diagram of Figure 7.1

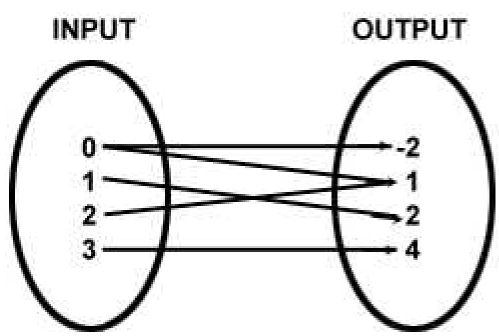


Figure 7.1

When a relation R is defined from a set A into the same set A then there are three useful properties to look at:

Reflexive Property:

A relation R on A is said to be **reflexive** if every element of A is related to itself. In notation, $a\ R\ a$ for all $a \in A$. Examples of reflexive relations include:

- "is equal to" (equality)
- "is a subset of" (set inclusion)
- "is less than or equal to" and "is greater than or equal to" (inequality)
- "divides" (divisibility).

An example of a non reflexive relation is the relation "is the father of" on a set of people since no person is the father of themself.

When looking at an arrow diagram, a relation is reflexive if every element of A has an arrow pointing to itself. For example, the relation in Figure 7.2 is a reflexive relation.

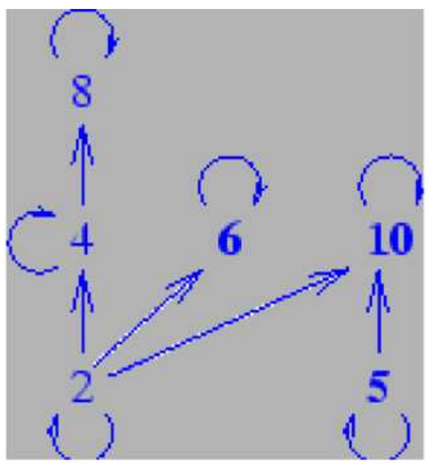


Figure 7.2

Symmetric Property

A relation R on A is **symmetric** if given $a\ R\ b$ then $b\ R\ a$. For example, "is married to" is a symmetric relation, while, "is less than" is not.The relation "is the sister of" is not symmetric on a set that contains a brother and sister but would be symmetric on a set of females.

The arrow diagram of a symmetric relation has the property that whenever there is a directed arrow from a to b, there is also a directed arrow from b to a. See Figure 7.3.

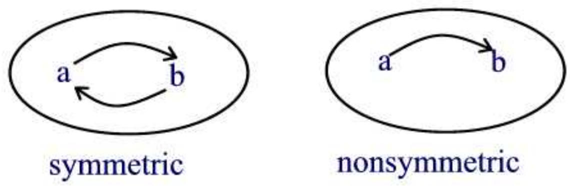


Figure 7.3

Transitive Property

A relation R on A is **transitive** if given $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$. Examples of reflexive relations include:
• "is equal to" (equality)
• "is a subset of" (set inclusion)
• "is less than or equal to" and "is greater than or equal to" (inequality)
• "divides" (divisibility).
On the other hand, "is the mother of" is not a transitive relation, because if Alice is the mother of Brenda, and Brenda is the mother of Claire, then Alice is not the mother of Claire.
The arrow diagram of a transitive relation has the property that whenever there are directed arrows from a to b and from b to c then there is also a directed arrow from a to c. See Figure 7.4.

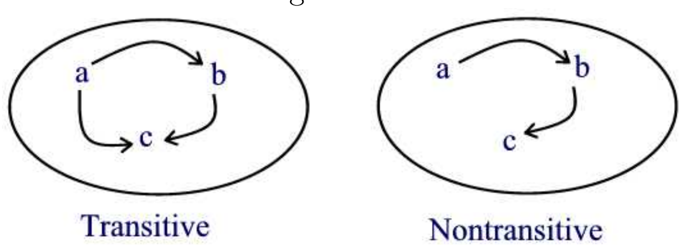
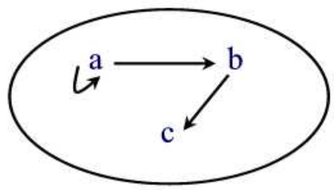


Figure 7.4

A relation that is reflexive, symmetric, and transitive is called an **equivalence relation** on A. Examples of equivalence relations include
• The equality ("=") relation between real numbers or sets.
• The relation "is similar to" on the set of all triangles.
• The relation "has the same birthday as" on the set of all human beings.
On the other hand, the relation " $\subseteq$ " is not an equivalence relation on the set of all subsets of a set A since this relation is not symmetric.

Practice Problems

Problem 7.1
Express the relation given in the arrow diagram below in its ordered-pair representation.



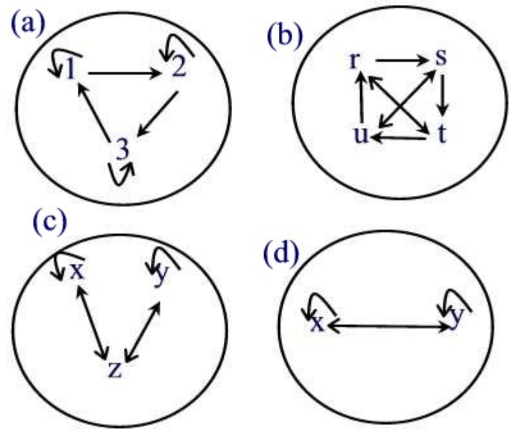
Problem 7.2
Consider the relation "is a factor of" from the set $A = \{2, 3, 4, 5, 6\}$ to the set $B = \{6, 8, 10, 12\}$. Make an arrow diagram of this relation.

Problem 7.3
Determine whether the relations represented by the following sets of ordered pairs are reflexive, symmetric, or transitive. Which of them are equivalence relations?
(a) $\{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)\}$
(b) $\{(1, 2), (1, 3), (2, 3), (2, 1), (3, 2), (3, 1)\}$
(c) $\{(1, 1), (1, 3), (2, 2), (3, 2), (1, 2)\}$
(d) $\{1, 1), (2, 2), (3, 3)\}$.

Problem 7.4
Determine whether the relations represented by the following sets of ordered pairs are reflexive, symmetric, or transitive. Which of them are equivalence relations?
(a) "less than" on the set $\mathbb{N}$
(b) "has the same shape as" on the set of all triangles
(c) "is a factor of" on the set $\mathbb{N}$
(d) "has the same number of factors as" on the set $\mathbb{N}$.

Problem 7.5
List all the ordered pairs of each of the following relations on the sets listed. Which, if any, is an equivalence relation?
(a) "has the same number of factors as" on the set $\{1, 2, 3, 4, 5, 6\}$
(b) "is a multiple of " on the set $\{2, 3, 6, 8, 10, 12\}$
(c) "has more factors than" on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Problem 7.6
Determine whether the relations represented by the following diagrams are reflexive, symmetric, or transitive. Which relations are equivalence relations?



Problem 7.7
Consider the relations R on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ defined by the equation $a + b = 11$. Determine all the ordered pairs (a, b) that satisfy the equation. Is this relation an equivalence relation?

Problem 7.8
True or false?
(a) ”If a is related to b then b is related to a” is an example of a reflexive relation.
(b) The ordered pair $(6, 24)$ satisfies the relation ”is a factor of”.

Problem 7.9
Let R be a relation on the set $A = \{a, b, c\}$. As a list of ordered pairs the relation has five elements. One of the element is (a, b). What are the remaining elements if R is both reflexive and symmetric?

Problem 7.10
If the relation $\{(1, 2), (2, 1), (3, 4), (2, 4), (4, 2), (4, 3)\}$ on the set $\{1, 2, 3, 4\}$ is to be altered to have the properties listed, what other ordered pairs, if any, are needed?
(a) Reflexive (b) Symmetric (c) Transitive (d) Reflexive and transitive.

Functions
Note that the definition of a relation does not say that each element from A

needs to be associated with one (or more) elements from B. It is sufficient if some associations between elements of A and B are defined. In contrast, there is the definition of a function:
A relation is a **function** if and only if every element of A occurs once and only once as a first element of the relation. That is, if every input of A has exactly one output in B. We call A the **domain** and B the **codomain.**

Example 7.2
Let $A = \{1, 2, 3, 4\}, B = \{14, 7, 234\}, C = \{a, b, c\}$, and $\mathbb{R} = \textit{real numbers}$. Define the following relations:
(a) R_1 is the relation between A and B that associates the pairs

$$1\ R\ 234, 2\ R\ 7, 3\ R\ 14, 4\ R\ 234, 2\ R\ 234$$

(b) R_2 is the relation between A and C given by $\{(1, c), (2, b), (3, a), (4, b)\}$
(c) R_3 is the relation between A and C given by $\{(1, a), (2, a), (3, a)\}$.
Which of those relations are functions ?

Solution.
(a) R_1 is not a function since the element 2 is associated with two elements of B, namely 7 and 234.
(b) R_2 is a function since every member of A is associated to exactly one member of B. Note that members of A can be associated to same elements of B.
(c) R_3 is not a function since the element 4 from the domain A has no element associated with it.■

Functions can be named using **function notation.** For example, the function represented symbolically by the equation:

$$y = x^2 + 1$$

might be named $f(x)$. In this case, the equation would be written as:

$$f(x) = x^2 + 1.$$

Note that the parentheses in the notation f(x) do not indicate multiplication. f(x) is "f of x", not "f times x".
With this notation, we define the **range** of f to be the set $\{(x, f(x))|x \in A\}$.

Example 7.3
In stores that sell athletic shoes of various kinds, the cost of doing business includes fixed expenses C_0(like rent and pay for employees) and variable expenses m (like the number of pairs of shoes bought from manufacturers). Operating cost of any store would be a function of those two main factors. Express this function using function notation. Use n to denote the number of shoes bought from the manufacturer.

Solution.
If $C(n)$ denote the total cost of manufacturing n shoes than $C(n) = mn+C_0$.■

Example 7.4
Find $f(2)$ if $f(x) = 3x - 4$.

Solution.
Replacing x by 2 to obtain $f(2) = 3(2) - 4 = 2$.■

Describing and Visualizing Functions
Functions as Machines
You can make an analogy between a function and a machine (like a meat grinder). The purpose of this analogy is to link together the abstract symbols used in function notation with a mechanical device that you are already very familiar with. If you ever get stuck on or confused by some function notation, try to think of what each symbol present would represent in "meat grinder terms."
• x - this is the unprocessed meat that goes into the meat grinder.
• f - this is the name of the machine that is being used (the meat grinder itself)
• $f(x)$ - this is the stuff (ground meat) that comes out of the machine.
Function notation is represented pictorially in Figure 7.5.

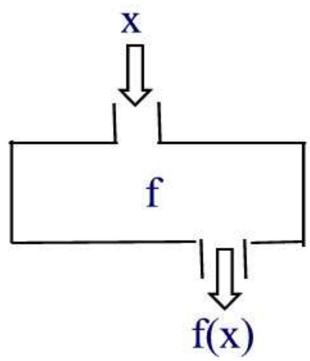
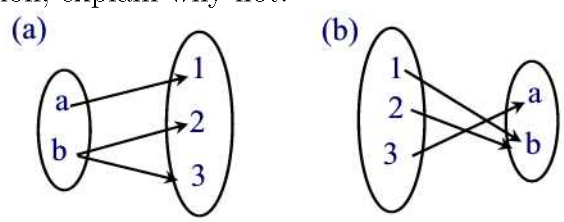


Figure 7.5

Functions as Arrow Diagrams
When the domain and codomain of a function are finite sets then one can represent the function by an arrow diagram. Remember that an arrow diagram represents a function if exactly one arrow must leave each element of the domain and point to one element in the codomain.

Example 7.5
Which of the following arrow diagrams represent functions? If one does not represent a function, explain why not.



Solution.
(a) The diagram does not represent a function since two arrows leave the same element b.
(b) The diagram represents a function since exactly one arrow leaves every element in the domain.■

Functions as Formulas(Symbolic Form)
Consider, for example, a circle of radius r, where the variable r is any positive integer. The formula $A(r) = \pi r^2$ defines the function area that expresses the area of the circle as a function of the radius r.

Functions in Tabular Form

Functions can be represented by tables. For example, the following table gives the grades of three students on a math quiz.

Student	Grade
Mark	8
Stve	7
Mary	10

Functions as Ordered Pairs

If the domain of the function is finite then one can represent the function by listing all the ordered pairs. If the domain is infinite then the function can be represented by ordered pairs using the set-builder notation. For example, the function that squares every real number can be represented by the set

$$\{(x, x^2)|x \in \mathbb{R}\}.$$

Example 7.6

Which of the following are functions from x to y? Assuming that the entire set of ordered pairs is given.

(a) $\{(1,2),(2,2),(3,4),(4,5)\}$

(b) $\{((1,3),(5,1),(5,2),(7,9)\}$.

Solution.

(a) The set satisfies the definition of a function.

(b) This is not a function since the element 5 is associated to two outputs 1 and 2.■

Functions as Graphs

A function whose domain and range are sets of numbers can be graphed on a set of x- and y-axes: if $f(x) = y$, plot the points (x, y) for all x in the domain of the function. If the domain is finite then the graph consists of dots whereas if the domain is infinite then the graph is usually a curve. For example, the graph of the function $\{(1,2),(2,2),(3,4),(4,5)\}$ is given in Figure 7.6(a) whereas the graph of the function $y = 2x$ is given in Figure 7.6(b).

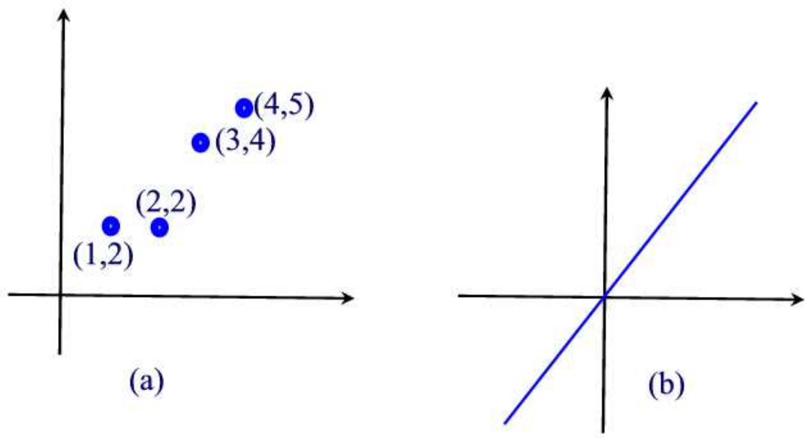


Figure 7.6

Next, suppose that the graph of a relationship between two quantities x and y is given. To say that y is a function of x means that for each value of x there is exactly one value of y. Graphically, this means that each vertical line must intersect the graph at most once. Hence, to determine if a graph represents a function one uses the following test:

Vertical Line Test: A graph is a function if and only if every vertical line crosses the graph at most once.

According to the vertical line test and the definition of a function, if a vertical line cuts the graph more than once, the graph could not be the graph of a function since we have multiple y values for the same x-value and this violates the definition of a function.

Example 7.7
Which of the graphs (a), (b), (c) in Figure 7.7 represent y as a function of x?

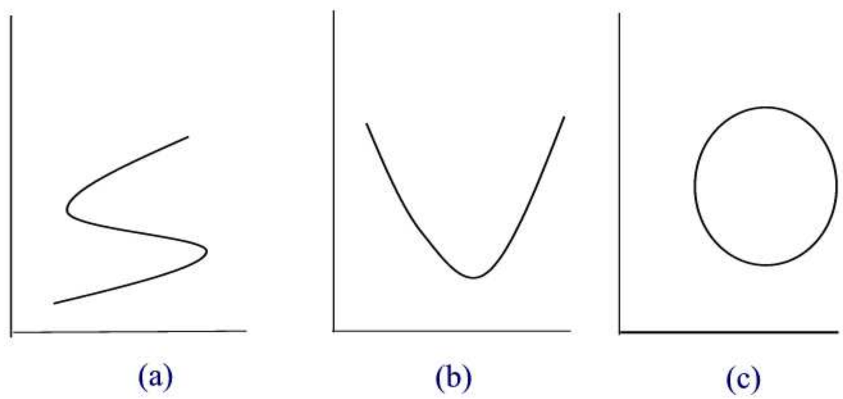


Figure 7.7

Solution.
By the vertical line test, (b) represents a function whereas (a) and (c) fail to represent functions since one can find a vertical line that intersects the graph more than once.■

Practice Problems

Problem 7.11
List the ordered pairs for these functions using the domain specified. Find the range for each function.
(a) $C(t) = 2t^3 - 3t$, with domain $\{0, 2, 4\}$
(b) $a(x) = x + 2$, with domain $\{1, 2, 9\}$
(c) $P(n) = \left(\frac{n+1}{n}\right)$, with domain $\{1, 2, 3\}$.

Problem 7.12
Find the value of $\frac{f(x+h)-f(x)}{h}$ given that $f(x) = x^2$.

Problem 7.13
Given $f(x) = -x^2 + 2x + 6$, find f(-4).

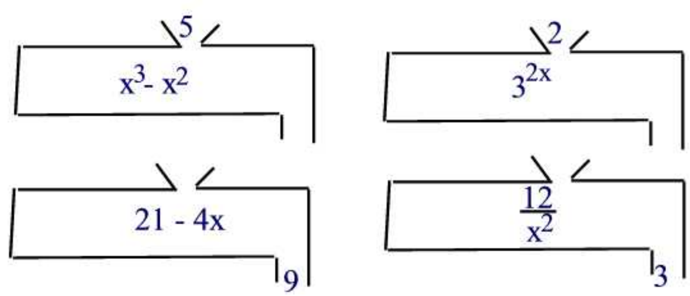
Problem 7.14
A function f on the set of real numbers $\mathbb{R}$ is defined as

$$f(x) = (3x + 2)/(x - 1).$$

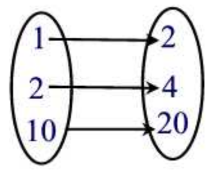
Find:
(a) the domain of f
(b) the range of f
(c) the image of -2 under f
(d) x when $f(x) = -3$.

Problem 7.15
Which of the following relations, listed as ordered pairs, could belong to a function? For those that cannot, explain why not.
(a) $\{(7, 4), (6, 3), (5, 2), (4, 1)\}$
(b) $\{((1, 1), (1, 2), (3, 4), (4, 4)\}$
(c) $\{(1, 1), (2, 1), (3, 1), (4, 1)\}$
(d) $\{(a, b), (b, b), (d, e), (b, c), (d, f)\}$.

Problem 7.16
Using the function machines, find all possible missing whole-number inputs and outputs.



Problem 7.17
The following functions are expressed in one of the following forms: a formula, an arrow diagram, a table, or a set of ordered pairs. Express each function in each of the other forms.
(a) $f(x) = x^3 - x$ for $x \in \{0, 1, 4\}$.
(b) $\{(1, 1), (4, 2), (9, 3)\}$
(c)



(d)

x	f(x)
5	55
6	66
7	77

Problem 7.18
(a) The function $f(n) = \frac{9}{5}n + 32$ can be used to convert degrees Celsius to degrees Fahrenheit. Calculate, $f(0), f(100), f(5)$, and $f(-40)$.
(b) The function $g(n) = \frac{5}{9}(n-32)$ can be used to convert degrees Fahrenheit to degrees Celsius. Calculate, $g(32), g(212), g(104)$, and $g(-40)$.
(c) Is there a temperature where the degrees Celsius equals the degrees Fahrenheit? If so, what is it?

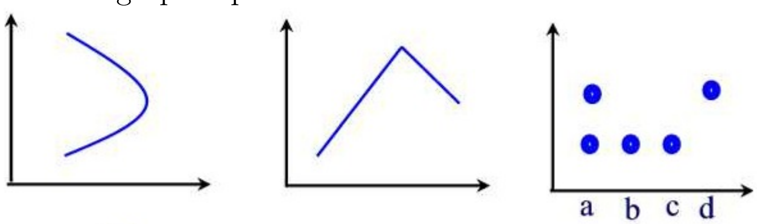
Problem 7.19
A fitness club charges an initiation fee of \$85 plus \$35 per month.
(a) Write a formula for a function, $C(x)$, that gives the total cost for using the fitness club facilities after x months.
(b) Calculate $C(18)$ and explain in words its meaning.
(c) When will the total amount spent by a club member first exceed \$1000?

Problem 7.20
If the interest rate of a \$1000 savings account is 5% and no additional money is deposited, the amount of money in the account at the end of t years is given by the function $a(t) = (1.05)^t \cdot 1000$.
(a) Calculate how much will be in the account after 2 years, 5 years, and 10 years.
(b) What is the minimum number of years that it will take to more than double the account?

Problem 7.21
A function has the formula $P(N) = 8n - 50$. The range for P is $\{46, 62, 78\}$. What is the domain?

Problem 7.22
Which of the following assignments creates a function?
(a) Each student in a school is assigned a teacher for each course.
(b) Each dinner in a restaurant is assigned a price.
(c) Each person is assigned a birth date.

Problem 7.23
Tell whether each graph represents a function.



(a) (b) (c)

8 Addition and Subtraction of Whole Numbers

In this section we introduce the operations of addition and subtraction of whole numbers $W = \{0, 1, 2, \cdots\}$ and discuss their properties. This will be done by means of two models: a set model and a number line model.

Addition and Its Properties
Finding the sum of two numbers is one of the first mathematical ideas a child encounters after learning the concept of whole numbers.

•The Set Model of Whole Number Addition
The idea of combining sets (that is, union) is used to define addition. An example is combining 5 frogs with three frogs to obtain a total of 8 frogs. See Figure 8.1.

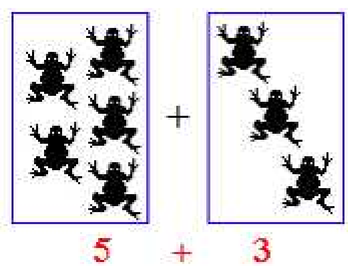


Figure 8.1

Thus, to find "5+3", find two disjoint sets, one with five elements and one with three elements, form their union, and then count the total number of elements. This suggests the following general definition of addition: If a set A contains a elements and a set B contains b elements with $A \cap B = \emptyset$ then $a + b$ is the number of elements in $A \cup B$.
The expression "a+b" is called the **sum** and the numbers a and b are called the **addends** or **summands**.

Example 8.1
Find the sum of 2 and 4.

Solution.
Let $A = \{a, b\}$ and $B = \{c, d, e, f\}$. Then $A \cup B = \{a, b, c, d, e, f\}$. Hence,

$2 + 4 = n(A \cup B) = 6$, where $n(A \cup B)$ denotes the number of elements in the set $A \cup B$.■

Because the sum of two whole numbers is again a whole number then we call addition a **binary operation.** Similarly, multiplication of whole numbers is a binary operation. However, subtraction and division of whole numbers are not binary operations since for example it is possible for the difference or ratio of two whole numbers not to be a whole number such as $2 - 3 = -1$ and $\frac{3}{2} = 1.5$.

•Number Line Model for Addition
On the number line, whole numbers are geometrically interpreted as distances. Addition can be visualized as combining two distances to get a total distance. The result of adding 3 and 5 is shown in Figure 8.2.

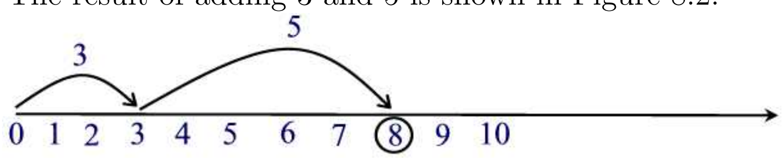


Figure 8.2

Example 8.2
Josh has 4 feet of red ribbon and 3 feet of white ribbon. How many feet of ribbon does he have altogether?

Solution.
According to Figure 8.3, Josh has a total of 7 feet of ribbon.■

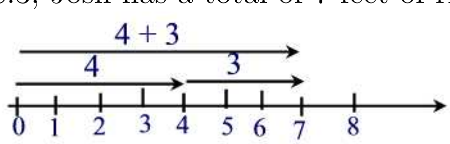


Figure 8.3

Next we examine some fundamental properties of addition of whole numbers that can be helpful in simplifying computations.

Closure Property
The sum of any two whole numbers is also a whole number, so we say that the set of whole numbers is **closed** under the operation of addition.

Commutative Property
If a and b are any two whole numbers then $a + b = b + a$.
This follows clearly from the fact that the sets $A \cup B$ and $B \cup A$ are equal.

Associative Property
If a, b, and c are three whole numbers then $a + (b + c) = (a + b) + c$.
This follows from the fact that $A \cup (B \cup C) = (A \cup B) \cup C$.

Identity Property for Addition
If a is a whole number then $a + 0 = 0 + a = a$.
This follows from the fact that $A \cup \emptyset = \emptyset \cup A = A$.

The addition properties are very useful when adding several whole numbers, since we are permitted to rearrange the order of the addends and the order in which pairs of addends are summed.

Example 8.3
Which properties justifies each of the following statements?
(a) $8 + 3 = 3 + 8$
(b) $(7 + 5) + 8 = 7 + (5 + 8)$.

Solution.
(a) Commutative property.
(b) Associative property.■

Example 8.4
Justify each equality below

$$\begin{array}{rcll} (20+2)+(30+8) & = & 20+[2+(30+8)] & (i) \\ & = & 20+[(30+8)+2] & (ii) \\ & = & 20+[30+(8+2)] & (iii) \\ & = & (20+30)+(8+2) & (iv) \end{array}$$

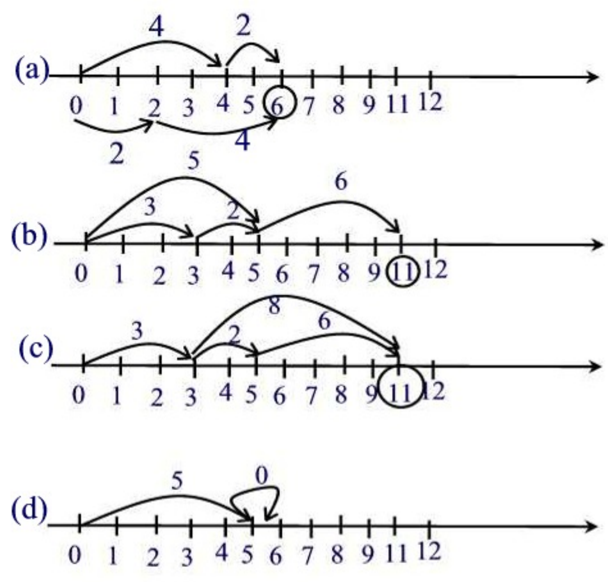
Solution.
(i) Associative property (ii) commutative property (iii) associative property (iv) associative property.■

Practice Problems

Problem 8.1
Explain whether the following sets are closed under addition. If the set is not closed, give an example of two elements in the set whose sum is not in the set.
(a) $\{0\}$
(b) $\{0, 3, 6, 9, 12, \cdots\}$
(c) $\{1, 2, 3, 4, 5, \cdots\}$
(d) $\{x \in W | x > 10\}$.

Problem 8.2
Each of the following is an example of one of the properties for addition of whole numbers. Identify the property illustrated:
(a) $1 + 5 = 5 + 1$
(b) $(1 + 5) + 7 = 1 + (5 + 7)$
(c) $(1 + 5) + 7 = (5 + 1) + 7$.

Problem 8.3
What properties of whole number addition are shown below?



Problem 8.4
Let $A = \{a, b, c\}, B = \{d, e\}, C = \{d, b, f\}$.
(a) Find $n(A \cup B), n(A \cup C)$, and $n(B \cup C)$.
(b) In which case is the number of elements in the union is not the sum of the number of elements in the individual sets?

Problem 8.5
Let $n(A) = 5, n(B) = 8$, and $n(A \cup B) = 10$. What can you say about $n(A \cap B)$?

Problem 8.6
The set C contains 2 and 3 and is closed under addition.
(a) What whole numbers must be in C?
(b) What whole numbers may not be in C?
(c) Are there any whole numbers definitely not in C?

Problem 8.7
Find 3 + 5 using a number line.

Problem 8.8
For which of the following pairs of sets is it true that $n(D)+n(E) = n(D \cup E)$?
(a) $D = \{1, 2, 3, 4\}, E = \{7, 8, 9, 10\}$
(b) $D = \emptyset, E = \{1\}$
(c) $D = \{a, b, c, d\}, E = \{d, c, b, a\}$.

Problem 8.9
Each of the following is an example of one of the properties for addition of whole numbers. Fill in the blank to complete the statement, and identify the property.
(a) $5 + \_\_\_\_ = 5$
(b) $7 + 5 = \_\_\_\_ + 7$
(c) $(4 + 3) + 6 = 4 + (\_\_\_\_ + 6)$
(d) $(4 + 3) + 6 = \_\_\_\_ + (4 + 3)$
(e) $(4 + 3) + 6 = (3 + \_\_\_\_) + 6$
(f) $2 + 9$ is a ____ number.

Problem 8.10
A first grader works out $6 + \_\_\_\_ = 10$ by counting forward on a number line. "I start at 6 and count up to 10. That's 6,7,8,9,10. The answer is 5."
(a) What is the child confused about?
(b) How would you help the child understand the correct procedure?

Subtraction of Whole Numbers
Subtraction of whole numbers can be modeled in several ways including the set (take-away) model, the missing-addend model, the comparison model, and the number-line model.

•Take-Away Model
You remember that in addition, a first set of objects is added to a second set of objects. For subtraction, a second set of objects is **taken away** from a first set of objects. For example, suppose that we have five circles and take away 2 of them, as shown in Figure 8.4. We record this process as 5 - 2 = 3.

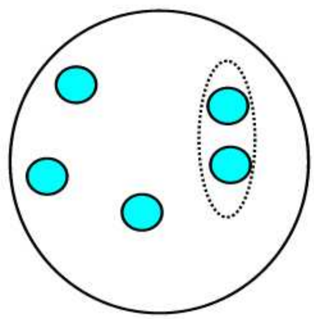


Figure 8.4

Using sets, we can state the above approach as follows: Let a and b be two whole numbers and A and B be two sets such that $B \subseteq A, a = n(A)$, and $b = n(B)$. Then

$$a - b = n(A - B)$$

where $A - B = \{x \in A \ and \ x \notin B\}$.

•The Missing-Addend Model
This model relates subtraction and addition. In this model, given two whole numbers a and b we would like to find the whole number c such that $c+b = a$. We call c the **missing-addend** and its value is $c = a - b$.
Cashiers often use this model. For example, if the bill for a movie is \$8 and you pay \$10, the cashier might calculate the change by saying "8 and 2 is 10."
From this model one can define subtraction of whole numbers as follows: Let a and b be two whole numbers with $a \geq b$. Then $a - b$ is the unique number

such that $c + b = a$.

•The Comparison Model
A third approach to consider subtraction is by using a comparison model. Suppose Jean has 5 circles and Peter has 3 circles and we want to know how many more circles Jean has than Peter. We can pair Peter circles with some of Jean circles, as shown in Figure 8.5, and determine that Jean has 2 more circles than Peter. We also write $5 - 3 = 2$.

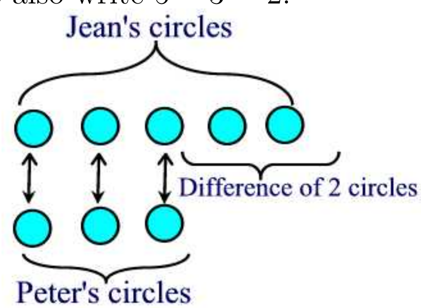


Figure 8.5

•The Number-Line Model
Subtraction can also be modeled on a number line, as shown in Figure 8.6

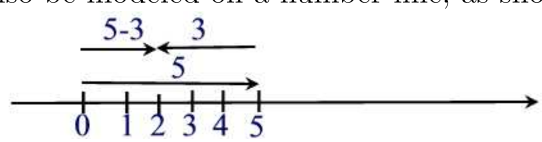


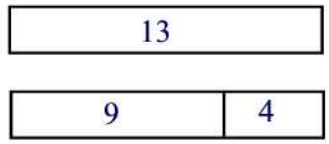
Figure 8.6

Practice Problems

Problem 8.11
Rewrite each of the following subtraction problems as an equivalent addition problem:
(a) $21 - 7 = x$ (b) $x - 119 = 213$ (c) $213 - x = 119$.

Problem 8.12
Explain how the following model can be used to illustrate each of the following addition and subtraction facts:

(a) $9 + 4 = 13$ (b) $4 + 9 = 13$ (c) $4 = 13 - 9$ (d) $9 = 13 - 4$.



Problem 8.13
Identify the conceptual model of subtraction that best fits these problems.
(a) Mary got 43 pieces of candy trick-or-treating on Halloween. Karen got 36 pieces. How many more pieces of candy does Mary have than Karen?
(b) Mary gave 20 pieces of her 43 pieces of candy to her sick brother, Jon. How many pieces of candy does Mary have left?
(c) Karen's older brother, Ken, collected 53 pieces of candy. How many more pieces of candy would Karen need to have as many as Ken?
(d) Ken left home and walked 10 blocks east along Grand Avenue trick-or-treating. The last 4 blocks were after crossing Main Street. How far is Main Street from Ken's house?

Problem 8.14
Let $A = \{a, b, c\}, B = \{d, e\}, C = \{d, b, f\}$.
(a) Find $n(A \cup B), n(A \cup C)$, and $n(B \cup C)$.
(b) In which case is the number of elements in the union is not the sum of the number of elements in the individual sets?

Problem 8.15
Jeff must read the last chapter of his book. It begins on the top of page 241 and ends at the bottom of page 257. How many pages must he read?

Problem 8.16
There is a nonempty subset of the whole numbers that is closed under subtraction. Find this subset.

Problem 8.17
For each of the following determine whole numbers x, y, and z that make the statement true.
(a) $x - 0 = 0 - x = x$.
(b) $x - y = y - x$
(c) $(x - y) - z = x - (y - z)$.

Problem 8.18
The property "If $a+c = b+c$ then $a = b$" is called the **additive cancellation property**. Is this property true for all whole numbers? If it is, how would you convince your students that it is always true? If not, give a counterexample.

Problem 8.19
How would you use the number line to show a child that $5 - 2 = 3$?

Problem 8.20
Make a drawing that shows $8 - 2 = 6$ using
(a) take-away approach
(b) comparison approach
(c) number line approach.

Problem 8.21
A first grader works out $8 - 3$ by counting back on a number line. "I start at 8 and go back 3. That's 8, 7, 6. The answer is 6."
(a) What is the child confused about?
(b) How would you help the child understand the correct procedure?

Problem 8.22
Represent the following algebraic expressions, using the variable x.
(a) The difference between 10 and a number.
(b) A number is increased by 2.
(c) The sum of a number and 6.

Problem 8.23
Solve the following using addition and subtraction:" John starts off with $A. He buys food for $F and clothes for $C, and then he receives a paycheck for $P. Write an expression representing the total amount of money he has now."

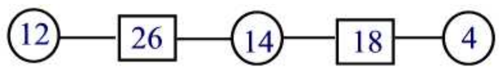
Problem 8.24
Under what conditions $(a - b) - c$ is a whole number?

Problem 8.25
If $a - b = c$ then $a - (b + 1) =$ ____.

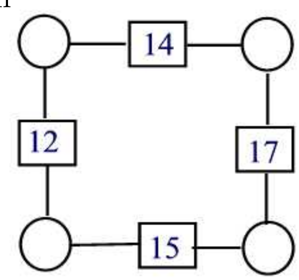
Problem 8.26
Consider the following problem:"Dad just hung up 3 shirts from the laundry in the closet next to his other shirts. Now there are 10 shirts in the closet. How many were there before?"

Problem 8.27
How could you measure 1 oz of syrup using only a 4-oz container and a 7-oz container?

Problem 8.28
A **number chain** is created by adding and subtracting. The number in each square is the sum of the numbers that are next to it on both sides as shown in the figure below.



Consider the number chain



(a) Fill in the circles with numbers that work.
(b) If possible, find a second solution.
(c) Start with X in the upper left-hand circle and use algebra to fill in all the circles.
(d) What does your answer in part (c) tell you about the relationship between the numbers in opposite corners?

Problem 8.29
A child in a first-grade class asks,"What does 2 - 3 equal?" How would you respond?

9 Multiplication and Division of Whole Numbers

Multiplication of Whole Numbers
In this section, we use four models to discuss multiplication: the repeated-addition model, the array model, the Cartesian-product model, and the tree diagram model. Also, we investigate the properties of multiplication.

•Repeated-Addition Model
For any whole numbers a and b,

$$a \times b = \underbrace{b + b + \cdots + b}_{a \ addends}, \quad where\ a \neq 0.$$

If $a = 0$, then $0 \times b = 0$.
The number $a \times b$, read a times b, is called the **product** of a and b. The numbers a and b are called **factors.** Another notation for $a \times b$ is $a \cdot b$.

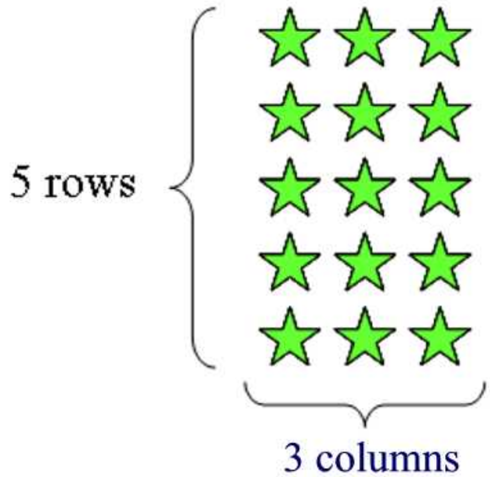
Example 9.1
Misha has an after-school job at a local bike factory. Each day she has a 3-mile round-trip walk to the factory. At her job she assembles 4 hubs and wheels. How many hubs and wheels does she assemble in 5 afternoons?

Solution.
This problen can be answered by repeated addition. Misha assembles

$$4 + 4 + 4 + 4 + 4 = 20 = 5 \times 4$$

hubs and wheels.■

•The Rectangular Array Model
Let a and b be whole numbers, the product $a \times b$ is defined to be the number of elements in a rectangular array having a rows and b columns. For example, 5×3 is equal to the number of stars in the following array



Example 9.2
Suppose Lida, as part of her biology research, planted 5 rows of bean seeds and each row contains 8 seeds. How many seeds did she plant in her rectangular plot?

Solution.
Counting the seeds in the rectangular plot we find 40 seeds or 5×8. ■

•The Cartesian Product Model
Let a and b be any two whole numbers. Pick sets A and B such that $n(A) = a$ and $n(B) = b$. Then $a \times b$ is the number of ordered pairs in the Cartesian product $A \times B$, i.e.

$$a \times b = n(A \times B)$$

where

$$A \times B = \{(a, b)|a \in A \ and \ b \in B\}.$$

For example, to compute 3×2, we pick sets $A = \{a, b, c\}$ and $B = \{x, y\}$. Then

$$A \times B = \{(a, x), (b, x), (c, x), (a, y), (b, y), (c, y)\}.$$

Hence, $3 \times 2 = 6$.

•The Multiplication Tree Model
This model is also known as the **multiplication counting principle** and plays an important role in probability theory. To find $a \times b$ we find the total

number of different pairs formed by pairing any object from one set with any object from a second set. Let's consider the following problem: You are at a carnival. One of the carnival games asks you to pick a door and then pick a curtain behind the door. There are 3 doors and 4 curtains behind each door. How many choices are possible for the player? The multiplication tree shown in Figure 9.1 represents the product 3×4.

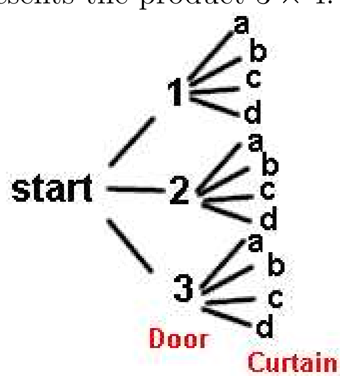


Figure 9.1

Properties of Whole Numbers Multiplication

Properties of whole-number operations make it easier to memorize basic facts and do certain computations. For example, if you learn $7 \times 9 = 63$ then you know what 9×7 equals.

Closure property

The product of any two whole numbers is still a whole number.

Commutative property

For any two whole numbers a and b, $a \times b = b \times a$

Associative property

For any three whole numbers a, b, and c,

$$a \times (b \times c) = (a \times b) \times c.$$

Identity property

For any whole number a, we have

$$a \times 1 = a = 1 \times a.$$

Distributive Property of Multiplication over addition
For any whole numbers a, b, and c,

$$a \times (b + c) = a \times b + a \times c.$$

Distributive property of Multiplication over subtraction
Let a, b, and c be whole numbers with $b \geq c$, then

$$a \times (b - c) = a \times b - a \times c.$$

Multiplication property of Zero.
For any whole number a, we have $a \times 0 = 0$.

Zero Product Property
For any whole numbers a and b, if $a \times b = 0$, then either $a = 0$ or $b = 0$.

Example 9.3
Use the properties of multiplication to justify the formula

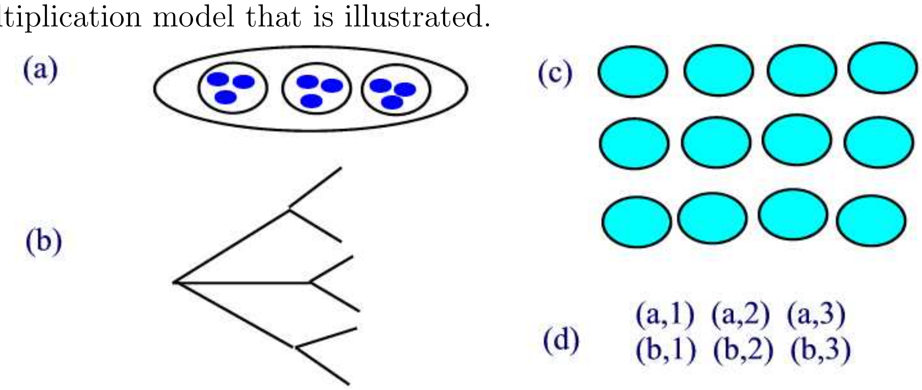
$$(a + b)(c + d) = ac + ad + bc + bd.$$

Solution.

$$\begin{aligned}
(a + b)(c + d) &= (a + b)c + (a + b)d \quad \textit{distributive property} \\
&= ac + bc + ad + bd \quad \textit{distributive property} \\
&= ac + ad + bc + bd \quad \textit{commutative property} \blacksquare
\end{aligned}$$

Practice Problems

Problem 9.1
What multiplication fact is illustrated in each of these diagrams? Name the multiplication model that is illustrated.



Problem 9.2
Illustrate 4×6 using each of the following models.
(a) set model (repeated addition)
(b) rectangular array model
(c) Cartesian product model
(d) multiplication tree.

Problem 9.3
Which of the following sets are closed under multiplication? Why or why not?
(a) $\{2, 4\}$
(b) $\{0, 2, 4, 6, \cdots\}$
(c) $\{5, 7, 9, 11, \cdots\}$
(d) $\{0, 2^0, 2^1, 2^2, 2^3, \cdots\}$.

Problem 9.4
What properties of whole number multiplication justify these equations?
(a) $4 \times 9 = 9 \times 4$
(b) $4 \times (6 + 2) = 4 \times 6 + 4 \times 2$
(c) $0 \times 12 = 0$
(d) $5 \times (9 \times 11) = (5 \times 9) \times 11$
(e) $7 \times 3 + 7 \times 8 = 7 \times (3 + 8)$.

Problem 9.5
Rewrite each of the following expressions using the distributive property for multiplication over addition or subtraction. Your answer should contain no parentheses.
(a) $4 \times (60 + 37)$
(b) $3 \times (29 + 30 + 6)$
(c) $a \times (7 - b + z)$.

Problem 9.6
Each situation described below involves a multiplication problem. In each case state whether the problem situation is best represented by the repeated-addition model, the rectangular array model, or the Cartesian product model, and why. Then write an appropriate equation to fit the situation.

(a) At the student snack bar, three sizes of beverages are available: small, medium, and large. Five varieties of soft drinks are available: cola, diet cola, lemon-lime, root beer, and orange. How many different choices of soft drink does a student have, including the size that may be selected?

(b)At graduation students file into the auditorium four abreast. A parent seated near the door counts 72 rows of students who pass him. How many students participated in the graduation exercise?

(c) Kirsten was in charge of the food for an all-school picnic. At the grocery store she purchased 25 eight-packs of hot dog buns for 70 cents each. How much did she spend on the hot dog buns?

Problem 9.7
A stamp machine dispenses twelve 32 cents stamps. What is the total cost of the twelve stamps?

Problem 9.8
What properties of multiplication make it easy to compute these values mentally?
(a) $7 \times 19 + 3 \times 19$
(b) $36 \times 15 - 12 \times 45$.

Problem 9.9
Using the distributive property of multiplication over addition we can factor as in $x^2 + xy = x(x + y)$. Factor the following:
(a) $xy + x^2$
(b) $47 \times 99 + 47$
(c) $(x + 1)y + (x + 1)$
(d) $a^2b + ab^2$.

Problem 9.10
Using the distributive property of multiplication over addition and subtraction to show that
(a) $(a + b)^2 = a^2 + 2ab + b^2$
(b) $(a - b)^2 = a^2 - 2ab + b^2$
(c) $(a - b)(a + b) = a^2 - b^2$.

Problem 9.11
Find all pairs of whole numbers whose product is 36.

Problem 9.12
A new model of car is available in 4 exterior colors and 3 interior colors. Use a tree diagram and specific colors to show how many color schemes are possible for the car?

Problem 9.13
Is $x \times x$ ever equal to x? Explain your answer.

Problem 9.14
Describe all pairs of numbers whose product and sum are the same.

Problem 9.15
The operation $\odot$ is defined on the set $S = \{a, b, c\}$ by the following **Cayley's table**. For example, $a \odot c = c$.

$\odot$	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

(a) Is S closed under $\odot$?
(b) Is $\odot$ commutative?
(c) Is $\odot$ associative?
(d) Is there an identity for $\odot$ on S? If yes, what is it?

Division of Whole Numbers

In this section, we discuss division using three models: the repeated-subtraction model, the set (partition) model, and the missing factor model.

•Repeated-subtraction approach
For any whole numbers a and b, $a \div b$ is the maximum number of times that b objects can be successively taken away from a set of a objects (possibly with a remainder).

Example 9.4
Suppose we have 18 cookies and want to package them in cookie boxes that hold 6 cookies each. How many boxes are needed?

Solution.
If one box is filled then we would have $18 - 6 = 12$ cookies left. If one more box is filled, then there are $12-6 = 6$ cookies left. Finally, we could place the last 6 cookies in a third box. This discussion can be summarized by writing

$$18 - 6 - 6 - 6 = 0.$$

We have found by repeated subtraction that $18 \div 6 = 3.$ ■

•The Partition Model
If a and b are whole numbers, then $a \div b$ is the number of objects in each group when a objects are separated into b equal groups.

Example 9.5
Suppose we have 18 cookies and want to give an equal number of cookies to each of three friends:Bob, Dean, and Charlie. How many should each person receive?

Solution.
If we draw a picture, we can see that we can partition the 18 cookies into 3 sets, with an equal number of cookies in each set. Figure 9.2 shows that each friend receives 6 cookies. ■

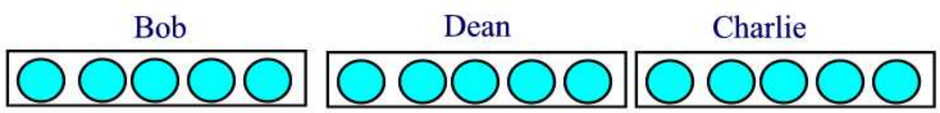


Figure 9.2

Remark 9.1
The difference between the above two approaches is very subtle when we deal with whole numbers, but it will become more apparent when we divide decimals or fractions.

•The Missing-Factor Model
Let a and b be two whole numbers with $b \neq 0$. Then $a \div b = c$ if and only if there exists a <u>unique</u> whole number c such that $a = b \times c$.
The symbol $a \div b$ is read "a divided by b", where a is called the **dividend,** b is called the **divisor**, and c is called the **quotient** or the **missing-factor.**

Example 9.6
Suppose you have 78 tiles. Describe how to illustrate $78 \div 13$ with the tiles, using each of the three models above.

Solution.
(a) **Repeated subtraction:** $78 - 13 - 13 - 13 - 13 - 13 - 13 = 0$. Thus, $78 \div 13 = 6$.
(b) **Partition:** Partition the tiles into 13 equal-sized parts. Since each part contains exactly 6 tiles then $78 \div 13 = 6$.
(c) Since $13 \times 6 = 78$ then $78 \div 13 = 6$.■

Division Properties of Zero
1. If a is a nonzero whole number then since $a \times 0 = 0$ then $0 \div a = 0$.
2. Since any whole number c satisfies the equality $0 = c \times 0$, that is c is not unique, then $0 \div 0$ is undefined.
3. If a is a nonzero whole number then there is no whole number c such that $c \times 0 = a$. That is, $a \div 0$ is undefined.

Division with Remainders: The Division Algorithm
Just as subtraction of whole numbers is not closed so is division of whole numbers. To see this, consider the division problem $27 \div 6$. There is no whole number c such that $6 \times c = 27$ so $27 \div 6$ is not defined in the whole numbers. That is, the set of whole numbers is not closed under division.
By allowing the possibility of a remainder, we can extend the division operation. Using repeated subtraction, four groups of 6 can be removed from 27. This leaves 3, which is too few to form another group of 6. This can be written as

$$27 = 4 \times 6 + 3.$$

Here the 4 is called the **quotient** and 3 is the **remainder.** In general, we have

The division algorithm
If a and b are whole numbers with $b \neq 0$, then there exist unique whole numbers q and r such that

$$a = b \times q + r, \quad \textit{where } 0 \leq r < b.$$

Example 9.7
Find the quotient and the remainder when $57 \div 9$.

Solution.
Since $57 = 6 \times 9 + 3$ then $q = 6$ and $r = 3$.■

Practice Problems

Problem 9.16
Rewrite each of the following division problems as a multiplication problem.
(a) $48 \div 6 = 8$ (b) $24 \div x = 12$ (c) $a \div b = x$.

Problem 9.17
Show, that each of the following is false when x, y, and z are replaced by whole numbers. Give an example (other than dividing by zero) where each statement is false.

(a) $x \div y$ is a whole number
(b) $x \div y = y \div x$
(c) $x \div (y \div z) = (x \div y) \div z$
(d) $x \div (y + z) = x \div y + x \div z$.

Problem 9.18
Find the quotient and the remainder for each division.
(a) $7 \div 3$ (b) $3 \div 7$ (c) $7 \div 1$ (d) $1 \div 7$ (e) $15 \div 4$.

Problem 9.19
How many possible remainders (including zero) are there when dividing by the following numbers?
(a) 2 (b) 12 (c) 62 (d) 23.

Problem 9.20
Which of the following properties hold for division of whole numbers?
(a) Closure (b) Commutativity (c) Associativity (d) Identity.

Problem 9.21
A square dancing contest has 213 teams of 4 pairs each. How many dancers are participating in the contest?

Problem 9.22
Discuss which of the three conceptual models of division-repeated subtraction, partition, missing factor-best corresponds to the following problems.

More than one model may fit.

(a) Preston owes \$3200 on his car. If his payments are \$200 a month, how many months will preston make car payments?
(b) An estate of \$76,000 is to be split among 4 heirs. How much can each heir expect to inherit?
(c) Anita was given a grant of \$375 to cover expenses on her trip. She expects that it will cost her \$75 a day. How many days can she plan to be gone?

Problem 9.23
Solve for the unknown whole number in the following expressions:
(a) When y is divided by 5 the resulting quotient is 5 and the remainder is 4.
(b) When 20 is divided by x the resulting quotient is 3 and the remainder is 2.

Problem 9.24
Place parentheses, if needed, to make each of the following equations true:
(a) $4 + 3 \times 2 = 14$
(b) $9 \div 3 + 1 = 4$
(c) $5 + 4 + 9 \div 3 = 6$
(d) $3 + 6 - 2 \div 1 = 7$.

Problem 9.25
A number leaves remainder 6 when divided by 10. What is the remainder when the number is divided by 5?

Problem 9.26
Is $x \div x$ always equal to 1? Explain your answer.

Problem 9.27
Find infinitely many whole numbers that leave remainder 3 upon division by 5.

Problem 9.28
Steven got his weekly paycheck. He spent half of it on a gift for his mother. Then he spent \$8 on a pizza. Now he has \$19. How much was his paycheck?

10 Ordering and Exponents of Whole Numbers

Ordering Whole Numbers
In Section 5, we used the concept of a set and the concept of one-to-one correspondence to define **less than** relations. In this section we want to define ”¡” using addition of whole numbers.
A number line can be used to describe **less than** and **greater than** relations on the set of whole numbers. For example, in Figure 10.1, notice that 4 is to the left of 7 on the number line. We say that ”four is less than seven” and we write $4 < 7$. We cn also say that ”seven is greater than 4” and write $7 > 4$.

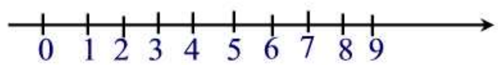


Figure 10.1

Note that since 4 is to the left of 7, there is a whole number that can be added to 4 to get 7, namely 3. Thus, $4 < 7$ since $4 + 3 = 7$. We can generalize this discussion to introduce the following deifnition of ”less than”.
For any whole numbers a and b, we say that a is **less than** b, denoted by $a < b$, if and only if there is a unique nonzero whole number n such that $a + n = b$. Similarly, we say that ”a is greater than” b and write $a > b$ if and only if there is a unique nonzero whole number n such that $a - n = b$.
Sometimes equality is combined with inequalities, greater than and less than, to give the relations **greater than** or **less than or equal to** denoted by $\geq$ and $\leq$ respectively. Thus, $a \leq b$ means $a < b$ or $a = b$.

Example 10.1
Find the nonzero whole number n in the definition of $<$ and $>$ that verifies the following statements:
(a) $12 < 31$ (b) $53 > 37$.

Solution.
(a) Since $12 + 19 = 31$ then $n = 19$.
(b) Since $53 - 16 = 37$ then $n = 16$.■

Properties of "less than" of Whole Numbers

• **Transitive Property:** If a, b, and c are whole numbers such that $a < b$ and $b < c$ then $a < c$. See Figure 10.2.
To verify this property formally, we let n be a whole number such that $a+n = b$ and m a whole number such that $b+m = c$. Then $a+(n+m) = c$. But $n+m$ is a whole number since W is closed under addition. By the definition of "less than" we conclude that $a < c$.

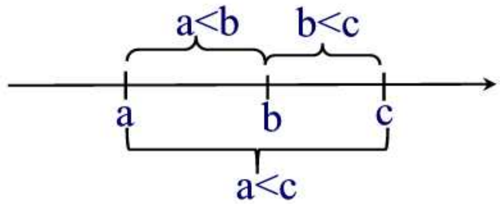


Figure 10.2

• **Addition Property:** If a and b are two whole numbers such that $a < b$ then $a+c < b+c$ for any whole number c.
To verify this property, let n be a whole number such that $a+n = b$. Then

$$\begin{array}{rcll} (a+c)+n & = & a+(c+n) & (+\ \textit{is associative}) \\ & = & a+(n+c) & (+\ \textit{is commutative}) \\ & = & (a+n)+c & (+\ \textit{is associative}) \\ & = & b+c & \end{array}$$

From the definition of "less than" we conclude that $a+c < b+c$.

Remark 10.1
The addition property of inequality also allows us to subtract the same number from both sides of an inequality because subtraction is defined in terms of addition.

• **Multiplication Property:** If a and b are whole numbers such that $a < b$ then for any nonzero whole number c we have $ac < bc$.
To see this, let n be a whole number such that $a+n = b$. Then $(a+n)c = bc$ or $ac+nc = bc$ since multiplication is distributive with repsect to addition. Also, since W is closed under multiplication we see that $cn \in \mathbb{N}$. By the definition of "less than" we conclude that $ac < bc$.

Practice Problems

Problem 10.1
Using the definition of $<$ and $>$ given in this section, write four inequality statements based on the fact that $2 + 8 = 10$.

Problem 10.2
The statement $a < x < b$ is equivalent to writing $a < x$ and $x < b$ and is called a **compound inequality.** Suppose that a, x, and b are whole numbers such that $a < x < b$. Is it is always true that for any whole number c we have $a + c < x + c < b + c$?

Problem 10.3
Find nonzero whole number n in the definition of "less than" that verifies the following statements.
(a) $17 < 26$ (b) $113 > 49$.

Problem 10.4
If $a < x < b$, where a, x, b are whole numbers, and c is a nonzero whole number, is it always true that $ac < xc < bc$?

Problem 10.5
True or false?
(a) $0 \geq 0$ (b) $0 < 0$ (c) $3 < 4$ (d) $2 \times 3 + 5 < 8$.

Problem 10.6
Write an inequality that describe each situation.

(a) The length of a certain rectangle must be 4 meters longer than the width, and the perimeter must be at least 120 meters.
(b) Fred made a 76 on the midterm exam. To get a B, the average of his mid-term and his final exam must be between 80 and 90.

Problem 10.7
Find all the whole numbers x such that $3 + x < 8$.

Problem 10.8
Find all the whole numbers x such that $3x < 12$.

Problem 10.9
Complete the following statement: If $x - 1 < 2$ then $x <$ ___.

Problem 10.10
Complete the following statement: If $x + 3 < 3x + 5$ then $3x + 9 <$ _____.

Exponents of Whole Numbers
Instead of writing $3 \cdot 3 \cdot 3 \cdot 3$ we can follow a notation introduced by Descartes and write 3^4. This operation is called "taking three to the fourth power." The general definition is described as follows.
Let a and n be two whole numbers with $n \neq 0$. Then a to the **nth power**, written a^n, is defined by

$$a^1 = 1, \quad if\ n = 1$$

and

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n\ factors}, \quad if\ n > 1.$$

We call a the **base**, n the **exponent** or **power**, and a^n is called an **exponential.**

Example 10.2
Rewrite using a single exponent.
(a) $7^4 \cdot 7^2$ (b) $6^3 \cdot 6^5$.

Solution.
(a) $7^4 \cdot 7^2 = (7 \cdot 7 \cdot 7 \cdot 7) \cdot (7 \cdot 7) = 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^6$.
(b) $6^3 \cdot 6^5 = (6 \cdot 6 \cdot 6) \cdot (6 \cdot 6 \cdot 6 \cdot 6 \cdot 6) = 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^8$.■

Properties of Exponentials
• Let a, m, and n be whole numbers with $m \neq 0$ and $n \neq 0$. Then $a^m \cdot a^n = a^{m+n}$.
To see this, note that $a^m \cdot a^n = (\underbrace{a \cdot a \cdots a}_{m\ factors})(\underbrace{a \cdot a \cdots a}_{n\ factors}) = \underbrace{a \cdot a \cdots a}_{m+n\ factors} = a^{m+n}$.
• Let a, m, and n be whole numbers with $m \neq 0$ and $n \neq 0$. Then $(a^m)^n = a^{m \cdot n}$.
To see this, note that $(a^m)^n = \underbrace{a^m \cdot a^m \cdots a^m}_{n\ factors} = \underbrace{\underbrace{a \cdot a \cdots a}_{m\ factors} \cdot \underbrace{a \cdot a \cdots a}_{m\ factors} \cdots \underbrace{a \cdot a \cdots a}_{m\ factors}}_{n\ factors} =$

$\underbrace{a \cdot a \cdots a}_{m \cdot n\ factors} = a^{m \cdot n}$.

• Let a, b, and n be whole numbers with $n \neq 0$. Then $(a \cdot b)^n = a^n \cdot b^n$.
To see this, note that $a^n \cdot b^n = \underbrace{a \cdot a \cdots a}_{n\ factors} \cdot \underbrace{b \cdot b \cdots b}_{n\ factors} = \underbrace{(ab) \cdot (ab) \cdots (ab)}_{n\ factors} = (a \cdot b)^n$.

Example 10.3
Compute the following products and powers, expressing your answers in the form of a single exponential a^n.
(a) $7^4 \cdots 7^3$ (b) $2^3 \cdot 5^3$ (c) $3^2 \cdot 5^2 \cdot 7^2$ (d) $(3^2)^5$.

Solution.
(a) $7^4 \cdot 7^3 = 7^{4+3} = 7^7$.
(b) $2^3 \cdot 5^3 = (2 \cdot 5)^3 = 10^3$.
(c) $3^2 \cdot 5^2 \cdot 7^2 = (2 \cdot 5 \cdot 7)^2 = 70^2$.
(d) $(3^2)^5 = 3^{10}$.■

If the formula $a^m \cdot a^n = a^{m+n}$ were extended to allow $m = 0$, it would state that $a^0 \cdot a^n = a^{0+n} = a^n$. This suggets that it is reasonable to define

$$a^0 = 1, \quad for\ a \neq 0.$$

What if $a = 0$? If we look at the two patterns $3^0 = 1, 2^0 = 1, 1^0 = 1$ and $0^3 = 0, 0^2 = 0, 0^1 = 0$ then the first one suggests that $0^0 = 1$ whereas the second one suggests that 0^0. To avoid such an inconsistency, 0^0 is undefined.
We close this section by considering the division of exponentials. Let a, m, and n be whole numbers with $m \geq n, m \neq 0$, and $n \neq 0$. Since $a^{m-n} \cdot a^n = a^{(m-n)+n} = a^m$ then from the definition of division we see that

$$a^m \div a^n = a^{m-n}$$

or using the bar notation for division we have

$$\frac{a^m}{a^n} = a^{m-n}.$$

Example 10.4
Rewrite the following division with a single exponential.
(a) $5^7 \div 5^3$ (b) $7^8 \div 7^5$.

Solution.
(a) $5^7 \div 5^3 = 5^{7-3} = 5^4$.
(b) $7^8 \div 7^5 = 7^{8-5} = 7^3$.■

Practice Problems

Problem 10.11
Rewrite the following products using exponentials.
(a) $3 \cdot 3 \cdot 3 \cdot 3$
(b) $2 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2$
(c) $a \cdot b \cdot a \cdot b$.

Problem 10.12
Rewrite each with a single exponent.
(a) $5^3 \cdot 5^4$
(b) $3^{12} \div 3^2$
(c) $2^7 \cdot 5^7$
(d) $8 \cdot 2^5$
(e) $25^3 \div 5^2$
(f) $9^2 \cdot 12^3 \cdot 2$

Problem 10.13
Find a whole number x.
(a) $3^7 \cdot 3^x = 3^{13}$
(b) $(3^x)^4 = 3^{20}$
(c) $3^x \cdot 2^x = 6^x$.

Problem 10.14
The price of a candy bar doubled every five years. Suppose that the price continued to double every five years and that the candy bar cost 25 cents in 2000.
(a) What would the price of the candy bar be in the year 2015?
(b) What would the price be in the year 2040?
(c) Write an expression representing the price of the candy bar after n five years.

Problem 10.15
Pizzas come in four different sizes, each with or without a choice of up to four ingredients. How many ways are there to order a pizza?

Problem 10.16
Write each of the following in expanded form, i.e. without exponents.
(a) $(2x)^5$ (b) $2x^5$.

11 Whole Numbers: Mental Arithmetic and Estimation

So far we have been focusing on paper-and-pencil strategies for doing arithmetic with whole numbers. In this section we focus on two other tools, namely, mental arithmetic and computational estimation.

Mental Arithmetic

Mental arithmetic is the process of producing an answer to a computation without using any computational aids such as calculators, computers, tables, etc. We consider some of the most common strategies for performing arithmetic operations mentally on whole numbers.

Mental Addition

• **Left-to-Right Approach:** In this model, to add 347 and 129, we first add the hundreds (300 + 100) then the tens (40 + 20), and then the ones (7+9), to obtain 476.

• **Compensation:** To find the sum $67 + 29$, we add $67 + 30$ (since 30 is the next multiple of 10 greater than 29) to obtain 97 and then we subtract 1 from 97 to compensate for the extra 1 that was added to obtain $97 - 1 = 96$.

• **Using Compatible Numbers:** Compatible numbers are numbers whose sums are easy to calculate mentally. For example, if we are trying to find the sum $130 + 50 + 70 + 20 + 50$ we will add first the numbers 130 and 70 to obtain 200, the numbers 50 and 50 to obtain 100, then the numbers 200 and 100 to obtain 300 and finally we add 20 to 300 to obtain 320.

• **Breaking Up and Bridging:** To find the sum $67 + 36$ we find first the sum $67 + 30 = 97$ and then the sum $97 + 6 = 103$.

Mental Subtraction

• **Left-to-Right:** To find the difference $93 - 38$ we find first $90 - 30 = 60$, then $8 - 3 = 5$ and finally $60 - 5 = 55$. To find the difference, $47 - 32$, we find first $40 - 30 = 10$, then $7 - 2 = 5$ and finally $10 + 5 = 15$.

• **Breaking Up and Bridging:** To find the difference $67 - 36$ we first find $67 - 30 = 37$ and then $37 - 6 = 31$.

• **Compensation** To find $47 - 29$ is the same as finding $48 - 30 = 18$. That is we add the same number to both addends. This is known as the **equal addition method** for subtraction.

• **Drop the Zeros:** To find $8700 - 500$ we find $87 - 5 = 82$ and then add the two zeros to the right of 82 to obtain 8200.
• **Compatible Numbers:** To find the difference $170 - 50 - 30 - 50$ we can find first $170 - 30 = 140$, then $50 + 50 = 100$ and finally $140 - 100 = 40$.

Mental Multiplication
• **Compatible Numbers:** To find the product $2 \times 9 \times 5 \times 20 \times 5$ we can rearrange the product in the form $9 \times (2 \times 5) \times (20 \times 5) = 9 \times 10 \times 100 = 9000$.
• **Left-to-Right:** To find the product 3×123 we can look at it as the expression $3 \times 100 + 3 \times 20 + 3 \times 3 = 1012$.
• **Multiplying Powers of 10:** To find the product $12,000 \times 110,000$ we multiply the numbers $12 \times 11 = 132$ and then add 7 zeros to the right to obtain $1,320,000,000$.

Mental Division
• **Compatible Numbers** To find $105 \div 3$ we look for two numbers that are divisible by 3 and whose sum is 105, namely, 90 and 15. We then divide both numbers by 3 to obtain $90 \div 3 = 30$ and $15 \div 3 = 5$. Finally add the quotients to obtain $105 \div 3 = 30 + 5 = 35$.

Practice Problems

Problem 11.1
Perform each of the following computations mentally and explain what technique you used to find the answer.
(a) $40 + 160 + 29 + 31$
(b) $3679 - 474$
(c) $75 + 28$
(d) $2500 - 700$.

Problem 11.2
Compute each of the following mentally.
(a) $180 + 97 - 23 + 20 - 140 + 26$
(b) $87 - 42 + 70 - 38 + 43$.

Problem 11.3
Use compatible numbers to compute each of the following mentally.
(a) $2 \cdot 9 \cdot 5 \cdot 6$

(b) $5 \cdot 11 \cdot 3 \cdot 20$
(c) $82 + 37 + 18 + 13$.

Problem 11.4
Use compensation to compute each of the following mentally.
(a) $85 - 49$
(b) $87 + 33$
(c) $19 \cdot 6$.

Problem 11.5
A car trip took 8 hours of driving at an average of 62 mph. Mentally compute the total number of miles traveled. Describe your method.

Problem 11.6
Perform these calculations from left to right.
(a) $425 + 362$
(b) $572 - 251$
(c) $3 \cdot 342$
(d) $47 + 32 + 71 + 9 + 26 + 32$.

Problem 11.7
Calculate mentally using properties of operations, i.e. commutative, associative, distributive.
(a) $(37 + 25) + 43$
(b) $47 \cdot 15 + 47 \cdot 85$
(c) $(4 \times 13) \times 25$
(d) $26 \cdot 24 - 21 \cdot 24$.

Problem 11.8
Find each of the following differences using compensation method.
(a) $43 - 17$
(b) $132 - 96$
(c) $250 - 167$.

Problem 11.9
Calculate mentally.
(a) $58,000 \times 5,000,000$
(b) $7 \times 10^5 \times 21,000$
(c) $5 \times 10^3 \times 7 \times 10^7 \times 4 \times 10^5$.

Problem 11.10
Show the steps for three different ways to compute mentally $93 + 59$.

Problem 11.11
Show the steps for three different ways to compute mentally $134 - 58$.

Problem 11.12
Show the steps to compute mentally $(500)^3$.

Problem 11.13
A restaurant serves launch to 90 people per day. Show the steps to mentally compute the number of people served lunch in 31 days.

Problem 11.14
There is a shortcut for multiplying a whole number by 99. For example, consider 15×99.
(a) Why does $15 \times 99 = (15 \times 100) - (15 \times 1)$?
(b) Compute 15×99 mentally, using the formula in part (a)
(c) Compute 95×99 mentally, using the same method.

Problem 11.15
(a) Develop a shortcut for multiplying by 25 mentally in a computation such as 24×25.
(b) Compute 44×25 using the same shortcut.

Problem 11.16
(a) Develop a shortcut for multiplying by 5 mentally in a computation such as 27×5.
(b) Compute 42×5 using the same shortcut

Problem 11.17
A fifth grader computes 29×12 as follows: $30 \times 12 = 360$ and $360 - 12 = 348$. On what property is the child's method based?

Estimation
Computational estimation is the process of forming an approximate answer to a numerical solution. Estimation strategies can be used to tell whether answers are reasonable or not. An **estimate** is a number close

to an exact answer. We consider two estimation strategies

•Range Estimation
Consider the operation 294×53. The product $300 \times 60 = 18000$ is an **overestimate** of 294×53 whereas the product $200 \times 50 = 10000$ is an **underestimate**. We say that the product 294×53 is in the **range** from 10000 to 18000.

Example 11.1
Find a range estimate for the sum $3741 + 1252$.

Solution.
The lower estimate or underestimate is 4000 and the upper estimate or overestimate is 6000. So the sum is in the range from 4000 to 6000.■

•Front-End Strategy
Front-end (leading digit) estimation is especially useful in addition. We consider three types of such estimate. Consider the problem of estimating the sum $4589 + 398$.
(1) One-Column Front End Estimation Draw a line seperating the leading digit(s). Add the numbers to the left of the column and ignore the numbers to the right. Thus, $4589 + 398 \approx 4000$. The symbol $\approx$ means "is approximated by". Similarly, $372 + 53 + 417 \approx 300 + 0 + 400 = 700$.
(2) Two-Column Front End Estimation This one improves the previous estimate. To see how this strategy works, consider the sum $372 + 53 + 417$. Draw a line seperating the first two leading digits. Add the numbers to the left of the line and ignore the ones to the right. Thus, $372 + 53 + 417 \approx 370 + 50 + 410 = 830$.
(3) Front-End with Adjustments This method enhances the one-column front-end methods. After adding the numbers to the left of the line, adjust the answer by considering the digits in the next column to the right. For example, to estimate $498 + 251$ we first do $400 + 200 = 600$ and then $98 + 51 \approx 100 + 50 = 150$. Thus, $498 + 251 \approx 600 + 150 = 750$.

Example 11.2
Estimate using the method indicated.
(a) 503×813 using one-column front-end method
(b) 1200×35 using range estimation

(c) $4376 - 1889$ using two-column front-end
(d) $3257 + 874$ using front-end adjustment.

Solution.
(a) $503 \times 813 \approx 500 \times 800 = 400000$.
(b) 1200×35 is in the range from $1200 \times 30 = 36000$ to $1200 \times 40 = 48000$.
(c) $4376 - 1889 \approx 4300 - 1800 = 2500$.
(d) We first do $3000 + 800 = 3800$ and then $250 + 80 = 330$ so that $3257 + 874 \approx 3800 + 330 = 4100$.■

Estimation by Rounding
Suppose you are asked to ROUND a number to the nearest ten, to the nearest hundred, to the nearest thousand, and so forth. In that case, underline the digit in the place you are asked to round to and then follow these steps:
• Look one place to the right of the number you have underlined. If the number to the right is a 5 or higher then the number you underlined will increase by one. The rest of the numbers to the right will become ZERO.
• If the number one place to the right of the number you underlined is 4 or less then the number you underlined will stay the same and all numbers to the right will become ZERO.

Example 11.3
Round $467 + 221$ to the nearest
(a) hundred
(b) ten

Solution.
(a) Since $467 \approx 500$ and $221 \approx 200$ then $467 + 221 \approx 700$.
(b) In this case, $467 \approx 470$ and $221 \approx 220$ so that $467 + 221 \approx 690$.■

Practice Problems

Problem 11.18
Round 235,476 to the nearest
(a) ten thousand
(b) thousand
(c) hundred.

Problem 11.19
Round each of these to the position indicated.
(a) 947 to the nearest hundred.
(b) 27,462,312 to the nearest million.
(c) 2461 to the nearest thousand.

Problem 11.20
Rounding to the left-most digit, calculate approximate values for each of the following:
(a) $681 + 241$
(b) $678 - 431$
(c) 257×364
(d) $28,329 \div 43$.

Problem 11.21
Using rounding to the left-most digit, estimate the following products.
(a) 2748×31 (b) 4781×342 (c) $23,247 \times 357$.

Problem 11.22
Round each number to the position indicated.
(a) 5280 to the nearest thousand
(b) 115,234 to the nearest ten thousand
(c) 115,234 to the nearest hundred thousand
(d) 2,325 to the nearest ten.

Problem 11.23
Use front-end estimation with adjustment to estimate each of the following:
(a) $2215 + 3023 + 5987 + 975$
(b) $234 + 478 + 987 + 319 + 469$.

Problem 11.24
Use range estimation to estimate each of the following.
(a) $22 \cdot 38$ (b) $145 + 678$ (c) $278 + 36$.

Problem 11.25
Tom estimated $31 \cdot 179$ in the three ways shown below.
(i) $30 \cdot 200 = 6000$
(ii) $30 \cdot 180 = 5400$
(iii) $31 \cdot 200 = 6200$
Without finding the actual product, which estimate do you think is closer to the actual product? Explain.

Problem 11.26
About 3540 calories must be burned to lose 1 pound of body weight. Estimate how many calories must be burned to lose 6 pounds.

Problem 11.27
A theater has 38 rows and 23 seats in each row. Estimate the number of seats in the theater and tell how you arrived at your estimate.

Problem 11.28
Use estimation to tell whether the following calculator answers are reasonable. Explain why or why not.
(a) $657 + 542 + 707 = 543364$
(b) $26 \times 47 = 1222$.

Problem 11.29
Estimate the sum

$$87 + 45 + 37 + 22 + 98 + 51$$

using compatible numbers.

Problem 11.30
clustering is a method of estimating a sum when the numbers are all close to one value. For example, $3648+4281+3791 \approx 3 \cdot 4000 = 12,000$. Estimate the following using clustering.
(a) $897 + 706 + 823 + 902 + 851$
(b) $36,421 + 41,362 + 40,987 + 42,621$.

Problem 11.31
Estimate each of the following using (i) range estimation, (ii) one-column front-end estimation (iii) two-column fron-end estimation, and (iv) front-end with adjustment.
(a) $3741 + 1252$
(b) $1591 + 346 + 589 + 163$
(c) $2347 + 58 + 192 + 5783$.

Problem 11.32
Estimate using compatible number estimation.
(a) 51×212 (b) $3112 \div 62$ (c) 103×87.

12 Algorithms for Addition and Subtraction of Whole Numbers

In the previous section we discussed the mental arithmetic of whole numbers. In this section we discuss algorithms for performing pencil-and-paper computations. By an **algorithm** we mean a systematic step by step procedure used to find an answer to a calculation.

Algorithms for Adding Whole Numbers
If we are asked to find the value of the sum $28 + 45$ using pencil-and-paper we will proceed as shown in Figure 12.1.

$$\begin{array}{r} {\scriptstyle 1} \\ 28 \\ +\ 45 \\ \hline 73 \end{array}$$

Figure 12.1

Why does such procedure work?Why "carry" the "1"?Why add by columns? To many people, these procedures remain a great mystery-we add this way because we were told to-it's simply done by rote with no understanding. So we would like to introduce to children this standard algorithm of addition.
Children in general learn abstract notions by first experiencing them concretely with devices they can actually see, touch, and manipulate. The use of concrete teaching aids-such as base-ten blocks-helps provide insight into the creation of the standard algorithm for addition.
We now use base-ten blocks to help develop the algorithm for addition of whole numbers. Suppose we want to find the sum $34 + 27$. We show this computation with a concrete model in Figure 12.2(a), with the expanded algorithm in Figure 12.2(b) and the standard algorithm in Figure 12.2(c).

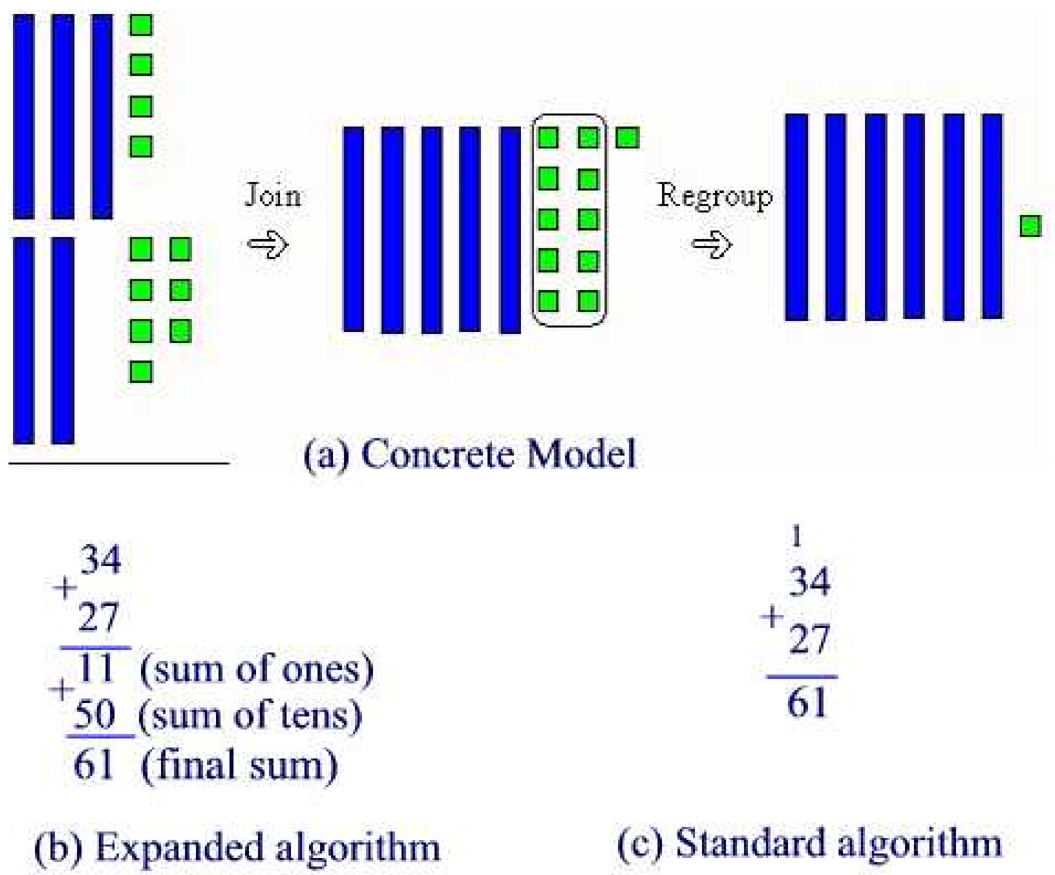


Figure 12.2

A more formal justification for this addition where properties of addition are applied is the following:

$$\begin{array}{rcll}
34+27 & = & (3\times 10+4)+(2\times 10+7) & \textit{expanded form}\\
 & = & (3\times 10+2\times 10)+(4+7) & \textit{associative and commutative properties}\\
 & = & (3\times 10+2\times 10)+11 & \\
 & = & (3+2+1)\times 10+1 & \textit{distributive property}\\
 & = & 6\times 10+1=61 & \textit{simplified form}
\end{array}$$

Next, we explore a couple of algorithms that have been used throughout history.

Lattice Algorithm

We explore this algorithm by looking at an example such as the one given in Figure 12.3.

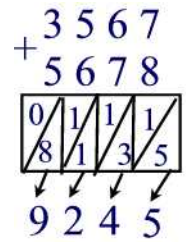


Figure 12.3

To use this algorithm you add single digit number to a single digit number from right to left and record the results in a lattice as shown. Then the sums are added along the diagonal.

Scratch Algorithm

Consider the sum shown in Figure 12.4.

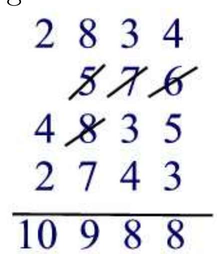


Figure 12.4

Begin by adding from the top down in the units column. When you add a digit that makes your sum 10 or more, scratch out the digit as shown and make a mental note of the units digit of your present sum. Start with the digit noted and continue adding and scratching until you have completed the units column, writing down the units digit of the last sum as the units digit of the answer as shown. Now, count the number of scratches in the units column and, starting with this number, add on down the tens column repeating the scratch process as you go. Continue the entire process until all the columns have been added. This gives the desired answer.

Practice Problems

Problem 12.1
Use the addition expanded algorithm as discussed in this section to perform the following additions:
(a) $23+44$ (b) $57+84$ (c) $324+78$

Problem 12.2
Use base ten blocks to represent the sum $279+84$.

Problem 12.3
State the property that justifies each of the following steps.

$$\begin{array}{rcll}
36+52 & = & (3\cdot 10+6)+(5\cdot 10+2) & ____ \\
 & = & 3\cdot 10+[6+(5\cdot 10+2)] & ____ \\
 & = & 3\cdot 10+[(6+5\cdot 10)+2] & ____ \\
 & = & 3\cdot 10+[(5\cdot 10+6)+2] & ____ \\
 & = & 3\cdot 10+[5\cdot 10+(6+2)] & ____ \\
 & = & (3\cdot 10+5\cdot 10)+(6+2) & ____ \\
 & = & (3+5)\cdot 10+(6+2) & ____ \\
 & = & 8\cdot 10+8 & ____ \\
 & = & 88 &
\end{array}$$

Problem 12.4
Find the missing digits.

(a)	(b)	(c)
_437	_721	38_1
2_1	901_	24_3
+347_	+71_3	+512_
6_94	_0_26	_ _5_9

Problem 12.5
Julien Spent one hour and 45 minutes mowing the lawn and two hours and 35 minutes trimming the hedge and some shrubs. How long did he work all together?

Problem 12.6
Compute the sum $38+97+246$ using scratch addition.

Problem 12.7
Find the sum 3 hr 36 min 58 sec + 5 hr 56 min 27 sec.

Problem 12.8
Compute the following sums using the lattice method.
(a) $482 + 269$ (b) $567 + 765$.

Problem 12.9
Larry, Curly, and Moe each add incorrectly as follows.

Larry: $\begin{array}{r} 29 \\ +83 \\ \hline 1012 \end{array}$ Curly: $\begin{array}{r} {}^{2}\!29 \\ +83 \\ \hline 121 \end{array}$ Moe: $\begin{array}{r} 29 \\ +83 \\ \hline 102 \end{array}$

How would you explain their mistakes to each of them?

Problem 12.10
A **palindrome** is any number that reads the same backward as forward, for example, 121 and 2332. Try the following. Begin with any number. Is it a palindrome? If not, reverse the digits and add this new number to the old one. Is the result a palindrome?If not, repeat the above procedure until a palindrome is obtained. For example, suppose you start with 78. Because 78 is not a palindrome, we add $78 + 87 = 165$. Since 165 is not a palindrome we add $165 + 561 = 726$. Since 726 is not a palindrome we add $726 + 627 = 1353$. Since 1353 is not a palindrome we add $1353 + 3531 = 4884$ which is a palindrome. Try this method with the following numbers:
(a) 93 (b) 588 (c) 2003.

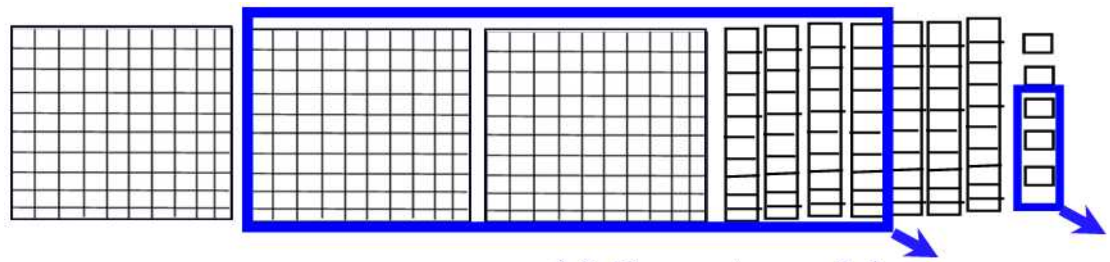
Problem 12.11
Another algorithm for addition uses the so-called **partial sums**. The digits in each column are summed and written on separate lines as shown below.

$$\begin{array}{r} 632 \\ +798 \\ \hline 10 \\ 12 \\ 13 \\ \hline 1430 \end{array}$$

Using this method, compute the following sums:
(a) $598 + 396$ (b) $322 + 799 + 572$.

Algorithms for Subtracting Whole Numbers

As with addition, base-ten blocks can provide a concrete model for subtraction. Suppose we want to find the difference $243 - 375$. Figure 12.5(a) shows the computation with a concrete model, Figure 12.5(b) with an expanded algorithm, and Figure 12.5(c) with the standard algorithm.



(a) Concrete model

$$\begin{array}{r} 375 \\ -\ 243 \\ \hline 2 \\ 30 \\ 100 \\ \hline 132 \end{array}$$

(b) Expanded algorithm

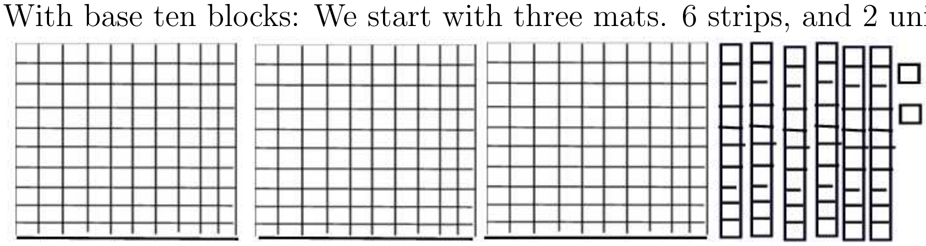
$$\begin{array}{r} 375 \\ -\ 243 \\ \hline 132 \end{array}$$

(c) Standard algorithm

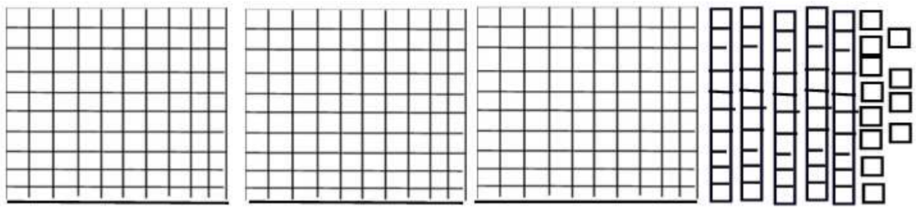
Figure 12.5

Example 12.1 *(Subtracting with exchanging)*
Use the three algorithms dicussed above to subtract 185 from 362.

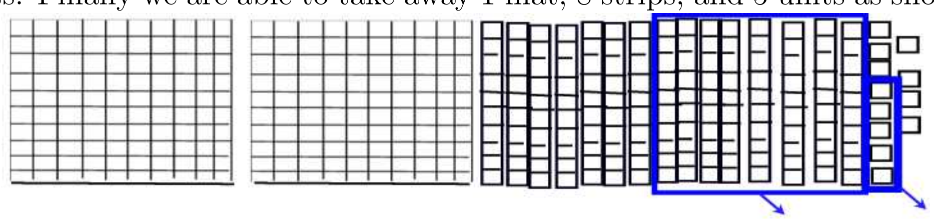
Solution
(a) With base ten blocks: We start with three mats. 6 strips, and 2 units.



We want to take away 1 mat, 8 strips, and 5 units. Since we cannot pick up 5 units from our present arrangement, we exchange a strip for 10 units to obtain 3 mats, 5 strips, and 12 units.



We can now take away 5 units, but we still cannot pick up 8 strips. Therefore, we exchange a mat for 10 strips to obtain 2 mats, 15 strips, and 12 units. Finally we are able to take away 1 mat, 8 strips, and 5 units as shown.



(b) Expanded algorithm

$$\begin{array}{rccc} & 3 & 6 & 2 \\ - & 1 & 8 & 5 \\ \hline & 3 & 5 & (12) \\ - & 1 & 8 & 5 \\ \hline & 2 & (1\,5) & (12) \\ - & 1 & 8 & 5 \\ \hline & 1 & 7 & 7 \end{array}$$

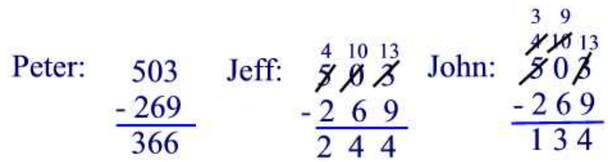
(c) Standard Algorithm

$$\begin{array}{rccc} & \overset{2}{\cancel{3}} & \overset{{}^{1}5}{\cancel{6}} & {}^{1}2 \\ - & 1 & 8 & 5 \\ \hline & 1 & 7 & 7 \end{array}$$

Practice Problems

Problem 12.12
Sketch the solution to $42 - 27$ using base-ten blocks.

Problem 12.13
Peter, Jeff, and John each perform a subtraction incorrectly as follows:



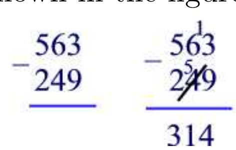
How would you explain their mistakes to each one of them?

Problem 12.14
Find the difference 5 *hr* 36 *min* 38 *sec* − 3 *hr* 56 *min* 58 *sec*.

Problem 12.15
In subtracting 462 from 827, the 827 must be regrouped as _____ hundreds, _____ tens, and _____ ones.

Problem 12.16
Suppose you add the same amount to both numbers of a subtraction problem. What happens to the answer? Try the following.
(a) What is 86 − 29?
(b) Add 11 to both numbers in part (a) and subtract. Do you obtain the same number?

Problem 12.17
The **equal-addition algorithm** has been used in some US schools in the past 60 years. The property developed in the preceding problem is the basis for this algorithm. For example, in computing 563 − 249, one needs to add 10 to 3. To compensate, one adds 10 to 249. Then the subtraction can be done without regrouping as shown in the figure below.



Compute the difference 1464 − 687 using the equal-addition algorithm.

Problem 12.18
Sketch the solution to 275 − 136 using base ten blocks.

Problem 12.19
Use the expanded algorithm to perform the following:
(a) 78 − 35 (b) 75 − 38 (c) 414 − 175

Problem 12.20
Fill in the missing digits.

(a)
$$\begin{array}{r} \_\,3\,\_ \\ -\;2\,\_\,1 \\ \hline 594 \end{array}$$

(b)
$$\begin{array}{r} 3\,\_\,\_\,4 \\ -\;\_\,3\,4\,6 \\ \hline 1\,7\,5\,\_ \end{array}$$

(c)
$$\begin{array}{r} 6\,3\,\_\,\_\,4 \\ -\;2\,\_\,1\,2\,\_ \\ \hline \_\,6\,2\,0\,9 \end{array}$$

Problem 12.21
After her dad gave her her allowance of 10 dollars, Ellie had 25 dollars and 25 cents. After buying a sweater for 14 dollars and 53 cents, including tax, how much money did Ellie have left?

Problem 12.22
A hiker is climbing a mountain that is 6238 feet high. She stops to rest at 4887 feet. How many more feet must she climb to reach the top?

13 Algorithms for Multiplication and Division of Whole Numbers

In this section, we discuss algorithms of whole numbers multiplication and division.

Algorithms for Whole Numbers Multiplication
Similar to addition and subtraction, a developemnt of our **standard multiplication algorithm** is shown in Figure 13.1.

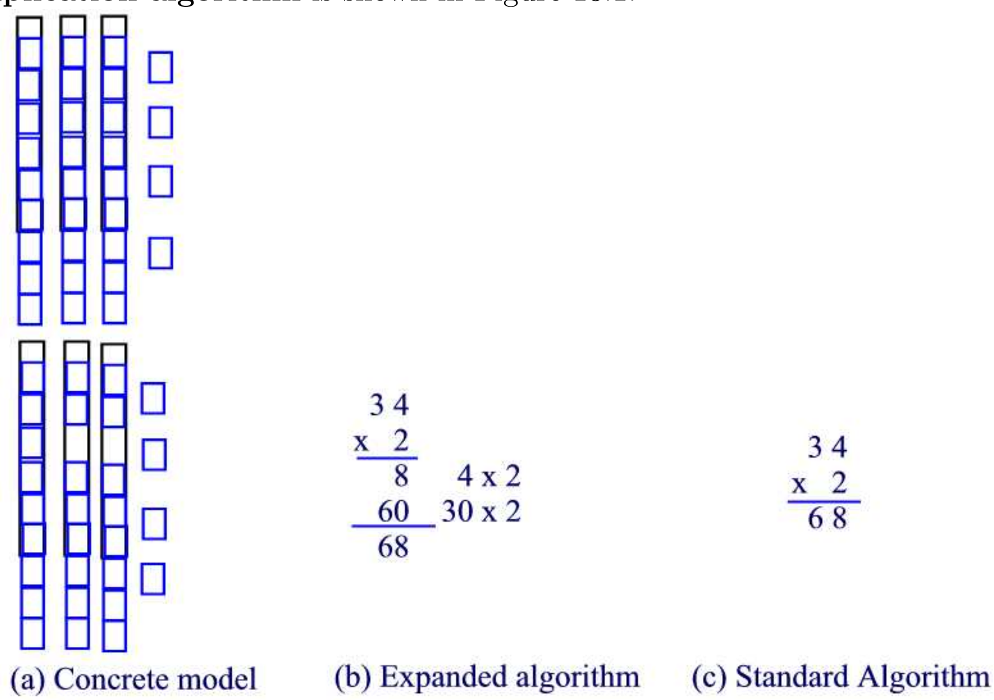


Figure 13.1

Whole number properties help justify the standard procedure:

$$\begin{array}{rcll} 34 \times 2 & = & (30+4)\times 2 & \textit{Expanded notation} \\ & = & (30\times 2)+(4\times 2) & \textit{Distributivity} \\ & = & 60+8 & \textit{multiplication} \\ & = & 68 & \textit{addition} \end{array}$$

Example 13.1
Perform 35×26 using the expanded algorithm.

Solution.

$$
\begin{array}{rl}
35 & \\
\times\ 26 & \\
\hline
30 & 5 \times 6 \\
180 & 30 \times 6 \\
100 & 5 \times 20 \\
600 & 30 \times 20 \\
\hline
910 &
\end{array}
$$

Lattice Multiplication

Figure 13.2 illustrates the steps of this algorithm in computing 27×34.

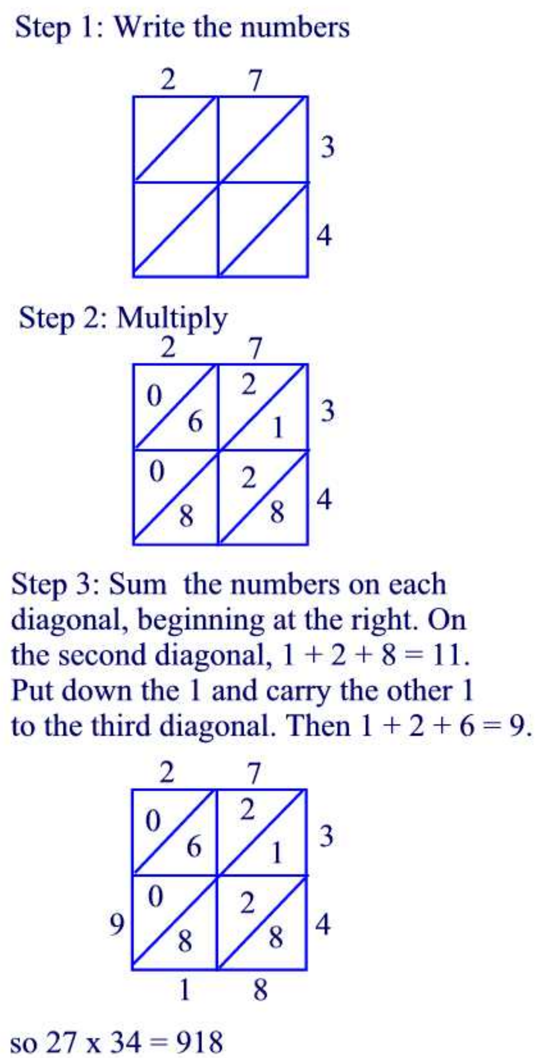


Figure 13.2

The Russian Peasant Algorithm

This algorithm employs halving and doubling. Remainders are ignored when halving. The algorithm for 33×47 is shown in Figure 13.3.

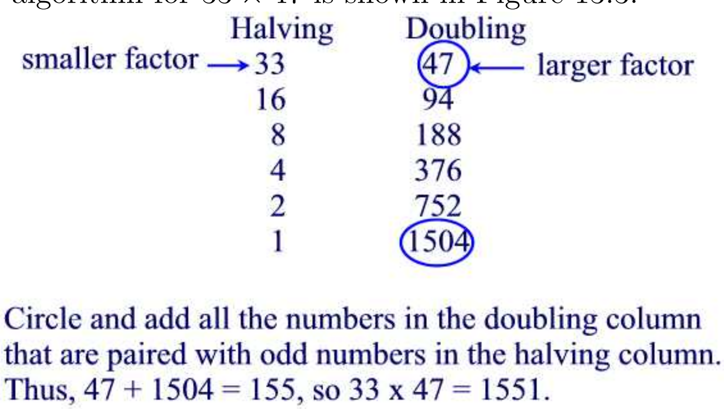


Figure 13.3

Practice Problems

Problem 13.1

(a) Compute 83×47 with the expanded algorithm.
(b) Compute 83×47 with the standard algorithm.
(c) What are the advantages and disadvantages of each algorithm?

Problem 13.2

Suppose you want to introduce a fourth grader to the standard algorithm for computing 24×4. Explain how to find the product with base-ten blocks. Draw a picture.

Problem 13.3

In multiplying 62×3, we use the fact that $(60 + 2) \times 3 = (60 \times 3) + (2 \times 3)$. What property does this equation illustrate?

Problem 13.4

(a) Compute 46×29 with lattice multiplication.
(b) Compute 234×76 with lattice multiplication.

Problem 13.5
Show two other ways besides the standard algorithm to compute 41×26.

Problem 13.6
Four fourth graders work out 32×15. Tell whether each solution is correct. If so, what does the child understand about multiplication? If the answer is wrong, what would you tell the child about how to solve the problem?
(a) 32×10 is 320. Add half of 320, which is 160. You get 480.

```
(b)  32
   x 15
   ----
    160
    32
   ----
    480
```

```
(c)  32
   x 15
   ----
    160
     32
   ----
    192
```

(d) 32×15 is the same as 16×30, which is 480.

Problem 13.7
Compute 18×127 using the Russian peasant algorithm.

Problem 13.8
What property of the whole numbers justifies each step in this calculation?

$$
\begin{array}{rcll}
17 \cdot 4 & = & (10+7)\cdot 4 & \textit{Expanded notation} \\
 & = & 10\cdot 4 + 7\cdot 4 & \underline{\hspace{2em}} \\
 & = & 10\cdot 4 + 28 & \textit{multiplication} \\
 & = & 10\cdot 4 + (2\cdot 10 + 8) & \textit{expanded notation} \\
 & = & 4\cdot 10 + (2\cdot 10 + 8) & \underline{\hspace{2em}} \\
 & = & (4\cdot 10 + 2\cdot 10) + 8 & \underline{\hspace{2em}} \\
 & = & (4+2)\cdot 10 + 8 & \underline{\hspace{2em}} \\
 & = & 60 + 8 & \textit{multiplication} \\
 & = & 68 & \textit{addition}
\end{array}
$$

Problem 13.9
Fill in the missing digit in each of the following.

$$
\text{(a)}\quad \begin{array}{r} 4\_6 \\ \times\ \ 783 \\ \hline 1\_78 \\ 3408 \\ \_982 \\ \hline 3335\_8 \end{array}
\qquad\qquad
\text{(b)}\quad \begin{array}{r} 327 \\ \times\ \ 9\_1 \\ \hline 327 \\ 1\_08 \\ \_9\_3 \\ \hline 30\_\_07 \end{array}
$$

Problem 13.10
Complete the following table:

a	b	ab	a+b
	56	3752	
32			110
		270	33

Problem 13.11
Find the products of the following and describe the pattern that emerges.
(a)

$$
\begin{array}{ccc}
1 & \times & 1 \\
11 & \times & 11 \\
111 & \times & 111 \\
1111 & \times & 1111
\end{array}
$$

(b)

$$
\begin{array}{ccc}
99 & \times & 99 \\
999 & \times & 999 \\
9999 & \times & 9999
\end{array}
$$

Algorithms for Whole Numbers Division
As in the previous operations, we will develop the standard algorithm of division by starting from a concrete model. We consider three algorithms: base ten blocks, repeated-subtraction (or scaffold), and standard division (also known as the long division algorithm).
Figure 13.4 shows how to compute $53 \div 4$ with base ten blocks, expanded

algorithm, and standard algorithm.

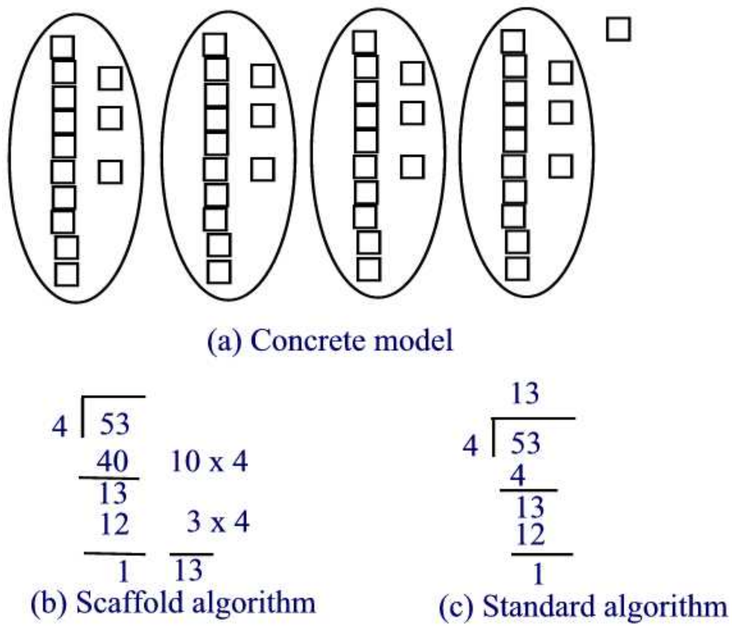


Figure 13.4

As just shown in Figure 13.4 (b), various multiples of 4 are subtracted successively from 53 (or the resulting difference) until a remainder less than 4 is found. The key to this method is how well one can estimate the appropriate multiples of 4.
The scaffold algorithm is useful either as a transitional algorithm to the standard algorithm or an alternative for students who have been unable to learn the standard algorithm.

Example 13.2
Find the quotient and the remainder of the division $1976 \div 32$ using the scaffold method.

Solution.

$$
\begin{array}{r|l l}
32 & \overline{1976} & \\
 & \underline{1920} & 60 \times 32 \\
 & 56 & \\
 & \underline{32} & \underline{1 \times 32} \\
 & 24 & 61
\end{array}
$$

Quotient = 61
Remainder = 24

Practice Problems

Problem 13.12
Sketch how to use base ten blocks to model the operation $673 \div 4$.

Problem 13.13
Use the standard algorithm to find the quotient and the remainder of the division $354 \div 29$.

Problem 13.14
Perform each of the following divisions by the scaffold method.
(a) $7425 \div 351$ (b) $6814 \div 23$

Problem 13.15
Two fourth graders work out $56 \div 3$. Tell whether each solution is correct. If so, what does the child understand about division? In each case, tell what the child understands about division?
(a) How many 3s make 56? Ten 3s make 30. That leaves 26. That will take 8 more 3s, and 2 are left over. So the quotient is 18 and the remainder is 2.
(b) Twenty times 3 is 60. That is too much. Take off two 3s. That makes eighteen 3s and 2 extra. Thus, the quotient is 18 and the remainder is 2.

Problem 13.16
Suppose you want to introduce a fourth grader to the standard algorithm for computing $246 \div 2$. Explain how to find the the quotient with base ten blocks.

Problem 13.17

A fourth grader works out $117 \div 6$ as follows. She finds $100 \div 6$ and $17 \div 6$. She gets $16 + 2 = 18$ sixes and 9 left over. Then $9 \div 6$ gives 1 six with 3 left over. So the quotient of the division $117 \div 6$ is 19 and the remainder is 3.
(a) Tell how to find $159 \div 7$ with the same method.
(b) How do you think this method compares to the standard algorithm?

Problem 13.18

Find the quotient and the remainder of $8569 \div 23$ using a calculator.

Problem 13.19

(a) Compute $312 \div 14$ with the repeated subtraction algorithm.
(b) Compute $312 \div 14$ with the standard algorithm.
(c) What are the advantages and disadvantages of each algorithm?

Problem 13.20

Using a calculator, Ralph multiplied by 10 when he should have divided by 10. The display read 300. What should the correct answer be?

Problem 13.21

Suppose $a = 131 \times 4789 + 200$. What is the quotient and the remainder of the division of a by 131?

14 Arithmetic Operations in Bases Other Than Ten

The base-ten arithmetic algorithms discussed in the previous two sections also work in other bases. In this section we apply the algorithms to base five.

Addition in Base Five
In base five the digits used are 0,1,2,3, and 4. Using blocks one can easily build the following addition table.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

All numerals in the table are written in base five with subscripts omitted.

Example 14.1
Compute $12_{five} + 31_{five}$ using blocks.

Solution.
Figure 14.1 shows how to compute $342_{five} + 134_{five}$ using blocks.■

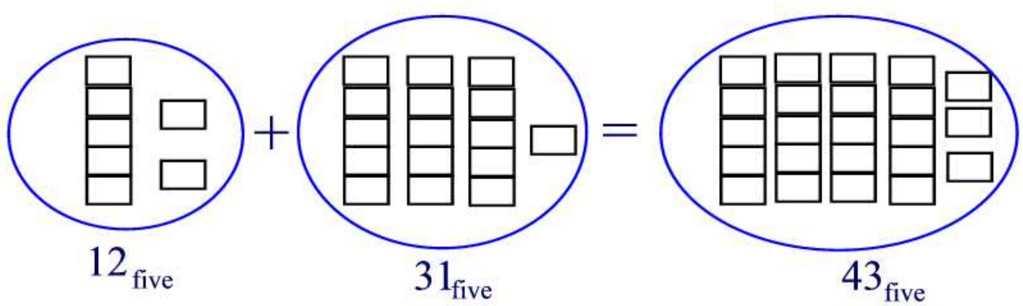
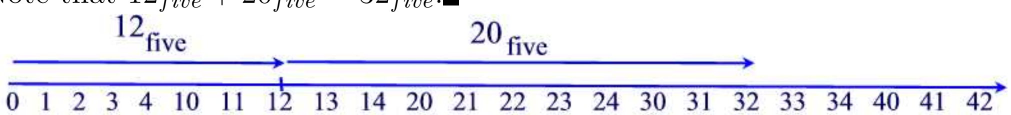


Figure 14.1

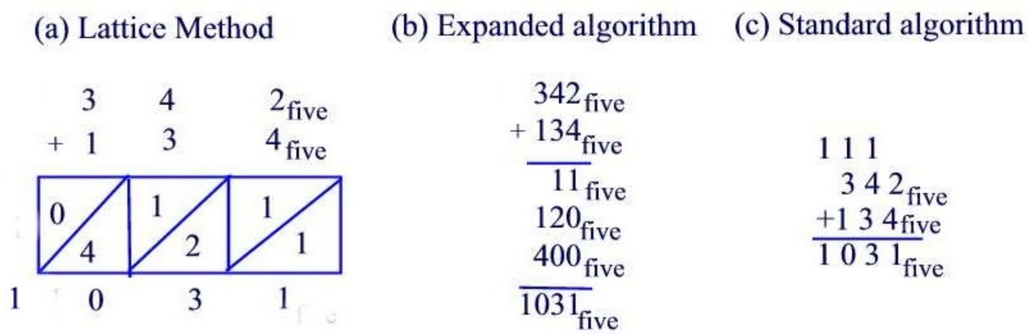
Example 14.2
Use a base five line to illustrate $12_{five} + 20_{five}$.

Solution.
Note that $12_{five} + 20_{five} = 32_{five}$.■



Example 14.3
Compute the sum $342_{five} + 134_{five}$
(a) using the lattice algorithm
(b) using the expanded algorithm
(c) using the standard algorithm.

Solution.



To check that the answer to the above addition is correct, we convert everything to base 10 where we feel comfortable.

$$\begin{array}{rcccr} 342_{five} & = & 3\cdot 5^2 + 4\cdot 5 + 2 & = & 97 \\ 134_{five} & = & 1\cdot 5^2 + 3\cdot 5 + 4 & = & 44 \\ 1031_{five} & = & 1\cdot 5^3 + 0\cdot 5^2 + 3\cdot 5 + 1 & = & 141 \end{array}$$

The result is confirmed since $97 + 44 = 141$.■

Practice Problems

Problem 14.1
Compute the sum $13_{five} + 22_{five}$ using
(a) base five blocks
(b) expanded algorithm

(c) lattice algorithm
(d) standard algorithm.

Problem 14.2
Perform the following computations.

$$\text{(a)}\ \begin{array}{r} 23_{\text{five}} \\ +34_{\text{five}} \\ \hline \end{array} \qquad \text{(b)}\ \begin{array}{r} 312_{\text{five}} \\ +132_{\text{five}} \\ \hline \end{array} \qquad \text{(c)}\ \begin{array}{r} 432_{\text{five}} \\ +233_{\text{five}} \\ \hline \end{array}$$

Problem 14.3
Complete the following base eight addition table.

+	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

Problem 14.4
Compute $132_{eight} + 66_{eight}$.

Problem 14.5
Computers use base two since it contains two digits, 0 and 1, that correspond to electronic switches in the computer being off or on. In this base, $101_{two} = 1 \cdot 2^2 + 0 \cdot 2 + 1 = 5_{ten}$.
(a) Construct addition table for base two.
(b) Write 1101_{two} in base ten.
(c) Write 123_{ten} in base two.
(d) Compute $1011_{two} + 111_{two}$.

Problem 14.6
For what base b would $32_b + 25_b = 57_b$?

Problem 14.7
(a) Construct an addition table for base four.
(b) Compute $231_{four} + 121_{four}$.

Problem 14.8
Use blocks to illustrate the sum $41_{six} + 33_{six}$.

Problem 14.9
Use an expanded algorithm to compute $78_{nine} + 65_{nine}$.

Problem 14.10
Create a base seven number line and illustrate the sum $13_{seven} + 5_{seven}$.

Problem 14.11
Construct an addition table in base seven.

Problem 14.12
Use the lattice method to compute the following sums.
(a) $46_{seven} + 13_{seven}$.
(b) $13_{four} + 23_{four}$.

Subtraction in Base Five
The development of subtraction in base five from concrete to abstract is shown in Figure 14.2.

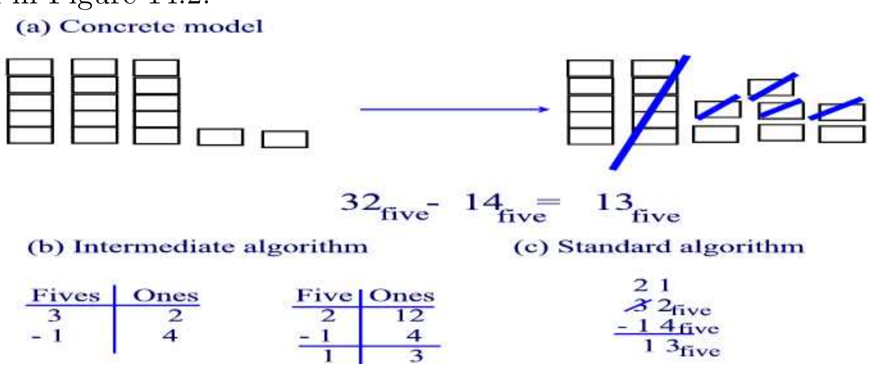
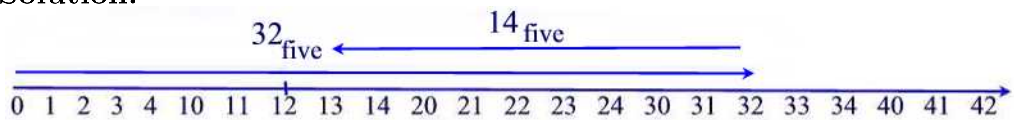


Figure 14.2

Example 14.4
Use base five number line to illustrate $32_{five} - 14_{five}$.

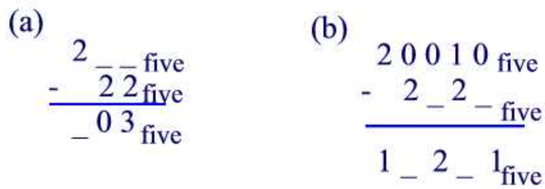
Solution.



Practice Problems

Problem 14.13
Perform the following subtractions:
(a) $1101_{two} - 111_{two}$
(b) $43_{five} - 23_{five}$
(c) $21_{seven} - 4_{seven}$.

Problem 14.14
Fill in the missing numbers.



Problem 14.15
Use blocks for the appropriate base to illustrate the following problems.
(a) $555_{seven} - 66_{seven}$ (b) $3030_{four} - 102_{four}$.

Problem 14.16
Use both the intermediate algorithm (discussed in Figure 14.2) and the standard algorithm to solve the following differences.
(a) $31_{four} - 12_{four}$ (b) $1102_{four} - 333_{four}$.

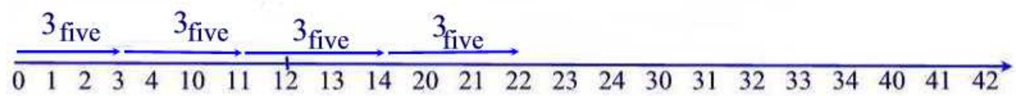
Problem 14.17
Use base five number line to illustrate the difference $12_{five} - 4_{five}$.

Multiplication in Base Five
Next, consider the multiplication algorithm. A base-five multiplication table will be helpful.

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

Example 14.5
Calculate $4_{five} \times 3_{five}$ using base five number line.

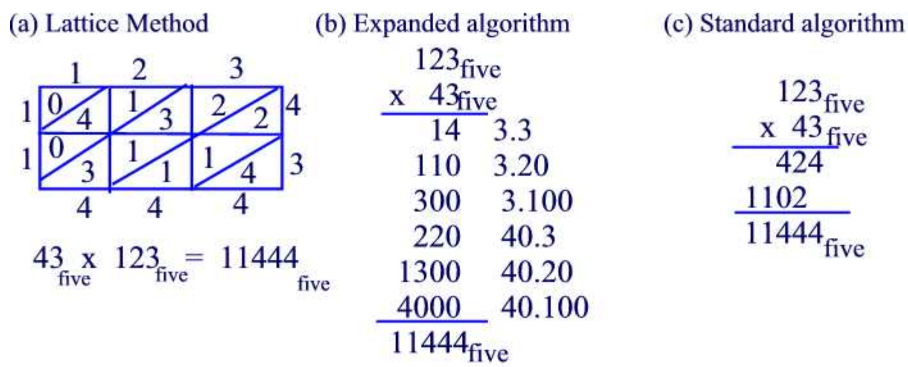
Solution.



So $4_{five} \times 3_{five} = 22_{five}$■

Example 14.6
Calculate $43_{five} \times 123_{five}$ using
(a) the lattice method for multiplication
(b) the expanded algorithm
(c) the standard algorithm.

Solution.



Practice Problems

Problem 14.18
Create a base seven number line to illustrate $6_{seven} \times 3_{seven}$.

Problem 14.19
Find the following products using the lattice method, the expanded algorithm, and the standard algorithm.
(a) $31_{four} \times 2_{four}$ (b) $43_{five} \times 3_{five}$

Division in Base Five

Long division in base five can be dome with a long division analogous to the base ten algorithm. The ideas behind the algorithms for division can be developed by using repeated subtraction. For example, $3241_{five} \div 43_{five}$ is computed by means of repeated-subtraction technique in Figure 14.3(a) and by means of the conventional algorithm in Figure 14.3(b). Thus, $3241_{five} \div 43_{five} = 34_{five}$ with remainder 14_{five}.

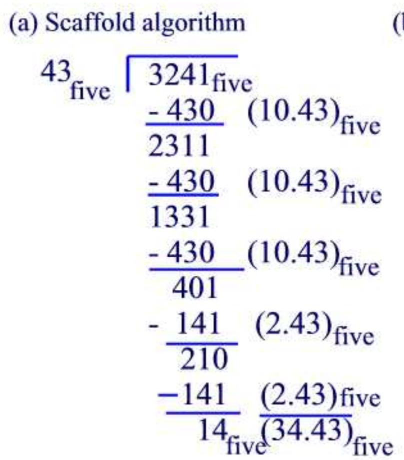


Figure 14.3

Practice Problems

Problem 14.20

Perform the following divisions:

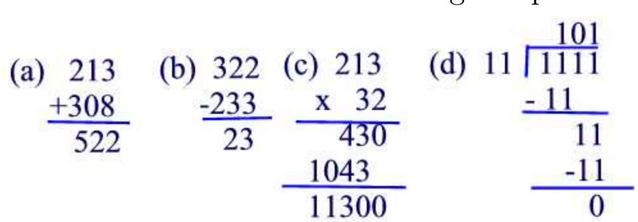
(a) $32_{five} \div 4_{five}$

(b) $143_{five} \div 3_{five}$

(c) $10010_{two} \div 11_{two}$.

Problem 14.21

For what possible bases are each of the following computations correct?



Problem 14.22
(a) Compute $121_{five} \div 3_{five}$ with repeated subtraction algorithm.
(b) Compute $121_{five} \div 3_{five}$ with long division algorithm.

Problem 14.23
(a) Compute $324_{five} \div 4_{five}$ with repeated subtraction algorithm.
(b) Compute $324_{five} \div 4_{five}$ with long division algorithm.

Problem 14.24
(a) Compute $1324_{seven} \div 6_{seven}$ with repeated subtraction algorithm.
(b) Compute $1324_{seven} \div 6_{seven}$ with long division algorithm.

Problem 14.25
Solve the following problems using the missing-factor definition of division, that is, $a \div b = c$ if and only if $b \cdot c = a$.(Hint: Use a multiplication table for the appropriate base).
(a) $21_{four} \div 3_{four}$ (b) $23_{six} \div 3_{six}$ (c) $24_{eight} \div 5_{eight}$

Problem 14.26
Sketch how to use base seven blocks to illustrate the operation $534_{seven} \div 4_{seven}$.

15 Prime and Composite Numbers

Divides, Divisors, Factors, Multiples
In section 13, we considered the division algorithm: If a and b are whole numbers with $b \neq 0$ then there exist unique numbers q and r such that

$$a = bq + r, \quad 0 \leq r < b.$$

Of special interest is when $r = 0$. In this case, $a = bq$. We say that b **divides** a or b is a **divisor** of a. Also, we call b a **factor** of a and we say that a is a **multiple** of b. When b divides a we will write $a \mid b$.
If b does not divide a we will write $b \not| a$. For example, $2 \not| 3$.

Example 15.1
List all the divisors of 12.

Solution.
The divisors (or factors) of 12 are 1, 2, 3, 4, 6, 12 since $12 = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4$.■

Next, we discuss some of the properties of "|".

Theorem 15.1
Let a, k, m, n be whole numbers with $a \neq 0$.
(a) If $a|m$ and $a|n$ then $a|(m+n)$.
(b) If $a|m$ and $a|n$ and $m \geq n$ then $a|(m-n)$.
(c) If $a|m$ then $a|km$.

Proof.
(a) Since $a|m$ and $a|n$ then we can find unique whole numbers b and c such that $m = ba$ and $n = ca$. Adding these equalities we find $m + n = a(b + c)$. But the set of whole numbers is closed under addition so that $b + c$ is also a whole number. By the definition of "|" we see that $a|(m + n)$.
(b) Similar to part (a) where $m + n$ is replaced by $m - n$.
(c) Since $a|m$ then $m = ba$ for some unique whole number b. Multiply both sides of this equality by k to obtain $km = (kb)a$. Since the set of whole numbers is closed with respect to multiplication then kb is a whole number. By the definition of "|" we have $a|km$.■

Practice Problems

Problem 15.1
(a) The number $162 = 2 \cdot 3^4$. How many different divisors does 162 have?
(b) Try the same process with $225 = 3^2 \cdot 5^2$.
(c) Based on your results in parts (a) - (b), if p and q are prime numbers and $a = p^m \cdot q^n$ then how many different divisors does n have?

Problem 15.2
(a) List all the divisors of 48.
(b) List all the divisors of 54.
(c) Find the largest common divisor of 48 and 54.

Problem 15.3
Let $a = 2^3 \cdot 3^1 \cdot 7^2$.
(a) Is $2^2 \cdot 7^1$ a factor of a? Why or why not?
(b) Is $2^1 \cdot 3^2 \cdot 7^1$ a factor of a? Why or why not?
(c) How many different factors does a possess?
(d) Make an orderly list of all the factors of a.

Problem 15.4
If n, b, and c are nonzero whole numbers and $n|bc$, is it necessarily the case that $n|b$ or $n|c$? Justify your answer.

Problem 15.5
Which of the following are true or false? Justify your answer in each case.
(a) $n|0$ for every nonzero whole number n.
(b) $0|n$ for every nonzero whole number n.
(c) $0|0$.
(d) $1|n$ for every whole number n.
(e) $n|n$ for every nonzero whole number n.

Problem 15.6
Find the least nonzero whole number divisible by each nonzero whole number less than or equal to 12.

Problem 15.7
If $42|n$ then what other whole numbers divide n?

Problem 15.8
If $2N = 2^6 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7$, explain why $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ is a factor of N.

Prime and Composite Numbers
Any whole number a greater than 2 has at least two different factors, namely a and 1 since $a = 1 \cdot a$. If a and 1 are the only distinct factors of a then we call a a **prime** number. That is, a prime number is a number with only two distinct divisors 1 and the number itself. Examples of prime numbers are 2, 3, 5, 7, etc.
A number that is not prime is called **composite.** Thus, a composite number is a number that has more than two divisors. Examples of composite numbers are: 4, 6, 8, 9, etc.
The number 1 is called the **unit**. It is neither prime nor composite.

Example 15.2
List all the prime numbers less than 20.

Solution.
The prime numbers less than 20 are: 2, 3, 5, 7, 11, 13, 17, 19.■

Prime Factorization
Composite numbers can be expressed as the product of 2 or more factors greater than 1. For example, $260 = 26 \cdot 10 = 5 \cdot 52 = 26 \cdot 2 \cdot 5 = 2 \cdot 2 \cdot 5 \cdot 13$. When a composite number is written as the product of prime factors such as $260 = 2 \cdot 2 \cdot 5 \cdot 13$ then this product is referred to as the **prime factorization.**
Two procedures for finding the prime factorization of a number:
The Factor-Tree Method: Figure 15.1 shows two factor-trees for 260.

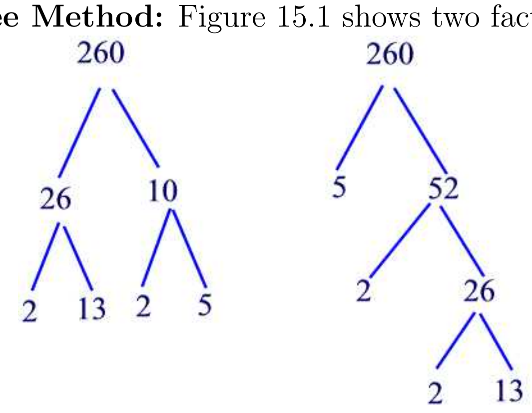


Figure 15.1

Note that a number can have different trees. However, all of them produce the same prime factorization except for order in which the primes appear in the products.
The **Fundamental Theorem of Arithmetic** also known as the **Unique Factorization Theorem** states that in general, if order of the factors are disregarded then the prime factorization is unique. More formally we have

<u>Fundamemtal Theorem of Arithmetic</u>
Every whole number greater than 1 can be expressed as the product of different primes in one and only one way apart from order.

The primes in the prime factorization are typically listed in increasing order from left to right and if a prime appears more than once, exponential notation is used. Thus, the prime factorization of 260 is $260 = 2^2 \cdot 5 \cdot 13$.

Prime-Divisor Method Besides the factor-tree method there is another method known as the "prime-divisor method". In this method, try all prime numbers in increasing order as divisors, beginning with 2. Use each prime number as a divisor as many times as needed. This method is illustrated in Figure 15.2

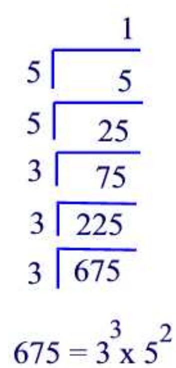
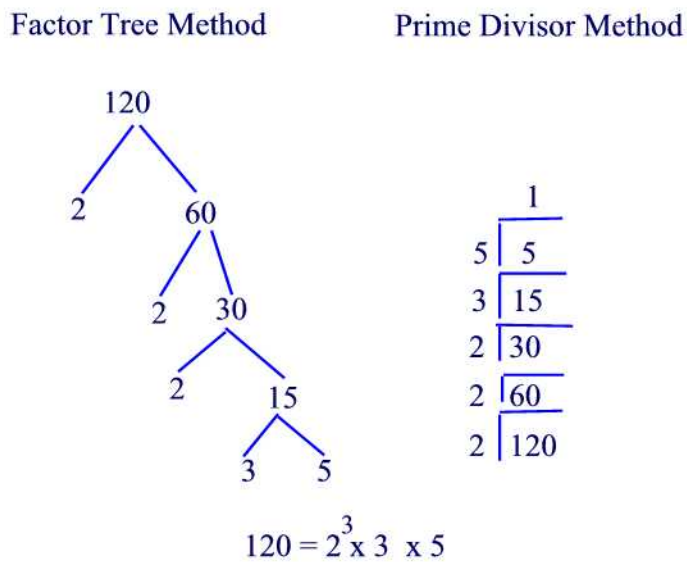


Figure 15.2

Example 15.3
Find the prime factorization of 120 using the two methods described above.

Solution.



Practice Problems

Problem 15.9
Eratosthenes, a Greek mathematician, developed the **Sieve of Eratosthenes** about 2200 years ago as a method for finding all prime numbers less than a given number. Follow the directions to find all the prime numbers less than or equal to 50.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50				

(a) Copy the list of numbers.
(b) Cross out 1 because 1 is not prime.
(c) Circle 2. Count 2s from there, and cross out $4, 6, 8, \cdots, 50$ because all

these numbers are divisible by 2 and therefore are not prime.
(d) Circle 3. Count 3s from there, and cross out all numbers not already crossed out because these numbers are divisible by 3 and therefore are not prime.
(e) Circle the smallest number not yet crossed out. Count by that number, and cross out all numbers that are not already crossed out.
(f) Repeat part (e) until there are no more numbers to circle. The circled numbers are the prime numbers.
(g) List all the prime numbers between 1 and 50.

Problem 15.10
List all prime numbers between 1 and 100 using the Sieve of Eratosthenes.

Problem 15.11
Extend the Sieve of Eratosthenes to find all the primes less than 200.

Problem 15.12
Write the prime factorizations of the following.
(a) 90 (b) 3155 (c) 84.

Problem 15.13
Find the prime factorization using both the factor-tree method and the prime divisor method.
(a) 495 (b) 320.

Problem 15.14
Twin primes are any two consecutive odd numbers, such as 3 and 5, that are prime. Find all the twin primes between 101 and 140.

Problem 15.15
(a) How many different divisors does $2^5 \cdot 3^2 \cdot 7$ have?
(b) Show how to use the prime factorization to determine how many different factors 148 has.

Problem 15.16
Construct factor trees for each of the following numbers.
(a) 72 (b) 126 (c) 264 (d) 550

Problem 15.17
Use the prime divisors method to find all the prime factors of the following numbers.
(a) 700 (b) 198 (c) 450 (d) 528

Problem 15.18
Determine the prime factorizations of each of the following numbers.
(a) 48 (b) 108 (c) 2250 (d) 24750

Problem 15.19
Show that if 1 were considered a prime number then every number would have more than one prime factorization.

Problem 15.20
Explain why $2^3 \cdot 3^2 \cdot 25^4$ is not a prime factorization and find the prime factorization of the number.

Determining if a Given Number is a Prime
How does one determine if a given whole number is a prime? To answer this question, observe first that if n is composite say with two factors b and c then one of its factor must be less than $\sqrt{n}$. For if not, that is, if $b > \sqrt{n}$ and $c > \sqrt{n}$ then

$$n = bc > \sqrt{n} \cdot \sqrt{n} = n,$$

that is $n > n$ which is impossible. Thus, if n is composite then either $b \leq \sqrt{n}$ or $c \leq \sqrt{n}$ or alternatively $b^2 \leq n$ or $c^2 \leq n$.
The above argument leads to the following test for prime numbers.

Theorem 15.2 *(Primality Test)*
If every prime factor of n is greater than $\sqrt{n}$ then n is composite. Equivalently, if there is a prime factor p of n such that $p^2 \leq n$ then n is prime.

Example 15.4
(a) Is 397 composite or prime?
(b) Is 91 composite or prime?

Solution.
(a) The possible primes p such that $p^2 \leq 397$ are 2,3,5,7,11,13,17, and 19. None of these numbers divide 397. So 397 is composite.

(b) The possible primes such that $p^2 \leq 91$ are 2, 3, 5, and 7. Since $7|91$ then by the above theorem 91 is prime.■

Practice Problems

Problem 15.21
Classify the following numbers as prime, composite or neither.
(a) 71 (b) 495 (c) 1

Problem 15.22
Without computing the results, explain why each of the following numbers will result in a composite number.
(a) $3 \times 5 \times 7 \times 11 \times 13$
(b) $(3 \times 4 \times 5 \times 6 \times 7 \times 8) + 2$
(c) $(3 \times 4 \times 5 \times 6 \times 7 \times 8) + 5$

Problem 15.23
To determine that 431 is prime, what is the minimum set of numbers you must try as divisors?

Problem 15.24
Use the Primality Test to classify the following numbers as prime or composite.
(a) 71 (b) 697 (c) 577 (d) 91.

Problem 15.25
What is the greatest prime you must consider to test whether 5669 is prime?

16 Tests of Divisibility

Sometimes it is handy to know if one number is divisible by another just by looking at it or by performing a simple test. The purpose of this section is to discuss some of the rules of divisibility. Such rules have limited use except for mental arithmetic. We will test the divisibility of a number by the numbers 2 through 11, excluding 7.
Two important facts are needed in this section. The first one states that if $a|m$ and $a|n$ then $a|(m+n)$, $a|(m-n)$, and $a|km$ for any whole numbers a, m, n, k with $a \neq 0$. The second fact is about the expanded representation of a number in base 10. That is, if $n = d_{k-1} \cdots d_2 d_1 d_0$ is a whole number with k digits then

$$n = d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1 + d_0.$$

Divisibility Tests for 2, 5, and , 10
The divisibility tests for 2, 5, and 10 are grouped together because they all require checking the unit digit of the whole number.

Theorem 16.1
(a) $n = d_{k-1} \cdots d_2 d_1 d_0$ is divisible by 2 if and only if $2|d_0$, i.e., $d_0 \in \{0, 2, 4, 6, 8\}$. That is, n is divisible by 2 if and only if the unit digit is either 0, 2, 4, 6, or 8.
(b) $n = d_{k-1} \cdots d_2 d_1 d_0$ is divisible by 5 if and only if $5|d_0$, i.e., $d_0 \in \{0, 5\}$. That is, n is divisible by 5 if and only if the unit digit is either 0 or 5.
(c) $n = d_{k-1} \cdots d_2 d_1 d_0$ is divisible by 10 if and only if $10|d_0$, i.e., $d_0 = 0$.

Proof.
(a) Suppose that n is divisible by 2. Since $2|10^i$ for $1 \leq i \leq k-1$ then 2 divides the sum $(d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1)$ and the difference $n - (d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1) = d_0$. That is, $2|d_0$.
Conversely, suppose that $2|d_0$. Since $2|(d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1)$ then $2|(d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1 + d_0)$. That is, $2|n$.
(b) The exact same proof of part (a) works by replacing 2 by 5.
(c) The exact same proof of part (a) works by replacing 2 by 10.■

Example 16.1
Without dividing, determine whether each number below is divisible by 2, 5 and/or 10.
(a) 8,479,238 (b) 1,046,890 (c) 317,425.

Solution.
(a) Since the unit digit of 8,479,238 is 8 then this number is divisible by 2 but not by 5 or 10.
(b) Since the unit digit of 1,046,890 is 0 then this number is divisible by 2, 5, and 10.
(c) Since the unit digit of 317,425 is 5 then this number is divisible by 5 but not by 2 or 10.■

Divisibility Tests for 3 and 9
The divisibility tests for 3 and 9 are grouped together because they both require computing the sum of the digits.

Theorem 16.2
(a) $n = d_{k-1}\cdots d_2d_1d_0$ is divisible by 3 if and only if $3|(d_{k-1}+\cdots+d_2+d_1+d_0)$.
(b) $n = d_{k-1}\cdots d_2d_1d_0$ is divisible by 9 if and only if $9|(d_{k-1}+\cdots+d_2+d_1+d_0)$.

Proof.
(a) Suppose that $3|n$. Write $[9]_i = 99\cdots 9$ where the 9 repeats i times. For example, $[9]_3 = 999$. With this notation we have $10^i = [9]_i + 1$. Hence

$$\begin{aligned} d_{k-1}\times 10^{k-1} + \cdots + d_2\times 10^2 + d_1\times 10^1 + d_0 &= \\ d_{k-1}\times([9]_{k-1}+1) + \cdots + d_2([9]_2+1) + d_1([9]_1+1) + d_0 &= \\ d_{k-1}\times[9]_{k-1} + \cdots + d_2\times[9]_2 + d_1\times[9]_1 + (d_{k-1}+\cdots+d_2+d_1+d_0) & \end{aligned}$$

Since $3|[9]_i$ for any i then $3|(d_{k-1}\times[9]_{k-1}+\cdots+d_2\times[9]_2+d_1\times[9]_1)$. Therefore, $3|[n-(d_{k-1}\times[9]_{k-1}+\cdots+d_2\times[9]_2+d_1\times[9]_1)]$. That is, $3|(d_{k-1}+\cdots+d_2+d_1+d_0)$.
Conversely, suppose that $3|(d_{k-1}+\cdots+d_2+d_1+d_0)$. Then $3|[(d_{k-1}\times[9]_{k-1}+\cdots+d_2\times[9]_2+d_1\times[9]_1)+(d_{k-1}+\cdots+d_2+d_1+d_0)]$. That is, $3|n$.
(b) The same exact proof of (a) works by replacing 3 by 9 since $9|[9]_i$.■

Example 16.2
Use the divisibility rules to determine whether each number is divisible by 3 or 9.
(a) 468,172 (b) 32,094.

Solution.
(a) Since $4+6+8+1+7+2 = 28$ which is divisible by 3 then 468,172 is divisible by 3. Since $9 \not| 28$ then 468,172 is not divisible by 9.

(b) Since $3+2+0+9+4=18$ and $3|18, 9|18$ then 32,094 is divisible by both 3 and 9.■

Divisibility by 4 and 8
The following theorem deals with the divisibility by 4 and 8.

Theorem 16.3
(a) $n = d_{k-1}\cdots d_2d_1d_0$ is divisible by 4 if and only if $4|(d_1d_0)$ where (d_1d_0) is the number formed by the last two digits of n.
(b) $n = d_{k-1}\cdots d_2d_1d_0$ is divisible by 8 if and only if $8|(d_2d_1d_0)$ where $(d_2d_1d_0)$ is the number formed by the last three digits of n.

Proof.
(a) Suppose that $4|n$. Write n in the form

$$n = d_{k-1}\times 10^{k-1} + \cdots + d_2 \times 10^2 + (d_1d_0).$$

Since $4|10^i$ for $2 \le i \le k-1$ then $4|(d_{k-1}\times 10^{k-1} + \cdots + d_2 \times 10^2)$. Hence, $4|(n - d_{k-1}\times 10^{k-1} + \cdots + d_2 \times 10^2) = (d_1d_0)$.
Conversely, suppose that $4|(d_1d_0)$. Since $4|(d_{k-1}\times 10^{k-1} + \cdots + d_2 \times 10^2)$ then $4|(d_{k-1}\times 10^{k-1} + \cdots + d_2 \times 10^2 + (d_1d_0)) = n$.
(b) The proof is similar to (a) and is omitted.■

Example 16.3
Use divisibility rules to test each number for the divisibility by 4 and 8.
(a) 1344 (b) 410,330

Solution.
(a) Since $4|44$ then $4|1344$. Since $8|344$ then $8|1344$.
(b) Since $4 \not| 30$ then $4 \not| 410,330$. Similarly, since $8 \not| 330$ then $8 \not| 410,330$■

Divisibility by 6
The divisibility by 6 follows from the following result.

Theorem 16.4
Let a and b be two whole numbers having only 1 as a common divisor and n a nonzero whole number. $a|n$ and $b|n$ if and only if $ab|n$.

Proof.
Suppose that $ab|n$. Then there is a unique nonzero whole number k such that $n = k(ab)$. Using associativity of multiplication we can write $n = (ka)b$. This means that $b|n$. Similarly, $n = (kb)a$. That is, $a|n$.
Conversely, suppose that $a|n$ and $b|n$. Write the prime factorizations of $a, b,$ and n.

$$\begin{array}{rcl} a & = & p_1^{t_1}p_2^{t_2}\cdots p_k^{t_k} \\ b & = & p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k} \\ n & = & p_1^{w_1}p_2^{w_2}\cdots p_k^{w_k} \end{array}$$

where $p_1, p_2, \cdots, p_k$ are distinct prime factors. Thus

$$ab = p_1^{t_1+s_1}p_2^{t_2+s_2}\cdots p_k^{t_k+s_k}.$$

Since $a|n$ then $t_1 \leq w_1, t_2 \leq w_2, \cdots, t_k \leq w_k$. Similarly, since $b|n$ then $s_1 \leq w_1, s_2 \leq w_2, \cdots, s_k \leq w_k$. Now, since a and b have no common divisor different from 1 then if $t_1 \neq 0$ then $s_1 = 0$ otherwise p_1 becomes a common divisor. Similarly, if $s_1 \neq 0$ then we must have that $t_1 = 0$. This shows that $s_i + t_i$ is either equal to s_i or to t_i. Hence, $s_1 + t_1 \leq w_1, s_2 + t_2 \leq w_2, \cdots, s_k + t_k \leq w_k$. We conclude from this that $ab|n$.■

If we let $a = 2$ and $b = 3$ in the previous theorem and use the fact that $6 = 2 \times 3$ we obtain the following result.

Theorem 16.5
A nonzero whole number n is divisible by 6 if and only if n is divisible by both 2 and 3.

Example 16.4
Use divisibility rules to test each number for the divisibility by 6.
(a) 746,988 (b) 4,201,012

Solution.
(a) Since the unit digit is 8 then the given number is divisible by 2. Since $7 + 4 + 6 + 9 + 8 + 8 = 42$ and $3|42$ then $6|746,988$.
(b) The given number is divisible by 2 since it ends with 2. However, $4 + 2 + 0 + 1 + 0 + 1 + 2 = 10$ which is not divisible by 3 then $6 \not| 4,201,012$.■

Divisibility by 11

Theorem 16.6
A nonzero whole number is divisible by 11 if and only if the difference of the sums of the digits in the even and odd positions in the number is divisible by 11.

Proof.
For simplicity we will proof the theorm for $n = d_4d_3d_2d_1d_0$. In this case, note that $10 = 11 - 1, 100 = 99 + 1, 1000 = 1001 - 1$, and $10000 = 9999 + 1$. The numbers 11, 99, 1001, and 9999 are all divisible by 11. Thus, n can be written in the following form

$$n = 11 \cdot q + d_4 - d_3 + d_2 - d_1 + d_0.$$

Now, suppose that $11|n$. Since $11|11 \cdot q$ then $11|(n - 11 \cdot q)$ i.e., $11|(d_4 - d_3 + d_2 - d_1 + d_0)$.
Conversely, suppose that $11|(d_4 - d_3 + d_2 - d_1 + d_0)$. Since $11|11 \cdot q$ then $11|(11 \cdot q + d_4 - d_3 + d_2 - d_1 + d_0)$, i.e., $11|n$.■

Example 16.5
Is the number 57, 729, 364, 583 divisible by 11?

Solution.
Since $(3 + 5 + 6 + 9 + 7 + 5) - (8 + 4 + 3 + 2 + 7) = 35 - 24 = 11$ and 11 is divisible by 11 then the given number is divisible by 11.■

Practice Problems

Problem 16.1
Using the divisibility rules discussed in this section, explain whether 6,868,395 is divisible by 15.

Problem 16.2
The number a and b are divisible by 5.
(a) Is $a + b$ divisible by 5?Why?
(b) Is $a - b$ divisible by 5?Why?
(c) Is $a \times b$ divisible by 5?Why?
(d) Is $a \div b$ divisible by 5?Why?

Problem 16.3
If 21 divides n, what other numbers divide n?

Problem 16.4
Fill each of the following blanks with the greatest digit that makes the statement true:
(a) $3|74\_$
(b) $9|83\_45$
(c) $11|6\_55$.

Problem 16.5
When the two missing digits in the following number are replaced, the number is divisible by 99. What is the number?

$$85\_\ \_1.$$

Problem 16.6
Without using a calculator, test each of the following numbers for divisibility by 2, 3, 4, 5, 6, 8, 9, 10, 11.
(a) 746,988
(b) 81,342
(c) 15,810
(d) 4,201,012
(e) 1,001
(f) 10,001.

Problem 16.7
There will be 219 students in next year's third grade. If the school has 9 teachers, can we assign each teacher the same number of students?

Problem 16.8
Three sisters earn a reward of \$37,500 for solving a mathematics problem. Can they divide the money equally?

Problem 16.9
What three digit numbers are less than 130 and divisible by 6?

Problem 16.10
True or false? If false, give a counter example.

(a) If a number is divisible by 5 then it is divisible by 10
(b) If a number is not divisible by 5 then it is not divisible by 10
(c) If a number is divisible by 2 and 4 then it is divisible by 8
(d) If a number is divisible by 8 then it is divisible by 2 and 4
(e) If a number is divisible by 99 then it is divisible by 9 and 11.

Problem 16.11
Test each number for divisibility by 2, 3, and 5. Do the work mentally.
(a) 1554 (b) 1999 (c) 805 (d) 2450

Problem 16.12
Are the numbers of the previous problem divisible by
(a)0 (b) 10 (c) 15 (d) 30

Problem 16.13
Is 1,927,643,001,548 divisible by 11? Explain.

Problem 16.14
At a glance, determine the digit d so that the number 87,543,24d is divisible by 4. Is there more than one solution?

Problem 16.15
Determine the digit d so that the number 6,34d,217 is divisible by 11.

Problem 16.16
Find the digit d so that the number 897,650,243,28d is divisible by 6.

Problem 16.17
(a) Determine whether 97,128 is divisible by 2,4 and 8.
(b) Determine whether 83,026 is divisible by 2,4, and 8.

Problem 16.18
Use the divisibility tests to determine whether each of the following numbers is divisible by 3 and divisible by 9.
(a) 1002 (b) 14,238

Problem 16.19
The store manager has an invoice of 72 four-function calculators. The first and last digits on the receipt are illegible. The manager can read $_67.9_.
What are the missing digits, and what is the cost of each calculator?

Problem 16.20
The number 57,729,364,583 has too many digits for most calculator to display. Determine whether this number is divisible by each of the following.
(a) 2 (b) 3 (c) 5 (d) 6 (e) 8 (f) 9 (g) 10 (h) 11

17 Greatest Common Factors and Least Common Multiples

Consider the following concrete problem: An architect is designing an elegant display room for art museum. One wall is to be covered with large square marble tiles. To obtain the desired visual effect, the architect wants to use the largest tiles possible. If the wall is to be 12 feet high and 42 feet long, how large can the tiles be?
The length of the size of a tile must be a factor of both the height and the length of the wall. But the factors of 12 are 1, 2, 3, 4, 6, and 12, and those of 42 are 1, 2, 3, 6, 7, 14, 21, and 42. Thus, the tile size must be chosen from the list 1, 2, 3, 6, the set of common factors. Since 6 is the largest common factor then the tiles must measure 6 feet on a side.
Consideration like these lead to the notion of the greatest common factor of two nonzero whole numbers.

Greatest Common Factors
The **Greatest Common Factor** is the largest whole number that divides evenly two or more nonzero whole numbers. The greatest common factor of a and b will be denoted by GCF(a,b).
There are three ways for finding GCF(a,b). The first one uses sets, the second one uses prime factorizations, and the third one uses the division algorithm.

• Set Intersection Method
In this method we list all of the factors of each number, then list the common factors and choose the largest one.

Example 17.1
Find GCF(36,54).

Solution.
Let F_{36} denote the set of factors of 36. Then

$$F_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

Similarly,

$$F_{54} = \{1, 2, 3, 6, 9, 18, 27, 54\}$$

Thus,

$$F_{36} \cap F_{54} = \{1, 2, 3, 6, 9, 18\}.$$

So, GCF(36,54) = 18.$\blacksquare$

• **Prime Factorization**
In this method we list the prime factors, then multiply the common prime factors.

Example 17.2
Find GCF(36,54) using prime factorization.

Solution.
Writing the prime factorization of both 36 and 54 we find

$$\begin{array}{rcl} 36 & = & 2 \times 2 \times 3 \times 3 \\ 54 & = & 2 \times 3 \times 3 \times 3 \end{array}$$

Notice that the prime factorizations of 36 and 54 both have one 2 and two 3s in common. So, we simply multiply these common prime factors to find the greatest common factor. That is, $GCF(36, 54) = 2 \times 3 \times 3 = 18.$$\blacksquare$

Remark 17.1
In general, if $a = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$ and $b = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$ then

$$GCF(a, b) = p_1^{min\{s_1,t_1\}} p_2^{min\{s_2,t_2\}} \cdots p_k^{min\{s_k,t_k\}} \blacksquare$$

To discuss the third method we proceed as follows. Let a and b be whole numbers with $a \geq b$. Then by the division algorithm (See Section 13) we can find unique whole numbers q and r such that

$$a = bq + r, \quad 0 \leq r < b.$$

Theorem 17.1

$$GCF(a, b) = GCF(b, r).$$

Proof.
If c is a common factor of a and b then $c|a$ and $c|b$. Thus, $c|a$ and $c|bq$. Hence, $c|(a - bq)$ which means $c|r$. Since $c|b$ and $c|r$ then c is a common factor of b and r. Similarly, if d is a common factor of b and r. Then $d|b$ and $d|r$. So $d|r$ and $d|bq$. Hence, $d|(bq + r) = a$. This says that d is also a common factor of a and b. It follows that the pairs of numbers $\{a, b\}$ and $\{b, r\}$ have the same common factors. This implies that they have the same greatest common factor.$\blacksquare$

• **Euclidean Algorithm**
Let's find the greatest common factor of 36 and 54 using the division algorithm.

$$\begin{array}{rcll} 54 & = & 36 \times 1 + 18 & GCF(54,36) = GCF(36,18) \\ 36 & = & 18 \times 2 + 0 & GCF(36,18) = GCF(18,0) \end{array}$$

Thus, GCF(54,36) = GCF(36,18) = GCF(18,0)=18. Hence, to find the GCF of two numbers, apply the above theorem repeatedly until a remainder of zero is obtained. The final divisor that leads to the zero remainder is the GCF of the two numbers.

Example 17.3
Use the three methods discussed above to find $GCF(42,24)$.

Solution.
Set Intersection Method:

$$F_{42} = \{1,2,3,6,7,14,21,42\}$$

and

$$F_{24} = \{1,2,3,4,6,8,12,24\}$$

Thus,

$$F_{42} \cap F_{24} = \{1,2,3,6\}$$

so that $GCF(42,24) = 6$.

Prime Factorization:
Since $42 = 2 \cdot 3 \cdot 7$ and $24 = 2 \cdot 2 \cdot 2 \cdot 3$ then $GCF(42,24) = 2 \cdot 3 = 6$.
Euclidean Algorithm:

$$\begin{array}{rcll} 42 & = & 24 \cdot 1 + 18 & GCF(42,24) = GCF(24,18) \\ 24 & = & 18 \cdot 1 + 6 & GCF(24,18) = GCF(18,6) \\ 18 & = & 6 \cdot 3 + 0 & GCF(18,6) = GCF(6,0) = 6 \end{array}$$

Thus, $GCF(42,24) = 6.$ ■

Least Common Multiple
The least common multiple is useful when adding or subtracting fractions. These two operations require what is called finding the **common denominator** which turns out to be the least common multiple.

The **least common multiple** of a and b, denoted by LCM(a,b), is the smallest nonzero whole number that is a multiple of both a and b.
As with the GCFs, there are three different ways for finding LCM(a,b): the set intersection method, the prime factorization method, and the build-up method.

• **Set Intersection Method**
In this method list the nonzero multiples of each number until a first common multiple appears. This number is the LCM(a,b).

Example 17.4
Find LCM(12,8).

Solution.
Let M_8 and M_{12} denote the set of nonzero multiples of 8 and 12 respectively. Then

$$M_8 = \{8, 16, 24, \cdots\}$$

and

$$M_{12} = \{12, 24, \cdots\}$$

Thus, LCM(8,12) = 24.■

• **Prime Factorization Method**
To find the LCM using this method we first find the prime factorization of each number. Then take each of the primes that are factors of <u>either</u> of the given numbers. The LCM is the product of these primes, each raised to the greatest power of the prime that occurs in either of the prime factorizations. That is, if

$$a = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$$

and

$$b = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$$

then

$$LCM(a,b) = p_1^{max\{s_1,t_1\}} p_2^{max\{s_2,t_2\}} \cdots p_k^{max\{s_k,t_k\}}$$

We illustrate the above method in the following example.

Example 17.5
Find LCM(2520,10530).

Solution.
Writing the prime factorization of each number we find

$$\begin{aligned} 2520 &= 2^3 \cdot 3^2 \cdot 5 \cdot 7 \\ 10530 &= 2 \cdot 3^4 \cdot 5 \cdot 13 \end{aligned}$$

So $LCM(2520, 10530) = 2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 13 = 294,840.$ ■

•Euclidean Algorithm
The following theorem is useful in finding the LCM of two numbers a and b when their prime factorization is not easy to find. We find the GCF(a,b) using the Euclidean algorithm. The LCM is found by dividing the product $a \cdot b$ by the GCF.

Theorem 17.2
For any two nonzero whole numbers a and b we have

$$LCM(a,b)GCF(a,b) = a \cdot b.$$

Proof.
The justification of this theorem uses the prime factorizations of a and b. So write

$$\begin{aligned} a &= p_1^{s_1} \cdot p_2^{s_2} \cdots p_k^{s_k} \\ b &= p_1^{t_1} \cdot p_2^{t_2} \cdots p_k^{t_k} \end{aligned}$$

But

$$\begin{aligned} a \cdot b &= p_1^{s_1+t_1} \cdot p_2^{s_2+t_2} \cdots p_k^{s_k+t_k} \\ LCM(a,b) &= p_1^{max\{s_1,t_1\}} \cdot p_2^{max\{s_2,t_2\}} \cdots p_k^{max\{s_k,t_k\}} \\ GCF(a,b) &= p_1^{min\{s_1,t_1\}} \cdot p_2^{min\{s_2,t_2\}} \cdots p_k^{min\{s_k,t_k\}} \end{aligned}$$

Hence,

$$LCM(a,b) \cdot GCF(a,b) = p_1^{max\{s_1,t_1\}+min\{s_1,t_1\}} \cdot p_2^{max\{s_2,t_2\}+min\{s_2,t_2\}} \cdots p_k^{max\{s_k,t_k\}+min\{s_k,t_k\}}.$$

But for any index i, we have

$$max\{s_i,t_i\} + min\{s_i,t_i\} = s_i + t_i.$$

Thus,

$$LCM(a,b) \cdot GCF(a,b) = a \cdot b$$ ■

Example 17.6
Find LCM(731,952).

Solution.
Using the Euclidean algorithm one will find that $GCF(731, 952) = 17$. Thus, by the above theorem

$$LCM(731, 952) = \frac{731 \times 952}{17} = 40,936. \blacksquare$$

Practice Problems

Problem 17.1
Find the GCF and LCM for each of the following using the set intersection method.
(a) 18 and 20
(b) 24 and 36
(c) 8, 24, and 52
(d) 7 and 9.

Problem 17.2
Find the GCF and LCM for each of the following using the prime factorization method.
(a) 132 and 504
(b) 65 and 1690
(c) 900, 96, and 630
(d) 108 and 360
(e) 11 and 19.

Problem 17.3
Find the GCF and LCM for each of the following using the Euclidean algorithm method.
(a) 220 and 2924
(b) 14,595 and 10,856
(c) 122,368 and 123,152.

Problem 17.4
Find the LCM using any method.
(a) 72, 90, and 96
(b) 90, 105, and 315.

Problem 17.5
Find the LCM of the following numbers using Theorem 17.2.
(a) 220 and 2924
(b) 14,595 and 10,856
(c) 122,368 and 123,152.

Problem 17.6
If a and b are nonzero whole numbers such that $GCF(a, b) = 1$ then we say that a and b are **relatively prime.** Determine whether the following pairs of numbers are relatively prime.
(a) 7 and 19
(b) 27 and 99
(c) 8 and 6
(d) 157 and 46.

Problem 17.7
(a) Draw a Venn diagram showing the factors and common factors of 10 and 24.
(b) What is the greatest common factor of 10 and 24?

Problem 17.8
Suppose that $a = 2 \cdot 3^2 \cdot 7^3$ and $GCF(a, b) = 2 \cdot 3^2 \cdot 7$. Give two possible values of b.

Problem 17.9
To find the GCF and LCM of three or more nonzero whole numbers the prime factorization method is the most desirable.
(a) Find the GCF and the LCM of $a = 2^2 \cdot 3^1 \cdot 5^2, b = 2^1 \cdot 3^3 \cdot 5^1, c = 3^2 \cdot 5^3 \cdot 7^1$.
(b) Is it necessarily true that $LCM(a, b, c) \cdot GCF(a, b, c) = a \cdot b \cdot c$?

Problem 17.10
Use the method of intersection to find LCM(18,24,12) and GCF(18,24,12).

Problem 17.11
Find all whole numbers x such that GCF(24,x)=1 and $1 \leq x \leq 24$.

Problem 17.12
George made enough money by selling candy bars at 15 cents each to buy several cans of pop at 48 cents each. If he had no money left over, what is the fewest number of candy bars he could have sold?

Problem 17.13
In the set $\{18, 96, 54, 27, 42\}$, find the pair(s) of numbers with the greatest GCF and the pair(s) with the smallest LCM.

Problem 17.14
Which is larger GCF(a,b) or LCM(a,b)?

Problem 17.15
Suppose that a and 10 are relatively prime. Find all the possible values of a that are less than 10.

Problem 17.16
LCM(24,x)=168 and GCF(24,x)=2. Find x.

Problem 17.17
(a) Show that for any nonzero whole numbers a and b with $a \geq b$ we have GCF(a,b)=GCF(a-b,b).
(b) Use part (a) to find GCF(546,390).

Problem 17.18
Suppose that $a = 2^3 \cdot 5^2 \cdot 7^3$, $GCF(a, b) = 2 \cdot 5^2 \cdot 7$, and $LCM(a, b) = 2^3 \cdot 3^3 \cdot 5^4 \cdot 7^3$. Find the value of b.

Problem 17.19
Suppose 0 were included as a possible multiple in the definition of LCM. What would be the LCM of any two whole numbers?

Problem 17.20
Assume a and b are nonzero whole numbers. Answer the following:
(a) If GCF(a,b) = 1, find LCM(a,b).
(b) Find GCF(a,a) and LCM(a,a).
(c) Find GCF(a^2,a) and LCM(a^2,a).
(d) If $a|b$, find GCF(a,b) and LCM(a,b).
(e) If a and b are two primes, find GCF(a,b) and LCM(a,b).
(f) What is the relationship between a and b if GCF(a,b) = a?
(g) What is the relationship between a and b if LCM(a,b) = a?

18 Fractions of Whole Numbers

Consider the following problem: Suppose that a class of twenty students took a math test and only 5 students made a passing grade. To describe such a situation one will say that one fourth of the class passed and three-fourth failed. The class is considered as a unit. We "break" this unit into four groups each consisting of five students. One of the group consists of those students who passed the test and the other three consist of those students who failed the test. Consideration like these lead to the introduction of fractions.
The word *fraction* comes from the Latin word *fractius* which means "to break". When an object is divided into an equal number of parts then each part is called a **fraction.**
There are different ways of writing a fraction. For example, two fifths of an object can be written as
• a common fraction: $\frac{2}{5}$
• a decimal 0.4
• a percentage 40%
We will learn about percentages and decimals later.

Now, let us have a closer look at the common fraction: $\frac{a}{b}$.
• The number a is called the **numerator**, derived from the Latin word *numeros*, meaning "number", and represents the number of parts in consideration.
• The number b is called the **denominator**, derived from the Latin word *denominare*, meaning "namer", and represents how many equal parts in the unit. Keep in mind that this number can never be zero since division by zero is undefined.

Example 18.1
Show that any whole number is a fraction.

Solution.
If a is a whole number then we can write a as the fraction $\frac{a}{1}$.■

Different Types of Fractions
There are 3 different types of fractions:

• **Proper Fractions**
Proper fractions have the numerator part smaller than the denominator part, for example $\frac{2}{5}$.
• **Improper Fractions**
Improper fractions have the numerator part greater or equal to the denominator part, for example $\frac{7}{6}$.
• **Mixed Fractions**
Mixed fractions have a whole number plus a fraction, for example, $3\frac{1}{2} = 3+\frac{1}{2}$.

Pictorial Representation of a Fraction
Several physical and pictorial representations are useful in the elementary school classroom to illustrate fraction concepts. We consider four different pictorial representations of a fraction.
• **Colored Regions**
A shape is chosen to represent the unit and is then subdivided into equal parts. A fraction is visualized by coloring some of the parts as shown in Figure 18.1.

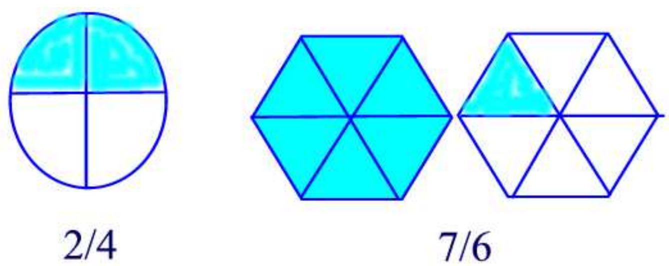


Figure 18.1

• **The Set Model**
Figure 18.2 shows a set of 7 apples that contains a subset of 3 that are wormy. Therefore, we would say that $\frac{3}{7}$ of the apples are wormy.

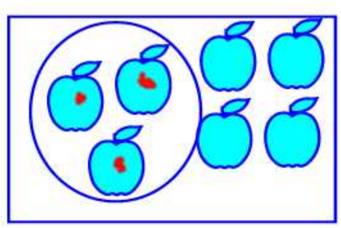


Figure 18.2

• **Fraction Strips**
Here the unit is defined by a rectangular strip. A fraction $\frac{a}{b}$ is modeled by shading a parts of the b equally sized subrectangles. Sample fraction strips are shown in Figure 18.3.

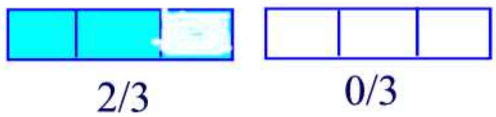


Figure 18.3

• **The Number-Line Model**
A fraction such as $\frac{5}{4}$ is assigned to a point along the number line by subdividing the interval $[0, 1]$ into four equal parts, and then counting off 5 of these lengths to the right of 0 as shown in Figure 18.4.

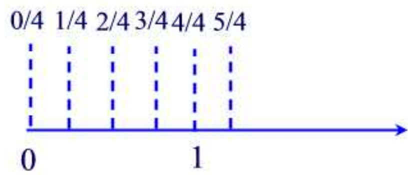
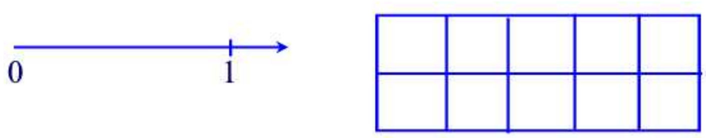


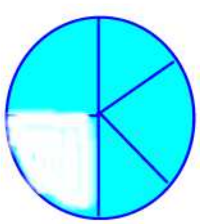
Figure 18.4

Practice Problems

Problem 18.1
Explain how to complete each diagram so that it shows $\frac{3}{10}$.

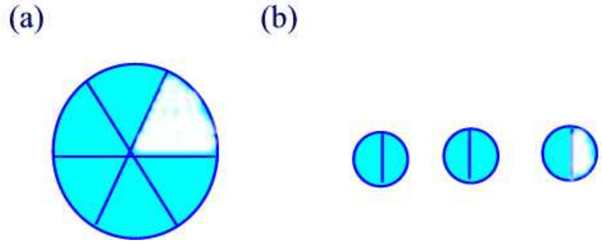


Problem 18.2
A child shows $\frac{4}{5}$ as



What is wrong with the diagram?

Problem 18.3
What fraction is represented by the shaded parts?



Problem 18.4
Depict the fraction $\frac{4}{6}$ with the following models.
(a) Colored region model
(b) Set model
(c) Fraction strip model
(d) Number-line model.

Problem 18.5
Express the following quantities by a fraction placed in the blank space.
(a) 20 minutes is ____ of an hour.
(b) 30 seconds is ____ of a minute.
(c) 5 days is ____ of a week.
(d) 25 years is ____ of a century.
(e) A quarter is ____ of a dollar.
(f) 3 eggs is ____ of a dozen.

Problem 18.6
Three fifths of a class of 25 students are girls. How many are girls?

Problem 18.7
The Independent party received one-eleventh of the 6,186,279 votes cast. How many votes did the party receive?

Equivalent or Equal Fractions
Equivalent fractions are fractions that have the same value or represent the same part of an object. If a pie is cut into two pieces, each piece is also one-half of the pie. If a pie is cut into 4 pieces, then two pieces represent the same amount of pie that 1/2 did. We say that 1/2 is **equivalent** or **equal** to 2/4 and we write $\frac{1}{2} = \frac{2}{4}$.

The Fundamental Law of Fractions describes the general relationship between equivalent fractions.

<u>The Fundamental Law of Fractions</u>
For any fraction $\frac{a}{b}$ and any nonzero whole number c we have

$$\frac{a \cdot c}{b \cdot c} = \frac{a \div c}{b \div c} = \frac{a}{b}.$$

Example 18.2
Show that the fraction $\frac{6}{14}$ is equivalent to $\frac{9}{21}$.

Solution.
Since $\frac{6}{14} = \frac{3 \cdot 2}{7 \cdot 2} = \frac{3}{7}$ and $\frac{9}{21} = \frac{3 \cdot 3}{7 \cdot 3} = \frac{3}{7}$ then $\frac{6}{14} = \frac{9}{21}$.■

The following theorem shows that two fractions are equivalent if and only if their cross-products are equal.

Theorem 18.1
$\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

Proof.
Suppose first that $\frac{a}{b} = \frac{c}{d}$. Since $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$ then we must have $\frac{ad}{bd} = \frac{bc}{bd}$. But this is true only when $ad = bc$.
Conversely, if $ad = bc$ then $\frac{ad}{bd} = \frac{bc}{bd}$. By the Fundamental Law of Fractions we have $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$. Thus, $\frac{a}{b} = \frac{c}{d}$.■

Example 18.3
Find a value for x so that $\frac{12}{42} = \frac{x}{210}$.

Solution.
By the above theorem we must have $42 \cdot x = 210 \times 12$. But $210 \times 12 = 60 \times 42$ so that $x = 60$.■

Simplifying Fractions
When a fraction $\frac{a \cdot c}{b \cdot c}$ is replaced with $\frac{a}{b}$, we say that $\frac{a \cdot c}{b \cdot c}$ has been **simplified.** We say that a fraction $\frac{a}{b}$ is in **simplest form** (or **lowest terms**) if a and b have no common divisor greater than 1. For example, the fraction $\frac{3}{7}$.
We write a fraction $\frac{a}{b}$ in simplest form by dividing both a and b by the GCF(a,b).

Example 18.4
Find the simplest form of each of the following fractions.
(a) $\frac{240}{72}$ (b) $\frac{399}{483}$.

Solution.
(a) First, we find GCF(240,72). Since $240 = 2^4 \cdot 3 \cdot 5$ and $72 = 2^3 \cdot 3^2$ then $GCF(240, 72) = 2^3 \cdot 3 = 24$. Thus,

$$\frac{240}{72} = \frac{240 \div 24}{72 \div 24} = \frac{10}{3}.$$

(b) Since $399 = 3 \cdot 7 \cdot 19$ and $483 = 3 \cdot 7 \cdot 23$ then $GCF(399, 483) = 3 \cdot 7 = 21$. Thus,

$$\frac{399}{483} = \frac{399 \div 21}{483 \div 21} = \frac{19}{23}.\blacksquare$$

Example 18.5
Simplify the fraction $\frac{54}{72}$.

Solution.
Since $GCF(54, 72) = 18$ then

$$\frac{54}{72} = \frac{54 \div 18}{72 \div 18} = \frac{3}{4}\blacksquare$$

Practice Problems

Problem 18.8
Show that $\frac{3}{5} = \frac{6}{10}$.

Problem 18.9
Use drawings of fractions strips to show that $\frac{3}{4}$, $\frac{6}{8}$, and $\frac{9}{12}$ are equivalent.

Problem 18.10
Write each fraction in simplest form.
(a) $\frac{168}{464}$ (b) $\frac{xy^2}{xy^3z}$.

Problem 18.11
Two companies conduct surveys asking people if they favor stronger controls on air pollution. The first company asks 1,500 people, and the second asks 2,000 people. In the first group, 1,200 say yes. Make up results for the second group that would be considered equivalent.

Problem 18.12
Find four different fractions equivalent to $\frac{4}{9}$.

Problem 18.13
Fill in the missing number to make the fractions equivalent.
(a) $\frac{4}{5} = \frac{}{30}$ (b) $\frac{6}{9} = \frac{2}{}$.

Problem 18.14
Rewrite the following fractions in simplest form.
(a) $\frac{84}{144}$ (b) $\frac{208}{272}$

Problem 18.15
Find the prime factorizations of the numerators and denominators of these fractions and use them to express the fractions in simplest form.
(a) $\frac{96}{288}$ (b) $\frac{2520}{378}$.

Problem 18.16
If a fraction is equal to $\frac{3}{4}$ and the sum of the numerator and denominator is 84, what is the fraction?

Problem 18.17
Determine if each of the following is correct.
(a) $\frac{ab+c}{b} = a + c$
(b) $\frac{a+b}{a+c} = \frac{b}{c}$
(c) $\frac{ab+ac}{ac} = \frac{b+c}{c}$.

Problem 18.18
If $\frac{a}{b} = \frac{c}{b}$. what must be true?

Problem 18.19
Solve for x.
(a) $\frac{2}{3} = \frac{x}{16}$

(b) $\frac{3}{x} = \frac{3x}{x^2}$.

Problem 18.20
Rewrite as a mixed number in simplest for.
(a) $\frac{525}{96}$ (b) $\frac{1234}{432}$.

Problem 18.21
I am a proper fraction. The sum of my numerator and denominator is one-digit square. Their product is a cube. What fraction am I?

Comparing and Ordering Fractions
If we place the fractions 2/7 and 5/7 on the fraction number line we notice that 2/7 is to the left of 5/7. This suggests the following definition.
We say that $\frac{a}{b}$ is **less than** $\frac{c}{b}$, and we write $\frac{a}{b} < \frac{c}{b}$, if and only if $a < c$.
The above definition compares fractions with the same denominator. What about fractions with unlike denominators? To compare fractions with unlike denominators, say $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq d$, we compare the fractions $\frac{ad}{bd}$ and $\frac{bc}{bd}$ since $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$. By the above definition, it follows that $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$. This establishes a proof of the following theorem.

Theorem 18.2
If a, b, c, d are whole numbers with $b \neq 0, d \neq 0$ then $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$.

Example 18.6
Compare the fractions $\frac{7}{8}$ and $\frac{9}{11}$.

Solution.
Since $7 \cdot 11 > 8 \cdot 9$ then $\frac{9}{11} < \frac{7}{8}$.$\blacksquare$

We conclude this section with the following question: Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$. Is there a fraction between these two fractions?
The answer is affirmative according to the following theorem.

Theorem 18.3 *(Density Property)*
If $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Proof.
Since $\frac{a}{b} < \frac{c}{d}$ then by Theorem 18.2 we have $ad < bc$. Add cd to both sides to obtain $ad + cd < bc + cd$. That is, $(a + c)d < (b + d)c$ or $\frac{a+c}{b+d} < \frac{c}{d}$.
Similarly, if we add ab to both sides of $ad < bc$ we find $ad + ab < bc + ab$ or $a(b + d) < b(a + c)$. Thus, $\frac{a}{b} < \frac{a+c}{b+d}$.$\blacksquare$

Example 18.7
Find a fraction between the fractions $\frac{9}{13}$ and $\frac{12}{17}$.

Solution.
Since $9 \cdot 17 < 12 \cdot 13$ then $\frac{9}{13} < \frac{12}{17}$. By the previous theorem we have

$$\frac{9}{13} < \frac{9+12}{13+17} < \frac{12}{17}$$

or

$$\frac{9}{13} < \frac{21}{30} < \frac{12}{17}. \blacksquare$$

Practice Problems

Problem 18.22
Show that
(a) $\frac{1}{3} < \frac{2}{3}$ (b) $\frac{5}{8} > \frac{3}{8}$.

Problem 18.23
Compare the pairs of fractions.
(a) $\frac{7}{8}$ and $\frac{3}{4}$ (b) $\frac{4}{9}$ and $\frac{7}{15}$.

Problem 18.24
You have two different recipes for making orange juice from concentrate. The first says to mix 2 cups of concentrate with 6 cups of water. The second says to mix 3 cups of concentrate with 8 cups of water. Which recipe will have a stronger orange flavor?

Problem 18.25
A third grader says that $\frac{1}{4}$ is less than $\frac{1}{5}$ because 4 is less than 5. What would you tell the child?

Problem 18.26
Find a fraction between $\frac{3}{4}$ and $\frac{7}{8}$.

Problem 18.27
Order the following fractions from least to greatest.
(a) $\frac{2}{3}$ and $\frac{7}{12}$.
(b) $\frac{2}{3}, \frac{5}{6}, \frac{29}{36}$, and $\frac{8}{9}$.

Problem 18.28
Compare $2\frac{4}{5}$ and $2\frac{3}{6}$.

Problem 18.29
If $\frac{a}{b} < 1$, compare the fractions $\frac{c}{d}$ and $\frac{ac}{bd}$.

Problem 18.30
Find a fraction between $\frac{5}{6}$ and $\frac{83}{100}$.

19 Addition and Subtraction of Fractions

You walked $\frac{1}{5}$ of a mile to school and then $\frac{3}{5}$ of a mile from school to your friend's house. How far did you walk altogether? How much farther was the second walk than the first?
To solve the first problem, you would add $\frac{1}{5}$ and $\frac{3}{5}$. To solve the second problem, you would subtract $\frac{1}{5}$ from $\frac{3}{5}$. Adding and subtracting fractions are the subjects of this section.

Addition of Fractions
Pictures and manipulatives help understand the process of finding the sum of two fractions. Let's consider the question of finding $\frac{1}{5} + \frac{3}{5}$. This sum is illustrated in two ways in Figure 19.1. The colored region and the number-line models both show that $\frac{1}{5} + \frac{3}{5} = \frac{4}{5}$.

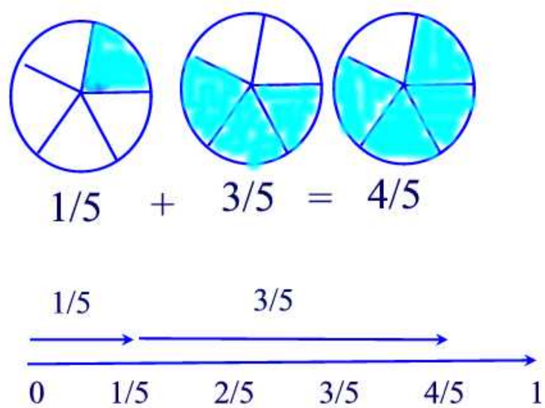


Figure 19.1

The above models suggest the following definition of fractions with like denominators:
The sum of two fractions with like denominators is found by adding the numerators and dividing the result by the common denominator. Symbolically

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

Example 19.1
Find the value of the sum $\frac{3}{8} + \frac{1}{8}$ and write the answer in simplest form.

Solution.
By the definition above we have

$$\frac{3}{8}+\frac{1}{8}=\frac{3+1}{8}=\frac{4}{8}=\frac{4\cdot 1}{4\cdot 2}=\frac{1}{2}.\blacksquare$$

Next we consider adding two fractions with unlike denominators. This is done by rewriting the fractions with a common denominator. The common denomintor is referred to as the **least common denominator** and is nothing else then the least common multiple of both denominators. For example, the least common denominator for $\frac{1}{4}$ and $\frac{2}{3}$ is the LCM(4,3), which is 12. Thus,

$$\begin{array}{ccc}\frac{1}{4}+\frac{2}{3} & = & \frac{1\cdot 3}{4\cdot 3}+\frac{2\cdot 4}{3\cdot 4}\\ & = & \frac{3}{12}+\frac{8}{12}=\frac{11}{12}\end{array}$$

The procedure just described can be modeled with fraction strips as shown in Figure 19.2

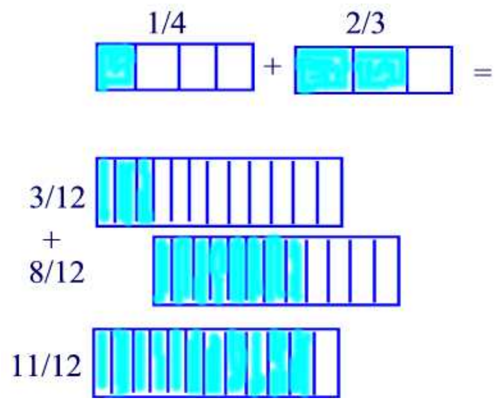


Figure 19.2

The general rule for adding fractions with unlike denominators is as follows. To add $\frac{a}{b}+\frac{c}{d}$ where $b\neq 0, d\neq 0$ and $b\neq d$:
(1) Rename each fraction with an equivalent fraction using the least common denominator, that is, LCM(a,b).
(2) Add the new fractions using the addition rule for like denominators.

Example 19.2
Find $\frac{3}{4}+\frac{5}{6}+\frac{2}{3}$.

Solution.
The least common denominator is $LCM(4,6,3) = 4 \cdot 3 = 12$. Thus,

$$\begin{aligned}
\tfrac{3}{4} + \tfrac{5}{6} + \tfrac{2}{3} &= \tfrac{3\cdot 3}{4\cdot 3} + \tfrac{5\cdot 2}{6\cdot 2} + \tfrac{2\cdot 4}{3\cdot 4} \\
&= \tfrac{9}{12} + \tfrac{10}{12} + \tfrac{8}{12} = \tfrac{9+10+8}{12} \\
&= \tfrac{27}{12} \blacksquare
\end{aligned}$$

Example 19.3
The sum of a whole number and a fraction is most often written as a **mixed number**. For example, $2 + \frac{3}{4} = 2\frac{3}{4}$.
(a) Express $3\frac{2}{5}$ as an improper fraction.
(b) Express $\frac{36}{7}$ as a mixed number.

Solution.
(a) We have: $3\frac{2}{5} = 3 + \frac{2}{5} = \frac{15}{5} + \frac{2}{5} = \frac{17}{5}$.
(b) We have $\frac{36}{7} = \frac{35}{7} + \frac{1}{7} = 5 + \frac{1}{7} = 5\frac{1}{7}$.$\blacksquare$

Example 19.4
Compute $2\frac{3}{4} + 4\frac{2}{5}$.

Solution.
The least common denominator is $LCM(4,5) = 4 \cdot 5 = 20$. Thus,

$$\begin{aligned}
2\tfrac{3}{4} + 4\tfrac{2}{5} &= 2 + 4 + \tfrac{3}{4} + \tfrac{2}{5} \\
&= 6 + \tfrac{15}{20} + \tfrac{8}{20} \\
&= 6 + \tfrac{23}{20} = 7\tfrac{3}{20} \blacksquare
\end{aligned}$$

Properties of Addition of Fractions
The following properties of addition can be used to simplify computations.

Theorem 19.1
(a) Closure: The sum of two fractions is a fraction.
(b) Commutativity: Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions. Then

$$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}.$$

(c) Associativity: Let $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ be any fractions. Then

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right).$$

(d) Additive identity: 0 is the additive identity since for any fraction $\frac{a}{b}$ we have
$$\frac{a}{b} + 0 = \frac{a}{b}.$$

Proof.
We will prove parts (a) and (b) and leave the rest for the reader.
(a) This follows from $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and the fact that both the numerator and denominator are whole numbers.
(b)By the definition of addition of fractions and the fact that addition and multiplication of whole numbers are commutative we have

$$\begin{array}{rcl} \frac{a}{b} + \frac{c}{d} & = & \frac{ad+bc}{bd} \\ & = & \frac{bc+ad}{bd} = \frac{cb+da}{db} \\ & = & \frac{c}{d} + \frac{a}{b} \blacksquare \end{array}$$

Practice Problems

Problem 19.1
If one of your students wrote $\frac{1}{4} + \frac{2}{3} = \frac{3}{7}$, how would you convince him or her that this is incorrect?

Problem 19.2
Use the colored region model to illustrate these sums, with the unit given by a circular disc.
(a) $\frac{2}{5} + \frac{6}{5}$ (b) $\frac{2}{3} + \frac{1}{4}$.

Problem 19.3
Find $\frac{1}{6} + \frac{1}{4}$ using fraction strips.

Problem 19.4
Represent each of these sums with a number-line diagram.
(a) $\frac{1}{8} + \frac{3}{8}$ (b) $\frac{2}{3} + \frac{1}{2}$.

Problem 19.5
Perform the following additions. Express each answer in simplest form.
(a) $\frac{2}{7} + \frac{3}{7}$ (b) $\frac{3}{8} + \frac{11}{24}$ (c) $\frac{213}{450} + \frac{12}{50}$.

Problem 19.6
A child thinks $\frac{1}{2} + \frac{1}{8} = \frac{2}{10}$. Use fraction strips to explain why $\frac{2}{10}$ cannot be the answer.

Problem 19.7
Compute the following without a calculator.
(a) $\frac{5}{12}+\frac{3}{8}$ (b) $\frac{1}{a}+\frac{2}{b}$.

Problem 19.8
Compute the following without a calculator.
(a) $\frac{2}{15}+\frac{1}{21}$ (b) $\frac{3}{2n}+\frac{4}{5n}$.

Problem 19.9
Compute $5\frac{3}{4}+2\frac{5}{8}$.

Problem 19.10
Solve mentally:
(a)$\frac{1}{8}+?=\frac{5}{8}$
(b) $4\frac{1}{8}+x=10\frac{3}{8}$.

Problem 19.11
Name the property of addition that is used to justify each of the following equations.
(a) $\frac{3}{7}+\frac{2}{7}=\frac{2}{7}+\frac{3}{7}$
(b) $\frac{4}{15}+0=\frac{4}{15}$
(c) $\left(\frac{2}{5}+\frac{3}{5}\right)+\frac{4}{7}=\frac{2}{5}+\left(\frac{3}{5}+\frac{4}{7}\right)$
(d) $\frac{2}{5}+\frac{3}{7}$ is a fraction.

Problem 19.12
Find the following sums and express your answer in simplest form.
(a) $\frac{3}{7}+\frac{7}{3}$
(b) $\frac{8}{9}+\frac{1}{12}+\frac{3}{16}$
(c) $\frac{8}{31}+\frac{4}{51}$
(d) $\frac{143}{1000}+\frac{759}{100,000}$.

Problem 19.13
Change the following mixed numbers to fractions.
(a) $3\frac{5}{6}$ (b) $2\frac{7}{8}$ (c) $7\frac{1}{9}$.

Problem 19.14
Use the properties of fraction addition to calculate each of the following sums mentally.
(a) $\left(\frac{3}{7}+\frac{1}{9}\right)+\frac{4}{7}$
(b) $1\frac{9}{13}+\frac{5}{6}+\frac{4}{13}$
(c) $\left(2\frac{2}{5}+3\frac{3}{8}\right)+\left(1\frac{4}{5}+2\frac{3}{8}\right)$

Problem 19.15
Change the following fractions to mixed numbers.
(a) $\frac{35}{3}$ (b) $\frac{49}{6}$ (c) $\frac{17}{5}$

Problem 19.16
(1) Change each of the following to mixed numbers: (a) $\frac{56}{3}$ (b) $\frac{293}{100}$.
(2) Change each of the following to a fraction of the form $\frac{a}{b}$ where a and b are whole numbers: (a) $6\frac{3}{4}$ (b) $7\frac{1}{2}$.

Problem 19.17
Place the numbers 2, 5, 6, and 8 in the following boxes to make the equation true:

$$\frac{\square}{\square} + \frac{\square}{\square} = \frac{23}{24}$$

Problem 19.18
A clerk sold three pieces of one type of ribbon to different customers. One piece was $\frac{1}{3}$ yard long, another $2\frac{3}{4}$yd long, and the third was $3\frac{1}{2}$yd. What was the total length of that type of ribbon sold?

Problem 19.19
Karl wants to fertilize his 6 acres. If it takes $8\frac{2}{3}$ bags of fertilizer for each acre, how much fertilizser does he need to buy?

Subtraction of Fractions
You walked $\frac{3}{6}$ of a mile to school and then $\frac{7}{6}$ of a mile from school to your friend's house. How much farther was the second walk than the first? This problem requires subtracting fractions with like denominators, i.e. $\frac{7}{6} - \frac{3}{6}$. Figure 19.3 shows how the take-away, measurement, and missing-addend models of the subtraction operation can be illustrated with colored regions, the number line, and fraction strips. In each case we see that $\frac{7}{6} - \frac{3}{6} = \frac{4}{6}$.

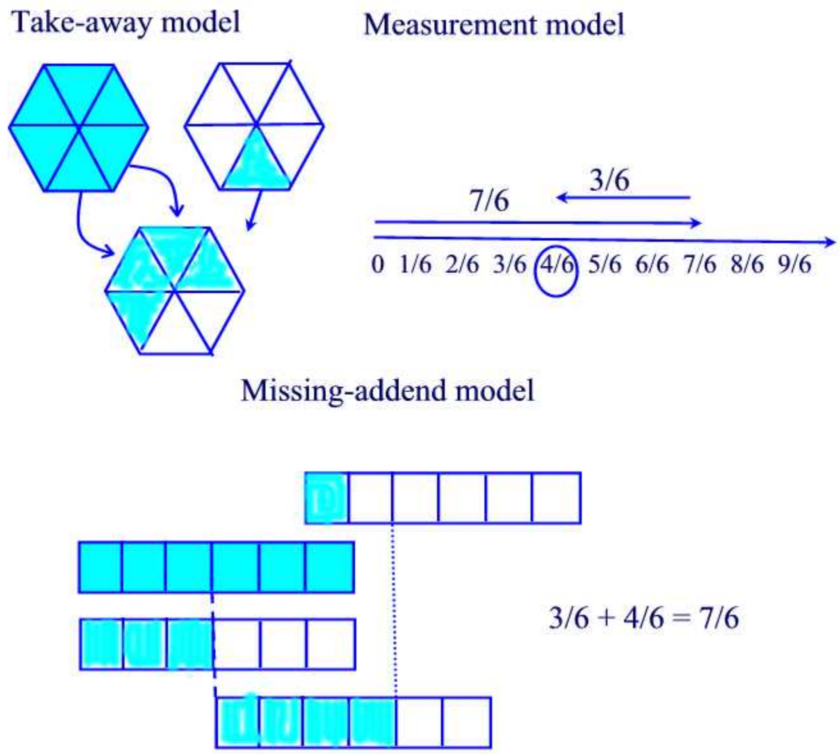


Figure 19.3

In general, subtraction of fractions with like denominators is determined as follows:

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}, \quad a \geq c.$$

Now, to subtract fractions we unlike denominators we proceed as follows: Suppose that $\frac{a}{b} \geq \frac{c}{d}$, i.e. $ad \geq bc$. Then

$$\begin{array}{rcl} \frac{a}{b} - \frac{c}{d} & = & \frac{ad}{bd} - \frac{bc}{bd} \\ & = & \frac{ad-bc}{bd} \end{array}$$

We summarize this result in the following theorem.

Theorem 19.2
If $\frac{a}{b}$ and $\frac{c}{d}$ are any fractions with $\frac{a}{b} \geq \frac{c}{d}$ then $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$.

Example 19.5
Find each difference.
(a) $\frac{5}{8} - \frac{1}{4}$ (b) $5\frac{1}{3} - 2\frac{3}{4}$.

Solution.
(a) Since LCM(4,8) = 8 then

$$\frac{5}{8} - \frac{1}{4} = \frac{5}{8} - \frac{2}{8} = \frac{3}{8}.$$

(b) Since LCM(3,4) = 12 then

$$\begin{array}{rcl} 5\frac{1}{3} - 2\frac{3}{4} & = & (5 + \frac{1}{3}) - (2 + \frac{3}{4}) \\ & = & (\frac{70}{12} + \frac{4}{12}) - (\frac{24}{12} + \frac{9}{12}) \\ & = & \frac{74}{12} - \frac{33}{12} = \frac{31}{12} = 2\frac{7}{12} \blacksquare \end{array}$$

Practice Problems

Problem 19.20
Use fraction strips, colored regions, and number-line models to illustrate $\frac{2}{3} - \frac{1}{4}$.

Problem 19.21
Compute these differences, expressing each answer in simplest form.
(a) $\frac{5}{8} - \frac{2}{8}$ (b) $\frac{3}{5} - \frac{2}{4}$ (c) $2\frac{2}{3} - 1\frac{1}{3}$.

Problem 19.22
(a) Find the least common denominator of $\frac{7}{20}$ and $\frac{3}{28}$.
(b) Compute $\frac{7}{20} - \frac{3}{28}$.

Problem 19.23
Compute the following without a calculator.
(a) $\frac{5}{12} - \frac{1}{20}$ (b) $\frac{5}{6c} - \frac{3}{4c}$.

Problem 19.24
Compute $10\frac{1}{6} - 5\frac{2}{3}$.

Problem 19.25
Compute $90\frac{1}{3} - 32\frac{7}{9}$.

Problem 19.26
Solve mentally: $3\frac{9}{10} - ? = 1\frac{3}{10}$.

Problem 19.27
Fill in each square with either a + sign or a - sign to complete each equation correctly.

$$\text{(a) } 1\frac{1}{4} \square \frac{1}{4} \square \frac{3}{4} = \frac{3}{4}$$
$$\text{(b) } 1\frac{7}{8} \square \frac{1}{4} \square \frac{3}{8} = 1\frac{1}{4}$$

Problem 19.28
On a number-line, demonstrate the following differences using the take-away approach.
(a) $\frac{5}{12} - \frac{1}{12}$ (b) $\frac{2}{3} - \frac{1}{4}$.

Problem 19.29
Perform the following subtractions.
(a) $\frac{9}{11} - \frac{5}{11}$ (b) $\frac{4}{5} - \frac{3}{4}$ (c) $\frac{21}{51} - \frac{7}{39}$.

Problem 19.30
Which of the following properties hold for fraction subtraction?
(a) Closure (b) Commutative (c) Associative (d) Identity

Problem 19.31
Rafael ate one-fourth of a pizza and Rocco ate one-third of it. What fraction of the pizza did they eat?

Problem 19.32
You planned to work on a project for about $4\frac{1}{2}$ hours today. If you have been working on it for $1\frac{3}{4}$ hours, how much more time will it take?

Problem 19.33
Martin bought $8\frac{3}{4}$yd of fabric. She wants to make a skirt using $1\frac{7}{8}$yd, pants using $2\frac{3}{8}$yd, and a vest using $1\frac{2}{3}$yd. How much fabric will be left over?

Problem 19.34
A recipe requires $3\frac{1}{2}c$ of milk. Don put in $1\frac{3}{4}c$ and emptied the container. How much more milk does he need to put in?

Problem 19.35
A class consists of $\frac{2}{5}$ freshmen, $\frac{1}{4}$ sophomore, and $\frac{1}{10}$ juniors. What fraction of the class is seniors?

Problem 19.36
Sally, her brother, and another partner own a pizza restaurant. If Sally owns $\frac{1}{3}$ and her brother owns $\frac{1}{4}$ of the restaurant, what part does the third partner own?

20 Multiplication and Division of Fractions

You are planning a chicken dinner. Each person will eat about $\frac{1}{4}$ of a chicken? How many chicken will you need for 3 people? To solve this problem, you would multiply $\frac{1}{4}$ by 3. Multiplying fractions is the first topic of this section.

Multiplication of Fractions of Whole Numbers

To motivate the definition of multiplication of fractions, we use the interpretations of multiplication as repeated addition. Using repeated addition, we can write

$$3 \cdot \left(\frac{3}{4}\right) = \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{9}{2} = 2\frac{1}{4}.$$

You can see this calculation by using fraction strips as shown in Figure 20.1

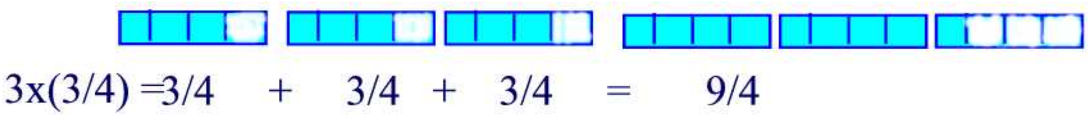


Figure 20.1

Example 20.1

Use number line to illustrate the product $\left(\frac{3}{4}\right) \cdot 3$.

Solution.

The value of $\left(\frac{3}{4}\right) \cdot 3$ is shown in Figure 20.2.

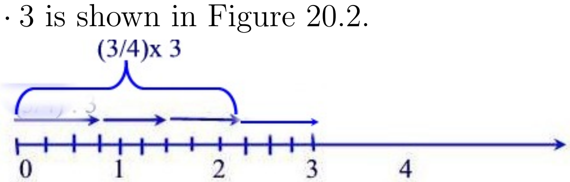


Figure 20.2

Next, we consider what happens when neither factor in the product is a whole number. Figure 20.3 shows how area models are used to illustrate the product $\frac{1}{2} \cdot \frac{3}{5}$.

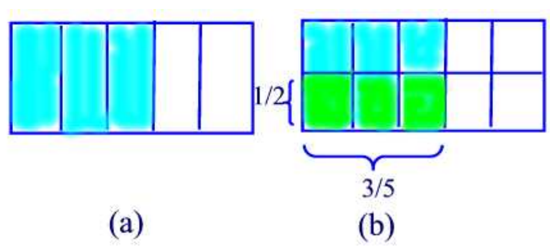


Figure 20.3

Figure 20.3(a) shows a one-unit rectangle seperated into fifths, with $\frac{3}{5}$ shaded. To find $\frac{1}{2}$ of $\frac{3}{5}$, we divide the shaded portion of the rectangle into two equal parts. The results would be the green portion of Figure 20.3(b). However, the green portion represents $\frac{3}{10}$ of the one-unit rectangle. Thus

$$\frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10} = \frac{1 \cdot 3}{2 \cdot 5}.$$

This discussion leads to the following definition of multiplication of fractions: If $\frac{a}{b}$ and $\frac{c}{d}$ are any two fractions then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Example 20.2
Compute the following products and express the answers in simplest form.
(a) $\frac{3}{4} \cdot \frac{28}{15}$ (b) $2\frac{1}{3} \cdot 7\frac{2}{5}$.

Solution.
(a) $\frac{3}{4} \cdot \frac{28}{15} = \frac{3 \cdot 28}{4 \cdot 15} = \frac{84}{60} = \frac{7 \cdot 12}{5 \cdot 12} = \frac{7}{5}$.
(b) $2\frac{1}{3} \cdot 7\frac{2}{5} = \frac{7}{3} \cdot \frac{37}{5} = \frac{259}{15} = 17\frac{4}{5}$.■

It is a good habit when finding the product of two fractions to simplify each fraction while carrying on the multiplication as shown in the following example.

Example 20.3
Compute and simplify: $\frac{50}{15} \cdot \frac{39}{55}$.

Solution.
We have

$$\frac{50}{15} \times \frac{39}{55} = \frac{\overset{10}{\cancel{50}}}{\underset{3}{\cancel{15}}} \times \frac{39}{55} = \frac{\overset{2}{\cancel{10}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{13}{\cancel{39}}}{\underset{11}{\cancel{55}}} = \frac{26}{11}$$

Properties of Multiplication
The properties of fraction multiplication are analogous to the properties of addition of fractions. These properties are summarized in the following theorem.

Theorem 20.1
Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be any fractions. Then we have the following:
Closure: The product of two fractions is a fraction.
Commutativity: $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$.
Associativity: $\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f}$.
Identity: $\frac{a}{b} \cdot 1 = \frac{a}{b} = 1 \cdot \frac{a}{b}$.
Inverse: $\frac{a}{b} \cdot \frac{b}{a} = 1$. We call $\frac{b}{a}$ the **reciprocal** of $\frac{a}{b}$ or the **multiplicative inverse** of $\frac{a}{b}$.
Distributivity: $\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$.

Example 20.4
Find the inverse of the following numbers.
(a) $\frac{3}{7}$ (b) 0 (c) $2\frac{3}{5}$.

Solution.
(a) The inverse of $\frac{3}{7}$ is $\frac{7}{3}$.
(b) 0 has no multiplicative inverse since $\frac{a}{0}$ is undefined.
(c) Since $2\frac{3}{5} = \frac{13}{5}$ then the multiplicative inverse of $2\frac{3}{5}$ is $\frac{5}{14}$.■

The multiplicative inverse property is useful for solving equations involving fractions.

Example 20.5
Find $x : \frac{3}{7}x = \frac{5}{8}$.

Solution.

$$\begin{array}{rcll}
\frac{3}{7}x & = & \frac{5}{8} & \\
\frac{7}{3} \cdot \left(\frac{3}{7}x\right) & = & \frac{7}{3} \cdot \frac{5}{8} & (Multiplication) \\
\left(\frac{7}{3} \cdot \frac{3}{7}\right) \cdot x & = & \frac{7}{3} \cdot \frac{5}{8} & (Associativity) \\
1 \cdot x & = & \frac{35}{24} & (Multiplicative\ inverse) \\
x & = & \frac{35}{24} & (Multiplicative\ identity) \blacksquare
\end{array}$$

In summary, we adopt the following steps to multiply fractions:

- Change any mixed numbers to common fractions.
- Cancel any factors common to both the numerator and denominator.
- Multiply the remaining terms in the numerator and in the denominator.

• Write the answer either as a fraction or as a mixed number.

Practice Problems

Problem 20.1
Use rectangular area models to illustrate the following multiplications.
(a) $2 \times \frac{3}{5}$ (b) $\frac{3}{2} \times \frac{3}{4}$ (c) $1\frac{2}{3} \times 2\frac{1}{4}$

Problem 20.2
A rectangular plot of land is $2\frac{1}{4}$ miles wide and $3\frac{1}{2}$ miles long. What is the area of the plot, in square miles? Draw a sketch that verifies your answer.

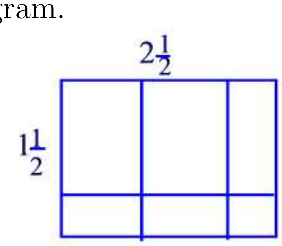
Problem 20.3
Show how to compute $4 \times \frac{1}{5}$ using repeated addition.

Problem 20.4
A child who does not know the multiplication rule for fractions wants to compute $\frac{1}{5} \times \frac{1}{3}$. Explain how to compute $\frac{1}{5} \times \frac{1}{3}$ using area models.

Problem 20.5
Show how to work $3\frac{1}{2} \times 4\frac{1}{8}$ with an area diagram.

Problem 20.6
Show how to work $2\frac{1}{6} \times 3\frac{1}{3}$ with an area diagram.

Problem 20.7
Consider the following diagram.



Find $2\frac{1}{2} \times 1\frac{1}{2}$ from the diagram.

Problem 20.8
Find mentally.
(a) $40 \times 6\frac{1}{8}$ (b) $\frac{1}{4} \times 8\frac{1}{2}$.

Problem 20.9
Use a rectangular region to illustrate each of the following products:
(a) $\frac{3}{4} \cdot \frac{1}{3}$ (b) $\frac{2}{5} \cdot \frac{1}{3}$.

Problem 20.10
Use the distributive property to find each product.
(a) $4\frac{1}{2} \times 2\frac{1}{3}$ (b) $3\frac{1}{3} \times 2\frac{1}{2}$

Problem 20.11
When you multiply a number by 3 and then subtract $\frac{7}{18}$, you get the same result as when you multiply the number by 2 and add $\frac{5}{12}$. What is the number?

Division of Fractions of Whole Numbers
Joe buys 4 lb of low-fat Swiss cheese. He and his wife eat a total of $\frac{1}{2}$ lb of low-fat Swiss cheese. For how many days will the cheese last? This problem requires dividing 4 by $\frac{1}{2}$. Division of fractions is the second topic of this section.
The expression $4 \div \frac{1}{2}$ means "How many $\frac{1}{2}s$ does it take to make 4?" It takes two $\frac{1}{2}s$ to make each whole. By Figure 20.4, it takes eight $\frac{1}{2}s$ to make 4. So $4 \div \frac{1}{2} = 8$.

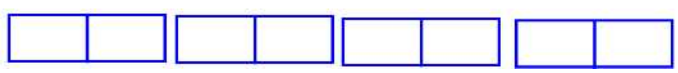
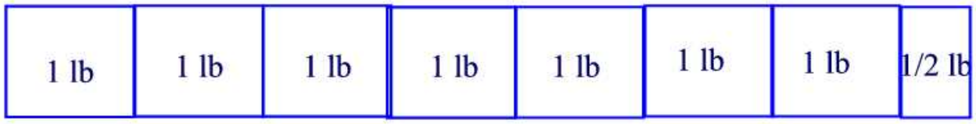


Figure 20.4

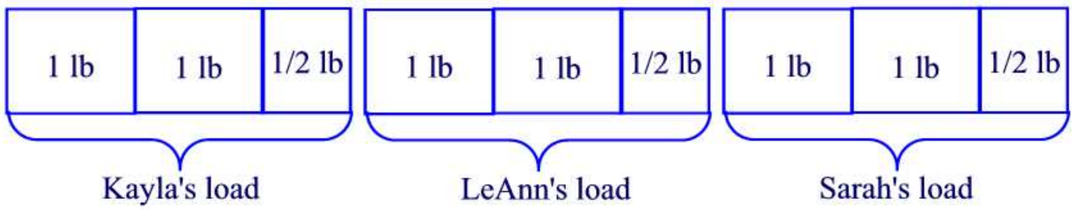
Notice that in the above problem the divident is a whole number and the divisor is a fraction. In the next example we consider a fraction whose dividend is a fraction and whose divisor is a whole number.

Example 20.6
Kayla, LeAnn, and Sarah are planning a weekend backpack trip and have a total of $7\frac{1}{2}$ pounds of freeze-dried food to carry. They want to share the load equally. How many pounds of food should each of them carry?

Solution.
The problem requires finding $7\frac{1}{2} \div 3$. First, we represent the $7\frac{1}{2}$ pounds of food with the following diagram.



Next, partition the food into three equal parts.



We see that each hiker should carry $2\frac{1}{2}$ pounds. That is $7\frac{1}{2} \div 3 = 2\frac{1}{2}$.■

The example below considers the case when both the dividend and the divisor are fractions.

Example 20.7
The new city park will have a $2\frac{1}{2}$acre grass playfield. Grass seed can be purchased in large bags, each sufficient to seed $\frac{3}{4}$ of an acre. How many bags are needed? Will there be some grass seed left over to keep on hand for reseeding worn spots in the field?

Solution.
The answer will be given by determining the number of $\frac{3}{4}$ acres in the $2\frac{1}{2}$acre field. That is, the problem requires finding $2\frac{1}{2} \div \frac{3}{4}$. In Figure 20.5, the field is partitioned into four regions. Three of these regions each cover $\frac{3}{4}$ of an acre, and therefore require a whole bag of seed. There is also a $\frac{1}{4}$ acre region that will use $\frac{1}{3}$ of a bag. Thus, a total of $3\frac{1}{3}$ bags of seed are needed for the playfield. Hence, $2\frac{1}{2} \div \frac{3}{4} = 3\frac{1}{3}$. We conclude that four bags of seeds should be ordered, which is enough for the initial seeding and leaves $\frac{2}{3}$ of a bag on hand for reseeding.■

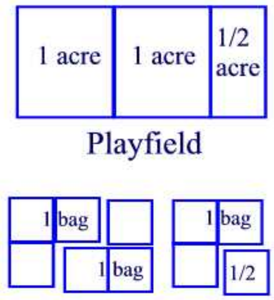


Figure 20.5

In the three examples above we notice the following:

$$\begin{array}{rcl} 4 \div \frac{1}{2} & = & \frac{4}{1} \times \frac{2}{1} = 8 \\ 7\frac{1}{2} \div 3 & = & \frac{15}{2} \times \frac{1}{3} = \frac{5}{2} = 2\frac{1}{2} \\ 2\frac{1}{2} \div \frac{3}{4} & = & \frac{5}{2} \times \frac{4}{3} = \frac{10}{3} = 3\frac{1}{3} \end{array}$$

This suggests the following definition for the division of two fractions:
Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions. Then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Using words, to find $\frac{a}{b} \div \frac{c}{d}$ multiply $\frac{a}{b}$ by the reciprocal of $\frac{c}{d}$.

Remark 20.1
After inverting, it is often simplest to "cancel" before doing the multiplication. Cancelling is dividing one factor of the numerator and one factor of the denominator by the same number.For example: $\frac{2}{9} \div \frac{3}{12} = \frac{2}{9} \times \frac{12}{3} = \frac{2\times 12}{9\times 3} = \frac{2\times 4}{3\times 3} = \frac{8}{9}$.

Estimation and Mental Math with Fractions
Estimation and mental math strategies were developed for whole numbers can also be used with fractions.

Example 20.8
Use mental math to find
(a) $(12 \cdot 25) \cdot \frac{1}{4}$ (b) $5\frac{1}{6} \cdot 12$ (c) $\frac{4}{5} \cdot 20$.

Solution.
(a) $(12 \cdot 25) \cdot \frac{1}{4} = 25 \cdot (12 \cdot \frac{1}{4}) = 25 \cdot 3 = 75$
(b) $5\frac{1}{6} \cdot 12 = (5 + \frac{1}{6}) \cdot 12 = 5 \cdot 12 + \frac{1}{6} \cdot 12 = 60 + 2 = 62$
(c) $\frac{4}{5} \cdot 20 = 4(\frac{1}{5} \cdot 20) = 4 \cdot 4 = 16.$■

Example 20.9
Estimate using the indicated technique.
(a) $5\frac{1}{8} \times 7\frac{5}{6}$ using range estimation
(b) $4\frac{3}{8} \times 9\frac{1}{16}$ using rounding.

Solution.
(a) $5\frac{1}{8} \times 7\frac{5}{6}$ is between $5 \times 7 = 35$ and $6 \times 8 = 48$.
(b) $4\frac{3}{8} \times 9\frac{1}{16} \approx 4\frac{1}{2} \times 9 = 36 + 4\frac{1}{2} = 40\frac{1}{2}.$■

We conclude this section by pointing out that division of fractions can be defined in term of the missing-factor model as follows:
Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions. Then $\frac{a}{b} \div \frac{c}{d} = \frac{e}{f}$ if and only if $\frac{a}{b} = \frac{e}{f} \times \frac{c}{d}$.

Example 20.10
Ted ismaking homemade root beer. The recipe he followed nearly fills a 5-gallon glass jug, and he estimates it contains $4\frac{3}{4}$ gallons of root beer. He is now ready to bottle his root beer. How many $\frac{1}{2}$-gallon bottles can he fill?

Solution.
Let x denote the number of half-gallon bottles required, where x will be allowed to be a fraction since we expect some bottle may be partially filled. We must then solve the equation

$$x \cdot \frac{1}{2} = 4\frac{3}{4}.$$

Using the missing-factor model we have

$$x = 4\frac{3}{4} \div \frac{1}{2} = \frac{19}{4} \div \frac{1}{2} = \frac{19}{4} \cdot 2 = \frac{19}{2} = 9\frac{1}{2}.$$

Ted will need 9 half-gallon bottles, and he will probably see if he can find a quart bottle to use.■

Practice Problems

Problem 20.12
Find the reciprocals of the following numbers.
(a) $\frac{3}{8}$ (b) $2\frac{1}{4}$ (c) 5

Problem 20.13
Compute these divisions, expressing your answer in simplest form.
(a) $\frac{2}{5} \div \frac{3}{4}$ (b) $2\frac{3}{8} \div 5$ (c) $2\frac{3}{8} \div 5\frac{1}{4}$.

Problem 20.14
Compute the fraction with simplest form that is equivalent to the given expression.
(a) $\left(\frac{3}{5} - \frac{3}{10}\right) \div \frac{6}{5}$ (b) $\left(\frac{2}{5} \div \frac{4}{15}\right) \cdot \frac{2}{3}$

Problem 20.15
A child has not learned yet the rule for dividing.
(a) Explain how to compute $2 \div \frac{1}{4}$ using a diagram.
(b) Explain how to compute $2 \div \frac{1}{4}$ using multiplication.

Problem 20.16
Solve the equation: $\frac{1}{2} \div x = 4$.

Problem 20.17
Compute and simplify.
(a) $\frac{5}{8} \div \frac{1}{2}$ (b) $\frac{x}{5} \div \frac{x}{7}$ (c) $10 \div \frac{4}{3}$

Problem 20.18
Solve mentally.
(a) $\frac{1}{2} \cdot x = 10$ (b) $4 \div \frac{1}{2} = y$

Problem 20.19
Last year a farm produced 1360 oranges. This year they produced $2\frac{1}{2}$ times as many oranges. How many oranges did they produce?

Problem 20.20
A wall is $82\frac{1}{2}$ inches high. It is covered with $5\frac{1}{2}$inch square tiles. How many tiles are in a vertical row from the floor to the ceiling?

Problem 20.21
A baker takes $\frac{1}{2}$ hour to decorate a cake. How many cakes can she decorate in H hours?

Problem 20.22
Four fifths of a class bring in food for a charity drive. Of those who brought food, one fourth brought canned soup. What fraction of the whole class brought canned soup?

Problem 20.23
Compute the following mentally. Find the exact answers.
(a) $3 \div \frac{1}{2}$ (b) $3\frac{1}{2} \div \frac{1}{2}$ (c) $3\frac{1}{4} \cdot 8$ (d) $9\frac{1}{5} \cdot 10$

Problem 20.24
Estimate the following.
(a) $5\frac{4}{5} \cdot 3\frac{1}{10}$ (b) $4\frac{1}{10} \cdot 5\frac{1}{8}$ (c) $20\frac{8}{9} \div 3\frac{1}{12}$

Problem 20.25
Five eighths of the students at Salem State College live in dormitories. If 6000 students at the college live in dormitories, how many students are there in the college?

Problem 20.26
Estimate using compatible numbers.
(a) $29\frac{1}{3} \times 4\frac{2}{3}$ (b) $57\frac{1}{5} \div 7\frac{4}{5}$.

21 Decimals

In most everyday applications, one encounters numbers written in decimal notation such as the price of a comodity, the Gross National Product, the diameter of an atom, etc. In this section, we introduce the concept of decimal numbers.
The word *decimal* comes from the Latin *decem*, meaning "ten". Thus, decimal numbers can be expressed as powers of ten. To be more precise, a decimal number such as 128.294 can be written in **expanded form** as

$$128.294 = 1 \times 10^2 + 2 \times 10^1 + 8 \times 10^0 + 2 \times \frac{1}{10} + 9 \times \frac{1}{10^2} + 4 \times \frac{1}{10^3}.$$

The dot in 128.294 is called the **decimal point**.
Decimal place value is an extension of whole number place value, and it has a symmetry as shown in the table of Figure 21.1

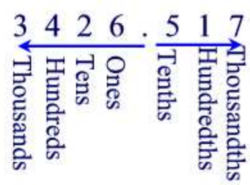


Figure 21.1

Remark 21.1
Note that every whole number is also a decimal number. For example, $15 = 15.0$.

Pictorial Representations of Decimals
For beginning grade school students, it is helpful to introduce decimals using pictorial representations. We can use base ten blocks and decide that 1 flat represents a unit, 1 long represents $\frac{1}{10}$ and 1 cube represents $\frac{1}{100}$ as shown in Figure 21.2.

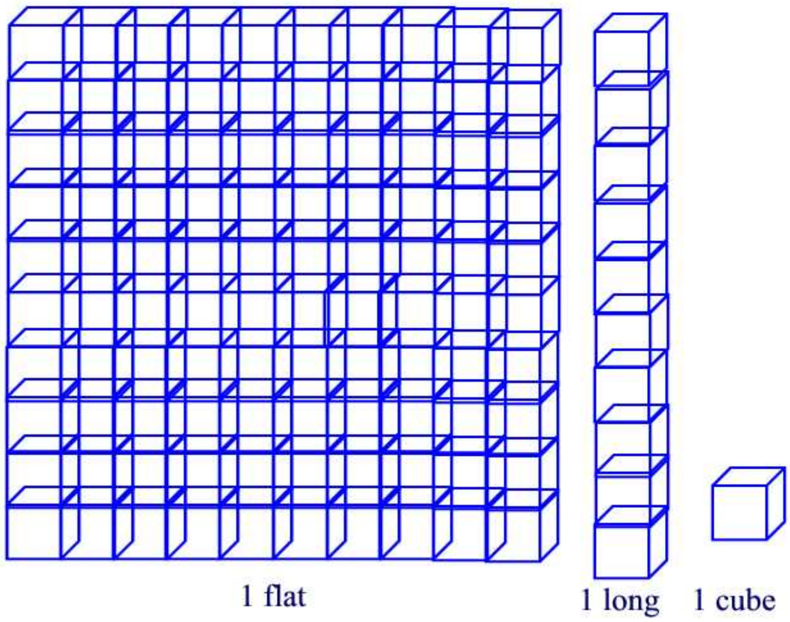


Figure 21.2

In this model, the number 1.23 is represented in Figure 21.3.

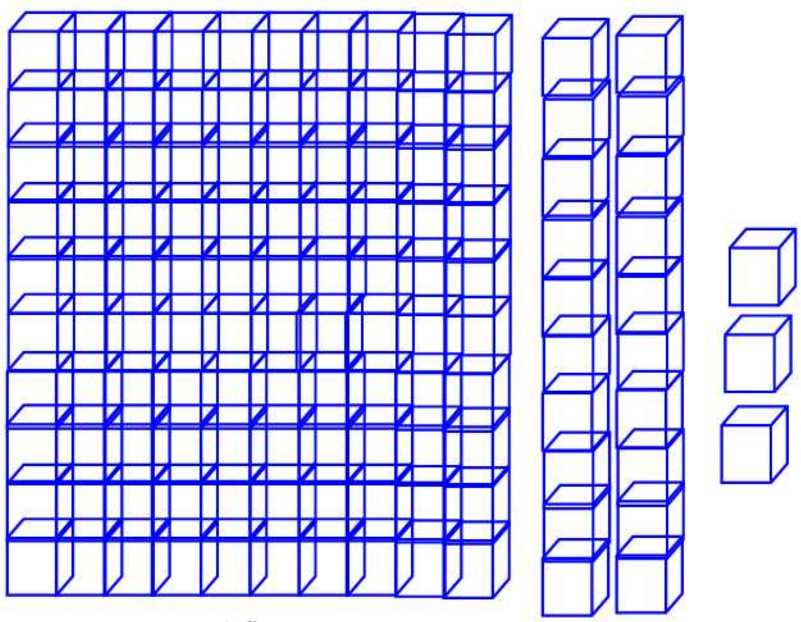


Figure 21.3

To represent a decimal such as 2.235, we can think of the block shown in Figure 21.4(a) as a unit. Then a flat represents $\frac{1}{10}$, a long represents $\frac{1}{100}$, and a cube represents $\frac{1}{1000}$. Using these objects, we show a representation of 2.235 in Figure 21.4(b).

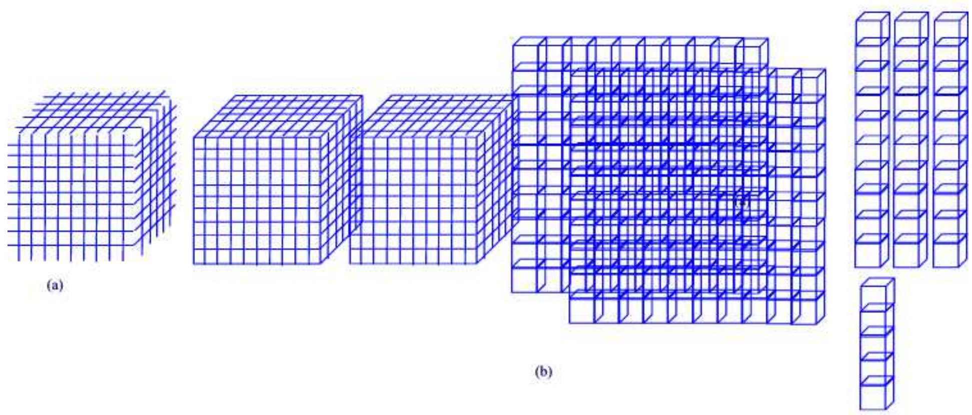


Figure 21.4

The number 2.235 is read "two and two hundred thirty five thousandths".

Example 21.1
Write the number 30.0012 in expanded form.

Solution.
The expanded form is

$$30.0012 = 3 \times 10^1 + 0 \times 10^0 + 0 \times \frac{1}{10} + 0 \times \frac{1}{10^2} + 1 \times \frac{1}{10^3} + 2 \times \frac{1}{10^4} \blacksquare$$

Multiplying and Dividing Decimals by Powers of Ten
Let's see the effect of multiplying a decimal number by a power of ten. Consider the number 25.723. writing this number in expanded form we find

$$25.723 = 2 \times 10 + 5 + 7 \times \frac{1}{10} + 2 \times \frac{1}{100} + 3 \times \frac{1}{1000}.$$

If we multiply this number by $100 = 10^2$ then using the distributive property we obtain

$$\begin{array}{rcc} (100) \times (25.723) & = & (100) \times (2 \times 10 + 5 + 7 \times \frac{1}{10} + 2 \times \frac{1}{100} + 3 \times \frac{1}{1000}) \\ & = & 100 \times 2 \times 10 + 100 \times 5 + 100 \times 7 \times \frac{1}{10} + 100 \times 2 \times \frac{1}{100} \\ & + & 100 \times 3 \times \frac{1}{1000} \\ & = & 2000 + 500 + 70 + 2 + 3 \times \frac{1}{10} = 2752.3 \end{array}$$

and the notational effect is to move the decimal point two places to the right. Note that 2 is the exponent of the power of 10 we are multiplying by and also the number of zeros in 100.
Now, let's try and divide 25.723 by 100. In this case, we have the following computation.

$$\begin{array}{rcc} 25.723 \div 100 & = & (2 \times 10 + 5 + 7 \times \frac{1}{10} + 2 \times \frac{1}{100} + 3 \times \frac{1}{1000}) \div 100 \\ & = & \frac{1}{100} \times 2 \times 10 + \frac{1}{100} \times 5 + \frac{1}{100} \times 7 \times \frac{1}{10} + \frac{1}{100} \times 2 \times \frac{1}{100} \\ & + & \frac{1}{100} \times 3 \times \frac{1}{1000} \\ & = & 2 \times \frac{1}{10} + 5 \times \frac{1}{100} + 7 \times \frac{1}{1000} + 2 \times \frac{1}{10000} + 3 \times \frac{1}{100000} \\ & = & 0.25723 \end{array}$$

so that the effect is to move the decimal point two places to the left. Summarizing we have the following rules:
(1) Multiplying a decimal number by 10^n, where $n \in W$, is the same as moving the decimal point n places to the right.
(2) Dividing a decimal number by 10^n, where $n \in W$, is the same as moving the decimal point n places to the left.

Example 21.2
Compute each of the following:
(a) $(10^3) \cdot (253.26)$ (b) $(253.26) \div 10^3$ (c) $(1000) \cdot (34.764)$ (d) $34.764 \div 1000$

Solution.
(a) $(10^3) \cdot (253.26) = 253,260.$

(b) $(253.26) \div 10^3 = 0.25326$
(c) $(1000) \cdot (34.764) = 34,764$
(d) $34.764 \div 1000 = 0.034764$■

Representing a Terminating Decimal as a Fraction
A number such as $0.33\ldots$ where the ellipsis dots indicate that the string of 3s continues without end is called a **nonterminating decimal.** On the contrary, a decimal like 24.357, which has finitely many digits to the right of the decimal point, is an example of a **terminating decimal.** Such a number can be written as a fraction. To see this, write first the expanded form

$$24.357 = 2 \times 10 + 4 + \frac{3}{10} + \frac{5}{100} + \frac{7}{1000}.$$

Now, find the common denominator and obtain

$$24.357 = \frac{24000}{1000} + \frac{300}{1000} + \frac{50}{1000} + \frac{7}{1000} = \frac{24,357}{1000}.$$

Note that the number of zeros at the bottom is just the number of digits to the right of the decimal point.
Now, what about converting a fraction into a terminating decimal number. Not all fractions have terminating decimal expansion. For example, $\frac{1}{3} = 0.33\cdots$. However, a fraction $\frac{a}{b}$, where the prime factorizations of b consists only of powers of 2 and 5, has a terminating decimal expansion. We illustrate this in the following example.

Example 21.3
Express $\frac{43}{1250}$ as a decimal number.

Solution.
$\frac{43}{1250} = \frac{43}{2 \cdot 5^4} = \frac{43 \cdot 2^3}{2^4 \cdot 5^4} = \frac{344}{10,000} = 0.0344.$■

Practice Problems

Problem 21.1
Write the following decimals in expanded form.
(a) 273.412 (b) 0.000723 (c) 0.020305

Problem 21.2
Write the following decimals as fractions in simplified form and determine the prime factorization of the denominator in each case.
(a) 0.324 (b) 0.028 (c) 4.25

Problem 21.3
Write these fractions as terminating decimals.
(a) $\frac{7}{20}$ (b) $\frac{18}{2^2 \cdot 5^4}$

Problem 21.4
If you move the decimal point in a number two places to the left, the value of the number is divided by ____ or multiplied by ____.

Problem 21.5
Write each of the following as a decimal number.
(a) Forty-one and sixteen hundredths
(b) Seven and five thousandths

Problem 21.6
You ask a fourth grader to add 4.21 + 18. The child asks,"Where is the decimal point in 18?" How would you respond?

Problem 21.7
A sign in a store mistakenly says that apples are sold for .89 cents a pound. What should the sign say?

Problem 21.8
Write each of the following numbers in expanded form.
(a) 0.023 (b) 206.06 (c) 0.000132 (d) 312.0103

Problem 21.9
Rewrite the following sums as decimals.
(a) $4 \times 10^3 + 3 \times 10^2 + 5 \times 10 + 6 + \frac{7}{10} + \frac{8}{10^2}$
(b) $4 \times 10^3 + \frac{6}{10} + \frac{8}{10^2}$
(c) $4 \times 10^4 + \frac{3}{10^2}$
(d) $\frac{2}{10} + \frac{4}{10^4} + \frac{7}{10^7}$

Problem 21.10
Write each of the following as numerals.
(a) Five hundred thirty six and seventy-six ten thousandths
(b) Three and eight thousandths
(c) Four hundred thirty-six millionths
(d) Five million and two tenths

Problem 21.11
Write each of the following terminating decimals in common fractions.
(a) 0.436 (b) 25.16 (c) 28.1902

Problem 21.12
Determine which of the following represent terminating decimals.
(a) $\frac{61}{2^2 \cdot 5}$ (b) $\frac{133}{625}$ (c) $\frac{26}{65}$

Problem 21.13
Explain how to use base-ten blocks to represent "two and three hundred fourty-five thousandths"

Problem 21.14
Write the following numbers in words.
(a) 0.013 (b) 68,485.532 (c) 0.0082 (d) 859.080509

Problem 21.15
Determine, without converting to decimals, which of the following fractions has a terminating decimal representation.
(a) $\frac{21}{45}$ (b) $\frac{326}{400}$ (c) $\frac{62}{125}$ (d) $\frac{54}{130}$

Problem 21.16
A student reads the number 3147 as "three thousand one hundred and forty-seven" What's wrong with this reading?

Problem 21.17
It is possible to write a decimal number in the form $M \times 10^n$ where $1 \leq M < 10$ and $n \in \{0, 1, 2, \cdots\}$. This is known as the **scientific notation**. Such a notation is useful in expressing large numbers. For example, $760,000,000,000 = 7.6 \times 10^9$. Write each of the following in scientific notation.
(a) 4326 (b) 1,000,000 (c) 64,020,000 (d) 71,000,000,000

Problem 21.18
Find each of the following products and quotients.
(a) $(6.75)(1,000,000)$ (b) $19.514 \div 100,000$ (c) $(2.96 \times 10^{16})(10^{12})$ (d) $\frac{2.96 \times 10^{16}}{10^{12}}$

Ordering Terminating Decimals
We use two methods for comparing two decimal numbers.

• **Fraction Method**
Two decimals can be ordered by converting each to fractions in the form $\frac{a}{b}$, where a and b are whole numbers, and determine which is greater. We illustrate this method in the next example.

Example 21.4
Compare the numbers 0.9 and 0.36.

Solution.
Writing each number as a fraction we find $0.9 = \frac{9}{10} frac90100$ and $0.36 = \frac{36}{100}$. Thus,

$$0.9 = \frac{90}{100} > \frac{36}{100} = 0.36\blacksquare$$

•**Place Value Method**
Ordering decimals with this method is much like ordering whole numbers. For example, to determine the larger of 247,761 and 2,326,447 write both numerals as if they had the same number of digits (by adding zeros when necessary); that is, write

$$0,247,761 \quad and \quad 2,326,447.$$

Next, start at the left and find the first place value where the face values are different and compare these two digits. The number containing the greater face value in this place is the greater of the two original numbers. In our example, the first place value from the left where the face values are different is in the "million" position. Since $0 < 2$ then $247,761 < 2,326,447$. The same process applies when comparing decimal numbers as shown in the next example.

Example 21.5
Compare 2.35714 and 2.35709

Solution.
The first digits from the left that differ are 1 and 0. Since $0 < 1$ then

$2.35709 < 2.35714$ ■

Practice Problems

Problem 21.19
Order the following decimals from greatest to lowest: 13.4919, 13.492, 13.49183, 13.49199.

Problem 21.20
If the numbers 0.804, 0.84 and 0.8399 are arranged on a number line, which is furthest to the right?

Problem 21.21
Which of the following numbers is the greatest: $100,000^3, 1000^5, 100,000^2$?

Problem 21.22
The five top swimmers in an event had the following times.

Emily	64.54 seconds
Molly	64.46 seconds
Martha	63.59 seconds
Kathy	64.02 seconds
Rhonda	63.54 seconds

List them in the order they placed.

Problem 21.23
Write the following numbers from smallest to largest: 25.412, 25.312, 24.999, 25.412412412...

Problem 21.24
Order the following from smallest to largest by changing each fraction to a decimal:$\frac{3}{5}, \frac{11}{18}, \frac{17}{29}$.

Mental Computation and Estimation
Some of the tools used for mental computations with whole numbers can be used to perform mental computations with decimals, as seen in the following:
- Using Compatible Numbers

$$7.91 + 3.85 + 4.09 + 0.15 = (7.91 + 4.09) + (3.85 + 0.15) = 12 + 4 = 16.$$

•**Using Properties**

$$17 \times 0.25 + 23 \times 0.25 = (17 + 23) \times 0.25 = 40 \times 0.25 = 10$$

where we used the dustributive property of multiplication over addition.
•**Compensation**

$$3.76 + 1.98 = 3.74 + 2 = 5.74$$

using additive compensation.

Computational estimations of operations on whole numbers and fractions can also be applied to estimate the results of decimal operations.

Example 21.6
Estimate $1.57 + 4.36 + 8.78$ using (i) range estimation, (ii) front-end adjustment, and (iii) rounding.

Solution.
Range: A low estimate is $1+4+8 = 13$ and a high estimate is $2+5+9 = 16$.
Front-end: Since $0.57 + 0.36 + 0.78 \approx 1.50$ then $1.57 + 4.36 + 8.78 \approx 13 + 1.50 = 14.50$.
Rounding: Rounding to the nearest whole number we obtain $1.57 + 4.36 + 8.78 \approx 2 + 4 + 9 = 15$.■

Practice Problems

Problem 21.25
Round 0.3678
(a) up to the next hundredth
(b) down to the preceding hundredth
(c) to the nearest hundredth.

Problem 21.26
Suppose that labels are sold in packs of 100.
(a) If you need 640 labels, how many labels would you have to buy?
(b) Does this application require rounding up, down, or to the "nearest"?

Problem 21.27
Mount Everest has an altitude of 8847.6 m and Mount Api has an altitude of 7132.1 m. How much higher is Mount Everest than Mount Api?
(a) Estimate using rounding.
(b) Estimate using the front-end strategy.

Problem 21.28
A 46-oz can of apple juice costs \$1.29. How can you estimate the cost per ounce?

Problem 21.29
Determine by estimating which of the following answers could not be correct.
(a) $2.13 - 0.625 = 1.505$
(b) $374 \times 1.1 = 41.14$
(c) $43.74 \div 2.2 = 19.88181818$.

Problem 21.30
Calculate mentally. Describe your method.
(a) $18.43 - 9.96$
(b) $1.3 \times 5.9 + 1.3 \times 64.1$
(c) $4.6 + (5.8 + 2.4)$
(d) $51.24 \div 10^3$
(e) 0.15×10^5

Problem 21.31
Estimate using the indicated techniques.
(a) $4.75 + 5.91 + 7.36$ using range and rounding to the nearest whole number.
(b) 74.5×6.1; range and rounding.
(c) $3.18 + 4.39 + 2.73$ front-end with adjustment.
(d) 4.3×9.7 rounding to the nearest whole number.

Problem 21.32
Round the following.
(a) 97.26 to the nearest tenth
(b) 345.51 to the nearest ten
(c) 345.00 to the nearest ten
(d) 0.01826 to the nearest thousandth
(e) 0.498 to the nearest tenth

22 Arithmetic Operations on Decimals

In this section we consider decimal arithmetic: addition, subtraction, multiplication, and division.

Addition

John bought two packages of cheese weighing 0.36 lb and 0.41 lb. What is the total weight of cheese that he bought? This problem requires finding the sum $0.36 + 0.41$ and calls for addition of decimal numbers.

One can use rectangular area model of addition of decimals just as was done to promote understanding of addition of whole nonzero whole numbers and fractions. The Figure 22.1 illustrates the $0.23 + 0.08$.

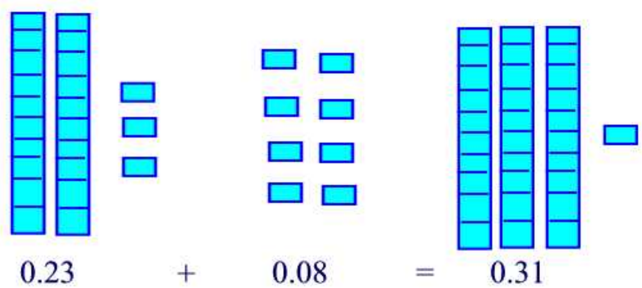


Figure 22.1

To add decimal numbers we are going to consider two approaches:

• **Fraction Approach**

To do some repair work at the Louvre in Paris, a plumber estimated that he needed 17.5 meters of copper tubing. After a plan change, he decided that he needed an additional 15.75 meters of tubing. How much tubing did the plumber purchase?

The total number of tubing purchased is the sum $17.5 + 15.75$. To find its value we convert each decimal number to a fraction and add the two resulting fractions as shown next.

$$\begin{array}{rcl} 17.5 + 15.75 & = & \frac{175}{10} + \frac{1575}{100} = \frac{1750}{100} + \frac{1575}{100} \\ & = & \frac{3325}{100} = 33.25 \; meters. \end{array}$$

Example 22.1

Compute: (a) $23.47 + 7.81$ (b) $351.42 + 417.815$

Solution.
(a) Converting to fractions and then adding to obtain

$$\begin{aligned} 23.47 + 7.81 &= \tfrac{2347}{100} + \tfrac{781}{100} \\ &= \tfrac{3128}{100} = 31.28 \end{aligned}$$

(b)

$$\begin{aligned} 351.42 + 417.815 &= \tfrac{35142}{100} + \tfrac{417815}{1000} \\ &= \tfrac{351420}{1000} + \tfrac{417815}{1000} \\ &= \tfrac{769,235}{1000} = 769.235 \blacksquare \end{aligned}$$

• **By Hand Approach**
To add decimals by hand, write the numbers in vertical style lining up the decimal points and then add essentially just as we add whole numbers. We illustrate this in next example.

Example 22.2
Compute, by hand, the sum $0.00094 + 80.183$.

Solution.

$$\begin{array}{r} 0.00094 \\ +\ 80.183 \\ \hline 80.18394 \end{array} \quad \blacksquare$$

Subtraction of Decimal Numbers
Pete find a CD for \$16.42 at one store and for \$16.98 at another store. He wants to know the difference in price. This situation calls for subtraction of decimal numbers.
The techniques used for addition works as well for subtraction.

Example 22.3
Compute: $16.98 - 16.42$

Solution.
Fraction Approach:

$$\begin{aligned} 16.98 - 16.42 &= \tfrac{1698}{100} - \tfrac{1642}{100} \\ &= \tfrac{56}{100} = 0.56 \end{aligned}$$

By Hand Approach:

$$\begin{array}{r} 16.98 \\ 16.42 \\ \hline 0.56 \end{array} \quad \blacksquare$$

Practice Problems

Problem 22.1
Perform the following by hand.
(a) $32.174 + 371.5$
(b) $0.057 + 1.08$
(c) $371.5 - 32.174$
(d) $1.08 - 0.057$

Problem 22.2
Use rectangular area model to represent the sum $0.18 + 0.24$

Problem 22.3
Use a rectangular area model to illustrate the difference $0.4 - 0.3$

Problem 22.4
A stock's price dropped from 63.28 per share to 27.45. What was the loss on a single share of the stock?

Problem 22.5
Make the sum of every row, column, and diagonal the same.

8.2		
3.7	5.5	
	9.1	2.8

Problem 22.6
Find the next three decimal numbers in each of the following arithmetic sequences.
(a) 0.9, 1.8, 2.7, 3.6, 4.5
(b) 0.3, 0.5, 0.7, 0.9, 1.1
(c) 0.2, 1.5, 2.8, 4.1, 5.4

Problem 22.7
Perform the following operations by hand.
(a) $38.52 + 9.251$
(b) $534.51 - 48.67$

Problem 22.8
Change the decimals in the previous exercise to fractions, perform the computations, and express the answers as decimals.

Multiplication of Decimals
Tom Swift wanted to try out his new Ferrari on a straight stretch of highway. If he drove at 91.7 miles per hour for 15 minutes, how far did he go? Since 15 minutes equals 0.25 hours and distance traveled equals rate times elapsed time, Tom traveled (91.7).(0.25) miles. For a more explicit answer, we need to be able to multiply decimals.
One can use rectangular area model of multiplication of decimals just as was done to promote understanding of multiplication of whole nonzero whole numbers and fractions. The Figure 22.2 illustrates the product 2.3×3.2.

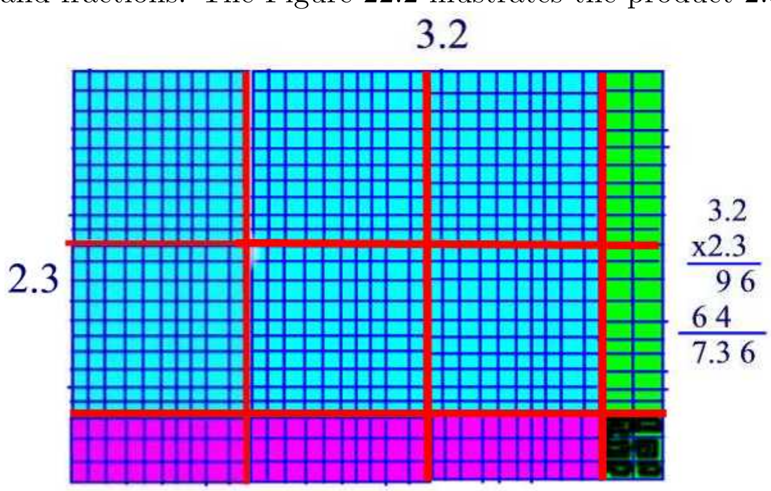


Figure 22.2

As we addition and subtraction we use the same two appraoches as above.

Example 22.4
Compute $(91.7) \cdot (0.25)$.

Solution.
Fraction Approach:

$$\begin{aligned}(91.7)\cdot(0.25) &= \frac{917}{10}\cdot\frac{25}{100}\\ &= \frac{22{,}925}{1000} = 22.925\end{aligned}$$

By Hand:

$$\begin{array}{r} 91.7 \\ \times\ 0.25 \\ \hline 4585 \\ 1834 \\ \hline 22.925 \end{array} \blacksquare$$

The algorithm of multiplying decimals by hand can be stated as follows: If there are m digits to the right of the decimal point in one number and n digits after the decimal point of a second number, multiply the two numbers, ignoring the decimals, and then place the decimal point in the product so that there are $m+n$ digits after the decimal point. Note that with decimal multiplication we do not need to align the decimal points as we do with addition and subtraction since multiplication of fractions does not require common denominator.

Division of Decimals
Suppose you fly your own plane. If you traveled 537.6 miles in 2.56 hours, how fast did you travel? Again, since distance traveled equals rate times elapsed time, then rate equals distance traveled divided by time. Therefore, we need to compute $537.6 \div 2.56$. This requires division of decimals.

Example 22.5
Show that $5.376 \div 2.56 = 53.76 \div 25.6 = 537.6 \div 256 = 5376 \div 2560$.

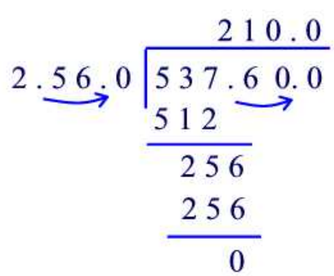
Solution.

$$\begin{aligned}
5.376 \div 2.56 &= \left(\tfrac{53.76}{10}\right) \div \left(\tfrac{25.6}{10}\right) \\
&= \tfrac{53.76}{10} \cdot \tfrac{10}{25.6} \\
&= 53.76 \div 25.6 = \left(\tfrac{537.6}{10}\right) \div \left(\tfrac{256}{10}\right) \\
&= \tfrac{537.6}{10} \cdot \tfrac{10}{256} \\
&= 537.6 \div 256 \\
&= \left(\tfrac{5376}{10}\right) \div \left(\tfrac{2560}{10}\right) \\
&= \tfrac{5376}{10} \cdot \tfrac{10}{2560} \\
&= 5376 \div 2560 \blacksquare
\end{aligned}$$

It follows that one can move the decimal points of the divident and the divisor the same number of digits to the right without affecting the original division. This leads to the following algorithm: Move the decimal points as much as necessary so that the divisor becomes a whole number. In this case, the division can be handled as with whole numbers division with the decimal point placed directly over the decimal point of the dividend. We illustrate this algorithm in the next example.

Example 22.6
Compute $537.6 \div 2.56$.

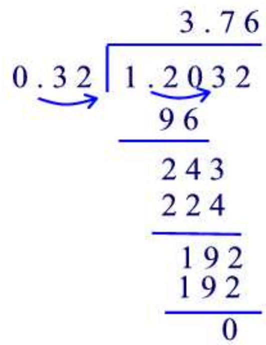
Solution.



$\blacksquare$

Example 22.7
Compute $0.32 \div 1.2032$.

Solution.



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Practice Problems

Problem 22.9
Perform the following multiplications and divisions by hand.
(a) $(37.1) \cdot (4.7)$
(b) $(3.71) \cdot (0.47)$
(c) $138.33 \div 5.3$
(d) $1.3833 \div 0.53$

Problem 22.10
Kristina bought pairs of gloves as Christmas presents for three of her best friends. If the gloves cost \$9.72 a pair, how much did she spend for these presents?

Problem 22.11
Yolanda also bought identical pairs of gloves for each of her four best friends. If her total bill was \$44.92, how much did each pair of gloves cost?

Problem 22.12
Show how to compute 2×0.18 using a rectangular area model.

Problem 22.13
The product 34.56×6.2 has the digits 214272. Explain how to place the decimal point by counting decimal places.

Problem 22.14
A runner burns about 0.12 calorie per minute per kilogram of body mass. How many calories does a 60-kg runner burn in a 10-minute run?

Problem 22.15
A fifth grader says 50×4.44 is the same as 0.50×4.44 which is 222. Is this right?

Problem 22.16
A fifth grader says $0.2 \times 0.3 = 0.6$
(a) Why do you think the child did the problem this way?
(b) What would you tell the child?

Problem 22.17
Show how to work out $0.6 \div 3$ with rectangular area model.

Problem 22.18
What do you multiply both numbers with to change $6.4 \div 0.32$ to $640 \div 32.$?

Problem 22.19
Which of the following are equal?
(a) $8 \div 0.23$ (b) $800 \div 0.0023$ (c) $80 \div 2.3$ (d) $0.8 \div 0.023$ (e) $80 \div 0.023$

Problem 22.20
A sixth grader divides 16 by 3 and gets 5.1
(a) How did the child obtain this answer?
(b) What concept doesn't the child understand?

Problem 22.21
Find the next three decimal numbers in the following geometric sequence: 1, 0.5, 0.25, 0.125

Problem 22.22
Perform the following operations using the algorithms of this section.
(a) 5.23×0.034
(b) $8.272 \div 1.76$

Problem 22.23
Mentally determine which of the following division problems have the same quotient.
(a) $1680 \div 56$ (b) $0.168 \div 0.056$ (c) $0.168 \div 0.56$

Problem 22.24
Perform the following calculations.
(a) $2.16 \times \frac{1}{3}$ (b) $2\frac{1}{5} \times 1.55$ (c) $16.4 \div \frac{4}{9}$.

Problem 22.25
We have seen that if the prime factorization of the numerator and the denominator of a fraction contains only 2s and 5s then the decimal representation is a terminating one. For example, $\frac{2}{5} = 0.4$. On the other hand, if the prime factorization have prime factors other than 2 and 5 then the decimal representation is nonterminating and repeating one. For example, $\frac{1}{3} = 0.\overline{3}$.
Write each of the following using a bar over the repetend.
(a) $0.7777\cdots$ (b) $0.47121212\cdots$ (c) 0.35 (d) $0.45315961596\cdots$

Problem 22.26
Write out the first 12 decimal places of each of the following.
(a) $0.\overline{3174}$ (b) $0.3\overline{174}$ (c) $0.31\overline{74}$

Problem 22.27
If a decimal number is nonterminating and repeating then one can rewrite it as a fraction. To see this, let $x = 0.\overline{34}$. Then $100x = 34 + 0.\overline{34}$. That is, $100x = 34 + x$ or $99x = 34$. Hence, $x = \frac{34}{99}$.
Use the above approach to express each of the following as a fraction in simplest form.
(a) $0.\overline{16}$ (b) $0.\overline{387}$ (c) $0.7\overline{25}$

23 Ratios and Proportions

Ratios

Ratios and proportions are a very important part of the middle-grades curriculum. Ratios are encountered in everyday life. For example, there may be a 2-to-3 ratio of Democrats to Republicans on a certain legislative committee, of the ratio of female students to male students is 3-to-5, etc.

A **ratio** compares two numbers using division. Some ratios gives **part-to–part** comparison. For example, the ratio of the number of students to one teacher. Ratios can also represent a **part-to–whole** or a **whole-to-part** comparisons. For example, if the ratio of boys to girls is 1-to-3 then the ratio of boys (part) to children (whole) is 1-to-4. We could also say that the ratio of children (whole) to boys (part) is 4-to-1.

Ratios are written either as a fraction $\frac{a}{b}$ or with the notation $a : b$ and are usually used to compare quantities. Hence, the order of a and b matters! A ratio like $\frac{8}{12}$ is read "the ratio is eight-to-twelve"

Example 23.1

Jeannine has a bag with 3 videocassettes, 4 marbles, 7 books, and 1 orange.

(a) What is the ratio of books to marbles?

(b) What is the ratio of videocassettes to the total number of items in the bag?

Solution.

(a) Expressed as a fraction, with the numerator equal to the first quantity and the denominator equal to the second, the answer would be 7/4. Two other ways of writing the ratio are 7 to 4, and 7:4.

(b) There are 3 videocassettes, and 3 + 4 + 7 + 1 = 15 items total. The answer can be expressed as 3/15, 3 to 15, or 3:15.■

Comparing Ratios

Ratios allow us to make clear comparisons when actual numbers sometimes make them more obscure. For example, at a basketball practice, Caralee made 27 of 45 free throws attempted and Sonja made 24 of 40 attempts. Which player appears to be the better foul shot shooter? For Carlee, saying that the ratio of shots made to shots tried is 27:45 amounts to saying that she made $\frac{3}{5}$ of her shots sine $\frac{27}{45} = \frac{3\cdot 9}{5\cdot 9} = \frac{3}{5}$.

Similarly, for sonja the ratio of shots made to ratio attempted is $\frac{24}{40} = \frac{3\cdot 8}{5\cdot 8} = \frac{3}{5}$.

This suggests that the two girls are equally capable at shooting foul shots. From the above discussion, we notice that when comparing ratios, write them as fractions. The ratios are equal if they are equal when written as fractions. But by Theorem 18.1, $\frac{a}{b} = \frac{c}{d}$ if and only if $a \times d = b \times c$.

Example 23.2
(a) Are the ratios 3 to 4 and 6:8 equal?
(b) Are the ratios 7:1 and 4:81 equal?

Solution.
(a) The ratios are equal if $\frac{3}{4} = \frac{6}{8}$. These are equal if their cross products are equal; that is, if $3 \times 8 = 4 \times 6$. Since both of these products equal 24, the answer is yes, the ratios are equal.
(b) Since $48 \times 7 \neq 1 \times 1$ then the two ratios are not equal.■

Proportions
A **proportion** is an equality between two ratios. The equality $3/4 = 6/8$ is an example of a proportion.
When one of the four numbers in a proportion is unknown, cross products may be used to find the unknown number. This is called **solving the proportion.** Question marks or letters are frequently used in place of the unknown number.
Remember that in a proportion the product of the **means** is equal to the product of the **extremes** as shown in Figure 23.1.

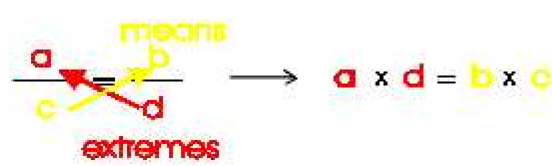


Figure 23.1

Example 23.3
Solve for $x : \frac{28}{49} = \frac{x}{21}$.

Solution.

$$\frac{28}{49} = \frac{x}{21} \Leftrightarrow 49x = (28)(21) \Leftrightarrow x = \frac{(28)(21)}{49} = 12. \blacksquare$$

Practice Problems

Problem 23.1
If two full time employees accomplish 20 tasks in a week, how many such tasks will 5 employees accomplish in a week?

Problem 23.2
A pipe transfers 236 gallons of fuel to the tank of a ship in 2 hours. How long will it take to fill the tank of the ship that holds 4543 gallons?

Problem 23.3
An I-beam 12 feet long weighs 52 pounds. How much does an I-beam of the same width weigh if it is 18 feet long?

Problem 23.4
Find the value of x.
(a) $\frac{16}{8} = \frac{x}{5}$ (b) $\frac{25}{15} = \frac{10}{x}$

Problem 23.5
A home has 2400 square feet of living space. The home also has 400 square feet of glassed window area. What is the ratio of glassed area to total square footage?

Problem 23.6
A model home you are looking at has a total square footage of 3,000 feet. It is stated that the ratio of glassed area to total square footage to be 1:10. How much glassed area is there?

Problem 23.7
Stan worked 5 hours on Monday for \$25. He worked 7 hours on Tuesday. Find his wage for that day.

Problem 23.8
Find these ratios. Write each ratio in the two different formats
(a) a 25 year-old man to his 45 year-old father
(b) a 1200 square foot house to a 4000 square foot house

Problem 23.9
If the ratio of saturated to unsaturated fatty acids in a cell membrane is 9 to 1, and there are a total of 85 billion fatty acid molecules, how many of them are saturated?

Problem 23.10
The lava output from the volcano in crater park has quadrupled over the past 30 days. If the lava output 30 days ago was 4 tons of rock per week, what is the output now?

Problem 23.11
The ratio of chocolate chips to raisins in one cookie is 5:4. If the recipe required 96 raisins, how many chocolate chips were used?

Problem 23.12
If the ratio of y to x is equal to 3 and the sum of y and x is 80, what is the value of y?

Problem 23.13
At a summer camp, there are 56 boys and 72 girls. Find the ratio of (a) boys to the total number of campers. (b) girls to boys.

Problem 23.14
The ratio of orange juice concentrate to water in a jug is 1:3. If there are 5 cups of concentrate in the jug, how much water was added?

Problem 23.15
(a) Tom works at Wegmans. He earns $27.30 for working 6.5 hours. How much will he earn working 20 hours?
(b) What is Toms hourly wage?

Problem 23.16
Michelle and Rachel are running a 26.2 mile marathon together as a team. They run in a ratio of $5:3$ respectively. How many miles do Michelle and Rachel run?

Problem 23.17
Express the following comparisons as ratios. Suppose a class has 14 redheads, 8 brunettes, and 6 blondes.

(a) What is the ratio of redheads to brunettes?
(b) What is the ratio of redheads to blondes?
(c) What is the ratio of blondes to brunettes?
(d) What is ratio of blondes to total students?

Problem 23.18
Express the following as ratios in fraction form and reduce.
a. 3 to 12
b. 25 to 5
c. 6 to 30
d. 100 to 10
e. 42 to 4
f. 7 to 30

Problem 23.19
Express each of the following ratios in fractional form then simplify.
(a) 5 cents to $2
(b) 12 feet to 2 yards
(c) 30 minutes to 2 hours
(d) 5 days to 1 year
(e) 1 dime to 1 quarter

Problem 23.20
Sandra wants to give a party for 60 people. She has a punch recipe that makes 2 gallons of punch and serves 15 people. How many gallons of punch should she make for her party?

24 Percent

Percents are very useful in conveying informtation. People hear that there is 60 percent chance of rain or that the interest rate on mortgage loans is up by 0.5 percent. Percent were devised for just such situations. Percent (from the Latin word *per centum* meaning "per hundred") is denoted by

$$n\% = \frac{n}{100}$$

where the symbol % indicates percent. Thus, a percent is a ratio that compares a number to 100.
Figure 24.1 illustrates a percent as a part of a whole.

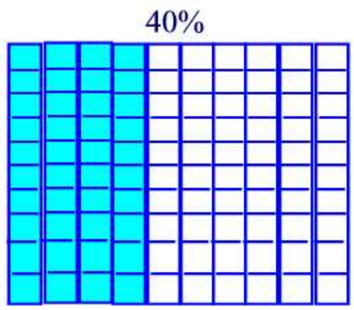


Figure 24.1

The concepts of percents, fractions, and decimals are interrelated so it is important to be able to convert among all three forms.

Converting Percents to Fractions
Do the following steps to convert a percent to a fraction:
For example: Convert 83% to a fraction.

• Remove the percent sign
• Make a fraction with the percent as the numerator and 100 as the denominator (e.g. 83/100)
• Reduce the fraction if needed.

Example 24.1
Express these percents as fractions in lowest terms.
(a) 60% (b) $66\frac{2}{3}\%$ (c) 125%

Solution.
(a) $60\% = \frac{60}{100} = \frac{3}{5}$.
(b) $66\frac{2}{3}\% = \frac{200}{3}\% = \frac{\frac{200}{3}}{100} = \frac{200}{3} \cdot \frac{1}{100} = \frac{2}{3}$.
(c) $125\% = \frac{125}{100} = \frac{5}{4}$.■

Converting Percents to Decimals
Since $n\%$ is defined as the fraction $\frac{n}{100}$ and the notational effect of dividing a number by 100 is to move the decimal point two places to the left, it is easy to write a given percent as a decimal.

Example 24.2
Express these percents as decimal.
(a) 40% (b) 12% (c) 127%

Solution.
(a) $40\% = 0.40$
(b) $12\% = 0.12$
(c) $127\% = 1.27$.■

Converting Decimals to Percents
To convert a decimal to a percent move the decimal point two places to the right and then add the % sign to the resulting number.

Example 24.3
Express these decimals as percents.
(a) 0.25 (b) $0.\overline{3}$ (c) 1.255

Solution.
(a) $0.25 = 25\%$
(b) $0.\overline{3} = 33.\overline{3} = 33\frac{1}{3}\%$
(C) $1.2555 = 125.55\%$■

Converting Fractions to Percents
A fraction is converted to a percent by using proportion. For example, to write $\frac{3}{5}$ as a percent, find the value of n in the following proportion:

$$\frac{3}{5} = \frac{n}{100}$$

Solving for n we find $n = 60$. Thus, $\frac{3}{5} = 60\%$

Example 24.4
Express these fractions as percents.
(a) $\frac{1}{8}$ (b) $\frac{1}{3}$ (c) $\frac{16}{5}$

Solution.
(a) Solving the proportion $\frac{1}{8} = \frac{n}{100}$ for n we find $n = 12.5$. Thus, $\frac{1}{8} = 12.5\%$.
(b) If $\frac{1}{3} = \frac{n}{100}$ then $n = \frac{100}{3} = 33\frac{1}{3}$. Thus, $\frac{1}{3} = 33\frac{1}{3}\%$
(c) If $\frac{16}{5} = \frac{n}{100}$ then $n = 320$ so that $\frac{16}{5} = 320\%$ ■

Applications Involving Percent
Use of percents is commonplace. Application problems that involve percents usually take one of the following forms:
1. Finding a percent of a number
2. Finding what percent one number is of another
3. Finding a number when a percent of that number is known
These three types of usages are illustrated in the next three examples.

Example 24.5 *(Calculating a Percentage of a Number)*
A house that sells for \$92,000 requires a 20% down payment. What is the amount of the down payment?

Solution.
The down payment is $(20\%) \cdot (92,000) = (0.2) \cdot (92,000) = \$18,400.$■

Example 24.6 *(Calculating What Percentage One Number is of Another)*
If Alberto has 45 correct answers on an 80-question test, what percent of his answers are correct?

Solution.
Alberto has $\frac{45}{80}$ of the answers correct. We want to convert this fraction to a percent. This requires solving the proportion $\frac{45}{80} = \frac{n}{100}$. Thus, $n = 56.25$. This says that 56.25% of the answers are correct■

Example 24.7 *(Calculating a Number when the Percent of that Number is Known)*
Paul scored 92% on his last test. If he got 23 questions right, how many problems were on the test?

Solution.
Let n be the total number of questions on the test. Then $(92\%)x = 23$. That is, $0.92x = 23$ or $x = \frac{23}{0.92}25$. So there were 25 questions on the test.■

Mental Math with Percents
Mental math may be helpful when working with percents. Two techniques follow:
• **Using Fraction Equivalents**
The table below gives several fraction equivalents.

Percent	25%	50%	75%	$33\frac{1}{3}\%$	$66\frac{2}{3}\%$	10%	5%	1%
Fraction Equivalent	1/4	1/2	3/4	1/3	2/3	1/10	1/20	1/100

Example 24.8
Compute mentally 50% of 80.

Solution.
50% of 80 is just $\left(\frac{1}{2}\right)80 = 40$■

•**Using a known Proportion**
Frequently, we may not know a percent of something, but a close percent of it as illustrated in the following exmaple.

Example 24.9
Find 55% 0f 62.

Solution.
we know that 50% of 62 is $\left(\frac{1}{2}\right)62 = 31$ and 5% of 62 is $\left(\frac{1}{20}\right)62 = 3.1$. Thus, 55% of 62 is $31 + 3.1 = 34.1$■

Estimation with Percents
Estimation with percents can be used to determine whether answers are reasonable. Following are two examples.

Example 24.10
Estimate 148% of 500.

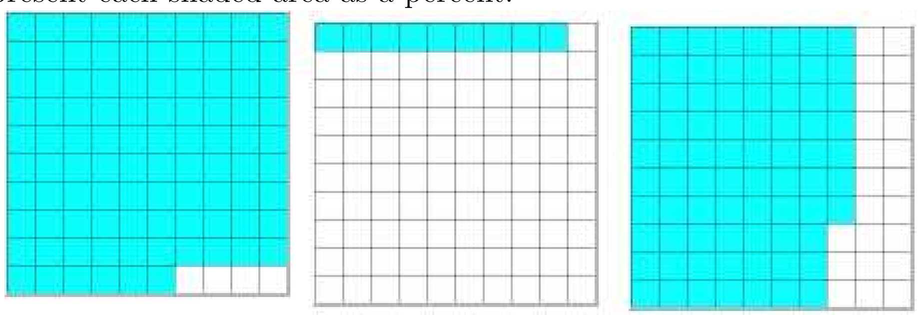
Solution.
Note that 148% of 500 is slightly less than 150% of 500 which is $1.5(500) = 750$. Thus 148% of 500 should be slightly less than 750.■

Example 24.11
Estimate 27$ of 598.

Solution.
27% of 598 is little less than 27% of 600. But 27% of 600 is the same as 30% of 600 minus 3% of 600. That is, 180 - 18 = 162. Hence, 27% of 598 is slightly less that 162.■

Practice Problems

Problem 24.1
Represent each shaded area as a percent.



Problem 24.2
Shade a rectangular area to represent 14%

Problem 24.3
Write each percent as a fraction and as a decimal.
(a) 34% (b) 180% (c) 0.06%

Problem 24.4
Write each decimal as a percent.
(a) 0.23 (b) 0.0041 (c) 24

Problem 24.5
Write each fraction as a percent.
(a) $\frac{1}{25}$ (b) $\frac{3}{8}$ (c) $1\frac{3}{4}$

Problem 24.6
A drink mix has 3 parts orange juice for every 2 parts of carbonated water.
(a) What fraction of the mix is carbonated water?
(b) What percent of the mix is orange juice?

Problem 24.7
Answer the following questions.
(a) What is 30% of 500?
(b) 25 is 40% of what number?
(c) 28 out of 40 is what percent?

Problem 24.8
Mentally compute
(a) 50% of 286 (b) 25% of 4000

Problem 24.9
This year, Nancy Shaw's salary increased from \$28,800 to \$32,256. What percent in crease is this?

Problem 24.10
A \$400 television is selling at a 25% discount. Mentally compute its sale price.

Problem 24.11
How could you compute mentally the exact value of each of the following?
(a) 75% of 12
(b) 70% of 210

Problem 24.12
In 2002, the voting-age populatin of the US was about 202 million, of which about 40% voted. Estimate the number of people who voted.

Problem 24.13
The Cereal Bowl seats 95,000. The stadium is 64% full for a certain game. Explain how to estimate the attendance mentally
(a) using rounding
(b) with compatible numbers.

Problem 24.14
Mentally convert each of the following to percent.
(a) $\frac{7}{28}$ (b) $\frac{72}{144}$ (c) $\frac{44}{66}$

Problem 24.15
Mentally estimate the number that should go in the blank to make each of these true.
(a) 27% of _____ equals 16.
(b) 4 is _____% of 7.5.
(c) 41% of 120 is equal to _____

Problem 24.16
Estimate
(a) 39% of 72   (b) 0.48% of 207   (c) 412% of 185

Problem 24.17
Order the following list from least to greatest: 13:25, $\frac{2}{25}$, 3%

Problem 24.18
Uncle Joe made chocolate chip cookies. Benjamin ate fifty percent of them right away. Thomas ate fifty percent of what was left. Ten cookies remain. How many cookies did Uncle Joe make?

Problem 24.19
Thomas won 90 percent of his wrestling matches this year and came in third at the state tournament. If he competed in 29 matches over the course of the season (including the state tournament), how many did he lose?

Problem 24.20
According to the statistics, the Megalopolis lacrosse team scores 25% of their goals in the first half of play and the rest during the second half. Thus, it seems that the coach's opinion that they are a "second half team" is correct. If they scored 14 goals in the first half this year, about how many did they score in the second half?

Problem 24.21
A sample of clay is found in Mongolia that contains aluminum, silicon, hydrogen, magnesium, iron, and oxygen. The amount of iron is equal to the amount of aluminum. If the clay is 20% silicon, 19% hydrogen, 10% magnesium and 24% oxygen, what is the percent iron?

Problem 24.22
Ms. Taylor wants to donate fourteen percent of her paycheck to the Mountain Springs Hospital for Children. If her paycheck is $801.00, how much should she send to the Mountain Springs Hospital for Children?

Problem 24.23
Alexis currently has an average of 94.7% on her three math tests this year. If one of her test grades was 91% and another was 97%, what was the grade of her third test?

Problem 24.24
Jennifer donated nine percent of the money she earned this summer to her local fire department. If she donated a total of $139 how much did she earn this summer?

Problem 24.25
If ten out of fifteen skinks have stripes and the rest don't, what percent of the skinks do not have stripes out of a population of 104 skinks? Round your answer to the nearest tenth of a percent.

Problem 24.26
Sixty-eight percent of the animals in Big Range national park are herbivores. If there are 794 animals in the park, how many are not herbivores? Round your answer to the nearest whole number.

Problem 24.27
There are a lot of reptiles at Ms. Floop's Reptile Park. She has snakes, lizards, turtles and alligators. If 27.8% of the reptiles are snakes, 18.2% are lizards, and 27% are alligators, what percent are turtles?

Problem 24.28
A soil sample from Mr. Bloop's farm was sent to the county agriculture department for analysis. It was found to consist of 22% sand, 24.7% silt, 29.7% clay, 7% gravel and the rest was humus. What percent of the sample was humus?

Problem 24.29
Attendance is up at the local minor league stadium this year. Last year there was an average of 3,010 fans per game. This year the average has been 4,655. What percent increase has occurred? Round your answer to the nearest hundredth of a percent.

Problem 24.30
If a baseball team begins the season with 5,000 baseballs, and at the end of the season they have 2,673, what percent of the balls are gone? Round your answer to the nearest tenth of a percent.

25 Solutions to Practice Problems

In each of the following problems write the equation that describes each situation. Do not solve the equation.

Problem 1.1
Two numbers differ by 5 and have a product of 8. What are the two numbers?

Solution.
Let x be one of the numbers. Then the second number is $x+5$. Since the product is 8 then the required equation is

$$x(x+5) = 8.\blacksquare$$

Problem 1.2
Jeremy paid for his breakfest with 36 coins consisting of nickels and dimes. If the bill was \$3.50, then how many of each type of coin did he use?

Solution.
Let x be the number of nickels. Then the number of dimes is $36-x$. The required equation is

$$0.05x + 0.1(36-x) = 3.50.\blacksquare$$

Problem 1.3
The sum of three consecutive odd integers is 27. Find the three integers.

Solution.
Let x be the first odd integer. Then $x+2$ is the second integer and $x+4$ is the third one. The required equation is

$$x + (x+2) + (x+4) = 27.\blacksquare$$

Problem 1.4
At an 8% sales tax rate, the sales tax Peter's new Ford Taurus was \$1,200. What was the price of the car?

Solution.
Let x be the original price of the car before tax. Then

$$0.08x = 1,200.\blacksquare$$

Problem 1.5
After getting a 20% discount, Robert paid \$320 for a Pioneer CD player for his car. What was the original price of the CD?

Solution.
Let x denote the original price of the CD. Then

$$x - 0.02x = 320.\blacksquare$$

Problem 1.6
The length of a rectangular piece of property is 1 foot less than twice the width. The perimeter of the property is 748 feet. Find the length and the width.

Solution.
Let x denote the width of the rectangular lot. Then the length is $2x - 1$. Since the perimeter of a rectangle is twice the length plus twice the width then the required equation is

$$2x + 2(2x - 1) = 748\blacksquare$$

Problem 1.7
Sarah is selling her house through a real estate agent whose commission rate is 7%. What should the selling price be so that Sarah can get the \$83,700 she needs to pay off the mortgage?

Solution.
Let x be the selling price of the house. Then

$$x - 0.07x = 83,700\blacksquare$$

Problem 1.8
Ralph got a 12% discount when he bought his new 1999 Corvette Coupe. If the amount of his discount was \$4,584, then what was the original price?

Solution.
Let x be the original price of the car before discount. Then the discount is found by the equation

$$0.012x = 4,584\blacksquare$$

Problem 1.9
Julia framed an oil painting that her uncle gave her. The painting was 4 inches longer than it was wide, and it took 176 inches of frame molding. What were the dimensions of the picture?

Solution.
Let x be the width of the frame. Then the length is $x+4$. Since the perimeter of the frame is 176 inches then

$$2x + 2(x + 4) = 176\blacksquare$$

Problem 1.10
If the perimeter of a tennis court is 228 feet and the length is 6 feet longer than twice the width, then what are the length and the width?

Solution.
Let x denote the width of the tennis court. Then the length is $2x + 6$. Since the perimeter is 228 feet then the required equation is

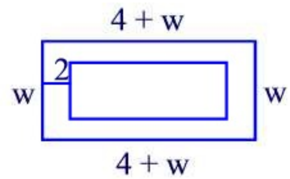
$$2x + 2(2x + 6) = 228\blacksquare$$

Problem 2.1
Susan made \$2.80 at her lemonade stand. She has 18 coins. What combination of coins does she have?

Solution.
If one of the coins is a quarter then the maximum value of the remaining 17 coins is \$1.70 giving a total of \$1.95. If three of the coins are quarters then then maximum value of the remaining 15 coins is \$1.50 giving a total of \$2.25. If seven of the coins were quarters then the maximum value of the remaining 11 coins is \$ 1.10 giving a total of \$2.85.

Problem 2.2
A rectangular garden is 4 feet longer than it is wide. Along the edge of the garden on all sides, there is a 2-foot gravel path. How wide is the garden if the perimeter of the garden is 28 feet? (Hint: Draw a diagram and use the guess and check strategy.)

Solution.
Let w denote the width of the rectangle. Then the length is $4 + w$ as shown in the figure below.



If $w = 2$ then the perimeter is $2 + 6 + 2 + 6 = 16$. If $w = 4$ then the perimeter is $4 + 8 + 4 + 8 = 24$. If $w = 5$ then $5 + 9 + 5 + 9 = 28$.■

Problem 2.3
There are two two-digit numbers that satisfy the following conditions:

(1) Each number has the same digits,
(2) the sum of digits in each number is 10,
(3) the difference between the two numbers is 54.
What are the two numbers?

Understanding the problem
The numbers 58 and 85 are two-digit numbers which have the same digits, and the sum of the digits is 13. Find two two-digit numbers such that the sum of the digits is 10 and both numbers have the same digits.
Devise a plan
Since there are only nine two-digit numbers whose digits have a sum of 10, the problem can be easily solved by guessing. What is the difference of your two two-digit numbers from part (a)? If this difference is not 54, it can provide information about your next guess.
Carry out the plan
Continue to guess and check. Which numbers has a difference of 54?
Looking back
This problem can be extended by changing the requirement that the sum of the two digits equal 10. Solve the problem for the case in which the digits have a sum of 12.

Solution.
Possible candidates are 19 and 91, 28 and 82, 37 and 73, 46 and 64. The only two numbers whose difference is 54 are 28 and 82.■

Problem 2.4
John is thinking of a number. If you divide it by 2 and add 16, you get 28. What number is John thinking of?

Solution.
If the number was 2 then the result would be $1 + 16 = 17$. If we try 4 then $2 + 16 = 18$. Let's try 8 then $4 + 16 = 20$. Let's try 16 then $8 + 16 = 24$. Try 20 to obtain $10 + 16 = 26$. Finally, if we try 24 we find $12 + 16 = 28$. So the number is 24.■

Problem 2.5
Place the digits 1, 2, 3, 4, 5, 6 in the circles in Figure 2.4 so that the sum of the three numbers on each side of the triangle is 12.

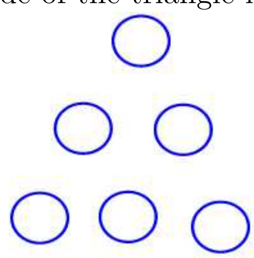
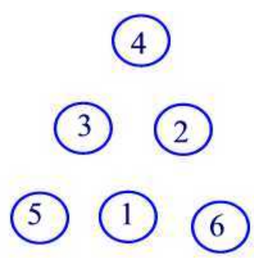


Figure 2.4

Solution.
By assigning the top circle 1, 2, 3, and so on we will end up with the following solution.



Problem 2.6
Carmela opened her piggy bank and found she had \$15.30. If she had only nickels, dimes, quarters, and half-dollars and an equal number of coins of each kind, how many coins in all did she have?

Solution.
If she had one coin of each type then the total would be

$$0.05 + 0.10 + 0.25 + 0.50 = 0.80$$

too small. Let's try 5 coins of each type. Then

$$0.25 + 0.50 + 1.25 + 2.50 = 4.50.$$

Now try 10 coins of each type.

$$0.50 + 1.00 + 2.50 + 5.00 = 9.00$$

Now we notice that 15 coins of each type will give a total of $9+4.50 = 13.50$. We are getting closer. let's try 17 coins of each type to obtain

$$0.85 + 1.7 + 4.25 + 8.50 = 15.30. \blacksquare$$

Problem 2.7
When two numbers are multiplied, their product is 759; but when one is subtracted from the other, their difference is 10. What are those two numbers?

Solution.
Let's try 20 and 10. Then $20-10 = 10$ and $20\times 10 = 200$ which is small. Let's try 21 and 11 then $21 \times 11 = 231$. Try 22 and 12 to obtain $22 \times 12 = 264$. Try 25 and 15 to obtain $25\times 15 = 375$. Next, try 30 and 20 to obtain $30\times 20 = 600$. We are getting closer. Try 35 and 25 to obtain $35 \times 25 = 875$ which is larger that what we want. Let's now try 33 and 23 then $33 \times 23 = 759$. Bingo.$\blacksquare$

Problem 2.8
Sandy bought 18 pieces of fruit (oranges and grapefruits), which cost \$4.62. If an orange costs \$0.19 and a grapefruit costs \$0.29, how many of each did she buy?

Solution.
If all 18 pieces were grapefruit then the total would be $18 \times 0.29 = 5.22$. For 16 grapfruits and 2 oranges the total would be $16\times 0.29+2\times 0.19 = 5.02$. For 14 grapefruits and 4 oranges the total would be $14\times 0.29+4\times 0.19 = 4.82$. For 12 grapefruits and 6 oranges the total would be $12 \times 0.29 + 6 \times 0.19 = 4.62.\blacksquare$

Problem 2.9
A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter," I count 24 heads and 80 feet. How many pigs and how many chickens are out there?"

Solution.
The number of pigs must be less than 20 since all 20 pigs give 80 legs. If the number of pigs is 18 and the number of chicken is 6 then the total number of legs is $18 \times 4 + 6 \times 2 = 90$ legs. Suppose now there were 14 pigs and 10 chicken then the total number of legs is $14 \times 4 + 10 \times 2 = 76$ which is less than 80. Let's take 15 pigs and 9 chicken. Then the total number of legs is $15 \times 4 + 9 \times 2 = 78$. Finally, let's take 16 pigs and 8 chicken. In this case the total number of legs is $16 \times 4 + 8 \times 2 = 80$ legs.■

Problem 2.10
At a benefit concert 600 tickets were sold and \$1,500 was raised. If there were \$2 and \$5 tickets, how many of each were sold?

Solution.
If all tickets sold were \$2 tickets then the total would be \$ 1,200. If 500 tickets were \$2 tickets and 100 tickets were \$ 5 tickets then the total would be \$1,500. ■

Problem 2.11
At a bicycle store, there were a bunch of bicycles and tricycles. If there are 32 seats and 72 wheels, how many bicyles and how many tricycles were there?

Solution.
We use guess and check. If there were 30 bicycles and 2 tricycles then the total number of wheels is 66. For 29 bicycles and 3 tricycles the total number of wheels is 67. For 26 bicycles and 6 tricycles the total number of wheels is 70 wheels. We are almost there. For 24 bicycles and 8 tricycles we get 72 wheels.■

Problem 2.12
If you have a bunch of 10 cents and 5 cents stamps, and you know that there are 20 stamps and their total value is \$1.50, how many of each do you have?

Solution.
We use the guess and check strategy. If we have 10 stamps of each type this will give us a total of \$1.50.■

Problem 2.13
A dog's weight is 10 kilograms plus half its weight. How much does the dog weigh?

Solution.
Let w be the weight of the dog. Then

$$w = \frac{w}{2} + 10.$$

Note that w must be even. By cheking even numbers we find $w = 20$ kilograms.■

Problem 2.14
The measure of the largest angle of a triangle is nine times the measure of the smallest angle. The measure of the third angle is equal to the difference of the largest and the smallest. What are the measures of the angles?(Recall that the sum of the measures of the angles in a triangle is $180°$)

Solution.
Let x denote the measure of the smallest angle. Then the largest number has measure $9x$ and the third angle has measure $9x - x = 8x$. Thus, we have the equation $x + 8x + 9x = 180$ or $18x = 180$. This gives $x = 10°$. The largest angle is then $9x = 90°$ and the third angle is $8x = 80°$■

Problem 2.15
The distance around a tennis court is 228 feet. If the length of the court is 6 feet more than twice the width, find the dimensions of the tennis court.

Solution.
Let w denote the width of the tennis court. Then the length is $2w + 6$. Since the perimeter is 228 feet then we have the equation $2w + 2(2w + 6) = 228$. Simplifying this equation we find $4w + 12 = 228$ or $4w = 216$. So $w = 54$ feet. Hence, the length of the rectangle is $2w + 6 = 108 + 6 = 114$ feet.■

Problem 2.16
The floor of a square room is covered with square tiles. Walking diagonally across the room from corner to corner, Susan counted a total of 33 tiles on the two diagonals. What is the total number of tiles covering the floor of the room?

Solution.
If x denotes the number of tiles on each diagonal then $2x - 1 = 33$. (We subtract 1 since the two diagonal share one tile in the center of the square) Solving for x we find $2x = 34$ or $x = 17$. Since the room is square then the total number of tiles covering the floor is $17 \times 17 = 289$ tiles.■

Problem 2.17
In three years, Chad will be three times my present age. I will then be half as old as he. How old am I now?

Solution.
Let x be my present age. In three years Chad's age will be $3x$. But then my age will be $\frac{3x}{2}$. Thus,

$$x + 3 = \frac{3x}{2}.$$

We solve this equation as follows.

$$\begin{array}{rcl} x + 3 & = & \frac{3x}{2} \\ \frac{3x}{2} - x & = & 3 \\ \frac{x}{2} & = & 3 \\ x & = & 6 \end{array}$$

So my present age is 6 years old.■

Problem 2.18
A fish is 30 inches long. The head is as long as the tail. If the head was twice as long and the tail was its present, the body would be 18 inches long. How long is each portion of the fish?

Solution.
Let x be the present length of the tail. Then the head would be $2x$. This leads to the equation

$$x + 2x + 18 = 30.$$

We solve this equation as follows.

$$\begin{aligned}
3x + 18 &= 30 \\
3x &= 30 - 18 \\
3x &= 12 \\
x &= 4
\end{aligned}$$

Thus, the tail is 4 inches, the head is 4 inches and the body is 22 inches.■

Problem 2.19
Two numbers differ by 5 and have a product of 8. What are the two numbers?

Solution.
Let x be the first number. Then the second number is $x+5$ since they differ by 5. Thus,

$$x(x+5) = 8.$$

The two numbers are the solutions to the quadratic equation $x^2+5x-8=0$.■

Problem 2.20
Jeremy paid for his breakfest with 36 coins consisting of nickels and dimes. If the bill was $3.50, then how many of each type of coin did he use?

Solution.
Let x be the number of nickels. Then $36-x$ is the number of dimes. Thus,

$$0.05x + 0.1(36-x) = 3.50.$$

We solve this equation as follows.

$$\begin{aligned}
0.05x + 0.1(36-x) &= 3.50 \\
0.05x + 3.6 - 0.1x &= 3.50 \\
0.1x - 0.05x &= 3.60 - 3.50 \\
0.05x &= 0.10 \\
x &= \tfrac{0.10}{0.05} = 2
\end{aligned}$$

So he paid 2 nickels and 34 dimes.■

Problem 2.21
The sum of three consecutive odd integers is 27. Find the three integers.

Solution.
Let x be the first odd integer. Then $x+2$ is the second integer and $x+4$ is the third one. The required equation is

$$x + (x+2) + (x+4) = 27.$$

Thus, $3x+6 = 27$ or $3x = 21$ so that $x = 7$. The three numbers are 7, 9, and 11.■

Problem 2.22
At an 8% sales tax rate, the sales tax Peter's new Ford Taurus was \$1,200. What was the price of the car?

Solution.
Let x be the original price of the car before tax. Then

$$0.08x = 1,200.$$

We see that $x = \frac{1200}{0.08} = \$15,000.$■

Problem 2.23
After getting a 20% discount, Robert paid \$320 for a Pioneer CD player for his car. What was the original price of the car?

Solution.
Let x denote the original price of the CD. Then

$$x - 0.02x = 320.$$

Thus, $0.98x = 320$ so that $x = \frac{320}{0.98} \approx \$326.53$■

Problem 2.24
The length of a rectangular piece of property is 1 foot less than twice the width. The perimeter of the property is 748 feet. Find the length and the width.

Solution.
Let x denote the width of the rectangular lot. Then the length is $2x-1$. Since the perimeter of a rectangle is twice the length plus twice the width then the required equation is

$$2x + 2(2x-1) = 748.$$

Solving this equation for x we find

$$\begin{array}{rcl} 2x + 2(2x-1) & = & 748 \\ 6x - 2 & = & 748 \\ 6x & = & 750 \\ x & = & \frac{750}{6} = 125 \end{array}$$

So the the length is $2(125) - 1 = 249$ feet.$\blacksquare$

Problem 2.25
Sarah is selling her house through a real estate agent whose commission rate is 7%. What should the selling price be so that Sarah can get the \$83,700 she needs to pay off the mortgage?

Solution.
Let x be the selling price of the house. Then

$$x - 0.07x = 83,700.$$

Thus, $0.93 = 83700$ or $x = \frac{83700}{0.93} = \$90,000$$\blacksquare$

Problem 2.26
Ralph got a 12% discount when he bought his new 1999 Corvette Coupe. If the amount of his discount was \$4,584, then what was the original price?

Solution.
Let x be the original price of the car before discount. Then the discount is found by the equation

$$0.012x = 4,584.$$

Thus, $x = \frac{4584}{0.012} = \$38,200$$\blacksquare$

Problem 2.27
Julia framed an oil painting that her uncle gave her. The painting was 4 inches longer than it was wide, and it took 176 inches of frame molding. What were the dimensions of the picture?

Solution.
Let x be the width of the frame. Then the length is $x+4$. Since the perimeter of the frame is 176 inches then

$$2x + 2(x+4) = 176.$$

Solving this equation we find

$$\begin{aligned}
2x + 2(x+4) &= 176\\
4x + 8 &= 176\\
4x &= 168\\
x &= \tfrac{168}{4} = 42\ \textit{inches}
\end{aligned}$$

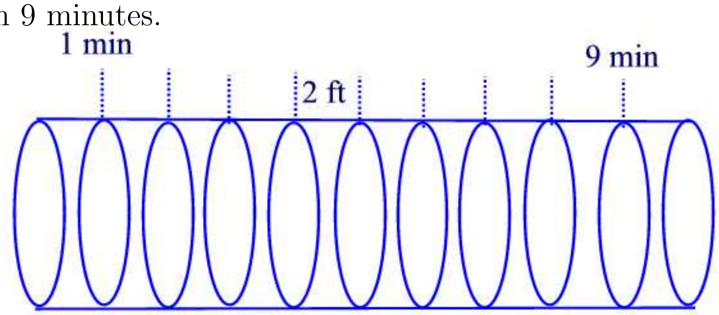
The dimensions of the frame are 42-by-46.■

Problem 2.28
If the perimeter of a tennis court is 228 feet and the length is 6 feet longer than twice the width, then what are the length and the width?

Solution.
This is just Problem 2.15■

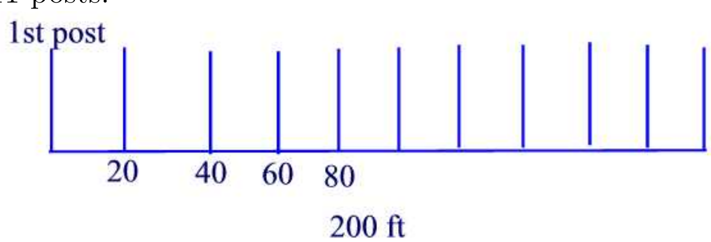
Problem 2.29
Bob can cut through a log in one minute. How long will it take Bob to cut a 20-foot log into 2-foot sections?

Solution.
It takes him 9 minutes.



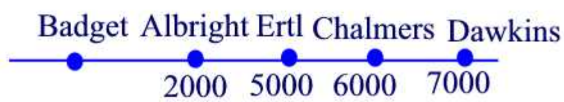
Problem 2.30
How many posts does it take to support a straight fence 200 feet long if a post is placed every 20 feet?

Solution.
It requires 11 posts.



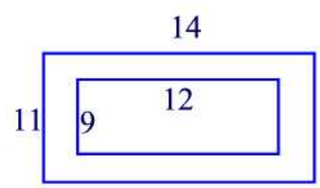
Problem 2.31
Albright, Badgett, Chalmers, Dawkins, and Earl all entered the primary to seek election to the city council. Albright received 2000 more votes than Badgett and 4000 fewer than Chalmers. Earl received 2000 votes fewer than Dawkins and 5000 votes more than Badgett. In what order did each person finish in the balloting?

Solution.



Problem 2.32
A 9-meter by 12-meter rectangular lawn has a concrete walk 1 meter wide all around it outside lawn. What is the area of the walk?

Solution.
The area of the walk is the area of the area of the rectangle of width 11 meters and length 14 meters minus the area of the lawn. That is, $14\times11-12\times9 = 56$ square meters.

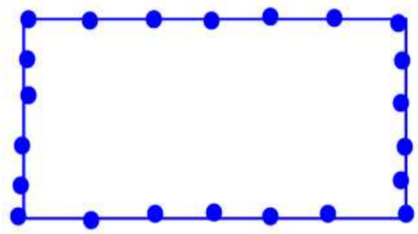


Problem 2.33
An elevator stopped at the middle floor of a building. It then moved up 4 floors and stopped. It then moved down 6 floors, and then moved up 10 floors and stopped. The elevator was now 3 floors from the top floor. How many floor does the building have?

Solution.
By drawing a diagram one can easily see that the building has 23 floors.■

Problem 2.34
In the Falkland Islands, south of Argentina, Armado, a sheepherder's son, is helping his father build a rectangular pen to keep their sheep from getting lost. The pen will be 24 meters long, 20 meters wide, and have a fence posts 4 meters apart. How many fence posts do they need?

Solution.
There are 23 posts.



Problem 2.35
Five people enter a racquetball tournment in which each person must play every other person exactly once. Determine the total number of games that will be played.

Solution.
Call the five players p_1, p_2, p_3, p_4, and p_5. Then we have the following games:$(p_1, p_2), (p_1, p_3), (p_1, p_4), ($
So the total number of games is 10 games.■

Problem 2.36
When two pieces of ropes are placed end to end, their combined length is 130 feet. When the two pieces are placed side by side, one is 26 feet longer than the other. What are the lengths of the two pieces?

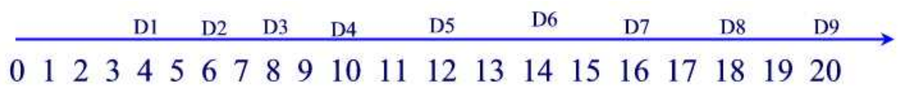
Solution.
Let x be the length of the shorter piece. Then $x + (x + 26) = 130$ or $2x + 26 = 130$. This implies that $2x = 104$ and therefore $x = \frac{104}{2} = 52$ feet. The length of the larger piece is 78 feet.■

Problem 2.37
There are 560 third- and fourth-grade students in Russellville elementary school. If there are 80 more third graders than fourth graders, how many third graders are there in the school?

Solution.
Let x be the number of fourth graders. Then the number of thirs graders is $x + 80$. But $x + (x + 80) = 560$ or $2x + 80 = 560$. This implies that $2x = 480$ and hence $x = 240$ fourth graders. The number of third graders is $240 + 80 = 320$ students.■

Problem 2.38
A well is 20 feet deep. A snail at the bottom climbs up 4 feet every day and slips back 2 feet each night. How many days will it take the snail to reach the top of the well?

Solution.
It takes 9 days for the snail to reach the top of the well.



Problem 2.39
Five friends were sitting on one side of a table. Gary set next to Bill. Mike sat next to Tom. Howard sat in the third seat from Bill. Gary sat in the third seat from Mike. Who sat on the other side of Tom?

Solution.

Problem 3.1
Sequences like $2, 5, 8, 11, \cdots$, where each term is the previous term increased by a constant, are called **arithmetic sequences**. Compute the sum of the following arithmetic sequence

$$1 + 7 + 13 + \cdots + 73.$$

Solution.
We use "look for a pattern" strategy. The numbers 1, 7, 13, ... form an arithmetic sequence where each number is the previous one increased by 6. We notice the following pattern:

$$\begin{array}{ccccc} 1+7 & = & 1+(1+1\cdot 6) & = & 2+\frac{(2)(2-1)}{2}\cdot 6 \\ 1+7+13 & = & 1+(1+1\cdot 6)+(1+2\cdot 6) & = & 3+\frac{3(3-1)}{2}\cdot 6 \\ 1+7+13+19 & = & 1+(1+1\cdot 6)+(1+2\cdot 6)+(1+3\cdot 6) & = & 4+\frac{4(4-1)}{2}\cdot 6 \end{array}$$

Since $73 = 1 + 12 \cdot 6$ then

$$1 + 7 + 13 + \cdots + 73 = 13 + \frac{(13)(13-1)}{2} \cdot 6 = 481. \blacksquare$$

Problem 3.2
Sequences like $1, 2, 4, 8, 16, \cdots$, where each term is the previous term multiplied by a constant, are called **geometric sequences**. Compute the sum of the following geometric sequence

$$1 + 2 + 4 + 8 + \cdots + 2^{100}.$$

Solution.
We use "look for a pattern" strategy. The numbers 1,2,4,8,... form a geometric sequence where each number is the previous one multiplied by 2. We notice the following pattern:

$$\begin{array}{ccc} 1 & = & 2^0 = \frac{2^1-1}{2-1} \\ 1+2 & = & 2^0 + 2^1 = \frac{2^2-1}{2-1} \\ 1+2+4 & = & 2^0 + 2^1 + 2^2 = \frac{2^3-1}{2-1} \end{array}$$

Thus,

$$1 + 2 + 4 + 8 + \cdots + 2^{100} = \frac{2^{101}-1}{2-1} = 2^{101} - 1 \blacksquare$$

Problem 3.3
(a) Fill in the blanks to continue this sequence of equations.

$$\begin{array}{ccc} 1 & = & 1 \\ 1+2+1 & = & 4 \\ 1+2+3+2+1 & = & 9 \\ \underline{\qquad\qquad\qquad} & = & \_ \\ \underline{\qquad\qquad\qquad} & = & \_ \end{array}$$

(b) Compute the sum

$$\begin{array}{ccl} 1 & + & 2 + 3 + \cdots + 99 + 100 + 99 \\ & + & \cdots + 3 + 2 + 1 = \underline{\qquad} \end{array}$$

(c) Compute the sum

$$\begin{array}{c} 1 + 2 + 3 + \cdots + (n-1) \\ +n + (n-1) + \cdots + 3 + 2 + 1 = \underline{\qquad} \end{array}$$

Solution.
(a)

$$\begin{array}{ccc} 1 & = & 1 \\ 1+2+1 & = & 4 \\ 1+2+3+2+1 & = & 9 \\ 1+2+3+4+3+2+1 & = & 16 \\ 1+2+3+4+5+4+3+2+1 & = & 25 \end{array}$$

Note that the sum is the square of the middle number on the list.
(b)

$$\begin{array}{rl} 1 + & 2+3+\cdots+99+100+99 \\ + & \cdots+3+2+1 = 100^2 = 10000 \end{array}$$

(c)

$$\begin{array}{c} 1+2+3+\cdots+(n-1) \\ +n+(n-1)+\cdots+3+2+1 = n^2 \blacksquare \end{array}$$

Problem 3.4
Find a pattern in the designs. How many squares will there be in the eighth design of your pattern?See Figure 3.2.

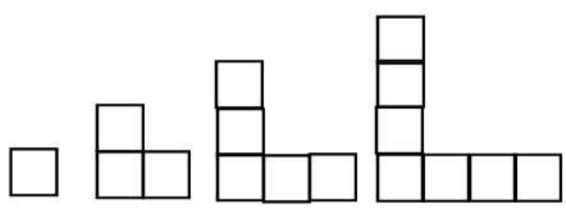


Figure 3.2

Solution.

$$\begin{array}{cc} \textit{Design} & \textit{Number of squares} \\ 1 & 1 = 2\cdot 1 - 1 \\ 2 & 3 = 2\cdot 2 - 1 \\ 3 & 5 = 2\cdot 3 - 1 \\ 4 & 7 = 2\cdot 4 - 1 \end{array}$$

It follows that the eighth design consists of $2 \cdot 8 - 1 = 15$ squares.$\blacksquare$

Problem 3.5
Find the sum of the first 100 even nonzero whole numbers.

Solution.
We use "look for a pattern" strategy. Note that the terms of the sequence form an arithmetic sequence where each term is the previous one increased by 2.

$$\begin{aligned}
2+4 &= 1\cdot 2+2\cdot 2 = 2+2(2+1)\\
2+4+6 &= 1\cdot 2+2\cdot 2+3\cdot 2 = 3(3+1)\\
2+4+6+8 &= 1\cdot 2+2\cdot 2+3\cdot 2+4\cdot 2 = 4(4+1)
\end{aligned}$$

It follows that

$$2+4+6+\cdots+100 = 50(50+1) = 2550\blacksquare$$

Problem 3.6
James began writing a book. At the end of the first week, he'd written 10 pages. By the end of the second week, he'd written 6 more pages, for a total of 16 pages. At the end of the third week, he had a total of 23 pages and by the end of the fourth week he had 31 pages completed in his book. If he continues writing at this same rate, how many pages will his book have at the end of the seventh week?

Solution.
We use "look for a pattern" strategy.

Number of weeks	*Number of pages*
1	10
2	$16 = 10+(6+0)$
3	$23 = 10+(6+0)+(6+1)$
4	$31 = 10+(6+0)+(6+1)+(6+2)$

Thus, after seven weeks he has written

$$10+(6+0)+(6+1)+(6+2)+(6+3)+(6+4)+(6+5) = 10+6.6+(1+2+3+4+5) = 61\ pages\blacksquare$$

Problem 3.7
Mary's five friends began an exercise group. They decided to walk along a trail each day. On the first day, they walked 2/3 of the trail. On the second day, they walked 3/5 of the trail. On the third day, they walked 4/7 and on the fourth day 5/9 of the trail. If this pattern continues, how far will Mary and her friends walk on the tenth day?

Solution.
We use "look for a pattern" strategy.

Day	*Distance walked*
1	$\frac{2}{3} = \frac{1+1}{2 \cdot 1+1}$
2	$\frac{3}{5} = \frac{1+2}{2 \cdot 2+1}$
3	$\frac{4}{7} = \frac{1+3}{2 \cdot 3+1}$
4	$\frac{5}{9} = \frac{1+4}{2 \cdot 4+1}$

It follows that on the tenth day they walked $\frac{1+10}{2 \cdot 10+1} = \frac{11}{21}$ of the trail.■

Problem 3.8
Patterns have been part of mathematics for a very long time. There are famous mathematicians who discovered patterns that are still used today. For example, Leonardo Fibonacci discovered the Fibonacci sequence. In this pattern, the first six numbers are: 1, 1, 2, 3, 5, 8. Work with a friend to find the next 5 numbers in this sequence. Write down the numbers that follow in the set and explain the pattern to your partner.

Solution.
We use "look for a pattern" strategy. We notice that starting from the second number, we see that a number on the list is the sum of the previous two numbers. Thus, we obtain the following list of Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \blacksquare$$

Problem 3.9
What is the units digit for 7^{3134}? (Hint: Work simpler problems to look for a pattern.)

Solution.
Let's first find some of the powers of 7.

$$\begin{array}{lcc} 7^2 & = 49 & \\ 7^3 & = & 343 \\ 7^4 & = & 2401 \\ 7^5 & = & 16807 \\ 7^6 & = & 117,649 \\ 7^7 & = & 823,543 \\ 7^8 & = & 5.764,801 \\ 7^9 & = & 40,353,607 \end{array}$$

Let x be the remainder of the division of the exponent by 4. Then we have the following.

x	*Units digit*
0	1
1	7
2	9
3	3

Since the remainder of the division of 3134 by 4 is 2 then the units digit of 7^{3134} is 9.■

Problem 3.10
Find these products: $7 \times 9, 77 \times 99, 777 \times 999$. Predict the product for $77,777 \times 99999$. What two numbers give a product of 77,762,223?

Solution.

$$\begin{array}{ccc} 7 \times 9 & = & 63 \\ 77 \times 99 & = & 7623(one\ 7, 6, 2, 3) \\ 777 \times 999 & = & 776223(two\ 7s, 6, two\ 2s, 3) \end{array}$$

From the above pattern we see that $77,777 \times 99,999 = 7777622223$. Also, $77762223 = 7777 \times 9999$.■

Problem 3.11
William is painting a design on a rug. He had time to paint a star, moon, sun, sun, moon, star, and moon before he had to quit. What shape will William paint next to finish the design?

Solution.
Assuming the pattern is star-moon-sun-sun-moon-star-moon then the next shape would be sun.■

Problem 3.12
Would you rather have \$100 a day for a month or \$1 on the first day and double it each day thereafter for a month?

Solution.
Suppose the month consists of 31 days. If you earn \$100 a day then at the

end of the month your income would be \$3,100. Let's say you choose the other option. Then we can construct the following table.

Day	*income*
1	1
2	$2 = 2^{2-1}$
3	$4 = 2^{3-1}$
4	$8 = 2^{4-1}$

Thus, by the end of the month your income would be $2^{31-1} = 2^{30}$ which is a huge number.■

Problem 3.13
Mrs. Lamp is sewing a quilt for her bed. She alternated the blocks in her quilt. One row in her quilt consists of 2 red blocks, 1 blue block, 2 yellow blocks, 1 green block, 2 orange blocks, 1 purple block. She started the second row and only made it to 1 yellow block. What color will come next in her quilt?

Problem 3.14
Take 25 marbles. Put them in 3 piles so an odd number is in each pile. How many ways can this be done?

Solution.
We use "make a list" strategy.

Pile 1	Pile 2	Pile 3
1	1	23
1	3	21
1	5	19
1	7	17
1	9	15
1	11	13
3	3	19
3	5	17
3	7	15
3	9	13
3	11	11
5	5	15
5	7	13
5	9	11
7	7	11
7	9	9

Problem 3.15

A rectangle has an area of 120 square centimeters. Its length and width are whole numbers. What are the possibilities for the two numbers? Which possibility gives the smallest perimeter?

Solution.

We use "make a list" strategy.

Length	Width	Perimeter
1	120	242
120	1	242
2	60	124
60	2	124
3	40	86
40	3	86
4	30	68
30	4	68
5	24	58
24	5	58
6	20	52
20	6	52
8	15	46
15	8	46
10	12	44
12	10	44

Problem 3.16
The product of two whole numbers is 96 and their sum is less than 30. What are possibilities for the two numbers?

Solution.
We use "make a list" strategy.

#1	#2
4	24
6	16
8	12

Problem 3.17
Lonnie has a large supply of quarters , dimes, nickels, and pennies. In how many ways could she make change for 50 cents?

Solution.
We use "make a list" strategy.

Pennies	Nickels	Dimes	Quarters
0	10	0	0
0	8	1	0
0	6	2	0
0	5	0	1
0	4	3	0
0	3	1	1
0	2	4	0
0	1	2	1
0	0	0	2
0	0	5	0
0	0	0	2
5	9	0	0
5	7	1	0
5	5	2	0
5	4	0	1
5	3	3	0
5	1	4	0
5	0	2	1
10	8	0	0
10	6	1	0
10	4	2	0
10	3	0	1
10	2	3	0
10	1	1	1
10	0	4	0
15	7	0	0
15	5	1	0
15	3	2	0
15	2	0	1
15	1	3	0
15	0	1	1
20	6	0	0
20	4	1	0
20	2	2	0
20	1	0	1
20	0	3	0
25	5	0	0
25	3	1	0

25	1	2	0
25	0	0	1
30	4	0	0
30	2	1	0
30	0	2	0
35	3	0	0
35	1	1	0
40	2	0	0
40	0	1	0
45	1	0	0
50	0	0	0

Problem 3.18
How many different four-digit numbers can be formed using the digits 1, 1, 9, and 9?

Solution.
We use "make a list" strategy.

Digit1	Digit2	Digit3	Digit4
1	1	1	1
1	1	1	9
1	1	9	1
1	1	9	9
1	9	1	1
1	9	1	9
1	9	9	1
1	9	9	9
9	1	1	1
9	1	1	9
9	1	9	1
9	1	9	9
9	9	1	1
9	9	1	9
9	9	9	1
9	9	9	9

Problem 3.19
Which is greater : \$5.00 or the total value of all combinations of three coins you can make using only pennies, nickels, dimes, and quarters?

Solution.
We use "make a list" strategy.

Pennies	Nickels	Dimes	Quarters	Sum(in $)
0	3	0	0	0.15
0	2	1	0	0.30
0	2	0	1	0.35
0	1	2	0	0.25
0	1	0	2	0.55
0	1	1	1	0.40
0	0	3	0	0.30
0	0	0	3	0.75
1	2	0	0	0.11
1	1	1	0	0.16
1	1	0	1	0.31
1	0	2	0	0.21
1	0	0	2	0.51
2	1	0	0	.07
2	0	1	0	0.12
2	0	0	1	0.27
3	0	0	0	0.03

The total sum is $4.84 ■

Problem 3.20
The Coffee Hut sold 5 small cups of coffee at $.75 each, 7 medium cups of coffee at $1.25 each, and 12 large cups of coffee at $1.50 each. What were the total sales of The Coffee Hut?

Solution.
The total sales were: $5 \times 0.75 + 7 \times 1.25 + 12 \times 1.50 = \$30.50$ ■

Problem 3.21
Chris decided to use his birthday money to buy some candy at The Sweet Shop. He bought 7 pieces of bubble gum for $.35 each, 3 candy bars for $1.25 each, and 2 bags of jellybeans for $3.35 each. How much money did Chris spend at The Sweet Shop?

Solution.
He spent: $7 \times 0.35 + 3 \times 1.25 + 2 \times 3.35 = \$12.90$ ■

Problem 3.22
Sean and Brad were at the candy store. Together they had \$15 total. They saw Gummie Worms that were \$3 per pound, War Heads were \$2 per pound and Lollie Pops were \$1 per pound. How many different combinations of candy could they buy for \$15.00?

Solution.
We use "make a list" strategy.

Gummie Worms	War Heads	Lollie Pops
0	0	15
0	1	13
1	0	12
0	2	11
0	3	9
2	0	9
0	4	7
2	1	7
3	0	6
0	5	5
2	2	5
1	4	4
3	1	4
0	6	3
4	0	3
2	3	3
1	5	2
3	2	2
0	7	1
2	4	1
4	1	1
1	6	0
3	3	0
5	0	0

Problem 3.23
Doug has 2 pairs of pants: a black pair and a green pair. He has 4 shirts: a white shirt, a red shirt, a grey shirt, and a striped shirt. How many different outfits can he put together?

Solution.
We use "make a list" strategy.

$$\begin{array}{cccc} BW & BR & BG & BS \\ GW & GR & GG & GS \blacksquare \end{array}$$

Problem 3.24
Ryan numbered his miniature race car collection according to the following rules:

1. It has to be a 3-digit number.
2. The digit in the hundreds place is less than 3.
3. The digit in the tens place is greater than 7.
4. The digit in the ones place is odd.

If Ryan used every possibility and each car had a different number, how many cars did Ryan have in his collection?

Solution.
We use "make a list" strategy.

Hundreds	Tens	Units
2	8	1
2	8	3
2	8	5
2	8	7
2	8	9
2	9	1
2	9	3
2	9	5
2	9	7
2	9	9
1	8	1
1	8	3
1	8	5
1	8	7
1	8	9
1	9	1
1	9	3
1	9	5
1	9	7
1	9	9
0	8	1
0	8	3
0	8	5
0	8	7
0	8	9
0	9	1
0	9	3
0	9	5
0	9	7
0	9	9

Problem 3.25
There will be 7 teams playing in the Maple Island Little League tournament. Each team is scheduled to play every other team once. How many games are scheduled for the tournament?

Solution.
We use "make a list" strategy.

$$\begin{array}{cccccc} T1T2 & T1T3 & T1T4 & T1T5 & T1T6 & T1T7 \\ & T2T3 & T2T4 & T2T5 & T2T6 & T2T7 \\ & & T3T4 & T3T5 & T3T6 & T3T7 \\ & & & T4T5 & T4T6 & T4T7 \\ & & & & T5T6 & T5T7 \\ & & & & & T6T7 \end{array}$$

Problem 3.26
How many different total scores could you make if you hit the dartboard shown with three darts?See Figure 3.3.

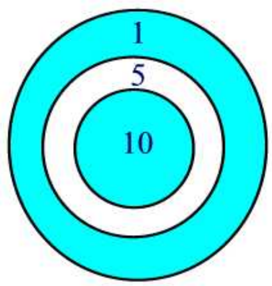


Figure 3.3

Solution.
We use "make a list" strategy.

1	5	10	Total
3	0	0	3
2	1	0	7
2	0	1	12
1	2	0	11
1	0	2	21
1	1	1	16
0	3	0	15
0	0	3	30
0	2	1	20
0	1	2	25

Problem 3.27
Sue and Ann earned the same amount of money, although one worked 6 days more than the other. If Sue earned \$36 per day and Ann earned \$60 per day, how many days did each work?

Solution.
We use "make a list" strategy.

# of Days	Sue	Ann
1	36	60
2	72	120
3	108	180
4	144	240
5	180	300
6	216	360
7	252	420
8	288	480
9	324	540
10	360	600
11	396	660
12	432	720
13	468	780
14	504	840
15	540	900

Thus Ann had to work 9 days and Sue had to work 15 days.■

Problem 3.28
A bank has been charging a monthly service fee of \$2 per checking accounts plus 15 cents for each check announces that it will change its monthly fee to \$3 and that each check will cost 8 cents. The bank claims the new plan will save the customer money. How many checks must a customer write per month before the new plan is cheaper than the old plan?

Solution.
We use "make a list" strategy.

# of Checks	Old Plan	New plan
1	2.15	3.08
2	2.30	3.16
3	2.45	3.24
4	2.60	3.32
5	2.75	3.40
6	2.90	3.48
7	3.05	3.56
8	3.20	3.64
9	3.35	3.72
10	3.50	3.80
11	3.65	3.88
12	3.80	3.96
13	3.95	4.04
14	4.10	4.12
15	4.25	4.20

Problem 3.29
Sasha and Francisco were selling lemonades for 25 cents per half cup and 50 cents per full cup. At the end of the day they had collected $15 and had used 37 cups. How many full cups and how many half cups did they sell?

Solution.

Half Cup	Full Cup	Total
0	37	18.50
1	36	18.25
2	35	18.00
3	34	17.75
4	33	17.50
5	32	17.25
6	31	17.00
7	30	16.75
8	29	16.50
9	28	16.25
10	27	16.00
11	26	15.75
12	25	15.50
13	24	15.25
14	23	15.00

Problem 3.30
Harold wrote to 15 people, and the cost of postage was \$4.08. If it cost 20 cents to mail a postcard and 32 cents to mail a letter, how many postcards did he write?

Solution.

# of Postcards	# of letters	Total Cost
1	14	4.68
2	13	4.56
3	12	4.44
4	11	4.32
5	10	4.20
6	9	4.08

Problem 3.31
I had some pennies, nickels, dimes, and quarters in my pocket. When I reached in and pulled out some change, I had less than 10 coins whose values was 42 cents. What are all the possibilities for the coins I had in my hand?

Solution.
Definitely you can have at most 1 quarter. If among the coins is a quarter then you are left with 17 cents, in this case, we have the following possibilities.

Pennies	Nickels	Dimes	Quarters	Total
2	3	0	1	42
2	1	1	1	42
7	2	0	1	42
7	0	1	1	42

If among the coins there is no quarter then at most you can have 4 dimes.

Pennies	Nickels	Dimes	Quarters	Total
2	0	4	0	42
2	2	3	0	42
2	4	2	0	42
2	6	1	0	42

Problem 3.32
There are 32 schools that participated in a statewide trivia tournament. In each round, one school played one match against another school and the winner continued on until 1 school remained. How many total matches were played?

Solution.
Let's construct the following table

# of Schools	4	8	16	32
# of matches	3	7	15	31

Note that the number of matches is the number of schools minus 1.■

Problem 3.33
A.J. plays baseball. There are 7 teams in his league. For the baseball season, each team plays each of the other teams twice. How many games are in a season?

Solution.
We construct the following table.

# of teams	2	3	4	5	6	7
# of games	2	6	12	20	30	42

Note that when we multiply two consecutive team numbers we obtain the number of games of the larger team. That is, $2 \times 3 = 6, 3 \times 4 = 12, 4 \times 5 = 20, etc.$■

Problem 3.34
A total of 28 handshakes were exchanged at a party. Each person shook hands exactly once with each of the others. How many people were present at the party?

Solution.
We construct the following table.

# of people	2	3	4	5	6
# of handshakes	1	3	6	10	15

Note that if n people were present then the number of handshakes is

$$1 + 2 + 3 + \cdots + n - 1.$$

This is an arithmetic sum which we encountered before. Its value is $\frac{n(n-1)}{2}$. We want to find the number of peaple with 28 handshakes. That is the value of n such that $\frac{n(n-1)}{2} = 28$. This implies that $n(n-1) = 56$. By guessing and checking we find $n = 8.\blacksquare$

Problem 3.35
Mike is paid for writing numbers on pages of a book. Since different pages require different numbers of digits, Mike is paid for writing each digit. In his last book, he wrote 642 digits. How many pages were in the book?

Solution.
We construct the following table.

# of pages	9	99	199	249	250
# of digits	9	189	489	639	642

Problem 3.36
A restaurant has 45 small square tables. Each table can seat only one person on each side. If the 45 tables are placed together to make one long table, how many people can sit there?

Solution.
We construct the following table.

# of tables	2	3	4	5	6
# of seats	6	8	10	12	14

Note that the number of seats is obtained by multiplying the number of tables by 2 and then increasing the reult by 2. Thus, with 45 tables we obtain $2(45) + 2 = 92$ seats.■

Problem 3.37
Drewby the goat loves green. Everything he has is green. He just built a brick wall and he's going to paint it green. The wall has 14 bricks across and is eleven bricks high. He is going to paint the front and back walls, and the sides that you can see. He is not going to paint the sides that touch one another. How many sides of the bricks will Drewby paint?

Solution.
We construct the following table.

Dimension	14×1	14×2	14×3	14×4	14×5
# of painted sides	44	72	104	134	164

Note that if the dimension is $14 \times n$ then the number of sides painted is $(2n + 1) \times 14 + 2n$. Thus, for 14×11 we obtain $23 \times 14 + 22 = 344$ sides ■

Problem 3.38
Three shapes-a circle, a rectangle, and a square-have the same area. Which shape has the smallest perimeter?

Solution.
Suppose the square has side 1. Then the rectangle can be of dimensions $2 \times \frac{1}{2}$. The circle must have radius $\frac{1}{\sqrt{\pi}}$. In this case, the perimeter of the square is 4, that of the rectangle is 5, and the circle is $2\sqrt{\pi}$. So the square has the smallest perimeter.■

Problem 3.39
How many palindromes are there between 0 and 1000? (A palindrome is a number like 525 that reads the same backward or forward.)

Solution.
We construct the following table.

Range	0 - 9	10 - 99	100 - 199	200 - 299	300 - 399
# of palindromes	10	9	10	10	10

Continuing in the table we find that the total number of palindromes between 0 and 1000 is

$$10 + 9 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 = 109. \blacksquare$$

Problem 3.40
Tony's restaurant has 30 small tables to be used for a banquet. Each table can seat only one person on each side. If the tables are pushed together to make one long table, how many people can sit at the table?

Solution.
See Problem 3.36 $\blacksquare$

Problem 4.1
Write a verbal description of each set.
(a) $\{4, 8, 12, 16, \cdots\}$
(b) $\{3, 13, 23, 33, \cdots\}$

Solution.
(a) {all counting numbers that are multiples of 4}
(b) {all counting numbers of the form $10n + 3$ where $0, 1, 2, \cdots$ } $\blacksquare$

Problem 4.2
Which of the following would be an empty set?
(a) The set of purple crows.
(b) The set of odd numbers that are divisible by 2.

Solution.
(a) Since there is no such thing as a purple crow then the set is empty set.
(b) Since only even numbers can be divisible by 2 then the given set is the empty set. $\blacksquare$

Problem 4.3
What two symbols are used to represent an empty set?

Solution.
The symbols $\emptyset$ and $\{\}$. $\blacksquare$

Problem 4.4
Each set below is taken from the universe $\mathbb{N}$ of counting numbers, and has been described either in words, by listing in braces, or with set-builder notation. Provide the two remaining types of description for each set.
(a) The set of counting numbers greater than 12 and less than 17
(b) $\{x|x = 2n \text{ and } n = 1, 2, 3, 4, 5\}$
(c) $\{3, 6, 9, 12, \cdots\}$

Solution.
(a) Roster notation:{13,14,15,16}. Roster notation:{n—$n \in \mathbb{N}$ and $12 < n < 17$}.
(b) Roster notation: $\{2, 4, 6, 8, 10\}$. Verbal Description: The set of the first five even counting numbers.
(c) Set-builder notation: {x—x=3n and $n \in \mathbb{N}$}. Verbal description: The set of all counting numbers that are multiples of 3. ■

Problem 4.5
Rewrite the following using mathematical symbols:

(a) P is equal to the set whose elements are a, b, c, and d.
(b) The set consisting of the elements 1 and 2 is a proper subset of $\{1, 2, 3, 4\}$.
(c) The set consisting of the elements 0 and 1 is not a subset of $\{1, 2, 3, 4\}$.
(d) 0 is not an element of the empty set.
(e) The set whose only element is 0 is not equal to the empty set.

Solution.
(a) $P = \{a, b, c, d\}$
(b) $\{1, 2\} \subset \{1, 2, 3, 4\}$
(c) $\{0, 1\} \not\subset \{1, 2, 3, 4\}$
(d) $0 \notin \emptyset$
(e) $\{0\} \neq \emptyset$. ■

Problem 4.6
Which of the following represent equal sets?

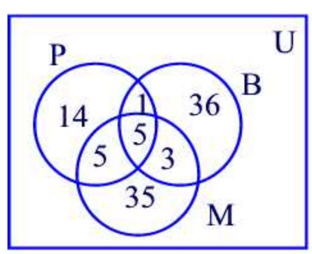
$$
\begin{array}{rclcrcr}
A & = & \{a,b,c,d\} & & B & = & \{x,y,z,w\} \\
C & = & \{c,d,a,b\} & & D & = & \{x \in \mathbb{N} | 1 \le x \le 4\} \\
E & = & \emptyset & & F & = & \{\emptyset\} \\
G & = & \{0\} & & H & = & \{\} \\
I & = & \{2n+1 | n \in W\} & where & W & = & \{0,1,2,3,\cdots\} \\
J & = & \{2n-1 | n \in \mathbb{N}\} & & & &
\end{array}
$$

Solution.
A = C; F=H; I = J. ■

Problem 4.7
In a survey of 110 college freshmen that investigated their high school backgrounds, the following information was gathered:
25 students took physics
45 took biology
48 took mathematics
10 took physics and mathematics
8 took biology and mathematics
6 took physics and biology
5 took all 3 subjects.

(a) How many students took biology but neither physics nor mathematics?
(b) How many students took biology, physics or mathematics?
(c) How many did not take any of the 3 subjects?

Solution.
We construct the following Venn diagram

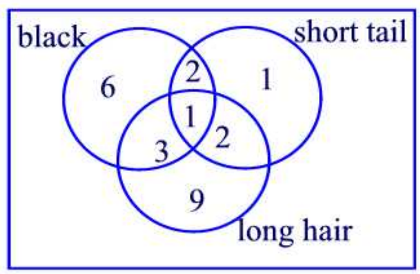


(a) 36 students too biology but neither physics or mathematics.
(b) The number of students who took physics, mathematics, or biology is:36

+ 1 + 3 + 5 + 5 + 24 + 35 = 99.
(c) 110 - 99 - 11 did not take any of the three subjects.■

Problem 4.8
Twenty-four dogs are in a kennel. Twelve of the dogs are black, six of the dogs have short tails, and fifteen of the dogs have long hair. There is only one dog that is black with a short tail and long hair. Two of the dogs are black with short tails and do not have long hair. Two of the dogs have short tails and long hair but are not black. If all of the dogs in the kennel have at least one of the mentioned characteristics, how many dogs are black with long hair but do not have short tails?Hint: Use Venn diagram.

Solution.
We construct the following Venn diagram.



According to this diagram, there must be 3 black dogs with long hair but do not have short tails.■

Problem 4.9
True or false?
(a) $7 \in \{6,7,8,9\}$ (b) $\frac{2}{3} \in \{1,2,3\}$
(c) $5 \notin \{2,3,4,6\}$ (d) $\{1,2,3\} \subseteq \{1,2,3\}$
(e) $\{1,2,5\} \subset \{1,2,5\}$ (f) $\emptyset \subseteq \{\}$
(g) $\{2\} \nsubseteq \{1,2\}$ (h) $\{1,2\} \nsubseteq \{2\}$.

Solution.
(a) True (b) False (c) True (d) True (e) False (f) True (g) False (h) True.■

Problem 4.10
Which of the following sets are equal?

(a) $\{5, 6\}$ (b) $\{5, 4, 6\}$
(c) Whole numbers greater than 3
(d) Whole numbers less than 7
(e) Whole numbers greater than 3 or less than 7
(f) Whole numbers greater than 3 and less than 8
(g) $\{e, f, g\}$
(h) $\{4, 5, 6, 5\}$

Solution.
(b) = (e) = (h) ■

Problem 4.11
Let $A = \{1, 2, 3, 4, 5\}, B = \{3, 4, 5\}$, and $C = \{4, 5, 6\}$. In the following insert $\in, \notin, \subseteq$, or $\not\subseteq$ to make a true statement.

(a) 2_____A (b) B_____A (c)C_____B (d) 6_____C.

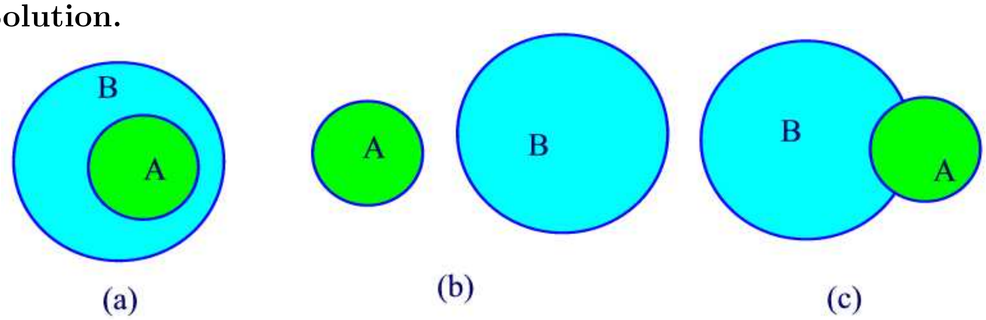
Solution.
(a) $2 \in A$ (b) $B \subseteq A$ (c) $C \not\subseteq B$ (d) $6 \in C$. ■

Problem 4.12
Rewrite the following expressions using symbols.
(a) A is a subset of B.
(b) The number 2 is not an element of set T.

Solution.
(a) $A \subseteq B$
(b) $2 \notin T$■

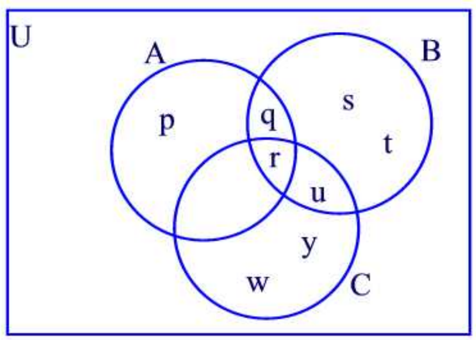
Problem 4.13
Draw Venn diagrams that represent sets A and B as described as follows:
(a) $A \subset B$ (b) $A \cap B = \emptyset$ (c) $A \cap B \neq \emptyset$.

Solution.



Problem 4.14
Let $U = \{p, q, r, s, t, u, v, w, x, y\}$ be the universe, and let $A = \{p, q, r\}, B = \{q, r, s, t, u\}$, and $C = \{r, u, w, y\}$. Locate all 10 elements of U in a three-loop Venn diagram, and then find the following sets:
(a) $A \cup C$ (b) $A \cap C$ (c) $\overline{B}$ (d) $A \cup \overline{B}$ (e) $A \cap \overline{C}$.

Solution.
(a) $A \cup C = \{p, q, r, u, w, y\}$
(b) $A \cap C = \{r\}$
(c) $\overline{B} = \{p, v, w, x, y\}$
(d) $A \cup \overline{B} = \{p, r, q, v, w, x, y\}$
(e) $A \cap \overline{C} = \{p, q, r\} \cap \{p, q, s, t, v, x\} = \{p, q\}$.■



Problem 4.15
If S is a subet of universe U, find each of the following:
(a) $S \cup \overline{S}$ (b) $\emptyset \cup S$ (c) $\overline{U}$ (d) $\overline{\emptyset}$ (e) $S \cap \overline{S}$.

Solution.
(a) $S \cup \overline{S} = U$
(b) $\emptyset \cup S = S$
(c) $\overline{U} = \emptyset$
(d) $\overline{\emptyset} = U$ (e) $S \cap \overline{S} = \emptyset$ ■

Problem 4.16
Answer each of the following:
(a) If A has five elements and B has four elements, how many elements are in $A \times B$?
(b) If A has m elements and B has n elements, how many elements are in $A \times B$?

Solution.
(a) An element in $A \times B$ is of the form (a, b) where $a \in A$ and $b \in B$. There are five possibilities for a. For each such possibility you have four possibilities for b. A total of $5 \times 4 = 20$ elements of $A \times B$
(b) Using the argument of part (a), there are $m \cdot n$ elements in $A \times B$ ■

Problem 4.17
Find A and B given that

$$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}.$$

Solution.
$A = \{1, 2\}$ and $B = \{1, 2, 3\}$■

Problem 4.18
Let $A = \{x, y\}, B = \{a, b, c\}$, and $C = \{0\}$. Find each of the following:
(a) $A \times B$
(b) $B \times \emptyset$
(c) $(A \cup B) \times C$
(d) $A \cup (B \times C)$.

Solution.
(a) $A \times B = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\}$
(b) $B \times \emptyset = \emptyset$
(c) $(A \cup B) \times C = \{x, y, a, b, c\} \times \{0\} = \{(x, 0), (y, 0), (a, 0), (b, 0), (c, 0)\}$
(d) $A \cup (B \times C) = \{x, y\} \cup \{(a, 0), (b, 0), (c, 0)\} = \{x, y, (a, 0), (b, 0), (c, 0)\}$■

Problem 4.19
For each of the following conditions, find $A - B$:

(a) $A \cap B = \emptyset$ (b) $B = U$ (c) $A = B$ (d) $A \subseteq B$.

Solution.
(a) $A - B = A$ since A and B have nothing in common so all the elements of A are still in A.
(b) $A - B = \emptyset$ since $A \subseteq U = B$.
(c) $A - B = \emptyset$
(d) $A - B = \emptyset$■

Problem 4.20
If $B \subseteq A$, find a simpler expression for each of the following:

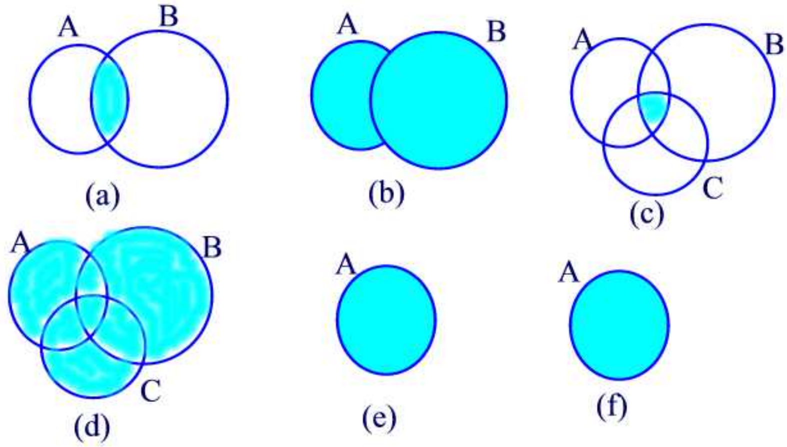
(a) $A \cap B$ (b) $A \cup B$ (c) $B - A$ (d) $B \cap \overline{A}$.

Solution.
(a) $A \cap B = B$
(b) $A \cup B = A$
(c) $B - A = \emptyset$
(d) $B \cap \overline{A} = \emptyset$ ■

Problem 4.21
Use a Venn diagram to decide whether the following pairs of sets are equal.

(a) $A \cap B$ and $B \cap A$
(b) $A \cup B$ and $B \cup A$
(c) $A \cap (B \cap C)$ and $(A \cap B) \cap C$
(d) $A \cup (B \cup C)$ and $(A \cup B) \cup C$
(e) $A \cup \emptyset$ and A
(f) $A \cup A$ and $A \cup \emptyset$.

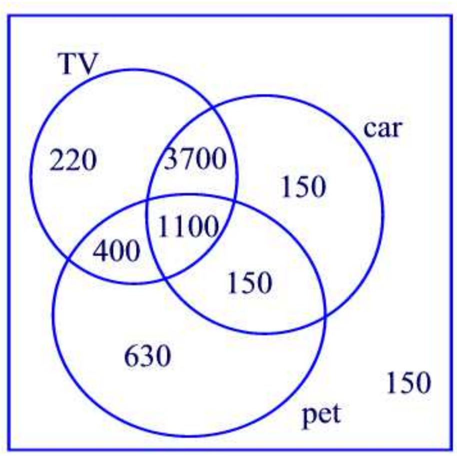
Solution.
(a) $A \cap B = B \cap A$
(b) $A \cup B = B \cup A$
(c) $A \cap (B \cap C) = (A \cap B) \cap C$
(d) $A \cup (B \cup C) = (A \cup B) \cup C$
(e) $A \cup \emptyset = A$
(f) $A \cup A = A \cup \emptyset$■



Problem 4.22
In a survey of 6500 people, 5100 had a car, 2280 had a pet, 5420 had a television set, 4800 had a TV and a car, 1500 had a TV and a pet, 1250 had a car and a pet, and 1100 had a TV, a car, and a pet.

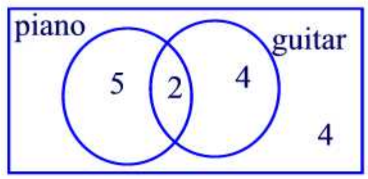
(a) How many people had a TV and a pet, but did not have a car?
(b) How many people did not have a pet or a TV or a car?

Solution.
See the Venn diagram below.
(a) 400 had a TV and a pet but with no car.
(b) 150 did not have a pet, a TV, or a car.■

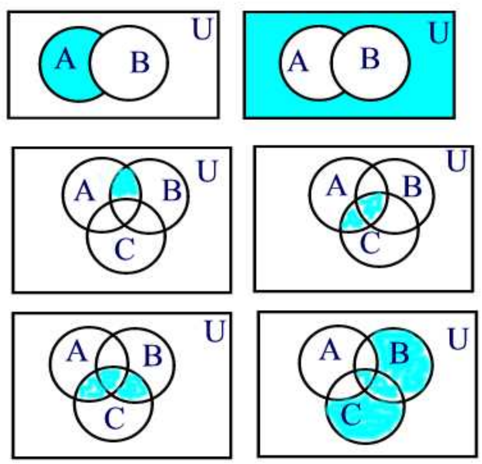


Problem 4.23
In a music club with 15 members, 7 people played piano, 6 people played guitar, and 4 people didn't play either of these instruments. How many people played both piano and guitar?

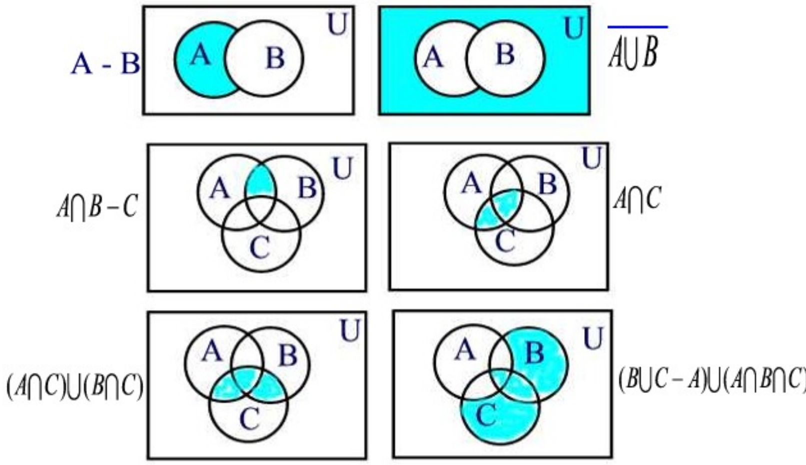
Solution.
According to the Venn diagram 2 play both piano and guitar■



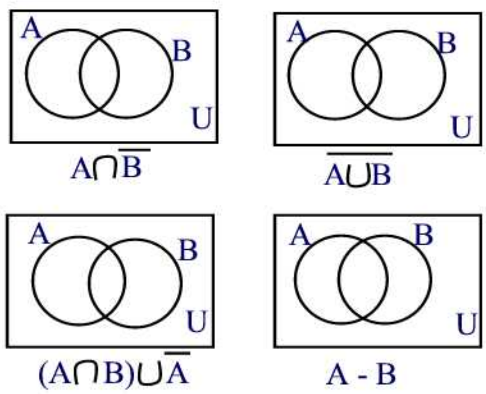
Problem 4.24
Use set notation to identify each of the following shaded region.



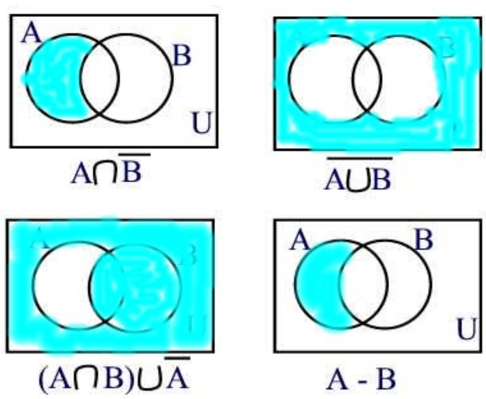
Solution.



Problem 4.25
In the following, shade the region that represents the given sets:



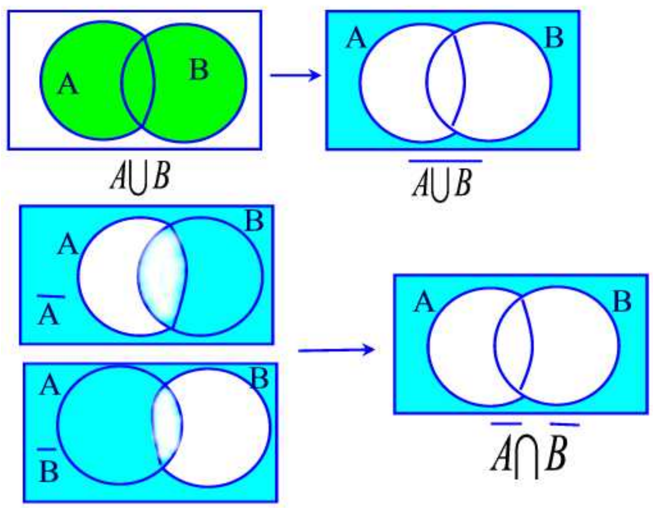
Solution.



Problem 4.26
Use Venn diagrams to show:

(a) $\overline{A \cup B} = \overline{A} \cap \overline{B}$
(b) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution.
We will do (a) and leave (b) for the reader.



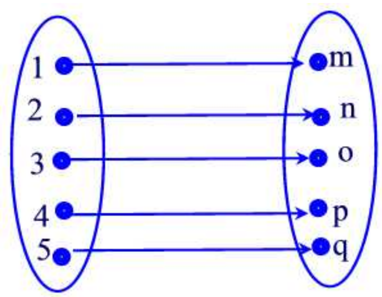
Problem 4.27
Let $G = \{n \in \mathbb{N} | n \text{ divides } 90\}$ and $D = \{n \in \mathbb{N} | n \text{ divides } 144\}$. Find $G \cap D$ and $G \cup D$.

Solution.
We have
$$G = \{1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90\}$$
and
$$D = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144\}$$
Thus,
$$G \cap D = \{1, 2, 3, 6, 9, 18\} \blacksquare$$

Problem 4.28
Which of the following pairs of sets can be placed in one-to-one correspondence?

(a) $\{1, 2, 3, 4, 5\}$ and $\{m, n, o, p, q\}$.
(b) $\{m, a, t, h\}$ and $\{f, u, n\}$.
(c) $\{a, b, c, d, e, f, \cdots, m\}$ and $\{1, 2, 3, \cdots, 13\}$.
(d) $\{x|x \textit{ is a letter in the word mathematics}\}$ and $\{1, 2, 3, \cdots, 11\}$.

Solution.
(a) The figure below illustrates a one-to-one correspondence between the given two sets.



(b) Since the set $\{m, a, t, h\}$ has four elements and the set $\{f, u, n\}$ has three elements then at least two of the elements of the first set share the same elements in $\{f, u, n\}$. According to the definition of one-to-one correspondence the given sets are not equaivalent.
(c) We can assign $a \to 1, b \to 2, c \to 3, \cdots, m \to 13$ so that the two sets are equivalent.
(c) The set $\{x|x \textit{ is a letter in the word mathematics}\} = \{m, a, t, h, e, i, c, s\}$ has eight elements. If no two letters are assigned the same number in $\{1, 2, 3, \cdots, 11\}$ then three elements in that set will be left without a preimage. Thus, these two sets are not equavalent.■

Problem 4.29
How many one-to-one correspondence are there between the sets $\{x, y, z, u, v\}$ and $\{1, 2, 3, 4, 5\}$ if in each correspondence

(a) x must correspond to 5?
(b) x must correspond to 5 and y to 1?
(c) x, y, and z correspond to odd numbers?

Solution.
(a) Suppose x is assigned the value 5. Then there are 4 possibilities to assign a value to y. One a value is assigned to y then there are 3 possibilities to assign a value to z, continuing there are two possibilities to assign a value to u and 1 possibility for v. Thus, the total number of one-to-one correspondence is $4 \cdot 3 \cdot 2 \cdot 1 = 24$.
(b) In this case, there are three possibilities to assign a value to z, two possibilities for u, and one possibility for v. Hence, the total number of one-to-one correspondence is $3 \cdot 2 \cdot 1 = 6$.
(c) There are three possibilities to assign an odd number to x. Then two possibilities to assign an odd number to y and one possibility to assign an odd number to z. Moreover, there are two possibilities to assign an even number to u and one possibility for v. Hence, the total number of one-to-one correspondence is $3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 = 12$■

Problem 4.30
True or false?

(a) The set $\{105, 110, 115, 120, \cdots\}$ is an infinite set.
(b) If A is infinite and $B \subseteq A$ then B is also infinite.
(c) For all finite sets A and B if $A \cap B = \emptyset$ then the number of elements in A plus the number of elements in B is equal to the number of elements in $A \cup B$.

Solution.
(a) There is a one-to-one correspondence between the given set and the set of counting numbers given by $105 \to 1, 110 \to 2, 115 \to 3, \cdots$. Since $\mathbb{N}$ is infinite then the given set is also infinte.
(b) False. The set $\{1, 2, 3\}$ is a finite subset of the infinite set $\mathbb{N} = \{1, 2, 3, \cdots\}$.
(c) For any two finite sets A and B the number of elements of $A \cup B$ is the number

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

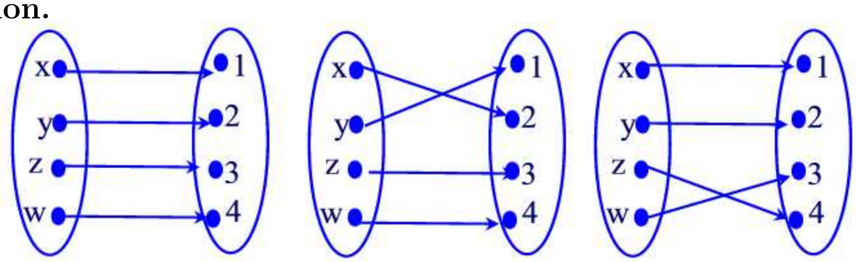
where $n(A)$ denotes the number of elements in the set A. In particular, if $A \cap B = \emptyset$ then $n(A \cap B) = 0$ and in this case

$$n(A \cup B) = n(A) + n(B).$$

Thus, the given statement is true.■

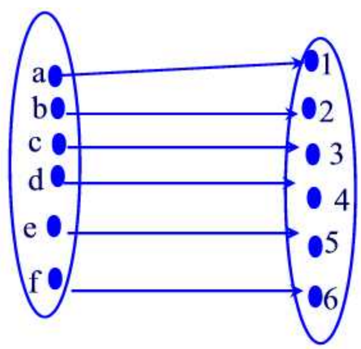
Problem 4.31
Show three different one-to-one correspondence between the sets $\{1, 2, 3, 4\}$ and $\{x, y, z, w\}$.

Solution.



Problem 4.32
Write a set that is equivalent but not equal to the set $\{a, b, c, d, e, f\}$.

Solution.



Problem 4.33
Determine which of the following sets are finite. For those sets that are finite, how many elements are in the set?

(a) $\{ears\ on\ a\ typical\ elephant\}$
(b) $\{1, 2, 3, \cdots, 99\}$
(c) Set of points belonging to a line segment.
(d) A closed interval.

Solution.
(a) A finite set with two elements.
(b) A finite set with 99 elements.

(c) A line segement is an infinte set. It consists of infinite number of points.
(d) A closed interval is in infinte set. For example, take the interval, [1,2]. Then all the numbers $1, 1.1, 1.11, 1.111, \cdots$ belong to this interval.$\blacksquare$

Problem 4.34
Decide whether each set is finite or infinite.

(a) the set of people named Lucky.
(b) the set of all perfect square numbers.

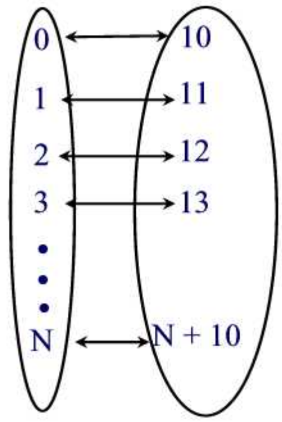
Solution.
(a) Finite set.
(b) $\{1, 2^2, 3^2, 4^2, \cdots\}$ is an infinite set.$\blacksquare$

Problem 4.35
How many one-to-one correspondence are possible between each of the following pairs of sets?

(a) Two sets, each having two elements (b) Two sets, each having three elements (c) Two sets, each having four elements (d) Two sets, each having N elements.

Solution.
(a) Consider the sets $\{a, b\}$ and $\{1, 2\}$. There are two possibilities to assign the a number to a. Once we assign a value for a then there is one possibility to assign a number to b. Hence, the total number of one-to-one correspondence is $2 \cdot 1 = 2$.
(b) Arguing as in part (a) we find $3 \cdot 2 \cdot 1 = 6$ one-to-one correspondence.
(c) $4 \cdot 3 \cdot 2 \cdot 1 = 24$ one-to-one correspondence.
(d) $N \cdot (N-1) \cdot (N-2) \cdots 3 \cdot 2 \cdot 1$ one-to-one correspondence.$\blacksquare$

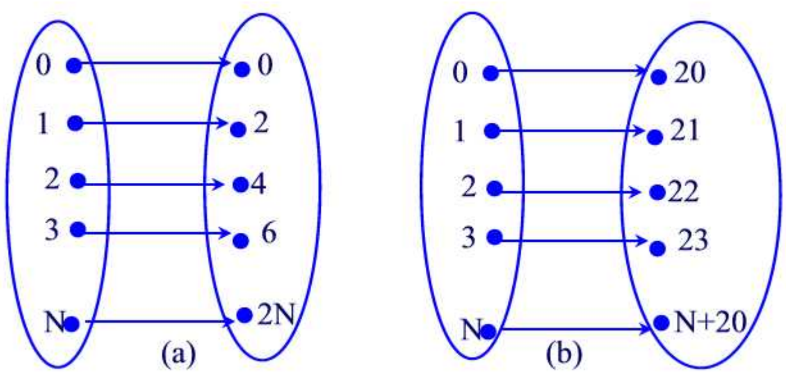
Problem 4.36
A set A is infinite if it can be put into a one-to-one correspondence with a proper subset of itself. For example, the set $W = \{0, 1, 2, 3, \cdots\}$ of whole numbers is infinite since it can be put in a one-to-one correspondence with its proper subset $\{10, 11, 12, \cdots\}$ as shown in the figure below.



Show that the following sets are infinite:

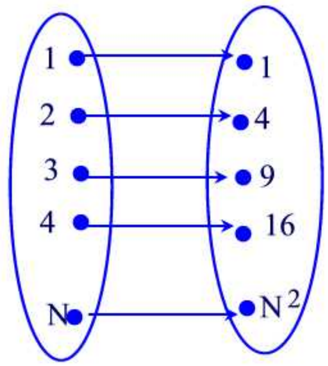
(a) $\{0, 2, 4, 6, \cdots\}$
(b) $\{20, 21, 22, \cdots\}$.

Solution.



Problem 4.37
Show that $\mathbb{N}$ and $S = \{1, 4, 9, 16, 25, \cdots\}$ are equivalent.

Solution.



Problem 5.1
Let A, B, and C be three sets such that $A \subset B \subseteq C$ and $n(B) = 5$.

(a) What are the possible values of $n(A)$?
(b) What are the possible values of $n(C)$?

Solution.
(a) $n(A) \in \{0, 1, 2, 3, 4\}$
(b) $n(C) \geq 5$■

Problem 5.2
Determine the cardinality of each of the following sets:

(a) $A = \{x \in \mathbb{N} | 20 \leq x < 35\}$
(b) $B = \{x \in \mathbb{N} | x + 1 = x\}$
(c) $C = \{x \in \mathbb{N} | (x - 3)(x - 8) = 0\}$.

Solution.
(a) $A = \{20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34\}$ so that $n(A) = 15$.
(b) There is no counting number that satisfies the equation $x + 1 = x$ so that $B = \emptyset$. Hence, $n(B) = 0$.
(c) The only two solutions to the equation $(x - 3)(x - 8) = 0$ are 3 and 8 so that $C = \{3, 8\}$. Hence, $n(C) = 2$.■

Problem 5.3
Let A and B be finite sets.

(a) Explain why $n(A \cap B) \leq n(A)$. (b) Explain why $n(A) \leq n(A \cup B)$.

Solution.
(a) The set $A \cap B$ is contained in the set A so that the number of elements in $A \cap B$ cannot exceed that of A. That is, $n(A \cap B) \leq n(A)$.
(b) The set A is contained in the set $A \cup B$ so that the number of elements in A cannot exceed that of $A \cup B$. That is, $n(A) \leq n(A \cup B)$■

Problem 5.4
Suppose B is a proper subset of C.

(a) If $n(C) = 8$, what is the maximum number of elements in B?
(b) What is the least possible elements of B?

Solution.
(a) If $A \subset C$ and $n(C) = 8$ then the maximum number of elements in B is 7.
(b) The least possible elements of B is zero. This occurs when $B = \emptyset$■

Problem 5.5
Suppose C is a subset of D and D is a subset of C.

(a) If $n(C) = 5$, find $n(D)$.
(b) What other relationships exist between C and D?

Solution.
(a) $n(D) = 5$.
(b) Since $C \subseteq D$ and $D \subseteq C$ then $C = D$■

Problem 5.6
Use the definition of less than to show each of the following:

(a) $2 < 4$ (b) $3 < 100$ (c) $0 < 3$.

Solution.
(a) Let $A = \{a, b\}$ and $B = \{a, b, c, d\}$. Then $n(A) = 2$ and $n(B) = 4$. Since

$A \subset B$ then $n(A) < n(B)$, i.e. 2¡4.
(b) Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, \cdots, 100\}$. Then $n(A) = 3$ and $n(B) = 100$. Since $A \subset B$ then $n(A) < n(B)$, i.e. $3 < 100$.
(c) Let $A = \emptyset$ and $B = \{1, 2, 3\}$. Then $n(A) = 0$ and $n(B) = 3$. Since $A \subset B$ then $n(A) < n(B)$, i.e. $0 < 3$■

Problem 5.7
If $n(A) = 4, n(B) = 5$, and $n(C) = 6$, what is the greatest and least number of elements in

(a) $A \cup B \cup C$ (b) $A \cap B \cap C$?

Solution.
(a) If $A \subset B \subset C$ then $n(A \cup B \cup C) = 6$. If $A \cap B \cap C = \emptyset$ then $n(A \cup B \cup C) = 15$. Thus, $6 \leq n(A \cup B \cup C) \leq 15$
(b) If $A \cap B \cap C = \emptyset$ then $n(A \cap B \cap C) = 0$. If $A \subset B \subset C$ then $n(A \cap B \cap C) = n(A) = 4$. Thus, $0 \leq n(A \cap B \cap C) \leq 4$■

Problem 5.8
True or false?If false give a counter example, i.e. an example that shows that the statement is false.

(a) If $n(A) = n(B)$ then $A = B$.
(b) If $n(A) < n(B)$ then $A \subset B$.

Solution.
(a) False. Let $A = \{1, 2\}$ and $B = \{a, b\}$. Then $n(A) = n(B)$ but $A \neq B$.
(b) False. Let $A = \{1\}$ and $B = \{a, b\}$. Then $n(A) < n(B)$ but $A \not\subset B$■

Problem 5.9
Suppose $n(A \cup B) = n(A \cap B)$. What can you say about A and B?

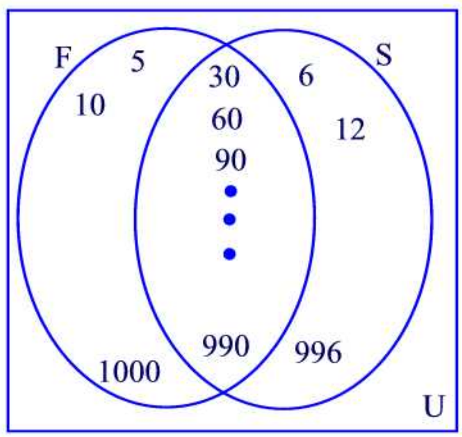
Solution.
Since $A \subseteq A \cup B$ then $n(A) \leq n(A \cup B)$. Similarly, since $A \cap B \subseteq A$ then $n(A \cap B) \leq n(A)$. It follows that $n(A \cap B) \leq A \leq n(A \cup B)$. So if $n(A \cap B) = n(A \cup B)$ then we must have $n(A) = n(A \cup B) = n(A \cap B)$. This implies that $A \subseteq B$■

Problem 5.10
Let $U = \{1, 2, 3, \cdots, 1000\}$, F be a subset of U consisting of multiples of 5 and S the subset of U consisting of multiples of 6.

(a) Find $n(S)$ and $n(F)$.
(b) Find $n(F \cap S)$.
(c) Label the number of elements in each region of a two-loop Venn diagram with universe U and subsets S and F.

Solution.
(a) Since $1000 \div 5 = 200$ then $F = \{5.\cdot 1, 5\cdot 2, \cdots, 5\cdot 200\}$. Thus, $n(F) = 200$. Similarly, since $1000 \div 6 = 166.66\cdots$ then $S = \{6\cdot 1, 6\cdot 2, \cdots, 6\cdot 166\}$ so that $n(S) = 166$.
(b) Since $F \cap S = \{30 \cdot 1, 30 \cdot 2, 30 \cdot 3, \cdots, 30 \cdot 33\}$ so that $n(F \cap S) = 33$.

(c)



Problem 5.11
Finish labeling the number of elements in the regions in the Venn diagram shown, where the subsets A, B, and C of the universe U satisfy the conditions listed. See Figure 5.2.

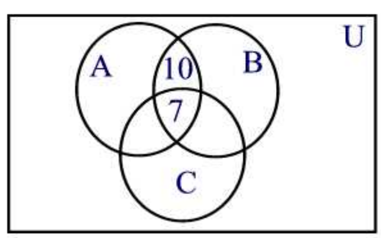
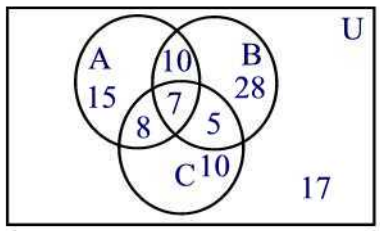


Figure 5.2

$$\begin{array}{rclrcl} n(U) & = & 100 & n(A) & = & 40 \\ n(B) & = & 50 & n(C) & = & 30 \\ n(A\cap B) & = & 17 & n(B\cap C) & = & 12 \\ n(A\cap C) & = & 15 & n(A\cap B\cap C) & = & 7 \end{array}$$

Solution.



Problem 5.12
Let $S = \{s, e, t\}$ and $T = \{t, h, e, o, r, y\}$. Find $n(S), n(T), n(S \cup T), n(S \cap T), n(S \cap \overline{T})$, and $n(\overline{S} \cap T)$.

Solution.
We have $n(S) = 3, n(T) = 6$. Since $S \cup T = \{s, e, t, h, o, r, y\}$ then $n(S \cup T) = 7$. Since $S \cap T = \{e, t\}$ then $n(S \cap T) = 2$. Since U is the universe of the english alphabet then $S \cap \overline{T} = \{s\}$ so that $n(S \cap \overline{T}) = 1$. Similarly, $\overline{S} \cap T = \{h, o, r, y\}$ so that $n(\overline{S} \cap T) = 4$■

Problem 5.13
Suppose that $n(A) = m$ and $n(B) = n$. Find $n(A \times B)$.

Solution.
An element of $A \times B$ is of the form (a, b) where $a \in A$ and $b \in B$. For each fixed a there are n possibilities for b. Since there are n possible values for a then the total number of ordered pairs (a, b) is $m \cdot n$. That is, $n(A \times B) = m \cdot n$■

Problem 5.14
Explain why $5 < 8$ using the definition of whole number inequality introduced in this section.

Solution.
Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Then $A \subset B$ and $n(A) < n(B)$. But $n(A) = 5$ and $n(B) = 8$ so that $5 < 8$■

Problem 5.15
Let A and B be two sets in the universe $U = \{a, b, c, \cdots, z\}$. If $n(A) = 12, n(B) = 14$, and $n(A \cup B) = 21$, find $n(A \cap B)$ and $n(A \cap \overline{B})$.

Solution.
Since $A \cup B$ consists of the elements of A and B with the common elements counted once then

$$n(A \cap B) = n(A) + n(B) - n(A \cup B) = 12 + 14 - 21 = 5.$$

Since $n(A) = 12$ with 5 of them belonging to B then only 7 does not belong to B. Thus, $n(A \cap \overline{B}) = 7$■

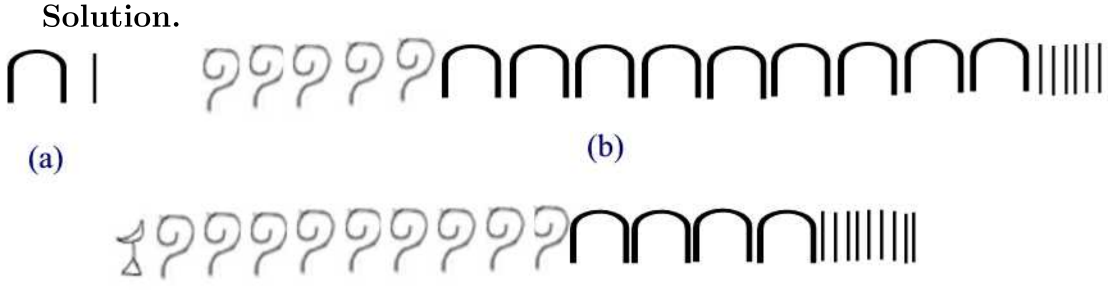
Problem 5.16
Suppose that $n(A \times B) = 21$. What are all the possible values of $n(A)$?

Solution.
The possible values of $n(A)$ are: 1, 3,7,21■

Problem 5.17
Write the following in Egyptian system.

(a) 11 (b) 597 (c) 1949.

Solution.



(a) (b) (c)

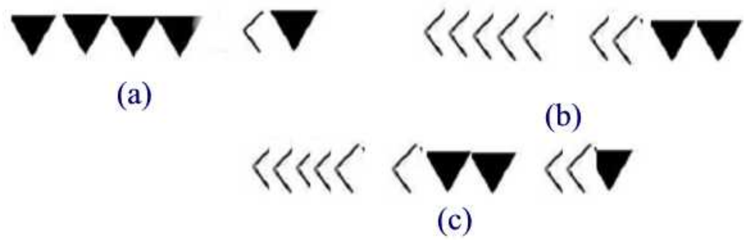
Problem 5.18
Write the following Roman notation using the subtraction principle as appropriate.

(a) 9 (b) 486 (c) 1945.

Solution.
(a) IX
(b) CDXXCVI
(c) MCMXLV ■

Problem 5.19
Write the following numbers in Babylonian system.

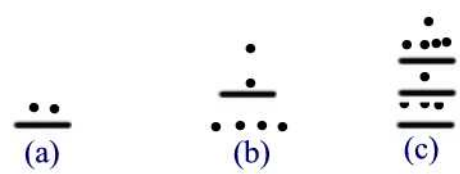
(a) 251 (b) 3022 (c) 18,741.

Solution.


Problem 5.20
Write the following numbers in Mayan System.

(a) 12 (b) 584 (c) 12,473.

Solution.
Note that $584 = 1 \cdot 18 \cdot 20 + 11 \cdot 20 + 4$ and $12473 = 1 \cdot 18 \cdot 20^2 + 14 \cdot 18 \cdot 20 + 11 \cdot 20 + 13$■



Problem 5.21
Write 2002, 2003, and 2004 in Roman numerals.

Solution.
2002 = MMII ; 2003 = MMIII; 2004=MMIV■

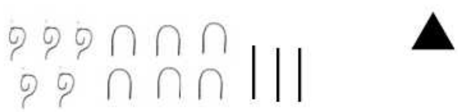
Problem 5.22
If the cornerstone represents when a building was built and it reads MCMXXII, when was this building built?

Solution.
MCMXXII = 1922■

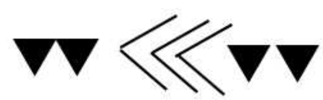
Problem 5.23
Write each of the following numbers in our numeration system, i.e. base 10.

(a) MDCCXXIX
(b) DCXCVII
(c) CMLXXXIV

(d) 

(e) 

(f) 

(g) 
(h) 
(i) 
(j) 

Solution.
(a) MDCCXXIX = 1000 + 500 + 200 + 20 + 9 = 1729
(b) DCXCVII = 500 + 100 + 90 + 7 = 697
(c) CMLXXXIV = 900 + 60 + 20 + 4 = 984■

Problem 5.24
Convert the Roman numeral DCCCXXIV to Babylonian numeral.

Solution.
First we convert to our numeration system: DCCCXXIV=500 + 300 + 24=824. Next we convert this to Babylonian numeral.

Problem 5.25
Write the following numbers in the given system.

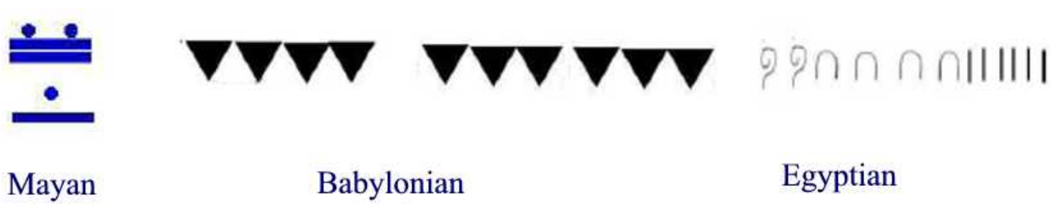
(a) Egyptian numeration:3275
(b) Roman numeration: 406
(c) Babylonian system: 8063
(d) Mayan numeration: 48

Solution.

Problem 5.26
Represent the number 246 in the Mayan, Babylonian, and Egyptian numeration systems.

(a) In which system is the greatest number of symbols required?
(b) In which system is the smallest number of symbols required?

Solution.



Mayan = 6 symbols, Babylonian = 10 symbols, Egyptian = 12
(a) Egyptian
(b) Mayan■

Problem 5.27
Some children go through reversal stage; that is they confuse 13 and 31, 27 and 72, 59 and 95. What numerals would give Roman children similar difficulties? How about Egyptian children?

Solution.
Roman children: VI and IV, XI and IX, DC and CD, etc.
The Egyptian system was not positional so there should not be a problem with reversal■

Problem 5.28
A newspaper advertisement introduced a new car as follows: IV Cams, XXXII Valves, CCLXXX Horsepower, coming December XXVI- the new 1999 Lincoln Mark VII. Write the Roman numeral that represents the year of the advertisement.

Solution.
MCMXCIX. The car was introduced in 1998■

Problem 5.29
After the credits of a film roll by, the Roman numeral MCMLXXXIX appears, representing the year in which the film was made. Express the year in our numeration system.

Solution.
MCMLXXXIX=1900 + 60 + 29 = 1989■

Problem 5.30
True of false?

(a) $|||$ is three in the tally system.
(b) $IV = VI$ in the Roman system.
(c) $\cap||| = |||\cap$

Solution.
(a) True.
(b) False. IV has numeral value 4 whereas VI has value 6.
(c) True. The Egyptian system was not positional so there should not be a problem with reversal■

Problem 6.1
Write each of the following numbers in expanded form.
(a) 70 (b) 746 (c) 840,001.

Solution.
(a) $70 = 7 \times 10^1 + 0 \times 10^0$
(b) $746 = 7 \times 10^2 + 4 \times 10^1 + 6 \times 10^0$
(c) $840,001 = 8 \times 10^5 + 4 \times 10^4 + 0 \times 10^3 + 0 \times 10^2 + 0 \times 10^1 + 1 \times 10^0$.■

Problem 6.2
Write each of the following expressions in standard place-value form. That is, $1 \times 10^3 + 2 \times 10^2 + 7 = 1207$.
(a) $5 \times 10^5 + 3 \times 10^2$.
(b) $8 \times 10^6 + 7 \times 10^4 + 6 \times 10^2 + 5$.
(c) $6 \times 10^7 + 9 \times 10^5$.

Solution.
(a) $5\times 10^5+3\times 10^2 = 5\times 10^5+0\times 10^4+0\times 10^3+3\times 10^2+0\times 10^1+0\times 10^0 = 500300$.
(b) $8 \times 10^6 + 7 \times 10^4 + 6 \times 10^2 + 5 = 8 \times 10^6 + 0 \times 10^5 + 7 \times 10^4 + 0 \times 10^3 + 6 \times 10^2 + 0 \times 10^1 + 5 \times 10^0 = 8070605$.
(c) $6 \times 10^7 + 9 \times 10^5 = 6 \times 10^7 + 0 \times 10^6 + 9 \times 10^5 + 0 \times 10^4 + 0 \times 10^3 + 0 \times 10^2 + 0 \times 10^1 + 0 \times 10^0 = 60900000$.■

Problem 6.3
Write the following numerals in words.
(a) $2,000,000,000$
(b) $87,000,000,000,000$
(c) $52,672,405,123,139$.

Solution.
(a) Two billion.
(b) Eighty seven trillion.
(c) Fifty two trillion six hundred seventy two billion four hundred five million one hundred twenty three thousand one hundred thirty nine■

Problem 6.4
Write each of the following base seven numerals in expanded notation.
(a) 15_{seven} (b) 123_{seven} (c) 5046_{seven}.

Solution.
(a) $15_{seven} = 1 \times 7^1 + 5 \times 7^0$
(b) $123_{seven} = 1 \times 7^2 + 2 \times 7^1 + 3 \times 7^0$
(c) $5046_{seven} = 5 \times 7^3 + 0 \times 7^2 + 4 \times 7^1 + 6 \times 7^0$■

Problem 6.5
Convert each base ten numeral into a numeral in the base requested.
(a) 395 in base eight
(b) 748 in base four
(c) 54 in base two.

Solution.
(a) Using repeated divisions by 8 we find

$$\begin{array}{rcl} 395 &=& 49 \times 8 + 3 \\ 49 &=& 6 \times 8 + 1 \\ 6 &=& 0 \times 8 + 6 \end{array}$$

Thus, $395_{ten} = 613_{eight}$
(b) Using repeated divisions by 4 we find

$$\begin{array}{rcl} 748 &=& 187 \times 4 + 0 \\ 187 &=& 46 \times 4 + 3 \\ 46 &=& 11 \times 4 + 2 \\ 11 &=& 2 \times 4 + 3 \\ 2 &=& 0 \times 4 + 2 \end{array}$$

Thus, $748_{ten} = 23230_{four}$
(c) Using repeated divisions by two we find

$$\begin{array}{rcl} 54 &=& 27 \times 2 + 0 \\ 27 &=& 13 \times 2 + 1 \\ 13 &=& 6 \times 2 + 1 \\ 6 &=& 3 \times 2 + 0 \\ 3 &=& 1 \times 2 + 1 \\ 1 &=& 0 \times 2 + 1 \end{array}$$

Thus, $54_{ten} = 110110_{two}$■

Problem 6.6
The base twelve numeration system has the following twelve symbols:0,1,2,3,4,5,6,7,8,9,T,E. Change each of the following numerals to base ten numerals.
(a) 142_{twelve} (b) 503_{twelve} (c) $T9_{twelve}$ (d) $ETET_{twelve}$.

Solution.
(a) $142_{twelve} = 1 \times 12^2 + 4 \times 12^1 + 2 \times 12^0 = 144 + 48 + 2 = 194_{ten}$
(b) $503_{twelve} = 5 \times 12^2 + 0 \times 12^1 + 3 \times 12^0 = 720 + 3 = 723_{ten}$
(c) $T9_{twelve} = 10 \times 12^1 + 9 \times 12^0 = 120 + 9 = 129_{ten}$
(d) $ETET_{twelve} = 11 \times 12^3 + 10 \times 12^2 + 11 \times 12^1 + 10 \times 12^0 = 19008 + 1440 + 132 + 10 = 20590_{ten}$■

Problem 6.7
Write each of the numerals in base six and base twelve.
(a) 128 (b) 74 (c) 2438.

Solution.
(a) We use repeated divisions by 6.

$$\begin{aligned} 128 &= 21 \times 6 + 2 \\ 21 &= 3 \times 6 + 3 \\ 3 &= 0 \times 6 + 3 \end{aligned}$$

Thus, $128_{ten} = 332_{six}$.
Using repeated divisions by 12 we find

$$\begin{aligned} 128 &= 10 \times 12 + 8 \\ 10 &= 0 \times 12 + 10 \end{aligned}$$

Thus, $128_{ten} = T8_{twelve}$
(b) We have

$$\begin{aligned} 74 &= 12 \times 6 + 2 \\ 12 &= 2 \times 6 + 0 \\ 2 &= 0 \times 6 + 2 \end{aligned}$$

Thus, $74_{ten} = 202_{six}$. Similarly, by repeated divisions by 12 we find

$$\begin{aligned} 74 &= 6 \times 12 + 2 \\ 6 &= 0 \times 12 + 6 \end{aligned}$$

Thus, $74_{ten} = 62_{twelve}$.
(c) We use repeated divisions by 6 and 12.

$$\begin{array}{rclcrcl} 2438 & = & 406 \times 6 + 2 & \quad & 2438 & = & 203 \times 12 + 2 \\ 406 & = & 67 \times 6 + 4 & & 203 & = & 16 \times 12 + 11 \\ 67 & = & 11 \times 6 + 1 & & 16 & = & 1 \times 12 + 4 \\ 11 & = & 1 \times 6 + 5 & & 1 & = & 0 \times 12 + 1 \\ 1 & = & 0 \times 6 + 1 & & & & \end{array}$$

Thus, $2438_{ten} = 15142_{six}$ and $2438_{ten} = 14E2_{twelve}$ ■

Problem 6.8
Convert the following base five numerals into base nine numerals.
(a) 12_{five} (b) 204_{five} (c) 1322_{five}.

Solution.
First we convert to base ten and then to base nine.
(a) $12_{five} = 1 \times 5^1 + 2 \times 5^0 = 52_{ten}$. Now using repeated division by 9 we find

$$\begin{array}{rcl} 52 & = & 4 \times 9 + 7 \\ 4 & = & 0 \times 9 + 4 \end{array}$$

Hence, $12_{five} = 47_{nine}$
(b) $204_{five} = 2 \times 5^2 + 0 \times 5^1 + 4 \times 5^0 = 54_{ten}$.

$$\begin{array}{rcl} 54 & = & 6 \times 9 + 0 \\ 6 & = & 0 \times 9 + 6 \end{array}$$

Thus, $204_{five} = 60_{nine}$
(c) $1322_{five} = 1 \times 5^3 + 3 \times 5^2 + 2 \times 5^1 + 2 \times 5^0 = 125 + 75 + 10 + 2 = 212_{ten}$.

$$\begin{array}{rcl} 212 & = & 23 \times 9 + 5 \\ 23 & = & 2 \times 9 + 5 \\ 2 & = & 0 \times 9 + 2 \end{array}$$

Hence, $1322_{five} = 255_{nine}$ ■

Problem 6.9
(a) How many different symbols would be necessary for a base twenty system?
(b) What is wrong with the numerals 85_{eight} and 24_{three}?

Solution.
(a) Twenty symbols.
(b) The symbols in base eight are $\{0, 1, 2, 3, 4, 5, 6, 7\}$ so 8 is not one of the symbols. Thus, the notation 85_{eight} is wrong. Similarly, the symbols in base three are $\{0, 1, 2\}$ so the notation 24_{three} is wrong$\blacksquare$

Problem 6.10
The set of even whole numbers is the set $\{0, 2, 4, 6, \cdots\}$. What can be said about the ones digit of every even number in the following bases?
(a) Ten (b) Four (c) Two (d) Five

Solution.
(a) The ones digit is one of the following: 0, 2, 4, 6, 8.
(b) By the division algorithm we have: $a = b \times 4 + r$ with $0 \leq r < 4$. Since a is even then the possible values for the ones digit are 0 and 2.
(c) Since $a = b \times 2 + r$ with $0 \leq r < 2$ and a is even then the only possible value for the ones digit is 0.
(d) Since $a = b \times 5 + r$ with $0 \leq r < 5$ and a is even then the only possible values of ones digit are 0, 2, and 4.$\blacksquare$

Problem 6.11
Translate the following numbers from one base to the other:
(a) $38_{ten} = \underline{\qquad}_{two}$.
(b) $63_{ten} = \underline{\qquad}_{two}$.

Solution.
(a) Using repeated divisions by 2 we find

$$\begin{array}{rcl} 38 & = & 19 \times 2 + 0 \\ 19 & = & 9 \times 2 + 1 \\ 9 & = & 4 \times 2 + 1 \\ 4 & = & 2 \times 2 + 0 \\ 2 & = & 1 \times 2 + 0 \\ 1 & = & 0 \times 2 + 1 \end{array}$$

Thus, $38_{ten} = 100110_{two}$
(b) Using repeated divisions by 2 we find

$$\begin{aligned}
63 &= 31 \times 2 + 1 \\
31 &= 15 \times 2 + 1 \\
15 &= 7 \times 2 + 1 \\
7 &= 3 \times 2 + 1 \\
3 &= 1 \times 2 + 1 \\
1 &= 0 \times 2 + 1
\end{aligned}$$

Thus, $63_{ten} = 111111_{two}$ ■

Problem 6.12
Translate the following numbers from one base to the other:
(a) $1101_{two} = \underline{\qquad}_{ten}$.
(b) $11111_{two} = \underline{\qquad}_{ten}$.

Solution.
(a) Using the expanded form we find

$$1101_{two} = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 8 + 4 + 1 = 13_{ten}$$

(b) $11111_{two} = 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 16 + 8 + 4 + 2 + 1 = 31_{ten}$ ■

Problem 6.13
The sum of the digits in a two-digit number is 12. If the digits are reversed, the new number is 18 greater than the original number. What is the number?

Solution.
We use the guess and checking technique. The possible numbers are 39, 93, 48,84, 57,75, and 66. Since $75 = 57 + 18$ then the number is 57 ■

Problem 6.14
State the place value of the digit 2 in each numeral.
(a) 6234 (b) 5142 (c) 2178

Solution.
(a) Hundreds
(b) units or ones
(c) Thousands ■

Problem 6.15
(a) Write out the first 20 base four numerals.
(b) How many base four numerals precede 2000_{four}?

Solution.
(a) $\{0, 1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33, 100, 101, 102, 103\}$
(b) Converting to base ten we find: $2000_{four} = 2\times4^3+0\times4^2+0\times4^1+0\times4^0 = 128_{ten}$. So there are 128 base four numerals preceding 2000_{four}■

Problem 6.16
True or false?
(a) $7_{eight} = 7_{ten}$ (b) $30_{four} = 30_{ten}$ (c) $8_{nine} = 8_{eleven}$ (d) $30_{five} = 30_{six}$

Solution. (a) True.
(b) Since $30_{four} = 3 \times 4^1 + 0 \times 4^0 = 12_{ten}$ then the statement is false.
(c) True.
(d) Since $30_{five} = 3 \times 5^1 + 0 \times 5^0 = 15_{ten}$ and $30_{six} = 3 \times 6^1 + 0 \times 6^0 = 18_{ten}$ then the statement is false■

Problem 6.17
If all the letters of the alphabet were used as our single-digit numerals, what would be the name of our base system?

Solution.
Base thirty six:$\{0, 1, 2, \cdots, 9, a, b, \cdots, z\}$■

Problem 6.18
Find the base ten numerals for each of the following.
(a) 342_{five} (b) $TE0_{twelve}$ (c) 101101_{two}

Solution.
(a) $342_{five} = 3 \times 5^2 + 4 \times 5^1 + 2 \times 5^0 = 97_{ten}$
(b) $TE0_{twelve} = 10 \times 12^2 + 11 \times 12^1 + 0 \times 12^0 = 1440 + 132 = 1572_{ten}$
(c) $101101_{two} = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 45_{ten}$■

Problem 6.19
The **hexadecimal** system is a base sixteen system used in computer programming. The system uses the symbols:0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F. Change each of the following hexadecimal numerals to base ten numerals.
(a) $213_{sixteen}$ (b) $1C2B_{sixteen}$ (c) $420E_{sixteen}$

Solution.
(a) $213_{sixteen} = 2 \times 16^2 + 1 \times 16^1 + 3 \times 16^0 = 531_{ten}$
(b) $1C2B_{sixteen} = 1 \times 16^3 + 11 \times 16^2 + 2 \times 16^1 + 11 \times 16^0 = 6955_{ten}$
(c) $420E_{sixteen} = 4 \times 16^3 + 2 \times 16^2 + 0 \times 16^1 + 14 \times 16^0 = 16910_{ten}$■

Problem 6.20
Write each of the following base ten numerals in base sixteen numerals.
(a) 375 (b) 2941 (c) 9520 (d) 24,274

Solution.
We use repeated divisions by 16. (a)

$$\begin{array}{rcl} 375 &=& 23 \times 16 + 7 \\ 23 &=& 1 \times 16 + 7 \\ 1 &=& 0 \times 16 + 1 \end{array}$$

So $375_{ten} = 177_{sixteen}$
(b)

$$\begin{array}{rcl} 2941 &=& 183 \times 16 + 13 \\ 183 &=& 11 \times 16 + 7 \\ 11 &=& 0 \times 16 + 11 \end{array}$$

So $2941_{ten} = B7D_{sixteen}$
(c)

$$\begin{array}{rcl} 9520 &=& 595 \times 16 + 0 \\ 595 &=& 37 \times 16 + 3 \\ 37 &=& 2 \times 16 + 5 \\ 2 &=& 0 \times 16 + 2 \end{array}$$

So $9520_{ten} = 2530_{sixteen}$
(d)

$$\begin{array}{rcl} 24274 &=& 1517 \times 16 + 2 \\ 1517 &=& 94 \times 16 + 13 \\ 94 &=& 5 \times 16 + 14 \\ 5 &=& 0 \times 16 + 5 \end{array}$$

So $24,274_{ten} = 5ED2_{sixteen}$■

Problem 6.21
Rod used base twelve to write the equation:

$$g36_{twelve} = 1050_{ten}.$$

What is the value of g?

Solution.
Converting 1050_{ten} to base twelve we find

$$\begin{array}{rcl} 1050 & = & 87 \times 12 + 6 \\ 87 & = & 7 \times 12 + 3 \\ 7 & = & 0 \times 12 + 7 \end{array}$$

So $1050_{ten} = 736_{twelve}$ so that $g = 7$■

Problem 6.22
For each of the following decimal numerals, give the place value of the underlined digit:
(a) $827,\underline{3}67$ (b) $8,421,0\underline{0}0$ (c) $9\underline{7},998$

Solution.
(a) Hundreds
(b) Tens
(c) Thousands■

Problem 6.23
A certain three-digit whole number has the following properties: The hundreds digit is greater than 7; the tens digit is an odd number; and the sum of the digits is 10. What could the number be?

Solution.
Write the number as xyz. We are given that x is either 8 or 9; y is either 1,3,5,7,9; and $x + y + z = 10$. If $x = 8$ then we have $8 + y + z = 10$ or $y + z = 2$. Since y is odd then the only possibility for y is 1 and this leads to $z = 1$. In this case we have the number 811. If $x = 9$ then $9 + y + z = 10$ so that $y + z = 1$. In this case the only value of y is 1 and so $z = 0$. But the sum of the digits of the number 810 does not add up to 10. Hence, the only answer is 811■

Problem 6.24
Find the number preceding and succeeding the number $EE0_{twelve}$.

Solution.
Converting to base ten we find $EEo_{twelve} = 11 \times 12^2 + 11 \times 12^1 + 0 \times 12^0 = 1716_{ten}$. The number preceding this is 1715_{ten} and the number succeding it

is 1717_{ten}. Now convert these numbers to base twelve by repeated divisions to obtain

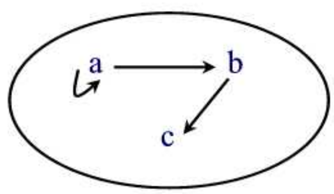
$$\begin{array}{rcl} 1715 &=& 142\times 12+11 \\ 142 &=& 11\times 12+10 \\ 11 &=& 0\times 12+11 \end{array}$$

and

$$\begin{array}{rcl} 1717 &=& 143\times 12+1 \\ 143 &=& 11\times 12+11 \\ 11 &=& 0\times 12+11 \end{array}$$

So $1715_{ten} = ETE_{twelve}$ and $1717_{ten} = EE1_{twelve}$ ■

Problem 7.1
Express the relation given in the arrow diagram below in its ordered-pair representation.

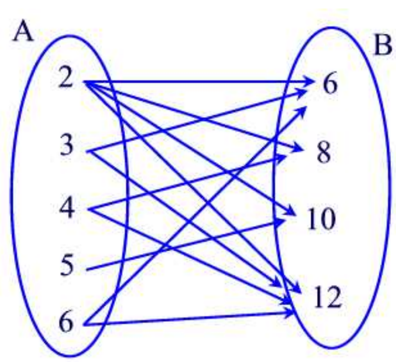


Solution.

$$R = \{(a,a),(a,b),(b,c)\} ■$$

Problem 7.2
Consider the relation "is a factor of" from the set $A = \{2,3,4,5,6\}$ to the set $B = \{6,8,10,12\}$. Make an arrow diagram of this relation.

Solution.



Problem 7.3
Determine whether the relations represented by the following sets of ordered pairs are reflexive, symmetric, or transitive. Which of them are equivalence relations?
(a) $R = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$
(b) $S = \{(1,2),(1,3),(2,3),(2,1),(3,2),(3,1)\}$
(c) $T = \{(1,1),(1,3),(2,2),(3,2),(1,2)\}$
(d) $U = \{1,1),(2,2),(3,3)\}$.

Solution.
(a) Reflexive; nonsymmetric since $(2,1) \in R$ but $(1,2) \notin R$; transitive.
(b) Nonreflexive since $(1,1) \notin S$; symmetric; nontransitive since $(1,3) \in S, (3,1) \in S$ but $(1,1) \notin S$.
(c) Nonreflexive since $(3,3) \notin T$; nonsymmetric since $(1,3) \in T$ but $(3,1) \notin T$; transitive.
(d) Reflexive, symmetric, transitive. So U is an equivalence relation■

Problem 7.4
Determine whether the relations represented by the following sets of ordered pairs are reflexive, symmetric, or transitive. Which of them are equivalence relations?
(a) "less than" on the set $\mathbb{N}$
(b) "has the same shape as" on the set of all triangles
(c) "is a factor of" on the set $\mathbb{N}$
(d) "has the same number of factors as" on the set $\mathbb{N}$.

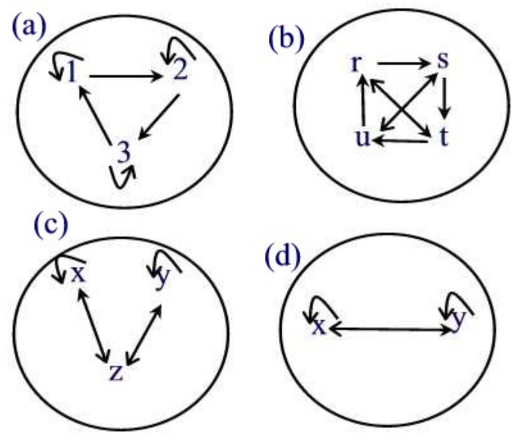
Solution.
(a) Nonreflexive since $a < a$ is false for any $a \in \mathbb{N}$. Nonsymmetric since $2 < 3$ but $3 \not< 2$. Transitive.
(b) Reflexive, symmetric, transitive. This is an equivalence relation.
(c) Reflexive. Nonsymmetric since 2 is a factor of 4 but 4 is not a factor of 2. Transitive.
(d) Reflexive. Symmetric. Transitive. So this relation is an equivalence relation■

Problem 7.5
List all the ordered pairs of each of the following relations on the sets listed. Which, if any, is an equivalence relation?

(a) "has the same number of factors as" on the set $\{1, 2, 3, 4, 5, 6\}$
(b) "is a multiple of " on the set $\{2, 3, 6, 8, 10, 12\}$
(c) "has more factors than" on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Solution.
(a) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 3), (3, 2), (2, 5), (5, 2), (3, 5), (5, 3)\}$.
R is reflexive. R is symmetric. R is transitive. R is an equivalence relation.
(b) $S = \{(2, 2), (3, 3), (6, 6), (8, 8), (10, 10), (12, 12), (2, 6), (2, 8), (2, 10), (2, 12), (3, 6), (3, 12), (6, 12)\}$. S is reflexive. S is not symmetric since $(2, 6) \in S$ but $(6, 2) \notin S$. S is transitive.
(c) $T = \{(2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 1), (8, 1), (4, 2), (4, 3), (4, 5), (4, 7), (6, 2), (6, 3), (6, 4), (6, 5), (6, 7), (8, 2), (8, 3), (8, 4), (8, 5)\}$. T is nonreflexive since $(1, 1) \notin T$. T is nonsymmetric since $(2, 1) \in T$ but $(1, 2) \notin T$. T is transitive■

Problem 7.6
Determine whether the relations represented by the following diagrams are reflexive, symmetric, or transitive. Which relations are equivalence relations?



Solution.
(a) Reflexive. Nonsymmetric since there is an arrow from 1 to 2 but no arrow from 2 to 1. Transitive.
(b) Nonreflexive since the elements are not in relation with themselves. Nonsymmetric since there is an arrow from s to t but no arrow from t to s. Transitive.
(c) Nonreflexive since z is not in relation with itself. Symmetric. Transitive.
(d) Reflexive, symmetric, and transitive. This is an equivalence relation■

Problem 7.7
Consider the relations R on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ defined by the equation $a + b = 11$. Determine all the ordered pairs (a, b) that satisfy the equation. Is this relation an equivalence relation?

Solution.

$$R = \{((1, 10), (10, 1), (2, 9), (9, 2), (3, 8), (8, 3), (4, 7), (7, 4), (5, 6), (6, 5)\}$$

R is not reflexive since $(1, 1) \notin R$. R is symmetric. R is not transitive since $(1, 10) \in R$ and $(10, 1) \in R$ but $(1, 1) \notin R$■

Problem 7.8
True or false?
(a) "If a is related to b then b is related to a" is an example of a reflexive relation.
(b) The ordered pair $(6, 24)$ satisfies the relation "is a factor of".

Solution.
(a) False. $R = \{(1, 2), (2, 1)\}$ is not reflexive.
(b) True. 6 is a factor of 24■

Problem 7.9
Let R be a relation on the set $A = \{a, b, c\}$. As a list of ordered pairs the relation has five elements. One of the element is (a, b). What are the remaining elements if R is both reflexive and symmetric?

Solution.
Since R is reflexive then we must have $(a, a) \in R, (b, b) \in R$, and $(c, c) \in R$. Since R is symmetric and $(a, b) \in R$ then $(b, a) \in R$. Hence,

$$R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}■$$

Problem 7.10
If the relation $\{(1, 2), (2, 1), (3, 4), (2, 4), (4, 2), (4, 3)\}$ on the set $\{1, 2, 3, 4\}$ is to be altered to have the properties listed, what other ordered pairs, if any, are needed?
(a) Reflexive (b) Symmetric (c) Transitive (d) Reflexive and transitive.

Solution.
(a) $\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(3,4),(2,4),(4,2),(4,3)\}$
(b) $\{(1,2),(2,1),(3,4),(2,4),(4,2),(4,3)\}$
(c) $\{(1,2),(2,1),(1,1),(3,4),(2,4),(4,2),(2,2),(4,3),(3,3),(4,4)\}$
(d) $\{(1,2),(2,1),(1,1),(3,4),(2,4),(4,2),(2,2),(4,3),(3,3),(4,4)\}$■

Problem 7.11
List the ordered pairs for these functions using the domain specified. Find the range for each function.
(a) $C(t) = 2t^3 - 3t$, with domain $\{0, 2, 4\}$
(b) $a(x) = x + 2$, with domain $\{1, 2, 9\}$
(c) $P(n) = \left(\frac{n+1}{n}\right)$, with domain $\{1, 2, 3\}$.

Solution.
(a) Since $C(0) = 0, C(2) = 10,$ and $C(4) = 116$ then $C = \{(0, 0), (2, 10), (4, 116)\}$. The range of C is $\{0, 10, 116\}$.
(b) $a = \{(1, 3), (2, 4), (9, 11)\}$. Range of a is $\{3, 4, 11\}$
(c) $p = \{(1, 2), (2, \frac{3}{2}), (3, \frac{4}{3})\}$. Range of p is $\{2, \frac{3}{2}, \frac{4}{3}\}$■

Problem 7.12
Find the value of $\frac{f(x+h)-f(x)}{h}$ given that $f(x) = x^2$.

Solution.
Since $f(x + h) = (x + h)^2 = (x + h)(x + h) = x^2 + 2xh + h^2$ and $f(x) = x^2$ then $f(x + h) - f(x) = 2xh + h^2 = h(2x + h)$. Thus.

$$\frac{f(x+h)-f(x)}{h} = \frac{h(2x+h)}{h} = 2x + h, \quad h \neq 0$$■

Problem 7.13
Given $f(x) = -x^2 + 2x + 6$, find f(-4).

Solution.
$f(-4) = -(-4)^2 + 2(-4) + 6 = -16 - 8 + 6 = -18$■

Problem 7.14
A function f on the set of real numbers $\mathbb{R}$ is defined as

$$f(x) = (3x + 2)/(x - 1).$$

Find:
(a) the domain of f
(b) the range of f
(c) the image of -2 under f
(d) x when $f(x) = -3$.

Solution.
(a) The domain of f consists of all numbers except 1 since replacing x by 1 will result of a division by zero which is not defined.
(b) For the range, one needs to write $y = \frac{3x+2}{x-1}$ and then find x in terms of y.

$$\begin{array}{rcl} \frac{3x+2}{x-1} & = & y \\ 3x+1 & = & (x-1)y \\ 3x+1 & = & xy-y \\ 3x-xy & = & -y-1 \\ x(3-y) & = & -y-1 \\ x & = & \frac{-y-1}{3-y} \end{array}$$

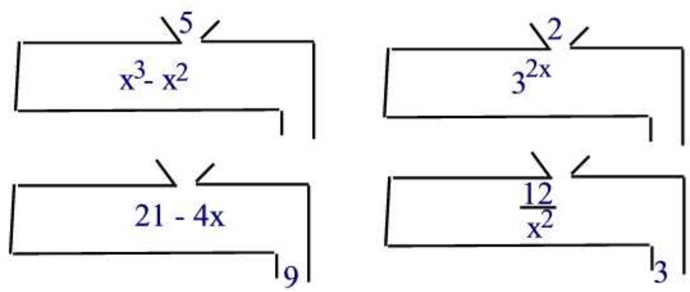
Thus, the range of f consists of all numbers except 3 since replacing y by 3 will result of a division by zero.
(c) $f(-2) = \frac{3(-2)+2}{-2-1} = \frac{-6+2}{-3} = \frac{-4}{-3} = \frac{4}{3}$.
(d)

$$\begin{array}{rcl} f(x) & = & -3 \\ \frac{3x+2}{x-1} & = & -3 \\ 3x+2 & = & -3(x-1) \\ 3x+2 & = & -3x+3 \\ 3x+3x & = & 3-2 \\ 6x & = & 1 \\ x & = & \frac{1}{6} \end{array}$$

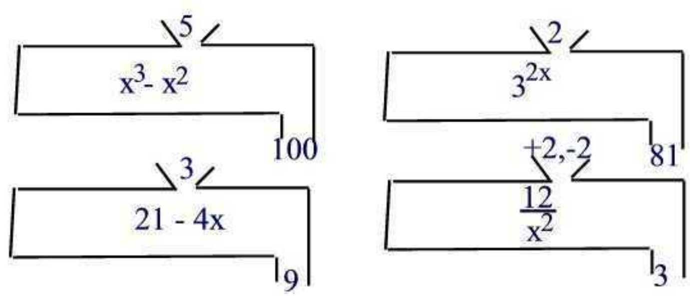
Problem 7.15
Which of the following relations, listed as ordered pairs, could belong to a function? For those that cannot, explain why not.
(a) $\{(7,4),(6,3),(5,2),(4,1)\}$
(b) $\{((1,1),(1,2),(3,4),(4,4)\}$
(c) $\{(1,1),(2,1),(3,1),(4,1)\}$
(d) $\{(a,b),(b,b),(d,e),(b,c),(d,f)\}$.

Solution.
(a) Since no two distinct second component share the same first component then the given relation is a function.
(b) Since the numbers 1 and 2 share the same first component 1 then this relation is not a function.
(c) This relation is a function.
(d) Since the input b has two distinct out b and c then the given relation is not a function.■

Problem 7.16
Using the function machines, find all possible missing whole-number inputs and outputs.

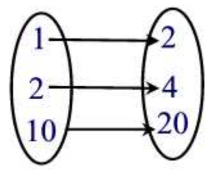


Solution.
Replacing x by 5 in $x^3 - x^2$ we find $125 - 25 = 100$.
Replacing x by 2 in $3^{2x} = 3^4 = 81$.
Since $21 - 4x = 9$ then $4x = 12$ and so $x = 3$.
Since $\frac{12}{x^2} = 3$ then $3x^2 = 12$ and $x^2 = \frac{12}{3} = 4$. Hence, $x = \pm 2$■



Problem 7.17
The following functions are expressed in one of the following forms: a formula, an arrow diagram, a table, or a set of ordered pairs. Express each function

in each of the other forms.
(a) $f(x) = x^3 - x$ for $x \in \{0, 1, 4\}$.
(b) $\{(1, 1), (4, 2), (9, 3)\}$
(c)



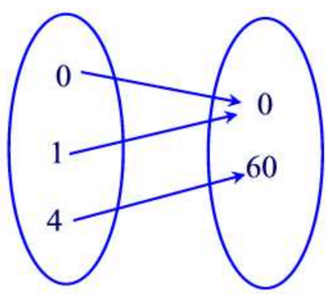
(d)

x	f(x)
5	55
6	66
7	77

Solution.
(a) A set of ordered pairs:$\{(0, 0), (1, 0), (4, 60)\}$.
A table:

x	0	1	4
f(x)	0	0	60

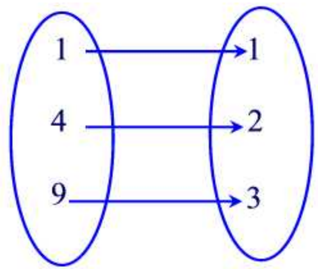
A diagram:



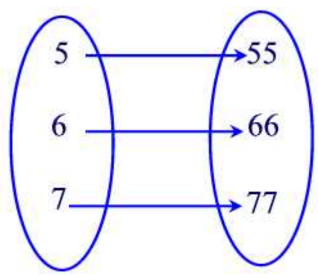
(b) A formula: $f(x) = \sqrt{x}$
A table:

x	1	4	9
hline f(x)	1	2	3

A diagram:



(c) A formula: $f(x) = 11x$.
A set of ordered pairs:$\{(5,55),(6,66),(7,77)\}$
A diagram:



Problem 7.18
(a) The function $f(n) = \frac{9}{5}n + 32$ can be used to convert degrees Celsius to degrees Fahrenheit. Calculate, $f(0), f(100), f(5)$, and $f(-40)$.
(b) The function $g(n) = \frac{5}{9}(n-32)$ can be used to convert degrees Fahrenheit to degrees Celsius. Calculate, $g(32), g(212), g(104)$, and $g(-40)$.
(c) Is there a temperature where the degrees Celsius equals the degrees Fahrenheit? If so, what is it?

Solution.
(a)$f(0) = 32°F, f(100) = \frac{9}{5}(100) + 32 = 212°F, f(5) = 41°F, f(-40) = -72 + 32 = -40°F$.
(b) $g(32) = 0°C, g(212) = 100°C, g(104) = 40°C, g(-40) = -40°C$.
(c) $-40°F = -40°C$■

Problem 7.19
A fitness club charges an initiation fee of \$85 plus \$35 per month.
(a) Write a formula for a function, $C(x)$, that gives the total cost for using

the fitness club facilities after x months.
(b) Calculate $C(18)$ and explain in words its meaning.
(c) When will the total amount spent by a club member first exceed \$1000?

Solution.
(a) $C(x) = 35x + 85$.
(b) $C(18) = 35(18) + 85 = \$715$. The total cost of using the facilities for 18 months is \$715
(c)

$$\begin{array}{rcl} C(x) & > & 1000 \\ 35x + 85 & > & 1000 \\ 35x & > & 915 \\ x & > & \frac{915}{35} \end{array}$$

Since $\frac{915}{35} \approx 26$ then one needs to use the facilities for more than 26 months■

Problem 7.20
If the interest rate of a \$1000 savings account is 5% and no additional money is deposited, the amount of money in the account at the end of t years is given by the function $a(t) = (1.05)^t \cdot 1000$.
(a) Calculate how much will be in the account after 2 years, 5 years, and 10 years.
(b) What is the minimum number of years that it will take to more than double the account?

Solution.
(a) $a(2) = 1000(1.05)^2 = \$1102.50; a(5) = 1000(1.05)^5 = \$1276.28; a(10) = 1000(1.05)^{10} = \$1628.89$
(b) We want to find t so that $a(t) > 2000$. That is, $(1.05)^t > 2$. By guessing and checking we find that $t = 15$■

Problem 7.21
A function has the formula $P(N) = 8n - 50$. The range for P is $\{46, 62, 78\}$. What is the domain?

Solution.
We have

$$\begin{array}{rcl} 8n - 50 & = & 46 \\ 8n & = & 96 \\ n & = & 12 \end{array} \qquad \begin{array}{rcl} 8n - 50 & = & 62 \\ 8n & = & 112 \\ n & = & 14 \end{array} \qquad \begin{array}{rcl} 8n - 50 & = & 78 \\ 8n & = & 128 \\ n & = & 16 \end{array}$$

Thus, the domain is $\{12, 14, 16\}$■

Problem 7.22
Which of the following assignments creates a function?
(a) Each student in a school is assigned a teacher for each course.
(b) Each dinner in a restaurant is assigned a price.
(c) Each person is assigned a birth date.

Solution.
(a) Since a student can be assigned to more than one class then the given rule is not a function.
(b) Since each dinner is assigned to only one price then the rule is a function.
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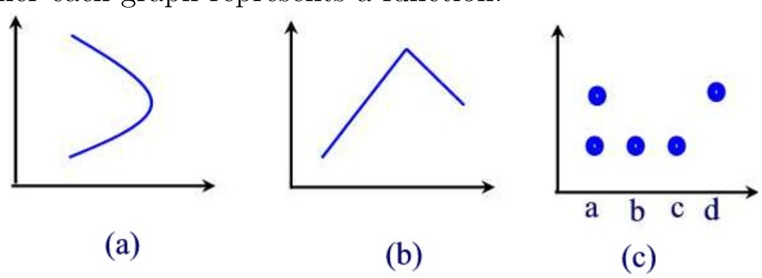
Problem 7.23
Tell whether each graph represents a function.



Solution.
(a) It is possible to find a vertical line that crosses the graph twice. Thus, this curve is not the graph of a function.
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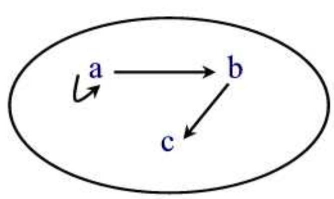
(c) Since a is assigned two different values then the graph is not the graph of a function■

Problem 8.1
Express the relation given in the arrow diagram below in its ordered-pair representation.

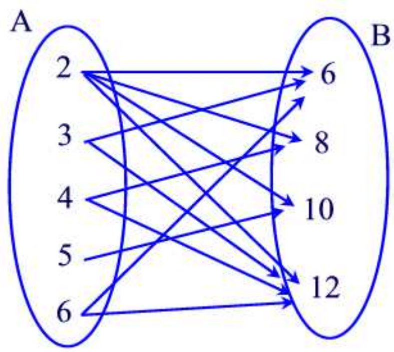


Solution.

$$R = \{(a,a),(a,b),(b,c)\}\blacksquare$$

Problem 8.2
Consider the relation "is a factor of" from the set $A = \{2,3,4,5,6\}$ to the set $B = \{6,8,10,12\}$. Make an arrow diagram of this relation.

Solution.



Problem 8.3
Determine whether the relations represented by the following sets of ordered pairs are reflexive, symmetric, or transitive. Which of them are equivalence relations?
(a) $R = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$
(b) $S = \{(1,2),(1,3),(2,3),(2,1),(3,2),(3,1)\}$
(c) $T = \{(1,1),(1,3),(2,2),(3,2),(1,2)\}$
(d) $U = \{1,1),(2,2),(3,3)\}$.

Solution.
(a) Reflexive; nonsymmetric since $(2,1) \in R$ but $(1,2) \notin R$; transitive.
(b) Nonreflexive since $(1,1) \notin S$; symmetric; nontransitive since $(1,3) \in S, (3,1) \in S$ but $(1,1) \notin S$.
(c) Nonreflexive since $(3,3) \notin T$; nonsymmetric since $(1,3) \in T$ but $(3,1) \notin T$; transitive.
(d) Reflexive, symmetric, transitive. So U is an equivalence relation$\blacksquare$

Problem 8.4
Determine whether the relations represented by the following sets of ordered

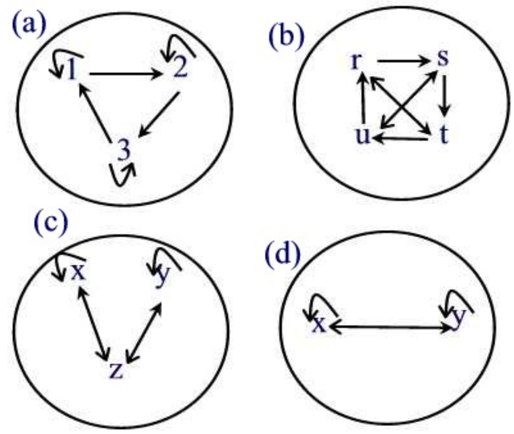
pairs are reflexive, symmetric, or transitive. Which of them are equivalence relations?
(a) "less than" on the set $\mathbb{N}$
(b) "has the same shape as" on the set of all triangles
(c) "is a factor of" on the set $\mathbb{N}$
(d) "has the same number of factors as" on the set $\mathbb{N}$.

Solution.
(a) Nonreflexive since $a < a$ is false for any $a \in \mathbb{N}$. Nonsymmetric since $2 < 3$ but $3 \not< 2$. Transitive.
(b) Reflexive, symmetric, transitive. This is an equivalence relation.
(c) Reflexive. Nonsymmetric since 2 is a factor of 4 but 4 is not a factor of 2. Transitive.
(d) Reflexive. Symmetric. Transitive. So this relation is an equivalence relation■

Problem 8.5
List all the ordered pairs of each of the following relations on the sets listed. Which, if any, is an equivalence relation?
(a) "has the same number of factors as" on the set $\{1, 2, 3, 4, 5, 6\}$
(b) "is a multiple of " on the set $\{2, 3, 6, 8, 10, 12\}$
(c) "has more factors than" on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Solution.
(a) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 3), (3, 2), (2, 5), (5, 2), (3, 5), (5, 3)\}$.
R is reflexive. R is symmetric. R is transitive. R is an equivalence relation.
(b) $S = \{(2, 2), (3, 3), (6, 6), (8, 8), (10, 10), (12, 12), (2, 6), (2, 8), (2, 10), (2, 12), (3, 6), (3, 12), (6, 12)\}$. S is reflexive. S is not symmetric since $(2, 6) \in S$ but $(6, 2) \notin S$. S is transitive.
(c) $T = \{(2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 1), (8, 1), (4, 2), (4, 3), (4, 5), (4, 7), (6, 2), (6, 3), (6, 4), (6, 5), (6, 7), (8, 2), (8, 3), (8, 4), (8, 5)\}$. T is nonreflexive since $(1, 1) \notin T$. T is nonsymmetric since $(2, 1) \in T$ but $(1, 2) \notin T$. T is transitive■

Problem 8.6
Determine whether the relations represented by the following diagrams are reflexive, symmetric, or transitive. Which relations are equivalence relations?



Solution.
(a) Reflexive. Nonsymmetric since there is an arrow from 1 to 2 but no arrow from 2 to 1. Transitive.
(b) Nonreflexive since the elements are not in relation with themselves. Nonsymmetric since there is an arrow from s to t but no arrow from t to s. Transitive.
(c) Nonreflexive since z is not in relation with itself. Symmetric. Transitive.
(d) Reflexive, symmetric, and transitive. This is an equivalence relation■

Problem 8.7
Consider the relations R on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ defined by the equation $a + b = 11$. Determine all the ordered pairs (a, b) that satisfy the equation. Is this relation an equivalence relation?

Solution.

$$R = \{((1, 10), (10, 1), (2, 9), (9, 2), (3, 8), (8, 3), (4, 7), (7, 4), (5, 6), (6, 5)\}$$

R is not reflexive since $(1, 1) \notin R$. R is symmetric. R is not transitive since $(1, 10) \in R$ and $(10, 1) \in R$ but $(1, 1) \notin R$■

Problem 8.8
True or false?
(a) "If a is related to b then b is related to a" is an example of a reflexive relation.
(b) The ordered pair $(6, 24)$ satisfies the relation "is a factor of".

Solution.
(a) False. $R = \{(1,2),(2,1)\}$ is not reflexive.
(b) True. 6 is a factor of 24$\blacksquare$

Problem 8.9
Let R be a relation on the set $A = \{a, b, c\}$. As a list of ordered pairs the relation has five elements. One of the element is (a, b). What are the remaining elements if R is both reflexive and symmetric?

Solution.
Since R is reflexive then we must have $(a, a) \in R, (b, b) \in R$, and $(c, c) \in R$. Since R is symmetric and $(a, b) \in R$ then $(b, a) \in R$. Hence,

$$R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}\blacksquare$$

Problem 8.10
If the relation $\{(1,2),(2,1),(3,4),(2,4),(4,2),(4,3)\}$ on the set $\{1,2,3,4\}$ is to be altered to have the properties listed, what other ordered pairs, if any, are needed?
(a) Reflexive (b) Symmetric (c) Transitive (d) Reflexive and transitive.

Solution.
(a) $\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(3,4),(2,4),(4,2),(4,3)\}$
(b) $\{(1,2),(2,1),(3,4),(2,4),(4,2),(4,3)\}$
(c) $\{(1,2),(2,1),(1,1),(3,4),(2,4),(4,2),(2,2),(4,3),(3,3),(4,4)\}$
(d) $\{(1,2),(2,1),(1,1),(3,4),(2,4),(4,2),(2,2),(4,3),(3,3),(4,4)\}\blacksquare$

Problem 8.11
List the ordered pairs for these functions using the domain specified. Find the range for each function.
(a) $C(t) = 2t^3 - 3t$, with domain $\{0, 2, 4\}$
(b) $a(x) = x + 2$, with domain $\{1, 2, 9\}$
(c) $P(n) = \left(\frac{n+1}{n}\right)$, with domain $\{1, 2, 3\}$.

Solution.
(a) Since $C(0) = 0, C(2) = 10$, and $C(4) = 116$ then $C = \{(0,0),(2,10),(4,116)\}$. The range of C is $\{0, 10, 116\}$.
(b) $a = \{(1,3),(2,4),(9,11)\}$. Range of a is $\{3, 4, 11\}$
(c) $p = \{(1,2),(2,\frac{3}{2}),(3,\frac{4}{3})\}$. Range of p is $\{2, \frac{3}{2}, \frac{4}{3}\}\blacksquare$

Problem 8.12
Find the value of $\frac{f(x+h)-f(x)}{h}$ given that $f(x) = x^2$.

Solution.
Since $f(x+h) = (x+h)^2 = (x+h)(x+h) = x^2 + 2xh + h^2$ and $f(x) = x^2$ then $f(x+h) - f(x) = 2xh + h^2 = h(2x+h)$. Thus.

$$\frac{f(x+h) - f(x)}{h} = \frac{h(2x+h)}{h} = 2x + h, \quad h \neq 0\blacksquare$$

Problem 8.13
Given $f(x) = -x^2 + 2x + 6$, find f(-4).

Solution.
$f(-4) = -(-4)^2 + 2(-4) + 6 = -16 - 8 + 6 = -18\blacksquare$

Problem 8.14
A function f on the set of real numbers $\mathbb{R}$ is defined as

$$f(x) = (3x + 2)/(x - 1).$$

Find:
(a) the domain of f
(b) the range of f
(c) the image of -2 under f
(d) x when $f(x) = -3$.

Solution.
(a) The domain of f consists of all numbers except 1 since replacing x by 1 will result of a division by zero which is not defined.
(b) For the range, one needs to write $y = \frac{3x+2}{x-1}$ and then find x in terms of y.

$$\begin{array}{ccc} \frac{3x+2}{x-1} & = & y \\ 3x+1 & = & (x-1)y \\ 3x+1 & = & xy - y \\ 3x - xy & = & -y - 1 \\ x(3-y) & = & -y - 1 \\ x & = & \frac{-y-1}{3-y} \end{array}$$

Thus, the range of f consists of all numbers except 3 since replacing y by 3 will result of a division by zero.

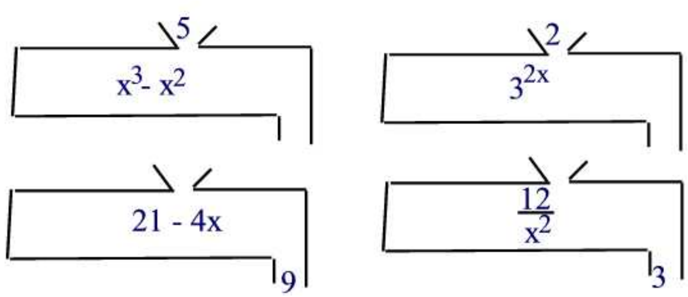
(c) $f(-2) = \frac{3(-2)+2}{-2-1} = \frac{-6+2}{-3} = \frac{-4}{-3} = \frac{4}{3}$.
(d)

$$\begin{array}{rcl} f(x) & = & -3 \\ \frac{3x+2}{x-1} & = & -3 \\ 3x+2 & = & -3(x-1) \\ 3x+2 & = & -3x+3 \\ 3x+3x & = & 3-2 \\ 6x & = & 1 \\ x & = & \frac{1}{6} \end{array}$$

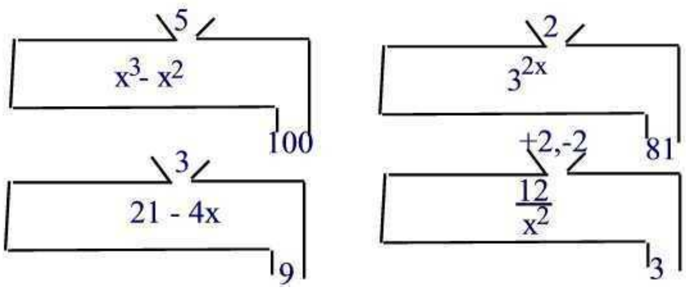
Problem 8.15
Which of the following relations, listed as ordered pairs, could belong to a function? For those that cannot, explain why not.
(a) $\{(7,4),(6,3),(5,2),(4,1)\}$
(b) $\{((1,1),(1,2),(3,4),(4,4)\}$
(c) $\{(1,1),(2,1),(3,1),(4,1)\}$
(d) $\{(a,b),(b,b),(d,e),(b,c),(d,f)\}$.

Solution.
(a) Since no two distinct second component share the same first component then the given relation is a function.
(b) Since the numbers 1 and 2 share the same first component 1 then this relation is not a function.
(c) This relation is a function.
(d) Since the input b has two distinct out b and c then the given relation is not a function.■

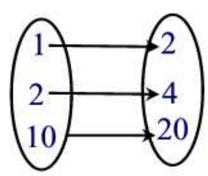
Problem 8.16
Using the function machines, find all possible missing whole-number inputs and outputs.



Solution.
Replacing x by 5 in $x^3 - x^2$ we find $125 - 25 = 100$.
Replacing x by 2 in $3^{2x} = 3^4 = 81$.
Since $21 - 4x = 9$ then $4x = 12$ and so $x = 3$.
Since $\frac{12}{x^2} = 3$ then $3x^2 = 12$ and $x^2 = \frac{12}{3} = 4$. Hence, $x = \pm 2$■



Problem 8.17
The following functions are expressed in one of the following forms: a formula, an arrow diagram, a table, or a set of ordered pairs. Express each function in each of the other forms.
(a) $f(x) = x^3 - x$ for $x \in \{0, 1, 4\}$.
(b) $\{(1, 1), (4, 2), (9, 3)\}$
(c)



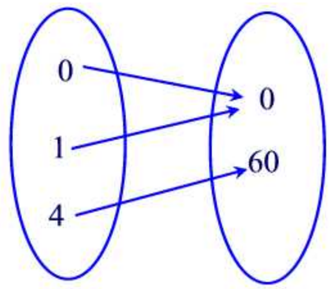
(d)

x	f(x)
5	55
6	66
7	77

Solution.
(a) A set of ordered pairs:$\{(0, 0), (1, 0), (4, 60)\}$.
A table:

x	0	1	4
f(x)	0	0	60

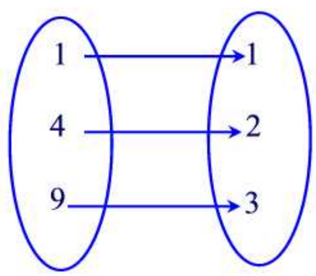
A diagram:



(b) A formula: $f(x) = \sqrt{x}$
A table:

x	1	4	9
hline f(x)	1	2	3

A diagram:



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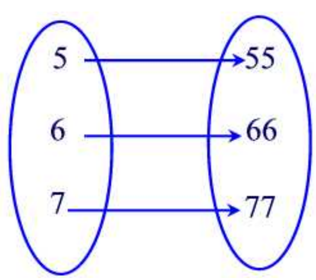

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$$\begin{array}{ccc} C(x) & > & 1000 \\ 35x + 85 & > & 1000 \\ 35x & > & 915 \\ x & > & \frac{915}{35} \end{array}$$

Since $\frac{915}{35} \approx 26$ then one needs to use the facilities for more than 26 months■

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Thus, the domain is $\{12, 14, 16\}$■

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Which of the following assignments creates a function?
(a) Each student in a school is assigned a teacher for each course.
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Solution.
(a) Since a student can be assigned to more than one class then the given rule is not a function.
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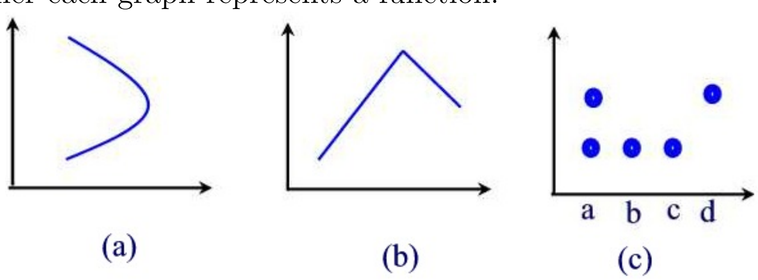
Tell whether each graph represents a function.



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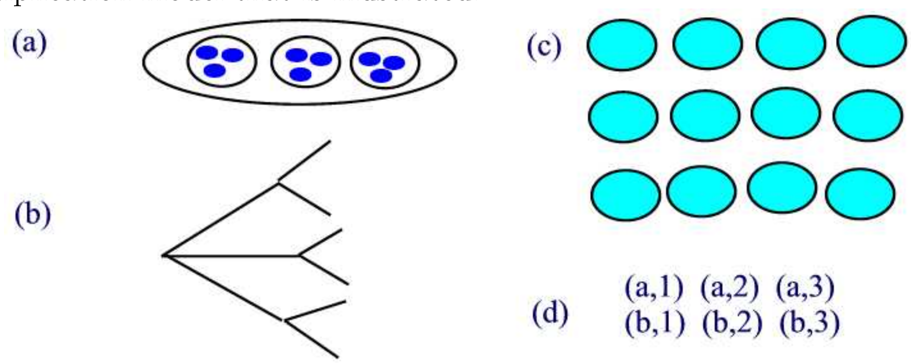
(a) It is possible to find a vertical line that crosses the graph twice. Thus, this curve is not the graph of a function.
(b) The graph satisfies the horizontal line test so we have a function.
(c) Since a is assigned two different values then the graph is not the graph of a function■

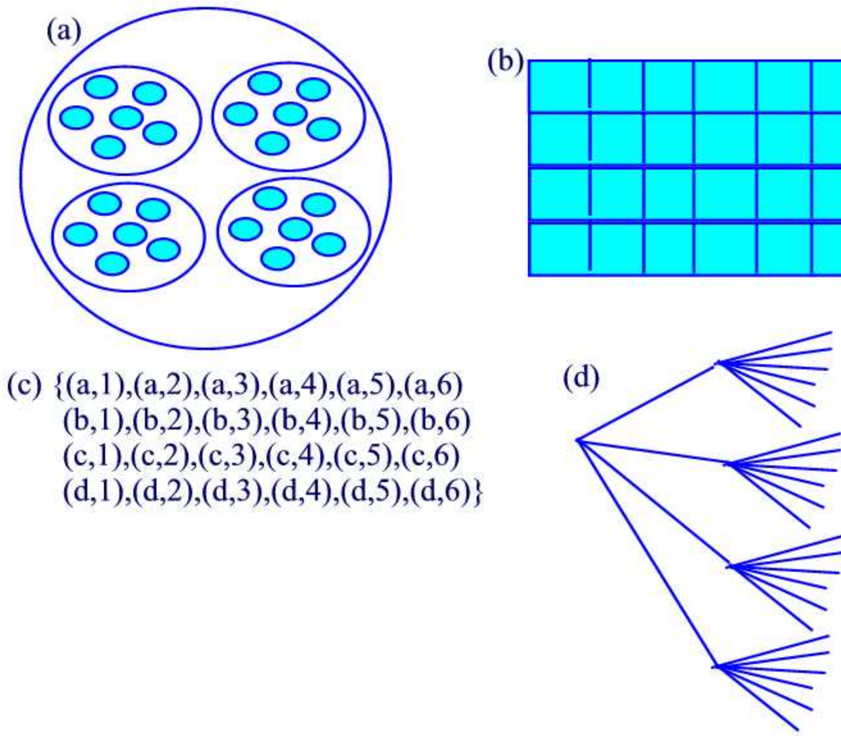
Problem 9.1
What multiplication fact is illustrated in each of these diagrams? Name the multiplication model that is illustrated.



Solution.
(a) Repeated addition model
(b) Rectangular array model
(c) Multiplication tree model
(d) Cartesian product model of multiplication ■

Problem 9.2
Illustrate 4×6 using each of the following models.
(a) set model (repeated addition)
(b) rectangular array model
(c) Cartesian product model
(d) multiplication tree.

Solution.



Problem 9.3
Which of the following sets are closed under multiplication? Why or why not?
(a) $\{2, 4\}$
(b) $\{0, 2, 4, 6, \cdots\}$
(c) $\{5, 7, 9, 11, \cdots\}$
(d) $\{0, 2^0, 2^1, 2^2, 2^3, \cdots\}$.

Solution.
(a) No, since $24 = 8 \notin \{2, 4\}$
(b) Yes, since the product of any two even whole numbers is again an even whole number
(c) Yes, since the product of any two odd numbers is again an odd number
(d) Yes, since $2^i \times 2^j = 2^{i+j}$■

Problem 9.4
What properties of whole number multiplication justify these equations?

(a) $4 \times 9 = 9 \times 4$
(b) $4 \times (6 + 2) = 4 \times 6 + 4 \times 2$
(c) $0 \times 12 = 0$
(d) $5 \times (9 \times 11) = (5 \times 9) \times 11$
(e) $7 \times 3 + 7 \times 8 = 7 \times (3 + 8)$.

Solution.
(a) Commutative
(b) Distributive over addition
(c) Multiplication by zero
(d) Associative
(e) Distributive over addition■

Problem 9.5
Rewrite each of the following expressions using the distributive property for multiplication over addition or subtraction. Your answer should contain no parentheses.
(a) $4 \times (60 + 37)$
(b) $3 \times (29 + 30 + 6)$
(c) $a \times (7 - b + z)$.

Solution.
(a) $4 \times 60 + 4 \times 37$
(b) $3 \times 29 + 3 \times 30 + 3 \times 6$
(c) $a \times 7 - a \times b + a \times z$■

Problem 9.6
Each situation described below involves a multiplication problem. In each case state whether the problem situation is best represented by the repeated-addition model, the rectangular array model, or the Cartesian product model, and why. Then write an appropriate equation to fit the situation.

(a) At the student snack bar, three sizes of beverages are available: small, medium, and large. Five varieties of soft drinks are available: cola, diet cola, lemon-lime, root beer, and orange. How many different choices of soft drink does a student have, including the size that may be selected?

(b)At graduation students file into the auditorium four abreast. A parent seated near the door counts 72 rows of students who pass him. How many students participated in the graduation exercise?

(c) Kirsten was in charge of the food for an all-school picnic. At the grocery store she purchased 25 eight-packs of hot dog buns for 70 cents each. How much did she spend on the hot dog buns?

Solution.
(a) Cartesian product, since the set of possibilities is

$$\{\text{small, medium, large}\} \times \{\text{cola, diet cola, lemon-lime, root beer, orange}\}$$

The total number of choices is: $3 \times 5 = 15$
(b) Rectangular array approach, since students from a moving array of 72 rows and 4 columns. The total number of choices is: $72 \times 4 = 288$
(c) Repeated addition approach, since the bill could be found by adding

$$70 ¢ + 70 ¢ + \cdots + 70¢ \blacksquare$$

Problem 9.7
A stamp machine dispenses twelve 32 cents stamps. What is the total cost of the twelve stamps?

Solution.
The total cost is $12 \times 0.32 = \$3.84$■

Problem 9.8
What properties of multiplication make it easy to compute these values mentally?
(a) $7 \times 19 + 3 \times 19$
(b) $36 \times 15 - 12 \times 45$.

Solution.
(a) Distributive over addition
(b) $36 \times 15 - 12 \times 45 = 36 \times 15 - 12 \times 3 \times 15 = 36 \times 15 - 36 \times 15 = 0$. Distributive over addition■

Problem 9.9
Using the distributive property of multiplication over addition we can factor as in $x^2 + xy = x(x + y)$. Factor the following:

(a) $xy + x^2$
(b) $47 \times 99 + 47$
(c) $(x + 1)y + (x + 1)$
(d) $a^2b + ab^2$.

Solution.
(a) $xy + x^2 = x(y + x)$
(b) $47 \times 99 + 47 = 47(99 + 1) = 47 \times 100$
(c) $(x + 1)y + (x + 1) = (x + 1)(y + 1)$
(d) $a^2b + ab^2 = ab(a + b)$■

Problem 9.10
Using the distributive property of multiplication over addition and subtraction to show that
(a) $(a + b)^2 = a^2 + 2ab + b^2$
(b) $(a - b)^2 = a^2 - 2ab + b^2$
(c) $(a - b)(a + b) = a^2 - b^2$.

Solution.
(a) $(a+b)^2 = (a+b)(a+b) = a(a+b)+b(a+b) = a^2+ab+ba+b^2 = a^2+2ab+b^2$ since ab = ba.
(b) $(a-b)^2 = (a-b)(a-b) = a(a-b)-b(a-b) = a^2-ab-ba+b^2 = a^2-2ab+b^2$ since ab = ba.
(c) $(a - b)(a + b) = a(a + b) - b(a + b) = a^2 + ab - ba - b^2 = a^2 - b^2$ ■

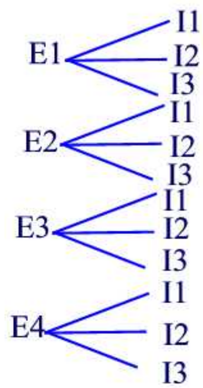
Problem 9.11
Find all pairs of whole numbers whose product is 36.

Solution.
{1, 36}, {2, 18}, {3, 12}, {4, 9}, {6, 6}■

Problem 9.12
A new model of car is available in 4 exterior colors and 3 interior colors. Use a tree diagram and specific colors to show how many color schemes are possible for the car?

Solution.
Let the exterior colors be E1,E2,E3,E4 and the interior colors be I1, I2, and

I3. Then we have the following tree model.■



Problem 9.13
Is $x \times x$ ever equal to x? Explain your answer.

Solution.
$x \times x = x$ only when $x = 0$ or $x = 1$ ■

Problem 9.14
Describe all pairs of numbers whose product and sum are the same.

Solution.
{0,0} and {2,2} ■

Problem 9.15
The operation $\odot$ is defined on the set $S = \{a, b, c\}$ by the following **Cayley's table**. For example, $a \odot c = c$.

$\odot$	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

(a) Is S closed under $\odot$?
(b) Is $\odot$ commutative?
(c) Is $\odot$ associative?
(d) Is there an identity for $\odot$ on S? If yes, what is it?

Solution.
(a) Since the entries of the table are all in S then S is closed under $\odot$.
(b) Because of symmetry with respect to the main diagonal the operation $\odot$ is commutative.
(c) Yes.
(d) There is an identity element which is a ■

Problem 9.16
Rewrite each of the following division problems as a multiplication problem.
(a) $48 \div 6 = 8$ (b) $24 \div x = 12$ (c) $a \div b = x$.

Solution.
(a) $86 = 48$
(b) $12x = 24$
(c) $xb = a$ ■

Problem 9.17
Show, that each of the following is false when $x, y,$ and z are replaced by whole numbers. Give an example (other than dividing by zero) where each statement is false.

(a) $x \div y$ is a whole number
(b) $x \div y = y \div x$
(c) $x \div (y \div z) = (x \div y) \div z$
(d) $x \div (y + z) = x \div y + x \div z$.

Solution.
(a) $1 \div 2 = 0.5$ which is not a whole number
(b) $2 \div 1 = 2$ and $1 \div 2 = 0.5$ so that $2 \div 16 = 1 \div 2$
(c) $3 \div (3 \div 2) = 3 \div 1.5 = 2$ and $(3 \div 3) \div 2 = 1 \div 2 = 0.5$
(d) $3 \div (1 + 2) = 3 \div 3 = 1$ and $3 \div 1 + 3 \div 2 = 3 + 1.5 = 4.5$ ■

Problem 9.18
Find the quotient and the remainder for each division.
(a) $7 \div 3$ (b) $3 \div 7$ (c) $7 \div 1$ (d) $1 \div 7$ (e) $15 \div 4$.

Solution.
(a) $7 = 3 \times 2 + 1$ so that $q = 2$ and $r = 1$
(b) $3 = 7 \times 0 + 3$ so that $q = 0$ and $r = 3$

(c) $7 = 1 \times 7 + 0$ so that $q = 7$ and $r = 0$
(d) $1 = 7 \times 0 + 1$ so that $q = 0$ and $r = 1$
(e) $15 = 4 \times 3 + 3$ so that $q = 4$ and $r = 3$■

Problem 9.19
How many possible remainders (including zero) are there when dividing by the following numbers?
(a) 2 (b) 12 (c) 62 (d) 23.

Solution.
Remember that by the division algorithm, when we write $a = b \times q + r$ then $0 \leq r < b$.
(a) 0,1
(b) $0, 1, 2, \cdots, 11$
(c) $0, 1, 2, \cdots, 61$
(d) $0, 1, 2, \cdots, 22$■

Problem 9.20
Which of the following properties hold for division of whole numbers?
(a) Closure (b) Commutativity (c) Associativity (d) Identity.

Solution.
(a) Does not hold since $1 \div 2 = 0.5$ which is not a whole number.
(b) Does not hold since $2 \div 1 \neq 1 \div 2$
(c) Does not hold since $3 \div (3 \div 2) \neq (3 \div 3) \div 2$
(d) Does not hold since $1 \div 2 \neq 2$■

Problem 9.21
A square dancing contest has 213 teams of 4 pairs each. How many dancers are participating in the contest?

Solution.
Each team consists of 8 dancers a total of $213 \times 8 = 1704$ dancers ■

Problem 9.22
Discuss which of the three conceptual models of division-repeated subtraction, partition, missing factor-best corresponds to the following problems. More than one model may fit.

(a) Preston owes \$3200 on his car. If his payments are \$200 a month, how many months will preston make car payments?
(b) An estate of \$76,000 is to be split among 4 heirs. How much can each heir expect to inherit?
(c) Anita was given a grant of \$375 to cover expenses on her trip. She expects that it will cost her \$75 a day. How many days can she plan to be gone?

Solution.
(a) Repeated subtraction model
(b) Partition model
(c) Missing factor model■

Problem 9.23
Solve for the unknown whole number in the following expressions:
(a) When y is divided by 5 the resulting quotient is 5 and the remainder is 4.
(b) When 20 is divided by x the resulting quotient is 3 and the remainder is 2.

Solution.
(a) We have $y = 5 \times q + 4$. By guessing and checking we see that $y = 29$
(b) We have $20 = 3 \times x + 2$. By guessing and checking we find $x = 8$■

Problem 9.24
Place parentheses, if needed, to make each of the following equations true:
(a) $4 + 3 \times 2 = 14$
(b) $9 \div 3 + 1 = 4$
(c) $5 + 4 + 9 \div 3 = 6$
(d) $3 + 6 - 2 \div 1 = 7$.

Solution.
(a) $(4 + 3) \times 2 = 14$
(b) $9 \div 3 + 1 = 4$. No parentheses needed
(c) $(5 + 4 + 9) \div 3 = 6$
(d) $3 + 6 - 2 \div 1 = 7$. No parentheses needed ■

Problem 9.25
A number leaves remainder 6 when divided by 10. What is the remainder when the number is divided by 5?

Solution.
Let a be the number. We are given that $a = 10 \times q + 6$. Then $a = (10 \times q + 5) + 1 = 5 \times q_0 + 1$ where $q_0 = 2q + 1$. Thus the remainder of the division of a by 5 is 1■

Problem 9.26
Is $x \div x$ always equal to 1? Explain your answer.

Solution.
This is always true except when $x = 0$. Division by 0 is undefined■

Problem 9.27
Find infinitely many whole numbers that leave remainder 3 upon division by 5.

Solution.
The number must be of the form $a = 5 \times q + 3$ where q is any whole number. Thus, by assigning the values $q = 0, 1, 2, \cdots$ we find

$$\{3, 8, 13, 18, 23, \cdots\}$$

■

Problem 9.28
Steven got his weekly paycheck. He spent half of it on a gift for his mother. Then he spent \$8 on a pizza. Now he has \$19. How much was his paycheck?

Solution.
Let x be the amount of his paycheck. Then $x \div 2 - 8 = 19$. Adding 8 to both sides we find $x \div 2 = 27$. This shows that $x = \$54$ ■

Problem 10.1
Using the definition of $<$ and $>$ given in this section, write four inequality statements based on the fact that $2 + 8 = 10$.

Solution.
$2 < 10, \quad 8 < 10, \quad 10 > 2$, and $10 > 8$■

Problem 10.2
The statement $a < x < b$ is equivalent to writing $a < x$ and $x < b$ and is called a **compound inequality.** Suppose that a, x, and b are whole numbers such that $a < x < b$. Is it is always true that for any whole number c we have $a + c < x + c < b + c$?

Solution.
If $a < x$ then there is a whole number d such that $a + d = x$. Add c to both sides to obtain $(a+c)+d = (a+x)$. This shows that $a+c < a+x$. Similarly, since $x < b$ then $x + c < b + c$ ■

Problem 10.3
Find nonzero whole number n in the definition of "less than" that verifies the following statements.
(a) $17 < 26$ (b) $113 > 49$.

Solution.
(a) $17 + 9 = 26$ so that $n = 9$
(b) $49 + 64 = 113$ so that $n = 64$■

Problem 10.4
If $a < x < b$, where a, x, b are whole numbers, and c is a nonzero whole number, is it always true that $ac < xc < bc$?

Solution.
If $a < x$ then there is a whole number n such that $a + n = x$. Multiply both sides by c to obtain $ac + cn = xc$. From the definition of inequality we see that $ac < xc$. Similarly, since $x < b$ then $xc < bc$■

Problem 10.5
True or false?
(a) $0 \geq 0$ (b) $0 < 0$ (c) $3 < 4$ (d) $2 \times 3 + 5 < 8$.

Solution.
(a) True, since $0 = 0$.
(b) False, since $0 = 0$.
(c) True, since $3 + 1 = 4$.
(d) True since $2 \times 3 + 5 = 11$ which is less than 8■

Problem 10.6
Write an inequality that describe each situation.

(a) The length of a certain rectangle must be 4 meters longer than the width, and the perimeter must be at least 120 meters.
(b) Fred made a 76 on the midterm exam. To get a B, the average of his mid-term and his final exam must be between 80 and 90.

Solution.
(a) Let W be the width of the rectangle and L its length then $L = 4 + W$. Since the perimeter is at least 120 meters then $2(W+L) \geq 120$ or $W+L \geq 60$.
(b) Let F be the grade of his final exam. Then $80 < \frac{76+F}{2} < 90$■

Problem 10.7
Find all the whole numbers x such that $3 + x < 8$.

Solution.
By guessing and checking, we find $x = 0, 1, 2, 3, 4$■

Problem 10.8
Find all the whole numbers x such that $3x < 12$.

Solution.
By guessing and checking we find $x = 0, 1, 2, 3$■

Problem 10.9
Complete the following statement: If $x - 1 < 2$ then $x <$ ___.

Solution.
If $x - 1 < 2$ then $x < 3$■

Problem 10.10
Complete the following statement: If $x + 3 < 3x + 5$ then $3x + 9 <$ _____.

Solution.
If $x + 3 < 3x + 5$ then $3x + 9 < 9x + 15$, since we are multiplying through by 3■

Problem 10.11
Rewrite the following products using exponentials.
(a) $3 \cdot 3 \cdot 3 \cdot 3$
(b) $2 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2$
(c) $a \cdot b \cdot a \cdot b$.

Solution.
(a) $3 \cdot 3 \cdot 3 \cdot 3 = 3^4$.
(b) $2 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 = 2^4 \cdot 3^2$.
(c) $a \cdot b \cdot a \cdot b = (ab)^2$ ■

Problem 10.12
Rewrite each with a single exponent.
(a) $5^3 \cdot 5^4$
(b) $3^{12} \div 3^2$
(c) $2^7 \cdot 5^7$
(d) $8 \cdot 2^5$
(e) $25^3 \div 5^2$
(f) $9^2 \cdot 12^3 \cdot 2$

Solution.
(a) $5^3 \cdot 5^4 = 5^{3+4} = 5^8$.
(b) $3^{12} \div 3^2 = 3^{12+2} = 3^{14}$.
(c) $2^7 \cdot 5^7 = (2 \cdot 5)^7 = 10^7$.
(d) $8 \cdot 2^5 = 2^3 \cdot 2^5 = 2^{3+5} = 2^8$.
(e) $25^3 \div 5^2 = (5^2)^3 \div 5^2 = 5^6 \div 5^2 = 5^{6-2} = 5^4$.
(f) $9^2 \cdot 12^3 \cdot 2 = (3^2)^2 \cdot (3 \cdot 4)^3 \cdot 2 = 3^4 \cdot 3^3 \cdot 4^3 \cdot 2 = 3^4 \cdot 3^3 \cdot 2^6 \cdot 2 = 3^7 \cdot 2^7 = (3 \cdot 2)^7 = 6^7$ ■

Problem 10.13
Find a whole number x.
(a) $3^7 \cdot 3^x = 3^{13}$
(b) $(3^x)^4 = 3^{20}$
(c) $3^x \cdot 2^x = 6^x$.

Solution.
(a) Since $3^7 \cdot 3^x = 3^{13}$ then $3^{7+x} = 3^{13}$. Thus, $7 + x = 13$ so that $x = 6$.
(b) Since $(3^x)^4 = 3^{20}$ then $3^{4x} = 3^{20}$ so that $4x = 20$. Hence, $x = 5$.
(c) Since $3^x \cdot 2^x = 6^x$ then $(3 \cdot 2)^x = 6^x$. This shows that x can be any whole number ■

Problem 10.14
The price of a candy bar doubled every five years. Suppose that the price continued to double every five years and that the candy bar cost 25 cents in 2000.
(a) What would the price of the candy bar be in the year 2015?
(b) What would the price be in the year 2040?
(c) Write an expression representing the price of the candy bar after n five years.

Solution.
(a) $((0.25 \cdot 2) \cdot 2) \cdot 2 = \$2.00$
(b) Since $40 \div 5 = 8$ then $0.25 \cdot 2^8 = \$64.00$
(c) $\$0.25 \cdot 2^n$ ■

Problem 10.15
Pizzas come in four different sizes, each with or without a choice of up to four ingredients. How many ways are there to order a pizza?

Solution.
For each size there are 15 different ways for ordering a pizza. Since we have four different sizes then there are $4 \times 15 = 60$ different ways of ordering a pizza■

Problem 10.16
Write each of the following in expanded form, i.e. without exponents.
(a) $(2x)^5$ (b) $2x^5$.

Solution.
(a) $(2x)^5 = (2x)(2x)(2x)(2x)(2x)$
(b) $2x^5 = 2 \cdot x \cdot x \cdot x \cdot x \cdot x$ ■

Problem 11.1
Perform each of the following computations mentally and explain what technique you used to find the answer.
(a) $40 + 160 + 29 + 31$
(b) $3679 - 474$
(c) $75 + 28$
(d) $2500 - 700$.

Solution.
(a) $40+160+29+31=(40+160)+(29+31)=200+60=260$ (compatible numbers)
(b) $3679-474=(3680-470)-5=3610-5=3605$ (breaking up and bridging)
(c) $75+28=(75+30)-2=105-2=103$ (breaking up and bridging)
(d) $2500-700$: first we find 25 - 7=18 and then add two zeros to obtain 1800 (drop the zeros technique)$\blacksquare$

Problem 11.2
Compute each of the following mentally.
(a) $180+97-23+20-140+26$
(b) $87-42+70-38+43$.

Solution.
(a) $180+97-23+20-140+26=(180-140)+(97-23)+(20+26)=40+74+46=40+120=160$
(b) $87-42+70-38+43=(87+43)-(42+38)+70=130-80+70=50+70=120$$\blacksquare$

Problem 11.3
Use compatible numbers to compute each of the following mentally.
(a) $2\cdot 9\cdot 5\cdot 6$
(b) $5\cdot 11\cdot 3\cdot 20$
(c) $82+37+18+13$.

Solution.
(a) $2\cdot 9\cdot 5\cdot 6=(9\cdot 6)\cdot(2\cdot 5)=54\cdot 10=540$
(b) $5\cdot 11\cdot 3\cdot 20=(11\cdot 3)\cdot(5\cdot 20)=33\cdot 100=3300$
(c) $82+37+18+13=(82+18)+(37+13)=100+50=150$$\blacksquare$

Problem 11.4
Use compensation to compute each of the following mentally.
(a) $85-49$
(b) $87+33$
(c) $19\cdot 6$.

Solution.
(a) $85-49=(85-50)+1=35+1=36$
(b) $87+33=90+30=120$
(c) $19\cdot 6=20\cdot 5+20-6=100+14=114$ $\blacksquare$

Problem 11.5
A car trip took 8 hours of driving at an average of 62 mph. Mentally compute the total number of miles traveled. Describe your method.

Solution.
The total number of miles traveled is 8×62 miles. To find the product, we use compensatio as follows.

$$\begin{array}{rcl} 8 \times 62 & = & 10 \times 60 + 10 \times 2 - 2 \times 60 - 2 \times 2 \\ & = & 600 + 20 - 120 - 4 \\ & = & (600 - 100) + (20 - 20) - 4) \\ & = & 500 - 4 = 496 \ miles \blacksquare \end{array}$$

Problem 11.6
Perform these calculations from left to right.
(a) $425 + 362$
(b) $572 - 251$
(c) $3 \cdot 342$
(d) $47 + 32 + 71 + 9 + 26 + 32$.

Solution.
(a) $425 + 362 = (400 + 300) + (20 + 60) + (5 + 2) = 700 + 80 + 7 = 787$
(b) $572 - 251 = (500 - 200) + (70 - 50) + (2 - 1) = 300 + 20 + 1 = 321$
(c) $3 \cdot 342 = 3 \cdot 300 + 3 \cdot 40 + 3 \cdot 2 = 900 + 120 + 6 = 1026$
(d) $47+32+71+9+26+32 = (40+30+70+20+30)+(7+2+1+9+6+2) = 190 + 27 = 217\blacksquare$

Problem 11.7
Calculate mentally using properties of operations, i.e. commutative, associative, distributive.
(a) $(37 + 25) + 43$
(b) $47 \cdot 15 + 47 \cdot 85$
(c) $(4 \times 13) \times 25$
(d) $26 \cdot 24 - 21 \cdot 24$.

Solution.
(a) $(37 + 25) + 43 = (25 + 37) + 43 = 25 + (37 + 43) = 25 + 80 = 105$
(b) $47 \cdot 15 + 47 \cdot 85 = 47 \cdot (15 + 85) = 47 \cdot 100 = 4700$
(c) $(4 \times 13) \times 25 = (13 \times 4) \times 25 = 13 \times (4 \times 25) = 13 \times 100 = 1300$
(d) $26 \cdot 24 - 21 \cdot 24 = (26 - 21) \cdot 24 = 5 \cdot 24 = 5 \cdot 20 + 5 \cdot 4 = 100 + 20 = 120\blacksquare$

Problem 11.8
Find each of the following differences using compensation method.
(a) $43 - 17$
(b) $132 - 96$
(c) $250 - 167$.

Solution.
(a) $43 - 17 = 40 - 20 + 6 = 20 + 6 = 26$
(b) $132 - 96 = 130 - 100 + 6 = 30 + 6 = 36$
(c) $250 - 167 = 250 - 170 - 3 = 80 - 3 = 77$ ■

Problem 11.9
Calculate mentally.
(a) $58,000 \times 5,000,000$
(b) $7 \times 10^5 \times 21,000$
(c) $5 \times 10^3 \times 7 \times 10^7 \times 4 \times 10^5$.

Solution.
(a) $58,000 \times 5,000,000$. We first find $58 \times 5 = 60 \times 5 - 2 \times 5 = 300 - 10 = 290$. Now we add nine zeros to this number to obtain $290,000,000,000$.
(b) $7 \times 10^5 \times 21,000 = 14,700,000,000$
(c) $5 \times 10^3 \times 7 \times 10^7 \times 4 \times 10^5 = 140 \times 10^{15}$ ■

Problem 11.10
Show the steps for three different ways to compute mentally $93 + 59$.

Solution.
$93 + 59 = (90 + 50) + (3 + 9) = 140 + 12 = 152$
$93 + 60 = 153$ and $153 - 1 = 152$ so that $93 + 59 = 152$
$93 + 50 = 143$ and $143 + 9 = 152$ so that $93 + 59 = 152$ ■

Problem 11.11
Show the steps for three different ways to compute mentally $134 - 58$.

Solution.
$134 - 50 = 84$ and $84 - 8 = 76$
$134 - 58 = 136 - 60 = 76$
$130 - 50 = 80$ and $8 - 4 = 4$ and $80 - 4 = 76$ ■

Problem 11.12
Show the steps to compute mentally $(500)^3$.

Solution.
$5^3 = 125$ and then add 6 zeroes: 125,000,000■

Problem 11.13
A restaurant serves launch to 90 people per day. Show the steps to mentally compute the number of people served lunch in 31 days.

Solution.
we want to compute 90×31. Using the property of distribution over addition we have $90 \times 31 = 90 \times 30 + 90 \times 1 = 2700 + 90 = 2790$■

Problem 11.14
There is a shortcut for multiplying a whole number by 99. For example, consider 15×99.
(a) Why does $15 \times 99 = (15 \times 100) - (15 \times 1)$?
(b) Compute 15×99 mentally, using the formula in part (a)
(c) Compute 95×99 mentally, using the same method.

Solution.
(a) Distribution of multiplication over subtraction.
(b) $15 \times 99 = (15 \times 100) - (15 \times 1) = 1500 - 15 = 1485$
(c) $95 \times 99 = 95 \times 100 - 95 \times 1 = 9500 - 95 = 9405$■

Problem 11.15
(a) Develop a shortcut for multiplying by 25 mentally in a computation such as 24×25.
(b) Compute 44×25 using the same shortcut.

Solution.
(a) $24 \times 25 = 20 \times 25 + 4 \times 25 = 500 + 100 = 600$
(b) $44 \times 25 = 40 \times 25 + 4 \times 25 = 1000 + 100 = 1100$■

Problem 11.16
(a) Develop a shortcut for multiplying by 5 mentally in a computation such as 27×5.
(b) Compute 42×5 using the same shortcut

Solution.
(a) $27 \times 5 = 20 \times 5 + 7 \times 5 = 100 + 35 = 135$
(b $42 \times 5 = 40 \times 5 + 2 \times 5 = 200 + 10 = 210$■

Problem 11.17
A fifth grader computes 29×12 as follows: $30 \times 12 = 360$ and $360 - 12 = 348$. On what property is the child's method based?

Solution.
Since 29 = 30 - 1 then the child is doing $29 \times 12 = (30 - 1) \times 12 = 30 \times 12 - 1 \times 12$. So he is using the property of distribution of multiplication over subtraction■

Problem 11.18
Round 235,476 to the nearest
(a) ten thousand
(b) thousand
(c) hundred.

Solution.
(a) $235,476 \approx 240,000$
(b) $235,476 \approx 235,000$
(c) $235,476 \approx 235,500$■

Problem 11.19
Round each of these to the position indicated.
(a) 947 to the nearest hundred.
(b) 27,462,312 to the nearest million.
(c) 2461 to the nearest thousand.

Solution.
(a) $947 \approx 900$
(b) $27,462,312 \approx 27,000,000$
(c) $2461 \approx 2000$■

Problem 11.20
Rounding to the left-most digit, calculate approximate values for each of the following:
(a) $681 + 241$

(b) $678 - 431$
(c) 257×364
(d) $28,329 \div 43$.

Solution.
(a) $681 + 241 \approx 700 + 200 = 900$
(b) $678 - 431 \approx 700 - 400 = 300$
(c) $257 \times 364 \approx 300 \times 400 = 12,000$
(d) $28,329 \div 43 \approx 28,000 \div 40 = 700$ ■

Problem 11.21
Using rounding to the left-most digit, estimate the following products.
(a) 2748×31 (b) 4781×342 (c) $23,247 \times 357$.

Solution.
(a) $2748 \times 31 \approx 3000 \times 30 = 90,000$
(b) $4781 \times 342 \approx 5000 \times 300 = 1,500,000$
(c) $23,247 \times 357 \approx 23,000 \times 400 = 9,200,000$ ■

Problem 11.22
Round each number to the position indicated.
(a) 5280 to the nearest thousand
(b) 115,234 to the nearest ten thousand
(c) 115,234 to the nearest hundred thousand
(d) 2,325 to the nearest ten.

Solution.
(a) $5280 \approx 5000$
(b) $115,234 \approx 120,000$
(c) $115,234 \approx 100,000$
(d) $2,325 \approx 2330$ ■

Problem 11.23
Use front-end estimation with adjustment to estimate each of the following:
(a) $2215 + 3023 + 5987 + 975$
(b) $234 + 478 + 987 + 319 + 469$.

Solution.
(a) We first do $2000 + 3000 + 5000 = 10,000$ and then $200 + 30 + 1000 + 70 =$

1300 so that $2215 + 3023 + 5987 + 975 \approx 11,300$
(b) We first do $200+400+900+300+400 = 2,200$ and then $30+80+20+70 = 200$ so that $234 + 478 + 987 + 319 + 469 \approx 2,200 + 200 = 2,400$■

Problem 11.24
Use range estimation to estimate each of the following.
(a) $22 \cdot 38$ (b) $145 + 678$ (c) $278 + 36$.

Solution.
(a) $22 \cdot 38$ ranges from $20 \times 30 = 600$ to $30 \times 40 = 1,200$
(b) $145 + 678$ ranges from $100 + 600 = 700$ to $200 + 700 = 900$
(c) $278 + 36$ ranges from $200 + 30 = 230$ to $300 + 40 = 340$■

Problem 11.25
Tom estimated $31 \cdot 179$ in the three ways shown below.
(i) $30 \cdot 200 = 6000$
(ii) $30 \cdot 180 = 5400$
(iii) $31 \cdot 200 = 6200$
Without finding the actual product, which estimate do you think is closer to the actual product? Explain.

Solution.
The second seems to be closer to the actual product since the factors are closer to the original numbers than the other two choices■

Problem 11.26
About 3540 calories must be burned to lose 1 pound of body weight. Estimate how many calories must be burned to lose 6 pounds.

Solution.
About $3600 \times 6 = 21,600$ calories must be burned■

Problem 11.27
A theater has 38 rows and 23 seats in each row. Estimate the number of seats in the theater and tell how you arrived at your estimate.

Solution.
For example, $40 \times 20 = 800$ seats or $40 \times 25 = 1000$ seats, 800 will be low and 1000 will be high■

Problem 11.28
Use estimation to tell whether the following calculator answers are reasonable. Explain why or why not.
(a) $657 + 542 + 707 = 543364$
(b) $26 \times 47 = 1222$.

Solution.
(a) If we use range then the upper value is $700 + 600 + 800 = 2100$ which is still less than 543,364.
(b) $25 \times 50 = 1250$ so the answer seems reasonable■

Problem 11.29
Estimate the sum

$$87 + 45 + 37 + 22 + 98 + 51$$

using compatible numbers.

Solution.
$87 + 45 + 37 + 22 + 98 + 51 \approx 90 + 50 + 40 + 20 + 100 + 50 = 350$ ■

Problem 11.30
clustering is a method of estimating a sum when the numbers are all close to one value. For example, $3648+4281+3791 \approx 3 \cdot 4000 = 12,000$. Estimate the following using clustering.
(a) $897 + 706 + 823 + 902 + 851$
(b) $36,421 + 41,362 + 40,987 + 42,621$.

Solution.
(a) $897 + 706 + 823 + 902 + 851 \approx 5 \times 800 = 4,000$
(b) $36,421 + 41,362 + 40,987 + 42,621 \approx 4 \times 40,000 = 160,000$■

Problem 11.31
Estimate each of the following using (i) range estimation, (ii) one-column front-end estimation (iii) two-column fron-end estimation, and (iv) front-end with adjustment.
(a) $3741 + 1252$
(b) $1591 + 346 + 589 + 163$
(c) $2347 + 58 + 192 + 5783$.

Solution.
(a) (i) Range: The range is from $3000 + 1000 = 4000$ to $4000 + 2000 = 6000$
(ii) $3741 + 1252 \approx 3000 + 1000 = 4000$
(iii) $3741 + 1252 \approx 3700 + 1200 = 4900$
(iv) We do first $3000 + 1000 = 4000$ and then $740 + 250 = 990$ so that $3741 + 1252 \approx 4000 + 990 = 4990$
(b) (i) Range: The range is from $1000 + 300 + 500 + 100 = 1900$ to $2000 + 400 + 600 + 200 = 3200$
(ii) $1591 + 346 + 589 + 163 \approx 1000$
(iii) $1591 + 346 + 589 + 163 \approx 1500 + 300 + 500 + 100 = 2400$
(iv) $1591 + 346 + 589 + 163 \approx 1000 + (590 + 350 + 590 + 160) = 2690$
(c) (i) Range: The range is from $2000 + 50 + 100 + 5000 = 7150$ to $3000 + 60 + 200 + 6000 = 9260$
(ii) $2347 + 58 + 192 + 5783 \approx 2000 + 5000 = 7000$
(iii) $2347 + 58 + 192 + 5783 \approx 2300 + 100 + 5700 = 8100$
(iv) We do first $2000 + 5000 = 7000$ and then $60 + 190 + 790 = 1080$ so that $2347 + 58 + 192 + 5783 \approx 7000 + 1080 = 8080$■

Problem 11.32
Estimate using compatible number estimation.
(a) 51×212 (b) $3112 \div 62$ (c) 103×87.

Solution.
(a) $51 \times 212 \approx 50 \times 200 = 10,000$
(b) $3112 \div 62 \approx 3000 \div 60 = 50$
(c) $103 \times 87 \approx 100 \times 87 = 8700$■

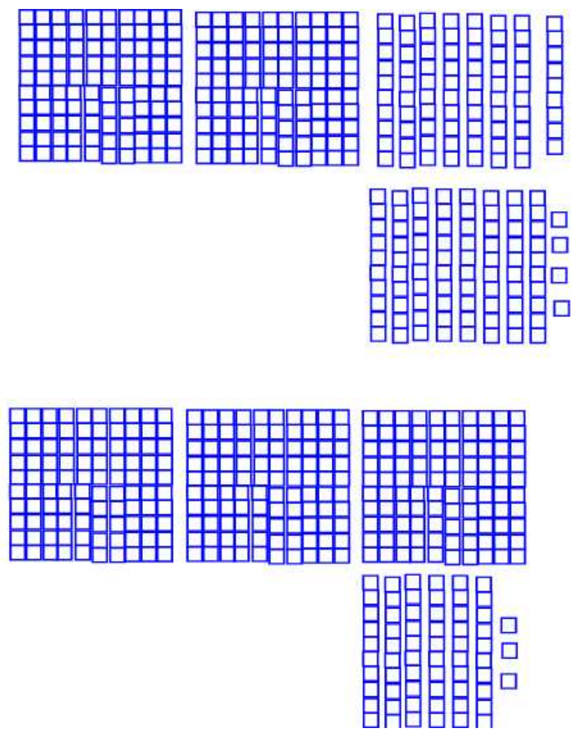
Problem 12.1
Use the addition expanded algorithm as discussed in this section to perform the following additions:
(a) $23 + 44$ (b) $57 + 84$ (c) $324 + 78$

Solution.

$$\text{(a)}\quad \begin{array}{r} 23 \\ +44 \\ \hline 7 \\ +60 \\ \hline 67 \end{array} \qquad \text{(b)}\quad \begin{array}{r} 57 \\ +\ 84 \\ \hline 11 \\ +130 \\ \hline 141 \end{array} \qquad \text{(c)}\quad \begin{array}{r} 324 \\ +\ \ 78 \\ \hline 12 \\ 90 \\ 300 \\ \hline 402 \end{array}$$

Problem 12.2
Use base ten blocks to represent the sum $279 + 84$.

Solution.



Problem 12.3
State the property that justifies each of the following steps.

$$\begin{array}{rcll}
36+52 & = & (3\cdot 10+6)+(5\cdot 10+2) & \underline{\qquad} \\
 & = & 3\cdot 10+[6+(5\cdot 10+2)] & \underline{\qquad} \\
 & = & 3\cdot 10+[(6+5\cdot 10)+2] & \underline{\qquad} \\
 & = & 3\cdot 10+[(5\cdot 10+6)+2] & \underline{\qquad} \\
 & = & 3\cdot 10+[5\cdot 10+(6+2)] & \underline{\qquad} \\
 & = & (3\cdot 10+5\cdot 10)+(6+2) & \underline{\qquad} \\
 & = & (3+5)\cdot 10+(6+2) & \underline{\qquad} \\
 & = & 8\cdot 10+8 & \underline{\qquad} \\
 & = & 88 &
\end{array}$$

Solution.

$$\begin{aligned}
36+52 &= (3\cdot 10+6)+(5\cdot 10+2) && \textit{expanded notation}\\
&= 3\cdot 10+[6+(5\cdot 10+2)] && \textit{associative}\\
&= 3\cdot 10+[(6+5\cdot 10)+2] && \textit{associative}\\
&= 3\cdot 10+[(5\cdot 10+6)+2] && \textit{commutative}\\
&= 3\cdot 10+[5\cdot 10+(6+2)] && \textit{associative}\\
&= (3\cdot 10+5\cdot 10)+(6+2) && \textit{associative}\\
&= (3+5)\cdot 10+(6+2) && \textit{distributive}\\
&= 8\cdot 10+8 && \textit{addition facts}\\
&= 88
\end{aligned}$$

Problem 12.4
Find the missing digits.

$$\text{(a)}\ \begin{array}{r} _437\\ 2_1\\ +\,347_\\ \hline 6_94 \end{array}
\qquad \text{(b)}\ \begin{array}{r} _721\\ 901_\\ +71_3\\ \hline _0_26 \end{array}
\qquad \text{(c)}\ \begin{array}{r} 38_1\\ 24_3\\ +512_\\ \hline _\,_5_9 \end{array}$$

Solution.

$$\text{(a)}\ \begin{array}{r} 2437\\ 281\\ +\ 3476\\ \hline 6194 \end{array}
\qquad \text{(b)}\ \begin{array}{r} 4721\\ 9012\\ +\,7193\\ \hline 20926 \end{array}
\qquad \text{(c)}\ \begin{array}{r} 3891\\ 2493\\ +5125\\ \hline 11509 \end{array}$$

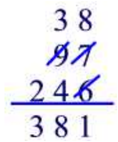
Problem 12.5
Julien Spent one hour and 45 minutes mowing the lawn and two hours and 35 minutes trimming the hedge and some shrubs. How long did he work all together?

Solution.

$$\begin{array}{l} \quad\ \text{1 hour, 45 minutes}\\ +\ \text{2 hours, 35 minutes}\\ \hline \quad\ \text{3 hours, 60 minutes} = \text{4 hours} \end{array}$$

Problem 12.6
Compute the sum $38+97+246$ using scratch addition.

Solution.

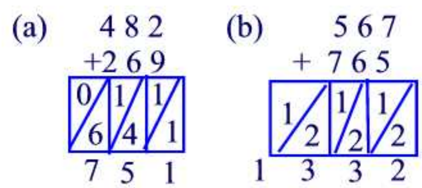


Problem 12.7
Find the sum 3 *hr* 36 *min* 58 *sec* + 5 *hr* 56 *min* 27 *sec*.

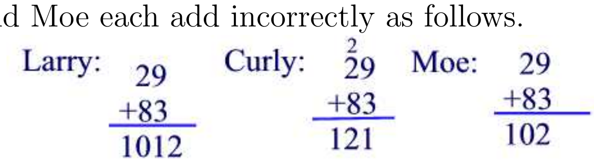
Solution.
3 *hr* 36 *min* 58 *sec* + 5 *hr* 56 *min* 27 *sec* = 8 *hr* 92 *min* 85 *min* = 8 *hr* 93 *min* 35 *sec* = 9 *hr* 33 *min* 35 *sec*■

Problem 12.8
Compute the following sums using the lattice method.
(a) $482 + 269$ (b) $567 + 765$.

Solution.



Problem 12.9
Larry, Curly, and Moe each add incorrectly as follows.



How would you explain their mistakes to each of them?

Solution.
Larry is not carrying properly; Curly carries the wrong digit; Moe forgets to Carry■

Problem 12.10
A **palindrome** is any number that reads the same backward as forward, for example, 121 and 2332. Try the following. Begin with any number.

Is it a palindrome? If not, reverse the digits and add this new number to the old one. Is the result a palindrome?If not, repeat the above procedure until a palindrome is obtained. For example, suppose you start with 78. Because 78 is not a palindrome, we add $78 + 87 = 165$. Since 165 is not a palindrome we add $165 + 561 = 726$. Since 726 is not a palindrome we add $726+627 = 1353$. Since 1353 is not a palindrome we add $1353+3531 = 4884$ which is a palindrome. Try this method with the following numbers:
(a) 93 (b) 588 (c) 2003.

Solution.
(a)

$$\begin{aligned} 93 + 39 &= 132 \\ 132 + 231 &= 363 \textit{ a palindrome} \end{aligned}$$

(b)

$$\begin{aligned} 588 + 885 &= 1473 \\ 1473 + 3741 &= 5214 \\ 5214 + 4125 &= 9339 \textit{ a palindrome} \end{aligned}$$

(c)

$$2003 + 3002 = 5005 \textit{ a palindrome}$$

Problem 12.11
Another algorithm for addition uses the so-called **partial sums**. The digits in each column are summed and written on separate lines as shown below.

$$\begin{array}{cccc} & 6 & 3 & 2 \\ + & 7 & 9 & 8 \\ \hline & & 1 & 0 \\ & 1 & 2 & \\ 1 & 3 & & \\ \hline 1 & 4 & 3 & 0 \end{array}$$

Using this method, compute the following sums:
(a) $598 + 396$ (b) $322 + 799 + 572$.

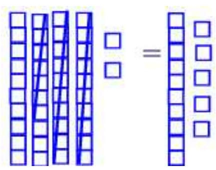
Solution.

$$
\begin{array}{lr}
\text{(a)} & 598 \\
 & +396 \\
\hline
 & 14 \\
 & 18 \\
 & 8 \\
\hline
 & 994
\end{array}
\qquad
\begin{array}{lr}
\text{(b)} & 322 \\
 & +799 \\
 & 572 \\
\hline
 & 13 \\
 & 18 \\
 & 15 \\
\hline
 & 1693
\end{array}
$$

Problem 12.12

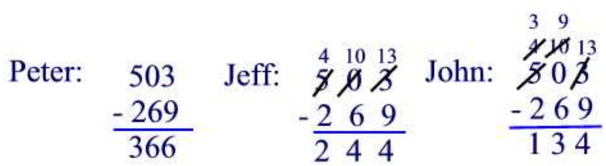
Sketch the solution to $42 - 27$ using base-ten blocks.

Solution.



Problem 12.13

Peter, Jeff, and John each perform a subtraction incorrectly as follows:



How would you explain their mistakes to each one of them?

Solution.

Peter is taking the smaller digit from the larger digit in each column; Jeff is not exchanging the tens digits properly; John is not exchanging the hundreds digits properly■

Problem 12.14

Find the difference 5 *hr* 36 *min* 38 *sec* − 3 *hr* 56 *min* 58 *sec*.

Solution.

$5\ hr\ 36\ min\ 38\ sec - 3\ hr\ 56\ min\ 58\ sec = 4\ hr\ 95\ min\ 98\ sec - 3\ hr\ 56\ min\ 58\ sec = 1 hr\ 39\ min\ 40\ sec$■

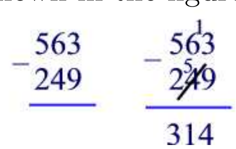
Problem 12.15
In subtracting 462 from 827, the 827 must be regrouped as _____ hundreds, _____ tens, and _____ ones.

Solution.
7 hundreds, 12 tens, and 7 ones■

Problem 12.16
Suppose you add the same amount to both numbers of a subtraction problem. What happens to the answer? Try the following.
(a) What is $86 - 29$?
(b) Add 11 to both numbers in part (a) and subtract. Do you obtain the same number?

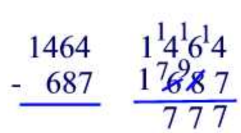
Solution.
(a) $86 - 29 = 57$
(b) $(86 + 11) - (29 + 11) = 97 - 40 = 57$ which is the same answer as in part (a)■

Problem 12.17
The **equal-addition algorithm** has been used in some US schools in the past 60 years. The property developed in the preceding problem is the basis for this algorithm. For example, in computing $563 - 249$, one needs to add 10 to 3. To compensate, one adds 10 to 249. Then the subtraction can be done without regrouping as shown in the figure below.



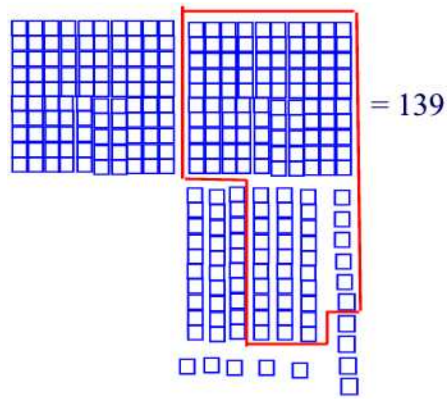
Compute the difference $1464 - 687$ using the equal-addition algorithm.

Solution.



Problem 12.18
Sketch the solution to $275 - 136$ using base-ten blocks.

Solution.



Problem 12.19
Use the expanded algorithm to perform the following:
(a) $78 - 35$ (b) $75 - 38$ (c) $414 - 175$

Solution.

$$\text{(a)}\quad \begin{array}{r} 78 \\ -35 \\ \hline 3 \\ 40 \\ \hline 43 \end{array} \qquad \text{(b)}\quad \begin{array}{r} 75 \\ -38 \\ \hline 3 \\ 30 \\ \hline 33 \end{array} \qquad \text{(c)}\quad \begin{array}{r} 414 \\ -175 \\ \hline 9 \\ 30 \\ 200 \\ \hline 239 \end{array}$$

Problem 12.20
Fill in the missing digits.

$$\text{(a)}\quad \begin{array}{r} \_3\_ \\ -\,2\_1 \\ \hline 594 \end{array} \qquad \text{(b)}\quad \begin{array}{r} 3\_\_4 \\ -\,\_346 \\ \hline 175\_ \end{array} \qquad \text{(c)}\quad \begin{array}{r} 63\_\_4 \\ -\,2\_12\_ \\ \hline \_6209 \end{array}$$

Solution.

$$\text{(a)}\quad \begin{array}{r} 835 \\ -\,241 \\ \hline 594 \end{array} \qquad \text{(b)}\quad \begin{array}{r} 3104 \\ -\,346 \\ \hline 2758 \end{array} \qquad \text{(c)}\quad \begin{array}{r} 63334 \\ -\,27125 \\ \hline 36209 \end{array}$$

Problem 12.21
After her dad gave her her allowance of 10 dollars, Ellie had 25 dollars and 25 cents. After buying a sweater for 14 dollars and 53 cents, including tax, how much money did Ellie have left?

Solution.
She is left with $\$25\ 25\ cents - \$14\ 53\ cents = \$24\ 125\ cents - \$14\ 53\ cents = \$10\ 72\ cents$■

Problem 12.22
A hiker is climbing a mountain that is 6238 feet high. She stops to rest at 4887 feet. How many more feet must she climb to reach the top?

Solution.
She must climb $6238 - 4887 = 1351$ feet to reach the top■

Problem 13.1
(a) Compute 83×47 with the expanded algorithm.
(b) Compute 83×47 with the standard algorithm.
(c) What are the advantages and disadvantages of each algorithm?

Solution.

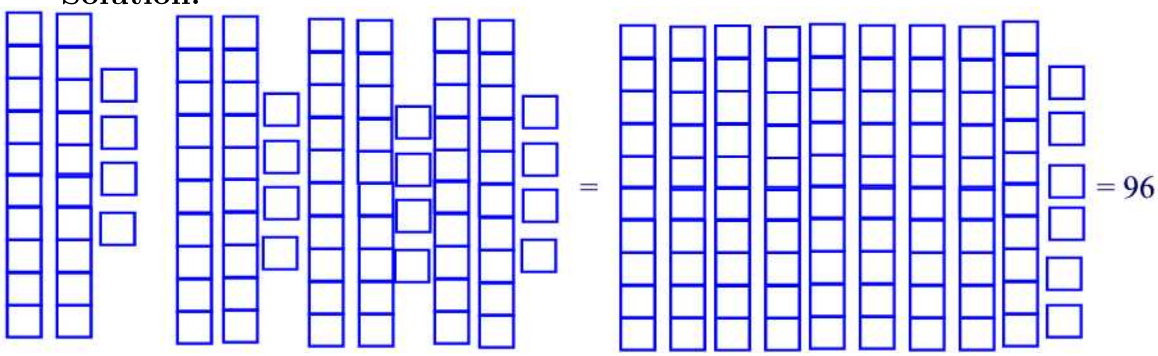
(a)
```
    83
  x 47
  ----
    21   7x3
   560   80x7
   120   3x40
  3200   80x40
  ----
  3901
```

(b)
```
    83
  x 47
  ----
   581
  332
  ----
  3901
```

(c) The expanded algorithm requires more multiplication skills then the standard algorithm. The standard algorithm is shorter to write and faster once it has been mastered.

Problem 13.2
Suppose you want to introduce a fourth grader to the standard algorithm for computing 24×4. Explain how to find the product with base-ten blocks. Draw a picture.

Solution.



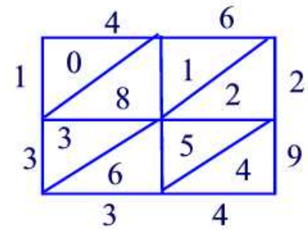
Problem 13.3
In multiplying 62×3, we use the fact that $(60 + 2) \times 3 = (60 \times 3) + (2 \times 3)$. What property does this equation illustrate?

Solution.
Distributive property of multiplication over addition■

Problem 13.4
(a) Compute 46×29 with lattice multiplication.
(b) Compute 234×76 with lattice multiplication.

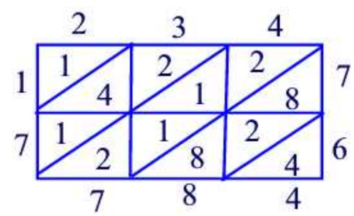
Solution.

(a)



46 x 29 = 1334

(b)



234 x 76 = 17784

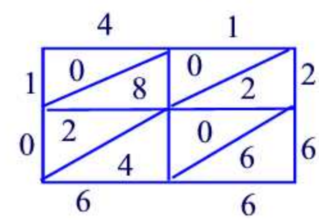
Problem 13.5
Show two other ways besides the standard algorithm to compute 41×26.

Solution.

Expanded Algorithm

$$\begin{array}{r} 41 \\ \times\, 26 \\ \hline 6 \\ 20 \\ 240 \\ 800 \\ \hline 1066 \end{array}$$

Lattice Algorithm



41 x 26 = 1066

Problem 13.6
Four fourth graders work out 32×15. Tell whether each solution is correct. If so, what does the child understand about multiplication? If the answer is wrong, what would you tell the child about how to solve the problem?
(a) 32×10 is 320. Add half of 320, which is 160. You get 480.

(b)
$$\begin{array}{r} 32 \\ \times\ 15 \\ \hline 160 \\ 32 \\ \hline 480 \end{array}$$

(c)
$$\begin{array}{r} 32 \\ \times\ 15 \\ \hline 160 \\ 32 \\ \hline 192 \end{array}$$

(d) 32×15 is the same as 16×30, which is 480.

Solution.
(a) Breaking apart the number 15 and then using the distributive property of multiplication over addition.
(b) Student is just using the standard algorithm of multiplication.
(c) Student is not using the standard algorithm properly. The digit 2 should be placed under 6 and not 0.
(d) $32 \times 15 = (30 + 2)(16 - 1) = 30 \times 16 - 30 + 2 \times 16 - 2 = 30 \times 16$. Student is using distributive property of multiplication over both addition and subtraction■

Problem 13.7
Compute 18×127 using the Russian peasant algorithm.

Solution.

	Halving	Doubling	
smaller factor →	18	127	← Larger factor
	9	254	
	4	508	
	2	1016	
	1	2032	

$$18 \times 127 = 254 + 2032 = 2286$$

Problem 13.8
What property of the whole numbers justifies each step in this calculation?

$$\begin{array}{rcc r}
17 \cdot 4 & = & (10+7) \cdot 4 & \textit{Expanded notation} \\
 & = & 10 \cdot 4 + 7 \cdot 4 & \textit{distributivr} \\
 & = & 10 \cdot 4 + 28 & \textit{multiplication} \\
 & = & 10 \cdot 4 + (2 \cdot 10 + 8) & \textit{expanded notation} \\
 & = & 4 \cdot 10 + (2 \cdot 10 + 8) & \underline{\qquad} \\
 & = & (4 \cdot 10 + 2 \cdot 10) + 8 & \underline{\qquad} \\
 & = & (4+2) \cdot 10 + 8 & \underline{\qquad} \\
 & = & 60 + 8 & \textit{multiplication} \\
 & = & 68 & \textit{addition}
\end{array}$$

Solution.

$$\begin{array}{rcc r}
17 \cdot 4 & = & (10+7) \cdot 4 & \textit{Expanded notation} \\
 & = & 10 \cdot 4 + 7 \cdot 4 & \underline{\qquad} \\
 & = & 10 \cdot 4 + 28 & \textit{multiplication} \\
 & = & 10 \cdot 4 + (2 \cdot 10 + 8) & \textit{expanded notation} \\
 & = & 4 \cdot 10 + (2 \cdot 10 + 8) & \textit{commutative} \\
 & = & (4 \cdot 10 + 2 \cdot 10) + 8 & \textit{associative} \\
 & = & (4+2) \cdot 10 + 8 & \textit{distributive} \\
 & = & 60 + 8 & \textit{multiplication} \\
 & = & 68 & \textit{addition} \blacksquare
\end{array}$$

Problem 13.9
Fill in the missing digit in each of the following.

$$\text{(a)}\ \begin{array}{r}
4_6 \\
\times\ \ 783 \\
\hline
1_78 \\
3408 \\
_982 \\
\hline
3335_8
\end{array}
\qquad
\text{(b)}\ \begin{array}{r}
327 \\
\times\ \ 9_1 \\
\hline
327 \\
1_08 \\
_9_3 \\
\hline
30_\,_07
\end{array}$$

Solution.

$$
\text{(a)}\quad
\begin{array}{r}
4\underline{2}6 \\
\times\ \ 783 \\
\hline
1\underline{2}78 \\
3408 \\
\underline{2}982 \\
\hline
3335\underline{5}8
\end{array}
\qquad\qquad
\text{(b)}\quad
\begin{array}{r}
327 \\
\times\ \ 9\underline{4}1 \\
\hline
327 \\
1\underline{2}08 \\
\underline{2}9\underline{4}3 \\
\hline
30\underline{6}\underline{7}07
\end{array}
$$

Problem 13.10

Complete the following table:

a	b	ab	a+b
	56	3752	
32			110
		270	33

Solution.

a	b	ab	a+b
67	56	3752	123
32	78	2496	110
15	18	270	33

Problem 13.11

Find the products of the following and describe the pattern that emerges.

(a)

$$
\begin{array}{ccc}
1 & \times & 1 \\
11 & \times & 11 \\
111 & \times & 111 \\
1111 & \times & 1111
\end{array}
$$

(b)

$$
\begin{array}{ccc}
99 & \times & 99 \\
999 & \times & 999 \\
9999 & \times & 9999
\end{array}
$$

Solution.

$$
\begin{array}{ccccr}
1 & \times & 1 & = & 1 \\
11 & \times & 11 & = & 121 \\
111 & \times & 111 & = & 12321 \\
1111 & \times & 1111 & = & 1234321
\end{array}
$$

If a number consists only of ones say, it has n ones then when multiplied by itself the result is

$$123\cdots n(n-1)(n-2)\cdots 321$$

(b)

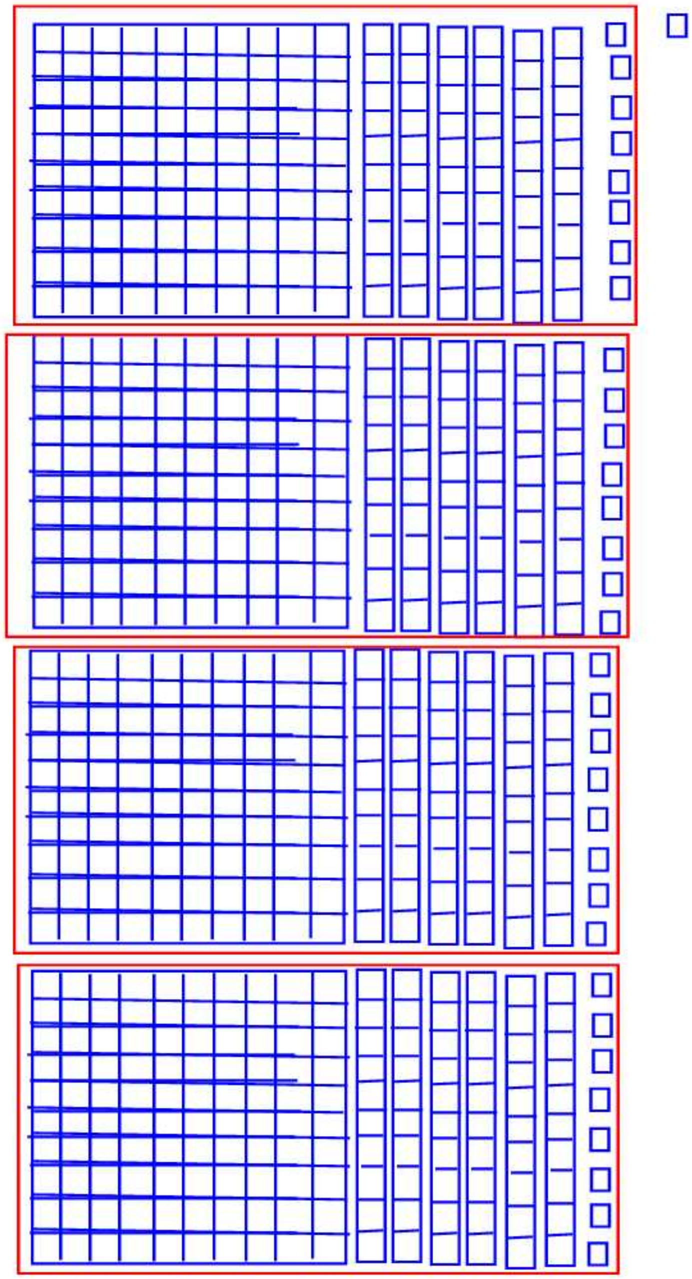
$$\begin{array}{ccccc} 99 & \times & 99 & = & 9801 \\ 999 & \times & 999 & = & 998001 \\ 9999 & \times & 9999 & = & 99980001 \end{array}$$

Thus, if a number consists of only nines, say of n nines then the product start with $n-1$ nines followed by an 8, then followed by $n-1$ zeros and ends with 1■

Problem 13.12

Sketch how to use base-ten blocks to model the operation $673 \div 4$.

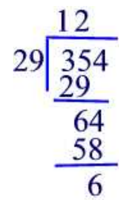
Solution.



It follows that the quotient is 168 and the remainder is 4■

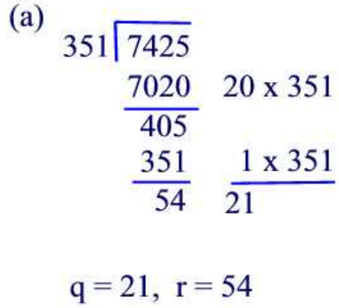
Problem 13.13
Use the standard algorithm to find the quotient and the remainder of the division $354 \div 29$.

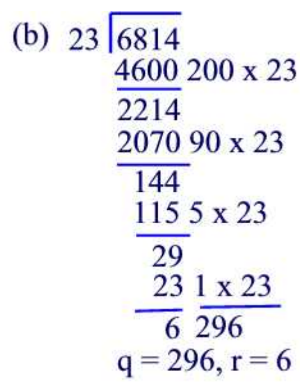
Solution.



Problem 13.14
Perform each of the following divisions by the scaffold method.
(a) $7425 \div 351$ (b) $6814 \div 23$

Solution.





Problem 13.15
Two fourth graders work out $56 \div 3$. Tell whether each solution is correct. If so, what does the child understand about division? In each case, tell what the child understands about division?
(a) How many 3s make 56? Ten 3s make 30. That leaves 26. That will take 8 more 3s, and 2 are left over. So the quotient is 18 and the remainder is 2.
(b) Twenty times 3 is 60. That is too much. Take off two 3s. That makes eighteen 3s and 2 extra. Thus, the quotient is 18 and the remainder is 2.

Solution.
(a) The child is using sort of repeated subtraction algorithm.
(b) The child is using standard algorithm■

Problem 13.16
Suppose you want to introduce a fourth grader to the standard algorithm for computing $246 \div 2$. Explain how to find the the quotient with base ten blocks.

Solution.
You want to divide 246 into 2 equal groups. We start with the hundreds so we put i hundred in each group. Next divide the 4 tens into two equal groups putting 2 tens in each group. Finally divide the 6 ones into two equal groups. Put 3 ones in each group. Hence, each group consists of one hundred, two tens and 3 ones so that the quotient is 123$\blacksquare$

Problem 13.17
A fourth grader works out $117 \div 6$ as follows. She finds $100 \div 6$ and $17 \div 6$. She gets $16 + 2 = 18$ sixes and 9 left over. Then $9 \div 6$ gives 1 six with 3 left over. So the quotient of the division $117 \div 6$ is 19 and the remainder is 3.
(a) Tell how to find $159 \div 7$ with the same method.
(b) How do you think this method compares to the standard algorithm?

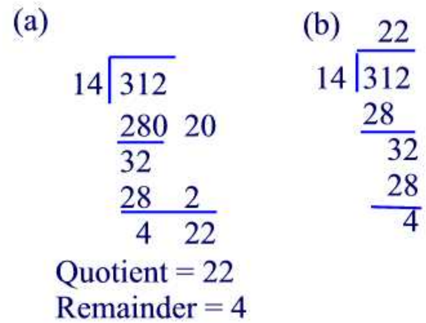
Solution.
(a) First we find $100 \div 7$ and $59 \div 7$ to get $14 + 8 = 22$ sevens with 5 left over. Thus, the quotient is 22 and the remainder is 5.
(b) This is more or less similar to the standard algorithm$\blacksquare$

Problem 13.18
Find the quotient and the remainder of $8569 \div 23$ using a calculator.

Solution.
Using a calculator ww find $8569 \div 23 \approx 372.56$ so that the quotient is 372. The remainder is $r = 8569 - 372 \times 23 = 13$ $\blacksquare$

Problem 13.19
(a) Compute $312 \div 14$ with the repeated subtraction algorithm.
(b) Compute $312 \div 14$ with the standard algorithm.
(c) What are the advantages and disadvantages of each algorithm?

Solution.



(c) Easier to understand place value in repeated subtraction; standard algorithm is faster■

Problem 13.20
Using a calculator, Ralph multiplied by 10 when he should have divided by 10. The display read 300. What should the correct answer be?

Solution.
Let x be the correct answer. Then $10x = 300$ so that $x = 30$■

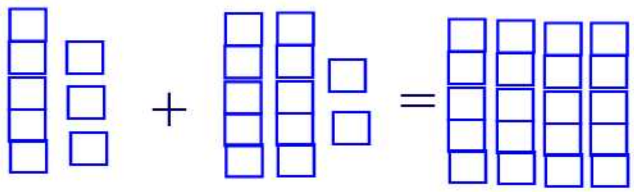
Problem 13.21
Suppose $a = 131 \times 4789 + 200$. What is the quotient and the remainder of the division of a by 131?

Solution.
Since $200 = 131 + 69$ then $a = 131 \times 4789 + 131 + 69 = 131 \times (4789 + 1) + 69 = 131 \times 4790 + 69$. Thus, the quotient is 131 and the remainder is 69■

Problem 14.1
Compute the sum $13_{five} + 22_{five}$ using
(a) base five blocks
(b) expanded algorithm
(c) lattice algorithm
(d) standard algorithm.

Solution.

(a)



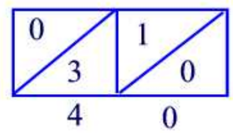
$13_{five} + 22_{five} = 40_{five}$

(b)

$$\begin{array}{r} 13_{five} \\ +\ 22_{five} \\ \hline 10_{five} \\ +\ 30_{five} \\ \hline 40_{five} \end{array}$$

(c)

$$\begin{array}{r} 13 \\ +\ 22 \end{array}$$



(d)

$$\begin{array}{r} 13_{five} \\ +\ 22_{five} \\ \hline 40_{five} \end{array}$$

Problem 14.2
Perform the following computations.

(a) $\begin{array}{r} 23_{five} \\ +34_{five} \\ \hline \end{array}$ (b) $\begin{array}{r} 312_{five} \\ +132_{five} \\ \hline \end{array}$ (c) $\begin{array}{r} 432_{five} \\ +233_{five} \\ \hline \end{array}$

Solution.

(a) $\begin{array}{r} 23_{five} \\ +34_{five} \\ \hline 112_{five} \end{array}$ (b) $\begin{array}{r} 312_{five} \\ +132_{five} \\ \hline 1011_{five} \end{array}$ (c) $\begin{array}{r} 432_{five} \\ +233_{five} \\ \hline 1220_{five} \end{array}$

Problem 14.3
Complete the following base eight addition table.

+	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

Solution.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

Problem 14.4
Compute $132_{eight} + 66_{eight}$.

Solution.
$132_{eight} + 66_{eight} = 220_{eight}$■

Problem 14.5
Computers use base two since it contains two digits, 0 and 1, that correspond to electronic switches in the computer being off or on. In this base, $101_{two} = 1 \cdot 2^2 + 0 \cdot 2 + 1 = 5_{ten}$.
(a) Construct addition table for base two.
(b) Write 1101_{two} in base ten.
(c) Write 123_{ten} in base two.
(d) Compute $1011_{two} + 111_{two}$.

Solution.
(a)

+	0	1
0	0	1
1	1	10

(b) $1101_{two} = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2 + 1 \times 2^0 = 8 + 4 + 1 = 13$
(c)

$$\begin{array}{rcl} 123 & = & 61 \times 2 + 1 \\ 61 & = & 30 \times 2 + 1 \\ 30 & = & 15 \times 2 + 0 \\ 15 & = & 7 \times 2 + 1 \\ 7 & = & 3 \times 2 + 1 \\ 3 & = & 1 \times 2 + 1 \\ 1 & = & 0 \times 2 + 1 \end{array}$$

Thus, $123_{ten} = 1111011_{two}$
(d) $1011_{two} + 111_{two} = 10010_{two}$ ■

Problem 14.6
For what base b would $32_b + 25_b = 57_b$?

Solution.
b is any whole number greater than 7■

Problem 14.7
(a) Construct an addition table for base four.
(b) Compute $231_{four} + 121_{four}$.

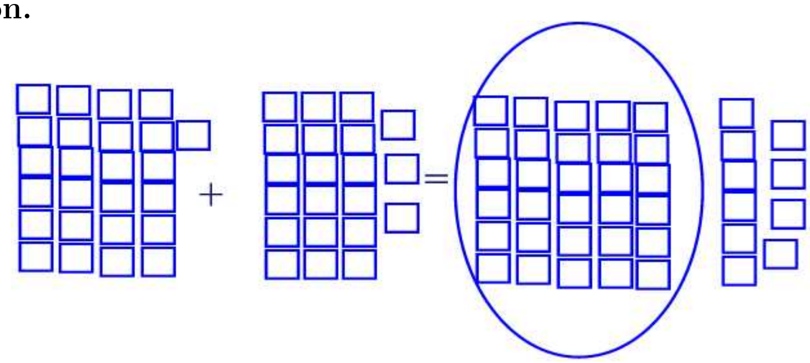
Solution.
(a)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	10
2	2	3	10	11
3	3	10	11	12

(b) $231_{four} + 121_{four} = 1012_{four}$■

Problem 14.8
Use blocks to illustrate the sum $41_{six} + 33_{six}$.

Solution.



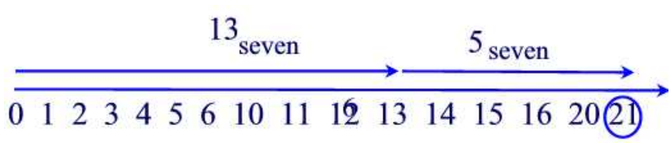
Problem 14.9
Use an expanded algorithm to compute $78_{nine} + 65_{nine}$.

Solution.

$$\begin{array}{r} 78_{\text{nine}} \\ +\ 65_{\text{nine}} \\ \hline 14 \\ 140 \\ \hline 154_{\text{nine}} \end{array}$$

Problem 14.10
Create a base seven number line and illustrate the sum $13_{seven} + 5_{seven}$.

Solution.



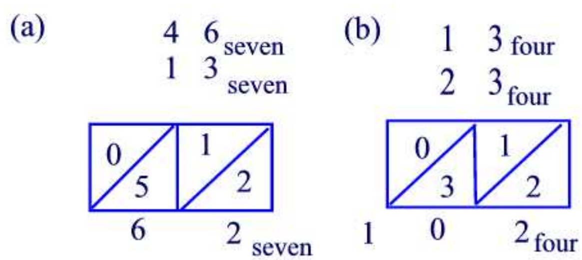
Problem 14.11
Construct an addition table in base seven.

Solution.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

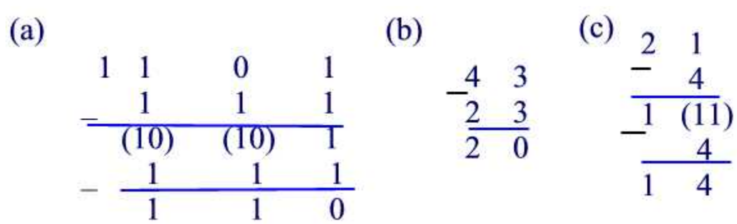
Problem 14.12
Use the lattice method to compute the following sums.
(a) $46_{seven} + 13_{seven}$.
(b) $13_{four} + 23_{four}$.

Solution.



Problem 14.13
Perform the following subtractions:
(a) $1101_{two} - 111_{two}$
(b) $43_{five} - 23_{five}$
(c) $21_{seven} - 4_{seven}$.

Solution.



Problem 14.14
Fill in the missing numbers.

(a)

$$\begin{array}{rl} 2\,\_\,\_ & _{five} \\ -\quad 2\,2 & _{five} \\ \hline \_\,0\,3 & _{five} \end{array}$$

(b)

$$\begin{array}{rl} 2\,0\,0\,1\,0 & _{five} \\ -\quad 2\,\_\,2\,\_ & _{five} \\ \hline 1\,\_\,2\,\_\,1 & _{five} \end{array}$$

Solution.

(a)

$$\begin{array}{rl} 2\,\underline{3}\,\underline{0} & _{five} \\ -\quad 2\,2 & _{five} \\ \hline \underline{2}\,0\,3 & _{five} \end{array}$$

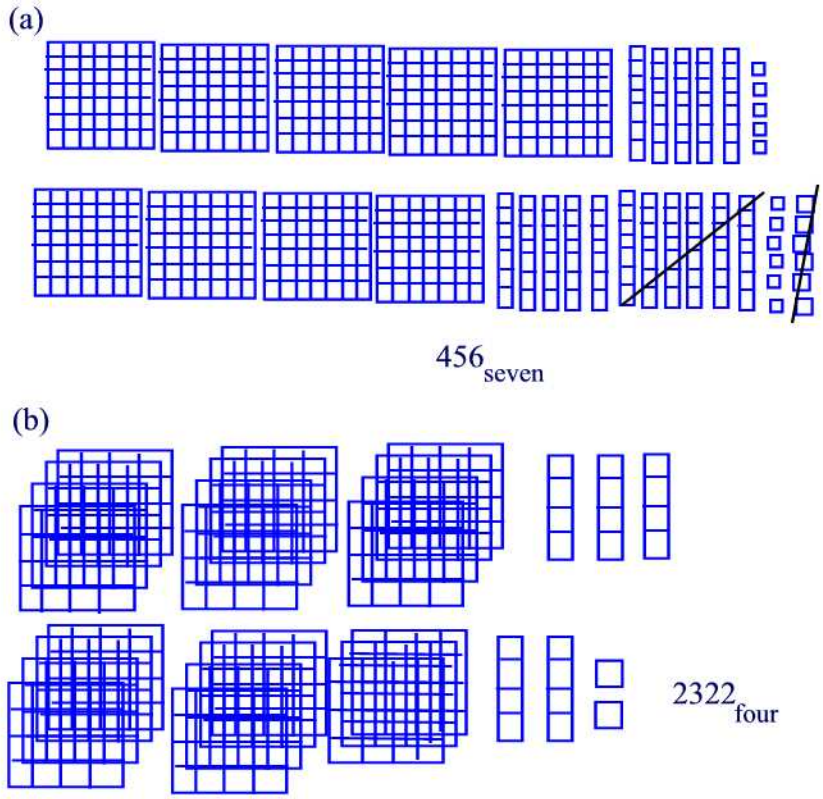
(b)

$$\begin{array}{rl} 2\,0\,0\,1\,0 & _{five} \\ -\quad 2\,\underline{0}\,2\,\underline{2} & _{five} \\ \hline 1\,\underline{0}\,2\,\underline{1}\,1 & _{five} \end{array}$$

Problem 14.15

Use blocks for the appropriate base to illustrate the following problems.
(a) $555_{seven} - 66_{seven}$ (b) $3030_{four} - 102_{four}$.

Solution.

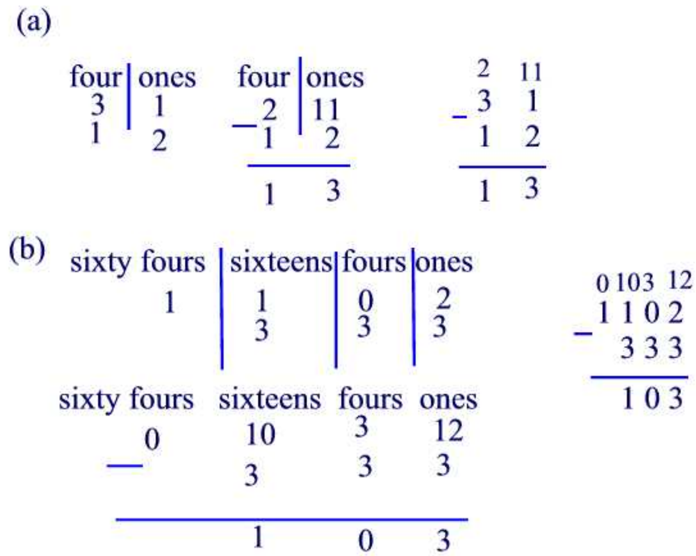


Problem 14.16

Use both the intermediate algorithm (discussed in Figure 14.2) and the standard algorithm to solve the following differences.

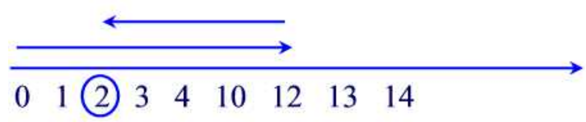
(a) $31_{four} - 12_{four}$ (b) $1102_{four} - 333_{four}$.

Solution.



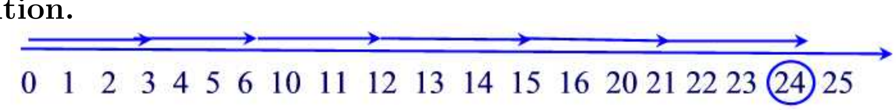
Problem 14.17
Use base five number line to illustrate the difference $12_{five} - 4_{five}$.

Solution.



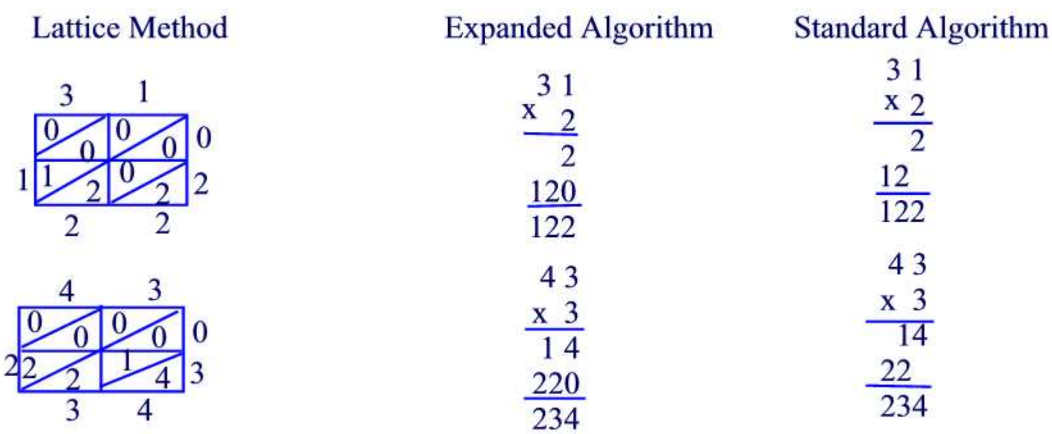
Problem 14.18
Create a base seven number line to illustrate $6_{seven} \times 3_{seven}$.

Solution.



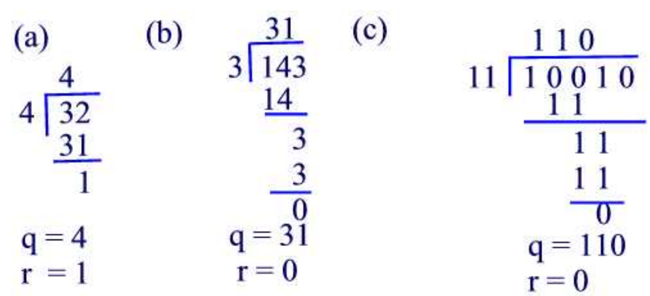
Problem 14.19
Find the following products using the lattice method, the expanded algorithm, and the standard algorithm.
(a) $31_{four} \times 2_{four}$ (b) $43_{five} \times 3_{five}$

Solution.

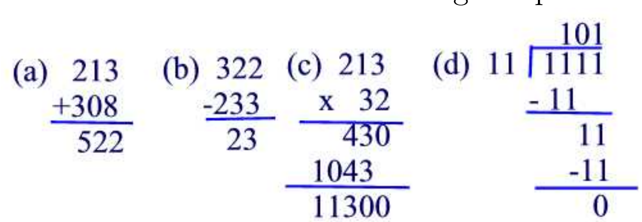


Problem 14.20
Perform the following divisions:
(a) $32_{five} \div 4_{five}$
(b) $143_{five} \div 3_{five}$
(c) $10010_{two} \div 11_{two}$.

Solution.



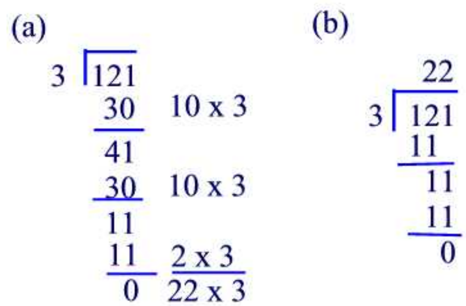
Problem 14.21
For what possible bases are each of the following computations correct?



Solution.
(a) Base nine
(b) Base four
(c) Base six
(d) Any base greater than or equal to two■

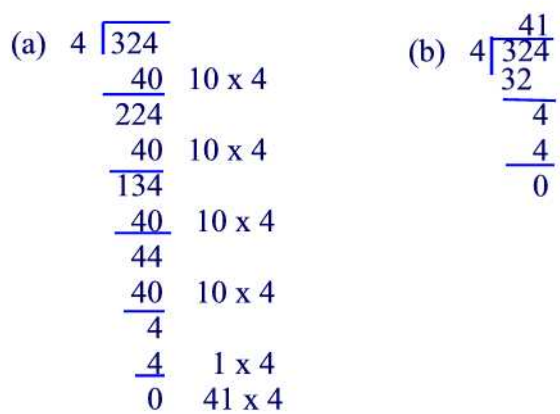
Problem 14.22
(a) Compute $121_{five} \div 3_{five}$ with repeated subtraction algorithm.
(b) Compute $121_{five} \div 3_{five}$ with long division algorithm.

Solution.



Problem 14.23
(a) Compute $324_{five} \div 4_{five}$ with repeated subtraction algorithm.
(b) Compute $324_{five} \div 4_{five}$ with long division algorithm.

Solution.



Problem 14.24
(a) Compute $1324_{seven} \div 6_{seven}$ with repeated subtraction algorithm.
(b) Compute $1324_{seven} \div 6_{seven}$ with long division algorithm.

Solution.

(a) $6\overline{)1324}$

600	100 x 6
424	
60	10 x 6
334	
60	10 x 6
244	
60	10 x 6
154	
60	10 x 6
64	
60	10 x 6
4	150 x 6

q = 150
r = 4

(b)

$$\begin{array}{r} 150 \\ 6\overline{)1324} \\ \underline{6} \\ 424 \\ \underline{42} \\ 4 \end{array}$$

Problem 14.25

Solve the following problems using the missing-factor definition of division, that is, $a \div b = c$ if and only if $b \cdot c = a$.(Hint: Use a multiplication table for the appropriate base).

(a) $21_{four} \div 3_{four}$ (b) $23_{six} \div 3_{six}$ (c) $24_{eight} \div 5_{eight}$

Solution.

(a) $21_{four} \div 3_{four} = c$ implies $3_{four} \cdot c = 21_{four}$. From the multiplication table of base four we find that $c = 3_{four}$

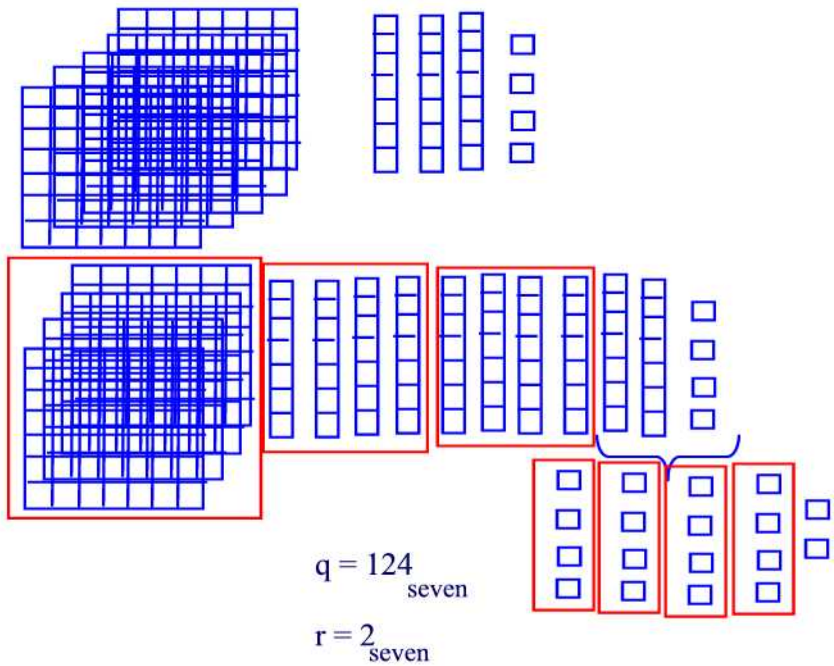
$23_{six} \div 3_{six} = c$ implies $3_{six} \cdot c = 23$. From the multiplication table of base six we find that $c = 5_{six}$.

(c) $24_{eight} \div 5_{eight} = c$ implies $5_{eight} \cdot c = 24_{eight}$. Using the multiplication table of base eight we find $c = 4_{eight}$■

Problem 14.26

Sketch how to use base seven blocks to illustrate the operation $534_{seven} \div 4_{seven}$.

Solution.



Problem 15.1
(a) The number $162 = 2 \cdot 3^4$. How many different divisors does 162 have?
(b) Try the same process with $225 = 3^2 \cdot 5^2$.
(c) Based on your results in parts (a) - (b), if p and q are prime numbers and $a = p^m \cdot q^n$ then how many different divisors does n have?

Solution.
(a) Let D_{162} be the set of divisors of 162. Then

$$D_{162} = \{1, 2, 3, 6, 9, 18, 27, 54, 81, 162\}$$

Thus, there are 10 different divisors of 162. Note that $10 = (1 + 1)(4 + 1)$
(b) We have

$$D_{225} = \{1, 3, 5, 9, 15, 25, 45, 75, 225\}$$

So there are 9 different divisors of 225. Note that $9 = (2 + 1)(2 + 1)$.
(c) From (a) and (b) we conclude that there are $(m + 1)(n + 1)$ different divisors of a■

Problem 15.2
(a) List all the divisors of 48.
(b) List all the divisors of 54.
(c) Find the largest common divisor of 48 and 54.

Solution.
(a) Since $48 = 2^4 \cdot 3$ then there are 10 different divisors of 48.

$$D_{48} = \{1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}$$

(b) Since $54 = 2 \cdot 3^3$ then there are 8 different divisors.

$$D_{54} = \{1, 2, 3, 6, 9, 18, 27, 54\}$$

(c) $LCD(48, 54) = 6$ since $D_{48} \cap D_{54} = \{1, 2, 3, 6\}$■

Problem 15.3
Let $a = 2^3 \cdot 3^1 \cdot 7^2$.
(a) Is $2^2 \cdot 7^1$ a factor of a? Why or why not?
(b) Is $2^1 \cdot 3^2 \cdot 7^1$ a factor of a? Why or why not?
(c) How many different factors does a possess?
(d) Make an orderly list of all the factors of a.

Solution.
(a) Yes, because $28 = 2^2 \cdot 7^1$, so all the prime factors of 28 appear in a and to at least as high a power.
(b) No, since the power of 3 in $2^1 \cdot 3^2 \cdot 7^1$ is higher than that of a
(c) There are $(3 + 1)(1 + 1)(2 + 1) = 24$ factors.
(d) First, note that $a = 1176$. Thus,

$$D_a = \{1, 2, 3, 4, 6, 7, 8, 12, 14, 21, 24, 28, 42, 49, 56, 84, 98, 147, 168, 196, 294, 392, 588, 1176\}$$

Problem 15.4
If n, b, and c are nonzero whole numbers and $n|bc$, is it necessarily the case that $n|b$ or $n|c$? Justify your answer.

Solution.
$8|2 \cdot 4$ but $8 \not| 2$ and $8 \not| 4$.■

Problem 15.5
Which of the following are true or false? Justify your answer in each case.
(a) $n|0$ for every nonzero whole number n.
(b) $0|n$ for every nonzero whole number n.
(c) $0|0$.
(d) $1|n$ for every whole number n.
(e) $n|n$ for every nonzero whole number n.

Solution.
(a) True, since $n = n \cdot 0$ for every counting number n
(b) False, since there are no nonzero while number m such that $0 \cdot m = n$.
(c) False. If $0|0$ then there is a unique counting number n such that $n \cdot 0 = 0$. But this is not the case since every counting number satisfies $n \cdot 0 = 0$.
(d) True, since for every whole number n we have $n = n \cdot 1$
(e) True, since $n = n \cdot 1$■

Problem 15.6
Find the least nonzero whole number divisible by each nonzero whole number less than or equal to 12.

Solution.
The number is $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$.■

Problem 15.7
If $42|n$ then what other whole numbers divide n?

Solution.
Since $42|n$ then there is a unique counting number q such that $n = 42q$. But $42 = 2 \cdot 3 \cdot 7$ so that the whole numbers: 2, 3, 7, 6, 14, 21 also divide n.■

Problem 15.8
If $2N = 2^6 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7$, explain why $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ is a factor of N.

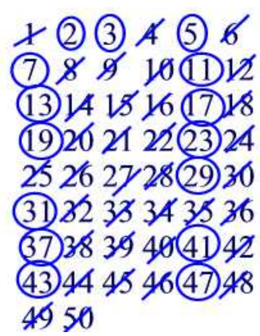
Solution.
Dividing by 2 we obtain $N = 2^5 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7 = (2^3 \cdot 5 \cdot 7 \cdot 11)(2^2 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11^6)$ so that $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)|N$.■

Problem 15.9
Eratosthenes, a Greek mathematician, developed the **Sieve of Eratosthenes** about 2200 years ago as a method for finding all prime numbers less than a given number. Follow the directions to find all the prime numbers less than or equal to 50.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50				

(a) Copy the list of numbers.
(b) Cross out 1 because 1 is not prime.
(c) Circle 2. Count 2s from there, and cross out $4, 6, 8, \cdots, 50$ because all these numbers are divisible by 2 and therefore are not prime.
(d) Circle 3. Count 3s from there, and cross out all numbers not already crossed out because these numbers are divisible by 3 and therefore are not prime.
(e) Circle the smallest number not yet crossed out. Count by that number, and cross out all numbers that are not already crossed out.
(f) Repeat part (e) until there are no more numbers to circle. The circled numbers are the prime numbers.
(g) List all the prime numbers between 1 and 50.

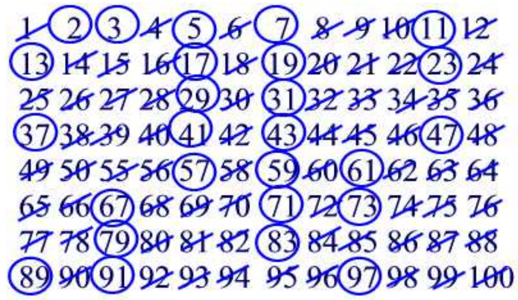
Solution.



The prime numbers are:2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47■

Problem 15.10
List all prime numbers between 1 and 100 using the Sieve of Eratosthenes.

Solution.



Problem 15.11
Extend the Sieve of Eratosthenes to find all the primes less than 200.

Solution.
Similar to the previous two problems■

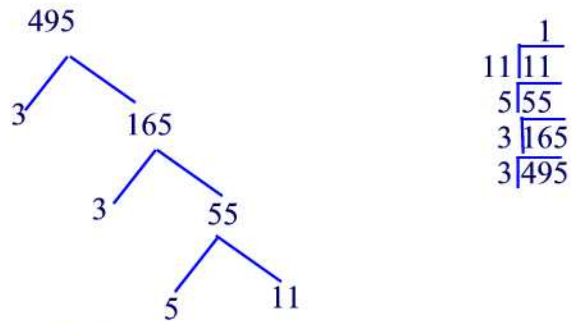
Problem 15.12
Write the prime factorizations of the following.
(a) 90 (b) 3155 (c) 84.

Solution.
(a) $90 = 2 \cdot 3^2 \cdot 5$
(b) $3155 = 5 \cdot 631$
(c) $84 = 2^3 \cdot 3 \cdot 7$■

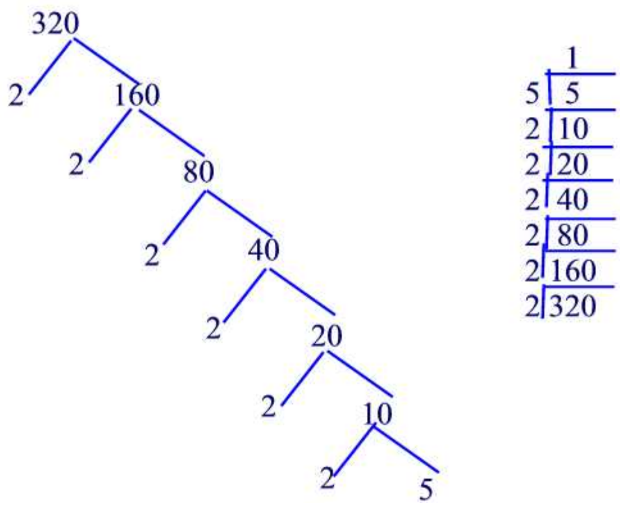
Problem 15.13
Find the prime factorization using both the factor-tree method and the prime divisor method.
(a) 495 (b) 320.

Solution.

(a)



(b)



Problem 15.14

Twin primes are any two consecutive odd numbers, such as 3 and 5, that are prime. Find all the twin primes between 101 and 140.

Solution.

101 and 103, 107 and 109, 137 and 139■

Problem 15.15

(a) How many different divisors does $2^5 \cdot 3^2 \cdot 7$ have?
(b) Show how to use the prime factorization to determine how many different factors 148 has.

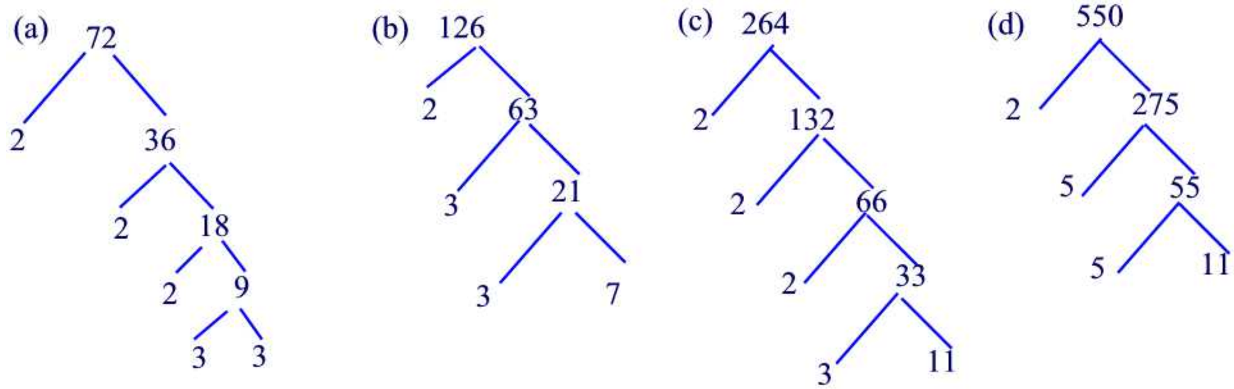
Solution.

(a) $(5+1)(2+1)(1+1) = 36$ different divisors.

(b) The prime factorization of 148 is $148 = 2^2 \cdot 37$. The number of different factors are $(2+1)(1+1) = 6$■

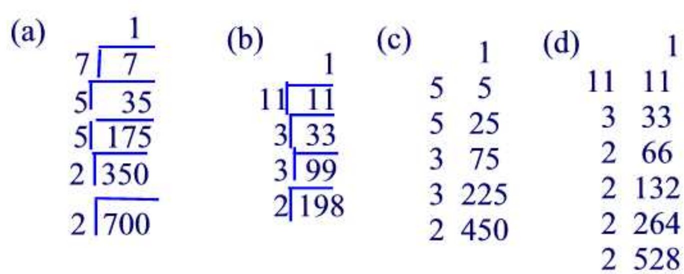
Problem 15.16
Construct factor trees for each of the following numbers.
(a) 72 (b) 126 (c) 264 (d) 550

Solution.



Problem 15.17
Use the prime divisors method to find all the prime factors of the following numbers.
(a) 700 (b) 198 (c) 450 (d) 528

Solution.



Problem 15.18
Determine the prime factorizations of each of the following numbers.
(a) 48 (b) 108 (c) 2250 (d) 24750

Solution.
(a) $48 = 2^4 \cdot 3$
(b) $108 = 2^3 \cdot 3^3$
(c) $2250 = 2 \cdot 3^2 \cdot 5^3$
(d) $24750 = 2 \cdot 3^2 \cdot 5^3 \cdot 11$■

Problem 15.19
Show that if 1 were considered a prime number then every number would have more than one prime factorization.

Solution.
Every number would have its usual factorization $1 \cdot p_1 \cdot p_2 \cdot \cdots \cdot p_n$ along with infinitely many other such factorizations with 1 being replaced by 1^n with n any counting number■

Problem 15.20
Explain why $2^3 \cdot 3^2 \cdot 25^4$ is not a prime factorization and find the prime factorization of the number.

Solution.
$2^3 \cdot 3^2 \cdot 25^4$ is not a prime factorization since 25 is not prime. The prime factorization is $2^3 \cdot 3^2 \cdot 5^8$■

Problem 15.21
Classify the following numbers as prime, composite or neither.
(a) 71 (b) 495 (c) 1

Solution.
(a) Prime.
(b) Composite since $5|495$
(c) Neither■

Problem 15.22
Without computing the results, explain why each of the following numbers will result in a composite number.
(a) $3 \times 5 \times 7 \times 11 \times 13$
(b) $(3 \times 4 \times 5 \times 6 \times 7 \times 8) + 2$
(c) $(3 \times 4 \times 5 \times 6 \times 7 \times 8) + 5$

Solution.
(a) The numbers 3, 5, 7, 11, 13 divide the given number.
(b) 2 divides the given number.
(c) 5 divides the given number■

Problem 15.23
To determine that 431 is prime, what is the minimum set of numbers you must try as divisors?

Solution.
Since $\sqrt{431} \approx 21$ then the minimum set of divisors to try is $\{2, 3, 5, 7, 11, 13, 17, 19\}$■

Problem 15.24
Use the Primality Test to classify the following numbers as prime or composite.
(a) 71 (b) 697 (c) 577 (d) 91.

Solution.
(a) Since $\sqrt{71} \approx 8$ and none of the numbers 2, 3, 5 and 7 divide 71 then 71 is prime
(b) Since $\sqrt{697} \approx 26$ and 14 and 17 divide 697 then this number is compisite.
(c) Since $\sqrt{577} \approx 24$ and none of the prime numbers less than 24 divide 577 then this number is prime.
(d) Since $\sqrt{91} \approx 9$ and $7|91$ then the number is composite■

Problem 15.25
What is the greatest prime you must consider to test whether 5669 is prime?

Solution.
Since $\sqrt{5669} \approx 75$ then the largest prime number to consider in the primality testing is 73■

Problem 16.1
Using the divisibility rules discussed in this section, explain whether 6,868,395 is divisible by 15.

Solution.
Since 3 and 5 are relatively prime, we just need to check that the number is divisible by both 3 and 5. Since the ones digit is 5 then the number is divisible by 5. Since $6 + 8 + 6 + 8 + 3 + 9 + 5 = 35$ and $3 \not| 35$ then the given number is not divisible by 15■

Problem 16.2
The number a and b are divisible by 5.
(a) Is $a+b$ divisible by 5?Why?
(b) Is $a-b$ divisible by 5?Why?
(c) Is $a \times b$ divisible by 5?Why?
(d) Is $a \div b$ divisible by 5?Why?

Solution.
Since $5|a$ and $5|b$ then there exist unique counting numbers q_1 and q_2 such that $a = 5q_1$ and $b = 5q_2$
(a) Since $a+b = 5(q_1+q_2)$ and $q_1+q_2 \in \mathbb{N}$ then $5|(a+b)$
(b) Assuming that $a-b \in W$ then $a-b = 5(q_1-q_2)$ with $q_1-q_2 \in \mathbb{N}$. Thus, $5|(a-b)$
(c) Since $ab = 5(5q_1q_2$ and $5q_1q_2 \in \mathbb{N}$ then $5|(a \times b)$
(d) That is not always true. For example, let $a = 5$ and $b = 10$ then $5|a$ and $5|b$ but $5 \not|(a \div b = 0.5)$■

Problem 16.3
If 21 divides n, what other numbers divide n?

Solution.
Since $21|n$ then $n = 21q$ where $q \in \mathbb{N}$. Since $21 = 3 \times 7$ then we can write $n = 3(7q)$ so that $3|n$ and $n = 7(3q)$ so that $7|n$■

Problem 16.4
Fill each of the following blanks with the greatest digit that makes the statement true:
(a) 3|74_
(b) 9|83_45
(c) 11|6_55.

Solution.
(a) 3|747
(b) 9|83745
(c) 11|6655■

Problem 16.5
When the two missing digits in the following number are replaced, the number is divisible by 99. What is the number?

85_ _1.

Solution.
Since the $99 = 11 \times 9$ and 9 and 11 are relatively prime then the required number must be divisible by 11 and 9. By trial and guessing the number is 85041■

Problem 16.6
Without using a calculator, test each of the following numbers for divisibility by 2, 3, 4, 5, 6, 8, 9, 10, 11.
(a) 746,988
(b) 81,342
(c) 15,810
(d) 4,201,012
(e) 1,001
(f) 10,001.

Solution.
(a) We will do (a) and leave the rest for the reader. Since the ones digit of $746,988$ is 8 then the number is divisible by 2. Since $7+4+6+9+8+8 = 42$ then the given number is divisible by 3 but not by 9. Since 88 is divisible by 4 then the given number is divisible by 4. Since the ones digit is not 0 nor 5 then the given number is not divisible by 5 or 10. Since the number is divisible by 2 and 3 then it is divisible by 6. Since 988 is not divisible by 8 then the given number is not divisible by 8. Since $(8+9+4)-(8+6+7) = 0$ then the given number is divisible by 11.
(b) 2, 3, 6, 9
(c) 2, 3, 5, 6, 10
(d) 2, 4
(e) 11
(f) none of them ■

Problem 16.7
There will be 219 students in next year's third grade. If the school has 9 teachers, can we assign each teacher the same number of students?

Solution.
Since $2 + 1 + 9 = 12$ then 219 is no divisible by 9. The answer is no■

Problem 16.8
Three sisters earn a reward of $37,500 for solving a mathematics problem. Can they divide the money equally?

Solution.
Since $3+7+5=15$ and $3|15$ then $3|37,500$. So the amount can be divided equally among the three sisters■

Problem 16.9
What three digit numbers are less than 130 and divisible by 6?

Solution.
The numbers are: 102, 108, 114, 120, and 126■

Problem 16.10
True or false? If false, give a counter example.
(a) If a number is divisible by 5 then it is divisible by 10
(b) If a number is not divisible by 5 then it is not divisible by 10
(c) If a number is divisible by 2 and 4 then it is divisible by 8
(d) If a number is divisible by 8 then it is divisible by 2 and 4
(e) If a number is divisible by 99 then it is divisible by 9 and 11.

Solution.
(a) False. $5|5$ by $10 \not| 5$.
(b) True.
(c) False. For exmaple, $2|12$ and $4|12$ but $8 \not| 12$
(d) True.
(e) True.■

Problem 16.11
Test each number for divisibility by 2, 3, and 5. Do the work mentally.
(a) 1554 (b) 1999 (c) 805 (d) 2450

Solution.
(a) Since the ones digit is 4 then the number is divisible by 2. Since $1+5+5+4=15$ and $3|15$ then the number is divisible by 3. Since the ones digit is neither 0 nor 5 then the number is not divisible by 5.
(b) Since the ones digit is 9 then the number is not divisible by 2. Since $1+99+9=28$ and $3 \not| |28$ then the number is not divisible by 3. Since the ones digit is neither 0 nor 5 then the number is not divisible by 5.
(c) Since the ones digit is 5 then the number is not divisible by 2. Since $8+0+5=13$ and $3 \not| 13$ then the number is not divisible by 3. Since the ones digit is 5 then the number is divisible by 5.

(d) Since the ones digit is 0 then the number is divisible by 2. Since $2+4+5+0=11$ and $3 \not| 11$ then the number is not divisible by 3. Since the ones digit is 0 then the number is divisible by 5.■

Problem 16.12
Are the numbers of the previous problem divisible by
(a)0   (b) 10   (c) 15   (d) 30

Solution.
(a) Division by 0 is undefined.
(b) only 2450
(c) None■

Problem 16.13
Is 1,927,643,001,548 divisible by 11? Explain.

Solution.
Since $(8+5+0+3+6+2+1)-(4+1+0+4+7+9) = 25-25 = 0$ then the number is divisible by 11■

Problem 16.14
At a glance, determine the digit d so that the number 87,543,24d is divisible by 4. Is there more than one solution?

Solution.
One possibility is 87,543,240. Other answers are 87,543,244, 87,543,248■

Problem 16.15
Determine the digit d so that the number 6,34d,217 is divisible by 11.

Solution.
The difference $(7+2+4+6)-(1+d+3)$ must be divisible by 11. This is true if $d = 4$■

Problem 16.16
Find the digit d so that the number 897,650,243,28d is divisible by 6.

Solution.
If d=0 then the given number is divisible by 2. However, 8 + 9 + 7 + 6 + 5 + 0 + 2 + 4 + 3 + 2 + 8 + 0 = 54 is not divisible by 3. If d = 4 then the number is both divisible by 2 and 3 and so by 6■

Problem 16.17
(a) Determine whether 97,128 is divisible by 2,4 and 8.
(b) Determine whether 83,026 is divisible by 2,4, and 8.

Solution.
(a) Since the ones digit is 8 then the number is divisible by 2. Since $4|28$ then the number is divisible by 4. Since $8|128$ then the number is divisible by 8.
(b) Since the ones digit is 6 then the number is divisible by 2. Since $4 \not| 26$ then the number is not divisible by 4. Since $8 \not| 26$ then the number is not divisible by 8.■

Problem 16.18
Use the divisibility tests to determine whether each of the following numbers is divisible by 3 and divisible by 9.
(a) 1002 (b) 14,238

Solution.
(a) $1 + 0 + 0 + 2 = 3$ so $3|1002$ but $9 \not| 1002$
(b) $1 + 4 + 2 + 3 + 8 = 18$ so $3|14238$ and $9|14238$■

Problem 16.19
The store manager has an invoice of 72 four-function calculators. The first and last digits on the receipt are illegible. The manager can read $_67.9_.
What are the missing digits, and what is the cost of each calculator?

Solution.
Since 8 and 9 are relatively prime then the total sale of the 72 calculators must be divisible by 8 and 9. That is _679_ cents must be divisible by 8 and 9. For this number to be divisible by 8 the ones digit must be 2 since $8|792$. Thus, we have _6792. Since the number must be divisible by 9 then we the sum of digits must be divisible by 9. That is, $d + 6 + 7 + 9 + 2$ or $d + 24$ must be divisible by 9. This gives $d = 3$. Thus, the total sales of the 72 calculators is \$367.92. The cost of each calculator is $367.92 \div 72 = \$5.11$■

Problem 16.20
The number 57,729,364,583 has too many digits for most calculator to display. Determine whether this number is divisible by each of the following.
(a) 2 (b) 3 (c) 5 (d) 6 (e) 8 (f) 9 (g) 10 (h) 11

Solution.
(a) Since the ones digit is 3 the number is not divisible by 2
(b) Since $5+7+7+2+9+3+6+4+5+8+3=59$ then the number is not divisible by 3
(c) Since the ones digit is neither 0 nor 5 then the number is not divisible by 5
(d) Since the number is not divisible by either 2 nor 3 then it is not divisible by 6
(e) Since $8 \not| 583$ then the number is not divisible by 8
(f) Since the sum of digits is 59 and $9 \not| 59$ then the given number is not divisible by 9
(g) Since the ones digit is not 0 then the number is not divisible by 10
(h) Since $(3+5+6+9+7+5)-(8+4+3+2+7)=35-24=11$ so the given number is divisible by 11■

Problem 17.1
Find the GCF and LCM for each of the following using the set intersection method.
(a) 18 and 20
(b) 24 and 36
(c) 8, 24, and 52
(d) 7 and 9.

Solution.
(a) Since $F_{18}=\{1,2,3,6,9,18\}$ and $F_{20}=\{1,2,4,5,10,20\}$ then $F_{18}\cap F_{20}=\{1,2\}$ so that GCF(18,20)=2. Next, we find the nonzero multiples of 18 and 20 : $M_{18}=\{18,36,54,72,90,108,126,144,162,180,\cdots\}$ and $M_{20}=\{20,40,60,80,100,120,140,160,180,\cdots\}$. Thus, LCM(18,20)=180
(b) We have $F_{24}=\{1,2,3,4,6,8,12,24\}$ and $F_{36}=\{1,2,3,4,9,12,18,36\}$. Thus, $F_{24}\cap F_{36}=\{1,2,3,4,12\}$ so that GCF(24,36)=12. Now, $M_{24}=\{24,48,72,\cdots\}$ and $M_{36}=\{36,72,\cdots\}$ so that LCM(24,36)=72.
(c) We have $F_8=\{1,2,4,8\}, F_{24}=\{1,2,3,4,6,8,12,24\}$, and $F_{52}=\{1,2,4,13,26,52\}$ then $F_8\cap F_{24}\cap F_{52}=\{1,2,4\}$ so that GCF(8,24,52)=4. Finding the nonzero multiples of 8,24, and 52 and then take the intersection of these sets we find $M_8\cap M_{24}\cap M_{52}=\{312,\cdots\}$ so that LCM(8,24,52)=312.
(d) $F_7\cap F_9=\{1,7\}\cap\{1,3,9\}=\{1\}$ so that GCF(7,9)=1 and $M_7\cap M_9=\{63,\cdots\}$ so that LCM(7,9)=63■

Problem 17.2
Find the GCF and LCM for each of the following using the prime factorization method.
(a) 132 and 504
(b) 65 and 1690
(c) 900, 96, and 630
(d) 108 and 360
(e) 11 and 19.

Solution.
(a) $132 = 2^2 \cdot 3 \cdot 11$ and $504 = 2^3 \cdot 3^2 \cdot 7$, $GCF(132, 504) = 2^2 \cdot 3 = 12$, $LCM(132, 504) = 2^3 \cdot 3^2 \cdot 7 \cdot 11 = 5544$.
(b) $65 = 5 \cdot 13, 1690 = 2 \cdot 5 \cdot 13^2, GCF(65, 1690) = 5 \cdot 13 = 65, LCM(65, 1690) = 2 \cdot 5 \cdot 13^2 = 1690$.
(c) $900 = 2^2 \cdot 3^2 \cdot 5^2, 96 = 2^5 \cdot 3, 630 = 2 \cdot 3^2 \cdot 5 \cdot 7, GCF(900, 96, 630) = 2 \cdot 3 = 6, LCM(900, 96, 630) = 2^5 \cdot 3^2 \cdot 5 \cdot 7 = 10080$.
(d) $108 = 2^2 \cdot 3^3, 360 = 2^3 \cdot 3^2 \cdot 5, GCF(108, 360) = 2^2 \cdot 3^2 = 36, LCM(108, 360) = 2^3 \cdot 3^3 \cdot 5 = 1080$.
(e) $GCF(11, 19) = 1$ and $LCM(11, 19) 11 \cdot 19 = 209$.

Problem 17.3
Find the GCF and LCM for each of the following using the Euclidean algorithm method.
(a) 220 and 2924
(b) 14,595 and 10,856
(c) 122,368 and 123,152.

Solution.
(a) Using the Euclidean algorithm repeatedly we find

$$\begin{aligned}
2924 &= 13 \times 220 + 64 \\
220 &= 3 \times 64 + 28 \\
64 &= 2 \times 28 + 8 \\
28 &= 3 \times 8 + 4 \\
8 &= 2 \times 4 + 0
\end{aligned}$$

Thus, GCF(220,2924)= 4 and $LCM(220, 2924) = \frac{220 \cdot 2924}{GCF(220,2924)} = \frac{220 \cdot 2924}{4} = 160820$.

(b)

$$\begin{array}{rcl} 14595 & = & 1 \times 10856 + 3739 \\ 10856 & = & 2 \times 3739 + 3378 \\ 3739 & = & 1 \times 3378 + 361 \\ 3378 & = & 9 \times 361 + 129 \\ 361 & = & 2 \times 129 + 3 \\ 129 & = & 43 \times 3 + 0 \end{array}$$

Thus, GCF(14595,10856)=3 and $LCM(14595, 10856) = \frac{14595 \cdot 10856}{3} = 52814440$

(c)

$$\begin{array}{rcl} 123152 & = & 1 \times 122368 + 784 \\ 122368 & = & 156 \times 784 + 64 \\ 784 & = & 12 \times 64 + 16 \\ 64 & = & 4 \times 16 + 0 \end{array}$$

Thus, GCF(123152,122368)=16 and $LCM(123152, 122368) = \frac{123152 \cdot 122368}{16} = 941866496$ ■

Problem 17.4
Find the LCM using any method.
(a) 72, 90, and 96
(b) 90, 105, and 315.

Solution.
(a) We will use the prime factorization method: $72 = 2^3 \cdot 3^2, 90 = 2 \cdot 3^2 \cdot 5, 96 = 2^5 \cdot 3$. Thus, $LCM(72, 90, 96) = 2^5 \cdot 3^2 \cdot 5 = 1440$.
(b) $90 = 2 \cdot 3^2 \cdot 5, 105 = 3 \cdot 5 \cdot 7, 315 = 3^2 \cdot 5 \cdot 7$. Thus, $LCM(90, 105, 315) = 2 \cdot 3^2 \cdot 5 \cdot 7 = 630$. ■

Problem 17.5
Find the LCM of the following numbers using Theorem 17.2.
(a) 220 and 2924
(b) 14,595 and 10,856
(c) 122,368 and 123,152.

Solution.
See Problem 17.3 ■

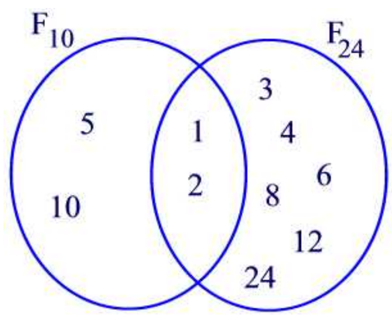
Problem 17.6
If a and b are nonzero whole numbers such that $GCF(a, b) = 1$ then we say

that a and b are **relatively prime.** Determine whether the following pairs of numbers are relatively prime.
(a) 7 and 19
(b) 27 and 99
(c) 8 and 6
(d) 157 and 46.

Solution.
(a) Since GCF(7,19) = 1 then 7 and 19 are relatively prime.
(b) Since GCF(27,99) = 9 then 27 and 99 are not relatively prime.
(c) Since GCF(8,6)=2 then 6 and 8 are not relatively prime.
(d) Since GCF(157,46)= 1 then 157 and 46 are relatively prime ■

Problem 17.7
(a) Draw a Venn diagram showing the factors and common factors of 10 and 24.
(b) What is the greatest common factor of 10 and 24?

Solution.



(a)
(b) GCF(10,24)=2 ■

Problem 17.8
Suppose that $a = 2 \cdot 3^2 \cdot 7^3$ and $GCF(a,b) = 2 \cdot 3^2 \cdot 7$. Give two possible values of b.

Solution.
Two possible values of b are: $b = 2 \cdot 2 \cdot 3^2 \cdot 7$ and $b = 2^3 \cdot 3^2 \cdot 7$ ■

Problem 17.9
To find the GCF and LCM of three or more nonzero whole numbers the prime factorization method is the most desirable.
(a) Find the GCF and the LCM of $a = 2^2 \cdot 3^1 \cdot 5^2, b = 2^1 \cdot 3^3 \cdot 5^1, c = 3^2 \cdot 5^3 \cdot 7^1$.
(b) Is it necessarily true that $LCM(a,b,c) \cdot GCF(a,b,c) = a \cdot b \cdot c$?

Solution.
(a) $GCF(a,b,c) = 3^1 \cdot 5^1 = 15$ and $LCM(a,b,c) = 2^2 \cdot 3^3 \cdot 5^3 \cdot 7^1 = 31,500$
(b) By part (a), $LCM(a,b,c) \cdot GCF(a,b,c) \neq a \cdot b \cdot c$ ■

Problem 17.10
Use the method of intersection to find LCM(18,24,12) and GCF(18,24,12).

Solution.
We have, $F_{18} = \{1,2,3,6,9,18\}, F_{24} = \{1,2,3,6,12,24\}$, and $F_{12} = \{1,2,3,4,6,12\}$ then $F_{18} \cap F_{24} \cap F_{12} = \{1,2,3,6\}$. Thus, GCF(18,24,12)=6. Now, $M_{18} \cap M_{24} \cap M_{12} = \{72, \cdots\}$ so that LCM(18,24,12)=72 ■

Problem 17.11
Find all whole numbers x such that GCF(24,x)=1 and $1 \leq x \leq 24$.

Solution.
Going through the list of numbers $\{1,2,3,\cdots,23,24\}$ we find $x \in \{1,5,7,11,13,17,19,23\}$ ■

Problem 17.12
George made enough money by selling candy bars at 15 cents each to buy several cans of pop at 48 cents each. If he had no money left over, what is the fewest number of candy bars he could have sold?

Solution.
Let x be the number of Candy bars and y be the number of cans of pop. Then $0.15x = 0.48y$ or $5x = 16y$. The smallest value of x so that x and y are both counting numbers is $x = 16$. So the number of candy bars sold is 16 ■

Problem 17.13
In the set $\{18, 96, 54, 27, 42\}$, find the pair(s) of numbers with the greatest GCF and the pair(s) with the smallest LCM.

Solution.
Writing the prime factorization of each number we obtain

$$\begin{aligned} 18 &= 2 \cdot 3^2 \\ 96 &= 2^5 \cdot 3 \\ 54 &= 2 \cdot 3^3 \\ 27 &= 3^3 \\ 42 &= 2 \cdot 3 \cdot 7 \end{aligned}$$

Thus,

$$\begin{array}{rclrcl}
GCF(18,96) & = & 6 & LCM(18,96) & = & 288 \\
GCF(18,54) & = & 18 & LCM(18,54) & = & 54 \\
GCF(18,27) & = & 9 & LCM(18,27) & = & 54 \\
GCF(18,42) & = & 6 & LCM(18,42) & = & 126 \\
GCF(96,54) & = & 6 & LCM(96,54) & = & 864 \\
GCF(96,27) & = & 3 & LCM(96,27) & = & 864 \\
GCF(96,42) & = & 6 & LCM(96,42) & = & 672 \\
GCF(54,27) & = & 27 & LCM(54,27) & = & 54 \\
GCF(54,42) & = & 6 & LCM(54,42) & = & 378 \\
GCF(27,42) & = & 3 & LCM(27,42) & = & 378
\end{array}$$

54 and 27 have the greatest GCF; $\{18,54\}, \{18,27\}, \{54,27\}$ have the smallest LCM ■

Problem 17.14
Which is larger GCF(a,b) or LCM(a,b)?

Solution.
If $a = p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$ and $b = p_1^{t_1}p_2^{t_2}\cdots p_k^{t_k}$ then

$$LCM(a,b) = p_1^{max\{s_1,t_1\}}p_2^{max\{s_2,t_2\}}\cdots p_k^{max\{s_k,t_k\}}$$

and

$$GCF(a,b) = p_1^{min\{s_1,t_1\}}p_2^{min\{s_2,t_2\}}\cdots p_k^{min\{s_k,t_k\}} \blacksquare$$

So that $LCM(a,b) > GCF(a,b)$ ■

Problem 17.15
Suppose that a and 10 are relatively prime. Find all the possible values of a that are less than 10.

Solution.
The possible values of a that are less than 10 and relatively prime with 10 are: 1, 3, 7, 9 ■

Problem 17.16
LCM(24,x)=168 and GCF(24,x)=2. Find x.

Solution.
We know that $LCM(24,x)\cdot GCF(24,x) = 24\cdot x$. Thus, $24\cdot x = 168\cdot 2$. Solving for x we find

$$x = \frac{168\cdot 2}{24} = 14 \blacksquare$$

Problem 17.17
(a) Show that for any nonzero whole numbers a and b with $a \ge b$ we have GCF(a,b)=GCF(a-b,b).
(b) Use part (a) to find GCF(546,390).

Solution.
(a) If c is a common factor of a and b then $a = cq_1$ and $b = cq_2$. Since $a \ge b$ then $q_1 \ge q_2$. Thus, $a - b = c(q_1 - q_2)$ so that c is also a common factor of b and $a - b$. Similarly, if c is a common factor of b and $a - b$ then $b = ck_1$ and $a - b = ck_2$ so that $a = (a - b) + b = c(k_1 + k_2)$. That is c is also a common factor of a and b. Hence, the pairs (a,b) and (a-b,b) have the same common factors so that GCF(a,b)=GCF(a-b,b).
(b)

$$\begin{array}{rcl} GCF(546,390) &=& GCF(546-390,390)\\ &=& GCF(156,390)\\ &=& GCF(390-156,156)\\ &=& GCF(234,156)\\ &=& GCF(234-156,156)\\ &=& GCF(78,156)\\ &=& GCF(156-78,78)\\ &=& GCF(78,78) = 78 \blacksquare \end{array}$$

Problem 17.18
Suppose that $a = 2^3{\cdot}5^2{\cdot}7^3, GCF(a,b) = 2{\cdot}5^2{\cdot}7$, and $LCM(a,b) = 2^3{\cdot}3^3{\cdot}5^4{\cdot}7^3$. Find the value of b.

Solution.
By Theorem 17. 2, we have $LCM(a,b)\cdot GCF(a,b) = a\cdot b$. That is,

$$2^3\cdot 3^3\cdot 5^4\cdot 7^3\cdot 2\cdot 5^2\cdot 7 = 2^3\cdot 5^2\cdot 7^3\cdot b$$

Thus,

$$b = \frac{2^4\cdot 3^3\cdot 5^6\cdot 7^4}{2^3\cdot 5^2\cdot 7^3} = 2\cdot 3^3\cdot 5^4\cdot 7 \blacksquare$$

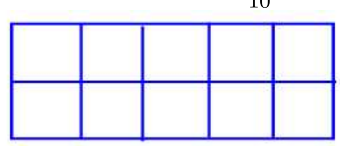
Problem 17.19
Suppose 0 were included as a possible multiple in the definition of LCM. What would be the LCM of any two whole numbers?

Solution.
The smallest common multiple of any two whole numbers would be zero ■

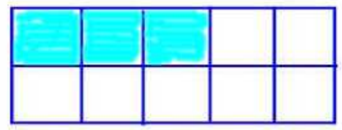
Problem 17.20
Assume a and b are nonzero whole numbers. Answer the following:
(a) If GCF(a,b) = 1, find LCM(a,b).
(b) Find GCF(a,a) and LCM(a,a).
(c) Find GCF(a^2,a) and LCM(a^2,a).
(d) If $a|b$, find GCF(a,b) and LCM(a,b).
(e) If a and b are two primes, find GCF(a,b) and LCM(a,b).
(f) What is the relationship between a and b if GCF(a,b) = a?
(g) What is the relationship between a and b if LCM(a,b) = a?

Solution.
(a) Since $LCM(a,b) \cdot GCF(a,b) = a \cdot b$ and GCF(a,b)=1 then $LCM(a,b) = a \cdot b$.
(b) GCF(a,a) = a and LCM(a,a)= a.
(c) If $a = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$ then $a = p_1^{2s_1} p_2^{2s_2} \cdots p_k^{2s_k}$ so that $GCF(a^2, a) = a$ and $LCM(a^2, a) = a^2$.
(d) GCF(a,b)=a and LCM(a,b) = b since $a|b$ so that $a \leq b$.
(e) GCF(a,b) = 1 and $LCM(a,b) = a \cdot b$.
(f) If GCF(a,b)=a then $a|b$.
(g) If LCM(a,b)=a then $b|a$ ■

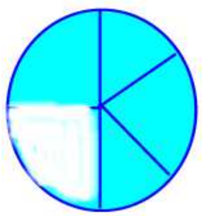
Problem 18.1
Explain how to complete each diagram so that it shows $\frac{3}{10}$.



Solution.



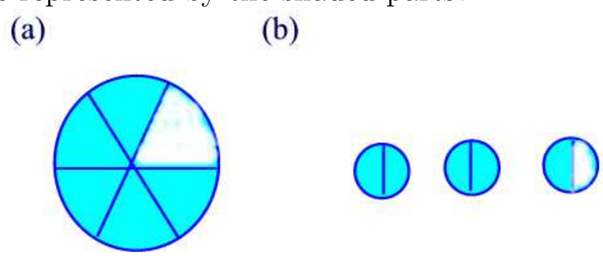
Problem 18.2
A child shows $\frac{4}{5}$ as



What is wrong with the diagram?

Solution.
The pie is not partitioned equaly ■

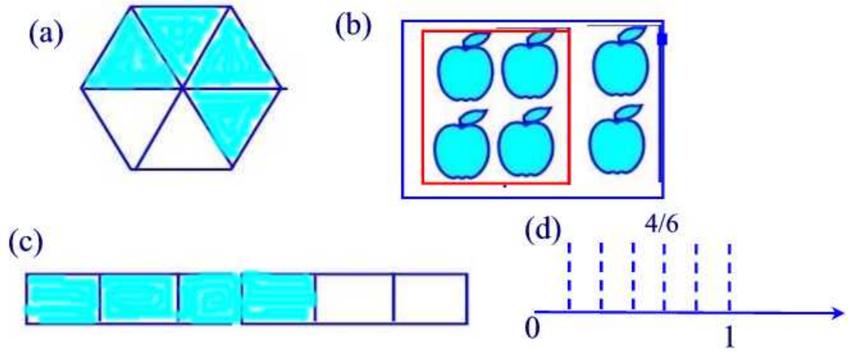
Problem 18.3
What fraction is represented by the shaded parts?



Solution.
(a) $\frac{1}{6}$ (b) $\frac{5}{2}$ ■

Problem 18.4
Depict the fraction $\frac{4}{6}$ with the following models.
(a) Colored region model
(b) Set model
(c) Fraction strip model
(d) Number-line model.

Solution.



Problem 18.5
Express the following quantities by a fraction placed in the blank space.
(a) 20 minutes is ____ of an hour.
(b) 30 seconds is ____ of a minute.
(c) 5 days is ____ of a week.
(d) 25 years is ____ of a century.
(e) A quarter is ____ of a dollar.
(f) 3 eggs is ____ of a dozen.

Solution.
(a) 20 minutes is $\frac{1}{3}$ of an hour.
(b) 30 seconds is $\frac{1}{2}$ of a minute.
(c) 5 days is $\frac{5}{7}$ of a week.
(d) 25 years is $\frac{1}{4}$ of a century.
(e) A quarter is $\frac{1}{4}$ of a dollar.
(f) 3 eggs is $\frac{1}{4}$ of a dozen ■

Problem 18.6
Three fifths of a class of 25 students are girls. How many are girls?

Solution.
Since one fifth is $\frac{25}{5} = 5$ then three fifths is $3 \times 5 = 15$. Thus, 15 students are female ■

Problem 18.7
The Independent party received one-eleventh of the 6,186,279 votes cast. How many votes did the party receive?

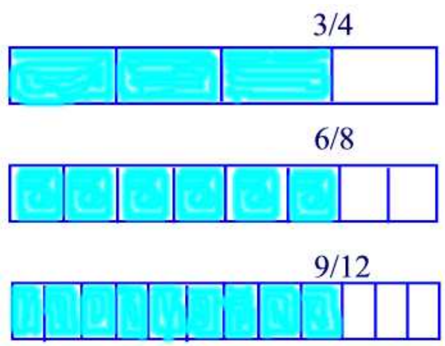
Solution.
The party received $\frac{6,186,279}{11} = 562,389$ ■

Problem 18.8
Show that $\frac{3}{5} = \frac{6}{10}$.

Solution.
Since $3 \times 10 = 5 \times 6$ then by Theorem 18.1 we must have $\frac{3}{5} = \frac{6}{10}$ ■

Problem 18.9
Use drawings of fractions strips to show that $\frac{3}{4}$, $\frac{6}{8}$, and $\frac{9}{12}$ are equivalent.

Solution.



Problem 18.10
Write each fraction in simplest form.
(a) $\frac{168}{464}$ (b) $\frac{xy^2}{xy^3z}$.

Solution.
(a) Since $168 = 2^3 \cdot 3 \cdot 7$ and $464 = 2^4 \cdot 29$ then GCF(168,464)$= 2^3 = 8$. Hence,

$$\frac{168}{464} = \frac{168 \div 8}{464 \div 8} = \frac{21}{58}$$

(b) $\frac{xy^2}{xy^3z} = \frac{1}{yz}$ ■

Problem 18.11
Two companies conduct surveys asking people if they favor stronger controls on air pollution. The first company asks 1,500 people, and the second asks 2,000 people. In the first group, 1,200 say yes. Make up results for the second group that would be considered equivalent.

Solution.
We have $\frac{1200}{1500} = \frac{4\cdot300}{5\cdot300} = \frac{4}{5}$. Since $2000 = 5 \cdot 400$ then for the two surveys to be equivalent $1600 = 4 \cdot 400$ in the second group must say yes ■

Problem 18.12
Find four different fractions equivalent to $\frac{4}{9}$.

Solution.
By the Fundamental Law of Fractions we have: $\frac{4}{9} = \frac{2\cdot4}{2\cdot9} = \frac{3\cdot4}{3\cdot9} = \frac{4\cdot4}{4\cdot9} = \frac{5\cdot4}{5\cdot9}$. Thus, the fractions $\frac{8}{18}, \frac{12}{27}, \frac{16}{36}$, and $\frac{20}{45}$ are all equivalent to $\frac{4}{9}$ ■

Problem 18.13
Fill in the missing number to make the fractions equivalent.
(a) $\frac{4}{5} = \frac{}{30}$ (b) $\frac{6}{9} = \frac{2}{}$.

Solution.
(a) Since $30 = 5 \cdot 6$ then $\frac{4}{5} = \frac{24}{30}$
(b) Since $6 = 2 \cdot 3$ and $9 = 3 \cdot 3$ then $\frac{6}{9} = \frac{2}{3}$ ■

Problem 18.14
Rewrite the following fractions in simplest form.
(a) $\frac{84}{144}$ (b) $\frac{208}{272}$

Solution.
(a) First, we find GCF(84,144). Writing the prime factorization we find $84 = 2^2 \cdot 3 \cdot 7$ and $144 = 2^4 \cdot 3^2$. Thus, GCF(84,144)=$2^2 \cdot 3 = 12$. Hence,

$$\frac{84}{144} = \frac{84 \div 12}{144 \div 12} = \frac{7}{12}$$

(b) Writing the prime factorization of 208 and 272 we find $208 = 2^4 \cdot 13$ and $272 = 2^4 \cdot 17$. Thus, GCF(208,272)=16. Hence,

$$\frac{208}{272} = \frac{208 \div 16}{272 \div 16} = \frac{13}{17} \blacksquare$$

Problem 18.15
Find the prime factorizations of the numerators and denominators of these fractions and use them to express the fractions in simplest form.
(a) $\frac{96}{288}$ (b) $\frac{2520}{378}$.

Solution.
(a) Since $96 = 2^5 \cdot 3$ and $288 = 2^5 \cdot 3^2$ then GCF(96,288) $= 2^5 \cdot 3 = 96$. Thus,

$$\frac{96}{288} = \frac{96 \div 96}{288 \div 96} = \frac{1}{3}$$

(b) Since $2520 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$ and $378 = 2 \cdot 3^3 \cdot 7$. Thus, GCF(2520,378)$=$ $2 \cdot 3^2 \cdot 7 = 126$. Thus

$$\frac{2520}{378} = \frac{2520 \div 126}{378 \div 126} = \frac{20}{3} \blacksquare$$

Problem 18.16
If a fraction is equal to $\frac{3}{4}$ and the sum of the numerator and denominator is 84, what is the fraction?

Solution.
Using the guessing and checking strategy together with The Fundamental Law of Fractions we see that

$$\frac{3}{4} = \frac{3 \cdot 12}{4 \cdot 12} = \frac{36}{48}$$

Note that $36 + 48 = 84$. Hence, the required fraction is $\frac{36}{48}$ $\blacksquare$

Problem 18.17
Determine if each of the following is correct.
(a) $\frac{ab+c}{b} = a + c$
(b) $\frac{a+b}{a+c} = \frac{b}{c}$
(c) $\frac{ab+ac}{ac} = \frac{b+c}{c}$.

Solution.
(a) This is false. Take $a = 1, b = 2$, and $c = 3$. Then $\frac{ab+c}{b} = \frac{5}{2}$ and $a + c = 4$.
(b) This is false. Take $a = 1, b = 2$, and $c = 3$. Then $\frac{a+b}{a+c} = \frac{3}{4}$ and $\frac{b}{c} = \frac{2}{3}$.
(c) This is correct by the Fundamental Law of Fractions since $\frac{b+c}{c} = \frac{(b+c)\cdot a}{c\cdot a} = \frac{ab+ac}{ac}$ $\blacksquare$

Problem 18.18
If $\frac{a}{b} = \frac{c}{b}$. what must be true?

Solution.
We must have $a = c$ $\blacksquare$

Problem 18.19
Solve for x.
(a) $\frac{2}{3} = \frac{x}{16}$

(b) $\frac{3}{x} = \frac{3x}{x^2}$.

Solution.
(a) By Theorem 18.1 we must have $3x = 32$. There is no whole number satisfying this equation.
(b) Again, using Theorem 18.1 we have $3x = 3x$. Any non zero whole number is a solution to this equation ■

Problem 18.20
Rewrite as a mixed number in simplest for.
(a) $\frac{525}{96}$ (b) $\frac{1234}{432}$.

Solution.
(a) $\frac{525}{96} = 5 + \frac{45}{96} = 5 + \frac{15}{32} = 5\frac{15}{32}$
(b) $\frac{1234}{432} = 2 + \frac{370}{432} = 2 + \frac{185}{216} = 2\frac{185}{216}$ ■

Problem 18.21
I am a proper fraction. The sum of my numerator and denominator is one-digit square. Their product is a cube. What fraction am I?

Solution.
By guessing and checking strategy we find $\frac{1}{8}$ ■

Problem 18.22
Show that
(a) $\frac{1}{3} < \frac{2}{3}$ (b) $\frac{5}{8} > \frac{3}{8}$.

Solution.
(a) Since $1 \cdot 3 < 2 \cdot 3$ then by Theorem 18.2 we must have $\frac{1}{3} < \frac{2}{3}$.
(b) Since $5 \cdot 8 > 8 \cdot 3$ then again by Theorem 18.2 we must have $\frac{5}{8} > \frac{3}{8}$ ■

Problem 18.23
Compare the pairs of fractions.
(a) $\frac{7}{8}$ and $\frac{3}{4}$ (b) $\frac{4}{9}$ and $\frac{7}{15}$.

Solution.
(a) Since $7 \cdot 4 > 8 \cdot 3$ then $\frac{7}{8} > \frac{3}{4}$.
(b) Since $4 \times 15 < 9 \times 7$ then $\frac{4}{9} < \frac{7}{15}$ ■

Problem 18.24
You have two different recipes for making orange juice from concentrate. The first says to mix 2 cups of concentrate with 6 cups of water. The second says to mix 3 cups of concentrate with 8 cups of water. Which recipe will have a stronger orange flavor?

Solution.
Reducing to the same denominator the first recipe consists of mixing 8 cups of concentrate with 24 cups of water whereas the second recipe consists of mixing 9 cups of concentrate with 24 cups of water. Hence, the first recipe has stronger orange flavor ■

Problem 18.25
A third grader says that $\frac{1}{4}$ is less than $\frac{1}{5}$ because 4 is less than 5. What would you tell the child?

Solution.
Reducing to the same denominator we see that $\frac{1}{4} = \frac{5}{20}$ and $\frac{1}{5} = \frac{4}{20}$. Thus, if a pie is partitioned into 20 equal slices then $\frac{1}{4}$ represents 5 slices of the pie whereas $\frac{1}{5}$ represent just 4 slices. Hence, $\frac{1}{4} > \frac{1}{5}$ ■

Problem 18.26
Find a fraction between $\frac{3}{4}$ and $\frac{7}{8}$.

Solution.
Since $3 \cdot 8 < 4 \cdot 7$ then $\frac{3}{4} < \frac{7}{8}$. By Theorem 18.3 we have

$$\begin{array}{ccccc} \frac{3}{4} & < & \frac{3+7}{4+8} & < & \frac{7}{8} \\ \frac{3}{4} & < & \frac{10}{12} & < & \frac{7}{8} \\ \frac{3}{4} & < & \frac{5}{6} & < & \frac{7}{8} \end{array} ■$$

Problem 18.27
Order the following fractions from least to greatest.
(a) $\frac{2}{3}$ and $\frac{7}{12}$.
(b) $\frac{2}{3}, \frac{5}{6}, \frac{29}{36}$, and $\frac{8}{9}$.

Solution.
(a) Since $2 \cdot 12 > 3 \cdot 7$ then $\frac{7}{12} < \frac{2}{3}$
(b) Reducing to the same denominator we find: $\frac{2}{3} = \frac{24}{36}, \frac{5}{6} = \frac{30}{36}, \frac{29}{36}, \frac{8}{9} = \frac{32}{36}$. Thus,

$$\frac{2}{3} < \frac{29}{36} < \frac{5}{6} < \frac{8}{9} \blacksquare$$

Problem 18.28
Compare $2\frac{4}{5}$ and $2\frac{3}{6}$.

Solution.
Since $2\frac{4}{5} = \frac{14}{5} = \frac{28}{10}$ and $2\frac{3}{6} = \frac{5}{2} = \frac{25}{10}$ then $2\frac{4}{5} < 2\frac{3}{6}$ $\blacksquare$

Problem 18.29
If $\frac{a}{b} < 1$, compare the fractions $\frac{c}{d}$ and $\frac{ac}{bd}$.

Solution.
Since $\frac{a}{b} < 1$ then $a < b$. Multiply both sides by cd to obtain $(ac)d < (bd)c$. By Theorem 18.2 we must have $\frac{ac}{bd} < \frac{c}{d}$ $\blacksquare$

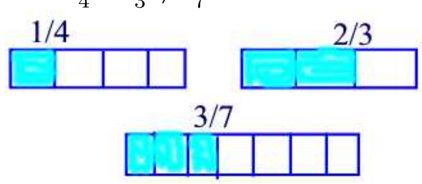
Problem 18.30
Find a fraction between $\frac{5}{6}$ and $\frac{83}{100}$.

Solution.
Since $5 \cdot 100 < 6 \cdot 83$ then $\frac{5}{6} < \frac{83}{100}$. By Theorem 18.3 we have

$$\begin{array}{ccccc} \frac{5}{6} & < & \frac{5+83}{6+100} & < & \frac{83}{100} \\ \frac{5}{6} & < & \frac{88}{106} & < & \frac{83}{100} \\ \frac{5}{6} & < & \frac{44}{53} & < & \frac{83}{100} \blacksquare \end{array}$$

Problem 19.1
If one of your students wrote $\frac{1}{4} + \frac{2}{3} = \frac{3}{7}$, how would you convince him or her that this is incorrect?

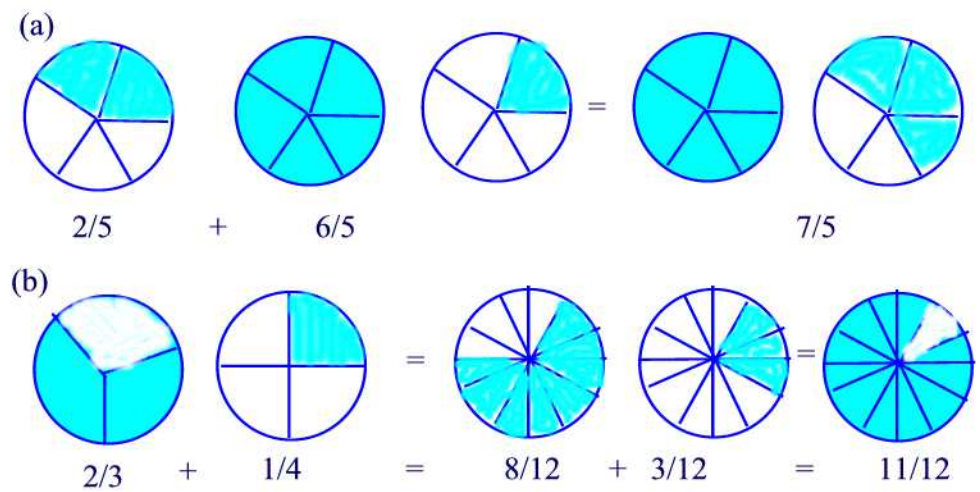
Solution.
Note that since $\frac{2}{3} > \frac{3}{7}$ then $\frac{1}{4} + \frac{2}{3} \neq \frac{3}{7}$.



Problem 19.2

Use the colored region model to illustrate these sums, with the unit given by a circular disc.
(a) $\frac{2}{5} + \frac{6}{5}$ (b) $\frac{2}{3} + \frac{1}{4}$.

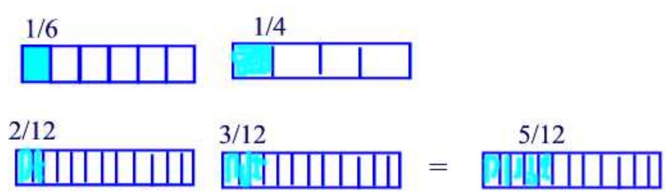
Solution.



Problem 19.3

Find $\frac{1}{6} + \frac{1}{4}$ using fraction strips.

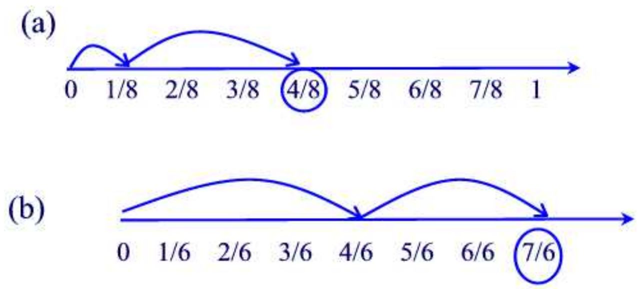
Solution.



Problem 19.4

Represent each of these sums with a number-line diagram.
(a) $\frac{1}{8} + \frac{3}{8}$ (b) $\frac{2}{3} + \frac{1}{2}$.

Solution.

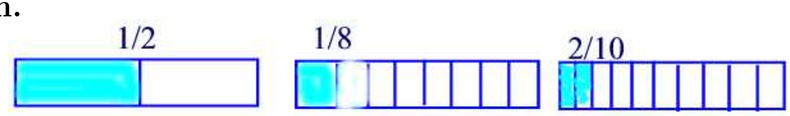


Problem 19.5
Perform the following additions. Express each answer in simplest form.
(a) $\frac{2}{7} + \frac{3}{7}$ (b) $\frac{3}{8} + \frac{11}{24}$ (c) $\frac{213}{450} + \frac{12}{50}$.

Solution.
(a) $\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$
(b) $\frac{3}{8} + \frac{11}{24} = \frac{9}{24} + \frac{11}{24} = \frac{20}{24} = \frac{5}{6}$
(c) $\frac{213}{450} + \frac{12}{50} = \frac{213}{450} + \frac{98}{450} = \frac{312}{450} = \frac{52}{75}$ ∎

Problem 19.6
A child thinks $\frac{1}{2} + \frac{1}{8} = \frac{2}{10}$. Use fraction strips to explain why $\frac{2}{10}$ cannot be the answer.

Solution.



Since $\frac{1}{2} > \frac{2}{10}$ then $\frac{1}{2} + \frac{1}{8} \neq \frac{2}{10}$ ∎

Problem 19.7
Compute the following without a calculator.
(a) $\frac{5}{12} + \frac{3}{8}$ (b) $\frac{1}{a} + \frac{2}{b}$.

Solution.
(a) Since LCM(12,8) = 24 then $\frac{5}{12} + \frac{3}{8} = \frac{10}{24} + \frac{9}{24} = \frac{19}{24}$
(b) $\frac{1}{a} + \frac{2}{b} = \frac{b}{ab} + \frac{2a}{ab} = \frac{2a+b}{ab}$ ∎

Problem 19.8
Compute the following without a calculator.
(a) $\frac{2}{15} + \frac{1}{21}$ (b) $\frac{3}{2n} + \frac{4}{5n}$.

Solution.
(a) Since $15 = 3 \times 5$ and $21 = 3 \times 7$ then $LCM(15, 21) = 3 \cdot 5 \cdot 7 = 105$. Thus, $\frac{2}{15} + \frac{1}{21} = \frac{14}{105} + \frac{5}{105} = \frac{19}{105}$.
(b) $\frac{3}{2n} + \frac{4}{5n} = \frac{15}{10n} + \frac{8}{10n} = \frac{23}{10n}$ ■

Problem 19.9
Compute $5\frac{3}{4} + 2\frac{5}{8}$.

Solution.
We have:
$$5\frac{3}{4} + 2\frac{5}{8} = \frac{23}{4} + \frac{21}{8} = \frac{46}{8} + \frac{21}{8} = \frac{67}{8} = 8\frac{3}{8} \blacksquare$$

Problem 19.10
Solve mentally:
(a)$\frac{1}{8} + x = \frac{5}{8}$
(b) $4\frac{1}{8} + x = 10\frac{3}{8}$.

Solution.
(a) Since the result has the denomirator 8 which is the same denominator than $\frac{1}{8}$ then $x = \frac{4}{8} = \frac{1}{2}$.
(b) Since $4\frac{1}{8} = 4 + \frac{1}{8} = \frac{33}{8}$ and $10\frac{3}{8} = 10 + \frac{3}{8} = \frac{83}{8}$ then $x = \frac{50}{8} = 6\frac{1}{4}$ ■

Problem 19.11
Name the property of addition that is used to justify each of the following equations.
(a) $\frac{3}{7} + \frac{2}{7} = \frac{2}{7} + \frac{3}{7}$
(b) $\frac{4}{15} + 0 = \frac{4}{15}$
(c) $\left(\frac{2}{5} + \frac{3}{5}\right) + \frac{4}{7} = \frac{2}{5} + \left(\frac{3}{5} + \frac{4}{7}\right)$
(d) $\frac{2}{5} + \frac{3}{7}$ is a fraction.

Solution.
(a) Commutative
(b) The zero or identity element for addition
(c) Associative
(d) Closure under addition ■

Problem 19.12
Find the following sums and express your answer in simplest form.
(a) $\frac{3}{7} + \frac{7}{3}$

(b) $\frac{8}{9}+\frac{1}{12}+\frac{3}{16}$
(c) $\frac{8}{31}+\frac{4}{51}$
(d) $\frac{143}{1000}+\frac{759}{100,000}$.

Solution.
(a) Since LCM(3,7) = 21 then $\frac{3}{7}+\frac{7}{3}=\frac{9}{21}+\frac{49}{21}=\frac{58}{21}$
(b) Since LCM(9,12,16)= 144 then $\frac{8}{9}+\frac{1}{12}+\frac{3}{16}=\frac{1408}{144}+\frac{12}{144}+\frac{27}{144}=\frac{1447}{144}$
(c) Since LCM(31,51)=1581 then $\frac{8}{31}+\frac{4}{51}=\frac{408}{1581}+\frac{124}{1581}=\frac{532}{1581}$
(d) $\frac{143}{1000}+\frac{759}{100,000}=\frac{14300}{100000}+\frac{759}{100000}=\frac{15059}{100,000}$ ■

Problem 19.13
Change the following mixed numbers to fractions.
(a) $3\frac{5}{6}$ (b) $2\frac{7}{8}$ (c) $7\frac{1}{9}$.

Solution.
(a) $3\frac{5}{6}=3+\frac{5}{6}=\frac{18}{6}+\frac{5}{6}=\frac{23}{6}$
(b) $2\frac{7}{8}=2+\frac{7}{8}=\frac{16}{8}+\frac{7}{8}=\frac{23}{8}$
(c) $7\frac{1}{9}=7+\frac{1}{9}=\frac{56}{9}+\frac{1}{9}=\frac{57}{9}$ ■

Problem 19.14
Use the properties of fraction addition to calculate each of the following sums mentally.
(a) $\left(\frac{3}{7}+\frac{1}{9}\right)+\frac{4}{7}$
(b) $1\frac{9}{13}+\frac{5}{6}+\frac{4}{13}$
(c) $\left(2\frac{2}{5}+3\frac{3}{8}\right)+\left(1\frac{4}{5}+2\frac{3}{8}\right)$

Solution.
(a) $\left(\frac{3}{7}+\frac{1}{9}\right)+\frac{4}{7}=\left(\frac{1}{9}+\frac{3}{7}\right)+\frac{4}{7}=\frac{1}{9}+\left(\frac{3}{7}+\frac{4}{7}\right)=\frac{1}{9}+1=1\frac{1}{9}$
(b) $1\frac{9}{13}+\frac{5}{6}+\frac{4}{13}=\left(\frac{22}{13}+\frac{4}{13}\right)+\frac{5}{6}=\frac{26}{13}+\frac{5}{6}=2+\frac{5}{6}=2\frac{5}{6}$
(c) $\left(2\frac{2}{5}+3\frac{3}{8}\right)+\left(1\frac{4}{5}+2\frac{3}{8}\right)=\left(2\frac{2}{5}+1\frac{4}{5}\right)+\left(3\frac{3}{8}+2\frac{3}{8}\right)=(3+\frac{6}{5})+(5+\frac{6}{8})=8+\frac{48}{40}+\frac{30}{40}=8+\frac{78}{40}=9+\frac{38}{40}=9+\frac{19}{20}=9\frac{19}{20}$ ■

Problem 19.15
Change the following fractions to mixed numbers.
(a) $\frac{35}{3}$ (b) $\frac{49}{6}$ (c) $\frac{17}{5}$

Solution.
Using division of whole numbers we find
(a) $\frac{35}{3}=11+\frac{2}{3}=11\frac{2}{3}$
(b) $\frac{49}{6}=8+\frac{1}{6}=8\frac{1}{6}$
(c) $\frac{17}{5}=3+\frac{2}{5}=3\frac{2}{5}$ ■

Problem 19.16
(1) Change each of the following to mixed numbers: (a) $\frac{56}{3}$ (b) $\frac{293}{100}$.
(2) Change each of the following to a fraction of the form $\frac{a}{b}$ where a and b are whole numbers: (a) $6\frac{3}{4}$ (b) $7\frac{1}{2}$.

Solution.
(1) (a) $\frac{56}{3} = 18\frac{2}{3}$ (b) $\frac{293}{100} = 2\frac{93}{100}$
(2) (a) $6\frac{3}{4} = 6 + \frac{3}{4} = \frac{27}{4}$ (b) $7\frac{1}{2} = 7 + \frac{1}{2} = \frac{15}{2}$ ■

Problem 19.17
Place the numbers 2, 5, 6, and 8 in the following boxes to make the equation true:

$$\frac{\square}{\square} + \frac{\square}{\square} = \frac{23}{24}$$

Solution.

$$\frac{\boxed{2}}{\boxed{6}} + \frac{\boxed{5}}{\boxed{8}} = \frac{23}{24}$$

Problem 19.18
A clerk sold three pieces of one type of ribbon to different customers. One piece was $\frac{1}{3}$ yard long, another $2\frac{3}{4}$yd long, and the third was $3\frac{1}{2}$yd. What was the total length of that type of ribbon sold?

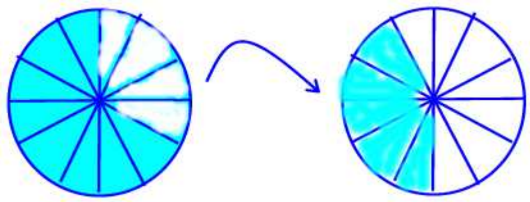
Solution.
The total length is $\frac{1}{3} + 2\frac{3}{4} + 3\frac{1}{2} = \frac{1}{3} + \frac{11}{4} + \frac{7}{2} = \frac{4}{12} + \frac{33}{12} + \frac{42}{12} = \frac{79}{12}$ ■

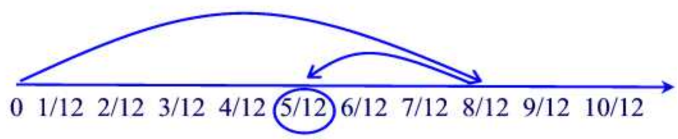
Problem 19.19
Karl wants to fertilize his 6 acres. If it takes $8\frac{2}{3}$ bags of fertilizer for each acre, how much fertilizser does he need to buy?

Solution.
Since $8\frac{2}{3} = \frac{26}{3}$ then Karl needs to buy $6\left(\frac{26}{3}\right) = 52$ bags of fertilizer ■

Problem 19.20
Use fraction strips, colored regions, and number-line models to illustrate $\frac{2}{3} - \frac{1}{4}$.

Solution.
First note that $\frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12}$.





Problem 19.21
Compute these differences, expressing each answer in simplest form.
(a) $\frac{5}{8} - \frac{2}{8}$ (b) $\frac{3}{5} - \frac{2}{4}$ (c) $2\frac{2}{3} - 1\frac{1}{3}$.

Solution.
(a) $\frac{5}{8} - \frac{2}{8} = \frac{3}{8}$
(b) $\frac{3}{5} - \frac{2}{4} = \frac{12}{20} - \frac{10}{20} = \frac{2}{20} = \frac{1}{10}$
(c) $2\frac{2}{3} - 1\frac{1}{3} = \frac{8}{3} - \frac{4}{3} = \frac{5}{3}$ ■

Problem 19.22
(a) Find the least common denominator of $\frac{7}{20}$ and $\frac{3}{28}$.
(b) Compute $\frac{7}{20} - \frac{3}{28}$.

Solution.
(a) Since $20 = 2^2 \cdot 5$ and $28 = 2^2 \cdot 7$ then the least common denominator is the $LCM(20, 28) = 2^2 \cdot 5 \cdot 7 = 140$.
(b) $\frac{7}{20} - \frac{3}{28} = \frac{49}{140} - \frac{15}{140} = \frac{34}{140} = \frac{17}{70}$ ■

Problem 19.23
Compute the following without a calculator.
(a) $\frac{5}{12} - \frac{1}{20}$ (b) $\frac{5}{6c} - \frac{3}{4c}$.

Solution.
(a) $\frac{5}{12} - \frac{1}{20} = \frac{25}{60} - \frac{3}{60} = \frac{22}{60} = \frac{11}{30}$
(b) $\frac{5}{6c} - \frac{3}{4c} = \frac{10}{12c} - \frac{9}{12c} = \frac{1}{12c}$ ■

Problem 19.24
Compute $10\frac{1}{6} - 5\frac{2}{3}$.

Solution.
We have:

$$\begin{aligned} 10\tfrac{1}{6} - 5\tfrac{2}{3} &= \tfrac{61}{6} - \tfrac{17}{3} \\ &= \tfrac{61}{6} - \tfrac{34}{6} = \tfrac{27}{6} = \tfrac{9}{2} = 4\tfrac{1}{2} \blacksquare \end{aligned}$$

Problem 19.25
Compute $90\frac{1}{3} - 32\frac{7}{9}$.

Solution.

$$\begin{aligned} 90\tfrac{1}{3} - 32\tfrac{7}{9} &= \tfrac{271}{3} - \tfrac{295}{9} \\ &= \tfrac{813}{9} - \tfrac{295}{9} = \tfrac{518}{9} \blacksquare \end{aligned}$$

Problem 19.26
Solve mentally: $3\frac{9}{10} - ? = 1\frac{3}{10}$.

Solution.
Since 3 - 2 = 1 and $\frac{9}{10} - \frac{6}{10} = \frac{3}{10}$ then the missing number is $2\frac{6}{10} = 2\frac{3}{5}$ $\blacksquare$

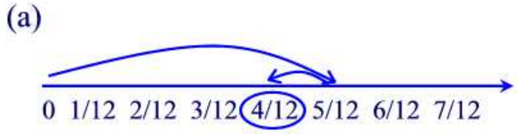
Problem 19.27
Fill in each square with either a + sign or a - sign to complete each equation correctly.

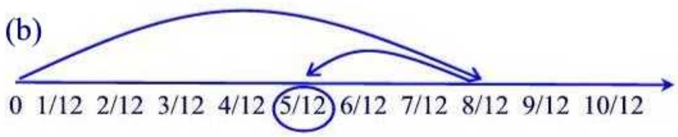
(a) $1\frac{1}{4} \square \frac{1}{4} \square \frac{3}{4} = \frac{3}{4}$
(b) $1\frac{7}{8} \square \frac{1}{4} \square \frac{3}{8} = 1\frac{1}{4}$

Solution.
(a) $1\frac{1}{4} + \frac{1}{4} - \frac{3}{4} = \frac{3}{4}$
(b) $1\frac{7}{8} - \frac{1}{4} - \frac{3}{8} = 1\frac{1}{4}$ $\blacksquare$

Problem 19.28
On a number-line, demonstrate the following differences using the take-away approach.
(a) $\frac{5}{12} - \frac{1}{12}$ (b) $\frac{2}{3} - \frac{1}{4}$.

Solution.





Problem 19.29
Perform the following subtractions.
(a) $\frac{9}{11} - \frac{5}{11}$ (b) $\frac{4}{5} - \frac{3}{4}$ (c) $\frac{21}{51} - \frac{7}{39}$.

Solution.
(a) $\frac{9}{11} - \frac{5}{11} = \frac{9-5}{11} = \frac{4}{11}$
(b) $\frac{4}{5} - \frac{3}{4} = \frac{16}{20} - \frac{15}{20} = \frac{1}{20}$
(c) $\frac{21}{51} - \frac{7}{39} = \frac{7}{17} - \frac{7}{39} = \frac{273}{663} - \frac{119}{663} = \frac{154}{663}$ ■

Problem 19.30
Which of the following properties hold for fraction subtraction?
(a) Closure (b) Commutative (c) Associative (d) Identity

Solution.
(a) No since $\frac{1}{3} - \frac{1}{2}$ is a not a fraction of whole numbers.
(b) No since $\frac{1}{3} - \frac{1}{2} \neq \frac{1}{2} - \frac{1}{3}$.
(c) No since $\frac{1}{2} - (\frac{1}{3} - \frac{1}{4}) \neq (\frac{1}{2} - \frac{1}{3}) - \frac{1}{4}$
(d) No since $\frac{1}{2} - 0 \neq 0 - \frac{1}{2}$ ■

Problem 19.31
Rafael ate one-fourth of a pizza and Rocco ate one-third of it. What fraction of the pizza did they eat?

Solution.
Since $\frac{1}{4} = \frac{3}{12}$ and $\frac{1}{3} = \frac{4}{12}$ then the fraction of the pizaa they ate is

$$\frac{1}{4} + \frac{1}{3} = \frac{7}{12} \blacksquare$$

Problem 19.32
You planned to work on a project for about $4\frac{1}{2}$ hours today. If you have been working on it for $1\frac{3}{4}$ hours, how much more time will it take?

Solution.
It will take an additional

$$4\frac{1}{2}-1\frac{3}{4}=\frac{9}{2}-\frac{7}{4}=\frac{18}{4}-\frac{7}{4}=\frac{11}{4}=2\frac{3}{4}\ hours\ \blacksquare$$

Problem 19.33
Martine bought $8\frac{3}{4}$yd of fabric. She wants to make a skirt using $1\frac{7}{8}$yd, pants using $2\frac{3}{8}$yd, and a vest using $1\frac{2}{3}$yd. How much fabric will be left over?

Solution.
She will be left with

$$\begin{aligned}
8\tfrac{3}{4}-\left(1\tfrac{7}{8}+2\tfrac{3}{8}+1\tfrac{2}{3}\right) &= \tfrac{35}{4}-\left(\tfrac{15}{8}+\tfrac{19}{8}+\tfrac{5}{3}\right)\\
&= \tfrac{35}{4}-\left(\tfrac{17}{4}+\tfrac{5}{3}\right)\\
&= \tfrac{35}{4}-\left(\tfrac{51}{12}+\tfrac{20}{12}\right)\\
&= \tfrac{35}{4}-\tfrac{71}{12}\\
&= \tfrac{105}{12}-\tfrac{71}{12}=\tfrac{34}{12}=\tfrac{17}{6}=2\tfrac{5}{6}\ yd\ \blacksquare
\end{aligned}$$

Problem 19.34
A recipe requires $3\frac{1}{2}c$ of milk. Don put in $1\frac{3}{4}c$ and emptied the container. How much more milk does he need to put in?

Solution.
He needs to put

$$3\frac{1}{2}-1\frac{3}{4}=\frac{7}{2}-\frac{7}{4}=\frac{14}{4}-\frac{7}{4}=\frac{7}{4}=1\frac{3}{4}c\ \blacksquare$$

Problem 19.35
A class consists of $\frac{2}{5}$ freshmen, $\frac{1}{4}$ sophomore, and $\frac{1}{10}$ juniors. What fraction of the class is seniors?

Solution.
The fractions of seniors is

$$\begin{aligned} 1-\left(\tfrac{2}{5}+\tfrac{1}{4}+\tfrac{1}{10}\right) &= 1-\left(\tfrac{8}{20}+\tfrac{5}{20}+\tfrac{2}{20}\right) \\ &= 1-\tfrac{3}{4}=\tfrac{1}{4}\ \blacksquare \end{aligned}$$

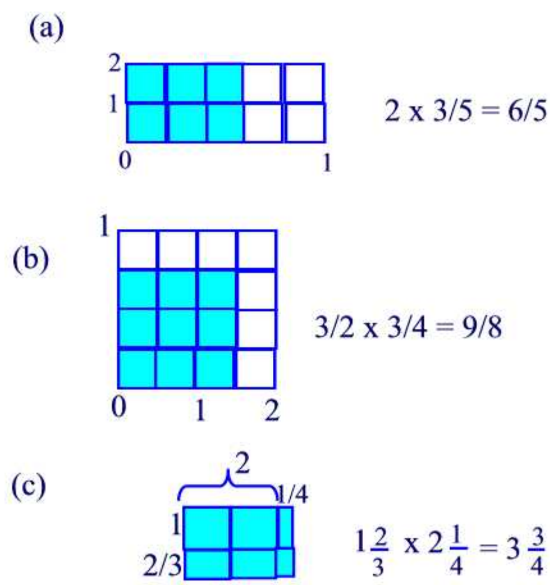
Problem 19.36
Sally, her brother, and another partner own a pizza restaurant. If Sally owns $\frac{1}{3}$ and her brother owns $\frac{1}{4}$ of the restaurant, what part does the third partner own?

Solution.
The third partner owns

$$1-\left(\frac{1}{3}+\frac{1}{4}\right)=1-\left(\frac{4}{12}+\frac{3}{12}\right)=1-\frac{7}{12}=\frac{5}{12}\ \blacksquare$$

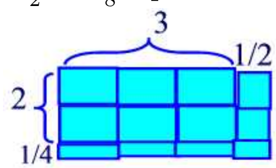
Problem 20.1
Use rectangular area models to illustrate the following multiplications.
(a) $2\times\frac{3}{5}$ (b) $\frac{3}{2}\times\frac{3}{4}$ (c) $1\frac{2}{3}\times 2\frac{1}{4}$

Solution.



Problem 20.2
A rectangular plot of land is $2\frac{1}{4}$ miles wide and $3\frac{1}{2}$ miles long. What is the area of the plot, in square miles? Draw a sketch that verifies your answer.

Solution.
The area of the plot is $2\frac{1}{4} \times 3\frac{1}{2} = 7\frac{7}{8}$ square miles. ■

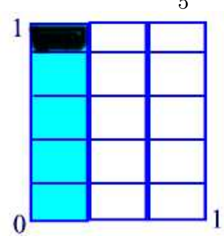


Problem 20.3
Show how to compute $4 \times \frac{1}{5}$ using repeated addition.

Solution.
We have: $4 \times \frac{1}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{4}{5}$. ■

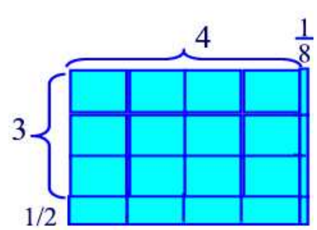
Problem 20.4
A child who does not know the multiplication rule for fractions wants to compute $\frac{1}{5} \times \frac{1}{3}$. Explain how to compute $\frac{1}{5} \times \frac{1}{3}$ using area models.

Solution.
According to the figure below we see that $\frac{1}{5} \times \frac{1}{3} = \frac{1}{15}$ ■



Problem 20.5
Show how to work $3\frac{1}{2} \times 4\frac{1}{8}$ with an area diagram.

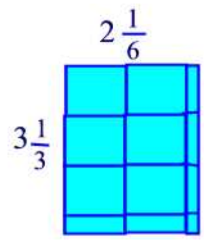
Solution.



It follows that $3\frac{1}{2} \times 4\frac{1}{8} = 14 + \frac{3}{8} + \frac{1}{16} = 14\frac{7}{16}$ ■

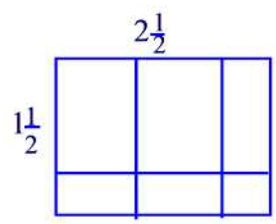
Problem 20.6
Show how to work $2\frac{1}{6} \times 3\frac{1}{3}$ with an area diagram.

Solution.



It follows that $2\frac{1}{6} \times 3\frac{1}{3} = 7 + \frac{1}{6} + \frac{1}{18} = 7\frac{2}{9}$ ■

Problem 20.7
Consider the following diagram.



Find $2\frac{1}{2} \times 1\frac{1}{2}$ from the diagram.

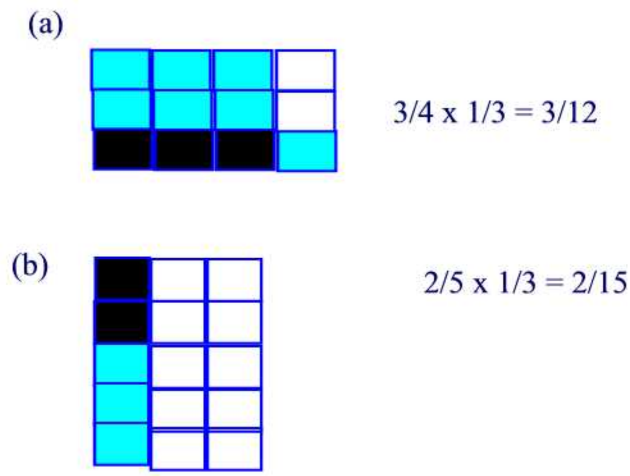
Solution.
From the diagram we see that $2\frac{1}{2} \times 1\frac{1}{2} = 3\frac{3}{4}$ ■

Problem 20.8
Find mentally.
(a) $40 \times 6\frac{1}{8}$ (b) $\frac{1}{4} \times 8\frac{1}{2}$.

Solution.
(a) $40 \times 6\frac{1}{8} = 40 \cdot 6 + \frac{40}{8} = 240 + 5 = 245$
(b) $\frac{1}{4} \times 8\frac{1}{2} = 8 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = 2 + \frac{1}{8} = 2\frac{1}{8}$ ■

Problem 20.9
Use a rectangular region to illustrate each of the following products:
(a) $\frac{3}{4} \cdot \frac{1}{3}$ (b) $\frac{2}{5} \cdot \frac{1}{3}$.

Solution.



Problem 20.10
Use the distributive property to find each product.
(a) $4\frac{1}{2} \times 2\frac{1}{3}$ (b) $3\frac{1}{3} \times 2\frac{1}{2}$

Solution.
(a)
$$\begin{aligned} 4\tfrac{1}{2} \times 2\tfrac{1}{3} &= \left(4 + \tfrac{1}{2}\right) \cdot \left(2 + \tfrac{1}{3}\right) \\ &= 4 \cdot 2 + 4 \cdot \tfrac{1}{3} + \tfrac{1}{2} \cdot 2 + \tfrac{1}{2} \cdot \tfrac{1}{3} \\ &= 8 + \tfrac{4}{3} + 1 + \tfrac{1}{6} \\ &= 10 + \tfrac{1}{3} + \tfrac{1}{6} = 10 + \tfrac{3}{6} = 10\tfrac{1}{2} \end{aligned}$$
(b)
$$\begin{aligned} 3\tfrac{1}{3} \times 2\tfrac{1}{2} &= \left(3 + \tfrac{1}{3}\right) \cdot \left(2 + \tfrac{1}{2}\right) \\ &= 6 + \tfrac{3}{2} + \tfrac{2}{3} + \tfrac{1}{6} \\ &= 7 + \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{1}{6} = 8\tfrac{1}{3} \blacksquare \end{aligned}$$

Problem 20.11
When you multiply a number by 3 and then subtract $\frac{7}{18}$, you get the same result as when you multiply the number by 2 and add $\frac{5}{12}$. What is the number?

Solution.
Let x be that number. Then we have $3x - \frac{7}{18} = 2x + \frac{5}{12}$. Add $-2x + \frac{7}{18}$ to both sides to obtain $x = \frac{5}{12} + \frac{7}{18} = \frac{15}{36} + \frac{14}{36} = \frac{29}{36}$ ■

Problem 20.12
Find the reciprocals of the following numbers.
(a) $\frac{3}{8}$ (b) $2\frac{1}{4}$ (c) 5

Solution.
(a) $\frac{8}{3}$
(b) Since $2\frac{1}{4} = \frac{9}{2}$ then its reciprocal is $\frac{2}{9}$
(c) $\frac{1}{5}$ ■

Problem 20.13
Compute these divisions, expressing your answer in simplest form.
(a) $\frac{2}{5} \div \frac{3}{4}$ (b) $2\frac{3}{8} \div 5$ (c) $2\frac{3}{8} \div 5\frac{1}{4}$.

Solution.
(a) $\frac{2}{5} \div \frac{3}{4} = \frac{2}{5} \times \frac{4}{3} = \frac{8}{15}$
(b) $2\frac{3}{8} \div 5 = \frac{19}{8} \div 5 = \frac{19}{8} \times \frac{1}{5} = \frac{19}{40}$
(c) $2\frac{3}{8} \div 5\frac{1}{4} = \frac{19}{8} \div \frac{21}{4} = \frac{19}{8} \times \frac{4}{21} = \frac{19}{42}$ ■

Problem 20.14
Compute the fraction with simplest form that is equivalent to the given expression.
(a) $\left(\frac{3}{5} - \frac{3}{10}\right) \div \frac{6}{5}$ (b) $\left(\frac{2}{5} \div \frac{4}{15}\right) \cdot \frac{2}{3}$

Solution.
Compute the fraction with simplest form that is equivalent to the given expression.
(a) $\left(\frac{3}{5} - \frac{3}{10}\right) \div \frac{6}{5} = \frac{3}{10} \div \frac{6}{5} = \frac{3}{10} \times \frac{5}{6} = \frac{1}{4}$
(b) $\left(\frac{2}{5} \div \frac{4}{15}\right) \cdot \frac{2}{3} = \left(\frac{2}{5} \times \frac{15}{4}\right) \cdot \frac{2}{3} = \frac{3}{2} \times \frac{2}{3} = 1$ ■

Problem 20.15
A child has not learned yet the rule for dividing.
(a) Explain how to compute $2 \div \frac{1}{4}$ using a diagram.
(b) Explain how to compute $2 \div \frac{1}{4}$ using multiplication.

Solution.
(a) $2 \div \frac{1}{4}$ means how many $\frac{1}{4}s$ does it take to make 2? It takes 8 quarters to make 2. So $2 \div \frac{1}{4} = 8$.
(b) $2 \div \frac{1}{4} = 2 \times \frac{4}{1} = 8$ ■

Problem 20.16
Solve the equation: $\frac{1}{2} \div x = 4$.

Solution.
We have $\frac{1}{2} \times \frac{1}{x} = 4$ so that $\frac{1}{2x} = 4$. Thus, $2x = \frac{1}{4}$. Divide both sides by 2 to obtain $x = \frac{1}{8}$ ■

Problem 20.17
Compute and simplify.
(a) $\frac{5}{8} \div \frac{1}{2}$ (b) $\frac{x}{5} \div \frac{x}{7}$ (c) $10 \div \frac{4}{3}$

Solution.
(a) $\frac{5}{8} \div \frac{1}{2} = \frac{5}{8} \times 2 = \frac{5}{4}$
(b) $\frac{x}{5} \div \frac{x}{7} = \frac{x}{5} \times \frac{7}{x} = \frac{7}{5}$
(c) $10 \div \frac{4}{3} = 10 \times \frac{3}{4} = \frac{15}{2}$ ■

Problem 20.18
Solve mentally.
(a) $\frac{1}{2} \cdot x = 10$ (b) $4 \div \frac{1}{2} = y$

Solution.
(a) Multiply both sides by 2 to obtain $x = 20$.
(b) $y = 4 \times 2 = 8$ ■

Problem 20.19
Last year a farm produced 1360 oranges. This year they produced $2\frac{1}{2}$ times as many oranges. How many oranges did they produce?

Solution.
They produced $2\frac{1}{2} \times 1360 = \frac{5}{2} \times 1360 = 3400$ oranges ■

Problem 20.20
A wall is $82\frac{1}{2}$ inches high. It is covered with $5\frac{1}{2}$inch square tiles. How many tiles are in a vertical row from the floor to the ceiling?

Solution.
There are $82\frac{1}{2} \div 5\frac{1}{2} = \frac{165}{2} \div \frac{11}{2} = \frac{165}{2} \times \frac{2}{11} = 15$ tiles $\blacksquare$

Problem 20.21
A baker takes $\frac{1}{2}$ hour to decorate a cake. How many cakes can she decorate in H hours?

Solution.
She can decorate $\frac{H}{2}$ cakes in H hours $\blacksquare$

Problem 20.22
Four fifths of a class bring in food for a charity drive. Of those who brought food, one fourth brought canned soup. What fraction of the whole class brought canned soup?

Solution.
The fraction of the class who brought canned soup is $\frac{1}{4} \times \frac{4}{5} = \frac{4}{20} = \frac{1}{5}$ $\blacksquare$

Problem 20.23
Compute the following mentally. Find the exact answers.
(a) $3 \div \frac{1}{2}$ (b) $3\frac{1}{2} \div \frac{1}{2}$ (c) $3\frac{1}{4} \cdot 8$ (d) $9\frac{1}{5} \cdot 10$

Solution.
(a) $3 \div \frac{1}{2} = 3 \times 2 = 6$
(b) $3\frac{1}{2} \div \frac{1}{2} = \frac{7}{2} \times 2 = 7$
(c) $3\frac{1}{4} \cdot 8 = \frac{13}{4} \times 8 = 26$
(d) $9\frac{1}{5} \cdot 10 = \frac{46}{5} \times 10 = 92$ $\blacksquare$

Problem 20.24
Estimate the following.
(a) $5\frac{4}{5} \cdot 3\frac{1}{10}$ (b) $4\frac{10}{1} \cdot 5\frac{1}{8}$ (c) $20\frac{8}{9} \div 3\frac{1}{12}$

Solution.
(a) $5\frac{4}{5} \cdot 3\frac{1}{10} = \frac{29}{5} \times \frac{31}{10} = \frac{899}{50}$
(b) $4\frac{1}{10} \cdot 5\frac{1}{8} = \frac{41}{10} \times \frac{41}{8} = \frac{451}{80}$
(c) $20\frac{8}{9} \div 3\frac{1}{12} = \frac{188}{9} \div \frac{37}{12} = \frac{188}{9} \times \frac{12}{37} = \frac{752}{111}$ $\blacksquare$

Problem 20.25
Five eighths of the students at Salem State College live in dormitories. If 6000 students at the college live in dormitories, how many students are there in the college?

Solution.
There are $6000 \div \frac{5}{8} = 6000 \times \frac{8}{5} = 9600$ students in the college ■

Problem 20.26
Estimate using compatible numbers.
(a) $29\frac{1}{3} \times 4\frac{2}{3}$ (b) $57\frac{1}{5} \div 7\frac{4}{5}$.

Solution.
(a) $30 \times 5 = 150$
(b) $56 \div 8 = 7$ ■

Problem 21.1
Write the following decimals in expanded form.
(a) 273.412 (b) 0.000723 (c) 0.020305

Solution.
(a) $273.412 = 2 \times 10^2 + 7 \times 10^1 + 3 \times 10^0 + 4 \times \frac{1}{10} + 1 \times \frac{1}{10^2} + 2 \times \frac{1}{10^3}$
(b) $0.000723 = 7 \times \frac{1}{10^4} + 2 \times \frac{1}{10^5} + 3 \times \frac{1}{10^6}$
(c) $0.020305 = 2 \times \frac{1}{10^2} + 3 \times \frac{1}{10^4} + 5 \times \frac{1}{10^6}$ ■

Problem 21.2
Write the following decimals as fractions in simplified form and determine the prime factorization of the denominator in each case.
(a) 0.324 (b) 0.028 (c) 4.25

Solution.
(a) $0.324 = \frac{324}{1000} = \frac{81}{250}, 250 = 2 \cdot 5^3$ (b) $0.028 = \frac{28}{1000} = \frac{7}{250}$ (c) $4.25 = \frac{425}{100} = \frac{9}{4}, 4 = 2^2$ ■

Problem 21.3
Write these fractions as terminating decimals.
(a) $\frac{7}{20}$ (b) $\frac{18}{2^2 \cdot 5^4}$

Solution.
(a) $\frac{7}{20} = 0.35$
(b) $\frac{18}{2^2 \cdot 5^4} = \frac{9}{1250} = 0.0072$ ■

Problem 21.4
If you move the decimal point in a number two places to the left, the value of the number is divided by ____ or multiplied by ____.

Solution.
If you move the decimal point in a number two places to the left, the value of the number is divided by 100 or multiplied by 10^{-2} ■

Problem 21.5
Write each of the following as a decimal number.
(a) Forty-one and sixteen hundredths
(b) Seven and five thousandths

Solution.
(a) 41.16
(b) 7.005 ■

Problem 21.6
You ask a fourth grader to add 4.21 + 18. The child asks,"Where is the decimal point in 18?" How would you respond?

Solution.
Every whole number is also a decimal number. The number 18 can be written as 18.0. Always remember that the decimal point is immediately to the right of the ones digit ■

Problem 21.7
A sign in a store mistakenly says that apples are sold for .89 cents a pound. What should the sign say?

Solution.
The sign should say \$0.89 a pound ■

Problem 21.8
Write each of the following numbers in expanded form.
(a) 0.023 (b) 206.06 (c) 0.000132 (d) 312.0103

Solution.
(a) $0.023 = 2 \times \frac{1}{100} + 3 \times \frac{1}{1000}$
(b) $206.06 = 2 \times 10^2 + 6 \times 10^0 + 6 \times \frac{1}{100}$
(c) $0.000132 = 1 \times \frac{1}{10000} + 3 \times \frac{1}{100000} + 2 \times \frac{1}{1000000}$
(d) $312.0103 = 3 \times 10^2 + 1 \times 10^1 + 2 \times 10^0 + 1 \times \frac{1}{100} + 3 \times \frac{1}{10000}$ ■

Problem 21.9
Rewrite the following sums as decimals.
(a) $4 \times 10^3 + 3 \times 10^2 + 5 \times 10 + 6 + \frac{7}{10} + \frac{8}{10^2}$
(b) $4 \times 10^3 + \frac{6}{10} + \frac{8}{10^2}$
(c) $4 \times 10^4 + \frac{3}{10^2}$
(d) $\frac{2}{10} + \frac{4}{10^4} + \frac{7}{10^7}$

Solution.
(a) $4 \times 10^3 + 3 \times 10^2 + 5 \times 10 + 6 + \frac{7}{10} + \frac{8}{10^2} = 4356.78$
(b) $4 \times 10^3 + \frac{6}{10} + \frac{8}{10^2} = 4060.08$
(c) $4 \times 10^4 + \frac{3}{10^2} = 40000.03$
(d) $\frac{2}{10} + \frac{4}{10^4} + \frac{7}{10^7} = 0.2004007$ ∎

Problem 21.10
Write each of the following as numerals.
(a) Five hundred thirty six and seventy-six ten thousandths
(b) Three and eight thousandths
(c) Four hundred thirty-six millionths
(d) Five million and two tenths

Solution.
(a) 536.0076
(b) 3.008
(c) 0.000436
(d) 5000000.2 ∎

Problem 21.11
Write each of the following terminating decimals in common fractions.
(a) 0.436 (b) 25.16 (c) 28.1902

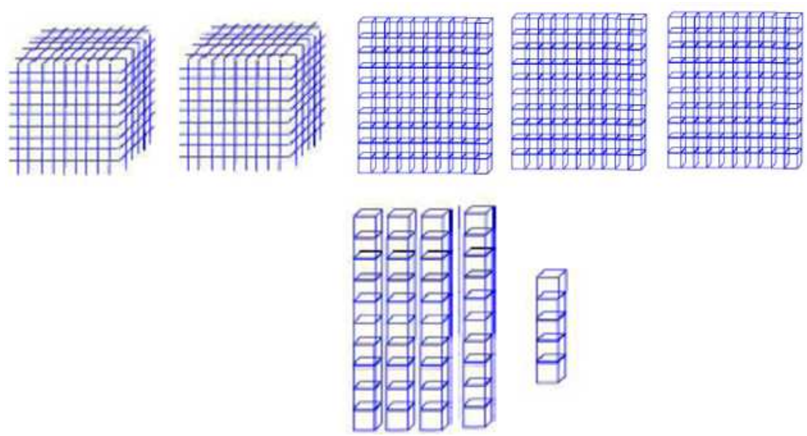
Solution.
(a) $0.436 = \frac{436}{1000} = \frac{109}{250}$
(b) $25.16 = \frac{2516}{100} = \frac{629}{25}$
(c) $28.1902 = \frac{281902}{10000} = \frac{140951}{5000}$ ∎

Problem 21.12
Determine which of the following represent terminating decimals.
(a) $\frac{61}{2^2 \cdot 5}$ (b) $\frac{133}{625}$ (c) $\frac{26}{65}$

Solution.
(a) $\frac{61}{2^2 \cdot 5}$ is a terminating decimal number since the denominator consists only of powers of 2 and 5.
(b) $\frac{133}{625} = \frac{133}{5^4}$ so a terminating decimal.
(c) $\frac{26}{65} = \frac{26}{5 \cdot 13} = \frac{2}{5}$ so the fraction is a terminating decimal number ■

Problem 21.13
Explain how to use base-ten blocks to represent "two and three hundred fourty-five thousandths"

Solution.



Problem 21.14
Write the following numbers in words.
(a) 0.013 (b) 68,485.532 (c) 0.0082 (d) 859.080509

Solution.
(a) Thirteen thousandths
(b) sixty-eight thousands four-hundred eighty-five and five hundred thrity two thousandths
(c) eighty-two ten thousandths
(d) eight-hundred fifty nine and eighty thousand five hundred nine millionths ■

Problem 21.15
Determine, without converting to decimals, which of the following fractions has a terminating decimal representation.
(a) $\frac{21}{45}$ (b) $\frac{326}{400}$ (c) $\frac{62}{125}$ (d) $\frac{54}{130}$

Solution.
Recall that a fraction whose denominator has a prime factorization consisting only of 2s and 5s is a terminating decimal number.
(a) $\frac{21}{45} = \frac{21}{5 \cdot 3^2} = \frac{7}{3 \cdot 5}$, a non-terminating decimal
(b) $\frac{326}{400} = \frac{326}{400} = \frac{326}{2^4 \cdot 5^2}$, a terminating decimal
(c) $\frac{62}{125} = \frac{62}{5^3}$, a terminating decimal
(d) $\frac{54}{130} = \frac{54}{2 \cdot 5 \cdot 13}$, non-terminating decimal ■

Problem 21.16
A student reads the number 3147 as "three thousand one hundred and forty-seven" What's wrong with this reading?

Solution.
The word "and" is used to seperate the whole part of a number from the decimal part. The give number should be read as "three thousand one hundred forty-seven" ■

Problem 21.17
It is possible to write a decimal number in the form $M \times 10^n$ where $1 \leq M < 10$ and $n \in \{0, 1, 2, \cdots\}$. This is known as the **scientific notation**. Such a notation is useful in expressing large numbers. For example, $760,000,000,000 = 7.6 \times 10^9$. Write each of the following in scientific notation.
(a) 4326 (b) 1,000,000 (c) 64,020,000 (d) 71,000,000,000

Solution.
(a) $4326 = 4.326 \times 10^3$
(b) $1,000,000 = 10^6$
(c) $64,020,000 = 6.402 \times 10^7$
(d) $71,000,000,000 = 7.1 \times 10^{10}$ ■

Problem 21.18
Find each of the following products and quotients.
(a) $(6.75)(1,000,000)$ (b) $19.514 \div 100,000$ (c) $(2.96 \times 10^{16})(10^{12})$ (d) $\frac{2.96 \times 10^{16}}{10^{12}}$

Solution.
(a) $(6.75)(1,000,000) = 6,750,000$
(b) $19.514 \div 100,000 = 0.00019514$
(c) $(2.96 \times 10^{16})(10^{12}) = 296 \times 10^{26}$
(d) $\frac{2.96 \times 10^{16}}{10^{12}} = 29,600$ ■

Problem 21.19
Order the following decimals from greatest to lowest: 13.4919, 13.492, 13.49183, 13.49199.

Solution.
$13.492 > 13.49199 > 13.4919 > 13.49183$ ■

Problem 21.20
If the numbers 0.804, 0.84, and 0.8399 are arranged on a number line, which is furthest to the right?

Solution.
$0.804 < 0.8399 < 0.84$ ■

Problem 21.21
Which of the following numbers is the greatest: $100,000^3, 1000^5, 100,000^2$? Justify your answer.

Solution.
Since $100000^3 = (10^5)^3 = 10^{15}, 1000^5 = (10^3)^5 = 10^{15}, 100000^2 = (10^5)^2 = 10^{10}$ then $100,000^3 = 1000^5$ is the largest in the list ■

Problem 21.22
The five top swimmers in an event had the following times.

Emily	64.54 seconds
Molly	64.46 seconds
Martha	63.59 seconds
Kathy	64.02 seconds
Rhonda	63.54 seconds

List them in the order they placed.

Solution.
First: Rhonda, Second: Martha, Third: Kathy, Fourth:Molly, and Fifth:Emily ■

Problem 21.23
Write the following numbers from smallest to largest: 25.412, 25.312, 24.999, 25.412412412...

Solution.
$24.999 < 25.312 < 25.412 < 25.412412412...$ ■

Problem 21.24
Order the following from smallest to largest by changing each fraction to a decimal:$\frac{3}{5}, \frac{11}{18}, \frac{17}{29}$.

Solution.
$\frac{3}{5} = 0.6, \frac{11}{18} = 0.6\overline{1}, \frac{17}{29} = 0.586.....$ Thus, $\frac{17}{29} < \frac{3}{5} < \frac{11}{18}$ ■

Problem 21.25
Round 0.3678
(a) up to the next hundredth
(b) down to the preceding hundredth
(c) to the nearest hundredth.

Solution.
(a) 0.37
(b) 0.35
(c) 0.37 ■

Problem 21.26
Suppose that labels are sold in packs of 100.
(a) If you need 640 labels, how many labels would you have to buy?
(b) Does this application require rounding up, down, or to the "nearest"?

Solution.
(a) You have to buy 7 packs or a total of 700 labels.
(b) Rounding up ■

Problem 21.27
Mount Everest has an altitude of 8847.6 m and Mount Api has an altitude of 7132.1 m. How much higher is Mount Everest than Mount Api?
(a) Estimate using rounding.
(b) Estimate using the front-end strategy.

Solution.
(a) $8847.6 - 7132.1 \approx 8800 - 7100 = 1700$ m.
(b) $8847.6 - 7132.1 \approx 8000 - 7000 = 1000$ m ■

Problem 21.28
A 46-oz can of apple juice costs \$1.29. How can you estimate the cost per ounce?

Solution.
$129 \div 46 \approx 120 \div 40 = 3$ cents $\blacksquare$

Problem 21.29
Determine by estimating which of the following answers could not be correct.
(a) $2.13 - 0.625 = 1.505$
(b) $374 \times 1.1 = 41.14$
(c) $43.74 \div 2.2 = 19.88181818$.

Solution.
(a) $2.13 - 0.625 \approx 2 - 0.5 = 1.5$
(b) $374 \times 1.1 \approx 374$
(c) $43.74 \div 2.2 \approx 437 \div 22 \approx 19.8$ $\blacksquare$

Problem 21.30
Calculate mentally. Describe your method.
(a) $18.43 - 9.96$
(b) $1.3 \times 5.9 + 1.3 \times 64.1$
(c) $4.6 + (5.8 + 2.4)$
(d) $51.24 \div 10^3$
(e) 0.15×10^5

Solution.
(a) $18.43 - 9.96 \approx 18.43 - 10 = 8.43$; equal additions
(b) $1.3 \times 5.9 + 1.3 \times 64.1 = 1.3 \times 70 = 91$; commutativity and distributivity
(c) $4.6 + (5.8 + 2.4) = 7 + 5.8 = 12.8$; commutativity and associativity
(d) $51.24 \div 10^3 = 0.05124$; powers of 10
(e) $0.15 \times 10^5 = 15,000$; powers of 10 $\blacksquare$

Problem 21.31
Estimate using the indicated techniques.
(a) $4.75 + 5.91 + 7.36$ using range and rounding to the nearest whole number.
(b) 74.5×6.1; range and rounding.
(c) $3.18 + 4.39 + 2.73$ front-end with adjustment.
(d) 4.3×9.7 rounding to the nearest whole number.

Solution.
(a) Lower range is 15 and upper range is 19; $5+6+7=18$
(b) Lower range is 420 and upper range is 560; $75 \times 6 = 450$
(c) The one-column front-end estimate is $3+4+2=9$
(d) $4 \times 10 = 40$ ■

Problem 21.32
Round the following.
(a) 97.26 to the nearest tenth
(b) 345.51 to the nearest ten
(c) 345.00 to the nearest ten
(d) 0.01826 to the nearest thousandth
(e) 0.498 to the nearest tenth

Solution.
(a) 97.3
(b) 350
(c) 350
(d) 0.018
(e) 0.5 ■

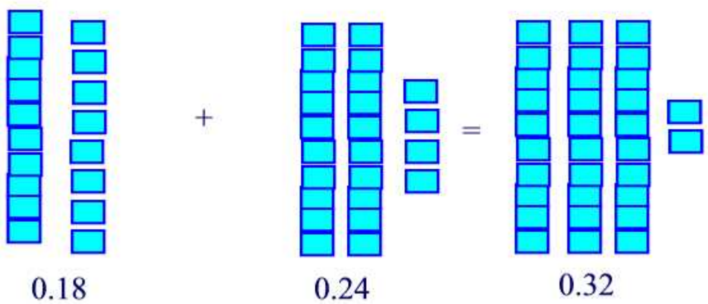
Problem 22.1
Perform the following by hand.
(a) $32.174 + 371.5$
(b) $0.057 + 1.08$
(c) $371.5 - 32.174$
(d) $1.08 - 0.057$

Solution.

$$\text{(a)}\quad \begin{array}{r} {}^{1} \\ 371.5 \\ +\ \ 32.174 \\ \hline 403.674 \end{array} \qquad \text{(b)}\quad \begin{array}{r} {}^{1} \\ 0.057 \\ +1.08 \\ \hline 1.137 \end{array} \qquad \text{(c)}\quad \begin{array}{r} 3\overset{6}{\cancel{7}}1.\overset{4}{\cancel{5}}\overset{9}{\cancel{0}}0 \\ -\ \ 32.174 \\ \hline 339.326 \end{array} \qquad \text{(d)}\quad \begin{array}{r} 1.0\overset{7}{\cancel{8}}0 \\ -\ 0.057 \\ \hline 1.023 \end{array}$$

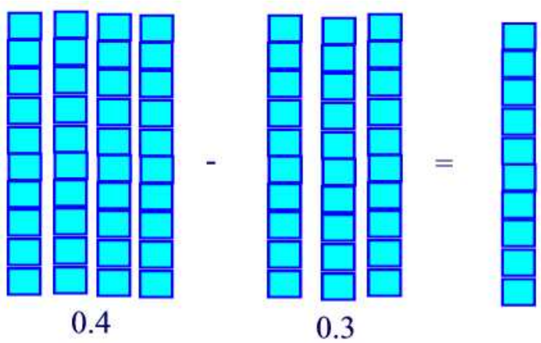
Problem 22.2
Use rectangular area model to represent the sum $0.18 + 0.24$

Solution.



Problem 22.3
Use a rectangular area model to illustrate the difference $0.4 - 0.3$

Solution.



Problem 22.4
A stock's price dropped from 63.28 per share to 27.45. What was the loss on a single share of the stock?

Solution.
The loss was $63.28 - 27.45 = 35.83$ ■

Problem 22.5
Make the sum of every row, column, and diagonal the same.

8.2		
3.7	5.5	
	9.1	2.8

Solution.

8.2	1.9	6.4
3.7	5.5	7.3
4.6	9.1	2.8

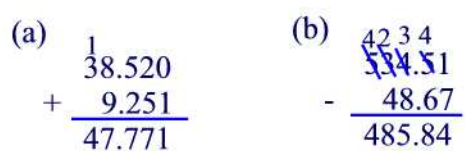
■

Problem 22.6
Find the next three decimal numbers in each of the following arithmetic sequences.
(a) 0.9, 1.8, 2.7, 3.6, 4.5
(b) 0.3, 0.5, 0.7, 0.9, 1.1
(c) 0.2, 1.5, 2.8, 4.1, 5.4

Solution.
(a) 0.9, 1.8, 2.7, 3.6, 4.5, 5.4, 6.3, 7.2
(b) 0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5,1.7
(c) 0.2, 1.5, 2.8, 4.1, 5.4, 6.7, 8.0, 9.3 ■

Problem 22.7
Perform the following operations by hand.
(a) $38.52 + 9.251$
(b) $534.51 - 48.67$

Solution.



Problem 22.8
Change the decimals in the previous exercise to fractions, perform the computations, and express the answers as decimals.

Solution.
(a) $38.52 + 9.251 = \frac{3852}{100} + \frac{9251}{1000} = \frac{38520}{1000} + \frac{9251}{1000} = \frac{47771}{1000} = 47.771$
(b) $534.51 - 48.67 = \frac{53451}{100} - \frac{4867}{100} = \frac{48584}{100} = 485.84$ ■

Problem 22.9
Perform the following multiplications and divisions by hand.
(a) $(37.1) \cdot (4.7)$
(b) $(3.71) \cdot (0.47)$
(c) $138.33 \div 5.3$
(d) $1.3833 \div 0.53$

Solution.

(a)
$$\begin{array}{r} 37.1 \\ \text{x}\ \ 4.7 \\ \hline 2597 \\ 1484 \\ \hline 174.37 \end{array}$$

(b
$$\begin{array}{r} 3.71 \\ \text{x}\ \ 0.47 \\ \hline 2597 \\ 1484 \\ \hline 1.7437 \end{array}$$

(c)
$$\begin{array}{r} 26.1 \\ 530\ \overline{)13833} \\ 1060 \\ \hline 3233 \\ 3180 \\ \hline 530 \\ 530 \\ \hline 0 \end{array}$$

(d) $1.3833 \div 0.53 = \frac{138.33}{100} \div \frac{5.3}{10} = \frac{138.33}{5.3} \times \frac{10}{100} = 26.1 \times 0.1 = 2.61$ ■

Problem 22.10
Kristina bought pairs of gloves as Christmas presents for three of her best friends. If the gloves cost \$9.72 a pair, how much did she spend for these presents?

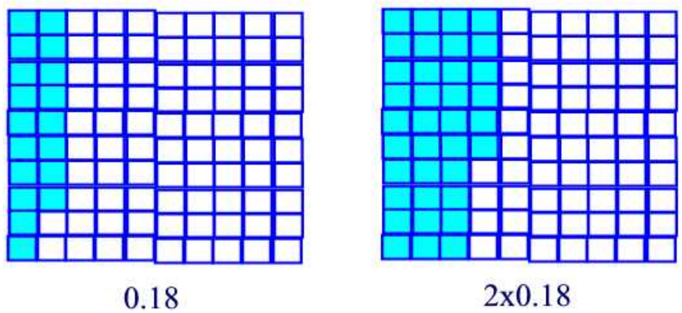
Solution.
She spent $9.72 \times 3 = \$29.16$ ■

Problem 22.11
Yolanda also bought identical pairs of gloves for each of her four best friends. If her total bill was \$44.92, how much did each pair of gloves cost?

Solution
Each pair of gloves cost $44.92 \div 4 = \$11.23$ ■

Problem 22.12
Show how to compute 2×0.18 using a rectangular area model.

Solution.



Problem 22.13
The product 34.56×6.2 has the digits 214272. Explain how to place the decimal point by counting decimal places.

Solution.
In 34.56×6.2, there are two decimal places in the first factor and one decimal place in the second factor. The product will have three decimal places: $34.56 \times 6.2 = 214.272$ ■

Problem 22.14
A runner burns about 0.12 calorie per minute per kilogram of body mass. How many calories does a 60-kg runner burn in a 10-minute run?

Solution.
$0.12 \times 60 \times 10 = 72$ calories ■

Problem 22.15
A fifth grader says 50×4.44 is the same as 0.50×444 which is 222. Is this right?

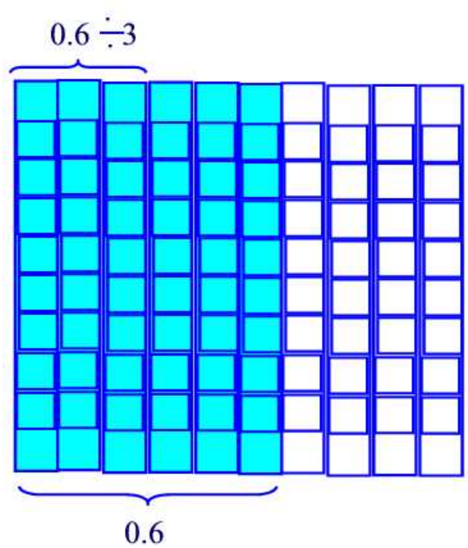
Solution.
It is correct ■

Problem 22.16
A fifth grader says $0.2 \times 0.3 = 0.6$
(a) Why do you think the child did the problem this way?
(b) What would you tell the child?

Solution.
(a) Since each factor has one decimal place so the child just multiplied 2 by 3 and used one decimal place for the answer.
(b) Since the firt factor had one decimal place and the second factor has also one decimal place then the product must have two decimal places. Hence, $0.2 \times 0.3 = 0.06$ ■

Problem 22.17
Show how to work out $0.6 \div 3$ with rectangular area model.

Solution.



Problem 22.18
What do you multiply both numbers with to change $6.4 \div 0.32$ to $640 \div 32$.?

Solution.
$6.4 \div 0.32 = \frac{640}{100} \div \frac{32}{100} = \frac{640}{100} \times \frac{100}{32} = 640 \div 32$ ■

Problem 22.19
Which of the following are equal?
(a) $8 \div 0.23$ (b) $800 \div 0.0023$ (c) $80 \div 2.3$ (d) $0.8 \div 0.023$ (e) $80 \div 0.023$

Solution.
We have: $8 \div 0.23 = 80 \div 2.3 = 0.8 \div 0.023$ ■

Problem 22.20
A sixth grader divides 16 by 3 and gets 5.1
(a) How did the child obtain this answer?
(b) What concept doesn't the child understand?

Solution.
(a) The child put the remainder after the decimal point.
(b) That the remainder represents a fraction of the divisor ■

Problem 22.21
Find the next three decimal numbers in the following geometric sequence: 1, 0.5, 0.25, 0.125

Solution.
Note that every number is the previous one multiplies by 0.5. Thus,

$$1, 0.5, 0.25, 0.125, 0.0625, 0.03125, 0.015625 \blacksquare$$

Problem 22.22
Perform the following operations using the algorithms of this section.
(a) 5.23×0.034
(b) $8.272 \div 1.76$

Solution.
(a) $5.23 \times 0.034 = \frac{5230}{1000} \times \frac{34}{1000} = \frac{177820}{1000000} = 0.17782$
(b) $8.272 \div 1.76 = \frac{8272}{1000} \div \frac{1760}{1000} = \frac{8272}{1760} = 4.7$ ■

Problem 22.23
Mentally determine which of the following division problems have the same quotient.
(a) $1680 \div 56$ (b) $0.168 \div 0.056$ (c) $0.168 \div 0.56$

Solution.
$0.168 \div 0.056 = \frac{1680}{10000} \div \frac{560}{1000} = \frac{1680}{560}$ so that $1680 \div 56 \neq 0.168 \div 0.056$. On the other hand, $0.168 \div 0.56 = \frac{1680}{10000} \div \frac{5600}{10000} = \frac{1680}{5600}$ so all the three divisions are different ■

Problem 22.24
Perform the following calculations.
(a) $2.16 \times \frac{1}{3}$ (b) $2\frac{1}{5} \times 1.55$ (c) $16.4 \div \frac{4}{9}$.

Solution.
(a) $2.16 \times \frac{1}{3} = \frac{216}{100} \times \frac{1}{3} = \frac{72}{100} = 0.72$
(b) $2\frac{1}{5} \times 1.55 = \frac{11}{5} \times \frac{155}{100} = 11 \times \frac{31}{100} = \frac{341}{100} = 3.41$
(c) $16.4 \div \frac{4}{9} = \frac{164}{10} \div \frac{4}{9} = \frac{164}{10} \times \frac{9}{4} = \frac{41}{10} \times 9 = \frac{361}{10} = 36.1$ ■

Problem 22.25
We have seen that if the prime factorization of the numerator and the denominator of a fraction contains only 2s and 5s then the decimal representation is a terminating one. For example, $\frac{2}{5} = 0.4$. On the other hand, if the prime factorizations have prime factors other than 2 and 5 then the decimal representation is nonterminating and repeating one. For example, $\frac{1}{3} = 0.\overline{3}$.
Write each of the following using a bar over the repetend.
(a) $0.7777\cdots$ (b) $0.47121212\cdots$ (c) 0.35 (d) $0.45315961596\cdots$

Solution.
(a) $0.7777\cdots = 0.\overline{7}$
(b) $0.47121212\cdots = 0.47\overline{12}$
(c) $0.35\overline{0}$
(d) $0.45315961596\cdots = 0.453\overline{1596}$ ■

Problem 22.26
Write out the first 12 decimal places of each of the following.
(a) $0.\overline{3174}$ (b) $0.3\overline{174}$ (c) $0.31\overline{74}$

Solution.
(a) $0.\overline{3174} = 0.317431743174$
(b) $0.3\overline{174} = 0.317417417417$
(c) $0.31\overline{74} = 0.317474747474$ ■

Problem 22.27
If a decimal number is nonterminating and repeating then one can rewrite it as a fraction. To see this, let $x = 0.\overline{34}$. Then $100x = 34 + 0.\overline{34}$. That is, $100x = 34 + x$ or $99x = 34$. Hence, $x = \frac{34}{99}$.
Use the above approach to express each of the following as a fraction in simplest form.
(a) $0.\overline{16}$ (b) $0.\overline{387}$ (c) $0.7\overline{25}$

Solution.
(a) Let $x = 0.\overline{16}$. Then $100x = 16 + x$ or $99x = 16$. Hence, $x = \frac{16}{99}$

(b) Let $x = 0.\overline{387}$ then $1000x = 387 + x$ or $999x = 387$. Thus, $x = \frac{387}{999} = \frac{43}{111}$.
(c) Note that $0.7\overline{25} = 0.7 + 0.0\overline{25}$. Let $x = 0.0\overline{25}$ then $100x = 2.5 + x$ or $99x = \frac{5}{2}$. Thus, $x = \frac{5}{198}$. Hence, $0.7\overline{25} = \frac{7}{10} + \frac{5}{198} = \frac{1436}{1980} = \frac{359}{495}$ ■

Problem 23.1
If two full time employees accomplish 20 tasks in a week, how many such tasks will 5 employees accomplish in a week?

Solution.
If x denotes the number of tasks accomplished by 5 employees in a week then we must have the proportion

$$\frac{x}{20} = \frac{5}{2}$$

Solving for x we find $x = 50$ tasks ■

Problem 23.2
A pipe transfers 236 gallons of fuel to the tank of a ship in 2 hours. How long will it take to fill the tank of the ship that holds 4543 gallons?

Solution.
Let x denotes the number of hours it takes to fill the ship with 4543 gallons. Then

$$\frac{x}{2} = \frac{4543}{236}$$

Multiply both sides by 2 to obtain $x = 38.5$ ■

Problem 23.3
An I-beam 12 feet long weighs 52 pounds. How much does an I-beam of the same width weigh if it is 18 feet long?

Solution.
Let x denote the weight of the 18 feet long I-beam. Then

$$\frac{x}{52} = \frac{18}{12}$$

Multiply both sides by 52 to obtain $x = 78$ pounds ■

Problem 23.4
Find the value of x.
(a) $\frac{16}{8} = \frac{x}{5}$ (b) $\frac{25}{15} = \frac{10}{x}$

Solution.
(a) Multiply both sides by 5 to obtain $x = 10$
(b) Multiply both sides by 15x to obtain $25x = 150$. Divide both sides by 25 to obtain $x = 6$ ■

Problem 23.5
A home has 2400 square feet of living space. The home also has 400 square feet of glassed window area. What is the ratio of glassed area to total square footage?

Solution.
The total square footage is 2800 square feet. Thus, the requires ratio is $400 : 2800$ or $1 : 7$ ■

Problem 23.6
A model home you are looking at has a total square footage of 3,000 feet. It is stated that the ratio of glassed area to total square footage to be 1:10. How much glassed area is there?

Solution.
Let x be the glassed area. Then $\frac{x}{3000} = \frac{1}{10}$. Solving for x we find $x = 300$ square feet ■

Problem 23.7
Stan worked 5 hours on Monday for \$25. He worked 7 hours on Tuesday. Find his wage for that day.

Solution.
His hourly wage is \$ 5. So his wage for Tuesday was $5 \times 7 = \$35$ ■

Problem 23.8
Find these ratios. Write each ratio in the two different formats
(a) a 25 year-old man to his 45 year-old father
(b) a 1200 square foot house to a 4000 square foot house

Solution.
(a) $25 : 45$ or $5 : 9$
(b) $1200 : 4000$ or $3 : 10$ ■

Problem 23.9
If the ratio of saturated to unsaturated fatty acids in a cell membrane is 9 to 1, and there are a total of 85 billion fatty acid molecules, how many of them are saturated?

Solution.
Let x be the saturated fatty acids. Then $\frac{x}{85{,}000{,}000} = \frac{9}{10}$. Solving for x we find $x = 76,500,000$. ■

Problem 23.10
The lava output from the volcano in crater park has quadrupled over the past 30 days. If the lava output 30 days ago was 4 tons of rock per week, what is the output now?

Solution.
$4 \times 4 = 16$ tons of rock per week ■

Problem 23.11
The ratio of chocolate chips to raisins in one cookie is 5:4. If the recipe required 96 raisins, how many chocolate chips were used?

Solution.
Let x be the number of chocolate chips used. Then $\frac{x}{96} = \frac{5}{4}$. Solving for x we find $x = 120$ ■

Problem 23.12
If the ratio of y to x is equal to 3 and the sum of y and x is 80, what is the value of y?

Solution.
We have $\frac{y}{x} = 3$ or $y = 3x$. Also, we have $x + y = 80$. So $3x + x = 80$ and solving for x we find $x = 20$. Thus, $y = 3(20) = 60$ ■

Problem 23.13
At a summer camp, there are 56 boys and 72 girls. Find the ratio of (a) boys to the total number of campers. (b) girls to boys.

Solution.
(a) 56 : 128 or 7 : 16
(b) 72 : 56 or 9 : 7 ■

Problem 23.14
The ratio of orange juice concentrate to water in a jug is 1:3. If there are 5 cups of concentrate in the jug, how much water was added?

Solution.
Let x be the number of caps of water. Then $\frac{5}{x} = \frac{1}{3}$. Solving for x we find $x = 15$ ∎

Problem 23.15
(a) Tom works at Wegmans. He earns \$27.30 for working 6.5 hours. How much will he earn working 20 hours?
(b) What is Toms hourly wage?

Solution.
(a) Let x be his earnings for the 20 hours of work. Then $\frac{x}{27.30} = \frac{20}{6.5}$. Solving for x we find $x = \$84$
(b) $\frac{27.30}{6.5} = \$4.20$ per hour ∎

Problem 23.16
Michelle and Rachel are running a 26.2 mile marathon together as a team. They run in a ratio of 5 : 3 respectively. How many miles do Michelle and Rachel run?

Solution.
Let x the number of miles Michelle run. Then 26.2 - x is the number of miles Rachel runs. But $\frac{x}{26.2-x} = \frac{5}{3}$. Cross-multiply to find $3x = 5(26.2 - x)$ or $3x = 131 - 5x$. Hence, $8x = 131$ or $x = \frac{131}{8} = 16.375$ miles ∎

Problem 23.17
Express the following comparisons as ratios. Suppose a class has 14 redheads, 8 brunettes, and 6 blondes.
(a) What is the ratio of redheads to brunettes?
(b) What is the ratio of redheads to blondes?
(c) What is the ratio of blondes to brunettes?
(d) What is ratio of blondes to total students?

Solution.
(a) 14 : 8 or 7 : 4
(b) 14 : 6 or 7 : 3
(c) 6 : 8 or 3 : 4
(d) 6 : 28 or 3 : 14 ∎

Problem 23.18
Express the following as ratios in fraction form and reduce.
a. 3 to 12
b. 25 to 5
c. 6 to 30
d. 100 to 10
e. 42 to 4
f. 7 to 30

Solution.
(a) $\frac{3}{12} = \frac{1}{4}$
(b) $\frac{25}{5} = 5$
(c) $\frac{6}{30} = \frac{1}{5}$
(d) $\frac{100}{10} = 10$
(e) $\frac{42}{4} = \frac{21}{2}$
(f) $\frac{7}{30}$ ■

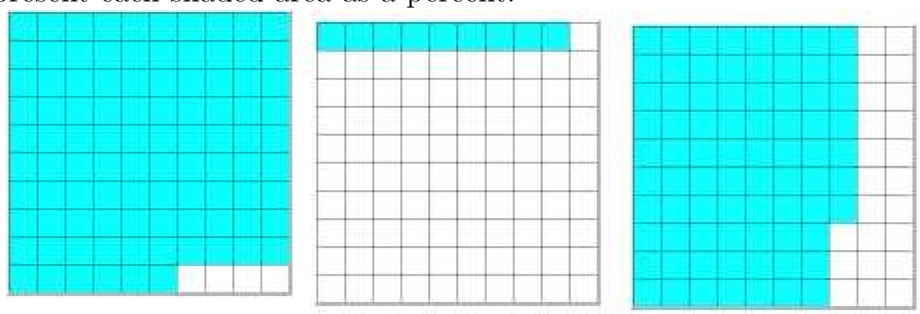
Problem 23.19
Express each of the following ratios in fractional form then simplify.
(a) 5 cents to $2
(b) 12 feet to 2 yards
(c) 30 minutes to 2 hours
(d) 5 days to 1 year
(e) 1 dime to 1 quarter

Solution.
(a) $\frac{5}{200} = \frac{1}{40}$
(b) $\frac{12}{6} = 2$
(c) $\frac{30}{120} = \frac{1}{40}$
(d) $\frac{5}{365} = \frac{1}{73}$
(e) $\frac{1}{5}$ ■

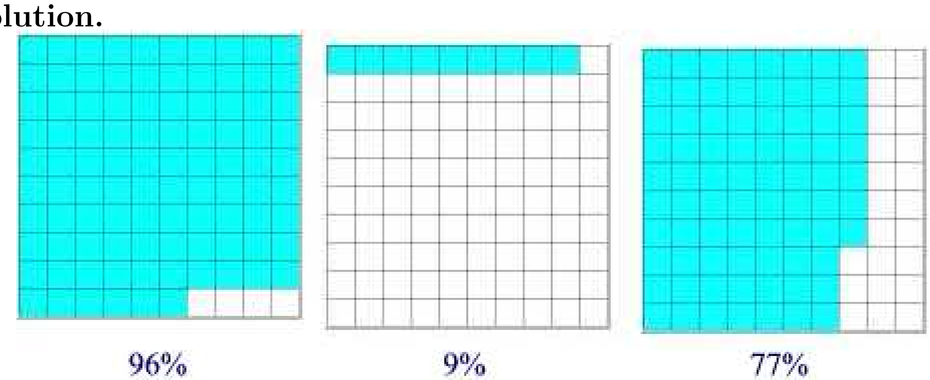
Problem 23.20
Sandra wants to give a party for 60 people. She has a punch recipe that makes 2 gallons of punch and serves 15 people. How many gallons of punch should she make for her party?

Solution.
Let x be the number of gallons of punch for serving 60 people. Then $\frac{x}{2} = \frac{60}{15}$. That is, $\frac{x}{2} = 4$. Hence, $x = 8$ gallons ■

Problem 24.1
Represent each shaded area as a percent.

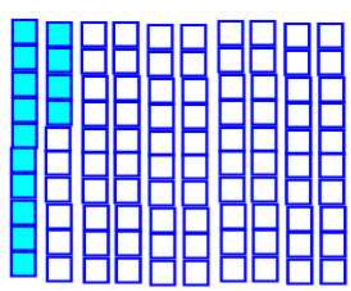


Solution.



Problem 24.2
Shade a rectangular area to represent 14%

Solution.



Problem 24.3
Write each percent as a fraction and as a decimal.
(a) 34% (b) 180% (c) 0.06$

Solution.
(a) $34\% = 0.34$
(b) $180\% = 1.8$
(c) $0.06\% = 0.0006$ ■

Problem 24.4
Write each decimal as a percent.
(a) 0.23 (b) 0.0041 (c) 24

Solution.
(a) $0.23 = 23\%$
(b) $0.0041 = 0.41\%$
(c) $24 = 2400\%$ ■

Problem 24.5
Write each fraction as a percent.
(a) $\frac{1}{25}$ (b) $\frac{3}{8}$ (c) $1\frac{3}{4}$

Solution.
(a) $\frac{1}{25} = 0.04 = 4\%$
(b) $\frac{3}{8} = 0.375 = 37.5\%$
(c) $1\frac{3}{4} = \frac{7}{4} = 1.75 = 175\%$ ■

Problem 24.6
A drink mix has 3 parts orange juice for every 2 parts of carbonated water.
(a) What fraction of the mix is carbonated water?
(b) What percent of the mix is orange juice?

Solution.
(a) $\frac{2}{5}$
(b) 60% ■

Problem 24.7
Answer the following questions.
(a) What is 30% of 500?
(b) 25 is 40% of what number?
(c) 28 out of 40 is what percent?

Solution.
(a) $0.3(500) = 159$
(b) $25 \div 0.4 = 62.5$
(c) $\frac{28}{40} = 0.7 = 70\%$ ■

Problem 24.8
Mentally compute
(a) 50% of 286 (b) 25% of 4000

Solution.
(a) 50% of 286 = 0.5(286) = 143
(b) 25% of 4000 = 0.25(4000) = 25(40) = 1000 ■

Problem 24.9
This year, Nancy Shaw's salary increased from $28,800 to $32,256. What percent increase is this?

Solution.
Since $\frac{32256-28800}{28800} = 0.12$ then the percent increase is 12$ ■

Problem 24.10
A $400 television is selling at a 25% discount. Mentally compute its sale price.

Solution.
Its sale price is 400 - 0.25(400) = $ 300 ■

Problem 24.11
How could you compute mentally the exact value of each of the following?
(a) 75% of 12
(b) 70% of 210

Solution.
(a) 10% of 12 is 1.2 so 5% is 0.6. Hence, (75%)(12) = 7(1.2) + 0.6 = 9
(b) (10%)(210) = 21 so (70%)(210) = 7(21) = 147 ■

Problem 24.12
In 2002, the voting-age populatin of the US was about 202 million, of which about 40% voted. Estimate the number of people who voted.

Solution.
$0.4(202) \approx 4(20) = 80$ million people ■

Problem 24.13
The Cereal Bowl seats 95,000. The stadium is 64% full for a certain game. Explain how to estimate the attendance mentally
(a) using rounding
(b) with compatible numbers.

Solution.
(a) $0.64(95000) = 64(950) = 60800$
(b) $0.64(95000) = 64(950) = 60(950) + 4(950) = 57000 + 3800 = 60800$ ■

Problem 24.14
Mentally convert each of the following to percent.
(a) $\frac{7}{28}$ (b) $\frac{72}{144}$ (c) $\frac{44}{66}$

Solution.
(a) $\frac{7}{28} = 0.25 = 25\%$
(b) $\frac{72}{144} = 0.5 = 50\%$
(c) $\frac{44}{66} = \frac{2}{3} = 66\%$ ■

Problem 24.15
Mentally estimate the number that should go in the blank to make each of these true.
(a) 27% of ______ equals 16.
(b) 4 is ______% of 7.5.
(c) 41% of 120 is equal to ______

Solution.
(a) 27% of 60 equals 16.
(b) 4 is 53% of 7.5.
(c) 41% of 120 is equal to 49.2 ■

Problem 24.16
Estimate
(a) 39% of 72 (b) 0.48% of 207 (c) 412% of 185

Solution.
(a) $(39\%)(72) \approx (40\%)(70) = 28$
(b) $(0.48\%)(207) \approx (0.5\%)(200) = 1$
(c) $(412\%)(185) \approx (410\%)(190) = 779$ ■

Problem 24.17
Order the following list from least to greatest: 13:25, $\frac{2}{25}$, 3%

Solution.
Since $13:25 = \frac{13}{25} = 0.52$, $\frac{2}{25} = 0.08$, and $3\% = 0.03$ then $3\% < \frac{2}{25} < 13:25$ ■

Problem 24.18
Uncle Joe made chocolate chip cookies. Benjamin ate fifty percent of them right away. Thomas ate fifty percent of what was left. Ten cookies remain. How many cookies did Uncle Joe make?

Solution.
Let x be the number of cookies made by Uncle Joe. Then $x - (0.5x + 0.5(0.5x)) = 10$. That is, $0.25x = 10$ so $x = \frac{10}{0.25} = 40$ cookies ■

Problem 24.19
Thomas won 90 percent of his wrestling matches this year and came in third at the state tournament. If he competed in 29 matches over the course of the season (including the state tournament), how many did he lose?

Solution.
He lost 10% of 29 that is about three matches. ■

Problem 24.20
According to the statistics, the Megalopolis lacrosse team scores 25% of their goals in the first half of play and the rest during the second half. Thus, it seems that the coach's opinion that they are a "second half team" is correct. If they scored 14 goals in the first half this year, about how many did they score in the second half?

Solution.
14 is 25% of 56. Thus, in the second half they scored $(75\%)(56) = 42$ goals ■

Problem 24.21
A sample of clay is found in Mongolia that contains aluminum, silicon, hydrogen, magnesium, iron, and oxygen. The amount of iron is equal to the amount of aluminum. If the clay is 20% silicon, 19% hydrogen, 10% magnesium and 24% oxygen, what is the percent iron?

Solution.
Since $20\% + 19\% + 10\% + 24\% = 73\%$ the percent of iron is $\frac{27\%}{2} = 13.5\%$ ■

Problem 24.22
Ms. Taylor wants to donate fourteen percent of her paycheck to the Mountain Springs Hospital for Children. If her paycheck is \$801.00, how much should she send to the Mountain Springs Hospital for Children?

Solution.
She should sent $(14\%)(801) = \$112.14$ ■

Problem 24.23
Alexis currently has an average of 94.7% on her three math tests this year. If one of her test grades was 91% and another was 97%, what was the grade of her third test?

Solution.
Let x be her grade on the third test. Then $91\% + 97\% + x = 3(94.7\%) = 284.1\%$. That is, $188\% + x = 284.1\%$ or $x = 284.1\% - 188\% = 96.1\%$ ■

Problem 24.24
Jennifer donated nine percent of the money she earned this summer to her local fire department. If she donated a total of \$139 how much did she earn this summer?

Solution.
Let x be the money she earned in the summer. Then $0.09x = 139$ so that $x = \frac{139}{0.09} \approx \$1544.45$ ■

Problem 24.25
If ten out of fifteen skinks have stripes and the rest don't, what percent of the skinks do not have stripes out of a population of 104 skinks? Round your answer to the nearest tenth of a percent.

Solution.
Out of 104 skinks there are $5 \times \frac{104}{15} \approx 35$ skinks do not have stripes. This is $\frac{35}{104} \approx 0.337 = 33.7\%$ ■

Problem 24.26
Sixty-eight percent of the animals in Big Range national park are herbivores. If there are 794 animals in the park, how many are not herbivores? Round your answer to the nearest whole number.

Solution.
There are $(32\%)(794) \approx 254$ animals that are not harbivores ■

Problem 24.27
There are a lot of reptiles at Ms. Floop's Reptile Park. She has snakes, lizards, turtles and alligators. If 27.8% of the reptiles are snakes, 18.2% are lizards, and 27% are alligators, what percent are turtles?

Solution.
The percent of turtles is 100% - (27.8% + 18.2% + 27%) = 27% ■

Problem 24.28
A soil sample from Mr. Bloop's farm was sent to the county agriculture department for analysis. It was found to consist of 22% sand, 24.7% silt, 29.7% clay, 7% gravel and the rest was humus. What percent of the sample was humus?

Solution.
The percent of humus is 100% - (22% + 24.7% + 29.7% + 7%) = 16.6% ■

Problem 24.29
Attendance is up at the local minor league stadium this year. Last year there was an average of 3,010 fans per game. This year the average has been 4,655. What percent increase has occurred? Round your answer to the nearest hundredth of a percent.

Solution.
The percent increase is $\frac{4655-3010}{3010} \approx 54.65\%$ ■

Problem 24.30

If a baseball team begins the season with 5,000 baseballs, and at the end of the season they have 2,673, what percent of the balls are gone? Round your answer to the nearest tenth of a percent.

Solution.

The percent of baseball lost is $\frac{5000-2673}{5000} \approx 46.5\%$ ■