SEMI-DECIDABLE EQUIVALENCE RELATIONS OBTAINED BY COMPOSITION AND LATTICE JOIN OF DECIDABLE EQUIVALENCE RELATIONS

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Abstract. Composition and lattice join (transitive closure of a union) of equivalence relations are operations taking pairs of decidable equivalence relations to relations that are semi-decidable, but not necessarily decidable. This article addresses the question, is *every* semi-decidable equivalence relation obtainable in those ways from a pair of decidable equivalence relations? It is shown that every semi-decidable equivalence relation, of which every equivalence class is infinite, is obtainable as both a composition and a lattice join of decidable equivalence relations having infinite equivalence classes. An example is constructed of a semi-decidable, but not decidable, equivalence relation having finite equivalence classes that can be obtained from decidable equivalence relations, both by composition and also by lattice join. Another example is constructed, in which such a relation cannot be obtained from decidable equivalence relations in either of the two ways.

1. Introduction

Pullback, intersection, composition and lattice join of equivalence relations are the operations by which, in practice, new equivalence relations are typically formed from antecedent ones.[1] All four operations preserve semi-decidability.[2,3] However, while pullback and intersection of decidable relations yield relations that are also decidable, composition and lattice join take pairs of decidable equivalence relations to relations that are not necessarily decidable. Since those two operations are defined by existential quantification (over elements of their arguments, for composition; and over sequences of those elements, for lattice join), an analogy with

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[1]The transitive closure of a union of equivalence relations is the join of those relations (and their intersection is their meet) in the lattice of equivalence relations, in which a refinement of a relation is ordered below it.

[2]In this article, *computable* and *decidable* describe the functions and relations, respectively, that have been called *recursive* in the older terminology adopted by Kleene [1952] and Rogers Jr. [1967]. *Partially computable* and *semi-decidable* describe objects that Rogers called *partially recursive* and *recursively enumerable.* Some authors also use the adjective, *positive,* and the acronym, *ceer,* to refer to a semi-decidable (or computably enumerable) equivalence relation. Concepts and results that will be introduced below without definition or proof can be found (in identical or transparently equivalent form) in the early chapters of Rogers' book. Although their article focuses on the specific topic of universal relations, Andrews et al. [2017], broadly cover the research literature on semi-decidable equivalence relations in their bibliography.

[3]In this assertion and the following one, a pullback by a computable function is assumed.

Kleene's projection theorem—that a set is semi-decidable iff it is defined by existential quantification over a decidable set—suggests the possibility that *every* semi-decidable equivalence relation might be obtainable as a composition or lattice join of decidable equivalence relations. This article investigates that conjecture. It is shown in proposition 4 that every semi-decidable equivalence relation, of which every equivalence class is infinite, is so obtainable from decidable equivalence relations having infinite equivalence classes. Proposition 5 specifies a semi-decidable, but not decidable, equivalence relation, having finite equivalence classes, that can be obtained from decidable equivalence relations in each of those two ways. An example is constructed also, in corollary 1 of proposition 6, of such a relation that cannot be obtained from decidable equivalence relations in either way.

2. Computability and decidability, equivalence relations

Let $\mathbb{N}$ denote $\{0, 1, 2 \dots\}$, let $\mathbb{N}_+$ denote $\{1, 2, 3 \dots\}$, and let $\mathbb{Z}$ denote the integers. In quantified formulae below, variables will range over $\mathbb{N}$. *Enumeration of R* will be used in this article to mean *computable function from $\mathbb{N}_+$ onto R.*[4] Recall that R is semi-decidable iff there is an enumeration of R.

For every $1 \leq i \leq n \in \mathbb{N}_+$, there is a computable function $\pi^n_i \colon \mathbb{N}^n \to \mathbb{N}$ such that, for every $e = (x_1, \dots, x_n) \in \mathbb{N}^n$,

$$\pi^n_i(e) = x_i \tag{1}$$

Due to this fact, there is no loss of generality in restricting attention to relations that are subsets of $\mathbb{N} \times \mathbb{N}$ rather than studying subsets of $\mathbb{N}^n \times \mathbb{N}^n$ in general. (Cf. Rogers Jr. [1967, pp. 64–66].) This simplification will be made henceforth in this article, recognizing that all results proved here can be generalized.

An *equivalence relation* on $\mathbb{N}$ is a reflexive, transitive, symmetric relation, represented as a subset of $\mathbb{N} \times \mathbb{N}$. Define an equivalence relation, E, to be *IC* if every equivalence class of E has the cardinality of $\mathbb{N}$, and to be *FC* if every equivalence class is finite.[5] Let $[i]_E$ (or simply $[i]$, if the meaning is clear) denote the equivalence class of i in E. The *partition corresponding to E* is the set of those equivalence classes.

Define the *field* of a binary relation, H, by[6]

$$\mathcal{F}(H) = \{i \mid \exists j\, (i, j) \in H \cup H^{-1}\} \tag{2}$$

The notion of an equivalence class can be extended to symmetric, transitive relations (with $[i]_H = \emptyset$ if $i \notin \mathcal{F}(H)$) by defining

$$[i]_H = \{j \mid (i, j) \in H\} \tag{3}$$

A symmetric, transitive relation, H, will be said to be IC if $[i]$ is infinite for every i in $\mathcal{F}(H)$. The following, obvious lemma will be used in section 7.

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[4]The choice to make $\mathbb{N}_+$ the domain, rather than $\mathbb{N}$, is motivated by the definition of a walk later in this section, where it will be convenient that $n \neq -n$ for all n in the domain of the enumeration.

[5]'FC', acronym for 'finite class', was introduced by Gao and Gerdes [2001]. Correspondingly, 'IC' stands for 'infinite class'.

[6]For $(i, j) \in \mathbb{N} \times \mathbb{N}$, define $(i, j)^{-1} = (j, i)$. For $R \subset \mathbb{N} \times \mathbb{N}$, define $R^{-1} = \{e^{-1} \mid e \in R\}$, the *converse relation of R.*

Lemma 1. *If H is a symmetric, transitive relation on $\mathbb{N}$, then $i \in \mathcal{F}(H)$ iff $(i,i) \in H$. If each of H and J is a symmetric, transitive relation on $\mathbb{N}$ and is IC, and if $\mathcal{F}(J) = \mathbb{N} \setminus \mathcal{F}(H)$, then $H \cup J$ is an IC equivalence relation on $\mathbb{N}$.*

Proofs below will require a formal definition of the transitive closure of a binary relation and the statement of an equivalent characterization of it (lemma 3) for symmetric, semi-decidable relations. Let R^+ denote the transitive closure of R. Define $R^{(1)} = R$ and $R^{(n+1)} = RR^{(n)}$.[7] Then

$$R^+ = \bigcup_{n \in \mathbb{N}_+} R^{(n)} \tag{4}$$

The following lemma, proved by showing inductively that the hypothesis entails that $R^{(n)} \subseteq R$, will be used in section 8.

Lemma 2. *Suppose that the equivalence class in E of every number is a singleton or a pair. If $R \subseteq E$ is reflexive, then $R^+ = R$. If $S \subseteq E$ is reflexive and symmetric, then S is an equivalence relation.*

Suppose that $R \subset \mathbb{N} \times \mathbb{N}$ is a semi-decidable relation enumerated by ε. For $k \in \mathbb{N}_+$, define

$$\tau(k) = \eta(-k) = \pi_1^2 \varepsilon(k) \text{ and } \eta(k) = \tau(-k) = \pi_2^2 \varepsilon(k) \tag{5}$$

That is, the ordered pairs in $R \cup R^{-1}$ are viewed as edges of a directed graph, ε is extended to $\mathbb{Z} \setminus \{0\}$ by defining $\varepsilon(-k) = \varepsilon(k)^{-1}$, and $x \in \mathbb{Z} \setminus \{0\}$ is interpreted as being a directed edge with tail $\tau(x)$ and head $\eta(x)$.

A *walk* (of length n) from i to j in R is a sequence, $(x_1, \ldots, x_n) \in (\mathbb{Z} \setminus \{0\})^n$ such that $\tau(x_1) = i$, $\eta(x_n) = j$, and, for $1 \le k < n$, $\eta(x_k) = \tau(k+1)$.

The next lemma follows straightforwardly by induction from these definitions.

Lemma 3. *If ε enumerates R, then $(i,j) \in (R \cup R^{-1})^+$ iff there is a walk from i to j in R. If there is a walk of length n, then $(i,j) \in (R \cup R^{-1})^{(n)}$.*

3. Coding a semi-decidable equivalence relation

Henceforth throughout this article, it will be assumed that

$$E \subseteq \mathbb{N}^2 \text{ is an equivalence relation and } \varepsilon \colon \mathbb{N}_+ \to E \text{ enumerates } E. \tag{6}$$

In this section, a computable injection, $\gamma \colon \mathbb{N}_+ \to \mathbb{N}_+$, will be defined that will be seen to encode E implicitly, where E is a semi-decidable, IC equivalence relation. Defining

$$\delta = \varepsilon\gamma, \tag{7}$$

Later in the article, δ will be shown to enumerate a decidable subset of E. The transitive closure of $\delta(\mathbb{N}) \cup (\delta(\mathbb{N}))^{-1}$, to be denoted by F, will be shown to be a decidable, IC equivalence relation. E will subsequently be characterized as FGF, where G is another such relation that is also derived from γ. Because $F \subset E$ and $G \subseteq E$, it follows that E is the join of F and G in the lattice of equivalence relations.

A computable function, $\gamma \colon \mathbb{N}_+ \to \mathbb{N}_+$, satisfying conditions (8) below will be called a *coding of ε,* and will be said to *code ε*. It will be called a *coding of E* iff it codes some enumeration of E. An equivalence relation, of which some

[7] RS denotes the composition of R and S.

enumeration has a coding, will be called *codable*. Proposition 1 will establish that a semi-decidable equivalence relation is codable iff *every* enumeration has a coding, and also iff the relation is IC.

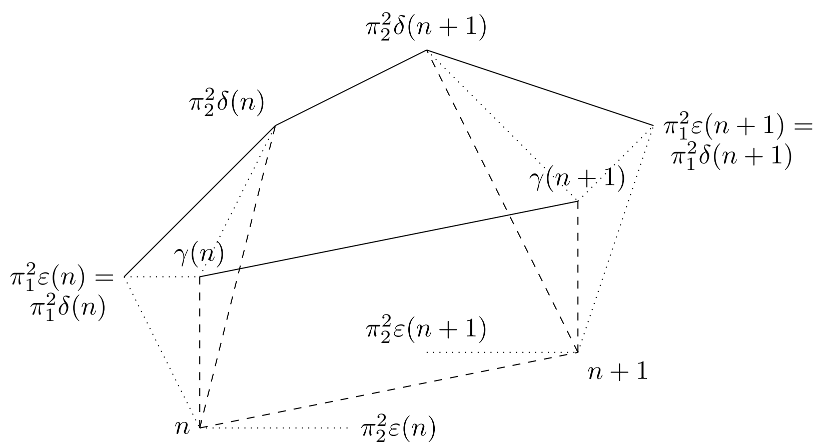
$$
\begin{gathered}
1 < \gamma(1) \qquad \pi_1^2\delta(1) = \pi_1^2\varepsilon(1) \qquad \max\{1, \pi_1^2\varepsilon(1)\} < \pi_2^2\delta(1) \\
\gamma(n) < \gamma(n+1) \qquad \pi_1^2\varepsilon(\gamma(n+1)) = \pi_1^2\varepsilon(n+1) \\
\max\{\pi_1^2\delta(n+1), \pi_2^2\delta(n)\} < \pi_2^2\delta(n+1))\}
\end{gathered} \tag{8}
$$

Lemma 4. *If $\gamma\colon \mathbb{N}_+ \to \mathbb{N}_+$, satisfies conditions (8), and if equation (7) defines δ, then γ and δ are computable.*

Proof. Conditions (8) define a partially computable function, γ, by recursion. By hypothesis, γ is total. A partially computable, total function is computable, so γ is computable. A composition of computable functions is computable, so δ is computable. $\square$

The relations specified in (8) are depicted in the following diagram, figure (9). The higher endpoint of a solid line segment is explicitly specified to be a larger number than the lower endpoint is. Additionally, from $1 < \gamma(1)$ and $\gamma(n) < \gamma(n+1)$, it follows by induction that $n < \gamma(n)$ and $n + 1 < \gamma(n + 1)$. By parallel reasoning, $n < \pi_2^2\delta(n)$ and $n + 1 < \pi_2^2\delta(n + 1)$.[8] Also $n < n + 1$. The heights of the opposite endpoints of the respective dashed line segments indicate these relative magnitudes. A dotted line segment indicates that its endpoints are related as argument and image of a function (which may be a composite function). The relative magnitudes of opposite endpoints of dotted line segments are not constrained by (8).

(9)



[8]These parallel conclusions regarding γ and δ will be restated in lemma 5.

Two crucial features of this diagram encapsulate the role that the construction of γ will play in the following proofs:

The relative magnitudes of $\pi_1^2\varepsilon(n)$ and $\pi_2^2\varepsilon(n)$ are indeterminate,

but $\pi_1^2\delta(n) < \pi_2^2\delta(n)$

The relative magnitudes of $\pi_2^2\varepsilon(n)$ and $\pi_2^2\varepsilon(n+1)$ are indeterminate,

but $\pi_2^2\delta(n) < \pi_2^2\delta(n+1)$

Lemma 5. *If γ codes ε, then both γ and $\pi_2^2\delta$ are strictly increasing functions. For all n, $n < \gamma(n)$ and $n < \pi_2^2\delta(n)$.*

Proof. Equation (8) requires explicitly that γ must be increasing, and also that $1 < \gamma(1)$. If $n < \gamma(n)$, then $n+1 < \gamma(n)+1 \leq \gamma(n+1)$, so γ satisfies $\forall n\ n < \gamma(n)$ by induction. Taking $k = \gamma(1)$ in the first equation and $k = \gamma(n+1)$ in the second equation of (8), those equations specify that $1 < \pi_2^2\varepsilon\gamma(1) = \pi_2^2\delta(1)$ and that $\pi_2^2\delta(n) < \pi_2^2\varepsilon\gamma(n+1) = \pi_2^2\delta(n+1)$. The second of these inequalities states that $\pi_2^2\delta$ is strictly increasing, and the two inequalities together imply by induction that $\forall n\ n < \pi_2^2\delta(n)$. $\square$

Lemma 6. *If some enumeration, ε, of an equivalence relation, E, has a coding, γ, then, for all $i \in \mathbb{N}_+$, $\{n \mid \pi_1^2\delta(n) = i\}$ is infinite, and E is IC. Specifically, where $\varepsilon(k) = (i,i)$, $\{\delta\gamma^n(k) \mid n \in \mathbb{N}\} = \{i\} \times \{\pi_2^2\delta\gamma^n(k) \mid n \in \mathbb{N}\}$ is an infinite subset of E.*

Proof. Assume that γ codes ε, an enumeration of E, and that $\delta = \varepsilon\gamma$. Let $i \in \mathbb{N}_+$, and let $\varepsilon(k) = (i,i)$. By induction, for all n, $(\pi_1^2\varepsilon\gamma^n(k), \pi_2^2\varepsilon\gamma^n(k)) = (i, \pi_2^2\varepsilon\gamma^n(k)) \in E$. By induction, invoking lemma 5, for all n, $\gamma^n(k) < \gamma(\gamma^n(k)) = \gamma^{n+1}(k)$ and consequently $\pi_2^2\delta(\gamma^n(k)) < \pi_2^2\delta(\gamma^{n+1}(k))$.[9]

Thus, for every i, $\{\pi_2^2\delta\gamma^n(k) \mid n \in \mathbb{N}\}$ is an infinite subset of $[i]$, where $\varepsilon(k) = (i,i)$. $\square$

Lemma 7. *If E is an IC equivalence relation with enumeration ε, then there exists a coding, γ, of ε.*

Proof. Define $\gamma\colon \mathbb{N}_+ \to \mathbb{N}_+$ as follows.

$$
\begin{aligned}
\gamma(1) &= \min\{k \mid 1 < k \text{ and } \pi_1^2\varepsilon(k) = \pi_1^2\varepsilon(1) \text{ and} \\
&\qquad \max\{1, \pi_1^2\varepsilon(1)\} < \pi_2^2\varepsilon(k)\} \\
\gamma(n+1) &= \min\{k \mid \gamma(n) < k \text{ and } \pi_1^2\varepsilon(k) = \pi_1^2\varepsilon(n+1) \\
&\qquad \text{and } \max\{\pi_1^2\varepsilon(k), \pi_2^2\delta(n)\} < \pi_2^2\varepsilon(k)\}
\end{aligned} \tag{10}
$$

By lemma 3, γ is computable if it is total. That γ is total is proved by induction on the hypothesis that $\gamma(n)$ converges. To prove the basis step, let $\pi_1^2\varepsilon(1) = i$ and note that the equivalence class of i is infinite. Therefore, for infinitely many $j > i$, $(i,j) \in E$. Since ε enumerates E, there are some $(i,j) \in E$ and $h > 1$ such that $(i,j) = \varepsilon(h)$, so $\{k \mid 1 < k \text{ and } \pi_1^2\varepsilon(1) = \pi_1^2\varepsilon(k) < \pi_2^2\varepsilon(k)\}$ is non empty. $\gamma(1)$ is defined to be the least element of this set, so $\gamma(1)$ converges.

The proof of the induction step is parallel. Suppose that $\gamma(n)$ converges. Let $\pi_1^2\varepsilon(n+1) = i$. The equivalence class of i is infinite, so, for infinitely many $j > \max\{\pi_1^2\varepsilon(k), \pi_2^2\delta(n)\}$, $(i,j) \in E$. For some such j, and for some $h > \gamma(n)$,

[9] For any mapping $f\colon \mathbb{N}_+ \to \mathbb{N}_+$, f^0 denotes the identity mapping and $f^{n+1} = ff^n$.

$(i,j) = \varepsilon(h)$. Thus $h \in \{k \mid \gamma(n) < k$ and $\pi_1^2\varepsilon(k) = \pi_1^2\varepsilon(n+1)$ and $\pi_2^2\varepsilon(k) > \max\{\pi_1^2\varepsilon(k), \pi_2^2\delta(n)\}\}$. Since $\{k \mid \pi_1^2\varepsilon(k) = \pi_1^2\varepsilon(n+1)$ and $\pi_2^2\varepsilon(k) > \max\{\pi_1^2\varepsilon(k), \pi_2^2\delta(n)\}\}$ is non empty, it has a least element, so $\gamma(n+1)$ converges. By the principle of induction, then, γ is total, and therefore γ is computable.

Clearly (10) specifies a function that satisfies definition (8) of a coding. □

Proposition 1. *For a semi-decidable equivalence relation, E, the following three conditions are equivalent.*

(11) *Some enumeration of E has a coding;*
(12) *E is IC;*
(13) *Every enumeration of E has a coding.*

Proof. Condition (11) implies condition (12) by lemma 6. Condition (12) implies condition (13) by lemma 7. Since every semi-decidable equivalence relation has an enumeration, condition (13) implies condition (11). □

4. Three relations defined from a coding

Throughout sections 4–7, it will be assumed that

(14) γ codes ε, an enumeration of E, an IC equivalence relation.

Two new relations—R_γ and S_γ—will be defined in this section, and they will be shown to be decidable. It will be shown that $R_\gamma \subseteq E$, $S_\gamma \subseteq E$, and $E = R_\gamma S_\gamma R_\gamma^{-1}$. A third relation, T_γ, will also be defined. It will be shown that T_γ is decidable, $T_\gamma \subseteq E$, and $S_\gamma \cup T_\gamma$ is symmetric and transitive, laying the groundwork for extending the result that $E = R_\gamma S_\gamma R_\gamma^{-1}$ to a result that every semi-decidable, IC equivalence relation the lattice join of decidable, IC, equivalence relations.

Define

$$R_\gamma = \delta(\mathbb{N}_+) \tag{15}$$

and

$$S_\gamma = \{(\pi_2^2\delta(m), \pi_2^2\delta(n)) \mid \varepsilon(n) = (\varepsilon(m))^{-1}\} \tag{16}$$

Lemma 8. *R_γ and S_γ are decidable.*

Proof. First, consider R_γ. Since $n < \pi_2^2\delta(n)$ (lemma 5),

$$R_\gamma = \{(i,j) \mid \exists n{<}j\ (i,j) = \delta(n)\} \tag{17}$$

Since δ is computable (lemma 4),

$$\{(i,j,k,n) \mid n < k \text{ and } (i,j) = \delta(n)\} \tag{18}$$

is a decidable relation. Decidable relations are closed under bounded quantification.[10] (Cf. Rogers Jr. [1967, p. 311].) Therefore

$$\{(i,j,k) \mid \exists n{<}k\ (i,j) = \delta(n)\} \tag{19}$$

is decidable. Let f be the characteristic function of $\{(i,j,k) \mid \exists n{<}k\ (i,j) = \delta(n)\}$, and let $g(i,j) = (i,j,j)$. Both f and g are computable, so fg, the characteristic function of R_γ, is computable. That is, R_γ is decidable.

[10]In the proofs of lemma 13 and proposition 5 below, it will be important that the bound may be the value of a computable function of variables of the relation. For example, if R is decidable and f is computable, then $S(x,y) \iff \exists z{<}f(x,y)\, R(x,y,z)$ defines a decidable relation, S.

The decision procedure for S_γ is similar. S_γ is defined by (16) which, setting $i = \pi^2_2\delta(m)$ and $j = \pi^2_2\delta(n)$, is equivalent to

$$S_\gamma = \{(i,j) \mid \exists m\, \exists n\, [i = \pi^2_2\delta(m)] \text{ and } j = \pi^2_2\delta(n) \text{ and } \varepsilon(n) = (\varepsilon(m))^{-1}\} \tag{20}$$

By lemma 5, the existential quantifiers in (20) can be replaced by bounded existential quantifiers, yielding

$$\begin{aligned} S_\gamma = \{(i,j) \mid \exists m{<}i\ \exists n{<}j\ [i = \pi^2_2\delta(m) \text{ and}\\ j = \pi^2_2\delta(n) \text{ and } \varepsilon(n) = (\varepsilon(m))^{-1}]\} \end{aligned} \tag{21}$$

The proof that S_γ is decidable from equation (21) is parallel to that for R_γ from equation (19). □

Lemma 9. *$R_\gamma \subseteq E$ and $S_\gamma \subseteq E$.*

Proof. For $e \in \mathbb{N}\times\mathbb{N}$, $e \in R_\gamma$ if and only if, for some n, $e = \delta(n)$. Thus there is some number (specifically, $m = \gamma(n)$), such that $e = \varepsilon(m)$, so $e \in E$.

If $(i,j) \in S_\gamma$, then, for some m, n, p, q,

$$i = \pi^2_2\delta(m) \quad \varepsilon(m) = (p,q) \qquad j = \pi^2_2\delta(n) \quad \varepsilon(n) = (q,p) \tag{22}$$

There are h and k such that $(h,i) = \delta(m)$ and $(k,j) = \delta(n)$. Since $\forall r\ \pi^2_1\delta(r) = \pi^2_1\varepsilon(r)$, $h = p$ and $k = q$. Therefore $(h,k) = (p,q) = \varepsilon(m) \in E$.

Since δ enumerates $R_\gamma \subseteq E$, $(h,i) \in E$. Since E is symmetric, $(i,h) \in E$. Since also $(h,k) \in E$ and E is transitive, $(i,k) \in E$. Finally, since also $(k,j) \in E$, $(i,j) \in E$. □

Proposition 2. $E = R_\gamma S_\gamma R_\gamma^{-1}$. *An IC equivalence relation is semi-decidable if, and only if, it is a composition of decidable relations.*

Proof. By lemma 9, $R_\gamma S_\gamma R_\gamma^{-1} \subseteq E$. To show that $E \subseteq R_\gamma S_\gamma R_\gamma^{-1}$, suppose that $e = (i,j) \in E$. Let $e = \varepsilon(m)$ and let $e^{-1} = \varepsilon(n)$. Let $(i,h) = \delta(m)$ and let $(j,k) = \delta(n)$. Then $(i,h) \in R_\gamma$, $(h,k) \in S_\gamma$, and $(k,j) \in R_\gamma^{-1}$, so $e \in R_\gamma S_\gamma R_\gamma^{-1}$.

That every composition of decidable relations is semi-decidable, is a routine result of computability theory.[11] An IC , semi-decidable equivalence relation is codable by proposition 1, so $E = R_\gamma S_\gamma R_\gamma^{-1}$ establishes that every IC , semi-decidable relation is a composition of decidable relations. □

To prove that a semi-decidable, IC equivalence relation is the lattice join of decidable equivalence relations, will require that there should be a decidable, transitive, symmetric relation, U, such that $S_\gamma \subseteq U \subseteq E$. This relation will be obtained by taking U to be the union of S_γ with the following relation.

$$T_\gamma = \{(i,j) \mid \exists m{<}i\ \exists n{<}j\ [i = \pi^2_2\delta(m) \text{ and } j = \pi^2_2\delta(n) \text{ and } \varepsilon(n) = \varepsilon(m)]\} \tag{23}$$

Also define the *identity relation,*

$$I = \{(n,n) \mid n \in \mathbb{N}\} \tag{24}$$

Lemma 10. *T_γ is decidable and $T_\gamma \subseteq E$. $S_\gamma \cup T_\gamma$ is symmetric and transitive. Thus $S_\gamma \cup T_\gamma \cup I \subseteq E$ is an equivalence relation. $S_\gamma \cup T_\gamma$ and $S_\gamma \cup T_\gamma \cup I$ are decidable relations.*

[11] Cf. Rogers Jr. [1967, problem 5-18]. Rogers uses 'relative product' to denote the composition of two relations.

Proof. Since S_γ and T_γ are subsets of E (lemma 9) and also $I \subseteq E$, $S_\gamma \cup T_\gamma \cup I \subseteq E$.

The proof that T_γ is decidable closely follows the corresponding proof for S_γ. To prove that $T_\gamma \subseteq E$, suppose that $(i, j) \in T_\gamma$.Then, for some m, n, p, q,

$$i = \pi_2^2\delta(m) \quad j = \pi_2^2\delta(n) \quad \varepsilon(m) = \varepsilon(n) = (p, q) \tag{25}$$

There are h and k such that $(h, i) = \delta(m)$ and $(k, j) = \delta(n)$. Since $\forall r\ \pi_1^2\delta(r) = \pi_1^2\varepsilon(r)$, $h = k = p$. That is, $(p, i) = \delta(m)$ and $(p, j) = \delta(n)$, so $(p, i) \in E$ and $(p, j) \in E$. Since E is symmetric and transitive, $(i.j) \in E$.

Substituting j and n for i and m respectively in equations (16) and (23) results in equivalent expressions. Thus S_γ and T_γ, and consequently $S_\gamma \cup T_\gamma$, are symmetric. To see that $S_\gamma \cup T_\gamma$ is transitive, suppose that $(i, j) \in S_\gamma \cup T_\gamma$ and $(j, k) \in S_\gamma \cup T_\gamma$. If exactly one of (i, j) and (j, k) is in S_γ, then $(i, k) \in S_\gamma$. Otherwise, $(i, k) \in T_\gamma$.

Decidable relations are closed under union. Since S_γ is decidable (lemma 8) and T_γ is also decidable (by parallel reasoning) and I is decidable (obvious), $S_\gamma \cup T_\gamma$ and $S_\gamma \cup T_\gamma \cup I$ are decidable. Since $S_\gamma \cup T_\gamma$ is symmetric and transitive, and taking their union with I preserves those properties and makes the resulting relation reflexive, $S_\gamma \cup T_\gamma \cup I$ is a decidable equivalence relation. □

5. $(R_\gamma \cup R_\gamma^{-1})^+$ is decidable

It will be shown that $(R_\gamma \cup R_\gamma^{-1})^+$ is a decidable, IC, equivalence relation. The first step is to show that it is decidable. By lemma 3, $(i, j) \in (R_\gamma \cup R_\gamma^{-1})^+$ iff there is a walk from i to j in R_γ.

Lemma 11. *Suppose that $i \neq j$ and that $(x_1, \ldots, x_n)$ is a walk of minimal length from i to j in R_γ, where δ enumerates R_γ. Then, for some k, $0 \leq k < n$ and $\forall t \leq k\ x_t < 0$ and $\forall t > k\ x_t > 0$.*

Proof. Suppose that $i \neq j$ and that $(x_1, \ldots, x_n)$ is a walk of minimal length from i to j in R_γ. If $\forall t\ x_t > 0$, then $k = 0$ satisfies the required condition. If $\forall t\ x_t < 0$, then $k = n$ satisfies the condition. If $\exists p\, \exists q\, [x_q < 0 \text{ and } x_p > 0]$, then define $q^* = \max\{q \mid x_q < 0\}$. If $\forall q < q^*\ x_q < 0$, then $k = q^*$ satisfies the condition. Otherwise, set $p^* = \max\{p \mid p < q^* \text{ and } x_p > 0\}$. By the definition of a walk, then, $\pi_2^2\delta(x_{p^*}) = \pi_2^2\delta(-x_{p^*+1})$. Since $\pi_2^2\gamma$ is strictly increasing, (lemma 5), $x_{p^*} = -x_{p^*+1}$. It follows that $n > 2$, since otherwise $i = \pi_1^2\delta(x_{p^*}) = \pi_1^2\delta(-x_{p^*+1}) = j$, contrary to hypothesis. Since $n > 2$, deleting x_{p^*} and x_{p^*+1} from the walk does not delete the entire walk, so $(x_1, \ldots, x_{p^*-1}, x_{p^*+2}, \ldots, x_n)$ is a walk from i to j in R_γ. This contradicts the minimality of the length of $(x_1, \ldots, x_n)$. Thus, since $\exists p\, [p < q^* \text{ and } x_p > 0]$ is impossible, $k = q^*$ satisfies the required condition. □

Computable functions can be defined that respectively associate numbers with integer sequences of arbitrary positive length and bound the smallest representing number of a sequence. Specifically,

Lemma 12. *There exist computable functions $\sigma\colon \mathbb{N}_+ \times \mathbb{N} \to \mathbb{Z}$ and $\beta\colon \mathbb{N}_+ \to \mathbb{N}_+$ such that, for every $n > 0$, $(x_1, \ldots, x_n) \in (\mathbb{Z} \setminus \{0\})^n$, $k \geq n$, and $k \geq \max\{|x_i| \mid 1 \leq i \leq n\}$,*

$$\exists z \leq \beta(k)\ \ [\sigma(z, 0) = n \text{ and } \forall i < n\ \sigma(z, i+1) = x_i] \tag{26}$$

Proof. Begin by defining an injection, $f\colon \bigcup_{n>0}(\mathbb{Z}\setminus\{0\})^n \to \mathbb{N}$, as follows. Represent x, a non-zero integer, by the string consisting of $|x|$ ocurrences of '0', preceded by '1' if $x<0$ and '11' if $x>0$. Represent a sequence of non-zero integers by the concatenation of their strings. If '$d^1\cdots d^n$' (a concatenation of n binary-digit strings, d^i, representing the respective x_i) is the binary-digit string representing $(x_1,\ldots,x_n)\in(\mathbb{Z}\setminus\{0\})^n$, and if '$d^1\cdots d^n$' is '$b_1$'$\cdots$'$b_t$' (where each '$b_k$' is '0' or '1') then define $f(x_1,\ldots,x_n)=\sum_{s<t} b_{n-s}\,2^s$.

Define $\sigma\colon \mathbb{N}_+\times\mathbb{N}\to\mathbb{Z}$ by

$$\sigma(z,i)=\begin{cases}0 & \text{if } z \text{ is not in the range of } f\text{, and}\\ & \quad\text{if } z=f(x_1,\ldots,x_n)\text{, then}\\ n & \text{if } i=0;\\ x_i & \text{if } 0<i\le n;\\ 0 & \text{if } i>n\end{cases} \tag{27}$$

This definition works because the set of numbers corresponding to nonzero-integer sequences, and also the relation, "x is the i^{th} element of the nonzero-integer sequence corresponding to z", are decidable. That is true, in turn, because occurrences of '10' and '11' occur exactly at the beginnings substrings of a concatenated string that represent the respective x_i and because the range of f is decidable.[12] The z satisfying (26) can be bounded because, if the binary-digit string, 'b_1'$\cdots$'b_t', encodes $(x_1,\ldots,x_n)$, then $t\le\sum_{i=1}^{n}(2+x_i)$, so $f(x_1,\ldots,x_n)\le\sum_{s=0}^{\sum_{i=1}^{n}(2+x_i)}2^s<2^{1+n(2+\max\{|x_i||1\le i\le n\})}$. That is, if $\beta(k)=2^{1+k(2+k)}$ and $n\le k$ and $\max\{|x_i|\mid 1\le i\le n\}\le k$, then $f(x_1,\ldots,x_n)<\beta(k)$ and $z=f(x_1,\ldots,x_n)$ witnesses (26). □

Lemma 13. $(R_\gamma\cup R_\gamma^{-1})^+$ *is a decidable relation*

Proof. R_γ is decidable by lemma 8, so $(R_\gamma\cup R_\gamma^{-1})$ is decidable. By lemma 3, $(i,j)\in(R_\gamma\cup R_\gamma^{-1})^+$ iff there is a walk from i to j in R_γ. Let $(x_1,\ldots,x_n)$ be such a walk of minimal length. By lemma 11, for some k, $0\le k<n$ and $\forall i\le k\ x_i<0$ and $\forall j>k\ x_j>0$. This means that $i=\pi_2^2\delta(-x_1)$; that, for $h\le k$, $\pi_1^2\delta(-x_h)=\pi_2^2\delta(-x_{h+1})$; that $\pi_1^2\delta(x_{k+1})=\pi_2^2\delta(-x_k)$; that, for $h\ge k$, $\pi_2^2\delta(x_h)=\pi_1^2\delta(x_{h+1})$; and that $\pi_2^2\delta(x_n)=j$. These facts, together with the fact that $\pi_1^2\delta(|x_h|)<\pi_2^2\delta(|x_h|)$ (definition (8)), have two implications. First, $\pi_2^2\delta(-x_h)\le i$ for $h\le k$ and $\pi_2^2\delta(-x_h)\le j$ for $h>k$. Thus, for all h, $\pi_2^2\delta(|x_h|)\le\max\{i,j\}\le i+j$. Second, $n\le i+j$. Thus, in view of lemma 12, it is clear that the set of (i,j) such that there is a walk from i to j in R_γ is defined from R_γ by the bounded-quantifier formula

$$\begin{gathered}\exists x\le\beta(i+j)\,[\tau(\sigma(x,1))=i \text{ and } \eta(\sigma(x,\sigma(x,0)))=j \text{ and}\\ \forall k<\sigma(x,0)-1\,[\eta(\sigma(x,k+1))=\tau(\sigma(x,k+2) \text{ and}\\ (\sigma(x,k+1),\sigma(x,k+2))\in R_\gamma\cup R_\gamma^{-1}]]\end{gathered} \tag{28}$$

Thus that set—which is $(R_\gamma\cup R_\gamma^{-1})^+$ by lemma 3—is decidable. □

[12] A string of digits represents a sequence of non-zero integers iff it satisfies three conditions: the first digit must be '1', the last digit must be '0', and there must be no substring of form '111'. These conditions can be expressed by bounded-quantifier formulae in base-2 arithmetic, so the range of f is decidable. A detailed proof of this decidability assertion would proceed along the general lines of Hájek and Pudlak [1998, definition V.3.5 and proposition V.3.30].

6. Decidable and semi-decidable IC relations

The main result of this section, proposition 3, will provide a partial answer to the question, under what conditions is a semi-decidable relation the lattice join of of decidable relations?

Lemma 14. $(R_\gamma \cup R_\gamma^{-1})^+$ *is a decidable, IC equivalence relation.*

Proof. Clearly $R_\gamma \cup R_\gamma^{-1}$ is symmetric, and the transitive closure of a symmetric relation is also symmetric as well as being transitive. For every i, there is some n such that $(i,i) = \varepsilon(n)$, so $\delta(n) \in R_\gamma$, $(\delta(n))^{-1} \in R_\gamma^{-1}$, $i = \pi_1^2\delta(n)$ and therefore $(i,i) \in R_\gamma R_\gamma^{-1} \subseteq (R_\gamma \cup R_\gamma^{-1})^+$. That is, $(R_\gamma \cup R_\gamma^{-1})^+$ is an equivalence relation. $(R_\gamma \cup R_\gamma^{-1})^+$ is IC by lemma 6, and is decidable by lemma 13. $\square$

Recall that, if F and G are equivalence relations on $\mathbb{N}$, then their join in the lattice of equivalence relations is defined by

$$F \vee G = (F \cup G)^+ \tag{29}$$

Proposition 3. *If E is a semi-decidable, IC equivalence relation, then there are decidable equivalence relations, F and G, such that F is IC and*

$$E = FGF = F \vee G \tag{30}$$

Specifically,

$$F = (R_\gamma \cup R_\gamma^{-1})^+ \qquad G = S_\gamma \cup T_\gamma \cup I \tag{31}$$

Proof. By definition (31) and lemma 14, F is a decidable, IC equivalence relation. By lemma 10, G is a decidable equivalence relation. They are both sub-relations of E so $F \vee G \subseteq E$. By proposition 2, $E = R_\gamma S_\gamma R_\gamma^{-1} \subseteq FGF \subseteq F \vee G \subseteq E$, which is equivalent to (30). $\square$

7. Decidable IC equivalence relations generate the semi-decidable IC equivalence relations

It is of interest to strengthen proposition 3 to assert that G, as well as F, is IC. That is,

Proposition 4. *If E is a semi-decidable, IC equivalence relation, then there are decidable, IC equivalence relations, F and G, such that condition (30) holds.*

Proposition 4 implies that the set of decidable, IC equivalence relations is large enough to generate the semi-decidable IC equivalence relations as a subset of the semigroup of binary relations, and as an upper semilattice.

If G is defined as in (31), then $[i]_G$ is not guaranteed to be infinite for $i \in \pi_2^2\delta(\mathbb{N}_+)$, and $[i]_G = \{i\}$ for $i \notin \pi_2^2\delta(\mathbb{N}_+)$.[13] There does not seem to be any ad hoc adjustment of the definition of G in (31) that will yield, for arbitrary γ, a decidable, IC equivalence relation that includes $S_\gamma \cup T_\gamma$, which will be required required in order to transform the proof of proposition 3 into a proof of proposition 4.

Proposition 3 and the results on which it depends have been proved by starting with an arbitrary coding, γ, of an arbitrary enumeration ε, of E. In order to obtain

[13] Since $\pi_2^2\delta$ is strictly increasing, and since, because E is IC, there are infinitely many n such that $\pi_2^2\varepsilon(n+1) = \pi_2^2\delta(1)$, $\mathbb{N}_+ \setminus \pi_2^2\delta(\mathbb{N}_+)$ is infinite.

the decidable, IC equivalence relation that is required to prove proposition 4, ε and γ will be constructed, each of which has a specific property. For every $e \in E$, $\varepsilon^{-1}(e)$ will be infinite. For each $i \notin \pi_2^2\delta(\mathbb{N}_+)$, $[i]_E \setminus \pi_2^2\gamma(\mathbb{N}_+)$ will be infinite. These two properties will ensure that F is IC and enable a decidable, IC, symmetric and transitive relation, H to be constructed that will play a cognate role to that of I in definition (31) of G.

Lemma 15. *There is an enumeration, ε, of E (a semi-decidable, coarse equivalence relation) such that*

$$\text{for all } (i,j) \in E,\ \varepsilon^{-1}(i,j) \text{ is infinite.} \tag{32}$$

Proof. The mapping $(m,n) \mapsto 2^m(2n+1)$ is a bijection of $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}_+$. Since E is semi-decidable, it has some enumeration, θ. An enumeration satisfying (32) is defined by $\varepsilon(2^m(2n+1)) = \theta(n+1)$. □

Lemma 16. *There are an enumeration, ε, of E (a semi-decidable, coarse equivalence relation), and codings, γ and ζ, of ε, such that ε satisfies (32) and*

$$\text{for all } n \in \mathbb{N}_+,\ \gamma(n) < \zeta(n) < \gamma(n+1) \tag{33}$$

Proof. The enumeration exists by lemma 15. Intuitively, the codings are constructed by interleaving two, concurrent recursions, each resembling the one used to prove lemma 7. At each stage, n, implicitly first $\gamma(n)$ is constructed and then $\zeta(n)$ is constructed and then the results are merged by use of a pairing function, $(i,j) \mapsto 2^i(2j+1)$. That is, define

$$\kappa(n) = 2^{\gamma(n)}(2\zeta(n)+1) \tag{34}$$

To describe this procedure within a single, recursive, definition of κ, define

$$\begin{gathered} \overline{\pi}_1(k) = \max\{m \mid\ 2^m|k\} \qquad \overline{\pi}_2(k) = [(k/\overline{\pi}_1(k)) - 1]/2 \\ \gamma = \overline{\pi}_1\kappa \qquad \zeta = \overline{\pi}_2\kappa \qquad \theta = \varepsilon\zeta \end{gathered} \tag{35}$$

Now, $\kappa\colon \mathbb{N}_+ \to \mathbb{N}_+$ will be defined so that γ and ζ, derived from κ via (35), will each satisfy the definition (10) of a coding, and so that they will be interleaved as specified in (33).

$$\begin{gathered} \kappa(1) = \min\{k \mid 1 < \min\{\overline{\pi}_1(k), \overline{\pi}_2(k)\} \text{ and} \\ \pi_1^2\varepsilon(1) = \pi_1^2\varepsilon(\overline{\pi}_1(k)) = \pi_1^2\varepsilon(\overline{\pi}_2(k)) \text{ and} \\ \max\{1, \pi_1^2\varepsilon(1)\} < \pi_2^2\varepsilon(\overline{\pi}_1(k)) < \pi_2^2\varepsilon(\overline{\pi}_1(k))\} \\ \kappa(n+1) = \min\{k \mid \zeta(n) < \overline{\pi}_1(k) < \overline{\pi}_2(k) \text{ and} \\ \pi_1^2\varepsilon(n+1) = \pi_1^2\varepsilon(\overline{\pi}_1(k)) = \pi_1^2\varepsilon(\overline{\pi}_2(k)) \text{ and} \\ \max\{\pi_1^2\varepsilon(\overline{\pi}_1(k)), \pi_1^2\varepsilon(\overline{\pi}_2(k)), \pi_2^2\theta(n)\} < \pi_2^2\varepsilon(\overline{\pi}_1(k)) < \pi_2^2\varepsilon(\overline{\pi}_2(k))\} \end{gathered} \tag{36}$$

By a parallel argument to the proof of lemma 7, the fact that E is IC implies that (36) defines a total function. Clearly γ and ζ, defined by (35) and (36), each satisfy definition (8) of a coding, and they are related according to (33). □

Lemma 17. *If γ codes ε, an enumeration of E that satisfies (32), then $S_\gamma \cup T_\gamma$ is an IC, symmetric and transitive relation.*

Proof. By lemma 10, $S_\gamma \cup T_\gamma$ is symmetric and transitive. Let $i \in \mathcal{F}(S_\gamma \cup T_\gamma)$. It must be shown that $[i]_{S_\gamma \cup T_\gamma}$ is infinite.

For some n, $i = \pi_2^2\delta(n)$. Let $\delta(n) = (j, i)$. Let $M = \varepsilon^{-1}(j, i)$. By (32), M is infinite. Since $\pi_2^2\delta$ is strictly increasing, $\pi_2^2\delta(M)$ is infinite. For each $m \in M$, $(\pi_2^2\delta(m), \pi_2^2\delta(n)) \in T_\gamma$, so $\pi_2^2\delta(M) \subseteq [i]_{T_\gamma} \subseteq [i]_{S_\gamma\cup T_\gamma}$, so $[i]_{S_\gamma \cup T_\gamma}$ is infinite. $\square$

The next step is to include $S_\gamma \cup T_\gamma$ in a decidable, IC, equivalence relation on $\mathbb{N}$. This will be done by appealing to lemma 1. Setting $H = S_\gamma \cup T_\gamma$, it is sufficient to find another relation that satisfies the conditions specified in the following lemma.

Lemma 18. *There is a sub-relation of E that is decidable, IC, symmetric and transitive, and has $\mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma)$ as its field.*

Proof. The following relation will be proved to satisfy the specified conditions.

$$(37) \qquad J = ((\mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma)) \times (\mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma)) \cap (R_\zeta \cup R_\zeta^{-1})^+$$

$\mathcal{F}(S_\gamma \cup T_\gamma)$ is Turing reducible to $S_\gamma \cup T_\gamma$ because $i \in \mathcal{F}(S_\gamma \cup T_\gamma) \iff (i, i) \in S_\gamma \cup T_\gamma$ (lemma 1). $R_\zeta \cup R_\zeta^{-1}$ is decidable (lemma 10), so $\mathcal{F}(S_\gamma \cup T_\gamma)$ is decidable. Therefore $(\mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma)) \times (\mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma))$ is decidable, also. $(R_\zeta \cup R_\zeta^{-1})^+$ is decidable (lemma 13), so J is decidable. $J \subseteq E$ because $(R_\zeta \cup R_\zeta^{-1})^+ \subseteq E$. Being an intersection of symmetric, transitive relations, J shares those properties.

It remains to be proved that J is IC and that J has $\mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma)$ as its field. A single argument establishes both of these facts. It follows directly from (37) that $\mathcal{F}(J) \cap \mathcal{F}(S_\gamma \cup T_\gamma) = \emptyset$, so, to prove that $\mathcal{F}(J) = \mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma)$, it is sufficient to prove that $\mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma) \subseteq \mathcal{F}(J)$. To that end, suppose that $i \notin \mathcal{F}(S_\gamma \cup T_\gamma)$, and suppose that $(i, i) = \varepsilon(k)$. By lemma 6 and (15), $\{\theta\zeta^n(k) \mid n \in \mathbb{N}\} = \{i\} \times \{\pi_2^2\theta\zeta^n(k) \mid n \in \mathbb{N}\}$ is an infinite subset of R_ζ, hence of $(R_\zeta \cup R_\zeta^{-1})^+$. Lemma 6 and condition (33) imply that $\{\pi_2^2\theta\zeta^n(k) \mid n \in \mathbb{N}\} \subseteq \mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma))$, so $\{i\} \times \{\pi_2^2\theta\zeta^n(k) \mid n \in \mathbb{N}\} \subseteq (\mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma)) \times (\mathbb{N} \setminus \mathcal{F}(S_\gamma \cup T_\gamma)$. Thus $\{i\} \times \{\pi_2^2\theta\zeta^n(k) \mid n \in \mathbb{N}\} \subseteq J$ and $[i]_J$ is infinite. Since $[i]_J \neq \emptyset$, $i \in \mathcal{F}(J)$. $\square$

The main result of this article, proposition 4, follows directly from preceding lemmas.

Proof of proposition 4. Let ε, γ, and ζ be functions such as are guaranteed by lemma 16 to exist. Define J according to (37), and also define

$$(38) \qquad F = (R_\gamma \cup R_\gamma^{-1})^+ \qquad H = S_\gamma \cup T_\gamma \qquad G = H \cup J$$

Lemmas 1, 13, 17, and 18 establish that $F \subseteq E$ and $G \subseteq E$ are decidable, IC, equivalence relations. That $E = FGF = F \vee G$ follows as in the proof of proposition 3. $\square$

8. semi-decidable FC relations

Two examples of FC relations that are semi-decidable but not decidable will now be constructed. One of the examples has a representation of form $E = F \vee G$, where F and G are decidable, while the other cannot be so represented.

Proposition 5. *There exist decidable equivalence relations, F and G, such that $F \vee G$ is a semi-decidable, FC equivalence relation that is not decidable, and $F \vee G = FGF$.*

Proof. Let $\theta\colon \mathbb{N}_+ \to A$ be an enumeration of A, a semi-decidable, but not decidable, subset of $\mathbb{N}$. Assume, in accordance with Rogers Jr. [1967, problem 5-2, p. 73], that θ enumerates A without repetitions. Define two equivalence relations, F and G, by

$$(i,j) \in F \iff \exists n \leq \max\{i,j\}\, [i = j \text{ or } \{i,j\} = \{2n, 3^{\theta(n)}\}] \tag{39}$$

and

$$(i,j) \in G \iff \exists n \leq \max\{i,j\}\, [i = j \text{ or } \{i,j\} = \{2n, 5^{\theta(n)}\}] \tag{40}$$

Since F and G are defined from equality and computable functions by bounded-quantifier sentences, they are decidable. They are symmetric and reflexive and, since θ is an enumeration without repetition, each equivalence class is either a singleton or a pair. Thus, by lemma 2, F and G are equivalence relations.

Since θ is bijective, the partition corresponding to F is $\{\{i,j\} \mid [i = j \text{ and } \neg\exists n\, [i = 2n \text{ or } i = 3^{\theta(n)}]] \text{ or } \exists n\, \{i,j\} = \{2n, 3^{\theta(n)}\}\}$ and the partition corresponding to G is $\{\{i,j\} \mid [i = j \text{ and } \neg\exists n [i = 2n \text{ or } i = 5^{\theta(n)}]] \text{ or } \exists n\, \{i,j\} = \{2n, 5^{\theta(n)}\}\}$

Suppose that $(i,j) \in FGF$. Then there are h and k such that $(i,h) \in F$, $(h,k) \in G$, and $(k,j) \in F$. If $i = h$, then $(i,j) \in GF$. If $i \neq j$, then either $h = k$ or $h \neq k$. If $h = k$, then $(i,j) \in FIF = F \subseteq FG$. If $h \neq k$, then, for some n, $i = 3^{\theta(n)}$ and $h = 2n$ and $k = 5^{\theta(n)}$. In that case, since $(k,j) \in F$, $j = k$, so $(i,j) \in FG$. Thus $FGF \subseteq FG \cup GF = FGI \cup IGF \subseteq FGF \cup FGF = FGF$. A parallel argument shows that $GFG \subseteq FG \cup GG \subseteq FGF$. Therefore $FGF \cup GFG \subseteq FG \cup GF \subseteq FGF$. That is,

$$FGF \cup GFG = FG \cup GF = FGF \tag{41}$$

Using the first identity in (41) to carry out the induction step, it is seen by induction that $\forall n\, [n \geq 2 \implies (F \cup G)^{(n)} = FG \cup GF]$. Then, by equations (4) and (29) and the second identity in (41), $F \vee G = FGF$. Clearly the partition corresponding to $F \vee G$ is $\{\{i,j,k\} \mid [i = j = k \text{ and } \neg\exists n\, [i = 2n \text{ or } i = 3^{\theta(n)} \text{ or } i = 5^{\theta(n)}]] \text{ or } \exists n\, \{i,j,k\} = \{2n, 3^{\theta(n)}, 5^{\theta(n)}\}\}$, so $F \vee G$ is FC.

Clearly $F \vee G$ is semi-decidable. But A is Turing reducible to $F \vee G$ by the equivalence $n \in A \iff (3^n, 5^n) \in F \vee G$. Therefore $F \vee G$ not decidable. □

Proposition 6. *There is a semi-decidable, FC equivalence relation, E, that is not the transitive closure of a decidable relation.*

Proof. Let A be a semi-decidable subset of $\mathbb{N}$ that is not decidable. Consider the semi-decidable equivalence relation, E, defined by

$$\begin{gathered}(x,y) \in E \iff [x = y \text{ or } [\min\{x,y\} \in 2\mathbb{N} \\ \text{and } \min\{x,y\}/2 \in A \text{ and } (y-x)^2 = 1]]\end{gathered} \tag{42}$$

E corresponds to the partition, the equivalence classes of which are defined by

$$[i] = \begin{cases} \{i\} & \text{if } 2n \leq i \leq 2n+1 \text{ and } n \notin A; \\ \{2n, 2n+1\} & \text{if } 2n \leq i \leq 2n+1 \text{ and } n \in A. \end{cases} \tag{43}$$

A is Turing reducible to E because $n \in A \iff \{2n, 2n+1\} \in E$, so E is not decidable.

A contradiction will be derived from the assumption that, for some decidable R, $E = R^+$. If so, then, since $(R \cup I)^+ = R \cup I$ by lemma 2, $E = R^+ \subseteq (R \cup I)^+ = R \cup I \subseteq E$. That is, $E = R \cup I$, which is impossible since $R \cup I$ is decidable but E is not decidable. □

Corollary 1. *There is a semi-decidable, FC equivalence relation, E, such that (a) there do not exist $n > 1$ and decidable equivalence relations, $H_1, \ldots, H_n$, that satisfy $E = \bigvee_{i \leq n} H_i$, and (b) there do not exist $n > 1$ and decidable equivalence relations, $H_1, \ldots, H_n$ (not necessarily distinct), such that $E = H_1 \cdots H_n$.*

Proof. Let E be defined by (42), with A being a non decidable, semi-decidable subset of $\mathbb{N}$. To prove (a), suppose that each H_i, for $1 \leq i \leq n$¡ is a decidable equivalence relation. Then $\bigcup_{i \leq n} H_i$ is decidable and $\bigvee_{i \leq n} H_i = (\bigcup_{i \leq n} H_i)^+$. By proposition 6, then, $E \neq \bigvee_{i \leq n} H_i$. To prove (b) by contradiction, suppose that $H_1, \ldots, H_n$ are decidable equivalence relations such that $E = H_1 \cdots H_n$. Because $I \subseteq H_j$ for every j, $H_i \subseteq H_1 \cdots H_n = E$ for every i. Thus $\bigcup_{i<n} H_i \subseteq E$. Also $\bigcup_{i<n} H_i$ is reflexive and symmetric, so, by lemma 2, $(\bigcup_{i<n} H_i)^+ = \bigcup_{i<n} H_i$. $H_1 \cdots H_n \subseteq (\bigcup_{i<n} H_i)^{(n)} \subseteq (\bigcup_{i<n} H_i)^+$, so $E \subseteq \bigcup_{i<n} H_i \subseteq E$. That is, $E = \bigcup_{i<n} H_i$, contradicting $\bigcup_{i<n} H_i$ being decidable but E not being so. □

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