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¹Each lesson is ONE day, and ONE day is considered a 45-minute period.

Precalculus and Advanced Topics • Module 4

Trigonometry

OVERVIEW

Trigonometry was introduced in Geometry through a study of right triangles. In Algebra II, work was conducted on extending basic trigonometry to the domain of all real numbers via the unit circle. This module revisits, unites, and further expands those ideas and introduces new tools for solving geometric and modeling problems through the power of trigonometry.

Topic A helps students recall how to use special triangles positioned within the unit circle to determine geometrically the values of sine, cosine, and tangent at special angles. The unit circle is then used to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x , where x is any real number (**F-TF.A.3**) and to explain the periodicity of the trigonometric functions and their symmetries (**F-TF.A.4**). Students develop the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems (**F-TF.C.9**) and to model geometric phenomena. Students also discuss the construction of tangent lines to circles (**G-C.A.4**) and revisit the geometric origins of the tangent function. Student exploration of tangents through a paper folding and compass activity is then used in modeling waves, waves traveling together, and wave patterns generated by musical instruments.

Students derive sophisticated applications of the trigonometric functions in Topic B including: the area formula for a general triangle, $A = \frac{1}{2}ab \sin(\theta)$ (**G-SRT.D.9**), the law of sines, the law of cosines, and Heron's formula. They use previous knowledge and apply their understanding of the Pythagorean theorem and oblique triangles to discover these formulas while analyzing patterns. Finally, as students investigate force diagrams and paths across rivers, they solve survey and elevation problems and revisit vectors (**G-SRT.D.10**, **G-SRT.D.11**).

The graphs of the trigonometric functions are revisited in Topic C. Students visualize these graphs with the aid of the appropriate software and briefly recall how changing various parameters of a trigonometric function affects its graph. Students extend their knowledge of inverse functions to trigonometric functions (**F-TF.B.6**) as they restrict domains to create inverse trigonometric functions. These inverse functions are then used to solve trigonometric equations, evaluate their solutions using technology, and interpret these solutions in the appropriate contexts (**F-TF.B.7**). Students determine viewing angle, line of sight, height of objects, and angle of elevation for inclined surfaces using inverse trigonometric functions and their periodic phenomena.

Focus Standards

Extend the domain of trigonometric functions using the unit circle.

- F-TF.A.3** (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$, and $\pi/6$, and use the unit circle to express the values of sine, cosines, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x , where x is any real number.
- F-TF.A.4** (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions.

- F-TF.B.6** (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
- F-TF.B.7** (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.*

Prove and apply trigonometric identities.

- F-TF.C.9²** (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Understand and apply theorems about circles.

- G-C.A.4** (+) Construct a tangent line from a point outside a given circle to the circle.

Apply trigonometry to general triangles.

- G-SRT.D.9** (+) Derive the formula $A = 1/2ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
- G-SRT.D.10** (+) Prove the Laws of Sines and Cosines and use them to solve problems.
- G-SRT.D.11** (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Foundational Standards

Extend the domain of trigonometric functions using the unit circle.

- F-TF.A.1** Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

²Students are now responsible for proofs of angle addition and subtraction formulas.

- F-TF.A.2** Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Model periodic phenomena with trigonometric functions.

- F-TF.B.5** Choose trigonometry functions to model periodic phenomena with specified amplitude, frequency, and midline.*

Prove and apply trigonometric identities.

- F-TF.C.8** Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Understand the concept of a function and use function notation.

- F-IF.A.1** Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x . The graph of f is the graph of the equation $y = f(x)$.
- F-IF.A.2** Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Analyze functions using different representations.

- F-IF.C.7** Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*
- e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Build new functions from existing functions.

- F-BF.B.4** Find inverse functions.
- a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example, $f(x) = 2x^3$ or $f(x) = x + 1/x - 1$ for $x \neq 1$.*
- b. (+) Verify by composition that one function is the inverse of another.
- c. (+) Read values of an inverse function from a graph or table, given that the function has an inverse.
- d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Define trigonometric ratios and solve problems involving right triangles.

- G-SRT.C.6** Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
- G-SRT.C.7** Explain and use the relationship between the sine and cosine of complementary angles.
- G-SRT.C.8** Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.*

Understand and apply theorems about circles.

- G-C.A.2** Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*

Prove geometric theorems.

- G-CO.C.10** Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180° ; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

Focus Standards for Mathematical Practice

- MP.3** **Construct viable arguments and critique the reasoning of others.** Students construct mathematical arguments as they explain their calculations of the area of triangles leading to a new formula for area, $\text{Area} = A = \frac{1}{2}ab \sin(\theta)$. Students explain the properties of trigonometric functions and explain construction of tangent lines. Students use the periodic nature of trigonometric functions to reason about their graphs and problem solve using inverse functions.
- MP.4** **Model with mathematics.** Students apply sum and difference formulas in the context of modeling sound waves using trigonometric functions. Students model using trigonometric functions applying tangent lines and laws of sines and cosines to solve surveying problems and revisit vectors. Students investigate viewing distance, line of sight, and viewing angle using inverse trigonometric functions as well as the angle of elevation for inclined surfaces.
- MP.5** **Use appropriate tools strategically.** Students see the unit circle as a tool to determine the values of trigonometric functions in terms of x and explain their periodicity and symmetry. Students use computer software and graphing calculators to graph trigonometric functions and their inverses. Students see trigonometric inverses, law of sines, law of cosines, and area formulas as tools in problem solving.

Terminology

Familiar Terms and Symbols³

- Cosine
- Function
- Period
- Radian Measure
- Sine
- Tangent
- Tangent Line
- Unit Circle
- Vector

Suggested Tools and Representations

- Compass
- Straightedge
- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software
- GeoGebra software

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic A	Constructed response with rubric	F-TF.A.3, F-TF.A.4, F-TF.C.9, G-C.A.4
End-of-Module Assessment Task	After Topic C	Constructed response with rubric	F-TF.B.6, F-TF.B.7, G-SRT.D.9, G-SRT.D.10, G-SRT.D.11

³These are terms and symbols students have seen previously.



Topic A

Trigonometric Functions

F-TF.A.3, F-TF.A.4, F-TF.C.9, G-C.A.4

Focus Standards:	F-TF.A.3	(+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$, and $\pi/6$, and use the unit circle to express the values of sine, cosines, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x , where x is any real number.
	F-TF.A.4	(+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.
	F-TF.C.9	(+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.
	G-C.A.4	(+) Construct a tangent line from a point outside a given circle to the circle.
Instructional Days: 6		
Lesson 1: Special Triangles and the Unit Circle (P) ¹		
Lesson 2: Properties of Trigonometric Functions (P)		
Lessons 3–4: Addition and Subtraction Formulas (P, P)		
Lesson 5: Tangent Lines and the Tangent Function (E)		
Lesson 6: Waves, Sinusoids, and Identities (P)		

Trigonometry was introduced in Geometry through a study of right triangles, and work on extending basic trigonometry to the domain of all real numbers via the unit circle was conducted in Algebra II. This module revisits, unites, and further expands those ideas and introduces new tools for solving geometric and modeling problems through the power of trigonometry. In Algebra II Module 2, the students were introduced to the unit circle and the trigonometric functions associated with it. Lesson 1 reviews these concepts, including the history behind the development of trigonometry. Students apply the unit circle and their knowledge of right triangles to find the values of sine, cosine, and tangent for rotations of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$ radians. They also examine the relationship between the sine, cosine, and tangent of θ and its relationship to sine, cosine, and tangent for $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$, allowing them to evaluate the trigonometric functions for values of θ in all four quadrants (F-TF.A.3). In Lesson 2, students continue to explore the relationship between trigonometric

¹Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

functions for rotations θ , examining the periodicity and symmetry of the sine, cosine, and tangent functions (**F-TF.A.4**). They also use the unit circle to define relationships between the sine and cosine functions. Lessons 3 and 4 extend students' knowledge of trigonometric functions and their properties as they use those properties to discover addition and subtraction formulas of trigonometric functions (**F-TF.C.9**). These formulas, $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$ and $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, are used to derive related formulas such as $\tan(\alpha + \beta)$, $\sin(2\alpha)$, and $\cos\left(\frac{\alpha}{2}\right)$. Lesson 5 involves constructing tangent lines to a given circle from a point outside the circle (**G-C.A.4**). In this lesson, students revisit the geometric origins of the tangent function and explore the standard by means of paper folding and compass constructions. Topic A concludes with Lesson 6, a modeling lesson. Students learn how sinusoids and trigonometric identities apply to waves and discover what happens when two traveling waves are combined to produce a standing wave pattern and how that model applies to sounds generated by a musical instrument. Students see the trigonometric addition and subtraction (**F-TF.C.9**) formulas as modeling tools in this lesson.

Topic A highlights MP.3 as students use the structure of the unit circle to develop and justify conjectures about the behavior of trigonometric functions. Similarly, students also reason to justify the steps for constructing tangent lines. MP.4 is also highlighted as students use the sum and difference formulas as tools for modeling sound waves. Students model standing waves produced by a guitar string and wave interference produced as two waves move together.



Lesson 1: Special Triangles and the Unit Circle

Student Outcomes

- Students determine the values of the sine, cosine, and tangent functions for rotations of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$ radians.
- Students use the unit circle to express the values of the sine, cosine, and tangent functions for $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$ for real-numbered values of θ .

Lesson Notes

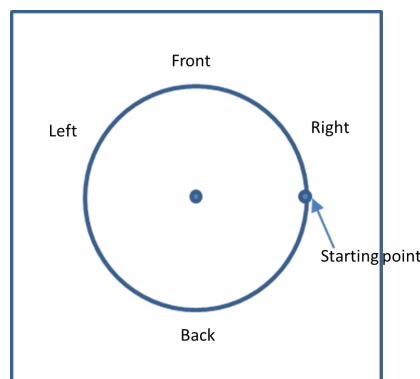
In Algebra II Module 2, students were introduced to the unit circle and the trigonometric functions associated with it. This lesson reviews these concepts, including the history behind the development of trigonometry. Students apply their knowledge of the unit circle and right triangles to find the values of sine, cosine, and tangent for rotations of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$ radians. They also examine the relationship between the sine, cosine, and tangent of θ and its relationship to sine, cosine, and tangent for $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$, allowing them to evaluate the trigonometric functions for values of θ in all four quadrants of the coordinate plane.

Classwork

Opening (7 minutes)

MP.4

In Algebra II Module 2, students modeled a Ferris wheel using a paper plate. In this lesson, they model a carousel, which they can use to help them recall their previous knowledge about the unit circle. Students should complete the task in pairs, and each pair should be given a paper plate, a brass fastener, and a sheet of cardstock that is large enough to be visible once the paper plate has been affixed to it. Students fasten the center of the paper plate to the cardstock using the brass fastener, which serves as the center of the carousel. Students then label the cardstock to indicate directionality (front, back, left, and right of the center). On their paper plates, students should also indicate the starting point for the ride, which should be a point on the plate directly to the right of the center of the carousel. This point represents the rider.



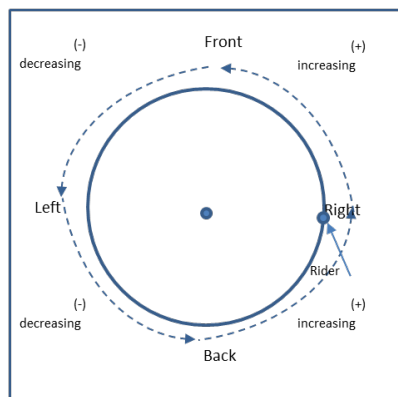
Each pair of students should be assigned to either group 1 or group 2. Students in group 1 should answer prompt 1, and students in group 2 should answer prompt 2.

1. Over the course of one complete turn, describe the position of the rider with respect to whether he is positioned to the left or to the right of the center of the carousel. Use as much detail as you can.
2. Over the course of one complete turn, describe the position of the rider with respect to whether she is positioned to the front or to the back of the center of the carousel. Use as much detail as you can.

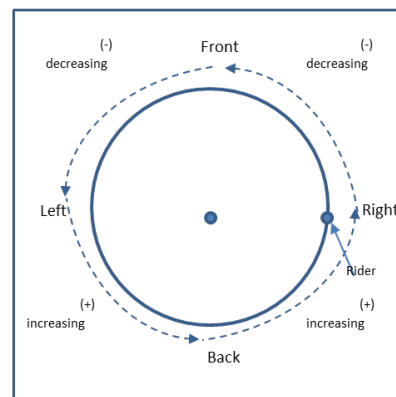
After a few minutes, several pairs should share their findings, which could be displayed on the board. Alternatively, a volunteer could record students' findings regarding the position of the rider on two charts (one for front/back and one for right/left). Students may or may not reference trigonometric functions, which is addressed explicitly later in the lesson. Likely student responses are shown:

- *The rider begins with a front/back position that is the same as the center of the carousel.*
- *As the carousel rotates counterclockwise, the rider moves in front of the center to a maximum value at a one-quarter turn. The rider then remains in front of the center point but decreases until she is again level with the center at one-half turn. As the carousel continues to rotate, the rider's position is behind the center, and the front/back value reaches its minimum at three-quarters of a turn. The rider continues to be behind the center, but the front/back position increases until the rider is again level at a full rotation.*
- *The maximum front/back distance from the center is the same and is equal to the radius of the carousel.*
- *The pattern of the front/back position of the rider repeats with every full turn. In other words, at one and one-quarter turns, the position of the rider is the same as it is for a one-quarter turn.*
- *The starting position of the rider is a maximum distance to the right of the center of the carousel. This distance is equal to the radius of the carousel.*
- *As the carousel rotates counterclockwise, the rider's position remains to the right of the center until, at a one-quarter turn, the rider is equidistant from the left and the right of the center. As the carousel continues to rotate, the rider's position is to the left of the center until he is again level with the center at a three-quarters turn. As the carousel continues to rotate, the rider's position is to the right of the center when it reaches a full turn.*
- *If "to the right" and "in front" are defined as positive directions (+) and "to the left" and "back" are defined as negative directions (-), the motions can be summarized in the diagrams:*

Summary of Front/Back Position of Rider



Summary of Left/Right Position of Rider

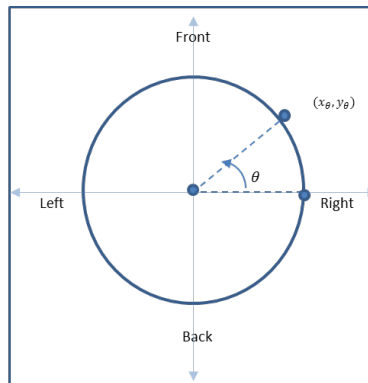


Discussion (8 minutes): Review of the Unit Circle and History of Trigonometry

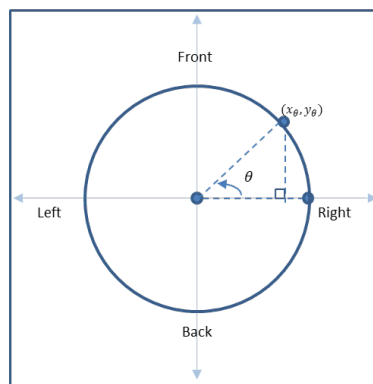
This brief discussion recounts what students learned in Algebra II about the unit circle and origins of trigonometric functions. This review helps students recall the properties of the unit circle, which they need to apply to find the value of trigonometric functions applied to specific rotation values.

- When you created the carousel models, why do you think we defined the rider's starting position as immediately to the right of the center and the direction of rotation to be counterclockwise?
 - *Answers will vary but may address that when they learned about the unit circle, the starting point was the point $(1,0)$ on the positive x -axis and a positive rotation was defined to be counterclockwise.*
- You were introduced to the origins of trigonometry in Algebra II. Hundreds of years ago, scientists were interested in the heights of the sun and other stars. Before we knew about the Earth's rotation about its axis and its orbit around the sun, scientists assumed that the sun rotated around the Earth in a motion that was somewhat circular. Why would they have defined the sun's motion as counterclockwise?
 - *Answers will vary, but some students might recall that the sun rises in the east and sets in the west, appearing to trace a counterclockwise path in the sky if the observer is facing north.*
- So our unit circle models the apparent counterclockwise rotation of the sun about the Earth, with the observer on Earth representing the center of the circle, the sun's position as it rises at the horizon as the initial position, and the radius of the circle defined as 1. We define θ as the rotation of the initial ray, which passes through the origin and the point $(1,0)$, to end up at the terminal ray. What units do we use to measure the rotation θ ?
 - *It is measured using degrees or radians.*
- And how do we define degrees and radians?
 - *A degree is one three hundred sixtieth of a full rotation; a radian represents the amount of rotation that θ undergoes so the length of the path traced by the initial ray from the positive x -axis to its terminal location is equal to the radius of the circle.*
- What is the radius of the unit circle?
 - 1
- How many radians are contained in a full rotation?
 - 2π
- Because of the direct relationship between the radius of the unit circle and radians, we use radians as our primary means of measuring rotations θ .

- Now that we have discussed θ as the amount of rotation that the initial ray undergoes in a counterclockwise direction, let's assign the radius of the carousel as 1 unit. Simulate a rotation of θ by marking a point on your plate that represents the position of the rider after a rotation of θ . Create a sketch on your carousel model to represent the rotation. For ease of notation, let's imagine that our carousel is superimposed on a coordinate plane, where the center is the origin and for now, rotation by θ produces an image point in the first quadrant. We can label the position of our rider given a rotation of θ as (x_θ, y_θ) :

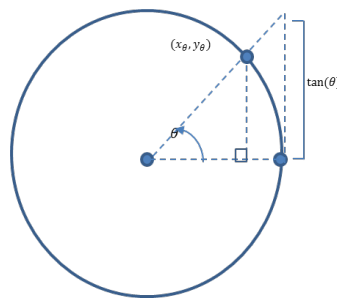


- How could we represent the front/back distance between the center of the carousel and our rider given the amount of rotation, θ ? Sketch this on your carousel model.
 - Answers may vary but should indicate that a vertical line segment could be drawn from the rider's position to the segment passing through the center and the starting position, and the distance represents the rider's distance in front of the carousel's center.
- And what is this distance?
 - y_θ



- Ancient scientists used the abbreviation "jyah" to refer to this distance. This abbreviation was converted from Sanskrit into the Arabic term "jiab" and then rewritten as the term "jaib," which was translated as the English term for cove, or sinus. This term was abbreviated into the term sine, which we are familiar with as a trigonometric function. Explain how our understanding of right triangle trigonometry demonstrates that $\sin(\theta) = y_\theta$.
 - Using right triangle trigonometry, $\sin(\theta) = \frac{y_\theta}{r}$, and since $r = 1$, $\sin(\theta) = y_\theta$.

- Now, Western scholars studying the height of the sun defined the segment representing the horizontal displacement of the sun as the “companion side” of the sine, which was shortened to the term cosine. How can we represent this distance on our model?
 - It is x_θ .
- And how does our understanding of right triangle trigonometry confirm that $\cos(\theta) = x_\theta$?
 - Using right triangle trigonometry, $\cos(\theta) = \frac{x_\theta}{r}$, and since $r = 1$, $\cos(\theta) = x_\theta$.
- Now you probably recall one additional core trigonometric function, the tangent function. The name is derived from the length of the line segment that has a point of tangency with the unit circle at the point $(1,0)$ and has as its end points the point $(1,0)$ and its point of intersection with the secant that passes from the origin through the terminal location of the ray with rotation, θ , as shown in the diagram.



- Use this diagram to determine the length $\tan(\theta)$. Share your solution with a partner.
 - Answers may vary but should address that the smaller and larger right triangles in the diagram are similar, and as such, the ratios of their corresponding leg lengths are equal: $\frac{\tan(\theta)}{1} = \frac{y_\theta}{x_\theta}$.
- How does our understanding of right triangle trigonometry flow from this definition of $\tan(\theta)$?
- In right triangle trigonometry, we defined

$$\tan(\theta) = \frac{\text{length of opposite side}}{\text{length of adjacent side}} = \frac{y_\theta}{x_\theta}.$$
- Now we have defined the unit circle, measurements of rotation about the circle, and the trigonometric functions associated with it. Let's use the unit circle to find the position of points on the unit circle for specific rotation values of θ .

Example 1 (6 minutes)

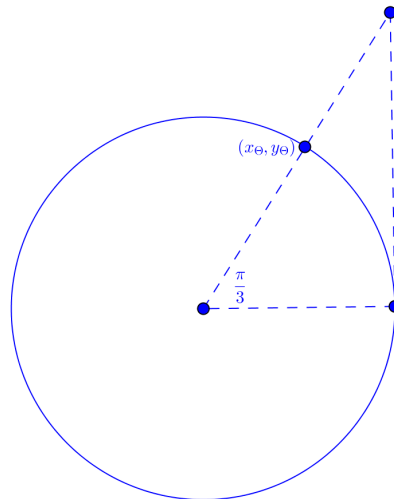
This example applies what students have learned about the unit circle to determine the values of the primary trigonometric functions for $\theta = \frac{\pi}{3}$. Students use a similar procedure to determine the trigonometric values for $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{4}$. They apply these results to determine the value of trigonometric functions for additional values of θ , including those outside Quadrant I.

Scaffolding:

- Encourage students to sketch the situation. Prompt them to draw in the altitude to the equilateral triangle in Example 1.
- Post an anchor chart with a right triangle and the definitions of $\sin(\theta)$ as the quotient of the lengths of the opposite side and the hypotenuse, $\cos(\theta)$ as the quotient of the lengths of the adjacent side and the hypotenuse, and $\tan(\theta)$ as the quotient of the lengths of the opposite and adjacent sides.

MP.7

- The diagram depicts the center of the carousel, the starting point of the rider, and the final position of the rider after rotating by $\frac{\pi}{3}$ radians, which is 60° . How can we find the values of the x - and y -coordinates of the rider's final position?



- We can use what we know about triangles and trigonometry to find the coordinates of the rider's final position.*
- What type of triangle is formed by the origin, the starting point, and the point representing the final position of the rider? How can you tell?
 - These three points form an equilateral triangle. Two of the sides represent radii of the unit circle, so their lengths are 1. The base angles theorem can be applied to determine that each angle of the triangle has a measure of 60° .*
- How can knowing the triangle formed by our three points is equilateral help us determine the values of the trigonometric functions?
 - If we draw the altitude of the triangle, we create two 30° – 60° – 90° triangles. We know that the horizontal line segment is bisected, so the value of $x_\theta = \frac{1}{2}$.*
- And how do we find the value of y_θ ?
 - We use the Pythagorean theorem, where we have a known leg length of $\frac{1}{2}$ and hypotenuse length of 1.*

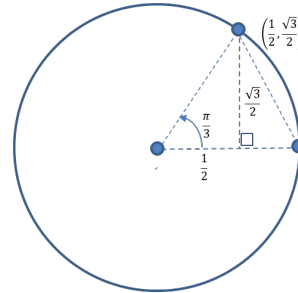
Example 1

Find the following values for the rotation $\theta = \frac{\pi}{3}$ around the carousel. Create a sketch of the situation to help you. Interpret what each value means in terms of the position of the rider.

a. $\sin(\theta)$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

For the rotation $\theta = \frac{\pi}{3}$, the rider is located $\frac{\sqrt{3}}{2}$ units in front of the center of the carousel.



b. $\cos(\theta)$

Based on the diagram shown in part (a), $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

For the rotation $\theta = \frac{\pi}{3}$, the rider is located $\frac{1}{2}$ unit to the right of the center of the carousel.

c. $\tan(\theta)$

Based on the diagram shown in part (a), $\tan\left(\frac{\pi}{3}\right) = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$.

For the rotation $\theta = \frac{\pi}{3}$, the ratio of the front/back position to the right/left position relative to the center of the carousel is $\sqrt{3}$.

MP.2

Exercise 1 (6 minutes)

Students should continue to use their paper plates to model the situations described in the exercise. They should be assigned to complete either part (a) or part (b) in their pairs from the opening activity. After a few minutes, each pair should explain their response to a pair assigned to a different part from them. Then, a few selected groups could share their results in a whole-class setting.

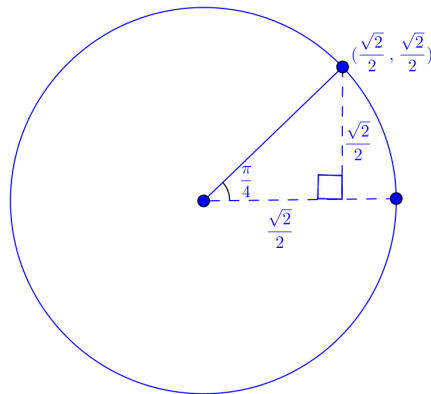
Exercise 1

Assume that the carousel is being safety tested, and a safety mannequin is the rider. The ride is being stopped at different rotation values so technicians can check the carousel's parts. Find the sine, cosine, and tangent for each rotation indicated, and explain how these values relate to the position of the mannequin when the carousel stops at these rotation values. Use your carousel models to help you determine the values, and sketch your model in the space provided.

Scaffolding:

Prompt students to draw a vertical line segment from the stopping point to the line segment representing the initial ray, and ask, "What are the measures of the angles of the resulting triangle?"

a. $\theta = \frac{\pi}{4}$

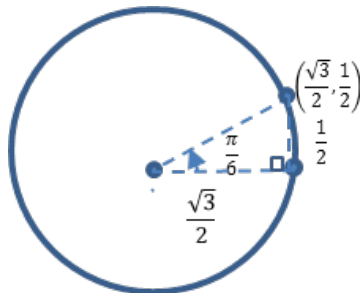


Since $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, the rider is approximately $\frac{\sqrt{2}}{2}$ units in front of the carousel's center when it stops.

Since $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, the rider is approximately $\frac{\sqrt{2}}{2}$ units to the right of the carousel's center when it stops.

Since $\tan\left(\frac{\pi}{4}\right) = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$, the front/back distance of the rider is equal to its right/left distance when it stops.

b. $\theta = \frac{\pi}{6}$



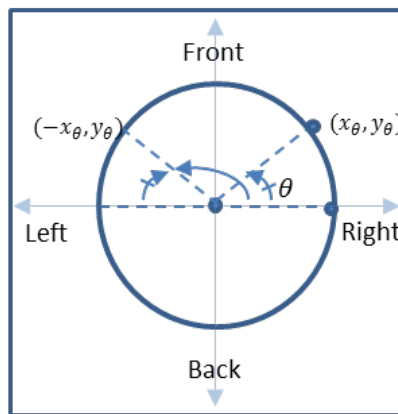
Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, the rider is approximately $\frac{1}{2}$ unit in front of the carousel's center when it stops.

Since $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, the rider is approximately $\frac{\sqrt{3}}{2}$ units to the right of the carousel's center when it stops.

Since $\tan\left(\frac{\pi}{6}\right) = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, the front/back to right/left ratio of the rider is $\frac{\sqrt{3}}{3}$ when it stops.

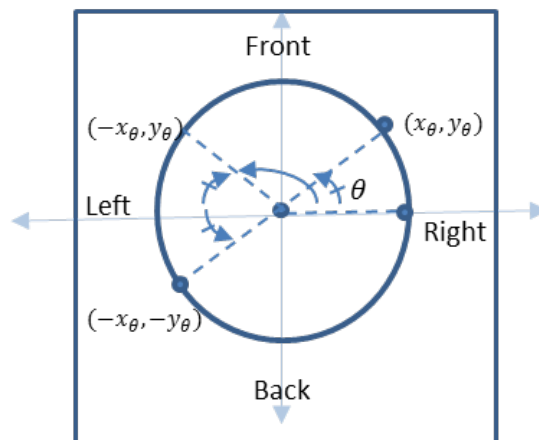
Discussion (5 minutes): Trigonometric Functions in All Four Quadrants

- We have just determined the results of applying the sine, cosine, and tangent functions to specific values of θ . All of these values, though, were restricted to the first quadrant. Let's see if our observations from the beginning of the lesson can help us expand our understanding of the effects of applying the trigonometric functions in the other quadrants. What do you recall about the front/back position of the rider as the carousel rotates counterclockwise?
 - *Answers will vary but might address that the rider's front/back position is 0 initially, increases to a maximum value (that we have defined as 1 unit forward) at one-quarter turn, and then decreases until it returns to 0 at one-half turn. It then becomes increasingly negative until it reaches a minimum position 1 unit behind the center of the carousel, and then it increases until it reaches a position of 0 after one full turn.*
- Refer to the sketch of θ in Quadrant I on your paper plate model. How do we represent the front/back position of a ray with rotation θ ?
 - *y_θ represents the front/back position.*
- Approximate another location on the model where the front/back position of a ray is equal to y_θ . Sketch a ray from the origin to this location. Describe the rotation of our initial ray that lands us at this location.
 - *Answers may vary but should indicate that the new terminal ray is located somewhere in Quadrant II, and the clockwise rotation between the negative x-axis and our new terminal ray is the same as the counterclockwise rotation between the positive x-axis and our original terminal ray in Quadrant I.*



- Let's try to confirm this using transformations. Describe the image that results if we reflect the original terminal ray in Quadrant I over the positive y-axis.
 - *The reflection creates a new terminal ray located in Quadrant II where the angle made with the negative x-axis and the ray is θ , and the ray intersects the unit circle at $(-x_\theta, y_\theta)$.*
- Now how can we determine the amount of rotation of this image? Explain.
 - *The rotation of this image is $(\pi - \theta)$ because the rotation is θ less than a one-half turn, and a one-half turn is π radians.*

- Based on our definitions from earlier in this lesson, what conclusions can we draw about $\sin(\pi - \theta)$, $\cos(\pi - \theta)$, and $\tan(\pi - \theta)$? Explain how you know.
 - Because the corresponding y -values are the same for θ and for $\pi - \theta$, $\sin(\pi - \theta) = y_\theta = \sin(\theta)$.
Because the corresponding x -values are opposites for θ and for $\pi - \theta$, $\cos(\pi - \theta) = -x_\theta = -\cos(\theta)$.
Because the corresponding x -values are opposites but y -values are the same for θ and for $-\theta$, $\tan(\pi - \theta) = \frac{y_\theta}{-x_\theta} = -\tan(\theta)$.
- Reflect the image of the ray in Quadrant II over the negative x -axis. Describe the new image.
 - The new image is a ray in Quadrant III where the measure of the angle between the image ray and the negative x -axis is θ . This new ray intersects the unit circle at $(-x_\theta, -y_\theta)$.



- How can we designate the amount of rotation of this image? Explain.
 - The rotation of this image is $(\pi + \theta)$ because the rotation is θ more than a one-half turn, which is π radians.
- And what are the values of $\sin(\pi + \theta)$, $\cos(\pi + \theta)$, and $\tan(\pi + \theta)$?
 - $\sin(\pi + \theta) = -y_\theta = -\sin(\theta)$
 $\cos(\pi + \theta) = -x_\theta = -\cos(\theta)$
 $\tan(\pi + \theta) = \frac{-y_\theta}{-x_\theta} = \tan(\theta)$
- And what conjectures can we make about the values of $\sin(2\pi - \theta)$, $\cos(2\pi - \theta)$, and $\tan(2\pi - \theta)$? Explain.
 - Rotation by $(2\pi - \theta)$ produces a reflection of the ray containing (x_θ, y_θ) over the positive x -axis, resulting in an image that intersects the unit circle at $(x_\theta, -y_\theta)$. This means that:
 - $\sin(2\pi - \theta) = -y_\theta = -\sin(\theta)$
 $\cos(2\pi - \theta) = x_\theta = \cos(\theta)$
 $\tan(2\pi - \theta) = \frac{-y_\theta}{x_\theta} = -\tan(\theta)$

Example 2 (2 minutes)

This example demonstrates how students can apply their discoveries relating rotations in the four quadrants to find trigonometric function values for specific θ in all four quadrants. The example should be completed in a whole-class setting, with students writing their responses on paper or on individual white boards.

- In part (a), what rotation is represented by $-\frac{\pi}{3}$?
 - *It is a clockwise rotation by $\frac{\pi}{3}$ radians.*
- What positive rotation produces the same terminal ray as rotation by $-\frac{\pi}{3}$?
 - $2\pi - \frac{\pi}{3}$
- How can we verify the sign of $\sin\left(-\frac{\pi}{3}\right)$?
 - *The rotation results in a point on the unit circle in Quadrant IV, and the y-coordinates of points in Quadrant IV are negative.*
- How can we verify the sign of $\tan\left(\frac{5\pi}{4}\right)$?
 - *The rotation results in a point on the unit circle in Quadrant III, and the x- and y-coordinates in Quadrant III are negative, which means that ratio of the coordinates is positive.*

Scaffolding:

- Advanced students could compute the values without further prompting.
- Advanced students could be challenged to evaluate trigonometric functions for θ exceeding 2π . For example, they could evaluate $\tan\left(\frac{25\pi}{4}\right)$.

Example 2

Use your understanding of the unit circle and trigonometric functions to find the values requested.

a. $\sin\left(-\frac{\pi}{3}\right)$

$$\sin\left(-\frac{\pi}{3}\right) = \sin\left(2\pi - \frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

b. $\tan\left(\frac{5\pi}{4}\right)$

$$\tan\left(\frac{5\pi}{4}\right) = \tan\left(\pi + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

Exercise 2 (4 minutes)

Students should complete the exercise independently. After a few minutes, they should verify their responses with a partner. At an appropriate time, selected students could share their answers.

Exercise 2

Use your understanding of the unit circle to determine the values of the functions shown.

a. $\sin\left(\frac{11\pi}{6}\right)$

$$\sin\left(\frac{11\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

b. $\cos\left(\frac{3\pi}{4}\right)$

$$\cos\left(\frac{3\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

c. $\tan(-\pi)$

$$\tan(-\pi) = \tan(\pi + 0) = \tan(0) = \frac{0}{1} = 0$$

Closing (2 minutes)

Ask students to work with a partner to respond to the following statement:

- Anna says that for any real number θ , $\sin(\theta) = \sin(\pi - \theta)$. Is she correct? Explain how you know.
 - Yes, Anna is correct. The point $(\cos(\theta), \sin(\theta))$ reflects across the y -axis to $(\cos(\pi - \theta), \sin(\pi - \theta))$. Since these two points have the same y -coordinate, $\sin(\pi - \theta) = \sin(\theta)$.

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 1: Special Triangles and the Unit Circle

Exit Ticket

1. Evaluate the following trigonometric expressions, and explain how you used the unit circle to determine your answer.

a. $\sin\left(\pi + \frac{\pi}{3}\right)$

b. $\cos\left(2\pi - \frac{\pi}{6}\right)$

2. Corinne says that for any real number θ , $\cos(\theta) = \cos(\theta - \pi)$. Is she correct? Explain how you know.

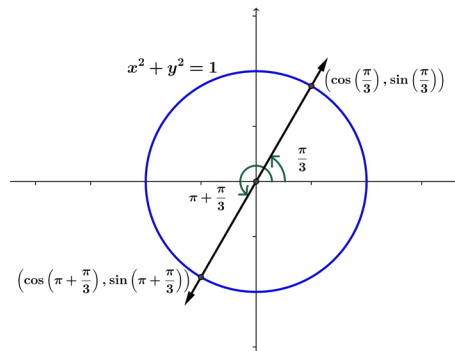
Exit Ticket Sample Solutions

1. Evaluate the following trigonometric expressions, and explain how you used the unit circle to determine your answer.

a. $\sin\left(\pi + \frac{\pi}{3}\right)$

$$\sin\left(\pi + \frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

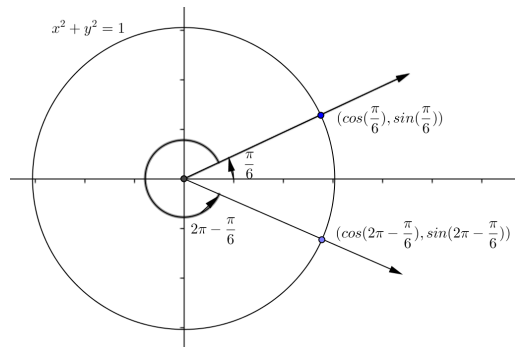
Because the point $\left(\cos\left(\pi + \frac{\pi}{3}\right), \sin\left(\pi + \frac{\pi}{3}\right)\right)$ is directly opposite the point $\left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right)$, we know that the values of $\sin\left(\frac{\pi}{3}\right)$ and $\sin\left(\pi + \frac{\pi}{3}\right)$ are opposites.



b. $\cos\left(2\pi - \frac{\pi}{6}\right)$

$$\cos\left(2\pi - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Because the point $\left(\cos\left(2\pi - \frac{\pi}{6}\right), \sin\left(2\pi - \frac{\pi}{6}\right)\right)$ is the reflection of the point $\left(\cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right)\right)$ across the x -axis, we know that the values of $\cos\left(\frac{\pi}{6}\right)$ and $\cos\left(2\pi - \frac{\pi}{6}\right)$ are equal.



2. Corinne says that for any real number θ , $\cos(\theta) = \cos(\theta - \pi)$. Is she correct? Explain how you know.

Yes, Corinne is correct. The point $(\cos(\theta), \sin(\theta))$ reflects across the y -axis to $(\cos(\pi - \theta), \sin(\pi - \theta))$. These two points have opposite x -coordinates, so $\cos(\pi - \theta) = -\cos(\theta)$. Since the cosine function is an even function, $\cos(\pi - \theta) = \cos(-(\theta - \pi)) = \cos(\theta - \pi)$.

Thus, $\cos(\theta - \pi) = \cos(\theta)$.

Problem Set Sample Solutions

1. Complete the chart below.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

2. Evaluate the following trigonometric expressions, and explain how you used the unit circle to determine your answer.

a. $\cos\left(\pi + \frac{\pi}{3}\right)$

$$\cos\left(\pi + \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

The rotation was $\frac{\pi}{3}$ more than π bringing the ray to the third quadrant where cosine is negative.

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \text{ so } -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}.$$

b. $\sin\left(\pi - \frac{\pi}{4}\right)$

$$\sin\left(\pi - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

The rotation was $\frac{\pi}{4}$ less than π bringing the ray to the second quadrant where sine is positive.

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \text{ so } \sin\left(\pi - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

c. $\sin\left(2\pi - \frac{\pi}{6}\right)$

$$\sin\left(2\pi - \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

The rotation was $\frac{\pi}{6}$ less than 2π bringing the ray to the fourth quadrant where sine is negative.

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \text{ so } \sin\left(2\pi - \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}.$$

d. $\cos\left(\pi + \frac{\pi}{6}\right)$

$$\cos\left(\pi + \frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

The rotation was $\frac{\pi}{6}$ more than π bringing the ray to the third quadrant where cosine is negative.

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \text{ so } -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

e. $\cos\left(\pi - \frac{\pi}{4}\right)$

$$\cos\left(\pi - \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

The rotation was $\frac{\pi}{4}$ less than π bringing the ray to the second quadrant where cosine is negative.

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \text{ so } -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

f. $\cos\left(2\pi - \frac{\pi}{3}\right)$

$$\cos\left(2\pi - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

The rotation was $\frac{\pi}{3}$ less than 2π bringing the ray to the fourth quadrant where cosine is positive.

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \text{ so } \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

g. $\tan\left(\pi + \frac{\pi}{4}\right)$

$$\tan\left(\pi + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

The rotation was $\frac{\pi}{4}$ more than π bringing the ray to the third quadrant where tangent is positive.

$$\tan\left(\frac{\pi}{4}\right) = 1, \text{ so } \tan\left(\frac{\pi}{4}\right) = 1.$$

h. $\tan\left(\pi - \frac{\pi}{6}\right)$

$$\tan\left(\pi - \frac{\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

The rotation was $\frac{\pi}{6}$ less than π bringing the ray to the second quadrant where tangent is negative.

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \text{ so } -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

i. $\tan\left(2\pi - \frac{\pi}{3}\right)$

$$\tan\left(2\pi - \frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$$

The rotation was $\frac{\pi}{3}$ less than 2π bringing the ray to the fourth quadrant where tangent is negative.

$$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}, \text{ so } -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}.$$

3. Rewrite the following trigonometric expressions in an equivalent form using $\pi + \theta$, $\pi - \theta$, or $2\pi - \theta$ and evaluate.

a. $\cos\left(\frac{\pi}{3}\right)$

$$\cos\left(2\pi - \frac{\pi}{3}\right) = \frac{1}{2}$$

b. $\cos\left(-\frac{\pi}{4}\right)$

$$\cos\left(2\pi - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

c. $\sin\left(\frac{\pi}{6}\right)$

$$\sin\left(\pi - \frac{\pi}{6}\right) = \frac{1}{2}$$

d. $\sin\left(\frac{4\pi}{3}\right)$

$$\sin\left(\pi + \frac{\pi}{3}\right) = \sin\left(2\pi - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

e. $\tan\left(-\frac{\pi}{6}\right)$

$$\tan\left(\pi - \frac{\pi}{6}\right) = \tan\left(2\pi - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

f. $\tan\left(-\frac{5\pi}{6}\right)$

$$\tan\left(\pi + \frac{\pi}{6}\right) = \tan\left(2\pi + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

4. Identify the quadrant of the plane that contains the terminal ray of a rotation by θ if θ satisfies the given conditions.

a. $\sin(\theta) > 0$ and $\cos(\theta) > 0$

Quadrant I

b. $\sin(\theta) < 0$ and $\cos(\theta) < 0$

Quadrant III

c. $\sin(\theta) < 0$ and $\tan(\theta) > 0$

Quadrant III

d. $\tan(\theta) > 0$ and $\sin(\theta) > 0$

Quadrant I

e. $\tan(\theta) < 0$ and $\sin(\theta) > 0$

Quadrant II

f. $\tan(\theta) < 0$ and $\cos(\theta) > 0$

Quadrant IV

g. $\cos(\theta) < 0$ and $\tan(\theta) > 0$

Quadrant III

h. $\sin(\theta) > 0$ and $\cos(\theta) < 0$

Quadrant II

5. Explain why $\sin^2(\theta) + \cos^2(\theta) = 1$.

For any real number θ the point $(\cos(\theta), \sin(\theta))$ lies on the unit circle with equation $x^2 + y^2 = 1$. Thus, we must have $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. With the shorthand notation $(\sin(\theta))^2 = \sin^2(\theta)$ and $(\cos(\theta))^2 = \cos^2(\theta)$, this gives $\sin^2(\theta) + \cos^2(\theta) = 1$.

6. Explain how it is possible to have $\sin(\theta) < 0$, $\cos(\theta) < 0$, and $\tan(\theta) > 0$. For which values of θ between 0 and 2π does this happen?

Because $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, if $\sin(\theta)$ and $\cos(\theta)$ are both negative, their quotient is positive. Thus, it is possible to have $\sin(\theta) < 0$, $\cos(\theta) < 0$, and $\tan(\theta) > 0$. This happens when the terminal ray of θ lies in the third quadrant, which is true for $\pi < \theta < \frac{3\pi}{2}$.

7. Duncan says that for any real number θ , $\tan(\theta) = \tan(\pi - \theta)$. Is he correct? Explain how you know.

No, Duncan is not correct. The terminal ray of rotation by θ and the terminal ray of rotation by $\pi - \theta$ are reflections of each other across the y-axis. Thus, $(\cos(\theta), \sin(\theta))$ is the reflection of $(\cos(\pi - \theta), \sin(\pi - \theta))$ across the y-axis. This means that $\cos(\pi - \theta) = -\cos(\theta)$ and $\sin(\pi - \theta) = \sin(\theta)$.

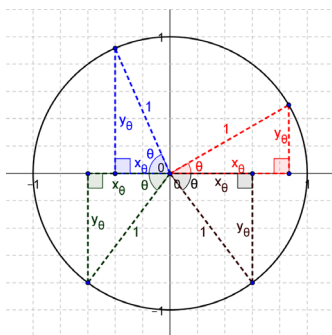
Thus,

$$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin(\theta)}{-\cos(\theta)} = -\tan(\theta).$$

We see that

$$\tan(\pi - \theta) \neq \tan(\theta).$$

8. Given the following trigonometric functions, identify the quadrant in which the terminal ray of θ lies in the unit circle shown below. Find the other two trigonometric functions of θ of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$.



a. $\sin(\theta) = \frac{1}{2}$ and $\cos(\theta) > 0$

Quadrant I; $\theta = \frac{\pi}{6}$

$$\sin(\theta) = \frac{1}{2}, \cos(\theta) = \frac{\sqrt{3}}{2}, \tan(\theta) = \frac{\sqrt{3}}{3}$$

b. $\cos(\theta) = -\frac{1}{2}$ and $\sin(\theta) > 0$

Quadrant II; $\theta = \frac{\pi}{3}$

$$\sin(\theta) = \frac{\sqrt{3}}{2}, \cos(\theta) = -\frac{1}{2}, \tan(\theta) = -\sqrt{3}$$

c. $\tan(\theta) = 1$ and $\cos(\theta) < 0$

Quadrant III; $\theta = \frac{\pi}{4}$

$$\sin(\theta) = -\frac{\sqrt{2}}{2}, \cos(\theta) = -\frac{\sqrt{2}}{2}, \tan(\theta) = 1$$

d. $\sin(\theta) = -\frac{\sqrt{3}}{2}$ and $\cot(\theta) < 0$

Quadrant IV; $\theta = \frac{\pi}{3}$

$$\sin(\theta) = -\frac{\sqrt{3}}{2}, \cos(\theta) = \frac{1}{2}, \tan(\theta) = -\sqrt{3}$$

e. $\tan(\theta) = -\sqrt{3}$ and $\cos(\theta) < 0$

Quadrant II; $\theta = \frac{\pi}{3}$

$$\sin(\theta) = \frac{\sqrt{3}}{2}, \cos(\theta) = -\frac{1}{2}, \tan(\theta) = -\sqrt{3}$$

f. $\sec(\theta) = -2$ and $\sin(\theta) < 0$

Quadrant III; $\theta = \frac{\pi}{3}$

$$\sin(\theta) = -\frac{\sqrt{3}}{2}, \cos(\theta) = -\frac{1}{2}, \tan(\theta) = \sqrt{3}$$

g. $\cot(\theta) = \sqrt{3}$ and $\csc(\theta) > 0$

Quadrant I; $\theta = \frac{\pi}{6}$

$$\sin(\theta) = \frac{1}{2}, \cos(\theta) = \frac{\sqrt{3}}{2}, \tan(\theta) = \frac{\sqrt{3}}{3}$$

9. Toby thinks the following trigonometric equations are true. Use $\theta = \frac{\pi}{6}, \frac{\pi}{4},$ and $\frac{\pi}{3}$ to develop a conjecture whether or not he is correct in each case below.

a. $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

Yes, he seems to be correct.

b. $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = \sin\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

Yes, he seems to be correct.

10. Toby also thinks the following trigonometric equations are true. Is he correct? Justify your answer.

a. $\sin\left(\pi - \frac{\pi}{3}\right) = \sin(\pi) - \sin\left(\frac{\pi}{3}\right)$

He is not correct because trigonometric functions are not linear.

$$\sin\left(\pi - \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\sin(\pi) - \sin\left(\frac{\pi}{3}\right) = 0 - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

b. $\cos\left(\pi - \frac{\pi}{3}\right) = \cos(\pi) - \cos\left(\frac{\pi}{3}\right)$

He is not correct because trigonometric functions are not linear.

$$\cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$\cos(\pi) - \cos\left(\frac{\pi}{3}\right) = -1 - \frac{1}{2} = -\frac{3}{2}$$

c. $\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{6}\right)$

He is not correct because trigonometric functions are not linear.

$$\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

$$\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{6}\right) = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$$

d. $\sin\left(\pi + \frac{\pi}{6}\right) = \sin(\pi) + \sin\left(\frac{\pi}{6}\right)$

He is not correct because trigonometric functions are not linear.

$$\sin\left(\pi + \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$\sin(\pi) + \sin\left(\frac{\pi}{6}\right) = 0 + \frac{1}{2} = \frac{1}{2}$$

e. $\cos\left(\pi + \frac{\pi}{4}\right) = \cos(\pi) + \cos\left(\frac{\pi}{4}\right)$

He is not correct because trigonometric functions are not linear.

$$\cos\left(\pi + \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\cos(\pi) + \cos\left(\frac{\pi}{4}\right) = -1 + \frac{\sqrt{2}}{2} = \frac{-2 + \sqrt{2}}{2}$$



Lesson 2: Properties of Trigonometric Functions

Student Outcomes

- Students explore the symmetry and periodicity of trigonometric functions.
- Students derive relationships between trigonometric functions using their understanding of the unit circle.

Lesson Notes

In the previous lesson, students reviewed the characteristics of the unit circle and used them to evaluate trigonometric functions for rotations of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ radians. They then explored the relationships between the trigonometric functions for rotations θ in all four quadrants and derived formulas to evaluate sine, cosine, and tangent for rotations $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$. In this lesson, we revisit the idea of periodicity of the trigonometric functions as introduced in Algebra II Module 1 Lesson 1. Students continue to explore the relationship between trigonometric functions for rotations θ , examining the periodicity and symmetry of the sine, cosine, and tangent functions. They also use the unit circle to explore relationships between the sine and cosine functions.

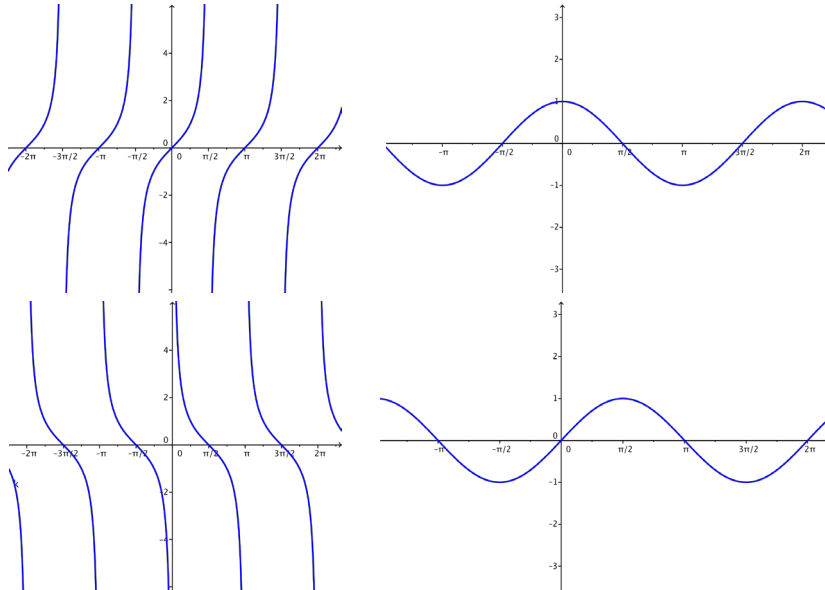
Classwork

Opening Exercise (3 minutes)

Students studied the graphs of trigonometric functions extensively in Algebra II but may not instantly recognize the graphs of these functions. The Opening Exercise encourages students to relate the graph of the sine, cosine, and tangent functions to rotations of a ray studied in the previous lesson. At the end of this exercise, allow students to provide justification for how they matched the graphs to the functions.

Opening Exercise

The graphs below depict four trigonometric functions. Identify which of the graphs are $f(x) = \sin(x)$, $g(x) = \cos(x)$, and $h(x) = \tan(x)$. Explain how you know.



The first graph is the graph of the tangent function $h(x) = \tan(x)$ because the range of the tangent function is all real numbers, and $\tan(0) = 0$, which rules out the third graph as a possibility.

The second graph (upper right) is the graph of the cosine function $g(x) = \cos(x)$ because the range of the cosine function is $[-1, 1]$, and $\cos(0) = 1$, ruling out the fourth graph as a possibility.

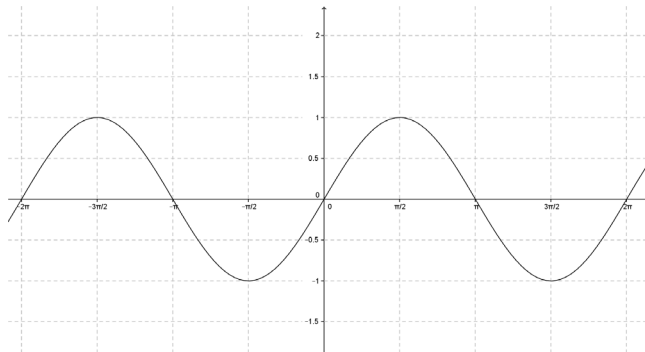
The fourth graph (bottom right) is the graph of the sine function $f(x) = \sin(x)$ because the range of the sine function is $[-1, 1]$, and $\sin(0) = 0$.

The third graph (bottom left) is the graph of the cotangent function. (This could be an extension problem for advanced students.)

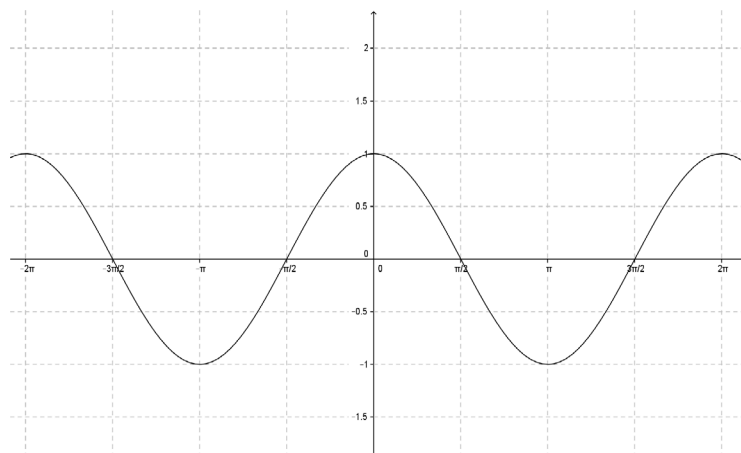
Discussion (7 minutes): Periodicity of Sine, Cosine, and Tangent Functions

This discussion helps students relate their graphs of the sine, cosine, and tangent functions to the unit circle. Students use the unit circle to determine the periodicity of the sine, cosine, and tangent functions, and they apply the periodicity to evaluate trigonometric functions for values of θ that are not between 0 and 2π .

- Let's look at the graph of the function $f(x) = \sin(x)$.

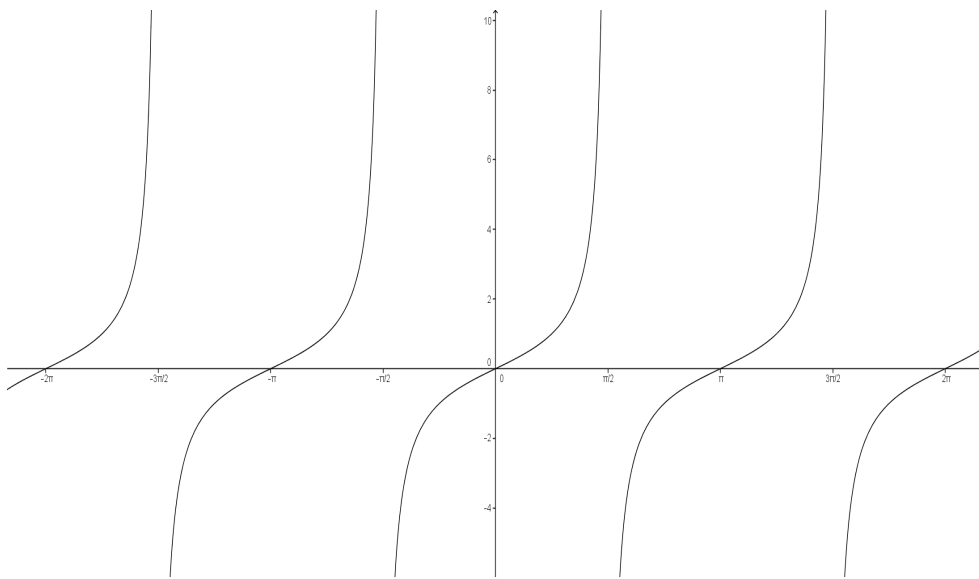


- How would you describe this graph to someone who has not seen it? Share your response with a partner.
 - Answers will vary but will probably address that there are x-intercepts at all integer multiples of π , that the graph is a wave whose height oscillates between -1 and 1 , and that the cycle repeats every 2π radians.*
- Let's look now at the graph of the function $g(x) = \cos(x)$.



- Compare and contrast this graph with the graph of $f(x) = \sin(x)$. Share your thoughts with your partner.
 - Answers will vary but will probably address that the graph of $g(x)$ appears to be the same as that of $f(x)$ shifted to the left by $\frac{\pi}{2}$ radians. In other words, the range of the function and the periodicity are the same for both graphs, but the x-intercepts are different.*
- The sketches illustrate that both the sine and cosine functions are **periodic**. Turn to your partner and describe what you remember about periodic functions in general and about the sine and cosine functions in particular.
 - Periodic functions repeat the same pattern every period; for the sine and cosine functions, the period is 2π .*
- How can we use the paper plate model of the unit circle to explain the periodicity of the sine and cosine functions? Discuss your reasoning in terms of the position of the rider on the carousel.
 - Answers may vary but should address that for any given rotation θ , the position of the rider on the carousel is (x_θ, y_θ) . If we want to represent the rotation $2\pi + \theta$, we would rotate the plate one full turn counterclockwise, and the position of the rider would again be (x_θ, y_θ) . This trend would continue for each additional rotation of 2π radians since a rotation of 2π radians represents a complete turn.*

- How does periodicity apply to negative rotational values (e.g., $\theta - 2\pi$)? Again, explain your reasoning in terms of the position of the rider on the carousel.
 - It would not affect the position of the rider. The rotation $\theta - 2\pi$ represents a full turn backwards (clockwise) from the position (x_θ, y_θ) , and the rider's position will still be the same as for a rotation θ .*
- And how does the pattern in the rider's position on the carousel relate to the periodicity of the sine and cosine functions?
 - We can see that both x_θ and y_θ repeat for every 2π radians of rotation. Since $\sin(\theta) = y_\theta$ and $\cos(\theta) = x_\theta$, we can conclude that $\sin(\theta) = \sin(2\pi n + \theta)$ and $\cos(\theta) = \cos(2\pi n + \theta)$ for all integer values of n .*
- We've determined that both $f(x) = \sin(x)$ and $g(x) = \cos(x)$ are periodic, and we've found formulas to describe the periodicity. What about $h(x) = \tan(x)$? Let's look at the graph of this function.



- Is the tangent function also periodic? Explain how you know.
 - Yes. Since the tangent function is the quotient of the sine and cosine functions, and the sine and cosine functions are periodic with the same period, the tangent function is also periodic.*
- How does the period of the tangent function compare with the periods of the sine and cosine functions?
 - The tangent function has a period of π radians, but the sine and cosine functions have periods of 2π radians.*
- Let's see if we can use the unit circle to explain this discrepancy. For rotation θ , how do we define $\tan(\theta)$?
 - $\tan(\theta) = \frac{y_\theta}{x_\theta}$
- Recall that in the previous lesson, the point on the unit circle that corresponded to a given rotation represented the position of a rider on a carousel with radius 1 unit. What does the ratio $\frac{y_\theta}{x_\theta}$ represent in terms of the position of the rider?
 - It is the ratio of the front/back position to the right/left position for rotation θ .*

- Let's examine this ratio as the carousel rotates. Use your unit circle model to examine the position of the carousel rider for a complete rotation. Use the front/back and right/left positions of the rider to determine how the value of $\tan \theta$ changes as θ increases from 0 to 2π radians. Share your findings with a partner.

Answers will vary but should address the following:

- The position of the rider starts at $(1,0)$, immediately to the right of center, at $\theta = 0$. Since $\tan(\theta) = \frac{y_\theta}{x_\theta}$, $\tan(0) = \frac{0}{1} = 0$. As the carousel rotates to $\frac{\pi}{2}$, the rider's front/back position increases from 0 to 1, and the right/left position decreases from 1 to 0. This means that $\tan \theta$ increases from 0 at $\theta = 0$ to infinity as θ approaches $\frac{\pi}{2}$.
 - The position of the rider is at $(0,1)$ at $\theta = \frac{\pi}{2}$. So, when $\theta = \frac{\pi}{2}$, $\tan(\theta) = \frac{1}{0}$, which is undefined. As the carousel rotates counterclockwise from $\theta = \frac{\pi}{2}$, the rider's front/back position decreases from 1 to 0, and the right/left position decreases from 0 to -1 . As the carousel rotates from $\theta = \frac{\pi}{2}$ to π , $\tan(\theta)$ increases from $-\infty$ to 0.
 - From $\theta = \pi$ to $\theta = \frac{3\pi}{2}$, the rider's front/back position decreases from 0 to -1 , and the right/left position increases from -1 to 0. This means that $\tan \theta$ increases from 0 at $\theta = \pi$ toward $+\infty$ at $\theta = \frac{3\pi}{2}$.
 - From $\theta = \frac{3\pi}{2}$ to $\theta = 2\pi$, the rider's front/back position increases from -1 to 0, and the right/left position increases from 0 to 1. This means that $\tan(\theta)$ increases from $-\infty$ at $\theta \rightarrow \frac{3\pi}{2}$ to 0 at $\theta = 2\pi$.
- How do your findings relate to the periodicity of the tangent function?
 - The values for $\tan(\theta)$ seem to repeat every π radians, as the sketch indicates.
- How can we explain this pattern based on our understanding of the unit circle and the definition of $\tan(\theta)$?
 - Answers will vary. An example of an acceptable response is shown: As our initial ray rotates through Quadrants I and III, the ratio of y_θ to x_θ is positive because the coordinates have the same sign. The ratio increases from 0 to $+\infty$ because the magnitudes of the y -coordinates increase, while the magnitudes of the x -coordinates decrease toward 0. Similarly, in Quadrants II and IV, the ratio of y_θ to x_θ is negative because the coordinates have opposite signs. The ratio should increase from $-\infty$ to 0 because the magnitudes of the y -coordinates decrease toward 0, while the magnitudes of the x -coordinates increase toward 1, which means the ratio of y_θ to x_θ becomes a smaller and smaller negative number until it reaches 0 at $\theta = 0$ and $\theta = 2\pi$.
- Great observations. How can we formalize these thoughts about the periodicity of the tangent function using a formula?
 - $\tan(\theta) = \tan(\theta + \pi n)$, where n is an integer.

Exercises 1–2 (5 minutes)

The following exercises reinforce the periodicity of the three primary trigonometric functions: sine, cosine, and tangent. Students should complete the exercises independently. After a few minutes, they could share their responses with a partner. Students could write their responses to Exercise 1 on individual white boards for quick checks or on paper. Exercise 2 should be discussed briefly in a whole-class setting.

Exercises 1–4

1. Use the unit circle to evaluate these expressions:

a. $\sin\left(\frac{17\pi}{4}\right)$

$$\sin\left(\frac{17\pi}{4}\right) = \sin\left(4\pi + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

b. $\cos\left(\frac{19\pi}{6}\right)$

$$\cos\left(\frac{19\pi}{6}\right) = \cos\left(2\pi + \frac{7\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right) = \cos\left(\pi + \frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

c. $\tan(450\pi)$

$$\tan(450\pi) = \tan(450\pi + 0) = \tan(0) = 0$$

2. Use the identity $\sin(\pi + \theta) = -\sin(\theta)$ for all real-numbered values of θ to verify the identity $\sin(2\pi + \theta) = \sin(\theta)$ for all real-numbered values of θ .

$$\sin(2\pi + \theta) = \sin(\pi + (\pi + \theta)) = -\sin(\pi + \theta) = -(-\sin(\theta)) = \sin(\theta)$$

Scaffolding:

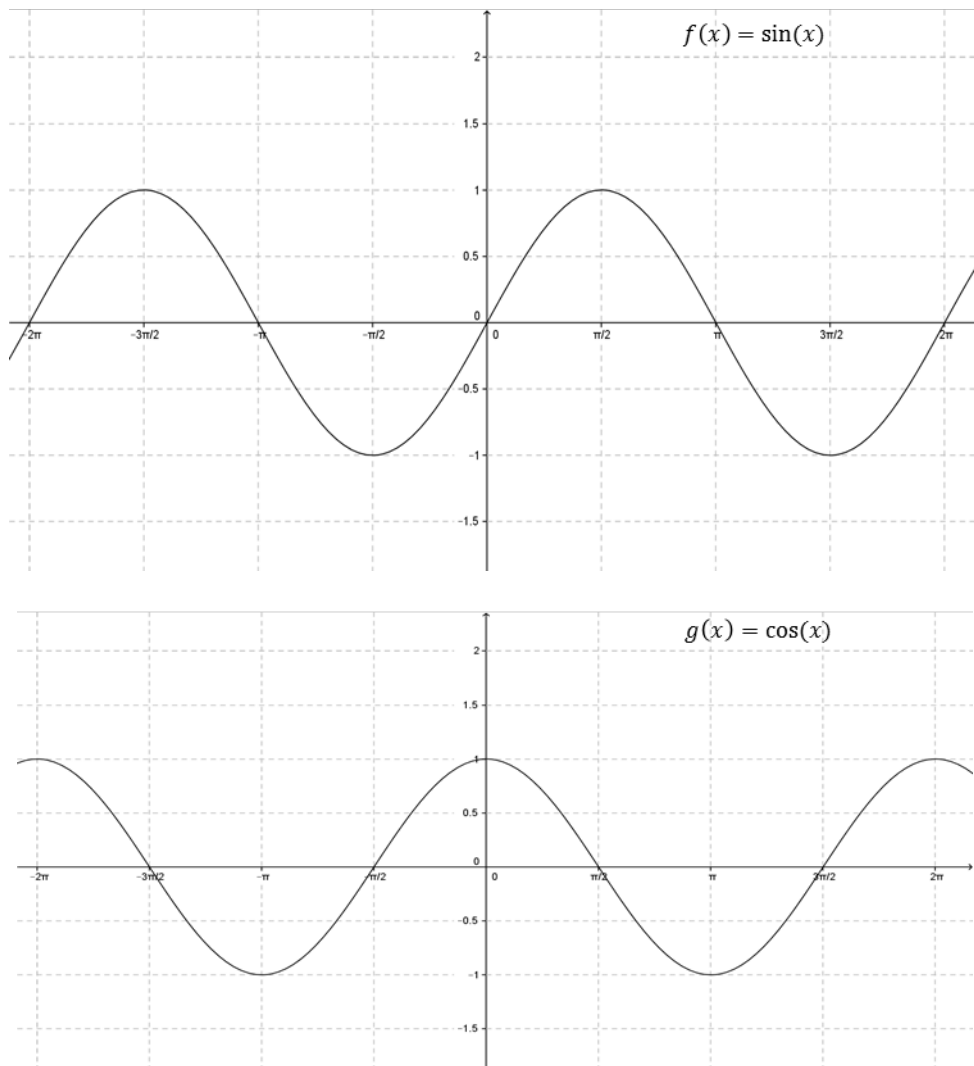
- The identity $\cos(\pi + \theta) = -\cos(\theta)$ can help students evaluate the function in Exercise 1 part (b).
- Prompt students by asking, “If we rewrite $\cos\left(\frac{7\pi}{6}\right)$ as $\cos(\pi + \theta)$, what is the value of θ ?”

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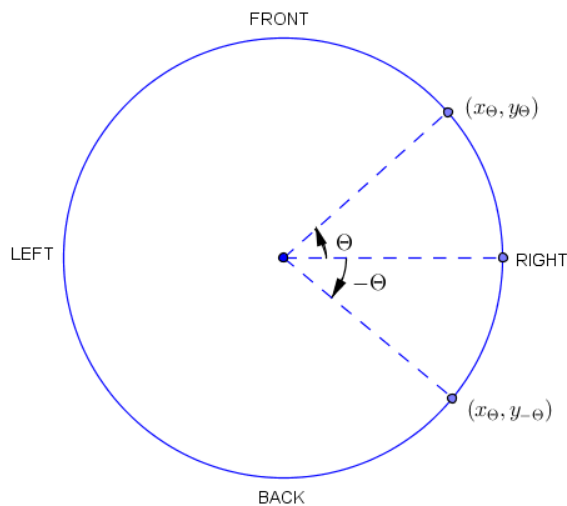
Discussion (3 minutes): Symmetry of Sine, Cosine, and Tangent Functions

This discussion addresses how the properties of the unit circle reveal symmetry in the graphs of the trigonometric functions. Students determine the evenness/oddness of the graphs for sine and cosine, which they in turn use to determine the evenness/oddness of the tangent function.

- Let's look again at the graphs of the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Describe the symmetry of the graphs.



- The graph of $f(x) = \sin(x)$ seems to be symmetric about the origin (or seems to have 180° rotational symmetry), and the graph of $g(x) = \cos(x)$ seems to be symmetric about the y-axis.
- Let's explore these apparent symmetries using the unit circle. We recall that for a rotation θ on our carousel, the rider's position is defined as (x_θ, y_θ) . Describe the position of a rider if the carousel rotates $-\theta$ radians.
 - A rotation by $-\theta$ is equivalent to a clockwise rotation by θ radians, which results in a position of $(x_\theta, -y_\theta)$.



- And how does this relate to our understanding of the symmetry of the sine and cosine functions?
 - Since $\sin(-\theta) = -y_\theta$, we have $\sin(-\theta) = -\sin(\theta)$. This means that $f(x) = \sin(x)$ is an odd function with symmetry about the origin. (If students do not suggest that the sine function is an odd function, remind them that an odd function is a function f for which $f(-x) = -f(x)$ for any x in the domain of f .)
 - Since $\cos(-\theta) = x_\theta$, we have $\cos(-\theta) = \cos(\theta)$. This means that $g(x) = \cos(x)$ is an even function with symmetry about the y -axis. (If students do not suggest that the cosine function is an even function, remind them that an even function is a function f for which $f(-x) = f(x)$ for any x in the domain of f .)
- Let's summarize these findings:
 - For all real numbers θ , $\sin(-\theta) = -\sin(\theta)$, and $\cos(-\theta) = \cos(\theta)$.

Exercises 3–4 (5 minutes)

Students should complete the exercises independently. Students could write their responses to Exercise 3 on individual white boards for quick checks or on paper. A few selected students should share their responses for Exercise 4 in a whole-class setting.

3. Use your understanding of the symmetry of the sine and cosine functions to evaluate these functions for the given values of θ .

a. $\sin\left(-\frac{\pi}{2}\right)$

$$\sin\left(-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$$

Scaffolding:

- Remind struggling students of the identity $\cos(2\pi - \theta) = \cos(\theta)$ to help them evaluate the function in Exercise 3 part (b).
- Have advanced students determine the symmetry for the graphs of the co-functions.

MP.7

MP.7

b. $\cos\left(-\frac{5\pi}{3}\right)$

$$\cos\left(-\frac{5\pi}{3}\right) = \cos\left(\frac{5\pi}{3}\right) = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

4. Use your understanding of the symmetry of the sine and cosine functions to determine the value of $\tan(-\theta)$ for all real-numbered values of θ . Determine whether the tangent function is even, odd, or neither.

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\frac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

The tangent function is odd.

Exploratory Challenge/Exercises 5–6 (10 minutes)

This challenge requires students to use the properties of the unit circle to derive relationships between the sine and cosine functions. Students should complete the challenge in pairs or small groups, with each group completing either Exercise 5 or Exercise 6. After a few minutes, students should review their findings with other pairs or groups assigned to the same exercise. A few groups should then share their responses in a whole-class setting.

Scaffolding:

Have advanced students verify additional identities based on the data collected in the tables; for example,

$$\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2} + \theta\right).$$

Exploratory Challenge/Exercises 5–6

5. Use your unit circle model to complete the table. Then use the completed table to answer the questions that follow.

θ	$\left(\frac{\pi}{2} + \theta\right)$	$\sin\left(\frac{\pi}{2} + \theta\right)$	$\cos\left(\frac{\pi}{2} + \theta\right)$
0	$\frac{\pi}{2}$	1	0
$\frac{\pi}{2}$	π	0	-1
π	$\frac{3\pi}{2}$	-1	0
$\frac{3\pi}{2}$	2π	0	1
2π	$\frac{5\pi}{2}$	1	0

- a. What does the value $\left(\frac{\pi}{2} + \theta\right)$ represent with respect to the rotation of the carousel?

It is a rotation by θ radians counterclockwise from the starting point $\frac{\pi}{2}$.

- b. What pattern do you recognize in the values of $\sin\left(\frac{\pi}{2} + \theta\right)$ as θ increases from 0 to 2π ?

Values of $\sin\left(\frac{\pi}{2} + \theta\right)$ follow the same pattern as values of $\cos(\theta)$.

- c. What pattern do you recognize in the values of $\cos\left(\frac{\pi}{2} + \theta\right)$ as θ increases from 0 to 2π ?

Values of $\cos\left(\frac{\pi}{2} + \theta\right)$ follow the same pattern as values of $-\sin(\theta)$.

- d. Fill in the blanks to formalize these relationships:

$$\sin\left(\frac{\pi}{2} + \theta\right) =$$

$$\cos\left(\frac{\pi}{2} + \theta\right) =$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos(\theta)$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$$

6. Use your unit circle model to complete the table. Then use the completed table to answer the questions that follow.

θ	$\left(\frac{\pi}{2} - \theta\right)$	$\sin\left(\frac{\pi}{2} - \theta\right)$	$\cos\left(\frac{\pi}{2} - \theta\right)$
0	$\frac{\pi}{2}$	1	0
$\frac{\pi}{2}$	0	0	1
π	$-\frac{\pi}{2}$	-1	0
$\frac{3\pi}{2}$	$-\pi$	0	-1
2π	$-\frac{3\pi}{2}$	1	0

- a. What does the value $\left(\frac{\pi}{2} - \theta\right)$ represent with respect to the rotation of a rider on the carousel?

It is a rotation by θ radians clockwise from a point directly in front of the center of the carousel.

- b. What pattern do you recognize in the values of $\sin\left(\frac{\pi}{2} - \theta\right)$ as θ increases from 0 to 2π ?

Values of $\sin\left(\frac{\pi}{2} - \theta\right)$ follow the same pattern as values of $\cos(\theta)$.

- c. What pattern do you recognize in the values of $\cos\left(\frac{\pi}{2} - \theta\right)$ as θ increases from 0 to 2π ?

Values of $\cos\left(\frac{\pi}{2} - \theta\right)$ follow the same pattern as values of $\sin(\theta)$.

- d. Fill in the blanks to formalize these relationships:

$$\sin\left(\frac{\pi}{2} - \theta\right) =$$

$$\cos\left(\frac{\pi}{2} - \theta\right) =$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

Exercise 7 (5 minutes)

Have students complete this exercise independently and then verify the solutions with a partner. Students should share their approaches to solving the problems as time permits.

Exercise 7

7. Use your understanding of the relationship between the sine and cosine functions to verify these statements.

a. $\cos\left(\frac{4\pi}{3}\right) = \sin\left(\frac{-\pi}{6}\right)$

$$\cos\left(\frac{4\pi}{3}\right) = \cos\left(\frac{\pi}{2} + \frac{5\pi}{6}\right) = -\sin\left(\frac{5\pi}{6}\right) = -\sin\left(\pi - \frac{5\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{-\pi}{6}\right)$$

b. $\cos\left(\frac{5\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right)$

$$\cos\left(\frac{5\pi}{4}\right) = \cos\left(\frac{\pi}{2} + \frac{3\pi}{4}\right) = -\sin\left(\frac{3\pi}{4}\right) = -\left(-\sin\left(\pi + \frac{3\pi}{4}\right)\right) = \sin\left(\frac{7\pi}{4}\right)$$

Closing (2 minutes)

Have students respond in writing to this prompt.

- Why do we only need to know values of $\sin(\theta)$ and $\cos(\theta)$ for $0 \leq \theta < 2\pi$ in order to find the sine or cosine of any real number?
 - *Because the sine and cosine functions are periodic with period 2π , we know that $\cos(\theta \pm 2\pi) = \cos(\theta)$ and $\sin(\theta \pm 2\pi) = \sin(\theta)$ for any real number θ . Thus, if x is any real number, and $x \geq 2\pi$, then we just subtract 2π as many times as is needed so that the result is between 0 and 2π , and then we can evaluate the sine and cosine. Likewise, if $x < 0$, then we add 2π as many times as needed so that the result is between 0 and 2π and then evaluate sine and cosine.*

Lesson Summary

For all real numbers θ for which the expressions are defined,

$$\sin(\theta) = \sin(2\pi n + \theta) \text{ and } \cos(\theta) = \cos(2\pi n + \theta) \text{ for all integer values of } n$$

$$\tan(\theta) = \tan(\pi n + \theta) \text{ for all integer values of } n$$

$$\sin(-\theta) = -\sin(\theta), \cos(-\theta) = \cos(\theta), \text{ and } \tan(-\theta) = -\tan(\theta)$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos(\theta) \text{ and } \cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta) \text{ and } \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

Exit Ticket (5 minutes)

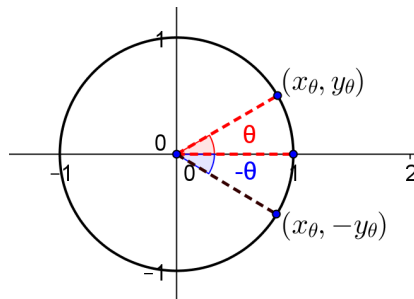
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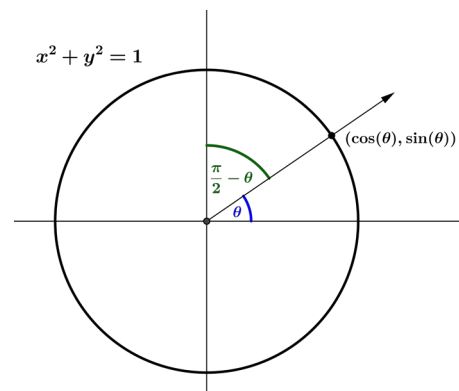
Lesson 2: Properties of Trigonometric Functions

Exit Ticket

- From the unit circle given, explain why the cosine function is an even function with symmetry about the y-axis, and the sine function is an odd function with symmetry about the origin.

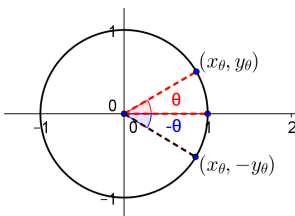


- Use the unit circle to explain why $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ for θ as shown in the figure to the right.



Exit Ticket Sample Solutions

- From the unit circle given, explain why the cosine function is an even function with symmetry about the y-axis, and the sine function is an odd function with symmetry about the origin.



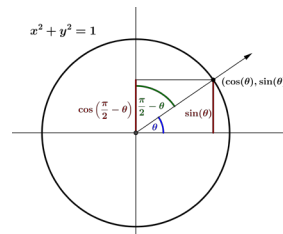
Suppose we rotate the point $(1, 0)$ by $-\theta$ radians, where $\theta \geq 0$. This gives the same x -coordinate as rotating by θ radians, so if $\theta \geq 0$, we have $\cos(-\theta) = \cos(\theta)$. Likewise, if $\theta < 0$ and we rotate the point $(1, 0)$ by $-\theta$ radians, then the resulting point has the same x -coordinate as rotation of $(1, 0)$ by θ radians. Thus, if $\theta < 0$ then $\cos(-\theta) = \cos(\theta)$. Since $\cos(-\theta) = \cos(\theta)$ for all real numbers θ , the cosine function is even and is symmetric about the y -axis.

Suppose we rotate the point $(1, 0)$ by $-\theta$ radians, where $\theta \geq 0$. This gives the opposite y -coordinate as rotating by θ radians, so if $\theta \geq 0$ we have $\sin(-\theta) = -\sin(\theta)$. Likewise, if $\theta < 0$ and we rotate the point $(1, 0)$ by $-\theta$ radians, then the resulting point has the opposite y -coordinate as rotation of $(1, 0)$ by θ radians. Thus, if $\theta < 0$ then $\sin(-\theta) = -\sin(\theta)$. Since $\sin(-\theta) = -\sin(\theta)$ for all real numbers θ , the sine function is odd and is symmetric about the origin.

- Use the unit circle to explain why $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ for θ as shown in the figure to the right.

The point where the terminal ray intersects the unit circle after rotation by θ has coordinates $(\cos(\theta), \sin(\theta))$. If we draw perpendicular lines from this point to the x -axis and y -axis, we create two triangles. The vertical legs of these triangles both have the same length. The triangle on the left shows that this length is $\cos\left(\frac{\pi}{2} - \theta\right)$, while the triangle on the right shows that this length is $\sin(\theta)$.

Thus, $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$.



Problem Set Sample Solutions

- Evaluate the following trigonometric expressions. Show how you used the unit circle to determine the solution.

a. $\sin\left(\frac{13\pi}{6}\right)$

$$\sin\left(\frac{13\pi}{6}\right) = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\frac{13\pi}{6} \text{ is a full rotation more than } \frac{\pi}{6} \text{ meaning } \sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$

b. $\cos\left(-\frac{5\pi}{3}\right)$

$$\cos\left(-\frac{5\pi}{3}\right) = \cos\left(-\left(2\pi - \frac{\pi}{3}\right)\right) = \cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$-\frac{5\pi}{3}$ is a full rotation less than $\frac{\pi}{3}$ meaning $\cos\left(-\frac{5\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

c. $\tan\left(\frac{25\pi}{4}\right)$

$$\tan\left(\frac{25\pi}{4}\right) = \tan\left(6\pi + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

$\frac{25\pi}{4}$ is three full rotations more than $\frac{\pi}{4}$ meaning $\tan\left(\frac{25\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$.

d. $\sin\left(-\frac{3\pi}{4}\right)$

$$\sin\left(-\frac{3\pi}{4}\right) = -\sin\left(\frac{3\pi}{4}\right) = -\sin\left(\pi - \frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$-\frac{3\pi}{4}$ is the same rotation as $\pi + \frac{\pi}{4}$ meaning $\sin\left(-\frac{3\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

e. $\cos\left(-\frac{5\pi}{6}\right)$

$$\cos\left(-\frac{5\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = \cos\left(\pi - \frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$-\frac{5\pi}{6}$ is the same rotation as $\pi + \frac{\pi}{6}$ meaning $\cos\left(-\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

f. $\sin\left(\frac{17\pi}{3}\right)$

$$\sin\left(\frac{17\pi}{3}\right) = \sin\left(4\pi + \frac{5\pi}{3}\right) = \sin\left(\frac{5\pi}{3}\right) = \sin\left(2\pi - \frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$\frac{17\pi}{3}$ is two full rotations more than $\frac{5\pi}{3}$ meaning $\sin\left(\frac{17\pi}{3}\right) = \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

g. $\cos\left(\frac{25\pi}{4}\right)$

$$\cos\left(\frac{25\pi}{4}\right) = \cos\left(6\pi + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$\frac{25\pi}{4}$ is three full rotations more than $\frac{\pi}{4}$ meaning $\cos\left(\frac{25\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

h. $\tan\left(\frac{29\pi}{6}\right)$

$$\tan\left(\frac{29\pi}{6}\right) = \tan\left(4\pi + \frac{5\pi}{6}\right) = \tan\left(\frac{5\pi}{6}\right) = \tan\left(\pi - \frac{\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

$\frac{29\pi}{6}$ is two full rotations more than $\frac{5\pi}{6}$ meaning $\tan\left(\frac{29\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.

i. $\sin\left(-\frac{31\pi}{6}\right)$

$$\sin\left(-\frac{31\pi}{6}\right) = -\sin\left(\frac{31\pi}{6}\right) = -\sin\left(4\pi + \frac{7\pi}{6}\right) = -\sin\left(\frac{7\pi}{6}\right) = -\sin\left(\pi + \frac{\pi}{6}\right) = -\left(-\sin\left(\frac{\pi}{6}\right)\right) = \frac{1}{2}$$

$-\frac{31\pi}{6}$ is the same as two full rotations in the clockwise direction plus $\frac{7\pi}{6}$ more, which is the same rotation as $\frac{5\pi}{6}$ meaning $\sin\left(-\frac{31\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

j. $\cos\left(-\frac{32\pi}{6}\right)$

$$\cos\left(-\frac{32\pi}{6}\right) = \cos\left(\frac{32\pi}{6}\right) = \cos\left(4\pi + \frac{8\pi}{6}\right) = \cos\left(\frac{4\pi}{3}\right) = \cos\left(\pi + \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

$-\frac{32\pi}{6}$ is the same as two full rotations in the clockwise direction plus $\frac{8\pi}{6}$ or $\frac{4\pi}{3}$ more, which is the same rotation as $\pi + \frac{\pi}{3}$ meaning $\cos\left(-\frac{32\pi}{6}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$.

k. $\tan\left(-\frac{18\pi}{3}\right)$

$$\tan\left(-\frac{18\pi}{3}\right) = -\tan\left(\frac{18\pi}{3}\right) = \tan(6\pi) = \tan(0) = 0$$

$-\frac{18\pi}{3}$ is three full rotations in the clockwise direction meaning $\tan\left(-\frac{18\pi}{3}\right) = \tan(0) = 0$.

2. Given each value of β below, find a value of α with $0 \leq \alpha \leq 2\pi$ so that $\cos(\alpha) = \cos(\beta)$ and $\alpha \neq \beta$.

a. $\beta = \frac{3\pi}{4}$

$$\frac{5\pi}{4}$$

b. $\beta = \frac{5\pi}{6}$

$$\frac{7\pi}{6}$$

c. $\beta = \frac{11\pi}{12}$

$$\frac{13\pi}{12}$$

d. $\beta = 2\pi$

$$0$$

e. $\beta = \frac{7\pi}{5}$

$$\frac{3\pi}{5}$$

f. $\beta = \frac{17\pi}{30}$
 $\frac{43\pi}{30}$

g. $\beta = \frac{8\pi}{11}$
 $\frac{14\pi}{11}$

3. Given each value of β below, find two values of α with $0 \leq \alpha \leq 2\pi$ so that $\cos(\alpha) = \sin(\beta)$.

a. $\beta = \frac{\pi}{3}$
 $\frac{\pi}{6}, \frac{11\pi}{6}$

b. $\beta = \frac{7\pi}{6}$
 $\frac{2\pi}{3}, \frac{4\pi}{3}$

c. $\beta = \frac{3\pi}{4}$
 $\frac{\pi}{4}, \frac{7\pi}{4}$

d. $\beta = \frac{\pi}{8}$
 $\frac{3\pi}{8}, \frac{13\pi}{8}$

4. Given each value of β below, find two values of α with $0 \leq \alpha \leq 2\pi$ so that $\sin(\alpha) = \cos(\beta)$.

a. $\beta = \frac{\pi}{3}$
 $\frac{\pi}{6}, \frac{5\pi}{6}$

b. $\beta = \frac{5\pi}{6}$
 $\frac{4\pi}{3}, \frac{5\pi}{3}$

c. $\beta = \frac{7\pi}{4}$
 $\frac{\pi}{4}, \frac{3\pi}{4}$

d. $\beta = \frac{\pi}{12}$
 $\frac{5\pi}{12}, \frac{7\pi}{12}$

5. Jamal thinks that $\cos\left(\alpha - \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{4}\right)$ for any value of α . Is he correct? Explain how you know.

Jamal is correct. Let $\theta = \alpha + \frac{\pi}{4}$. Then $\theta - \frac{\pi}{2} = \alpha - \frac{\pi}{4}$. We know that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ and that the cosine function is even, so we have $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$. Then $\cos\left(\alpha - \frac{\pi}{4}\right) = \sin\left(\alpha + \frac{\pi}{4}\right)$.

6. Shawna thinks that $\cos\left(\alpha - \frac{\pi}{3}\right) = \sin\left(\alpha + \frac{\pi}{6}\right)$ for any value of α . Is she correct? Explain how you know.

Shawna is correct. Let $\theta = \alpha + \frac{\pi}{6}$. Then $\theta - \frac{\pi}{2} = \alpha - \frac{\pi}{3}$. We know that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$ and that the cosine function is even, so we have $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$. Then $\cos\left(\alpha - \frac{\pi}{3}\right) = \sin\left(\alpha + \frac{\pi}{6}\right)$.

7. Rochelle looked at Jamal and Shawna's results from Problems 5 and 6 and came up with the conjecture below. Is she correct? Explain how you know.

$$\text{Conjecture: } \cos(\alpha - \beta) = \sin\left(\alpha + \left(\frac{\pi}{2} - \beta\right)\right).$$

Rochelle is also correct. Because $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$, we see that $\sin\left(\alpha + \left(\frac{\pi}{2} - \beta\right)\right) = \sin\left((\alpha - \beta) + \frac{\pi}{2}\right) = \cos(\alpha - \beta)$.

8. A frog is sitting on the edge of a playground carousel with radius 1 meter. The ray through the frog's position and the center of the carousel makes an angle of measure θ with the horizontal, and his starting coordinates are approximately $(0.81, 0.59)$. Find his new coordinates after the carousel rotates by each of the following amounts.

a. $\frac{\pi}{2}$

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin(\theta) = -0.59$$

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta) = 0.81$$

New position: $(-0.59, 0.81)$

b. π

$$\cos(\theta + \pi) = -\cos(\theta) = -0.81$$

$$\sin(\theta + \pi) = -\sin(\theta) = -0.59$$

New position: $(-0.81, -0.59)$

c. 2π

$$\cos(\theta + 2\pi) = \cos(\theta) = 0.81$$

$$\sin(\theta + 2\pi) = \sin(\theta) = 0.59$$

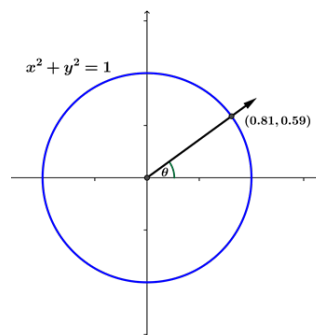
New position: $(0.81, 0.59)$

d. $-\frac{\pi}{2}$

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta) = 0.59$$

$$\sin\left(\theta - \frac{\pi}{2}\right) = -\cos(\theta) = -0.81$$

New position: $(0.59, -0.81)$



e. $-\pi$

$$\cos(\theta - \pi) = -\cos(\theta) = -0.81$$

$$\sin(\theta - \pi) = -\sin(\theta) = -0.59$$

New position: $(-0.81, -0.59)$

f. $\frac{\pi}{2} - \theta$

$$\cos\left(\theta + \left(\frac{\pi}{2} - \theta\right)\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\sin\left(\theta + \left(\frac{\pi}{2} - \theta\right)\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

New position: $(0, 1)$

g. $\pi - 2\theta$

$$\cos(\theta + (\pi - 2\theta)) = \cos(\pi - \theta) = -\cos(\theta) = -0.81$$

$$\sin(\theta + (\pi - 2\theta)) = \sin(\pi - \theta) = \sin(\theta) = 0.59$$

New position: $(-0.81, 0.59)$

h. -2θ

$$\cos(\theta - 2\theta) = \cos(-\theta) = \cos(\theta) = 0.81$$

$$\sin(\theta - 2\theta) = \sin(-\theta) = -\sin(\theta) = -0.59$$

New position: $(0.81, -0.59)$



Lesson 3: Addition and Subtraction Formulas

Student Outcomes

- Students prove the subtraction formula for cosine and use their understanding of the properties of the trigonometric functions to derive the remaining addition and subtraction formulas for the sine, cosine, and tangent functions.
- Students use the addition and subtraction formulas to evaluate trigonometric functions to solve problems.

Lesson Notes

In previous lessons, students have used the unit circle to examine the periodicity and symmetry of the sine, cosine, and tangent functions. They have also explored relationships between trigonometric functions and have derived formulas to evaluate the functions for various values of θ . In Algebra II Module 2 Lesson 17, the *teacher* led the students through a geometric proof that established the sum formula for sine, and the remaining sum and difference formulas followed. In this lesson, *students* use analytic methods to prove the subtraction formula for the cosine function and then extend the result to the remaining sum and difference formulas using their understanding of the periodicity and symmetry of the functions. Students apply the formulas to evaluate the trigonometric functions for specific values.

Classwork

Opening (3 minutes)

Remind students that they have used the periodicity and symmetry of trigonometric functions to evaluate the sine, cosine, and tangent functions for specific values of θ . Students should then respond to the following prompts. After sharing their thoughts with a partner, several students could discuss their suggestions.

- In the previous two lessons, we have been discussing the unit circle using a carousel model with a rider rotating counterclockwise on the outer edge of the carousel. Suppose the carousel rotates $\frac{\pi}{4}$ radians from its starting position and then stops. How can we determine the position of the rider when the carousel stops?
 - *The position is determined by $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$.*
- Instead, suppose that the carousel rotates $\frac{\pi}{3}$ radians from its starting position and then stops. How can we determine the position of the rider?
 - *The position is determined by $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{3}\right)$.*

- Now, suppose that the carousel rotates $\frac{7\pi}{12}$ radians from its starting position and then stops. How could we use what we have already discovered about the sine, cosine, and tangent functions to determine the position of the rider when the carousel stops? What additional information would we need to determine the position of the rider? Share your responses with a partner.
 - *Answers will vary but should address rewriting $\frac{7\pi}{12}$ as a sum or difference of values of θ for which we know the exact sine and cosine values (e.g., $\frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$). Additional information required might include determining a means to find the sine and cosine of a sum.*

Discussion (5 minutes)

This discussion addresses the fact that evaluating trigonometric functions for a sum of two values $\alpha + \beta$ is not equivalent to evaluating the trigonometric functions for each value α and β and then finding the sum. This discussion also address the utility of having a formula to compute the trigonometric functions of any sum or difference.

- One way to evaluate the position of the rider at $\theta = \frac{7\pi}{12}$ is to rewrite $\frac{7\pi}{12}$ as a sum of values for which we can calculate the exact positions, for example, $\frac{7\pi}{12} = \left(\frac{\pi}{4} + \frac{\pi}{3}\right)$. Perhaps we could find the position of our rider at $\theta = \frac{7\pi}{12}$ by adding the front/back positions and adding the right/left positions of the rider for $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{3}$.
- Determine whether this strategy is valid, and either justify why it is valid or provide a counterexample to demonstrate that it is not valid. Share your thoughts with a partner.
 - *Answers will vary, but students should determine that summing the position coordinates for $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{3}$ does not produce the position coordinates for $\theta = \frac{7\pi}{12}$. Counterexamples might include:*
We know that $\cos\left(\frac{7\pi}{12}\right) \neq \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{3}\right)$ because $\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{2}}{2} + \frac{1}{2}$, but $\cos\left(\frac{7\pi}{12}\right) < 0$ because rotation by $\frac{7\pi}{12}$ produces a terminal ray in Quadrant II where cosine is negative. We know that $\sin\left(\frac{7\pi}{12}\right) \neq \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)$ because $\sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}$, which exceeds the maximum possible value of 1 for the sine function.
- Now you may remember that in Algebra II we used geometry to establish the formulas for the sums and differences of sine and cosine. Here we are going to use a different approach to prove them.

Example 1 (7 minutes)

This example provides an analytic way to derive the formula for the cosine of a difference. The students then apply their understanding of the properties of the trigonometric functions to the cosine of a difference formula to derive the remaining addition and subtraction formulas and to solve problems. The example should be completed in a whole-class setting. Alternatively, students could complete it in pairs or small groups, and the results could be discussed in a whole-class setting.

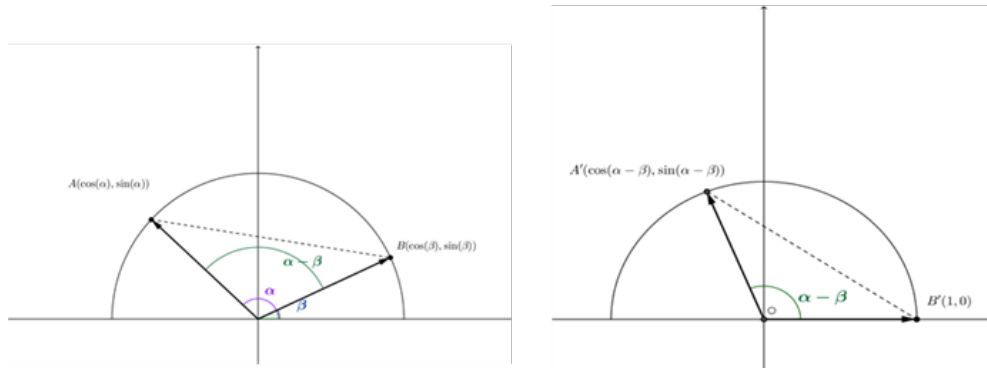
- What do the points A and B represent?
 - *Point A represents the position on the unit circle after the initial ray is rotated by the amount α , and point B represents the position on the unit circle after the initial ray is rotated by the amount β .*
- How can we find the distance between points A and B ?
 - *We can apply the distance formula.*
- Which gives us?
 - $AB = \sqrt{(\cos(\alpha) - \cos(\beta))^2 + (\sin(\alpha) - \sin(\beta))^2}$
- What is the expanded form of this expression?
 - $AB = \sqrt{\cos^2(\alpha) - 2\cos(\alpha)\cos(\beta) + \cos^2(\beta) + \sin^2(\alpha) - 2\sin(\alpha)\sin(\beta) + \sin^2(\beta)}$
- How can we simplify the expression under the radical?
 - *We can use the Pythagorean identity: $\sin^2(\alpha) + \cos^2(\alpha) = 1$ and $\sin^2(\beta) + \cos^2(\beta) = 1$. We can also factor -2 from the terms $-2\cos(\alpha)\cos(\beta) - 2\sin(\alpha)\sin(\beta)$, which results in the simplified expression $AB = \sqrt{2 - 2(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta))}$.*
- How does the procedure applied to part (b) compare to that in part (a)?
 - *The same procedure is applied but this time to the image points.*
- Why can we set AB equal to $A'B'$?
 - *$\overline{A'B'}$ is the image of \overline{AB} rotated by $-\beta$ about the origin, and distance is preserved under rotation, so the length of the segment is the same before and after rotation.*

Scaffolding:

- Provide a visual display of the distance formula.
- Show students the expressions for the lengths of \overline{AB} and $\overline{A'B'}$ and ask students what the different parts of the expression represent and where they come from in the diagram.
- Have advanced students derive the formula for $\cos(\beta - \alpha)$ using the same analytic method and explain why the formula is equivalent to that for $\cos(\alpha - \beta)$.

Example 1

Consider the figures below. The figure on the right is obtained from the figure on the left by rotating by $-\beta$ about the origin.



- a. Calculate the length of \overline{AB} in the figure on the left.

$$\begin{aligned} AB &= \sqrt{(\cos(\alpha) - \cos(\beta))^2 + (\sin(\alpha) - \sin(\beta))^2} \\ &= \sqrt{\cos^2(\alpha) - 2\cos(\alpha)\cos(\beta) + \cos^2(\beta) + \sin^2(\alpha) - 2\sin(\alpha)\sin(\beta) + \sin^2(\beta)} \\ &= \sqrt{\cos^2(\alpha) + \sin^2(\alpha) + \cos^2(\beta) + \sin^2(\beta) - 2\cos(\alpha)\cos(\beta) - 2\sin(\alpha)\sin(\beta)} \\ &= \sqrt{2 - 2(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta))} \end{aligned}$$

- b. Calculate the length of $\overline{A'B'}$ in the figure on the right.

$$\begin{aligned} A'B' &= \sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2} \\ &= \sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)} \\ &= \sqrt{\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta) + 1 - 2\cos(\alpha - \beta)} \\ &= \sqrt{2 - 2\cos(\alpha - \beta)} \end{aligned}$$

- c. Set AB and $A'B'$ equal to each other, and solve the equation for $\cos(\alpha - \beta)$.

$$\begin{aligned} \sqrt{2 - 2\cos(\alpha - \beta)} &= \sqrt{2 - 2(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta))} \\ (\sqrt{2 - 2\cos(\alpha - \beta)})^2 &= (\sqrt{2 - 2(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta))})^2 \\ 2 - 2\cos(\alpha - \beta) &= 2 - 2(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) \\ -2\cos(\alpha - \beta) &= -2(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) \\ \cos(\alpha - \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \end{aligned}$$

- Turn to your neighbor and explain how you know that AB and $A'B'$ are equal and what the formula you have just derived represents.
 - \overline{AB} and $\overline{A'B'}$ have the same length because they are images of each other under rotation, which preserves length. The formula just derived provides a way to calculate the cosine of a number for which the cosine is not readily known by expressing that number as a difference of two numbers for which the sine and cosine values are readily known.

Exercises 1–2 (7 minutes)

Students should complete the exercises in pairs. After an appropriate time, volunteers could display their solutions, and additional students could provide alternative solutions or offer counterarguments to refute a solution. Students establish the sum formula for sine in the Exit Ticket, but it is summarized here for reference.

Exercises 1–2

1. Use the fact that $\cos(-\theta) = \cos(\theta)$ to determine a formula for $\cos(\alpha + \beta)$.

$$\begin{aligned}
 \cos(\alpha + \beta) &= \cos(\alpha - \beta) \\
 &= \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) \\
 &= \cos(\alpha)\cos(\beta) + \sin(\alpha)(-\sin(\beta)) \\
 &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)
 \end{aligned}$$

2. Use the fact that $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$ to determine a formula for $\sin(\alpha - \beta)$.

$$\begin{aligned}
 \sin(\alpha - \beta) &= \cos\left(\frac{\pi}{2} - (\alpha - \beta)\right) \\
 &= \cos\left(\left(\frac{\pi}{2} - \alpha\right) + \beta\right) \\
 &= \cos\left(\frac{\pi}{2} - \alpha\right)\cos(\beta) - \sin\left(\frac{\pi}{2} - \alpha\right)\sin(\beta) \\
 &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)
 \end{aligned}$$

Scaffolding:

- Prompt struggling students for Exercise 1 by rewriting $\cos(\alpha + \beta)$ as $\cos(\alpha - (-\beta))$.
- Have advanced students determine the formulas without cueing.

Sum and Difference Formulas for the Sine and Cosine Functions

For all real numbers α and β ,

$$\begin{aligned}
 \cos(\alpha - \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\
 \cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\
 \sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\
 \sin(\alpha + \beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)
 \end{aligned}$$

Example 2 (5 minutes)

This example demonstrates how to apply the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ as the ratio of sine θ to cosine θ to derive the addition formula for tangent, which students use to evaluate the tangent function for exact values of θ . The example should be completed in pairs or small groups and then discussed in a whole-class setting.

- How can we use the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ to help us find an addition formula for tangent?
 - *Answers will vary but might indicate that we could rewrite $\tan(\alpha + \beta)$ as the quotient of $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, and we could apply the addition formulas for sine and cosine to determine a formula.*
- If we rewrite $\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$ and apply the addition formulas for sine and cosine, we find the formula $\tan(\alpha + \beta) = \frac{\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}$. How can we determine what to do next to try to find a more user-friendly expression for a tangent addition formula?
 - *Answers will vary but might include dividing by a common term so the expressions are written in terms of $\tan(\alpha)$ and $\tan(\beta)$. If students do not suggest this tactic, then suggest it to them.*
- What if we divide by $\cos(\alpha)\cos(\beta)$?
 - *Answers will vary but should address that the cosines in the numerators of each term in the expression will reduce with the cosines in $\cos(\alpha)\cos(\beta)$ so that each term can be written in terms of the tangent function or as 1.*
- Are there any restrictions on the values of α and β for $\tan(\alpha + \beta)$?
 - *Yes, α and β cannot sum to $\frac{\pi}{2} + \pi n$, where n is an integer because this would result in $\tan\left(\frac{\pi}{2} + \pi n\right)$, which we have previously determined to be undefined.*
- How can we verify this?
 - *Answers will vary. An example of an acceptable response is included. If $\alpha + \beta = \frac{\pi}{2}$, then $\beta = \frac{\pi}{2} - \alpha$ and $\tan(\alpha + \beta) = \frac{\sin(\alpha)\cos(\frac{\pi}{2} - \alpha) + \sin(\frac{\pi}{2} - \alpha)\cos(\alpha)}{\cos(\alpha)\cos(\frac{\pi}{2} - \alpha) - \sin(\alpha)\sin(\frac{\pi}{2} - \alpha)} = \frac{\sin(\alpha)\sin(\alpha) + \cos(\alpha)\cos(\alpha)}{\cos(\alpha)\sin(\alpha) - \sin(\alpha)\cos(\alpha)} = \frac{\sin^2(\alpha) + \cos^2(\alpha)}{0}$, which is undefined.*

Example 2

Use the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ to show that $\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$.

$$\begin{aligned}
 \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\
 &= \frac{\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} \\
 &= \left(\frac{\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} \right) \left(\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} \right) \\
 &= \frac{\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} + \frac{\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} - \frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\
 &= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}
 \end{aligned}$$

Exercises 3–5 (10 minutes)

Students should complete the exercises in pairs. After an appropriate time, volunteers could display their solutions. As time permits, students should be encouraged to show different approaches to finding the solutions.

Exercises 3–5

3. Verify the identity $\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$ for all $(\alpha - \beta) \neq \frac{\pi}{2} + \pi n$.

$$\begin{aligned}\tan(\alpha - \beta) &= \tan(\alpha + (-\beta)) \\ &= \frac{\tan(\alpha) + \tan(-\beta)}{1 - \tan(\alpha)\tan(-\beta)} \\ &= \frac{\tan(\alpha) + (-\tan(\beta))}{1 - \tan(\alpha)(-\tan(\beta))} \\ &= \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}\end{aligned}$$

4. Use the addition and subtraction formulas to evaluate the expressions shown.

a. $\cos\left(-\frac{5\pi}{12}\right)$

$$\begin{aligned}\cos\left(-\frac{5\pi}{12}\right) &= \cos\left(\frac{\pi}{4} - \frac{2\pi}{3}\right) \\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{2\pi}{3}\right) \\ &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

b. $\sin\left(\frac{23\pi}{12}\right)$

$$\begin{aligned}\sin\left(\frac{23\pi}{12}\right) &= \sin\left(\frac{9\pi}{4} - \frac{\pi}{3}\right) \\ &= \sin\left(\frac{9\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{9\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) - \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

Scaffolding:

Have advanced students verify using the relationship between the sine and cosine functions that

$$\cos\left(-\frac{5\pi}{12}\right) = -\sin\left(\frac{23\pi}{12}\right).$$

MP.3

c. $\tan\left(\frac{5\pi}{12}\right)$

$$\begin{aligned}\tan\left(\frac{5\pi}{12}\right) &= \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) \\ &= \frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{4}\right)} \\ &= \frac{\frac{\sqrt{3}}{3} + 1}{1 - \left(\frac{\sqrt{3}}{3}\right)(1)} \\ &= \frac{\sqrt{3} + 3}{3 - \sqrt{3}} \\ &= \frac{12 + 6\sqrt{3}}{6} \\ &= 2 + \sqrt{3}\end{aligned}$$

5. Use the addition and subtraction formulas to verify these identities for all real-number values of θ .

a. $\sin(\pi - \theta) = \sin(\theta)$

$$\sin(\pi - \theta) = \sin(\pi)\cos(\theta) - \sin(\theta)\cos(\pi) = 0(\cos(\theta)) - \sin(\theta)(-1) = \sin(\theta)$$

b. $\cos(\pi + \theta) = -\cos(\theta)$

$$\cos(\pi + \theta) = \cos(\pi)\cos(\theta) - \sin(\pi)\sin(\theta) = -1(\cos(\theta)) - 0(\sin(\theta)) = -\cos(\theta)$$

Sum and Difference Formulas for the Tangent Function

For all real numbers α and β for which the expressions are defined,

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)} \quad \text{and} \quad \tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}.$$

Closing (3 minutes)

Have students write a response to the prompts below and share their responses with a partner.

- What are the sum and difference formulas for sine and cosine?
 - $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
 - $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
 - $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
 - $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

- How can the addition and subtraction formulas be used to find the position of the rider from the beginning of the lesson?
 - To find the front/back position of the rider, we need to find $\sin\left(\frac{7\pi}{12}\right)$, which we could rewrite as the difference of two numbers, for example, $\sin\left(\frac{3\pi}{4} - \frac{\pi}{6}\right)$, and then apply the formula $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$ to find the position.
 - To find the right/left position of the rider, we need to find $\cos\left(\frac{7\pi}{12}\right)$, which we could rewrite as the difference of two numbers, for example, $\cos\left(\frac{3\pi}{4} - \frac{\pi}{6}\right)$, and then apply the formula $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ to find the position.

Lesson Summary

The sum and difference formulas for sine, cosine, and tangent are summarized below.

For all real numbers α and β for which the expressions are defined,

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 3: Addition and Subtraction Formulas

Exit Ticket

1. Prove that $\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \sin(\beta) \cos(\alpha)$.
2. Use the addition and subtraction formulas to evaluate the given trigonometric expressions.
 - a. $\sin\left(\frac{\pi}{12}\right)$
 - b. $\tan\left(\frac{13\pi}{12}\right)$

Exit Ticket Sample Solutions

1. Prove that $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$.

$$\begin{aligned}\sin(\alpha + \beta) &= \sin(\alpha - (-\beta)) \\ &= \sin(\alpha)\cos(-\beta) - \sin(-\beta)\cos(\alpha) \\ &= \sin(\alpha)\cos(\beta) - (-\sin(\beta))(\cos(\alpha)) \\ &= \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)\end{aligned}$$

2. Use the addition and subtraction formulas to evaluate the given trigonometric expressions.

a. $\sin\left(\frac{\pi}{12}\right)$

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) \\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

b. $\tan\left(\frac{13\pi}{12}\right)$

$$\begin{aligned}\tan\left(\frac{13\pi}{12}\right) &= \tan\left(\frac{3\pi}{4} + \frac{\pi}{3}\right) \\ &= \frac{\tan\left(\frac{3\pi}{4}\right) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan\left(\frac{3\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)} \\ &= \frac{-1 + \sqrt{3}}{1 - (-1)(\sqrt{3})} \\ &= \frac{-1 + \sqrt{3}}{1 + (\sqrt{3})} \\ &= \frac{-4 + 2\sqrt{3}}{-2} \\ &= 2 - \sqrt{3}\end{aligned}$$

Problem Set Sample Solutions

1. Use the addition and subtraction formulas to evaluate the given trigonometric expressions.

a. $\cos\left(\frac{\pi}{12}\right)$

$$\begin{aligned}\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

b. $\sin\left(\frac{\pi}{12}\right)$

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

c. $\sin\left(\frac{5\pi}{12}\right)$

$$\begin{aligned}\sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) \\ &= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

d. $\cos\left(-\frac{\pi}{12}\right)$

$$\begin{aligned}\cos\left(-\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) \\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) \\ &= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

e. $\sin\left(\frac{7\pi}{12}\right)$

$$\begin{aligned}\sin\left(\frac{7\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) \\ &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) \\ &= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

f. $\cos\left(-\frac{7\pi}{12}\right)$

$$\begin{aligned}\cos\left(-\frac{7\pi}{12}\right) &= \cos\left(-\frac{\pi}{4} - \frac{\pi}{3}\right) \\ &= \cos\left(-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(-\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) \\ &= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

g. $\sin\left(\frac{13\pi}{12}\right)$

$$\begin{aligned}\sin\left(\frac{13\pi}{12}\right) &= \sin\left(\frac{3\pi}{4} + \frac{\pi}{3}\right) \\ &= \sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) \\ &= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

h. $\cos\left(-\frac{13\pi}{12}\right)$

$$\begin{aligned}\cos\left(-\frac{13\pi}{12}\right) &= \cos\left(-\frac{\pi}{4} - \frac{\pi}{3}\right) \\ &= \cos\left(-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(-\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) \\ &= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

i. $\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)$

$$\begin{aligned}\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{6}\right) \\ &= \frac{1}{2}\end{aligned}$$

j. $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{6}\right)$

$$\begin{aligned}\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{6}\right) &= \sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

k. $\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right)$

$$\begin{aligned}\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right) &= \sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

l. $\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) - \sin\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right)$

$$\begin{aligned}\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) - \sin\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right) &= \cos\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

$$\begin{aligned} \text{m. } & \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{12}\right) \\ & \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{6}\right) \\ & = \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \text{n. } & \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{12}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{12}\right) \\ & \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{12}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{4}\right) \\ & = \frac{\sqrt{2}}{2} \end{aligned}$$

2. Figure 2 is obtained from Figure 1 by rotating the angle by α about the origin. Use the method shown in Example 1 to show that $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.

Figure 1

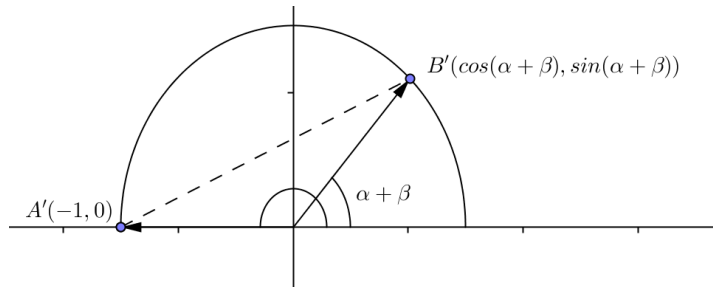
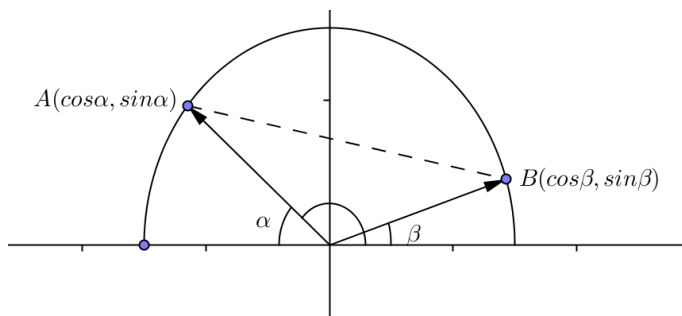


Figure 2



$$AB = \sqrt{(\cos(\beta) - \cos(\alpha))^2 + (\sin(\beta) - \sin(\alpha))^2}$$

$$A'B' = \sqrt{(\cos(\alpha + \beta) - (-1))^2 + (\sin(\alpha + \beta) - 0)^2}$$

$$AB = A'B'$$

$$\left(\sqrt{(\cos(\beta) - \cos(\alpha))^2 + (\sin(\beta) - \sin(\alpha))^2}\right)^2 = \left(\sqrt{(\cos(\alpha + \beta) - (-1))^2 + (\sin(\alpha + \beta) - 0)^2}\right)^2$$

$$\begin{aligned} (\cos(\beta) - \cos(\alpha))^2 + (\sin(\beta) - \sin(\alpha))^2 &= (\cos(\alpha + \beta) - (-1))^2 + (\sin(\alpha + \beta) - 0)^2 \\ \cos^2(\beta) - 2\cos(\beta)\cos(\alpha) + \cos^2(\alpha) + \sin^2(\beta) - 2\sin(\beta)\sin(\alpha) + \sin^2(\alpha) &= \cos^2(\alpha + \beta) + 2\cos(\alpha + \beta) + 1 + \sin^2(\alpha + \beta) \end{aligned}$$

$$2 - 2\cos(\beta)\cos(\alpha) - 2\sin(\beta)\sin(\alpha) = 2 + 2\cos(\alpha + \beta)$$

$$-\cos(\beta)\cos(\alpha) - \sin(\beta)\sin(\alpha) = \cos(\alpha + \beta)$$

$$\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \cos(\alpha + \beta)$$

3. Use the sum formula for sine to show that $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$.

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta)) = \sin(\alpha)\cos(-\beta) + \cos(\alpha)\sin(-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

4. Evaluate $\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$ to show $\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$. Use the resulting formula to show that $\tan(2\alpha) = \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)}$.

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}$$

Dividing the numerator and denominator by $\cos(\alpha)\cos(\beta)$ gives

$$\tan(\alpha + \beta) = \frac{\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} + \frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} - \frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

It then follows that

$$\tan(2\alpha) = \tan(\alpha + \alpha) = \frac{\tan(\alpha) + \tan(\alpha)}{1 - \tan(\alpha)\tan(\alpha)} = \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)}$$

5. Show $\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$.

$$\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}$$

Dividing the numerator and denominator by $\cos(\alpha)\cos(\beta)$ gives

$$\tan(\alpha - \beta) = \frac{\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} - \frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} + \frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

6. Find the exact value of the following by using addition and subtraction formulas.

a. $\tan\left(\frac{\pi}{12}\right)$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}} = 2 - \sqrt{3}$$

b. $\tan\left(-\frac{\pi}{12}\right)$

$$\tan\left(-\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{3}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}} = -2 + \sqrt{3}$$

c. $\tan\left(\frac{7\pi}{12}\right)$

$$\tan\left(\frac{7\pi}{12}\right) = \tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)} = \frac{1 + \sqrt{3}}{1 - \sqrt{3}} = -2 - \sqrt{3}$$

d. $\tan\left(-\frac{13\pi}{12}\right)$

$$\tan\left(-\frac{13\pi}{12}\right) = \tan\left(\frac{\pi}{4} - \frac{4\pi}{3}\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{4\pi}{3}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{4\pi}{3}\right)} = \frac{1 - \sqrt{3}}{1 + \sqrt{3}} = -2 + \sqrt{3}$$

e. $\frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{12}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{12}\right)}$

$$\frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{12}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{12}\right)} = \tan\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = \tan\left(\frac{4\pi}{12}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

f. $\frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{12}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{12}\right)}$

$$\frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{12}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{12}\right)} = \tan\left(\frac{\pi}{3} - \frac{\pi}{12}\right) = \tan\left(\frac{3\pi}{12}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

g. $\frac{\tan\left(\frac{\pi}{12}\right) + \tan\left(\frac{\pi}{12}\right)}{1 - \tan\left(\frac{\pi}{12}\right)\tan\left(\frac{\pi}{12}\right)}$

$$\frac{\tan\left(\frac{\pi}{12}\right) + \tan\left(\frac{\pi}{12}\right)}{1 - \tan\left(\frac{\pi}{12}\right)\tan\left(\frac{\pi}{12}\right)} = \tan\left(\frac{\pi}{12} + \frac{\pi}{12}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$



Lesson 4: Addition and Subtraction Formulas

Student Outcomes

- Students use the addition formulas to derive the double-angle formulas for sine, cosine, and tangent and evaluate trigonometric expressions for specific values of θ .
- Students use the double-angle formulas to derive the half-angle formulas for sine, cosine, and tangent and evaluate trigonometric expressions for specific values of θ .

Lesson Notes

In the previous lesson, students applied an analytic method to prove the subtraction formula for cosine. They then derived the remaining addition and subtraction formulas using their understanding of the periodicity and symmetry of the sine, cosine, and tangent functions. In this lesson, students continue to apply their understanding of the properties of the trigonometric functions to determine double and half-angle formulas for sine, cosine, and tangent, which they apply to evaluate trigonometric functions for specific input values.

Classwork

Opening (5 minutes)

- Let's return one more time to our carousel model of the unit circle. We know that a rider's position after a rotation θ is (x_θ, y_θ) , where $x_\theta = \cos(\theta)$ and $y_\theta = \sin(\theta)$. In the last lesson, we developed the sum and difference formulas for sine and cosine; that is, if we know $\sin(\alpha)$, $\sin(\beta)$, $\cos(\alpha)$, and $\cos(\beta)$, then we can find the values of $\sin(\alpha + \beta)$, $\sin(\alpha - \beta)$, $\cos(\alpha + \beta)$, and $\cos(\alpha - \beta)$.

Students should reflect on the following prompts. After a minute, they should share their reflections with a partner. Select several students to share their ideas; ask them to explain a strategy or display a procedure on the board.

- We know the position of the rider after rotating by $\theta = \frac{\pi}{3}$. How can we use the addition formulas to find her position after rotating by $\frac{2\pi}{3}$?
 - Answers will vary but might include using the formulas we have derived in previous lessons; for examples, we could rewrite $\sin(2\theta)$ as $\sin(\theta + \theta)$ and apply the formula for the sine of a sum to find the position of a rider after a rotation of 2θ .*
- We know the position of the rider after rotating by $\theta = \pi/4$. How can we determine his position after rotating by $\frac{\pi}{8}$?
 - Answers will vary but might include applying the angle sum formulas in reverse after writing $\sin(2\theta)$ as $\sin(\theta + \theta)$, using this particular angle measure, $\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{8} + \frac{\pi}{8}\right)$.*

MP.3
&
MP.7

Exercise 1 (7 minutes)

In this exercise, students derive the double-angle formula for sine using the addition formula. Allow students to struggle with this task before explaining that they apply the sum formulas with $2\theta = \theta + \theta$. This provides an opportunity for them to further develop their mathematical reasoning skills and to practice looking for structure. Students later apply the double-angle formulas to determine the half-angle formulas, which allows them to evaluate trigonometric functions for a larger number of input values.

Scaffolding:

Encourage advanced students to think about ways to write these formulas in terms of sine or cosine only.

The exercise should be completed in pairs and then discussed in a whole-class setting after a few minutes. Be sure to identify these formulas as the double-angle formulas for sine and cosine.

Exercises**1. Derive formulas for the following:****a. $\sin(2\theta)$**

$$\begin{aligned}\sin(2\theta) &= \sin(\theta + \theta) \\ &= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) \\ &= 2\sin(\theta)\cos(\theta)\end{aligned}$$

b. $\cos(2\theta)$

$$\begin{aligned}\cos(2\theta) &= \cos(\theta + \theta) \\ &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) \\ &= \cos^2(\theta) - \sin^2(\theta)\end{aligned}$$

MP.7

- Why was it helpful to rewrite 2θ as $(\theta + \theta)$ when deriving the double-angle formula?
 - While we don't know how to find the value of $\sin(2\theta)$ directly, we do know how to evaluate the sine of the sum $(\theta + \theta)$ using the formula $\sin(\theta + \theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) = 2\sin(\theta)\cos(\theta)$.
- Our formula for calculating $\cos(2\theta)$ is written in terms of $\cos^2(\theta)$ and $\sin^2(\theta)$. Is there a way to write this formula in terms of the cosine function only? Explain.
 - Yes. Since $\cos^2(\theta) + \sin^2(\theta) = 1$, we can substitute $\sin^2(\theta) = 1 - \cos^2(\theta)$. Then $\cos^2(\theta) - \sin^2(\theta) = \cos^2(\theta) - (1 - \cos^2(\theta)) = 2\cos^2(\theta) - 1$.
- Can we write $\cos(2\theta)$ in terms of the sine function only? Explain.
 - Yes. Since $\cos^2(\theta) + \sin^2(\theta) = 1$, we can substitute $\cos^2(\theta) = 1 - \sin^2(\theta)$. Then $\cos^2(\theta) - \sin^2(\theta) = (1 - \sin^2(\theta)) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$.

Exercises 2–3 (10 minutes)

Students should complete the exercises in pairs or small groups. Each pair or small group should verify one of the identities in Exercise 1. After a few minutes, volunteers could display their solutions. Other students should be allowed to offer alternative approaches or critiques. Exercise 2 could be completed in pairs or as part of a teacher-led discussion. The discussion of Exercise 2 could include a verification, using the unit circle, of the signs on the coordinates of the rider's position.

MP.3

2. Use the double-angle formulas for sine and cosine to verify these identities:

a. $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

$$\begin{aligned}\tan(2\theta) &= \frac{\sin(2\theta)}{\cos(2\theta)} \\&= \left(\frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta) - \sin^2(\theta)} \right) \left(\frac{\frac{1}{\cos^2(\theta)}}{\frac{1}{\cos^2(\theta)}} \right) \\&= \frac{\frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta)}}{\frac{\cos^2(\theta) - \sin^2(\theta)}{\cos^2(\theta)}} \\&= \frac{\frac{2\sin(\theta)\cos(\theta)}{\cos(\theta)\cos(\theta)}}{\frac{\cos^2(\theta)}{\cos^2(\theta)} - \frac{\sin^2(\theta)}{\cos^2(\theta)}} \\&= \frac{2\tan(\theta)}{1 - \tan^2(\theta)}\end{aligned}$$

b. $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$

$$\frac{1 - \cos(2\theta)}{2} = \frac{1 - (1 - 2\sin^2(\theta))}{2} = \frac{2\sin^2(\theta)}{2} = \sin^2(\theta)$$

c. $\sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta)$

$$\begin{aligned}\sin(3\theta) &= \sin(2\theta + \theta) \\ &= \sin(2\theta)\cos(\theta) + \cos(2\theta)\sin(\theta) \\ &= 2\sin(\theta)(\cos(\theta))(\cos(\theta)) + (1 - 2\sin^2(\theta))(\sin(\theta)) \\ &= 2\sin(\theta)\cos^2(\theta) + \sin(\theta) - 2\sin^3(\theta) \\ &= 2\sin(\theta)(1 - \sin^2(\theta)) + \sin(\theta) - 2\sin^3(\theta) \\ &= 2\sin(\theta) - 2\sin^3(\theta) + \sin(\theta) - 2\sin^3(\theta) \\ &= -4\sin^3(\theta) + 3\sin(\theta)\end{aligned}$$

3. Suppose that the position of a rider on the unit circle carousel is $(0.8, -0.6)$ for a rotation θ . What is the position of the rider after rotation by 2θ ?

$$x_{2\theta} = \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 0.8^2 - (-0.6)^2 = 0.64 - 0.36 = 0.28$$

$$y_{2\theta} = \sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2(-0.6)(0.8) = -0.96$$

The rider's position is $(0.28, -0.96)$.

Scaffolding:

- Point students to a visual representation of the definition of the tangent as $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.
- Cue students to look at the structure of the right side of the identity in Exercise 2 part (a) to help them simplify the expression on the left side. For instance, ask the students, “What term could we divide our expression by so the first term in the denominator reduces to 1?”
- Cue students to use a procedure similar to that used in Exercise 1 to verify the identity in Exercise 2 part (c).

MP.2

Discussion (5 minutes)

In this example, students use the double-angle formula for cosine to derive the half-angle formula for cosine. The half-angle formula provides students with an efficient means of evaluating trigonometric functions for a wider variety of inputs. The example should be completed as part of a teacher-led discussion.

- What are some of the limitations of the formulas we have derived so far in evaluating trigonometric expressions for specific inputs?
 - *Since we only determined the exact values of trigonometric functions for multiples of $\frac{\pi}{2}$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$, we are limited in the values of θ for which we can evaluate trigonometric expressions. In other words, we can only evaluate $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for values of θ that can be found by adding or subtracting multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$.*
- In the Opening, we thought about how we could find $\sin\left(\frac{\pi}{8}\right)$ and $\cos\left(\frac{\pi}{8}\right)$ since we know the values of sine and cosine at $\frac{\pi}{4}$. Do we have a way to evaluate sine and cosine at $\frac{\pi}{8}$?
 - *No, we do not yet have a way to evaluate sine and cosine at $\frac{\pi}{8}$.*
- Why might it be helpful to derive half-angle formulas for sine, cosine, and tangent?
 - *Answers will vary but should address that we could evaluate the trigonometric functions for more values of θ .*
- Recall that the double-angle formula for cosine is $\cos(2\theta) = 2\cos^2(\theta) - 1$. How can we use this formula to find a value for $\cos\left(\frac{\pi}{8}\right)$?
 - *Allow students to work with a partner and struggle with this question for a minute or two before revealing the answer.*
We can use the double-angle formula with $\cos\left(\frac{\pi}{4}\right) = \cos\left(2 \cdot \frac{\pi}{8}\right)$, which gives

$$\cos\left(\frac{\pi}{4}\right) = \cos\left(2 \cdot \frac{\pi}{8}\right) = 2\cos^2\left(\frac{\pi}{8}\right) - 1. \text{ Then we can solve } \frac{\sqrt{2}}{2} = 2\cos^2\left(\frac{\pi}{8}\right) - 1 \text{ for } \cos\left(\frac{\pi}{8}\right).$$
- This gives $\cos^2\left(\frac{\pi}{8}\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2} + 1\right)$. How do we know whether or not we need the positive or negative square root when we solve for $\cos\left(\frac{\pi}{8}\right)$?
 - *It will depend on the quadrant in which the initial ray lands after rotating by $\frac{\pi}{8}$. Cosine is positive in Quadrants I and IV and negative in Quadrants II and III. Since rotation by $\frac{\pi}{8}$ terminates in Quadrant I, the value of $\cos\left(\frac{\pi}{8}\right)$ is positive.*

Exercises 4–7 (10 minutes)

Students should complete the exercises in pairs, first working on the problems independently and then verifying their solutions with a partner. At an appropriate time, selected students should explain their solutions, including how they determined the signs for the position coordinates in Exercise 6 and the signs for the evaluated trigonometric functions in Exercise 7. As time permits, additional students should be encouraged to share alternative approaches to solving the problems; for example, there could be multiple ways to evaluate $\cos\left(\frac{\pi}{12}\right)$ in Exercise 7 part (b). Note: if calculators are available, students could use them to verify their solutions and approximate the values in Exercises 6 and 7. If they are not available, answers could be left in their exact form.

4. Use the double-angle formula for cosine to establish the identity $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta)+1}{2}}$.

Since $\theta = 2\left(\frac{\theta}{2}\right)$, the double-angle formula gives $\cos\left(2\left(\frac{\theta}{2}\right)\right) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$. Then we have

$$\begin{aligned}\cos(\theta) &= 2\cos^2\left(\frac{\theta}{2}\right) - 1 \\ 1 + \cos(\theta) &= 2\cos^2\left(\frac{\theta}{2}\right) \\ \frac{1 + \cos(\theta)}{2} &= \cos^2\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{\cos(\theta) + 1}{2}}\end{aligned}$$

5. Use the double-angle formulas to verify these identities:

a. $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$

Since $\theta = 2\left(\frac{\theta}{2}\right)$, the double-angle formulas give $\cos\left(2\left(\frac{\theta}{2}\right)\right) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$.

Then we have

$$\begin{aligned}\cos(\theta) &= 1 - 2\sin^2\left(\frac{\theta}{2}\right) \\ 1 - \cos(\theta) &= 2\sin^2\left(\frac{\theta}{2}\right) \\ \frac{1 - \cos(\theta)}{2} &= \sin^2\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1 - \cos(\theta)}{2}}\end{aligned}$$

b. $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \frac{\pm\sqrt{\frac{1 - \cos(\theta)}{2}}}{\pm\sqrt{\frac{\cos(\theta) + 1}{2}}} = \pm\sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$$

Scaffolding:

- Have advanced students derive a formula for the one-third angle for sine instead of verifying the identities in Exercise 5.
- Cue students in Exercise 7 by asking them, “Is there a way to use the addition or subtraction formulas to evaluate $\cos\left(\frac{\pi}{12}\right)$?”

6. The position of a rider on the unit circle carousel is $(0.8, -0.6)$ after a rotation by θ where $0 \leq \theta < 2\pi$. What is the position of the rider after rotation by $\frac{\theta}{2}$?

Given that $\cos(\theta)$ is positive and $\sin(\theta)$ is negative, the rider is located in Quadrant IV after rotation by θ , so $\frac{3\pi}{2} < \theta < 2\pi$. This means that $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$, which is in Quadrant II, so $\cos\left(\frac{\theta}{2}\right)$ is negative and $\sin\left(\frac{\theta}{2}\right)$ is positive.

$$x_{\frac{\theta}{2}} = \cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{\cos(\theta) + 1}{2}} = -\sqrt{\frac{0.8 + 1}{2}} \approx -0.95$$

$$y_{\frac{\theta}{2}} = \sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos(\theta)}{2}} = \sqrt{\frac{1 - 0.8}{2}} \approx 0.32$$

The rider's position is approximately $(-0.95, 0.32)$ after rotation by $\frac{\theta}{2}$.

7. Evaluate the following trigonometric expressions.

a. $\sin\left(\frac{3\pi}{8}\right)$

$$\sin\left(\frac{3\pi}{8}\right) = \sin\left(\frac{\frac{3\pi}{4}}{2}\right) = \sqrt{\frac{1 - \cos\left(\frac{3\pi}{4}\right)}{2}} = \sqrt{\frac{1 - \left(-\cos\left(\frac{\pi}{4}\right)\right)}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} \approx 0.92$$

b. $\tan\left(\frac{\pi}{24}\right)$

$$\tan\left(\frac{\pi}{24}\right) = \tan\left(\frac{\frac{\pi}{12}}{2}\right) = \sqrt{\frac{1 - \cos\left(\frac{\pi}{12}\right)}{1 + \cos\left(\frac{\pi}{12}\right)}}$$

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\tan\left(\frac{\pi}{24}\right) = \sqrt{\frac{1 - \frac{\sqrt{6} + \sqrt{2}}{4}}{1 + \frac{\sqrt{6} + \sqrt{2}}{4}}} = \sqrt{\frac{4 - \sqrt{6} - \sqrt{2}}{4 + \sqrt{6} + \sqrt{2}}} \approx 0.13$$

Closing (3 minutes)

Have students respond in writing to one or more of the prompts below:

- Write the double-angle formulas that we studied in this lesson for $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.
- Write the half-angle formulas that we studied in this lesson for $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$.
- How can our understanding of the trigonometric functions help us determine the position coordinates for a carousel rider on our unit circle model regardless of the size of the rotation θ ? Share your thoughts with a partner.

Answers will vary. Possible acceptable responses could include the following:

- We have found the exact position coordinates for rotations $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$.
- Our understanding of the unit circle allows us to determine the position coordinates for rotations that are multiples of $\frac{\pi}{2}$.
- The double- and triple-angle formulas enable us to find position coordinates for rotations that are multiples of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$.
- The half-angle and addition and subtraction formulas enable us to find position coordinates for many other rotational values.

Lesson Summary

The double-angle and half-angle formulas for sine, cosine, and tangent are summarized below.

For all real numbers θ for which the expressions are defined,

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2\cos^2(\theta) - 1 \\ &= 1 - 2\sin^2(\theta)\end{aligned}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{\cos(\theta) + 1}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$$

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 4: Addition and Subtraction Formulas

Exit Ticket

1. Show that $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$.
2. Evaluate $\cos\left(\frac{7\pi}{12}\right)$ using the half-angle formula, and then verify your solution using a different formula.

Exit Ticket Sample Solutions

1. Show that $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$.

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta + \theta) = \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\ &= (2\cos^2(\theta) - 1)(\cos(\theta)) - 2\sin(\theta)\cos(\theta)(\sin(\theta)) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2(1 - \cos^2(\theta))(\cos(\theta)) \\ &= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta) + 2\cos^3(\theta) \\ &= 4\cos^3(\theta) - 3\cos(\theta)\end{aligned}$$

2. Evaluate $\cos\left(\frac{7\pi}{12}\right)$ using the half-angle formula, and then verify your solution using a different formula.

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\frac{7\pi}{6}}{2}\right) = -\sqrt{\frac{\cos\left(\frac{7\pi}{6}\right) + 1}{2}} = -\sqrt{\frac{\frac{-\sqrt{3}}{2} + 1}{2}} \approx -0.26$$

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{2} - \sqrt{6}}{4} \approx -0.26$$

Problem Set Sample Solutions

1. Evaluate the following trigonometric expressions.

a. $2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

b. $\frac{1}{2}\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$

$$\frac{1}{4}\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{1}{8}$$

c. $4\sin\left(-\frac{5\pi}{12}\right)\cos\left(-\frac{5\pi}{12}\right)$

$$2\sin\left(-\frac{5\pi}{6}\right) = -2\sin\left(\frac{\pi}{6}\right) = -2\left(\frac{1}{2}\right) = -1$$

d. $\cos^2\left(\frac{3\pi}{8}\right) - \sin^2\left(\frac{3\pi}{8}\right)$

$$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

e. $2\cos^2\left(\frac{\pi}{12}\right) - 1$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

f. $1 - 2\sin^2\left(-\frac{\pi}{8}\right)$

$$\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

g. $\cos^2\left(-\frac{11\pi}{12}\right) - 2$

$$\frac{1}{2}\left(2\cos^2\left(-\frac{11\pi}{12}\right) - 1\right) - \frac{3}{2} = \frac{1}{2}\left(\cos\left(-\frac{11\pi}{6}\right)\right) - \frac{3}{2} = \frac{1}{2}\left(\cos\left(\frac{\pi}{6}\right)\right) - \frac{3}{2} = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{3}{2} = \frac{\sqrt{3}}{4} - \frac{3}{2}$$

h. $\frac{2\tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)}$

$$\tan\left(\frac{\pi}{4}\right) = 1$$

i. $\frac{2\tan\left(-\frac{5\pi}{12}\right)}{1 - \tan^2\left(-\frac{5\pi}{12}\right)}$

$$\tan\left(-\frac{5\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

j. $\cos^2\left(\frac{\pi}{8}\right)$

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{1 + \cos\left(\frac{\pi}{4}\right)}{2} = \frac{1 + \frac{\sqrt{2}}{2}}{2} = \frac{1}{2} + \frac{\sqrt{2}}{4}$$

k. $\cos\left(\frac{\pi}{8}\right)$

Rotation by $\theta = \frac{\pi}{8}$ terminates in Quadrant I; therefore, $\cos\left(\frac{\pi}{8}\right)$ has a positive value.

$$\cos\left(\frac{\pi}{8}\right) = \sqrt{\frac{1 + \cos\left(\frac{\pi}{4}\right)}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{2}}{2}}$$

l. $\cos\left(-\frac{9\pi}{8}\right)$

Rotation by $\theta = -\frac{9\pi}{8}$ terminates in Quadrant II; therefore, $\cos\left(-\frac{9\pi}{8}\right)$ has a negative value.

$$\cos\left(-\frac{9\pi}{8}\right) = -\sqrt{\frac{1 + \cos\left(-\frac{9\pi}{4}\right)}{2}} = -\sqrt{\frac{1 + \cos\left(\frac{\pi}{4}\right)}{2}} = -\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2 + \sqrt{2}}{2}}$$

m. $\sin^2\left(\frac{\pi}{12}\right)$

$$\sin^2\left(\frac{\pi}{12}\right) = \frac{1 - \cos\left(\frac{\pi}{6}\right)}{2} = \frac{1 - \frac{\sqrt{3}}{2}}{2} = \frac{1}{2} - \frac{\sqrt{3}}{4}$$

n. $\sin\left(\frac{\pi}{12}\right)$

Rotation by $\theta = \frac{\pi}{12}$ terminates in Quadrant I; therefore, $\sin\left(\frac{\pi}{12}\right)$ has a positive value.

$$\sin\left(\frac{\pi}{12}\right) = \sqrt{\frac{1 - \cos\left(\frac{\pi}{6}\right)}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 - \sqrt{3}}{4}} = \frac{\sqrt{2 - \sqrt{3}}}{2}$$

o. $\sin\left(-\frac{5\pi}{12}\right)$

Rotation by $\theta = -\frac{5\pi}{12}$ terminates in Quadrant IV; therefore, $\sin\left(-\frac{5\pi}{12}\right)$ has a negative value.

$$\sin\left(-\frac{5\pi}{12}\right) = -\sqrt{\frac{1 - \cos\left(-\frac{5\pi}{6}\right)}{2}} = -\sqrt{\frac{1 - \cos\left(\frac{\pi}{6}\right)}{2}} = -\sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = -\frac{\sqrt{2 - \sqrt{3}}}{2}$$

p. $\tan\left(\frac{\pi}{8}\right)$

Rotation by $\theta = \frac{\pi}{8}$ terminates in Quadrant I; therefore, $\tan\left(\frac{\pi}{8}\right)$ has a positive value.

$$\tan\left(\frac{\pi}{8}\right) = \sqrt{\frac{1 - \cos\left(\frac{\pi}{4}\right)}{1 + \cos\left(\frac{\pi}{4}\right)}} = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}}} = \sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}} = \sqrt{3 - 2\sqrt{2}} \approx 0.414$$

$$\tan\left(\frac{\pi}{8}\right) = \tan\left(\frac{\frac{\pi}{4}}{2}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{1 + \cos\left(\frac{\pi}{4}\right)} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2 + \sqrt{2}} \approx 0.414$$

$$\tan\left(\frac{\pi}{8}\right) = \tan\left(\frac{\frac{\pi}{4}}{2}\right) = \frac{1 - \cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = \frac{1 - \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2 - \sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1 \approx 0.414$$

q. $\tan\left(\frac{\pi}{12}\right)$

$$\tan\left(\frac{\pi}{12}\right) = \sqrt{\frac{1 - \cos\left(\frac{\pi}{6}\right)}{1 + \cos\left(\frac{\pi}{6}\right)}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}}} = \sqrt{\frac{2 - \sqrt{3}}{2 + \sqrt{3}}} = \sqrt{7 - 4\sqrt{3}} \approx 0.268$$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\frac{\pi}{6}}{2}\right) = \frac{\sin\left(\frac{\pi}{6}\right)}{1 + \cos\left(\frac{\pi}{6}\right)} = \frac{\frac{1}{2}}{1 + \frac{\sqrt{3}}{2}} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3} \approx 0.268$$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\frac{\pi}{6}}{2}\right) = \frac{1 - \cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3} \approx 0.268$$

r. $\tan\left(-\frac{3\pi}{8}\right)$

Rotation by $\theta = -\frac{3\pi}{8}$ terminates in Quadrant IV; therefore, $\tan\left(-\frac{3\pi}{8}\right)$ has a negative value.

$$\tan\left(-\frac{3\pi}{8}\right) = -\frac{1 - \cos\left(-\frac{3\pi}{4}\right)}{1 + \cos\left(-\frac{3\pi}{4}\right)} = -\frac{1 + \cos\left(\frac{\pi}{4}\right)}{1 - \cos\left(\frac{\pi}{4}\right)} = -\frac{1 + \frac{\sqrt{2}}{2}}{1 - \frac{\sqrt{2}}{2}} = -\frac{2 + \sqrt{2}}{2 - \sqrt{2}} = -\sqrt{3 + 2\sqrt{2}} \approx -2.414$$

$$\tan\left(-\frac{3\pi}{8}\right) = \tan\left(\frac{-\frac{3\pi}{4}}{2}\right) = \frac{\sin\left(-\frac{3\pi}{4}\right)}{1 + \cos\left(-\frac{3\pi}{4}\right)} = \frac{-\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{-\sqrt{2}}{2 + \sqrt{2}} \approx -2.414$$

$$\tan\left(-\frac{3\pi}{8}\right) = \tan\left(\frac{-\frac{3\pi}{4}}{2}\right) = \frac{1 - \cos\left(-\frac{3\pi}{4}\right)}{\sin\left(-\frac{3\pi}{4}\right)} = \frac{1 + \frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{2 + \sqrt{2}}{-\sqrt{2}} = -\sqrt{2} - 1 \approx -2.414$$

2. Show that $\sin(3x) = 3\sin(x)\cos^2(x) - \sin^3(x)$. (Hint: Use $\sin(2x) = 2\sin(x)\cos(x)$ and the sine sum formula.)

$$\begin{aligned}\sin(3x) &= \sin(x + (2x)) \\ &= \sin(x)\cos(2x) + \cos(x)\sin(2x) \\ &= \sin(x)[\cos^2(x) - \sin^2(x)] + \cos(x)[2\sin(x)\cos(x)] \\ &= \sin(x)\cos^2(x) - \sin^3(x) + 2\sin(x)\cos^2(x) \\ &= 3\sin(x)\cos^2(x) - \sin^3(x)\end{aligned}$$

3. Show that $\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x)$. (Hint: Use $\cos(2x) = \cos^2(x) - \sin^2(x)$ and the cosine sum formula.)

$$\begin{aligned}\cos(3x) &= \cos(x + (2x)) \\ &= \cos(x)\cos(2x) - \sin(x)\sin(2x) \\ &= \cos(x)[\cos^2(x) - \sin^2(x)] - \sin(x)[2\sin(x)\cos(x)] \\ &= \cos^3(x) - \cos(x)\sin^2(x) - 2\cos(x)\sin^2(x) \\ &= \cos^3(x) - 3\cos(x)\sin^2(x)\end{aligned}$$

4. Use $\cos(2x) = \cos^2(x) - \sin^2(x)$ to establish the following formulas.

a. $\cos^2(x) = \frac{1 + \cos(2x)}{2}$

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= \cos^2(x) - (1 - \cos^2(x)) \\ &= 2\cos^2(x) - 1\end{aligned}$$

Therefore, $\cos^2(x) = \frac{1 + \cos(2x)}{2}$.

b. $\sin^2(x) = \frac{1 - \cos(2x)}{2}$

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= (1 - \sin^2(x)) - \sin^2(x) \\ &= 1 - 2\sin^2(x)\end{aligned}$$

Therefore, $\sin^2(x) = \frac{1 - \cos(2x)}{2}$.

5. Jamia says that because sine is an odd function, $\sin\left(\frac{\theta}{2}\right)$ is always negative if θ is negative. That is, she says that for negative values of $\sin\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1 - \cos(\theta)}{2}}$. Is she correct? Explain how you know.

Jamia is not correct. Consider $\theta = -\frac{7\pi}{3}$. In this case, $\frac{\theta}{2} = -\frac{7\pi}{6}$, and rotation by $-\frac{7\pi}{6}$ terminates in Quadrant II. Thus, $\sin\left(-\frac{7\pi}{6}\right)$ is positive.

6. Ginger says that the only way to calculate $\sin\left(\frac{\pi}{12}\right)$ is using the difference formula for sine since $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$. Fred says that there is another way to calculate $\sin\left(\frac{\pi}{12}\right)$. Who is correct and why?

Fred is correct. We can use the half-angle formula with $\theta = \frac{\pi}{6}$ to calculate $\sin\left(\frac{\pi}{12}\right)$.

7. Henry says that by repeatedly applying the half-angle formula for sine we can create a formula for $\sin\left(\frac{\theta}{n}\right)$ for any positive integer n . Is he correct? Explain how you know.

Henry is not correct. Repeating this process will only give us formulas for $\sin\left(\frac{\theta}{2^k}\right)$ for positive integers k . There is no way to derive a formula for quantities such as $\sin\left(\frac{\theta}{5}\right)$ using this method.



Lesson 5: Tangent Lines and the Tangent Function

Student Outcomes

- Students construct a tangent line from a point outside a given circle to the circle (**G-C.A.4**).

Lesson Notes

This lesson is designed to address standard **G-C.A.4**, which involves constructing tangent lines to a given circle from a point outside the circle. The lesson begins by revisiting the geometric origins of the tangent function from Algebra II Module 2 Lesson 6. Students then explore the standard by means of construction by compass and straightedge and by paper folding. Students are provided several opportunities to create mathematical arguments to explain their constructions.

Classwork

Exercises 1–4 (4 minutes)

In this sequence of exercises, students recall the connections between the trigonometric functions and the geometry of the unit circle.

Exercises 1–12

The circle shown to the right is a unit circle, and the length of \widehat{DA} is $\frac{\pi}{3}$ radians.

- Which segment in the diagram has length $\sin\left(\frac{\pi}{3}\right)$?

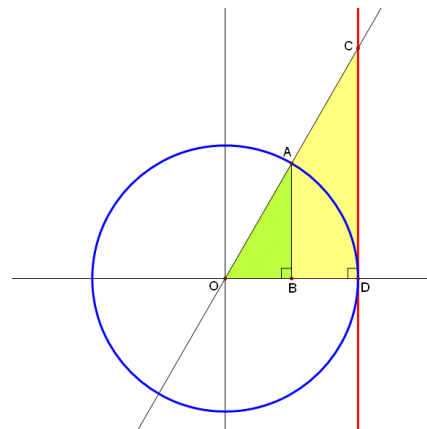
The sine of $\frac{\pi}{3}$ is the vertical component of point A, so \overline{AB} has length $\sin\left(\frac{\pi}{3}\right)$.

- Which segment in the diagram has length $\cos\left(\frac{\pi}{3}\right)$?

The cosine of $\frac{\pi}{3}$ is the horizontal component of point A, so \overline{OB} has length $\cos\left(\frac{\pi}{3}\right)$.

- Which segment in the diagram has length $\tan\left(\frac{\pi}{3}\right)$?

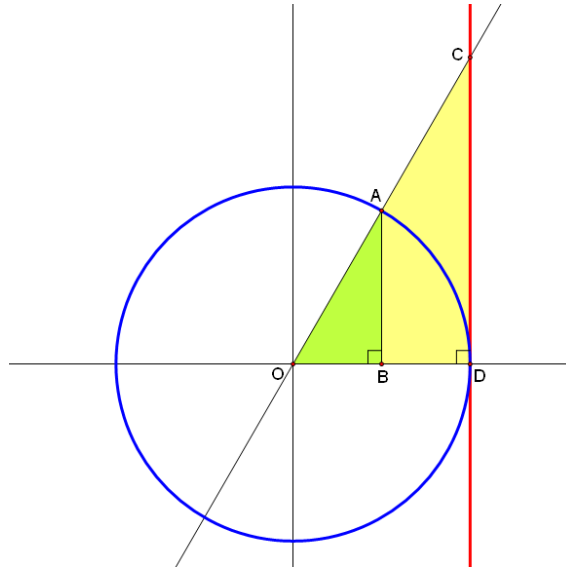
Since $\triangle OAB$ is similar to $\triangle OCD$, $\frac{CD}{1} = \frac{AB}{OB} = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}$. Thus, \overline{CD} has length $\tan\left(\frac{\pi}{3}\right)$.



4. Which segment in the diagram has length $\sec\left(\frac{\pi}{3}\right)$?

Since $\triangle OAB$ is similar to $\triangle OCD$, $\frac{OD}{OB} = \frac{OC}{OA}$. Since $OA = OD = 1$, we have $OC = \frac{1}{OB} = \frac{1}{\cos\left(\frac{\pi}{3}\right)}$. Thus, \overline{OC} has length $\sec\left(\frac{\pi}{3}\right)$.

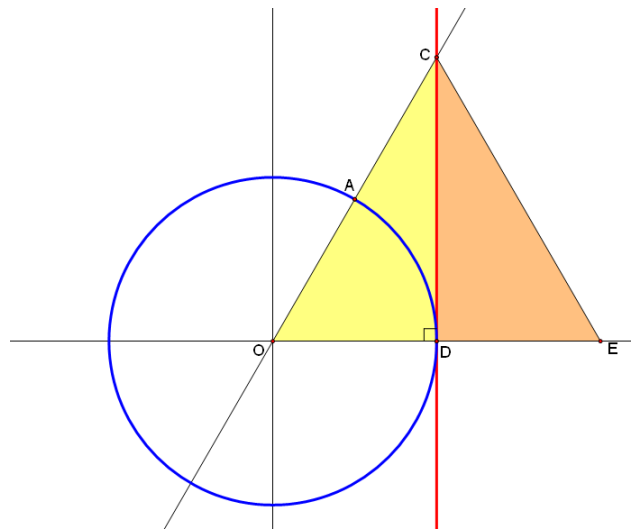
Discussion (7 minutes): Connections Between Geometry and Trigonometry



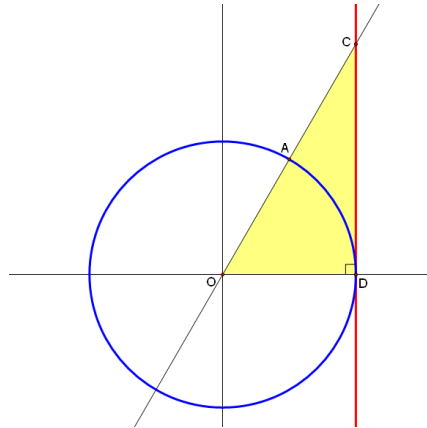
- Do you recall why the length of \overline{CD} is called the tangent of $\frac{\pi}{3}$? And do you recall why the length of \overline{OC} is called the secant of $\frac{\pi}{3}$? (Hint: Look at how lines containing \overline{CD} and \overline{OC} are related to the circle.)
 - The line containing \overline{CD} is tangent to the circle because it intersects the circle once, so it makes sense to refer to \overline{CD} as a tangent segment and to refer to the length CD as the tangent of $\frac{\pi}{3}$.
 - The line containing \overline{OC} is a secant line because it intersects the circle twice, so it makes sense to refer to \overline{OC} as a secant segment and to refer to the length OC as the secant of $\frac{\pi}{3}$.
- How can you be sure that the line containing \overline{CD} is, in fact, a tangent line?
 - We see that \overline{CD} is perpendicular to segment \overline{OD} at point D , and so \overline{CD} must be tangent to the circle.
- Explain how this diagram can be used to show that $\tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}$.
 - The triangles in the diagram have two pairs of congruent angles, so they are similar to each other. It follows that their sides are proportional, and so $\frac{\tan\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\cos\left(\frac{\pi}{3}\right)}$. If we multiply both sides by $\sin\left(\frac{\pi}{3}\right)$, we obtain the desired result.

MP.7

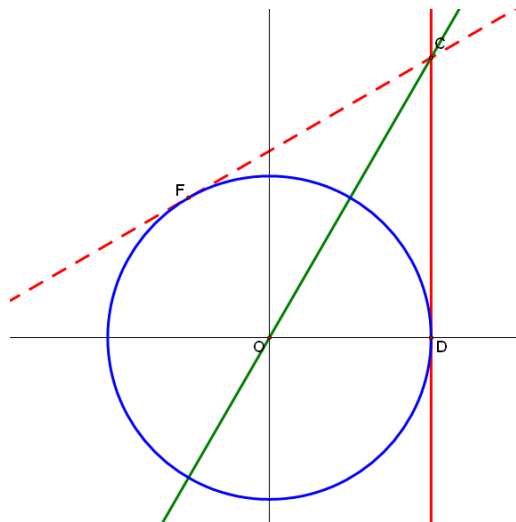
- Explain how this diagram can be used to show that $\sin^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) = 1$.
 - If we apply the Pythagorean theorem to $\triangle AOB$, we find that $(AB)^2 + (OB)^2 = (OA)^2$, so $\sin^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) = 1$.
- Which trigonometric identity can be established by applying the Pythagorean theorem to $\triangle COD$?
 - $\tan^2\left(\frac{\pi}{3}\right) + 1^2 = \sec^2\left(\frac{\pi}{3}\right)$
- What is the scale factor that relates the sides of $\triangle AOB$ to those of $\triangle COD$?
 - Point B is the midpoint of \overline{OD} , so the sides of $\triangle COD$ are twice the length of the corresponding sides of $\triangle AOB$.



- Explain how the diagram above can be used to show that $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.
 - Since the length of \widehat{DA} is $\frac{\pi}{3}$ radians, it follows that $\triangle COE$ is equilateral, and so its sides must each be 2 units long. Applying the Pythagorean theorem to $\triangle COD$, we find $1^2 + CD^2 = 2^2$, which means that $CD^2 = 3$, and so $CD = \sqrt{3}$. Thus, $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.
- This diagram contains a host of information about trigonometry. But can we take this diagram even further? Let's press on to some new territory. Much of the remaining discussion is devoted to the topic of constructions. Get ready to have some fun with paper folding and your compass and straightedge!

Exploratory Challenge 1 (7 minutes): Constructing Tangents via Paper-Folding

- Earlier we observed that \overrightarrow{CD} is tangent to the circle. Can you visualize another line through point C that is also tangent to the circle? Try to draw a second tangent line through point C , and label its point of intersection with the circle as point F .

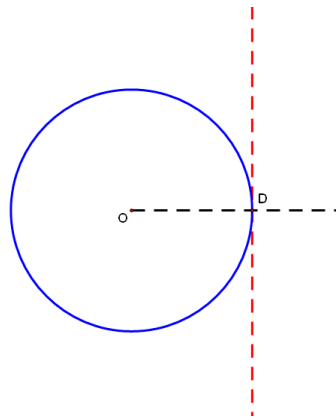


- First, let's examine this diagram through the lens of transformations. Can you see how to map \overrightarrow{CD} onto \overrightarrow{CF} ?
 - It appears that you could map \overrightarrow{CD} onto \overrightarrow{CF} by reflecting it across \overrightarrow{CO} .*
- Let's get some hands-on experience with this by performing the reflection in the most natural way imaginable—by folding a piece of paper.

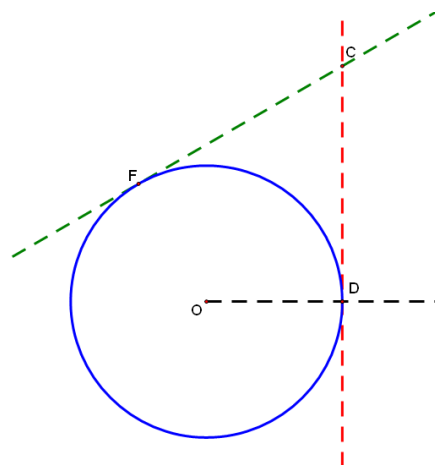
Give students an unlined piece of copy paper or patty paper, a ruler, and a compass.

The goal of this next activity is essentially to recreate the diagram above. Students take one tangent line and use a reflection to produce a second tangent line.

- Choose a point in the middle of your paper, and label it O . Use your compass to draw a circle with center O . Choose a point on that circle, and label it D . Use your straightedge to draw the line through O and D , extending it beyond the circle. Can you see how to fold the paper in such a way that the crease is tangent to the circle at point D ? Think about this for a moment.
 - *We want to create a line that is perpendicular to \overrightarrow{OD} at point D , so we fold the paper in such a way that the crease is on point D and \overrightarrow{OD} maps to itself.*



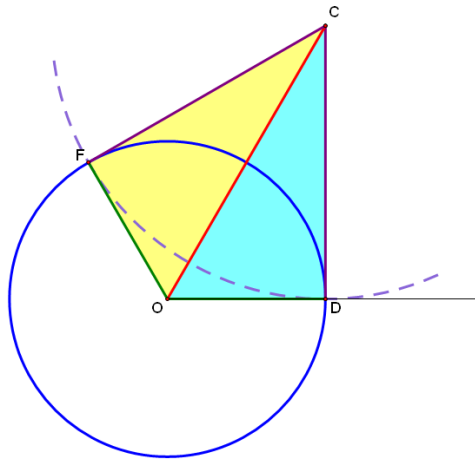
- Now choose a point on the crease, and label it C . Try to create a second tangent line through point C . Can you see how to do this?
 - *We just need to fold the paper so that the crease is on points C and O . We can see through the back of the paper where point D is, and this is where we mark a new point F , which is the other point of tangency. In other words, \overrightarrow{CF} is also tangent to the circle.*



- Do you think you could construct an argument that \overrightarrow{CF} is truly tangent to the circle? Think about this for a few minutes, and then share your thoughts with the students around you.
 - *A reflection is a rigid motion, preserving both distances and angles. Since \overrightarrow{CD} is tangent to the circle, we know that $\angle CDO$ is a right angle. And since the distance from O to D is preserved by the reflection, it follows that point F , which is the image of D , is also located on the circle because $OF = OD$, both of which are radii of the circle. This means that F is on the circle and that $\angle CFO$ is a right angle, which proves that \overrightarrow{CF} is indeed tangent to the circle.*

Exploratory Challenge 2 (5 minutes): Constructing Tangents Using a Compass

- Let's try to produce this tangent line without using paper-folding. Can you see how to produce this diagram using only a compass and a straightedge? Take a few minutes to explore this problem.
 - We use a straightedge to draw a line that is perpendicular to \overleftrightarrow{OD} at point D and choose a point C on this perpendicular line as before. Next, we use a compass to draw a circle around point C that contains point D . This circle will intersect the original circle at a point F , which is the desired point of tangency.

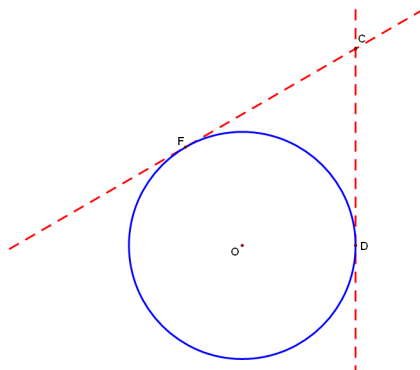


- Now make an argument that your construction produces a line that is truly tangent to the circle.
 - Each point on the circle around point O is the same distance from O , so $OF = OD$. Also, each point on the circle around point C is the same distance from point C , so $CF = CD$. Since $CO = CO$, $\triangle OCD \cong \triangle OCF$ by the SSS criterion for triangle congruence. Since $\angle CDO$ is a right angle, it follows by CPCTC that $\angle CFO$ is also a right angle. This proves that \overleftrightarrow{CF} is indeed tangent to the circle.

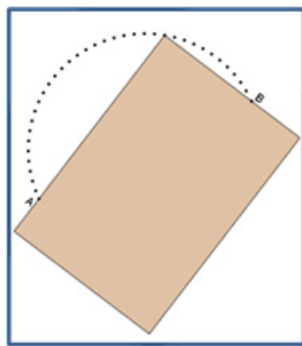
Exploratory Challenge 3 (7 minutes): Constructing the Tangents from an External Point

MP.1

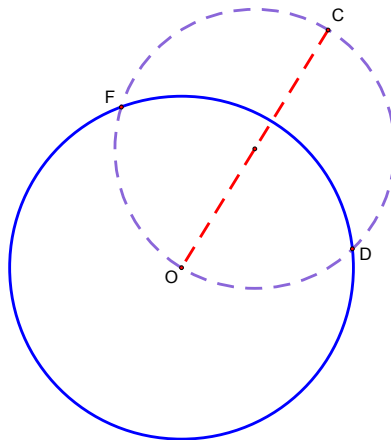
- Now let's change the problem slightly. Suppose we are given a circle and an external point. How might we go about constructing the lines that are tangent to the given circle and pass through the given point? Take several minutes to wrestle with this problem, and then share your thoughts with a neighbor.



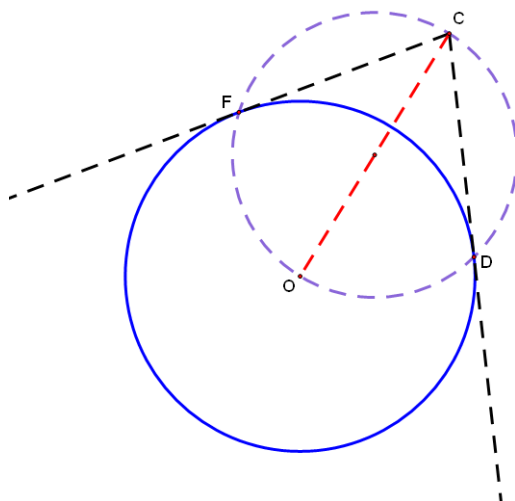
- This is a tough problem, isn't it? Let's see if we can solve it by using some logical reasoning. What do you know about the lines you want to create?
 - *We want the lines to be tangent to the circle, so they must be constructed in such a way that $\angle CFO$ and $\angle CDO$ are right angles.*
- Perhaps we can use this to our advantage. Let's consider the entire locus of points F such that $\angle CFO$ is a right angle. Can you describe this locus? Can you construct it? This is a challenge in its own right. Let's explore this challenge using a piece of paper, which comes with some built-in right angles.
- Take a blank piece of paper, and mark two points several inches apart; label them A and B . Now take a second piece of paper, and line up one edge on A and the adjacent edge on B . Mark the point where the corner of the paper is, then shift the corner to a new spot, keeping the edges against A and B . Quickly repeat this 20 or so times until a clear pattern emerges. Then describe what you see.



- *This locus of points is a circle with diameter \overline{AB} .*
- Now construct this circle using your compass.
 - *We just need to construct the midpoint of \overline{AB} , which is the center of the circle.*
- We are ready to apply this result to our problem involving tangents to a circle. Do you see how this relates to the problem of creating a tangent line?



- *By drawing a circle with diameter \overline{OC} , we create two points F and D with two important properties. First, they are both on the circle around O . Second, they are on the circle with diameter \overline{OC} , which means that $\angle OFC$ and $\angle ODC$ are right angles, just as we require for tangent lines.*

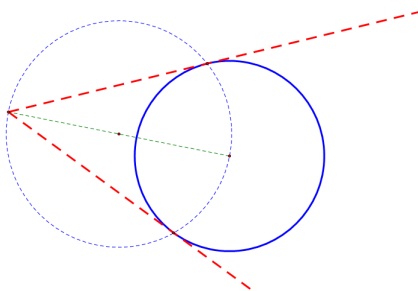


Exercises 5–7 (4 minutes)

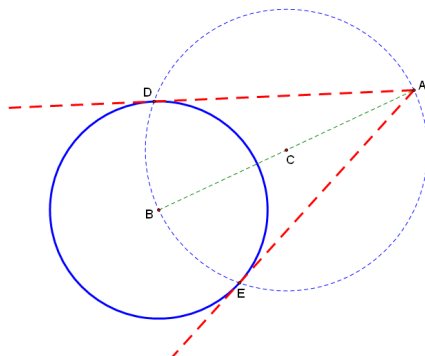
Instruct students to perform the following exercises and to compare their results with a partner.

5. Use a compass to construct the tangent lines to the given circle that pass through the given point.

Sample solution:

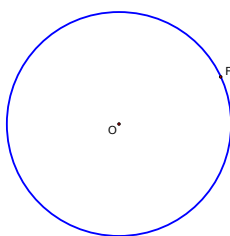


6. Analyze the construction shown below. Argue that the lines shown are tangent to the circle with center B .

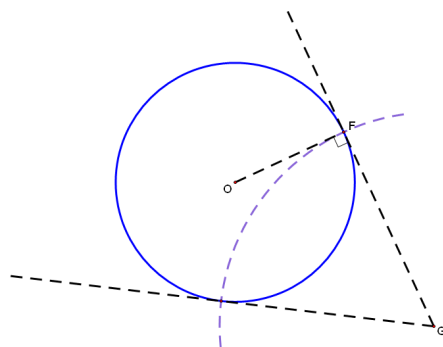


Since point D is on circle C , we know that $\angle BDA$ is a right angle. Since point D is on circle B and $\angle BDA$ is a right angle, it follows that \overline{AD} is tangent to circle B . In the same way, we can show that \overline{AE} is tangent to circle B .

7. Use a compass to construct a line that is tangent to the circle below at point F . Then choose a point G on the tangent line, and construct another tangent to the circle through G .



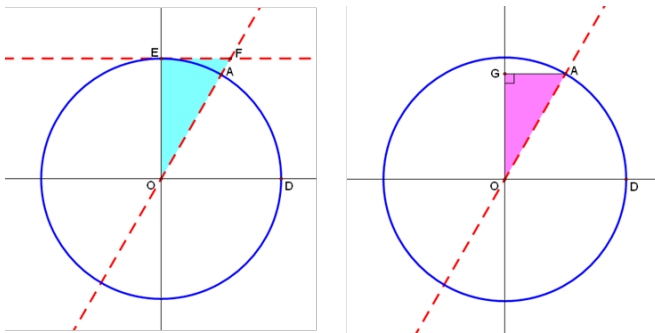
Sample solution:



Exercises 8–12 (5 minutes)

Instruct students to perform the following exercises and to compare their results with a partner.

8. The circles shown below are unit circles, and the length of \widehat{DA} is $\frac{\pi}{3}$ radians.



Which trigonometric function corresponds to the length of \overline{EF} ?

The length of \overline{EF} represents the cotangent of $\frac{\pi}{3}$.

9. Which trigonometric function corresponds to the length of \overline{OF} ?

The length of \overline{OF} represents the cosecant of $\frac{\pi}{3}$.

10. Which trigonometric identity gives the relationship between the lengths of the sides of $\triangle OEF$?

$$\cot^2\left(\frac{\pi}{3}\right) + 1^2 = \csc^2\left(\frac{\pi}{3}\right)$$

11. Which trigonometric identities give the relationships between the corresponding sides of $\triangle OEF$ and $\triangle OGA$?

$$\text{We have } \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{\cot\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} \text{ and } \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{\csc\left(\frac{\pi}{3}\right)}{1}.$$

12. What is the value of $\csc\left(\frac{\pi}{3}\right)$? What is the value of $\cot\left(\frac{\pi}{3}\right)$? Use the Pythagorean theorem to support your answers.

Let $x = EF$. Then we have $x^2 + 1^2 = (2x)^2 = 4x^2$, which means $3x^2 = 1$; therefore, $x = \sqrt{\frac{1}{3}}$. So $\cot\left(\frac{\pi}{3}\right) = \sqrt{\frac{1}{3}}$ and $\csc\left(\frac{\pi}{3}\right) = 2\sqrt{\frac{1}{3}}$.

Closing (2 minutes)

- If you are given a circle and an external point, how do you use a compass to construct the lines that pass through the given point and are tangent to the given circle? Use your notebook to write a summary of the main steps involved in this construction.
 - *Let's say the center of the circle is point A and the external point is B . First, we need to construct the midpoint of the segment joining A and B —call this point C . Next, we draw a circle around point C that goes through A and B . The points where this circle intersects the first circle are the points of tangency, so we just connect these points to point B to form the tangents.*

Exit Ticket (4 minutes)

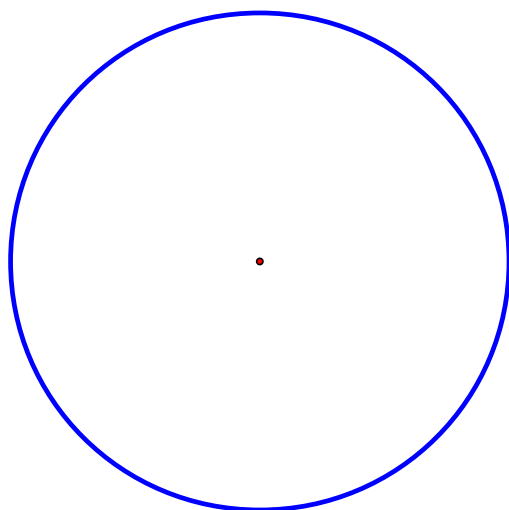
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Lesson 5: Tangent Lines and the Tangent Function

Exit Ticket

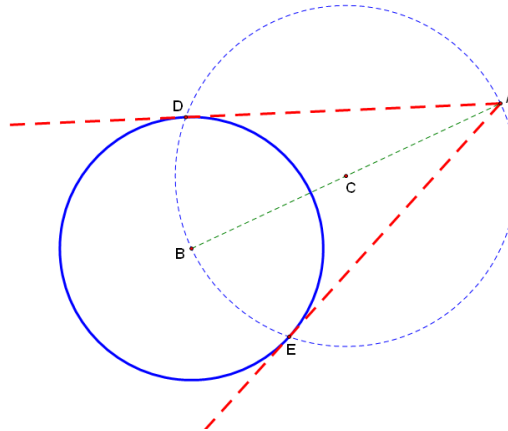
1. Use a compass and a straightedge to construct the tangent lines to the given circle that pass through the given point.



2. Explain why your construction produces lines that are indeed tangent to the given circle.

Exit Ticket Sample Solutions

1. Use a compass and a straightedge to construct the tangent lines to the given circle that pass through the given point.



2. Explain why your construction produces lines that are indeed tangent to the given circle.

Since points D and E are on circle C, $\angle BDA$ and $\angle BEA$ are right angles. Thus, \overrightarrow{AD} and \overrightarrow{AE} are tangent to circle B.

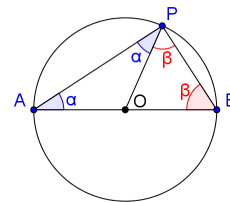
Problem Set Sample Solutions

1. Prove Thales' theorem: If A , B , and P are points on a circle where \overline{AB} is a diameter of the circle, then $\angle APB$ is a right angle.

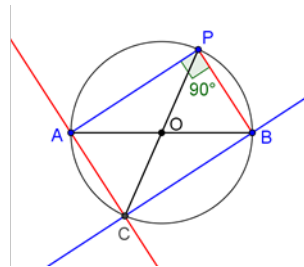
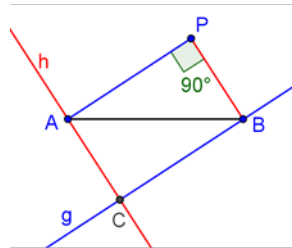
Since $OA = OP = OB$, $\triangle OPA$ and $\triangle OPB$ are isosceles triangles. Therefore, $m\angle OAP = m\angle OPA$, and $m\angle OPB = m\angle OBP$.

Let $m\angle OPA = \alpha$ and $m\angle OPB = \beta$. The sum of three internal angles of $\triangle APB$ equals 180° .

Therefore, $\alpha + (\alpha + \beta) + \beta = 180^\circ$, so $2\alpha + 2\beta = 180^\circ$, and $\alpha + \beta = 90^\circ$. Since $m\angle APB = \alpha + \beta$, we have $m\angle APB = 90^\circ$, so $\angle APB$ is a right angle.



2. Prove the converse of Thales' theorem: If \overline{AB} is a diameter of a circle and P is a point so that $\angle APB$ is a right angle, then P lies on the circle for which \overline{AB} is a diameter.



Construct the right triangle, $\triangle APB$.

Construct the line h that is parallel to \overline{PB} through point A .

Construct the line g that is parallel to \overline{AP} through point B .

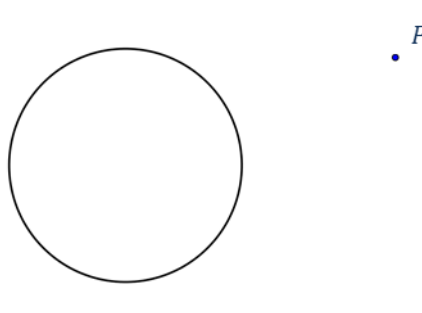
Let C be the intersection of lines h and g .

The quadrilateral $ACBP$ forms a parallelogram by construction.

By the properties of parallelograms, the adjacent angles are supplementary. Since $\angle APB$ is a right angle, it follows that angles $\angle CAP$, $\angle BCA$, and $\angle PBC$ are also right angles. Therefore, the quadrilateral $ACBP$ is a rectangle.

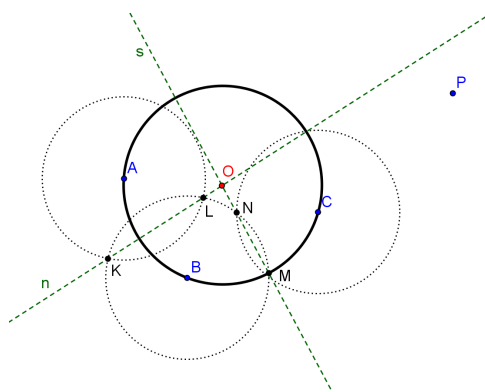
Let O be the intersection of the diagonals \overline{AB} and \overline{CP} . Then, by the properties of parallelograms, point O is the midpoint of \overline{AB} and \overline{CP} , so $OA = OB = OC = OP$. Therefore, O is the center of the circumscribing circle, and the hypotenuse of $\triangle APB$, \overline{AB} , is a diameter of the circle.

3. Construct the tangent lines from point P to the circle given below.

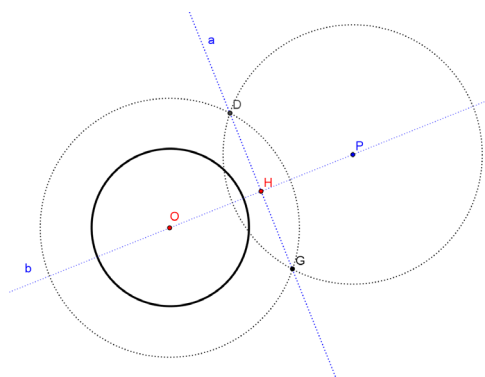


Mark any three points A , B , and C on the circle, and construct perpendicular bisectors of \overline{AB} and \overline{BC} .

Let O be the intersection of the two perpendicular bisectors.



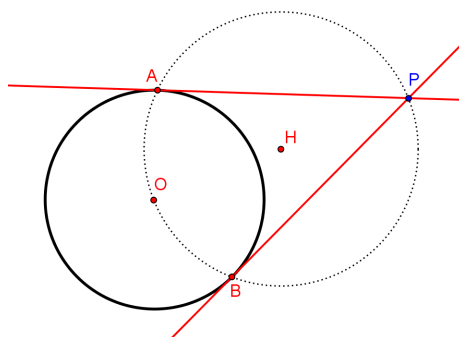
Construct the midpoint H of \overline{OP} .



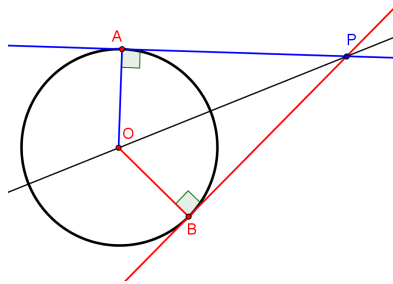
Construct a circle with center H and radius OH .

The circle centered at H will intersect the original circle O at points A and B .

Construct two tangent lines \overleftrightarrow{PA} and \overleftrightarrow{PB} .

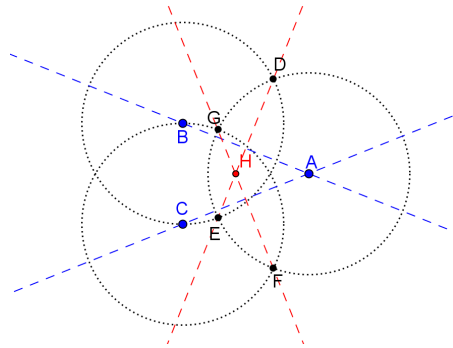


4. Prove that if segments from a point P are tangent to a circle at points A and B , then $\overline{PA} \cong \overline{PB}$.



Let P be a point outside of a circle with center O , and let A and B be points on the circle so that \overline{PA} and \overline{PB} are tangent to the circle. Then, $OA = OB$, $OP = OP$, and $m\angle OAP = m\angle OBP = 90^\circ$, so $\triangle PAO \cong \triangle PBO$ by the Hypotenuse Leg congruence criterion. Therefore, $\overline{PA} \cong \overline{PB}$ because corresponding parts of congruent triangles are congruent.

5. Given points A , B , and C so that $AB = AC$, construct a circle so that \overline{AB} is tangent to the circle at B and \overline{AC} is tangent to the circle at C .



Construct a perpendicular bisector of \overline{AB} .

Construct a perpendicular bisector of \overline{AC} .

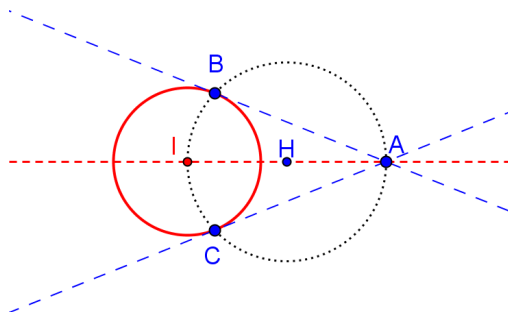
The perpendicular bisectors will intersect at point H .

Construct a line through points A and H .

Construct a circle with center H and radius \overline{HA} .

The circle centered at H will intersect \overline{HA} at I .

Construct a circle centered at I with radius \overline{IB} .





Lesson 6: Waves, Sinusoids, and Identities

Student Outcomes

- Students model waveforms using sinusoidal functions and understand how standing waves relate to the sounds generated by musical instruments. Students use trigonometric identities to rewrite expressions and understand wave interference and standing waves.

Lesson Notes

In this modeling lesson, students learn how sinusoidal functions and trigonometric identities apply to waves such as sound waves.

Much of the content of this lesson is based on the physics of waves. Two types of waves in physics are longitudinal and transverse waves. A slinky can be used to model both types of waves. The video clip <https://www.youtube.com/watch?v=SCtf-z4t9L8> illustrates two types of waveforms: longitudinal waves and transverse waves. Particles that oscillate perpendicular to the motion of the waves create a transverse wave. Particles that oscillate in the direction of the motion of the wave create a longitudinal wave. Sound, which is studied in this lesson, is a longitudinal wave.

Waves do not transport particles; they transport energy. A good model for this is to have your class do the wave. Have students stand in a line at the front of your classroom and raise their arms up and down in order. You can see the wave move through the line but no one's position is physically changed. This is an example of a transverse wave. You might ask the class (without actually doing it) how they could generate a longitudinal wave while standing in their line. This would require adjacent students giving each other a slight nudge horizontally.

The frequency f is how many times the particles in a wave oscillate back and forth in one unit of time. The period T is the length of time to create one back and forth oscillation. Thus,

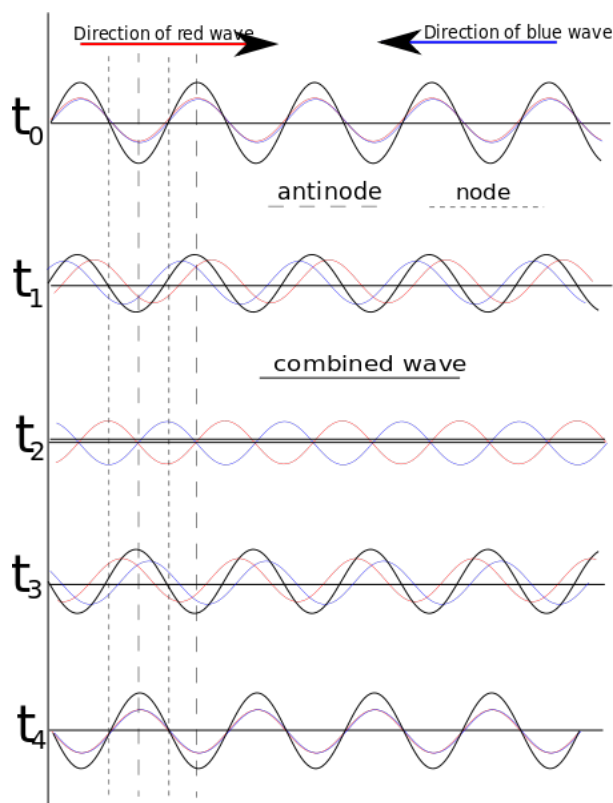
$$f = \frac{1}{T} \text{ and } T = \frac{1}{f}.$$

The wavelength λ is the horizontal distance between two adjacent troughs or crests of a transverse wave OR between two adjacent rarefactions or compressions of a longitudinal wave. When represented graphically as a sinusoid, troughs and crests (and rarefactions and compressions) are the extrema of the graph. Frequency and wavelength are related to the speed of the wave v by what is known as the wave equation where

$$v = \lambda \cdot f.$$

When the string on a guitar is plucked, a longitudinal wave is created that is reflected back on itself when it reaches the end of the string. When these waves interfere with one another in just the right way, a standing wave pattern is created. These standing wave patterns produce the musical tones that you hear.

Students study what happens when two waves are combined in different ways and then specifically discuss the effect of interference that produces a standing wave pattern. They use sum and difference identities to analyze the behavior of combinations of waves. Students should be encouraged to use technology as needed throughout this lesson. Waves can be modeled in graphing software such as Desmos or GeoGebra.



Web-based Resources:

- JAVA Applets for modeling transverse, longitudinal, and standing waves: <http://phy.hk/wiki/index.htm>
- Background Information on the Physics of Waves: <http://www.physicsclassroom.com/class/waves>
- Standing Waves: http://en.wikipedia.org/wiki/Standing_wave
- Information Specific to Sound: <http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html>

In Algebra II Module 2, students used sinusoidal equations to model periodic phenomena (**F-TF.B.5**). This lesson builds on that work and shows how wave interference can be modeled by addition of sinusoidal functions. In this lesson, students use addition and subtraction formulas as tools in the modeling process (**F-TF.C.9**) in Exercises 4–7.

MP.4

This lesson involves quite a bit of descriptive modeling and the modeling cycle. The problem is how to mathematically represent the waves that create the notes we hear on a musical instrument. Students simplify the problem and formulate graphs and equations that show wave interference and traveling waves to understand and interpret these effects. They validate their standing wave model using trigonometric identities and appropriate technology and report their understanding in the Closing and Exit Ticket.

Classwork

Opening (2 minutes)

Activate prior knowledge with the questions on the following page. Give students a minute to think about this, and then lead a short whole-group discussion.

- What are some real-world phenomena that can be represented with waves?
 - *Student answers will vary. Real-world phenomena that can be represented with waves include sound waves, waves at the beach, waves in a wave machine in a water park, microwaves, and radio waves.*
- How do you think waves are related to the study of trigonometry?
 - *Sinusoidal functions can be used to model waves.*

Opening Exercise (6 minutes)

The Opening Exercise shows graphs of several different sinusoidal functions that represent musical notes. The tones we hear when a note is played are created by the vibrations of the instrument. These vibrations create longitudinal waves that interfere with one another to produce the notes. Different tones are produced by varying the amplitude (loudness) and frequency (pitch) of the waves.

Use this exercise to review the features of the graphs of sinusoidal functions that students studied in Algebra II Module 2. This lesson can be completed primarily by sketching graphs by hand but could also be done using a graphing calculator or other graphing technology if students require more scaffolding. Since the focus is on modeling, technology is an appropriate tool for this investigation. The functions in the Opening Exercise are used to model wave interference later in the lesson.

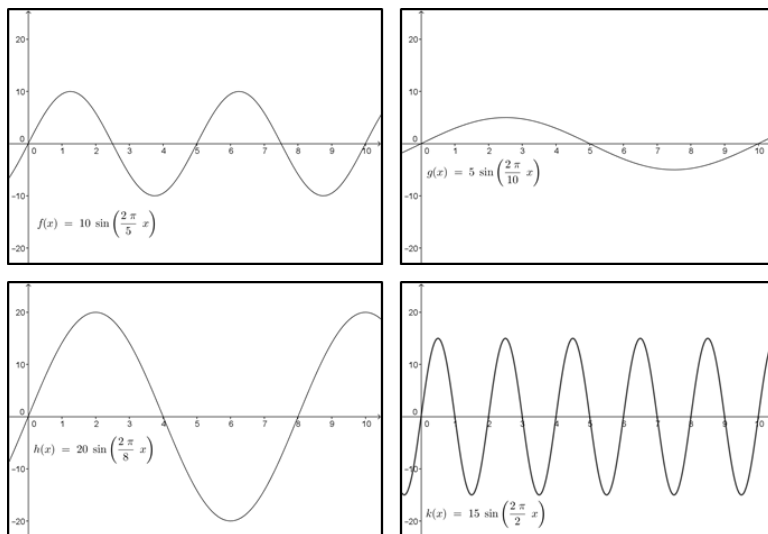
Scaffolding:

- If students are struggling to recall the features of graphs of sinusoidal functions, you may wish to revisit Algebra II Module 2 lessons that defined the features of the graphs of sinusoids.
- You can extend this lesson by reviewing how to sketch sinusoids and how to write the equations from the graphs of a sinusoid.
- Consider posting an anchor chart for student reference that shows the features of the graph of a sinusoid. You can use the Lesson Summary from Algebra II Module 2 Lesson 11 to create the anchor chart.

Opening Exercise

When you hear a musical note played on an instrument, the tones are caused by vibrations of the instrument. The vibrations can be represented graphically as a sinusoid. The amplitude is a measure of the loudness of the note, and the frequency is a measure of the pitch of the note. Recall that the frequency of a sinusoidal function is the reciprocal of its period. Louder notes have greater amplitude, and higher pitched notes have larger frequencies.

- a. State the amplitude, period, and frequency of each sinusoidal function graphed below.



Function f has amplitude 10, period 5, and frequency $\frac{1}{5}$.

Function g has amplitude 5, period 10, and frequency $\frac{1}{10}$.

Function h has amplitude 20, period 8, and frequency $\frac{1}{8}$.

Function k has amplitude 15, period 2, and frequency $\frac{1}{2}$.

- b. Order the graphs from quietest note to loudest note.

The quietest notes have lower amplitudes. The order from quietest note to loudest note would be g , f , k , and h .

- c. Order the graphs from lowest pitch note to highest pitch note.

Lower frequencies are lower pitch. The order from lowest pitch note to highest pitch note would be g , h , f , and k .

Discussion (5 minutes)

Define a wave and discuss the two types of waves: transverse and longitudinal.

Discussion

A wave is a disturbance moving through a medium that disrupts the particles that make up the medium. A medium can be any substance including solids, liquids, and gases. When a wave is present, the particles that make up the medium move about a fixed position. Energy is transferred between the particles, but the particles themselves always return to their fixed positions. This energy transfer phenomenon is a distinguishing feature of a wave.

One type of wave is a transverse wave where the particles oscillate perpendicular to the motion of the wave. Another type of wave is a longitudinal wave where the particles oscillate in the same direction as the motion of the wave. Sound waves are examples of longitudinal waves. The up and down motion of a buoy in the ocean as a wave passes is similar to a transverse wave.

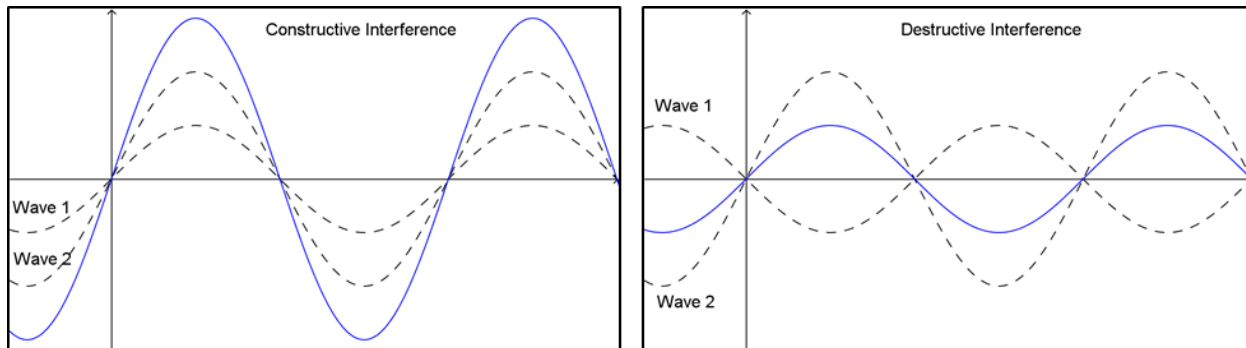
Show a short video clip or animation from one of the references listed at the beginning of this lesson that models the two types of waves, or physically model the two types of waves using a Slinky®. You can find additional resources for modeling waves by searching online.

Next, line up students in the front of your classroom and model a transverse wave by doing the wave like you might see at a sporting event. After doing the wave a few times, ask these questions.

- What was the medium that the wave traveled through? What were the particles? How did we represent the energy transfer in this wave?
 - *The line of people was the medium that the wave traveled through. Each person was a particle. The up and down movement of our arms represented the back and forth movement perpendicular to the motion of the wave.*
- How could we model a longitudinal wave in this lineup?
 - *We would have to bump into each other consecutively down the line.*

- Wave interference is what happens when two waves traveling through the same medium meet. The interference can be constructive or destructive. The result of wave interference can be represented mathematically as the sum of two wave equations.

The following diagrams can be used to help students understand constructive and destructive interference.



- What do you think constructive interference looks like? What about destructive interference?
 - When it's constructive interference, the amplitudes combine to produce a wave with larger amplitude. When it's destructive, the peaks of the graph cancel each other out.

Exercises 1–4 (10 minutes)

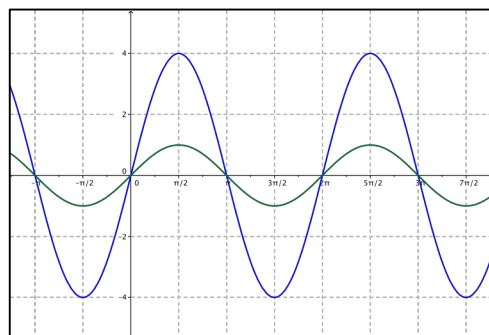
Students now model wave interference by adding selected points on the graphs of two functions. They use sum identities to show that the sum of two sinusoidal functions can be written in terms of a single sinusoidal function as long as the two functions have the same period. Note that the functions in Exercises 1 and 2 are the same functions that were graphed in the Opening Exercise. Blank graph paper with a trigonometric scale is provided at the end of the lesson if needed.

Exercise 4 provides an example of constructive interference.

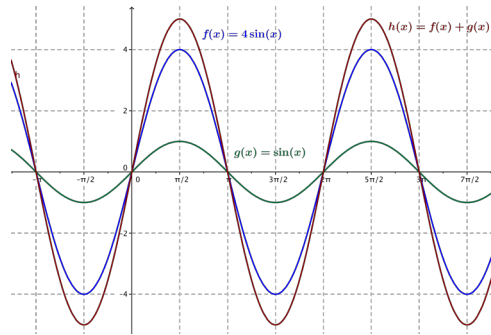
Exercises 1–7

When two musical notes are played simultaneously, wave interference occurs. Wave interference is also responsible for the actual sound of the notes that you hear.

- The graphs of two functions, f and g , are shown below.



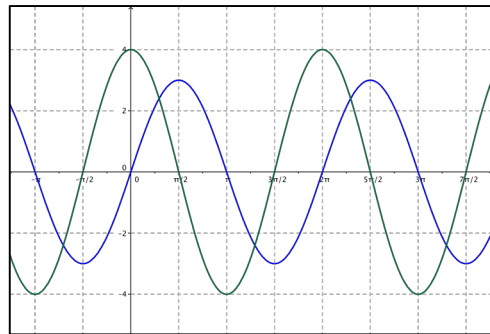
- a. Model wave interference by picking several points on the graphs of f and g and then using those points to create a graph of $h(x) = f(x) + g(x)$.



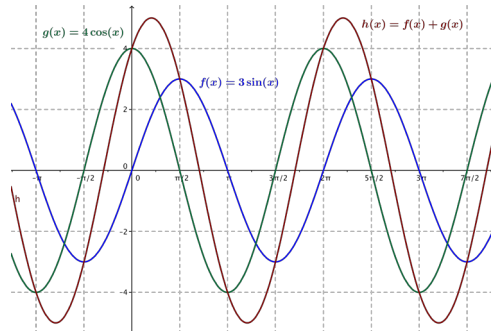
- b. What is a formula for h ? Explain how you got your answer.

A formula is $h(x) = 5\sin(x)$. Since $f(x) = \sin(x)$ and $g(x) = 4\sin(x)$, you can simply add the two expressions together.

2. The graphs of f and g are shown below.



- a. Model wave interference by picking several points on the graphs of f and g and then using those points to create a graph of $h(x) = f(x) + g(x)$.



- b. What is an approximate formula for h ? Explain how you got your answer.

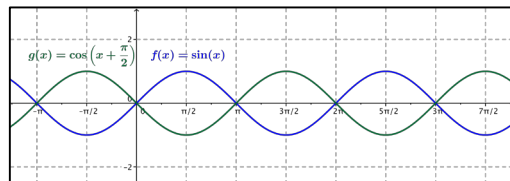
From the graph, the amplitude appears to be 5, the period appears to be 2π , and the graph looks like it has been shifted approximately $\frac{\pi}{4}$ units to the right. So $h(x) \approx 5 \cos\left(x - \frac{\pi}{4}\right)$.

In the sample response for Exercise 2 part (b), the phase shift is only approximate and is incorrect. Students will most likely NOT use an identity to determine the phase shift of the cosine graph precisely. Be sure to go through the algebraic steps below for finding the phase shift precisely during the debriefing period.

Exercise 3 below provides an example of destructive interference.

3. Let $f(x) = \sin(x)$ and $g(x) = \cos\left(x + \frac{\pi}{2}\right)$.

- a. Predict what the graph of the wave interference function $h(x) = f(x) + g(x)$ would look like in this situation.



The graphs are reflections of one another across the horizontal axis. Therefore, the wave interference function would be $h(x) = 0$ because the two functions would cancel each other out for every value of x .

- b. Use an appropriate identity to confirm your prediction.

$$\begin{aligned}\sin(x) + \cos\left(x + \frac{\pi}{2}\right) &= \sin(x) + \cos(x) \cos\left(\frac{\pi}{2}\right) - \sin(x) \sin\left(\frac{\pi}{2}\right) \\ &= \sin(x) + \cos(x) \cdot 0 - \sin(x) \cdot 1 \\ &= \sin(x) - \sin(x) \\ &= 0\end{aligned}$$

4. Show that in general, the function $h(x) = a \cos(bx - c)$ can be rewritten as the sum of a sine and cosine function with equal periods and different amplitudes.

$$\begin{aligned}f(x) &= a \cos(bx - c) \\ &= a(\cos(bx) \cos(c) + \sin(bx) \sin(c)) \\ &= a \cos(c) \cos(bx) + a \sin(c) \sin(bx)\end{aligned}$$

The two functions would be $f(x) = a \cos(c) \cos(bx)$ with amplitude $|a \cos(c)|$ and period $\left|\frac{2\pi}{b}\right|$ and

$g(x) = a \sin(c) \sin(bx)$ with amplitude $|a \sin(c)|$ and period $\left|\frac{2\pi}{b}\right|$. The only time the amplitudes would be equal is when $c = \frac{\pi}{4} + \pi n$ for integers n .

Discussion (5 minutes)

Before moving on to the next exercises, debrief student work by having one or two students present their graphs to the class or model the problem directly using GeoGebra or other graphing software that can be projected to the class. Then show students how to algebraically determine the equation of the function h in Exercise 2.

- In Exercise 2, how could you be sure that the function you wrote is correct?
 - *We would have to algebraically determine the values a and c .*
- In Exercise 3, we used an identity to show that $f(x) + g(x) = 0$. How could we show that $4 \cos(x) + 3 \sin(x) = a \cos(x - c)$ for some values of a and c ?
 - *Since one side is a sum, rewrite the right side of the equation using the sum identity for cosine.*

Start by rewriting using an identity:

$$a \cos(x - c) = a(\cos(x) \cos(c) + \sin(x) \sin(c)).$$

Then equate $4 \cos(x) + 3 \sin(x) = a(\cos(x) \cos(c) + \sin(x) \sin(c))$ by matching multiples of $\cos(x)$ and $\sin(x)$.

This gives $4 \cos(x) = a \cos(x) \cos(c)$ and $3 \sin(x) = a \sin(x) \sin(c)$. Since these equations are true for all values of x , they are true where $x \neq 0$, so we can divide by $\cos(x)$ or $\sin(x)$ as appropriate. We then have $a \cos(c) = 4$ and $a \sin(c) = 3$.

There are other ways to solve for the values of a and c besides the method modeled below. After rewriting the general cosine formula using the difference identity, let students work within their groups for a few minutes to determine the values of a and c .

Since $a \cos(c) = 4$ and $a \sin(c) = 3$, we know that $\tan(c) = \frac{3}{4}$.

Thus, $c = \tan^{-1}\left(\frac{3}{4}\right) \approx 0.64$.

Then, solving each equation for sine and cosine gives $\cos(c) = \frac{4}{a}$ and $\sin(c) = \frac{3}{a}$.

Since $\sin^2(c) + \cos^2(c) = 1$, we have $\frac{16}{a^2} + \frac{9}{a^2} = 1$; therefore, $a = 5$.

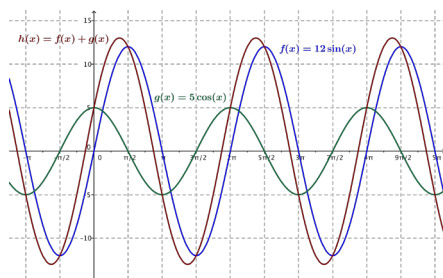
Substituting the values back into the equation gives us $h(x) \approx 5 \cos(x - 0.64)$.

Students should confirm using a graphing calculator that $4 \cos(x) + 3 \sin(x) \approx 5 \cos(x - 0.64)$ by graphing the functions $f(x) = 4 \cos(x) + 3 \sin(x)$ and $g(x) = 5 \cos(x - 0.64)$.

Exercises 5–7 (10 minutes)

5. Find an exact formula for $h(x) = 12\sin(x) + 5\cos(x)$ in the form $h(x) = a \cos(x - c)$. Graph $f(x) = 12\sin(x)$, $g(x) = 5\cos(x)$, and $h(x) = 12\sin(x) + 5\cos(x)$ together on the same axes.

$$\begin{aligned} 12\sin(x) + 5\cos(x) &= a \cos(x - c) \\ &= a(\cos(x)\cos(c) + \sin(x)\sin(c)) \\ &= a \cos(c)\cos(x) + a \sin(c)\sin(x) \end{aligned}$$



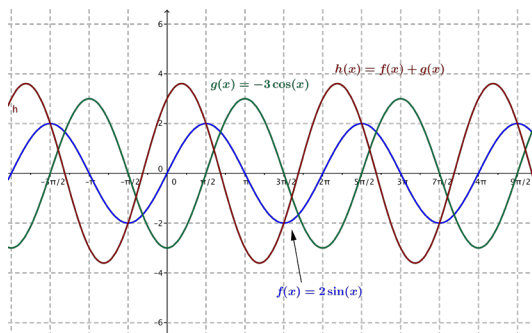
Then $12\sin(x) = a \sin(c) \sin(x)$ and $5\cos(x) = a \cos(c) \cos(x)$, so we have $\cos(c) = \frac{5}{a}$ and $\sin(c) = \frac{12}{a}$. Since $\sin^2(c) + \cos^2(c) = 1$, we have $\frac{144}{a^2} + \frac{25}{a^2} = 1$. Thus, $169 = a^2$ and $a = 13$. Then, $\tan(c) = \frac{12}{5}$, so $c = \arctan\left(\frac{12}{5}\right) \approx 1.176$. Thus, $h(x) = 13\cos(x - 1.176)$.

Scaffolding:

- This would be an appropriate point in the lesson to rely on technology for students who are still struggling to sketch graphs of sinusoidal functions quickly by hand.
- When using graphing technology, be sure to set up the graphics viewing window with the same scale and dimensions as the graphs shown in the student materials.

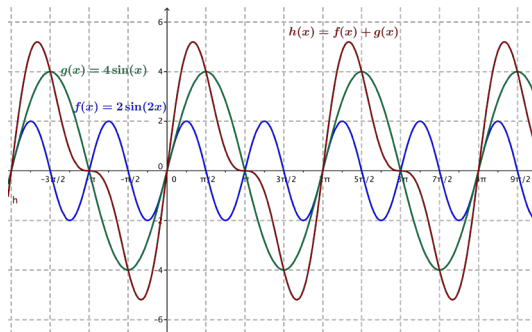
6. Find an exact formula for $h(x) = 2\sin(x) - 3\cos(x)$ in the form $h(x) = a \cos(x - c)$. Graph $f(x) = 2\sin(x)$, $g(x) = -3\cos(x)$, and $h(x) = 2\sin(x) - 3\cos(x)$ together on the same axes.

$$\begin{aligned} 2\sin(x) - 3\cos(x) &= a \cos(x - c) \\ &= a(\cos(x)\cos(c) + \sin(x)\sin(c)) \\ &= a \cos(c)\cos(x) + a \sin(c)\sin(x) \end{aligned}$$



Then, $2\sin(x) = a \sin(c) \sin(x)$ and $-3\cos(x) = a \cos(c)\cos(x)$, so we have $\cos(c) = \frac{-3}{a}$ and $\sin(c) = \frac{2}{a}$. Since $\sin^2(c) + \cos^2(c) = 1$, we have $\frac{9}{a^2} + \frac{4}{a^2} = 1$. Thus, $13 = a^2$ and $a = \pm\sqrt{13}$. Then, $\tan(c) = \frac{2}{-3}$, so $c = \arctan\left(\frac{2}{-3}\right) \approx -0.588$. Because $h(0) = -3$, and $\cos(-0.588) > 0$, we know that $a < 0$, and thus, $a = -\sqrt{13}$. Thus, $h(x) = -\sqrt{13}\cos(x + 0.588)$.

7. Can you find an exact formula for $h(x) = 2\sin(2x) + 4\sin(x)$ in the form $h(x) = a \sin(x - c)$? If not, why not? Graph $f(x) = 2\sin(2x)$, $g(x) = 4\sin(x)$, and $h(x) = 2\sin(2x) + 4\sin(x)$ together on the same axes.



Although it is periodic with period 2π , the graph of h is not the graph of a function $y = a \sin(x - c)$. Thus, we cannot write the function h in the specified form.

Closing (3 minutes)

To close this lesson, have students respond to the questions below in writing or with a partner. Review the Lesson Summary as time permits.

- What are the two types of waves, and how are they different?
 - *The two types of waves are transverse and longitudinal. Transverse waves vibrate perpendicular to the direction of travel. Longitudinal waves vibrate in the direction of travel.*
- How can we use sinusoidal functions to model waves?
 - *The waves repeat the displacement periodically, which we can model with a sine function.*
- How does the sum identity for sine help us understand the wave model?
 - *By rewriting the formula using the identity, we can better analyze the patterns of the nodes and anti-nodes of a wave caused by interference.*

Lesson Summary

A wave is displacement that travels through a medium. Waves transfer energy, not matter. There are two types of waves: transverse and longitudinal. Sound waves are an example of longitudinal waves.

When two or more waves meet, interference occurs and can be represented mathematically as the sum of the individual waves.

The sum identity for sine is useful for analyzing the features of wave interference.

Exit Ticket (4 minutes)

Name _____

Date _____

Lesson 6: Waves, Sinusoids, and Identities

Exit Ticket

1. Use appropriate identities to rewrite the wave equation shown below in the form $h(x) = a \cos(x - c)$.

$$h(x) = 6 \sin(x) + 8 \cos(x)$$

2. Rewrite $h(x) = \sqrt{3} \sin(x) + \cos(x)$ in the form $h(x) = a \cos(x - c)$.

Exit Ticket Sample Solutions

1. Use appropriate identities to rewrite the equation shown below in the form $h(x) = a \cos(x - c)$.

$$h(x) = 6 \sin(x) + 8 \cos(x)$$

$$6 \sin(x) + 8 \cos(x) = a \cos(c) \cos(x) + a \sin(c) \sin(x)$$

Then, we have $6 = a \sin(c)$ and $8 = a \cos(c)$, so $\sin(c) = \frac{6}{a}$ and $\cos(c) = \frac{8}{a}$. Since $\sin^2(c) + \cos^2(c) = 1$, we have $\frac{36}{a^2} + \frac{64}{a^2} = 1$, so $a = 10$. It follows that $\tan(c) = \frac{6}{8}$, so $c = \arctan\left(\frac{6}{8}\right) \approx 0.644$. Thus, $h(x) = 10 \cos(x - 0.644)$.

2. Rewrite $h(x) = \sqrt{3} \sin(x) + \cos(x)$ in the form $h(x) = a \cos(x - c)$.

$$c = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$$

$$a = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$h(x) = 2 \cos\left(x - \frac{\pi}{3}\right)$$

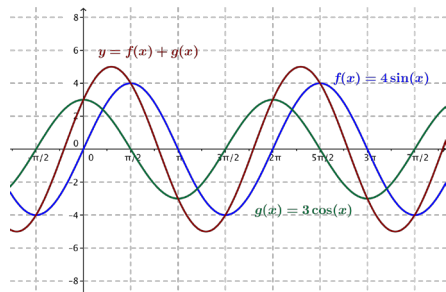
Problem Set Sample Solutions

1. Rewrite the sum of the following functions in the form $f(x) + g(x) = c \cos(x + k)$. Graph $y = f(x)$, $y = g(x)$, and $y = f(x) + g(x)$ on the same set of axes.

a. $f(x) = 4 \sin(x)$; $g(x) = 3 \cos(x)$

$$c = \sqrt{(4)^2 + (3)^2} = 5, a = \arctan\left(\frac{4}{3}\right) \approx 0.927$$

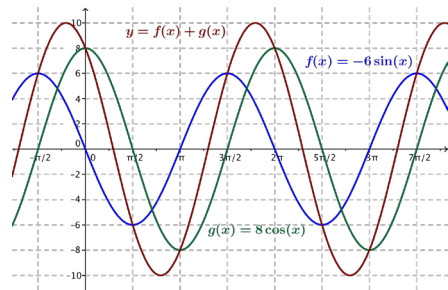
$$f(x) + g(x) = 5 \cos(x - 0.927)$$



b. $f(x) = -6 \sin(x); g(x) = 8 \cos(x)$

$$c = \sqrt{(-6)^2 + 8^2} = 10, a = \arctan\left(-\frac{6}{8}\right) \approx -0.644$$

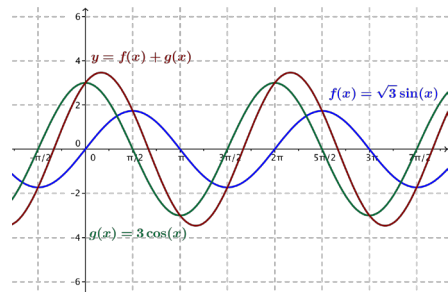
$$f(x) + g(x) = 10 \cos(x + 0.644)$$



c. $f(x) = \sqrt{3} \sin(x); g(x) = 3 \cos(x)$

$$c = \sqrt{(\sqrt{3})^2 + (3)^2} = 2\sqrt{3}, a = \arctan\left(\frac{\sqrt{3}}{3}\right) \approx 0.524$$

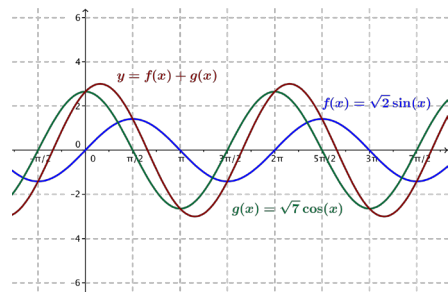
$$f(x) + g(x) = 2\sqrt{3} \cos(x - 0.524)$$



d. $f(x) = \sqrt{2} \sin(x); g(x) = \sqrt{7} \cos(x)$

$$c = \sqrt{(\sqrt{2})^2 + (\sqrt{7})^2} = 3, a = \arctan\left(\frac{\sqrt{2}}{\sqrt{7}}\right) \approx 0.491$$

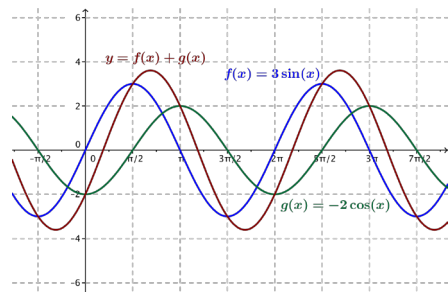
$$f(x) + g(x) = 3 \cos(x - 0.491)$$



e. $f(x) = 3 \sin(x)$; $g(x) = -2 \cos(x)$

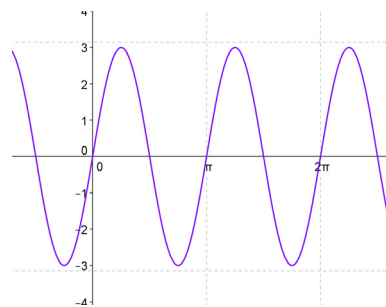
$$c = \sqrt{(-2)^2 + (3)^2} = \pm\sqrt{13}, a = \arctan\left(\frac{3}{-2}\right) \approx -0.983$$

$$f(x) + g(x) = -\sqrt{13} \cos(x + 0.983)$$



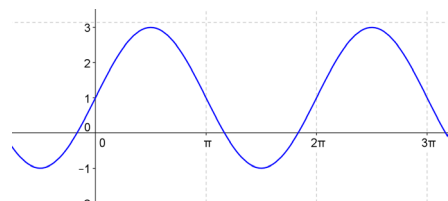
2. Find a sinusoidal function $f(x) = a \sin(bx + c) + d$ that fits each of the following graphs.

a.



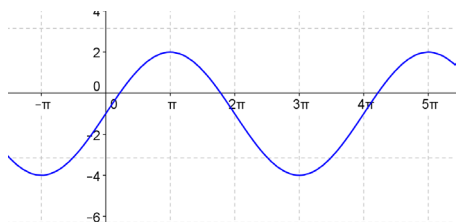
$$f(x) = 3 \sin(2x)$$

b.



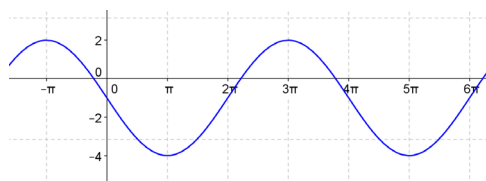
$$f(x) = 2 \sin(x) + 1$$

c.



$$f(x) = 3 \sin\left(\frac{x}{2}\right) - 1$$

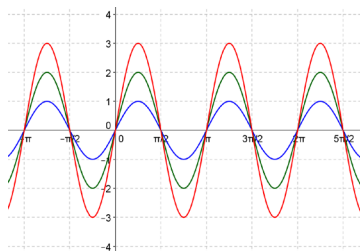
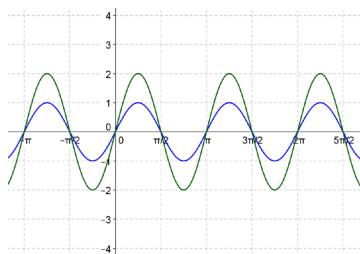
d.



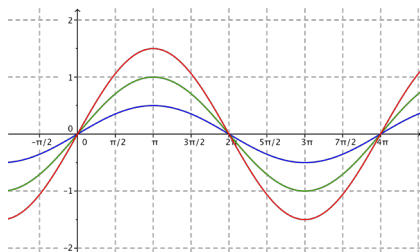
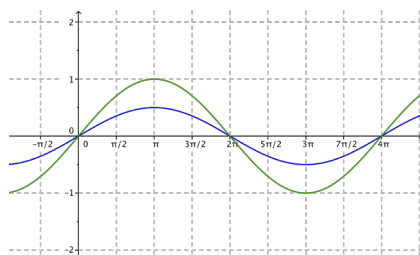
$$f(x) = -3 \sin\left(\frac{x}{2}\right) - 1$$

3. Two functions f and g are graphed below. Sketch the graph of the sum $f + g$.

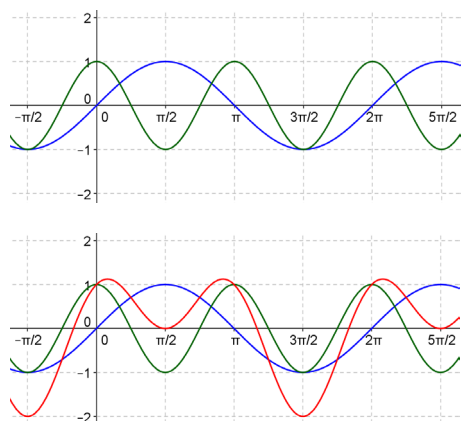
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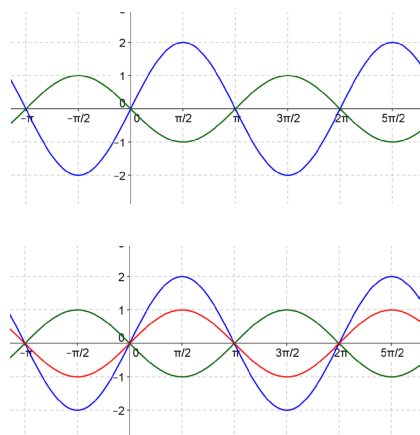
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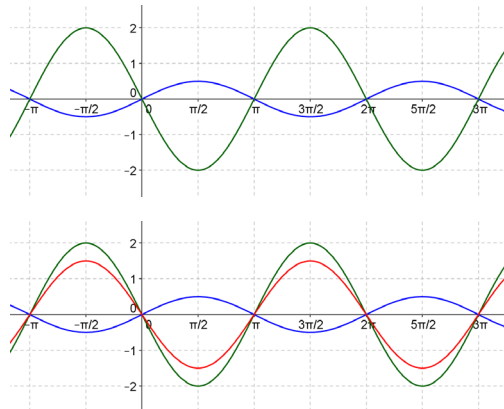
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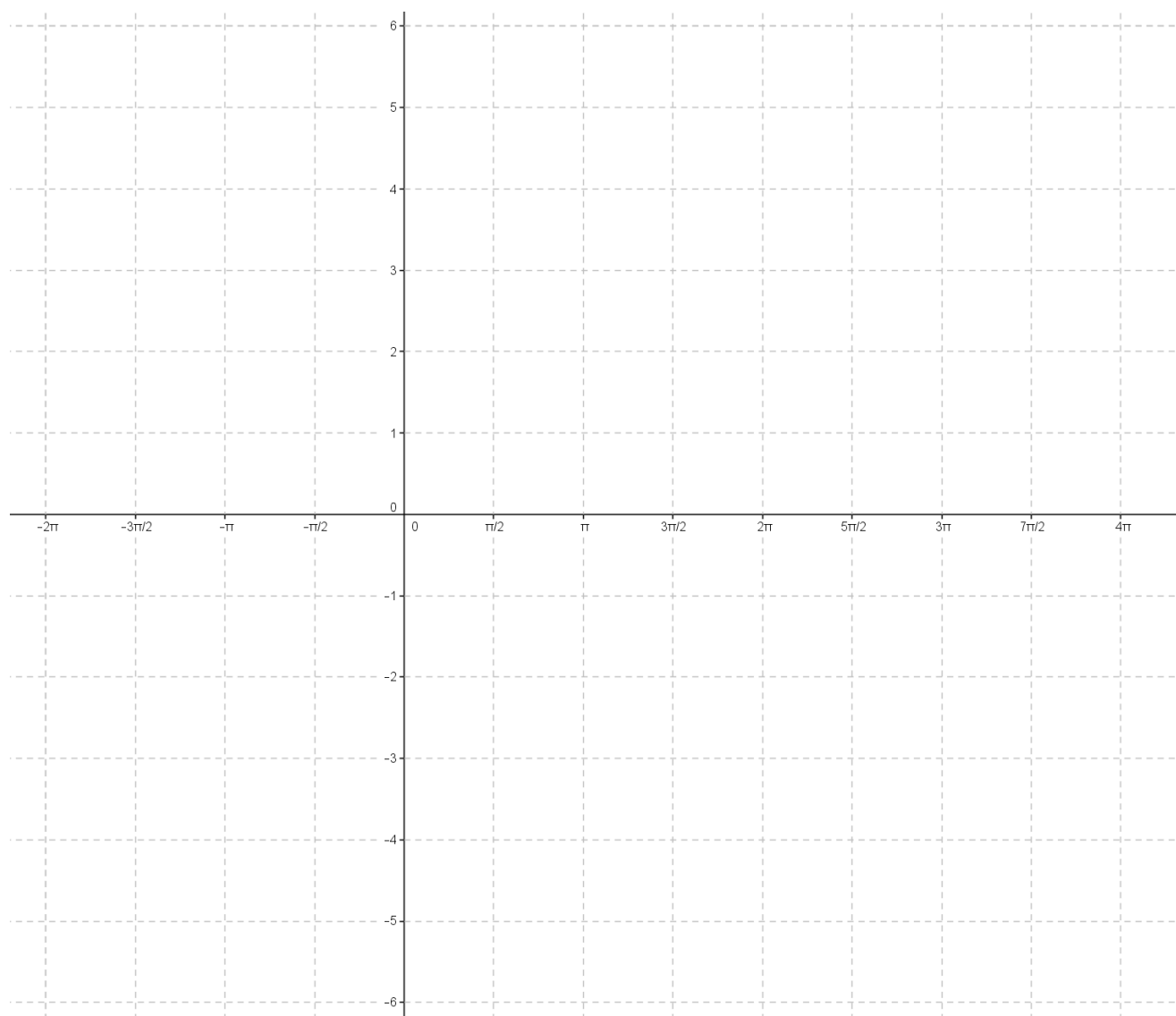
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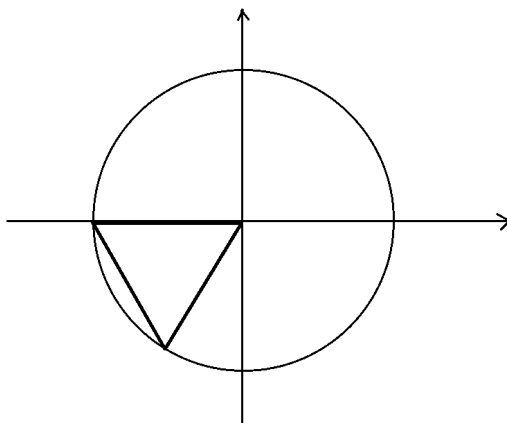
Blank Graph Paper with Trigonometric Scale



Name _____

Date _____

1. An equilateral triangle is drawn within the unit circle centered at the origin as shown.



Explain how one can use this diagram to determine the values of $\sin\left(\frac{4\pi}{3}\right)$, $\cos\left(\frac{4\pi}{3}\right)$, and $\tan\left(\frac{4\pi}{3}\right)$.

2. Suppose x is a real number with $0 < x < \frac{\pi}{4}$.
- a. Set $a = \sin(\pi - x)$, $b = \cos(\pi + x)$, $c = \sin(x - \pi)$, and $d = \cos(2\pi - x)$.
Arrange the values a , b , c , and d in increasing order, and explain how you determined their order.

b. Use the unit circle to explain why $\tan(\pi - x) = -\tan(x)$.

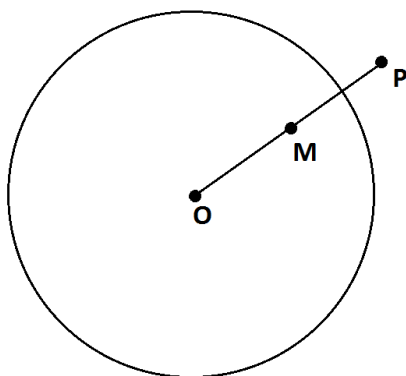
3.

a. Using a diagram of the unit circle centered at the origin, explain why $f(x) = \cos(x)$ is an even function.

- b. Using a diagram of the unit circle centered at the origin, explain why $\sin(x - 2\pi) = \sin(x)$ for all real values x .

- c. Explain why $\tan(x + \pi) = \tan(x)$ for all real values x .

4. The point P shown lies outside the circle with center O . Point M is the midpoint of \overline{OP} .



- a. Use a ruler and compass to construct a line through P that is tangent to the circle.

- b. Explain how you know that your construction does indeed produce a tangent line.

5. Each rectangular diagram below contains two pairs of right triangles, each having a hypotenuse of length 1. One pair of triangles has an acute angle measuring x radians. The other pair of triangles has an acute angle measuring y radians.

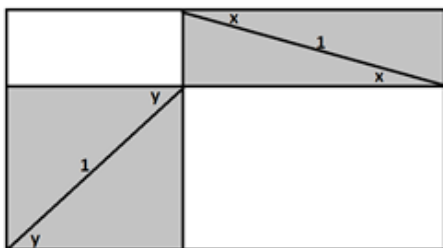


Figure 1

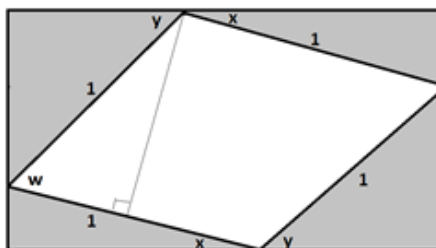


Figure 2

- a. Using Figure 1, write an expression, in terms of x and y , for the area of the non-shaded region.

- b. Figure 2 contains a quadrilateral which is not shaded and contains angle w . Write an expression, in terms of x and y , for the measure of angle w .
- c. Using Figure 2, write an expression, in terms of w , for the non-shaded area. Explain your work.

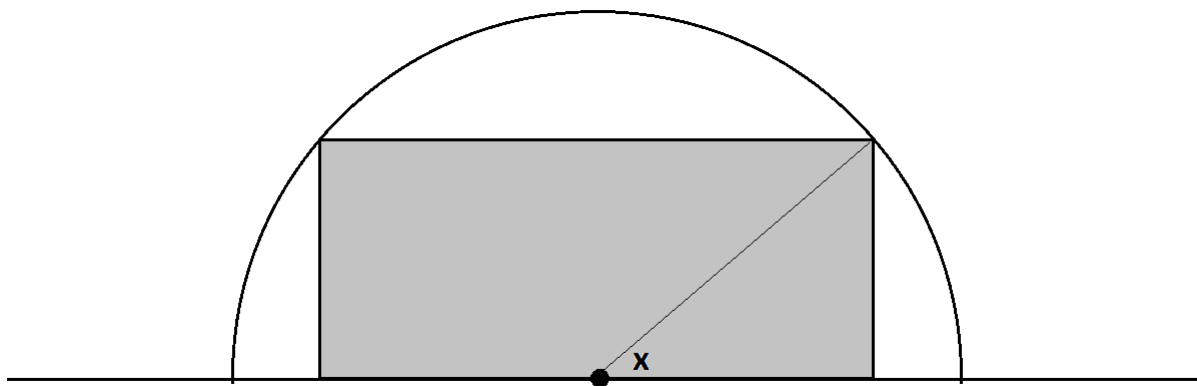
- d. Use the results of parts (a), (b), and (c) to show why $\sin(x + y) = \sin(x) \cos(y) + \sin(y) \cos(x)$ is a valid formula.

- e. Suppose α is a real number between $\frac{\pi}{2}$ and π and y is a real number between 0 and $\frac{\pi}{2}$. Use your result from part (d) to show the following:

$$\cos(\alpha + y) = \cos(\alpha) \cos(y) - \sin(\alpha) \sin(y).$$

Explain your work.

6. A rectangle is drawn in a semicircle of radius 3 with its base along the base of the semicircle as shown.



Find, to two decimal places, values for real numbers a and b so that $a \cos(x + b)$ represents the perimeter of the rectangle if the real number x is the measure of the angle shown.

A Progression Toward Mastery

Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem, OR an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	F-TF.A.3	Student shows little or no understanding of finding the angle.	Student shows some understanding of finding the angle and knows that the triangle is equilateral.	Student shows understanding of finding the angle using equilateral triangles and the Pythagorean theorem in his explanation but makes a mathematical error leading to an incorrect answer.	Student completely and correctly shows understanding of finding the angle by using equilateral triangles and the Pythagorean theorem in his explanation and makes no mathematical errors.
2	a F-TF.A.3	Student shows little or no understanding of the values of trigonometric functions.	Student correctly orders at least two of the values of a , b , c , or d .	Student orders a , b , c , and d correctly but lists them in increasing order. OR Student orders a , b , c , and d correctly and lists them in increasing order but with a missing or an incorrect explanation.	Student orders a , b , c , and d correctly and lists them in increasing order with a correct explanation.
	b F-TF.A.3	Student shows little or no understanding of the periodicity of trigonometric functions.	Student attempts to use periodicity of trigonometric functions but only sine or cosine is correct.	Student understands the periodicity of sine and cosine but does not relate it to tangent.	Student understands the periodicity of sine and cosine and correctly relates it to tangent.

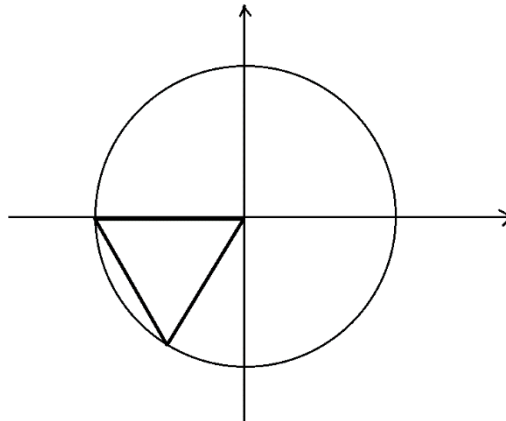
3	a F-TF.A.4	Student shows little or no understanding of the properties of trigonometric functions.	Student draws the unit circle diagram but not correctly.	Student draws the unit circle diagram correctly but does not clearly explain how symmetry guarantees this result.	Student draws the unit circle diagram correctly and clearly explains the result using symmetry.
	b F-TF.A.4	Student shows little or no understanding of the properties of trigonometric functions.	Student draws the unit circle diagram but not correctly.	Student draws the unit circle diagram correctly but does not clearly explain how periodicity guarantees this result.	Student draws the unit circle diagram correctly and clearly explains the result using periodicity.
	c F-TF.A.4	Student shows little or no understanding of the properties of trigonometric functions.	Student draws the unit circle diagram but not correctly.	Student draws the unit circle diagram correctly but does not clearly explain how symmetry guarantees this result.	Student draws the unit circle diagram correctly and clearly explains the result using symmetry.
4	a G-C.A.4	Student shows little or no understanding of tangent line construction.	Student completes one step of the construction correctly.	Student completes two steps of the construction correctly.	Student correctly constructs the tangent line to the circle.
	b G-C.A.4	Student shows little or no understanding of the tangent line to a circle.	Student attempts to explain why the line is tangent to the circle but with major mathematical errors.	Students explains why the line is tangent to the circle but does not explain that the tangent line is perpendicular to the radius at the point of tangency.	Students explains why the line is tangent to the circle and explains that the tangent line is perpendicular to the radius at the point of tangency.
5	a F-TF.C.9	Student shows little or no understanding of the area of the rectangle.	Student attempts to find the area but with major mathematical errors.	Student finds the correct area in terms of x and y , but the explanation is not complete.	Student correctly finds the area in terms of x and y , and the explanation is complete.
	b F-TF.C.9	Students shows little or no understanding of the angles in the diagram.	Student attempts to find the angle measure in terms of x and y but with major mathematical errors.	Student finds the correct angle measure in terms of x and y , but the explanation is not complete.	Student correctly finds the angle measure in terms of x and y , and the explanation is complete.
	c F-TF.C.9	Student shows little or no understanding of the area of the rectangle.	Student attempts to find the area but with major mathematical errors.	Student finds the correct area in terms of w , but the explanation is not complete.	Student correctly finds the area in terms of w , and the explanation is complete.
	d F-TF.C.9	Student shows little or no understanding of the sum of angles of a trigonometric function.	Student attempts to write a formula but with major mathematical errors.	Student writes the correct formula, but the explanation is not complete.	Student correctly writes the formula, and the explanation is complete.

	e F-TF.C.9	Student shows little or no understanding of the sum of angles of a trigonometric function.	Student attempts to prove the formula is valid but with major mathematical errors.	Student attempts to prove the formula is valid but with minor mathematical errors.	Student correctly proves the formula is valid, and the explanation is complete.
6	F-TF.C.9	Student shows little or no understanding of trigonometric functions or perimeter.	Student sets up the correct perimeter formula in terms of sine and cosine but does not complete the problem.	Student finds either the value of a or b correctly with supporting work.	Student finds the values of both a and b correctly with supporting work.

Name _____

Date _____

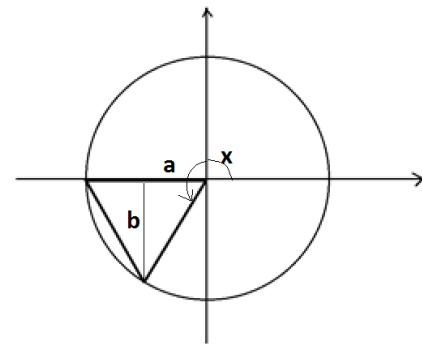
1. An equilateral triangle is drawn within the unit circle centered at the origin as shown.



Explain how one can use this diagram to determine the values of $\sin\left(\frac{4\pi}{3}\right)$, $\cos\left(\frac{4\pi}{3}\right)$, and $\tan\left(\frac{4\pi}{3}\right)$.

Interior angle of an equilateral triangle has a measure of $\frac{\pi}{3}$ and so the measure of the angle x shown is $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

Draw the altitude of the equilateral triangle, and mark the indicated lengths a and b as shown. We have that $\cos(x) = -a$ and $\sin(x) = -b$.



Now a is half the base of an equilateral triangle of side length 1, so $a = \frac{1}{2}$. By the Pythagorean theorem,

$b = \sqrt{(1)^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$. Thus, we have the following:

$$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$$

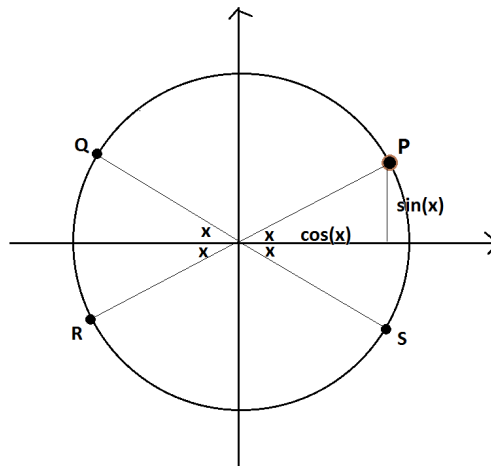
$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\tan\left(\frac{4\pi}{3}\right) = \frac{\sin\left(\frac{4\pi}{3}\right)}{\cos\left(\frac{4\pi}{3}\right)} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

2. Suppose x is a real number with $0 < x < \frac{\pi}{4}$.

- a. Set $a = \sin(\pi - x)$, $b = \cos(\pi + x)$, $c = \sin(x - \pi)$, and $d = \cos(2\pi - x)$. Arrange the values a , b , c , and d in increasing order, and explain how you determined their order.

Draw the unit circle and the point P on the circle with coordinates $(\cos(x), \sin(x))$. Because $0 < x < \frac{\pi}{4}$, we have that both $\sin(x)$ and $\cos(x)$ are positive numbers with $\sin(x) < \cos(x)$.



The points Q , R , and S on the diagram have coordinates:

$$Q = (\cos(\pi - x), \sin(\pi - x))$$

$$R = (\cos(\pi + x), \sin(\pi + x))$$

$$S = (\cos(2\pi - x), \sin(2\pi - x))$$

We also see from the diagram that $(\cos(x - \pi), \sin(x - \pi))$ is also the point R .

From the diagram we have

$$a = \sin(\pi - x) = \sin(x) \text{ (looking at the point } Q),$$

$$b = \cos(\pi + x) = -\cos(x) \text{ (looking at the point } R),$$

$$c = \sin(x - \pi) = -\sin(x) \text{ (looking at the point } R),$$

$$d = \cos(2\pi - x) = \cos(x) \text{ (looking at the point } S).$$

Since $\sin(x) < \cos(x)$ it follows that $b < c < a < d$.

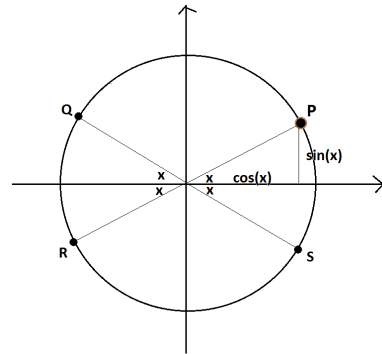
- b. Use the unit circle to explain why $\tan(\pi - x) = -\tan(x)$.

Look at the diagram of the four points on the unit circle from the previous question.

We are interested in $\tan(\pi - x) = \frac{\sin(\pi - x)}{\cos(\pi - x)}$. The point Q

has coordinates $(\cos(\pi - x), \sin(\pi - x))$ and we see, in relation to the point P, that $\sin(\pi - x) = \sin(x)$ and $\cos(\pi - x) = -\cos(x)$. Thus,

$$\tan(\pi - x) = \frac{\sin(\pi - x)}{\cos(\pi - x)} = \frac{\sin(x)}{-\cos(x)} = -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$



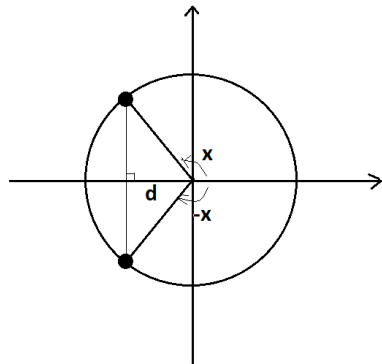
3.

- a. Using a diagram of the unit circle centered at the origin, explain why $f(x) = \cos(x)$ is an even function.

We need to show that $f(-x) = f(x)$, that is, $\cos(-x) = \cos(x)$ for all real numbers x .

Let d represent the length of the segment shown in the diagram. We see symmetry in the diagram.

From the diagram, $\cos(x) = -d$ and $\cos(-x) = -d$, that is, $\cos(x) = \cos(-x)$.

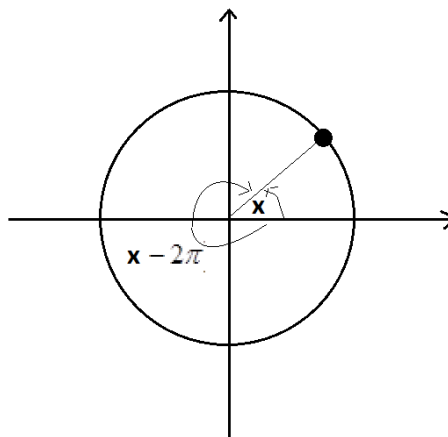


Note: Here we drew a diagram with x representing the measure of an obtuse angle. The same symmetry applies to all types of angles.

- b. Using a diagram of the unit circle centered at the origin, explain why $\sin(x - 2\pi) = \sin(x)$ for all real values x .

We see from a diagram that the points on the unit circle with coordinates $(\cos(x), \sin(x))$ and $(\cos(x - 2\pi), \sin(x - 2\pi))$ coincide.

Thus, $\sin(x - 2\pi) = \sin(x)$.

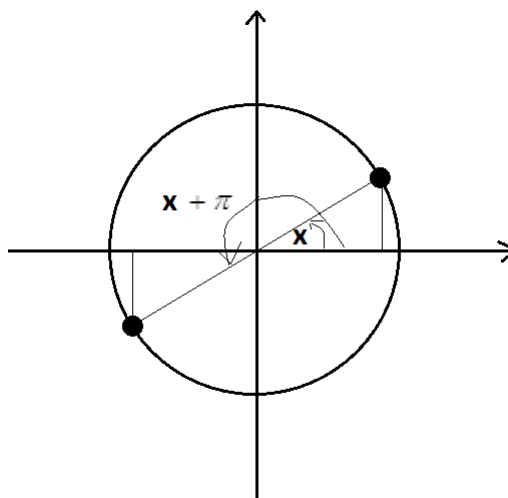


- c. Explain why $\tan(x + \pi) = \tan(x)$ for all real values x .

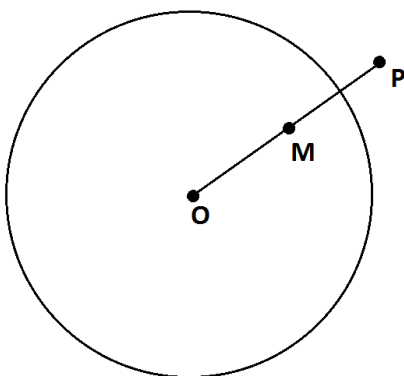
The symmetry of the following diagram shows that $\sin(x + \pi) = -\sin(x)$ and $\cos(x + \pi) = -\cos(x)$.

Thus,

$$\tan(x + \pi) = \frac{\sin(x + \pi)}{\cos(x + \pi)} = \frac{-\sin(x)}{-\cos(x)} = \frac{\sin(x)}{\cos(x)} = \tan(x).$$

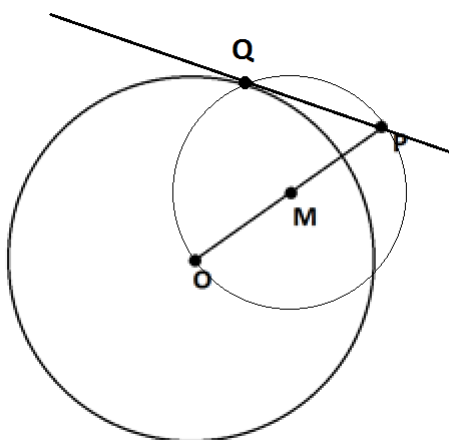


4. The point P shown lies outside the circle with center O . Point M is the midpoint of \overline{OP} .



- a. Use a ruler and compass to construct a line through P that is tangent to the circle.

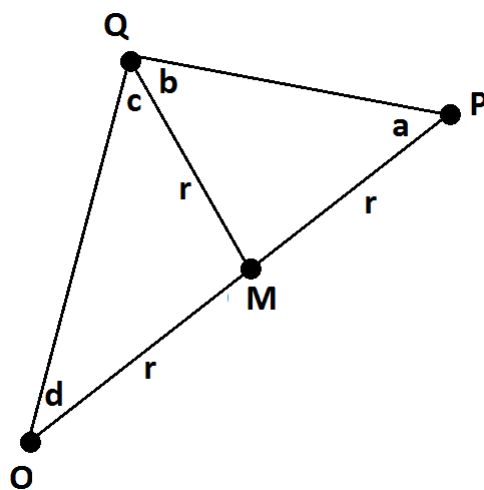
Draw a circle with center M and with radius set to the length OM . This new circle intersects the original circle at two points. Call one of those two points Q .



Draw the line through P and Q . This is a line through P and tangent to the circle.

- b. Explain how you know that your construction does indeed produce a tangent line.

Reason: Since we drew a circle with center M , we have $MO = MQ = MP$. Call this common length r . Thus, we have a diagram containing two isosceles triangles.



Mark the angles a , b , c , and d as shown.

Because they are base angles of an isosceles triangle, $a = b$.

Because they are base angles of an isosceles triangle, $d = c$.

Because angles in a triangle sum to π radians,

$$a + b + c + d = \pi.$$

That is, $2b + 2c = \pi$ giving

$$b + c = \frac{\pi}{2}.$$

Thus, the radius \overline{OQ} in the original circle meets \overline{PQ} at a right angle. It must be then that \overline{PQ} is indeed a tangent line to the original circle.

5. Each rectangular diagram below contains two pairs of right triangles, each having a hypotenuse of length 1. One pair of triangles has an acute angle measuring x radians. The other pair of triangles has an acute angle measuring y radians.

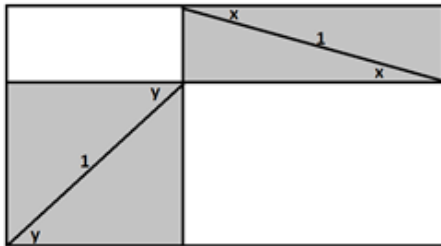


Figure 1

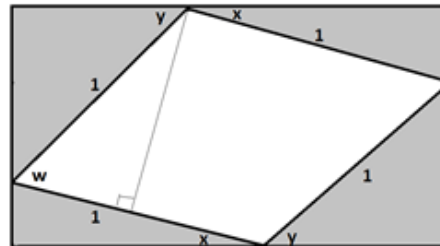
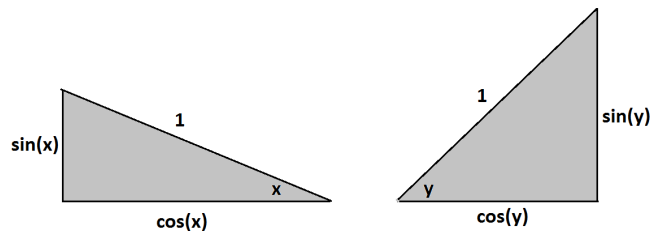


Figure 2

- a. Using Figure 1, write an expression, in terms of x and y , for the area of the non-shaded region.

The right triangles used in the diagrams have side lengths as shown:



In Figure 1, the area we seek is the sum of areas of two rectangles: one $\sin(x)$ -by- $\cos(y)$ rectangle and one $\sin(y)$ -by- $\cos(x)$ rectangle.

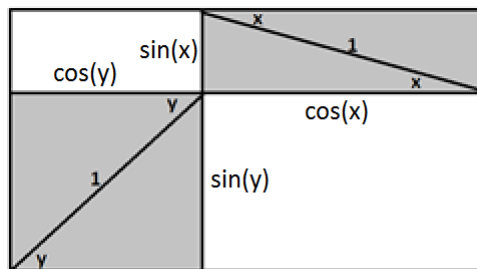
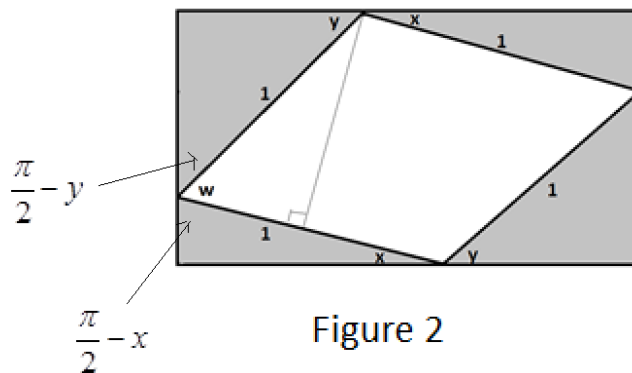


Figure 1

Thus, the area under consideration is given by $\sin(x)\cos(y) + \sin(y)\cos(x)$.

- b. Figure 2 contains a quadrilateral which is not shaded and contains angle w . Write an expression, in terms of x and y , for the measure of angle w .

In Figure 2, we see that three angles of measures $\frac{\pi}{2} - x$, w , and $\frac{\pi}{2} - y$ form a straight angle:



$$\text{Thus, } w = \pi - \left(\frac{\pi}{2} - x\right) - \left(\frac{\pi}{2} - y\right) = x + y.$$

- c. Using Figure 2, write an expression, in terms of w , for the non-shaded area. Explain your work.

We seek the area of a rhombus with a side length of 1. The area of a rhombus (or any parallelogram) is given by “base times height.” Here the base is 1.

Now w is an angle in a right triangle with hypotenuse 1. Noting this, we see that the height of our rhombus is $\sin(w)$. Thus, the area we seek is $1 \times \sin(w) = \sin(w)$.

- d. Use the results of parts (a), (b), and (c) to show why $\sin(x + y) = \sin(x) \cos(y) + \sin(y) \cos(x)$ is a valid formula.

The area inside the rectangle but outside the four right triangles is the same for both diagrams. Thus, our two different computations for this common area must be the same:

$$\sin(x) \cos(y) + \cos(x) \sin(y) = \sin(w).$$

Since $w = x + y$, we have the following formula:

$$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y).$$

- e. Suppose α is a real number between $\frac{\pi}{2}$ and π , and y is a real number between 0 and $\frac{\pi}{2}$. Use your result from part (d) to show the following:

$$\cos(\alpha + y) = \cos(\alpha) \cos(y) - \sin(\alpha) \sin(y).$$

Explain your work.

The formula we derived in part (d) is valid for any two real numbers x and y that represent the measures of acute angles. If $\frac{\pi}{2} < \alpha < \pi$, then $0 < \alpha - \frac{\pi}{2} < \frac{\pi}{2}$. Let's set $x = \alpha - \frac{\pi}{2}$. Then x and y are two real numbers representing the measures of acute angles, and so we have the following:

$$\sin(x + y) = \sin(x) \cos(y) + \cos(x) \sin(y).$$

This reads

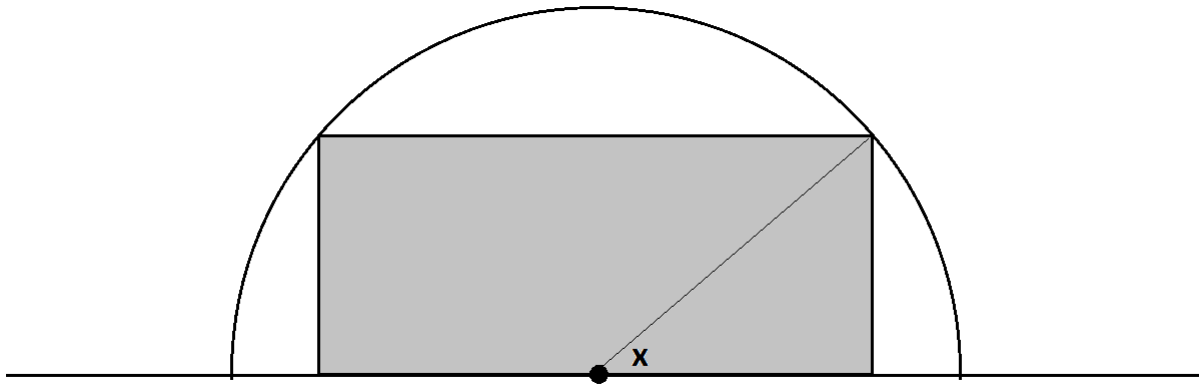
$$\sin\left(\alpha + y - \frac{\pi}{2}\right) = \sin\left(\alpha - \frac{\pi}{2}\right) \cos(y) + \cos\left(\alpha - \frac{\pi}{2}\right) \sin(y).$$

Using $\sin\left(\theta - \frac{\pi}{2}\right) = -\cos(\theta)$ and $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$, we see the following:

$$-\cos(\alpha + y) = -\cos(\alpha) \cos(y) + \sin(\alpha) \sin(y).$$

Multiplying through by -1 gives the result stated in the question.

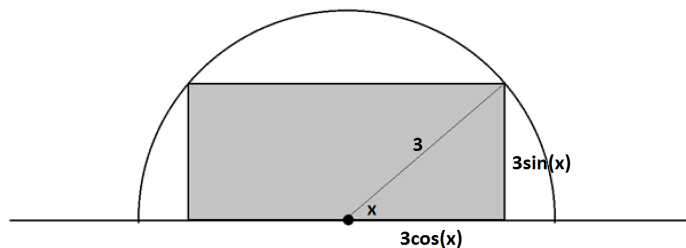
6. A rectangle is drawn in a semicircle of radius 3 with its base along the base of the semicircle as shown.



Find, to two decimal places, values for real numbers a and b so that $a \cos(x + b)$ represents the perimeter of the rectangle if the real number x is the measure of the angle shown.

We see that the perimeter of the rectangle is

$$2 \cdot 3 \sin(x) + 4 \cdot 3 \cos(x) = 6 \sin(x) + 12 \cos(x).$$



We wish to write this expression in the form $a \cos(x + b)$:

$$a \cos(x + b) = a \cos(x) \cos(b) - a \sin(x) \sin(b).$$

This suggests the following:

$$a \cos(b) = 12$$

$$-a \sin(b) = 6$$

Using $\sin^2(b) + \cos^2(b) = 1$ this gives $\frac{144}{a^2} + \frac{36}{a^2} = 1$

Therefore, $a = \sqrt{180} \approx 13.42$ is a candidate value for a .

Also,

$$-\tan(b) = \frac{-a \sin(b)}{a \cos(b)}$$

$$-\tan(b) = \frac{6}{12}$$

$$b = \tan^{-1}\left(-\frac{1}{2}\right)$$

$$b \approx -0.46$$

Thus b , is the measure of an angle in the fourth quadrant in this instance.

So let's examine $13.42 \cos(x - 0.46)$:

$$\begin{aligned} 13.42 \cos(x - 0.46) &= 13.42 \cos(x) \cos(-0.46) - 13.42 \sin(x) \sin(-0.46) \\ &\approx 13.42 \cdot \cos(x) \cdot 0.90 - 13.42 \cdot \sin(x) \cdot (-0.44) \\ &\approx 12 \cos(x) + 6 \sin(x) \end{aligned}$$

Therefore, the two numbers that gives us an expression, $a \cos(x - b)$, that represents the perimeter of the rectangle are $a = 13.42$ and $b = -0.46$.



Topic B

Trigonometry and Triangles

G-SRT.D.9, G-SRT.D.10, G-SRT.D.11

Focus Standards:	G-SRT.D.9	(+) Derive the formula $A = 1/2absin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
	G-SRT.D.10	(+) Prove the Laws of Sines and Cosines and use them to solve problems.
	G-SRT.D.11	(+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).
Instructional Days:	4	
Lesson 7:	An Area Formula for Triangles (E) ¹	
Lesson 8:	Law of Sines (E)	
Lesson 9:	Law of Cosines (E)	
Lesson 10:	Putting the Law of Cosines and the Law of Sines to Use (P)	

In Topic B, students derive sophisticated applications of the trigonometric functions: the area formula for a general triangle, $A = \frac{1}{2}ab \sin(\theta)$ (**G-SRT.D.9**), the law of sines and the law of cosines, and use them to solve problems (**G-SRT.D.10**, **G-SRT.D.11**).

Lesson 7 starts with triangles in the Cartesian plane as students discover that the height of a triangle can be calculated using the sine function for rotations that correspond to angles between 0° and 180° . They calculate the areas of all types of triangles and generalize their work to derive a formula for the area of any oblique triangle (a triangle with no right angles) (**G-SRT.D.9**). In Lessons 8 and 9, students continue their study of oblique triangles as they add the laws of sines and cosines to the set of tools they use to analyze the measurements of triangles (**G-SRT.D.10**). The law of sines is proven as students write equivalent expressions for the height of an oblique triangle. To prove the law of cosines, students build squares on the sides of an oblique triangle, then analyze the areas, building on their prior knowledge of the Pythagorean theorem. Students distinguish between triangles that can be solved by using the law of sines versus using the law of cosines in Lesson 10 and then, in groups, study and solve up to eight different application problems (**G-SRT.D.11**).

¹Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

In Topic B, students construct convincing arguments as they calculate the area of different types of triangles (MP.3) and see the laws of sines and cosines as tools for analyzing angle measurements in triangles (MP.5). In Lesson 10, students apply the laws of sines and cosines as they use vectors and oblique triangles to model a surveying problem and the motion of boats and airplanes (MP.4).



Lesson 7: An Area Formula for Triangles

Student Outcomes

- Students prove the formula $\text{Area} = \frac{1}{2}bc \sin(A)$ for a triangle. They reason geometrically and numerically to find the areas of various triangles.

Lesson Notes

This lesson starts with triangles in the Cartesian plane. Students discover that the height of a triangle can be calculated using the sine function for rotations that correspond to angles between 0° and 180° . They create convincing arguments as they calculate the areas of various triangles and eventually generalize their work to derive a formula for the area of any oblique triangle. The focus of this lesson is **G-SRT.D.9**, which calls for students to derive an area formula for a triangle. Students complete an Exploratory Challenge that leads them to this formula through a series of triangle area problems. In Geometry, students derived the formula $\text{Area} = \frac{1}{2}bc \sin(A)$ in Module 2 Lesson 31. However, in that lesson, the formula only applied to acute triangles because the trigonometry functions were only defined for acute angles.

This lesson begins by introducing the area formula and then making a connection to the definition of the sine function presented in Algebra II Module 2. In the Exploratory Challenge, students start with triangles drawn in the coordinate plane with one vertex located at the origin and then generalize specific examples to derive the formula. The standard calls for the construction of an auxiliary line through one vertex of a triangle that is perpendicular to the side opposite the vertex, which is included in Exercises 7 and 8. By defining the height of a triangle in terms of the sine function definition from Algebra II, we do not need to treat the case with an obtuse triangle as separate from an acute triangle when deriving the formula. You also may wish to review the Geometry Module 2 Lesson 31 to see how we proved the formula.

Prior to the lesson, consider building students' fluency with converting between radians and degrees and with the relationships within special triangles.

Students need a calculator for this lesson.

Classwork

Opening (2 minutes)

Lead a short discussion to activate prior knowledge about measuring the area and perimeter of geometric figures.

Scaffolding:

- This lesson has quite a bit of scaffolding built in for students working below grade level. Students begin with concrete examples that can be solved using $\text{Area} = \frac{1}{2}bh$ and make connections to the trigonometry functions via special triangle relationships. Additional scaffolds are provided by situating these triangles in a coordinate grid to reinforce measurement concepts.
- For students working above grade level, skip the first two exercises and begin with the Exploratory Challenge on Exercise 3. Students can more fully explore the last exercise in this lesson, which uses the area formula to develop a general formula for the area of a regular polygon.
- For early finishers of Exercises 1 and 2, have groups come up with a second way to determine the area of the triangles.

- How could you quickly estimate the area and circumference of the circle as well as the area and perimeter of the triangles shown in Exercise 1?
 - *To estimate the area, we could count the square units and estimate the partial square units since these figures are located in the coordinate plane. To estimate the perimeter of the triangles and the circumference of the circle, we could use a piece of string to measure the length around each figure and then see how many units long it is by comparing it to the number line shown on the coordinate grid.*
- What formulas could we use to precisely calculate the area and circumference of the circle and the area and perimeter of the triangle shown in Exercise 1? What dimensions are needed to use these formulas?
 - *The area formula is $A = \pi r^2$ where r is the radius of a circle. The circumference of a circle is $C = \pi d$ where d is the diameter of the circle. The area of a triangle is $A = \frac{1}{2} b \cdot h$ where b is the length of one side and h is the altitude of the triangle. We find the perimeter of a triangle by adding the measures of all the sides. We would need to know the lengths of three sides to find the perimeter.*

Exploratory Challenge/Exercises 1–6 (15 minutes): Triangles in Circles

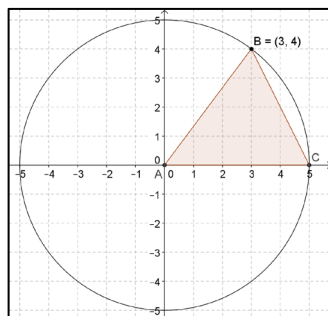
Organize your class into small groups, and have them work through the first few exercises in this Exploratory Challenge. These exercises are designed to scaffold students from concrete to more abstract examples to help them discover that they can determine an altitude of any triangle if they know an angle and a side adjacent to it using the sine function.

Exploratory Challenge/Exercises 1–6: Triangles in Circles

In this Exploratory Challenge, you will find the area of triangles with base along the positive x -axis and a third point on the graph of the circle $x^2 + y^2 = 25$.

1. Find the area of each triangle shown below. Show work to support your answer.

a.

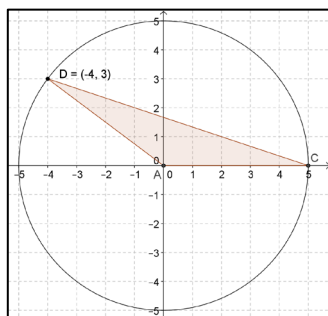


The base is given by the measure of \overline{AC} , which is 5 units. The height is the vertical distance from B to the horizontal axis, which is 4 units.

$$\frac{1}{2}(5)(4) = 10$$

The area is 10 square units.

b.



The base is 5 units and the height is 3 units.

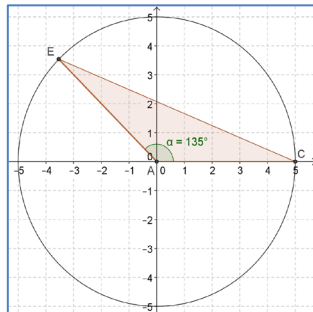
$$\frac{1}{2}(5)(3) = 7.5$$

The area is 7.5 square units.

If most of your students struggle on Exercise 1, consider pausing to review how to draw in the height of a triangle. Reinforce that the height is a segment from one vertex perpendicular to the opposite side (**G-SRT.D.9**). You may also discuss why it would make the most sense in these problems to use the measure of side \overline{AC} for the base. Students need to use special right triangle relationships to answer the problems in Exercise 2. These ideas were most recently revisited in Lessons 1 and 2 of this module.

2. Find the area of each of the triangles shown below. Show work to support your answer.

a.

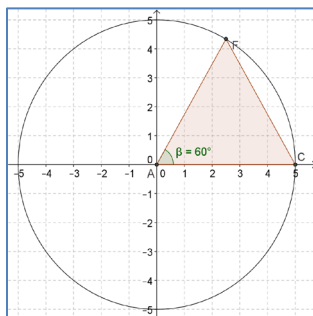


Draw a perpendicular line from E to the horizontal axis. The distance from E to the horizontal axis is a leg of a 45° - 45° - 90° right triangle whose sides are in a ratio $a: a: a\sqrt{2}$. Since $EA = 5$, the height of the triangle will be the solution to the equation $a\sqrt{2} = 5$. Thus, the height of triangle AEC is $\frac{5}{\sqrt{2}}$ or $\frac{5}{2}\sqrt{2}$ units.

$$\frac{1}{2}(5)\left(\frac{5}{2}\sqrt{2}\right) = \frac{25}{4}\sqrt{2}$$

The area is $\frac{25}{4}\sqrt{2}$ square units.

b.



Draw a perpendicular line from F to the horizontal axis. The height of the triangle is a leg of a 30° - 60° - 90° right triangle whose sides are the ratio $a: a\sqrt{3}: 2a$. Since $AF = 5$ units, the shorter leg will be $\frac{5}{2}$ units and the longer leg will be $\frac{5}{2}\sqrt{3}$ units. Thus, the height of the triangle is $\frac{5}{2}\sqrt{3}$ units.

$$\frac{1}{2}(5)\left(\frac{5}{2}\sqrt{3}\right) = \frac{25}{4}\sqrt{3}$$

The area is $\frac{25}{4}\sqrt{3}$ square units.

Before moving the class on to Exercise 3, have different groups of students present their solutions to the class. Focus the discussion on any different or unique approaches your students may have utilized. While the solutions above do not focus on using trigonometric ratios to determine the heights of the triangles, do not discount students that may have used this approach. Present solutions that use trigonometry last when you are discussing these problems.

The arguments presented in Exercise 3 part (a) focus on thinking about the circle shown as a dilation of the unit circle and applying the definitions of the sine and cosine functions covered in Algebra II Module 2. Some students may use the right trigonometry ratios they learned in Geometry Module 2 and may also specifically recall the formula they derived for area in Geometry Module 2 Lesson 31. The proof developed in Geometry relied on the definitions of the trigonometric functions that were defined for acute angles only. In this course, we extend use of the sine function to develop formulas for areas of triangles that include obtuse as well as acute triangles.

3. Joni said that the area of triangle AFC in Exercise 2 part (b) can be found using the definition of the sine function.

- a. What are the coordinates of point F in terms of the cosine and sine functions? Explain how you know.

The coordinates are the point on the unit circle given by $(\cos(\theta), \sin(\theta))$, where θ is the rotation of a ray from its initial position to its terminal position. This circle has a radius of 5 units so each point on this circle is a dilation by a factor of 5 of the points on the unit circle; thus, the coordinates are $(5 \cos(\theta), 5 \sin(\theta))$.

- b. Explain why the y -coordinate of point F is equal to the height of the triangle.

If \overline{AC} is the base of the triangle, then the height of the triangle is on a line perpendicular to the base that contains point F . The y -coordinate of any point in the Cartesian plane represents the distance from that point to the horizontal axis. Thus, the y -coordinate is equal to the height of this triangle.

- c. Write the area of triangle AFC in terms of the sine function.

The height is the y -coordinate of a point on the circle of radius 5 units. This y -coordinate is $5 \sin\left(\frac{\pi}{3}\right)$. Thus, the area in square units is given by

$$\begin{aligned}\text{Area} &= \frac{1}{2}(5)\left(5 \sin\left(\frac{\pi}{3}\right)\right) \\ &= \frac{5}{2}\left(\frac{5\sqrt{3}}{2}\right) \\ &= \frac{25}{4}\sqrt{3}\end{aligned}$$

- d. Does this method work for the area of triangle AEC ?

Yes. The angle 135° corresponds to a rotation of $\frac{3\pi}{4}$ radians. In square units, the area of triangle AEC is given by

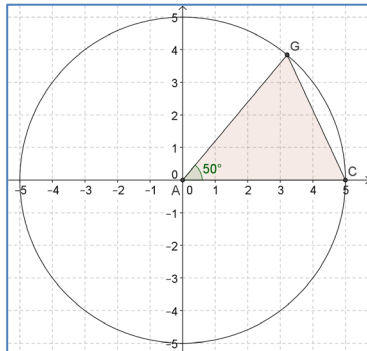
$$\begin{aligned}\text{Area} &= \frac{1}{2}(5)\left(5 \sin\left(\frac{3\pi}{4}\right)\right) \\ &= \frac{5}{2}\left(\frac{5\sqrt{2}}{2}\right) \\ &= \frac{25}{4}\sqrt{2}\end{aligned}$$

MP.3

The next two exercises require students to use the sine function to determine the measurement of the height of the triangles.

4. Find the area of the following triangles.

a.

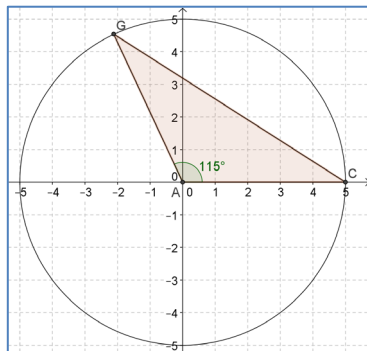


The base is 5 units, and if we draw a perpendicular line from point G to side \overline{AC} , then the distance from G to \overline{AC} is given by $5 \sin(50^\circ)$.

$$\frac{1}{2}(5)(5 \sin(50^\circ)) \approx 9.58$$

The area is approximately 9.58 square units.

b.



Drawing a perpendicular line from point G to the horizontal axis gives a height for this triangle that is equal to $5 \sin(115^\circ)$.

$$\frac{1}{2}(5)(5 \sin(115^\circ)) = 11.33$$

The area is approximately 11.33 square units.

Monitor groups as they work on Exercise 4, and move them on to Exercise 5. Hold a brief discussion if needed before proceeding to Exercise 5.

- How are all these problems similar?
 - They all have the same base along the horizontal axis. To determine the height of the triangle, we needed to draw an auxiliary line perpendicular to the base from the point on the circle that corresponded to the third vertex of the triangle. The y-coordinate of the point corresponded to the height of the triangle.
- What did you have to do differently in Exercise 4 to determine the height of the triangles?
 - The vertex on the circle did not correspond to easily recognized coordinates, nor could we use special right triangle ratios to determine the exact height. We had to use the sine function to determine the y-coordinate of the point that corresponded to the height of the triangle.

By noting these similarities and differences, students should now be able to generalize their work on the first four exercises.

5. Write a formula that will give the area of any triangle with vertices located at $A(0,0)$, $C(5,0)$, and $B(x,y)$ a point on the graph of $x^2 + y^2 = 25$ such that $y > 0$.

Area = $\frac{25}{2}\sin(\theta)$, where θ is the counterclockwise rotation of B from C about the origin.

6. For what value of θ will this triangle have maximum area? Explain your reasoning.

The triangle has maximum area when $\theta = \frac{\pi}{2}$ because the height is the greatest for this rotation since that is the rotation value when the height of the triangle is equal to the radius of the circle. For all other rotations, given that $y > 0$, the height is less than the radius.

MP.2
&
MP.8

Discussion (5 minutes)

The approach presented in the previous six exercises depended on the triangles being positioned in a circle. This was done to best utilize the definition of the sine function, which is required to derive the area formula for a triangle that is the focus of this lesson. This discussion helps students to see how to apply the same reasoning to circles of any size and then to triangles where the two given sides of the triangle are not equal.

- How would your approach to finding the area change if the triangles were constructed in a circle with a different radius?
 - *The base would equal the radius and the height would be the radius multiplied by the sine of the angle.*
- All of the triangles in the previous examples were isosceles triangles. Explain why.
 - *Two sides were radii of the circle. All radii of a circle are congruent.*
- How would your approach to finding the area change if the triangles were scalene?
 - *The base would change, but we could still use the sine function to find a value for the height of the triangle if we knew two sides and the angle in between them.*

Exploratory Challenge/Exercises 7–10 (15 minutes): Triangles in Circles

In these exercises, students extend their thinking and begin to generalize a process for finding the height of a triangle using the sine function. As you debrief Exercise 7, be sure to emphasize that the height from D to point \overline{AC} is still given by $4 \sin(65^\circ)$ because D can be thought of as a point on a circle with radius 4 units and center at the origin A . Changing the base to any other length will not affect the height of this triangle.

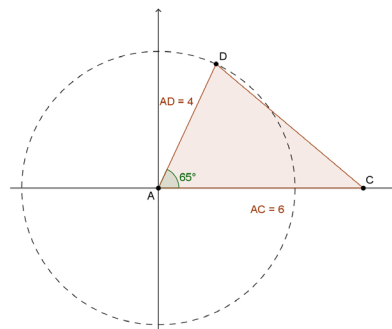
Exploratory Challenge/Exercises 7–10: Triangles in Circles

7. Find the area of the following triangle.

The height is $4 \sin(65^\circ)$ units, and the base is 6 units.

$$\frac{1}{2}(6)(4 \sin(65^\circ)) \approx 10.88$$

The area is approximately 10.88 square units.



This exercise asks students to generalize what they did in Exercise 7. Ask students to compare the variable symbols that represent the parts of the triangle shown below and how they relate back to the numbers in the previous exercise.

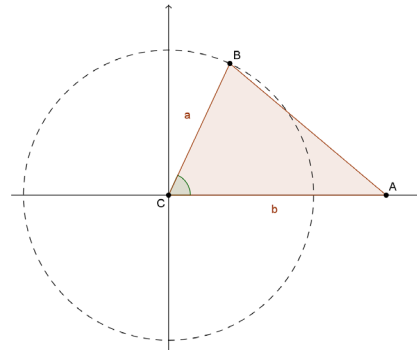
An oblique triangle is a triangle with no right angles. Students may need some introduction to this concept.

8. Prove that the area of any oblique triangle is given by the formula

$$\text{Area} = \frac{1}{2}ab\sin(C)$$

where a and b are adjacent sides of $\triangle ABC$ and C is the measure of the angle between them.

If we take B to be a point on a circle of radius a units centered at C , then the coordinates of B are given by $(a \cos(C), a \sin(C))$ where C is the measure of $\angle BCA$. Construct the height of $\triangle ABC$ from point B to side \overline{AC} .

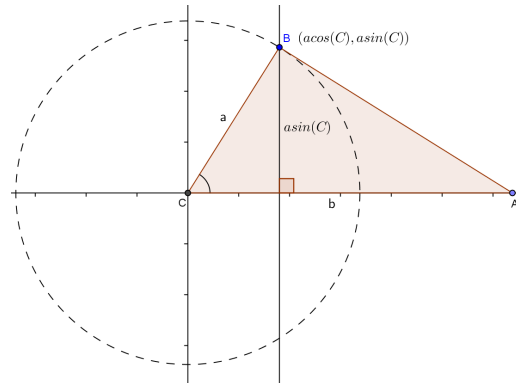


MP.3

Note the diagram has the auxiliary line drawn.

This height corresponds to the y-coordinate of B , which is $a \sin(C)$. The base is b units, and thus, the area is as follows:

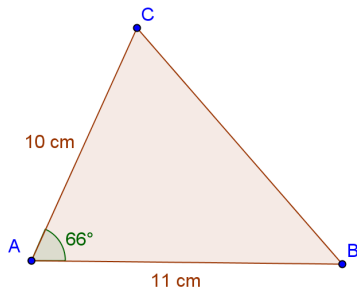
$$\begin{aligned} \text{Area} &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(b)(a \sin(C)) \\ &= \frac{1}{2}ab \sin(C). \end{aligned}$$



In the last exercises, students apply the formula that they just derived. If students are struggling to use the formula in Exercise 9 parts (b) and (c), remind them that these triangles could easily be rotated and translated to have the given angle correspond to the origin, which shows that the formula works regardless of the position of the triangle in a plane as long as two sides and the included angle are given.

9. Use the area formula from Exercise 8 to calculate the area of the following triangles.

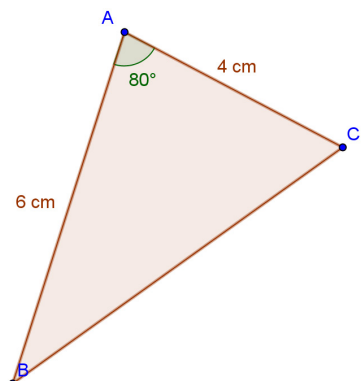
a.



$$\begin{aligned} \text{Area} &= \frac{1}{2}(10)(11) \sin(66^\circ) \\ \text{Area} &= 55 \sin(66^\circ) \\ \text{Area} &\approx 50.25 \end{aligned}$$

The area is 50.25 cm^2 .

b.



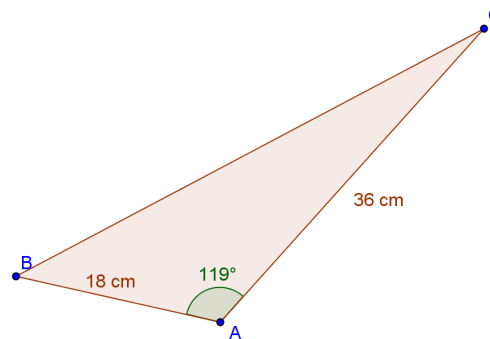
$$\text{Area} = \frac{1}{2}(6)(4)\sin(80^\circ)$$

$$\text{Area} = 12\sin(80^\circ)$$

$$\text{Area} \approx 11.82$$

The area is 11.82 cm².

c. A quilter is making an applique design with triangular pieces like the one shown below. How much fabric is used in each piece?



$$\text{Area} = \frac{1}{2}(18)(36)\sin(119^\circ)$$

$$\text{Area} = 18^2\sin(119^\circ)$$

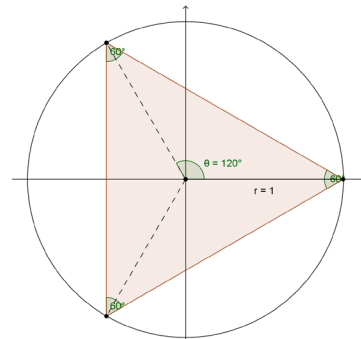
$$\text{Area} \approx 283.38$$

Each triangular piece is 283.38 cm² of fabric.

Exercise 10 can be used for early finishers or as an additional Problem Set exercise if time is running short. These problems are similar to the approach used by Archimedes to approximate the value of π in ancient times. Of course, he did not have the sine function at his disposal, but he did approximate the value of π by finding the area of regular polygons inscribed in a circle as the number of sides increased.

10. Calculate the area of the following regular polygons inscribed in a unit circle by dividing the polygon into congruent triangles where one of the triangles has a base along the positive x -axis.

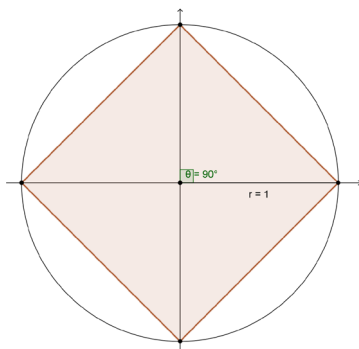
a.



$$3\left(\frac{1}{2}\right)(1)(1 \cdot \sin(120^\circ)) \approx 1.30$$

The area is approximately 1.30 square units.

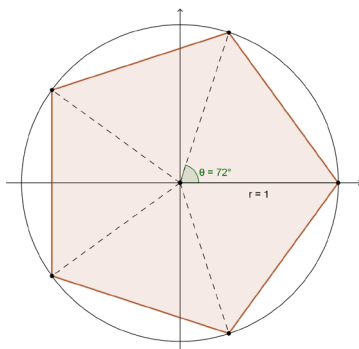
b.



$$4 \left(\frac{1}{2} \right) (1) (1 \cdot \sin(90^\circ)) \approx 2$$

The area is approximately 2 square units.

c.



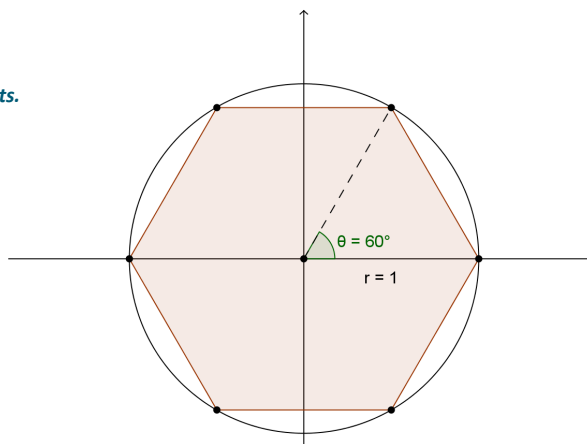
$$5 \left(\frac{1}{2} \right) (1) (1 \cdot \sin(72^\circ)) \approx 2.38$$

The area is approximately 2.38 square units.

d. Sketch a regular hexagon inscribed in a unit circle with one vertex at $(1, 0)$, and find the area of this hexagon.

$$6 \left(\frac{1}{2} \right) (1) (1 \cdot \sin(60^\circ)) \approx 2.59$$

The area is approximately 2.59 square units.



e. Write a formula that gives the area of a regular polygon with n sides inscribed in a unit circle if one vertex is at $(1, 0)$ and θ is the angle formed by the positive x -axis and the segment connecting the origin to the point on the polygon that lies in the first quadrant.

$$\text{Area} = \frac{n}{2} \sin\left(\frac{360^\circ}{n}\right)$$

MP.8

- f. Use a calculator to explore the area of this regular polygon for large values of n . What does the area of this polygon appear to be approaching as the value of n increases?

If we select $n = 10$, the area is 2.94 square units.

If we select $n = 30$, the area is 3.12 square units.

If we select $n = 100$, the area is 3.139 525 square units.

The area appears to be approaching π .

To further reinforce the solutions to part (f), ask students to calculate the area of the unit circle.

- What is the area of the unit circle?
 - *The area is π square units.*
- How does the area of a polygon inscribed in the circle compare to the area of the circle as the number of sides increases?
 - *As the number of sides increases, the polygon area would be getting closer to the circle's area.*

Another interesting connection can be made by graphing the related function and examining its end behavior. Students can confirm by graphing that the function, $f(x) = \frac{x}{2} \sin\left(\frac{2\pi}{x}\right)$ for positive integers x , appears to approach a horizontal asymptote of $y = \pi$ as the value of x increases.

Closing (4 minutes)

Use these questions as a quick check for understanding before students begin the Exit Ticket. You can encourage students to research other area formulas for oblique triangles as an extension to this lesson.

- Draw a triangle whose area can be calculated using the formula $\text{Area} = \frac{1}{2}ab \sin(C)$, and indicate on the triangle the parameters required to use the formula.
 - *Solutions will vary but should be a triangle with measurements provided for two sides and the angle formed by them.*
- Draw a triangle whose area CANNOT be calculated using this formula.
 - *Solutions will vary. One example would be an oblique triangle with three side measures given.*

Lesson Summary

The area of $\triangle ABC$ is given by the formula:

$$\text{Area} = \frac{1}{2}ab \sin(C)$$

where a and b are the lengths of two sides of the triangle and C is the measure of the angle between these sides.

Exit Ticket (4 minutes)

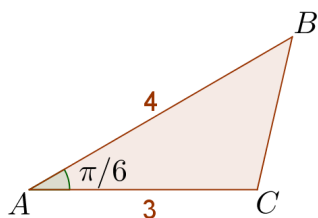
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Lesson 7: An Area Formula for Triangles

Exit Ticket

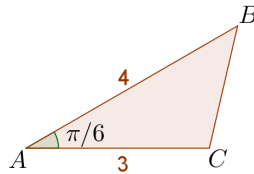
1. Find the area of $\triangle ABC$.



2. Explain why $\frac{1}{2}ab \sin(\theta)$ gives the area of a triangle with sides a and b and included angle θ .

Exit Ticket Sample Solutions

1. Find the area of $\triangle ABC$.

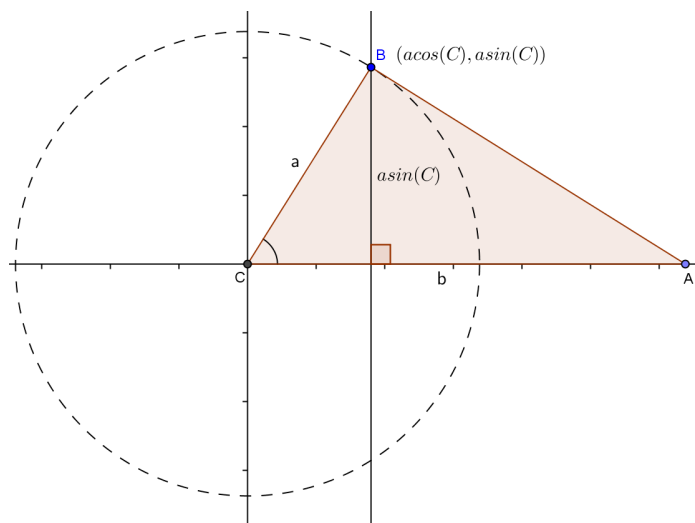


$$\text{Area} = \frac{1}{2}(3)(4) \sin\left(\frac{\pi}{6}\right) = 3$$

The area is 3 square units.

2. Explain why $\frac{1}{2}ab \sin(\theta)$ gives the area of a triangle with sides a and b and included angle θ .

In the diagram below, the height is the perpendicular line segment from point B to the base b . The length of this line segment is $a \sin(\theta)$, which is the y -coordinate of point B , a point on a circle of radius a with center at C as shown.



The area of a triangle is one-half the product of one side (the base) and the height to that side. If we let b be the base, then

$$\begin{aligned} \text{Area} &= \frac{1}{2}b \cdot a \sin(\theta) \\ &= \frac{1}{2}ab \sin(\theta). \end{aligned}$$

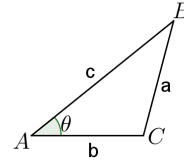
Problem Set Sample Solutions

1. Find the area of the triangle ABC shown to the right, with the following data:

a. $\theta = \frac{\pi}{6}$, $b = 3$, and $c = 6$.

$$\frac{1}{2} \left(b \cdot c \cdot \sin \left(\frac{\pi}{6} \right) \right) = \frac{1}{2} \left(18 \cdot \frac{1}{2} \right) = \frac{9}{2}$$

The area is $\frac{9}{2}$ square units.



b. $\theta = \frac{\pi}{3}$, $b = 4$, and $c = 8$.

$$\frac{1}{2} \left(b \cdot c \cdot \sin \left(\frac{\pi}{3} \right) \right) = \frac{1}{2} \left(32 \cdot \frac{\sqrt{3}}{2} \right) = 8\sqrt{3}$$

The area is $8\sqrt{3}$ square units.

c. $\theta = \frac{\pi}{4}$, $b = 5$, and $c = 10$.

$$\frac{1}{2} \left(b \cdot c \cdot \sin \left(\frac{\pi}{4} \right) \right) = \frac{1}{2} \left(50 \cdot \frac{\sqrt{2}}{2} \right) = \frac{25\sqrt{2}}{2}$$

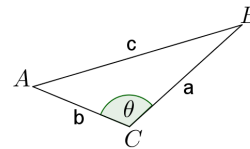
The area is $\frac{25\sqrt{2}}{2}$ square units.

2. Find the area of the triangle ABC shown to the right, with the following data:

a. $\theta = \frac{3\pi}{4}$, $a = 6$, and $b = 4$.

$$\frac{1}{2} \left(a \cdot b \cdot \sin \left(\frac{\pi}{4} \right) \right) = \frac{1}{2} \left(24 \cdot \frac{\sqrt{2}}{2} \right) = 6\sqrt{2}$$

The area is $6\sqrt{2}$ square units.



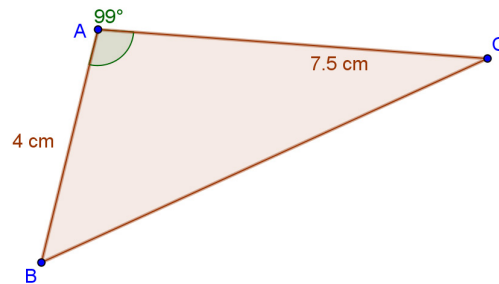
b. $\theta = \frac{5\pi}{6}$, $a = 4$, and $b = 3$.

$$\frac{1}{2} \left(a \cdot b \cdot \sin \left(\frac{\pi}{6} \right) \right) = \frac{1}{2} \left(12 \cdot \frac{1}{2} \right) = 3$$

The area is 3 square units.

3. Find the area of each triangle shown below. State the area to the nearest tenth of a square centimeter.

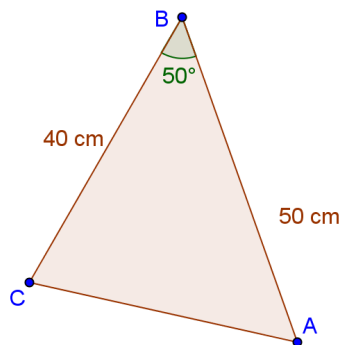
a.



$$A = \frac{1}{2} \cdot 4 \cdot 7.5 \cdot \sin(99^\circ) \approx 14.8$$

The area is approximately 14.8 sq. cm.

b.



$$A = \frac{1}{2} \cdot 40 \cdot 50 \cdot \sin(50^\circ) \approx 766.0$$

The area is approximately 766 sq. cm.

4. The diameter of the circle O in the figure shown to the right is $EB = 10$.

- a. Find the area of the triangle OBA .

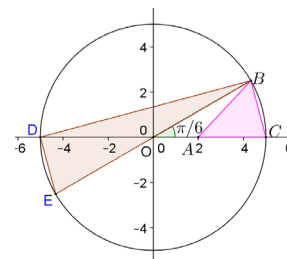
$$\frac{1}{2} \left(2 \cdot 5 \cdot \sin\left(\frac{\pi}{6}\right) \right) = \frac{1}{2} \left(2 \cdot 5 \cdot \frac{1}{2} \right) = \frac{5}{2}$$

The area is $\frac{5}{2}$ square units.

- b. Find the area of the triangle ABC .

$$\frac{1}{2} (bh) = \frac{1}{2} \left(3 \cdot \frac{5}{2} \right) = \frac{15}{4}$$

The area is $\frac{15}{4}$ square units.



- c. Find the area of the triangle DBO .

$$\frac{1}{2}(bh) = \frac{1}{2}\left(5 \cdot \frac{5}{2}\right) = \frac{25}{4}$$

The area is $\frac{25}{4}$ square units.

- d. Find the area of the triangle DBE .

The area of triangle DBE is the sum of the areas of triangles DBO , OBA , and ABC .

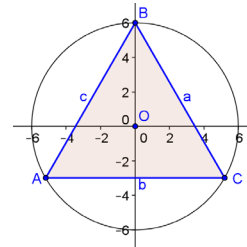
$$\frac{25}{4} + \frac{15}{4} + \frac{5}{2} = \frac{50}{4}$$

The area of triangle DBE is $\frac{50}{4}$ square units.

5. Find the area of the equilateral triangle ABC inscribed in a circle with a radius of 6.

$$3 \cdot \frac{1}{2} \left(6 \cdot 6 \cdot \sin\left(\frac{2\pi}{3}\right) \right) = \frac{54\sqrt{3}}{2}$$

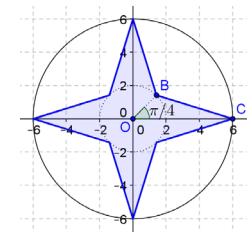
The area is $\frac{54\sqrt{3}}{2}$ square units.



6. Find the shaded area in the diagram below.

$$8 \cdot \frac{1}{2} \left(6 \cdot 2 \cdot \sin\left(\frac{\pi}{4}\right) \right) = 24\sqrt{2}$$

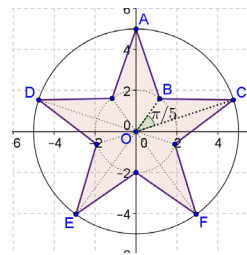
The area is $24\sqrt{2}$ square units.



7. Find the shaded area in the diagram below. The radius of the outer circle is 5; the length of the line segment OB is 2.

$$10 \cdot \frac{1}{2} \left(5 \cdot 2 \cdot \sin\left(\frac{\pi}{5}\right) \right) \approx 29.389$$

The area is 29.389 square units.

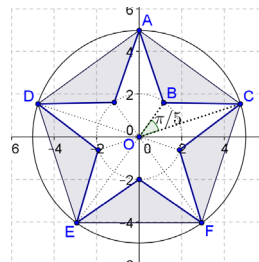


8. Find the shaded area in the diagram below. The radius of the outer circle is 5.

$$5 \cdot \frac{1}{2} \left(5 \cdot 5 \cdot \sin \left(\frac{2\pi}{5} \right) \right) = \frac{125}{2} \sin \left(\frac{2\pi}{5} \right) = 59.441$$

The area of the pentagon is 59.441 square units. From Problem 7, we have the area of the star is 29.389. The shaded area is the area of the pentagon minus the area of the star.

The shaded area is 30.052 square units.



9. Find the area of the regular hexagon inscribed in a circle if one vertex is at (2, 0).

$$6 \cdot \frac{1}{2} \left(2 \cdot 2 \cdot \sin \left(\frac{2\pi}{6} \right) \right) = 12 \cdot \sin \left(\frac{\pi}{3} \right) = 12 \cdot \frac{\sqrt{3}}{2} = 6\sqrt{3}$$

The area is $6\sqrt{3}$ square units.

10. Find the area of the regular dodecagon inscribed in a circle if one vertex is at (3, 0).

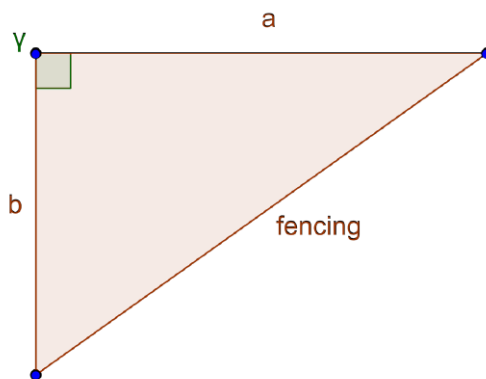
$$12 \cdot \frac{1}{2} \left(3 \cdot 3 \cdot \sin \left(\frac{2\pi}{12} \right) \right) = 54 \cdot \sin \left(\frac{\pi}{6} \right) = 54 \cdot \frac{1}{2} = 27$$

The area is 27 square units.

11. A horse rancher wants to add on to existing fencing to create a triangular pasture for colts and fillies. She has 1,000 feet of fence to construct the additional two sides of the pasture.

- a. What angle between the two new sides would produce the greatest area?

Our formula is $A = \frac{1}{2}ab \sin(\gamma)$. When a and b are constant, the equation is maximized when $\sin(\gamma)$ is maximized, which occurs at $\gamma = 90^\circ$. Thus, the greatest area would be when the angle is a 90° angle.



- b. What is the area of her pasture if she decides to make two sides of 500 ft. each and uses the angle you found in part (a)?

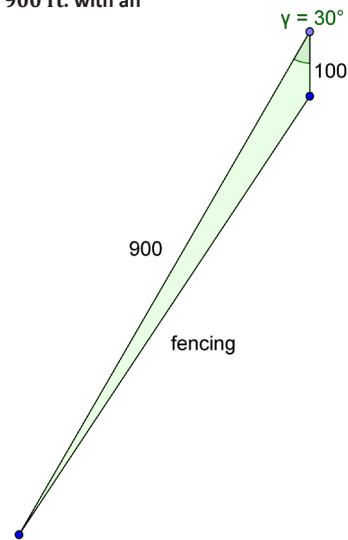
$$A = \frac{1}{2} \cdot 250\,000 \cdot 1 = 125\,000$$

The area would be 125,000 sq. ft.

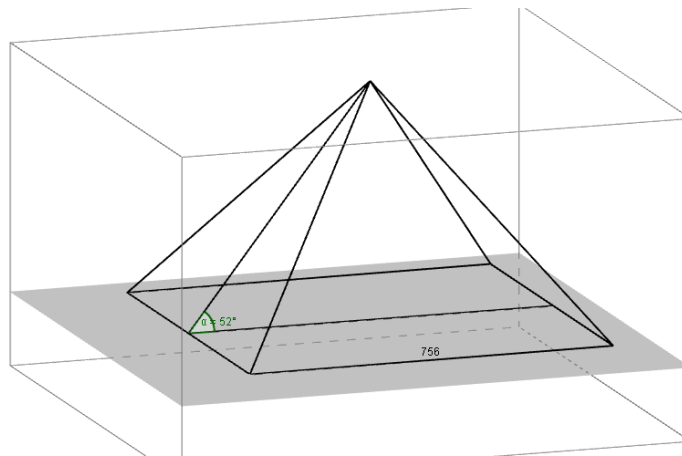
- c. Due to property constraints, she ends up using sides of 100 ft. and 900 ft. with an angle of 30° between them. What is the area of the new pasture?

$$\begin{aligned} A &= \frac{1}{2} \cdot 90000 \cdot \sin(30^\circ) \\ &= 45000 \cdot \frac{1}{2} \\ &= 22500 \end{aligned}$$

The area would be 22,500 sq. ft.



12. An enthusiast of Egyptian history wants to make a life-size version of the Great Pyramid using modern building materials. The base of each side of the Great Pyramid was measured to be 756 ft. long, and the angle of elevation is about 52° .
- a. How much material will go into the creation of the sides of the structure (the triangular faces of the pyramid)?



According to these measurements, each side of the Great Pyramid has a base of 756 ft. The Great Pyramid's height is in the middle of the triangle, so we can construct a right triangle of side $\frac{756}{2} = 378$ and the height of the pyramid, and use the cosine function.

$$\begin{aligned} \cos(52^\circ) &= \frac{378}{h} \\ h &= \frac{378}{\cos(52^\circ)} \\ h &\approx 614 \end{aligned}$$

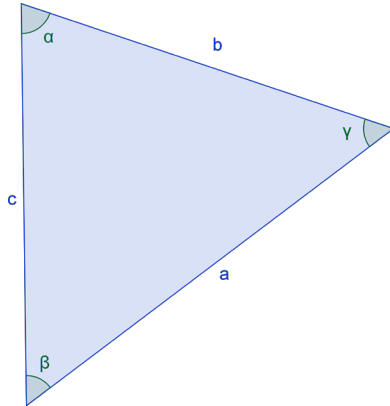
Area of one side: $\frac{1}{2} \cdot 614 \cdot 756 \approx 232\,092$. For four sides, the area is 928,368 sq. ft.

- b. If the price of plywood for the sides is \$0.75 per square foot, what is the cost of just the plywood for the sides?

The total price would be about \$696,276 for just the plywood going into the sides.

13. Depending on which side you choose to be the “base,” there are three possible ways to write the area of an oblique triangle, one being $A = \frac{1}{2}ab \sin(\gamma)$.

- a. Write the other two possibilities using $\sin(\alpha)$ and $\sin(\beta)$.



$$\frac{1}{2}bc \sin(\alpha)$$

and

$$\frac{1}{2}ac \sin(\beta)$$

- b. Are all three equal?

Yes, since each expression is equal to the area of the triangle, they are all equal to each other.

- c. Find $\frac{2A}{abc}$ for all three possibilities.

$$\frac{2A}{abc} = \frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

- d. Is the relationship you found in part (c) true for all triangles?

There was nothing special about the triangle we picked, so it should be true for all triangles.



Lesson 8: Law of Sines

Student Outcomes

- Students prove the law of sines and use it to solve problems (**G-SRT.D.10**).

Lesson Notes

In previous lessons, students developed tools for finding a missing side or a missing angle in a right triangle. In particular, students know how to use the Pythagorean theorem to find a missing side of a right triangle, and they know how to use the trigonometric functions to find a missing angle in a right triangle. In this lesson, students extend their knowledge to cover certain oblique triangles. Students develop a proof of the law of sines and apply it to solve problems. At the end of the lesson, students develop a second proof of the law of sines that uses properties of circles.

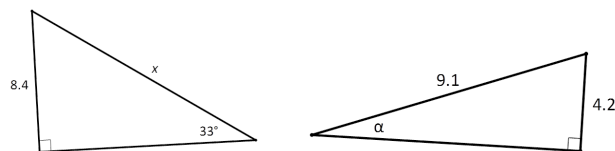
In the course of the lesson, students draw on their knowledge of the conditions that imply a unique triangle, which students know as the triangle congruence criteria (ASA, SSS, SAS). This topic was addressed in Geometry Module 1 Lessons 22, 24, and 25.

Classwork

Exercises 1–2 (2 minutes)

In these exercises, students apply their prior knowledge of right triangle trigonometry to solve problems.

Exercises



1. Find the value of x in the figure on the left.

$$\sin(33^\circ) = \frac{8.4}{x}$$

$$x = \frac{8.4}{\sin(33^\circ)} \approx 15.4$$

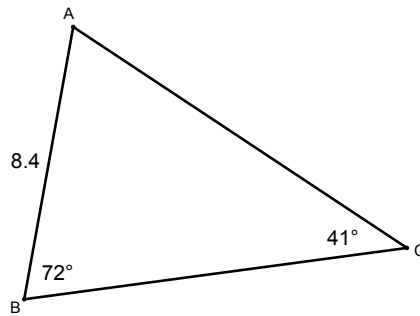
2. Find the value of α in the figure on the right.

$$\sin(\alpha) = \frac{4.2}{9.1}$$

$$\alpha \approx 27^\circ$$

Exploratory Challenge (9 minutes): Oblique Triangles

- In Exercises 1 and 2, what aspects of the triangle were you given, and what aspects did you find?
 - In Exercise 1, we were given a side and an angle in a right triangle. From these measurements, we determined the length of the hypotenuse.
 - In Exercise 2, we were given two sides in a right triangle. From these measurements, we determined the measure of one of the acute angles.



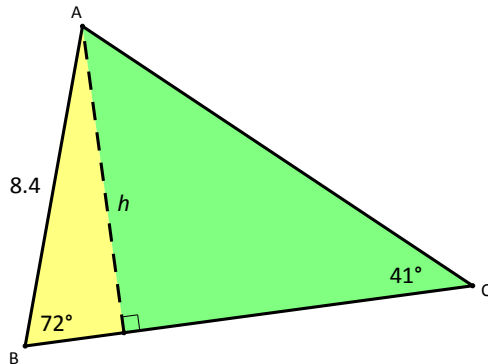
- Now examine the triangle above. What information is provided in this triangle?
 - We are given two angles and a non-included side.
- How is this triangle different from the ones presented in the Exercises 1 and 2?
 - The problems in Exercises 1 and 2 involved right triangles, but this triangle does not have a right angle.
- Do you think you should be able to determine the remaining measurements in this triangle? Why or why not? Think about this for a moment, then explain your reasoning to a neighbor.
 - The missing angle can be found easily using the fact that the sum of the angles is 180° .
 - The AAS criterion for triangle congruence tells us that only one triangle can be formed with the given information so the missing side measurements can be determined by the measurements that are given in the figure.
- See how many of the missing measurements you can determine. Take several minutes to explore this problem with the students around you.

Give students several minutes to work on this problem in groups of 3 or 4. Select one or more students to present their findings to the class.

MP.3

MP.1

Give students the opportunity to draw an altitude like the one shown below without being prompted to do so. If students do not see that an auxiliary line is necessary, use the cues in the scaffolding box on the right.



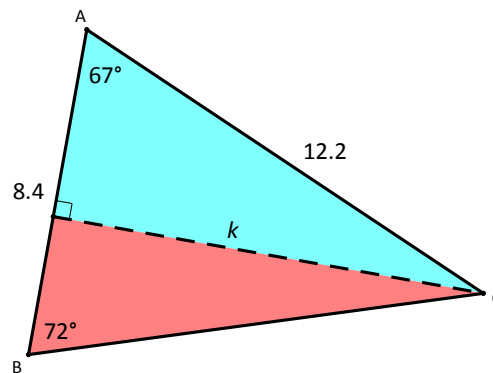
- When we draw the altitude to side \overline{BC} , we get two right triangles as shown above.

Looking at the yellow triangle, we get $\sin(72^\circ) = \frac{h}{8.4}$. This means that $h = 8.4 \sin(72^\circ)$.

Looking at the green triangle, we get $\sin(41^\circ) = \frac{h}{AC}$. This means that $h = AC \cdot \sin(41^\circ)$.

Thus we have $AC \cdot \sin(41^\circ) = 8.4 \sin(72^\circ)$, which means that $AC = \frac{8.4 \sin(72^\circ)}{\sin(41^\circ)} \approx 12.2$.

- To make 180° , the third angle must be 67° . When we draw in the altitude to side \overline{BA} , we get a different pair of right triangles as shown below.



Looking at the blue triangle, we get $\sin(67^\circ) = \frac{k}{12.2}$. This means that $k = 12.2 \sin(67^\circ)$.

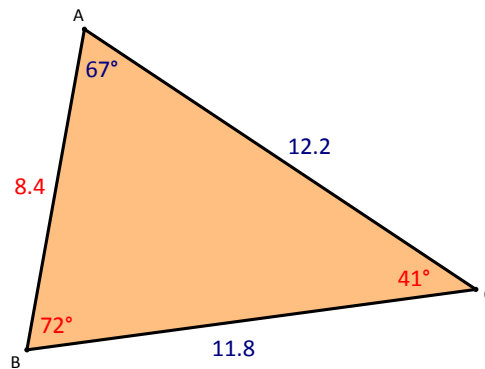
Looking at the red triangle, we get $\sin(72^\circ) = \frac{k}{BC}$. This means that $k = BC \cdot \sin(72^\circ)$.

Thus, we have $BC \cdot \sin(72^\circ) = 12.2 \sin(67^\circ)$, which means that $BC = \frac{12.2 \sin(67^\circ)}{\sin(72^\circ)} \approx 11.8$.

Scaffolding:

- If students need support with this task, prompt them with one or both of the following questions. The goal is to get students to realize that they can make progress by drawing in one of the altitudes, which allows them to capitalize on their knowledge of right triangles.
- What kind of auxiliary line could you draw here, and where could you draw it?
- It would be useful to be able to draw on our knowledge of right triangles. How might you create right triangles in this figure?

- Here is a picture of the triangle with the three original measurements, together with the three measurements we calculated above.

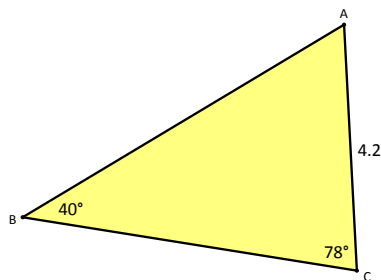


- To summarize, we can use what we know about right triangles to learn things about **oblique triangles**, that is, triangles that do not have a right angle.

Exercise 3 (5 minutes)

Instruct students to solve the following problem and then to compare their work with a partner. Select two students to write their work on the board to share with the class.

3. Find all of the measurements for the triangle below.

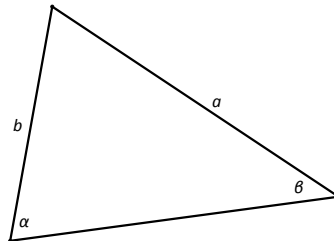


The measure of $\angle A$ is 62° . The length of side \overline{BA} is 6.4. The length of side \overline{BC} is 5.8.

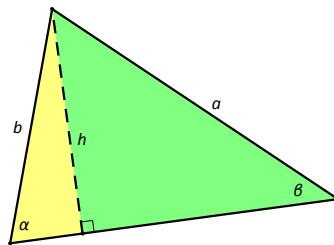
Discussion (5 minutes): Proving the Law of Sines

- Can you generalize your findings? Try to find an equation that shows the relationship between the quantities in the figure below.

Give students several minutes to work together. Select a student to share his or her work with the class.



- First we draw an altitude:



One expression for the altitude is $h = b \cdot \sin(\alpha)$.

Another expression for the altitude is $h = a \cdot \sin(\beta)$.

From these observations, it follows that $a \cdot \sin(\beta) = b \cdot \sin(\alpha)$.

We can use this equation to find a if we know the value of b and vice versa.

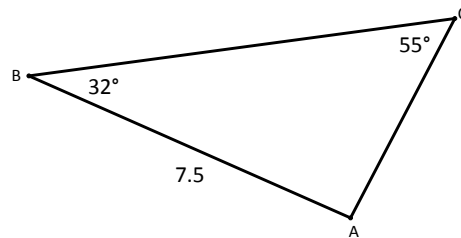
- The fact that $a \cdot \sin(\beta) = b \cdot \sin(\alpha)$ is called the **law of sines**. Since the sine function is a central feature of the formula, this name makes sense. Let's be explicit about the meaning of the symbols in this equation. How are these quantities related in the figure?
 - The side labeled a is opposite the angle labeled α , and the side labeled b is opposite the angle labeled β .
- By way of summary, let's make the reasoning absolutely clear. Why are the expressions $a \cdot \sin(\beta)$ and $b \cdot \sin(\alpha)$ equal to each other?
 - Each of these expressions represents the length of an altitude in the triangle. Since they represent the same thing, the expressions must be equal to each other.
- Now let's get some practice applying the law of sines. Keep in mind that you can always draw an altitude just like you did above, but perhaps it is also worth knowing that the equation $a \cdot \sin(\beta) = b \cdot \sin(\alpha)$ gives us a shortcut.
- When applying the law of sines, it is often easiest to work with the equation $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)}$. Can you see that this is equivalent to $a \cdot \sin(\beta) = b \cdot \sin(\alpha)$? Explain why these are equivalent.
 - By taking the equation $a \cdot \sin(\beta) = b \cdot \sin(\alpha)$ and dividing both sides by $\sin(\alpha) \cdot \sin(\beta)$, you get $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)}$.

MP.2

Exercises 4–5 (4 minutes)

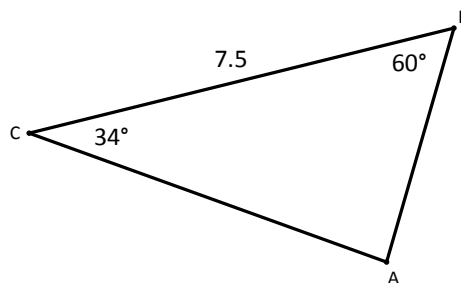
Instruct students to solve the following problems and then to compare their work with a partner. Select two students to write their work on the board to share with the class.

4. Find the length of side \overline{AC} in the triangle below.



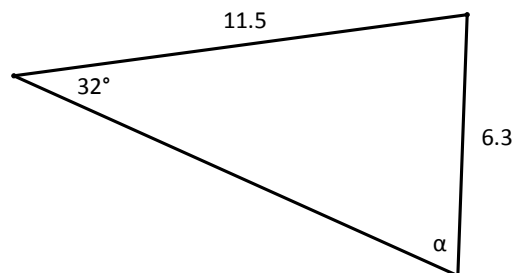
We have $\frac{AC}{\sin(32^\circ)} = \frac{7.5}{\sin(55^\circ)}$, which means $AC = \frac{7.5 \cdot \sin(32^\circ)}{\sin(55^\circ)} \approx 4.9$.

5. A hiker at point C is 7.5 kilometers from a hiker at point B; a third hiker is at point A. Use the angles shown in the diagram below to determine the distance between the hikers at points C and A.



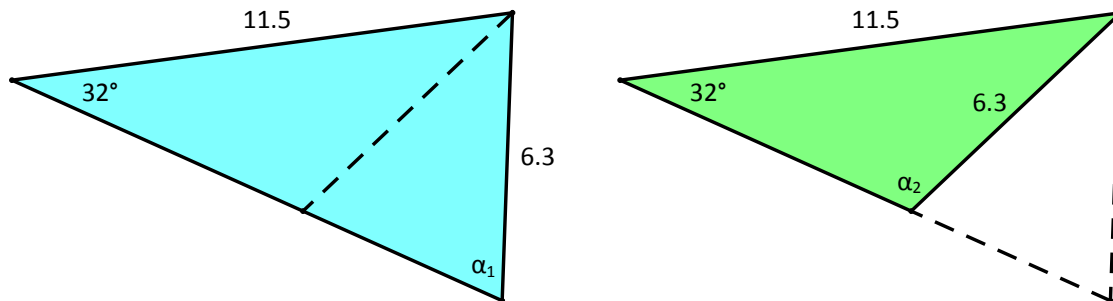
In order to apply the law of sines, first we need to find the third angle. The measure of the missing angle is 86°.

Now we can use the law of sines, which gives $\frac{AC}{\sin(60^\circ)} = \frac{7.5}{\sin(86^\circ)}$, which means $AC = \frac{7.5 \cdot \sin(60^\circ)}{\sin(86^\circ)} \approx 6.5$. Thus, the two hikers at points C and A are about 6.5 km apart.

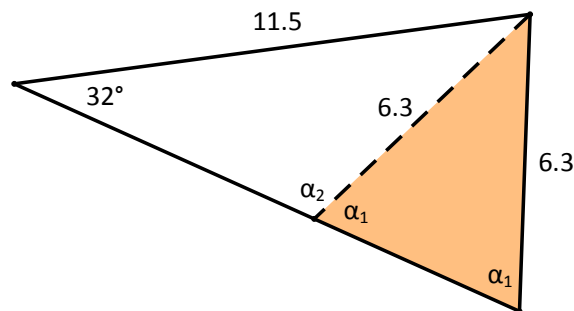
Discussion (7 minutes): Finding Angles

MP.3

- Examine the triangle on the previous page. What information is given to you in this triangle? How is this situation different from the ones we encountered earlier in the lesson?
 - We are given two sides and a non-included angle. In the previous examples, we were given two angles and a side.*
- Do you think you should be able to determine the remaining measurements in this triangle? Why or why not? Think about this for a moment, then explain your reasoning to a neighbor.
 - This is the SSA case. We have to allow for the possibility that there are two triangles with these specific measurements.*
- Let's have a look at the two triangles with these measurements:

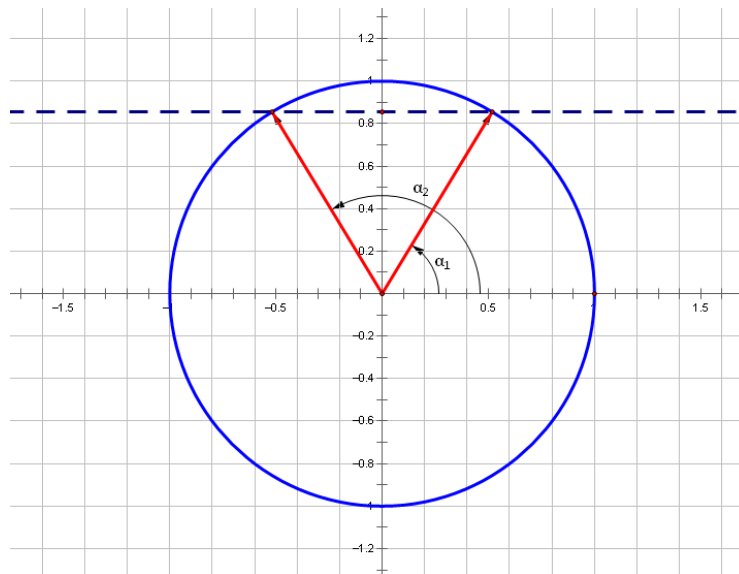


- Notice that both triangles have a side with length 11.5, an angle with measure 32° , and a side with length 6.3 that is opposite the 32° angle.
- As you can see in the figure, the angle across from the side with length 11.5 can be either acute or obtuse. What can we say about the relationship between α_1 and α_2 ? Take a moment to think about this, and then discuss what you notice with a partner.
 - The figure contains an isosceles triangle, so its base angles must be congruent.*

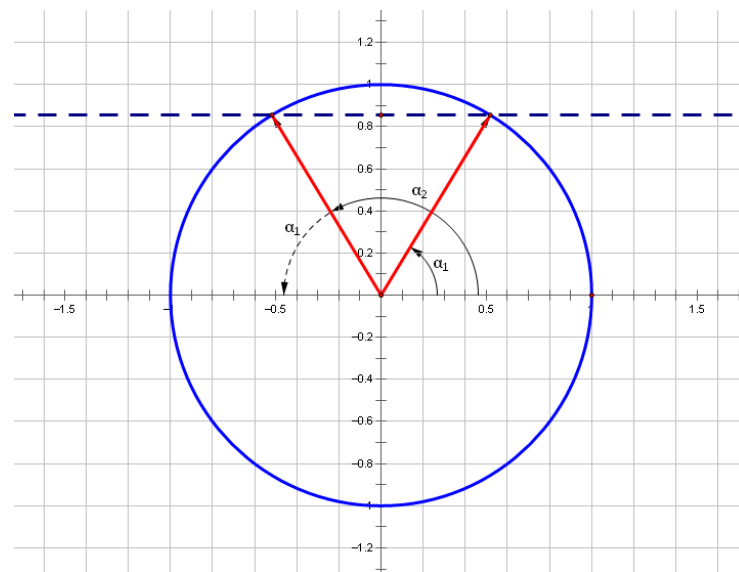


- Now we can see that $\alpha_1 + \alpha_2 = 180$.*
- Let's apply the law of sines and then see if we can determine how to get both of these angles.
 - We have $\frac{11.5}{\sin(\alpha)} = \frac{6.3}{\sin(32^\circ)}$, which gives $\sin(\alpha) = \frac{11.5 \cdot \sin(32^\circ)}{6.3} \approx 0.97$.*

- Okay, so now we want to find values of α for which $\sin(\alpha) \approx 0.97$. To see that there are indeed two such values, let's look at a diagram of the unit circle:

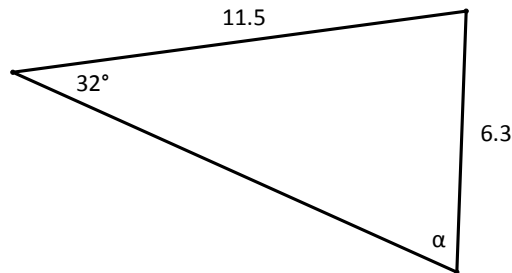


- Evidently there are two ways we can rotate the point $(1, 0)$ so that its image has a height equal to that of the dotted line.
- Can we see that $\alpha_1 + \alpha_2 = 180$ in this diagram as well?

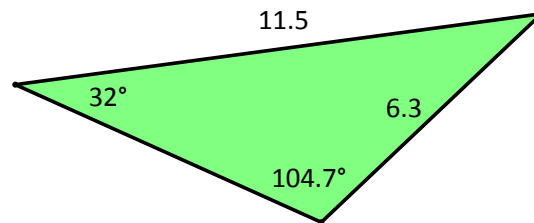
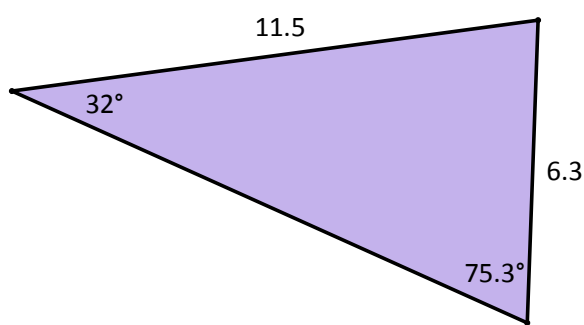


- Yes, the two angles labeled α_1 are congruent by a reflection across the y -axis. Thus, $\alpha_1 + \alpha_2 = 180$.
- Okay, now let's use a calculator to produce the two values for which $\sin(\alpha) = \frac{11.5 \cdot \sin(32^\circ)}{6.3}$.
 - We find that $\alpha_1 \approx 75.3^\circ$, so $\alpha_2 = 180 - 75.3 = 104.7$. $\alpha_2 = 104.7^\circ$

- Let's return now to the original problem: What are the two possible values of α in the figure below? Draw the two triangles with the measurements shown in the figure.



- We found that α could be 75.3° , which would give an acute triangle as shown on the left, but α could also be 104.7° , which would give the obtuse triangle shown on the right.



Exercises 6–7 (3 minutes)

Instruct students to solve the following problems and then to compare their work with a partner. Select two students to write their work on the board to share with the class.

6. Two sides of a triangle have lengths 10.4 and 6.4. The angle opposite 6.4 is 36° . What could the angle opposite 10.4 be?

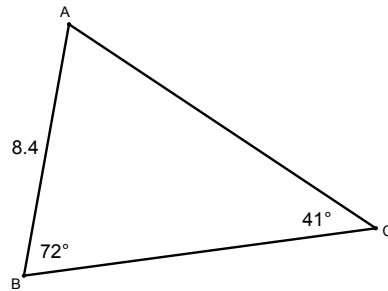
Let β represent the angle opposite 10.4. Then, we have $\frac{10.4}{\sin(\beta)} = \frac{6.4}{\sin(36^\circ)}$, which gives $\sin \beta \approx 0.955$ and $\beta = 72.8^\circ$. We could also have $\beta = 107.2^\circ$ because an angle and its supplement have the same sine value.

7. Two sides of a triangle have lengths 9.6 and 11.1. The angle opposite 9.6 is 59° . What could the angle opposite 11.1 be?

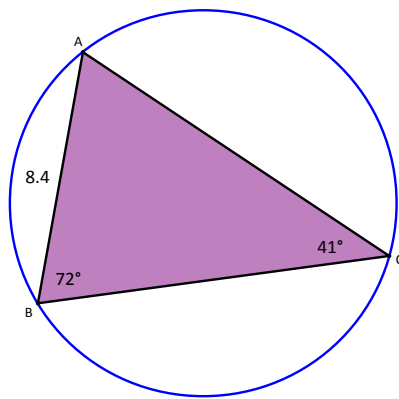
Let β represent the angle opposite 11.1. Then, we have $\frac{11.1}{\sin(\beta)} = \frac{9.6}{\sin(59^\circ)}$, which gives $\beta = 82.3^\circ$. We could also have $\beta = 97.7^\circ$.

Discussion (Optional): Circumscribed Circles

- One of the real joys of mathematics is that we can look at a problem through different lenses. Sometimes we develop new insights by using a different perspective, and this can be exciting! Let's return to the problem we solved earlier in the lesson.



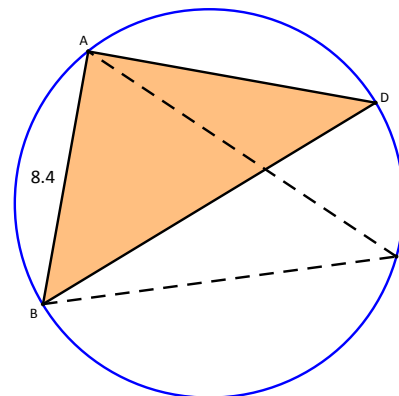
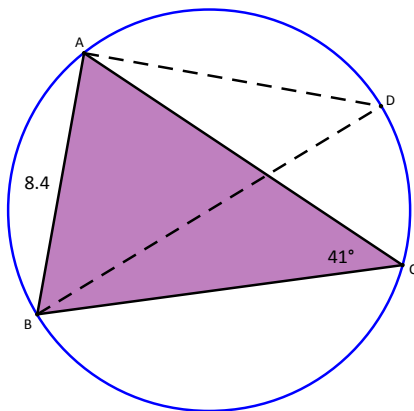
- Recall that we were able to calculate the length of side \overline{AC} by drawing the altitude to side \overline{BC} . Another way of finding the missing side lengths is to draw the circle that circumscribes the triangle:



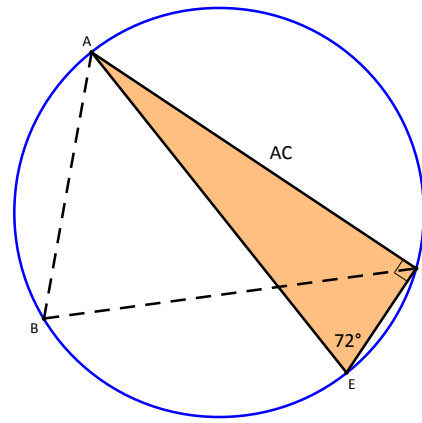
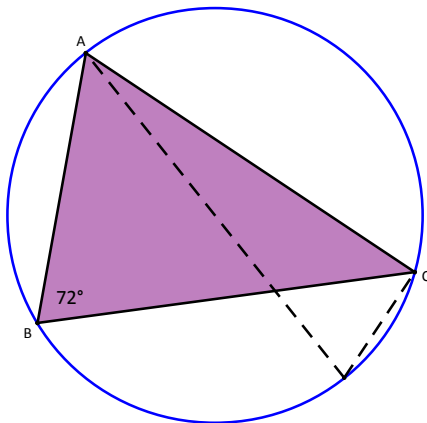
Scaffolding:

- Students may need to be reminded of Thales' theorem, which says that an angle inscribed in a semicircle must be a right angle.
- Students may need to be reminded that an angle inscribed in a circle has half the measure of its intercepted arc.

- Let's try to use what we know about circles to find \overline{AC} . The key here is to draw in the diameters one at a time.
- Remember, the inscribed angle that includes the diameter must be a right angle, so use that knowledge to create the diameters.



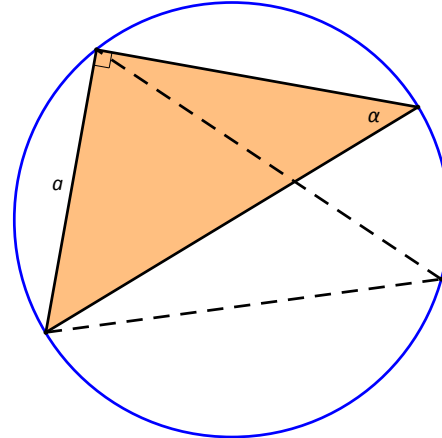
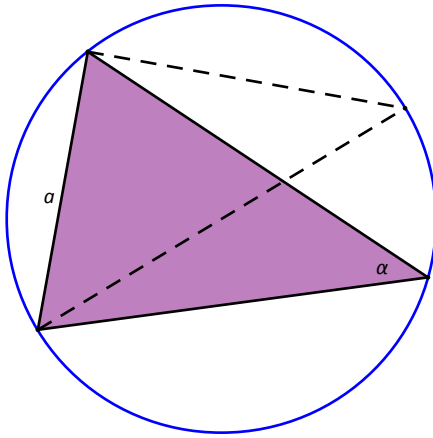
- Point D is the opposite end of the diameter through point B . What observations can you make about $\triangle ABD$? Take a minute to think about this. In particular, can you find the angles in $\triangle ABD$?
 - Since \overline{BD} is a diameter of the circle, it follows that $\triangle ABD$ has a right angle at A . We can also infer that $\angle D$ is 41° because it intercepts the same arc as $\angle C$.
- So we've distorted $\triangle ABD$ in such a way that it becomes a right triangle, but the distortion preserves the length of \overline{AB} and the measure of $\angle C$. Clever, isn't it? Now we can use right triangle trigonometry to describe the length of \overline{BD} . Can you see how to do this?
 - We have $\sin(41^\circ) = \frac{8.4}{BD}$. This means $BD = \frac{8.4}{\sin(41^\circ)}$.



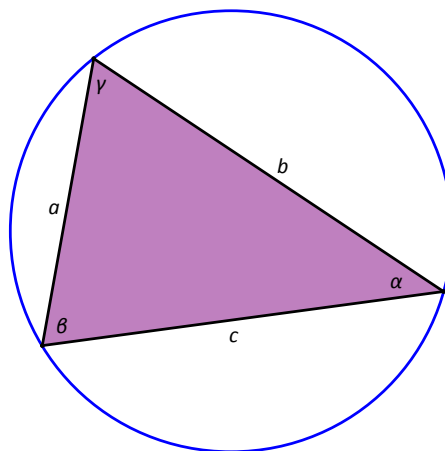
- Now let's run a diameter through point A and conduct a similar analysis.
 - Using the same reasoning as before, we conclude that $\triangle ACE$ is a right triangle and that $\angle E$ is equal to $\angle B$. Now we can describe the length of diameter \overline{AE} using the sine function: $\sin(72^\circ) = \frac{AC}{AE}$. This gives us $AE = \frac{AC}{\sin(72^\circ)}$.
- Now let's put it all together: We know that diameter has length $BD = \frac{8.4}{\sin(41^\circ)}$ and that diameter \overline{AE} satisfies $AE = \frac{AC}{\sin(72^\circ)}$. How can we use these observations to compute the length of side \overline{AC} ? Take a minute to really think about this!
 - The length of the diameter of a circle is constant, so we have $BD = AE$. This means that $\frac{8.4}{\sin(41^\circ)} = \frac{AC}{\sin(72^\circ)}$, and so $AC = \frac{8.4 \cdot \sin(72^\circ)}{\sin(41^\circ)} \approx 12.2$.
- Is this consistent with the result we found at the start of the lesson?
 - Yes, we got 12.2 in that case also.

Discussion (3 minutes): The Law of Sines via Circles

- Let's generalize the method involving circles.



- This pair of diagrams shows that the diameter, d , of the circle satisfies $\sin(\alpha) = \frac{a}{d}$. This gives $d = \frac{a}{\sin(\alpha)}$.
- What does this pair of diagrams show?
 - These diagrams show that the diameter, d , of the circle satisfies $\sin(\beta) = \frac{b}{d}$. This gives $d = \frac{b}{\sin(\beta)}$.
- And what do we get when we combine the two observations about the diameters?
 - Since the diameter of the circle is constant, we have $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)}$.
- In the same way, we can show that the diameter of the circle is described by the expression $\frac{c}{\sin(\gamma)}$.



- Thus, we can express the law of sines in this form: If sides a , b , and c are opposite angles α , β , and γ , respectively, then the measurements satisfy the equation $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$.

- Could we also write $\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$? Why or why not?
 - Yes. If three fractions are equal, then their reciprocals are also equal.
- It is worth knowing that it is sometimes convenient to set up a problem using this alternate form.

Closing (3 minutes)

- State the law of sines, and explain its uses.
 - The law of sines says that $\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$.
 - We can use this law to find the measurements of a triangle when we know two angles and one of the opposite sides or when we know two sides and one of the opposite angles.
- If b and c are sides of a triangle that are opposite angles β and γ , we can write $b \cdot \sin(\gamma) = c \cdot \sin(\beta)$. What do these two expressions represent geometrically?
 - Each of these expressions represents the length of an altitude of the triangle. Since the expressions represent the same segment, they must be equal to each other.
- If b and c are sides of a triangle that are opposite angles β and γ , we can write $\frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$. What do these two expressions represent geometrically?
 - Each of these expressions represents the diameter of the circle that circumscribes the triangle. Since the diameter of a circle is constant, the expressions must be equal to each other.

Exit Ticket (4 minutes)

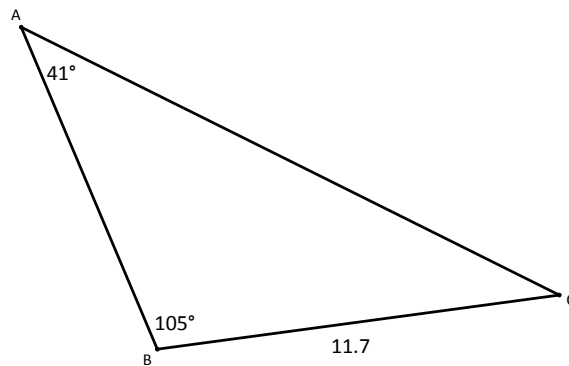
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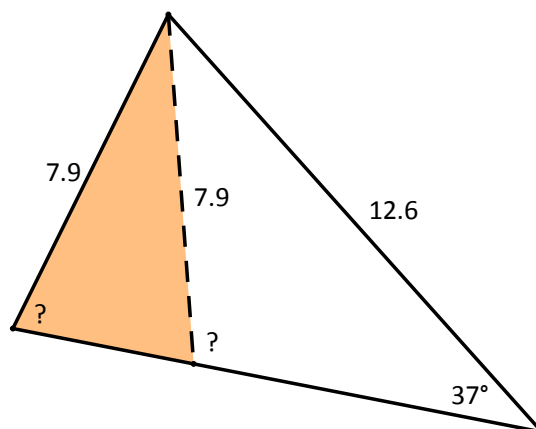
Lesson 8: Law of Sines

Exit Ticket

1. Find the length of side \overline{AC} in the triangle below.

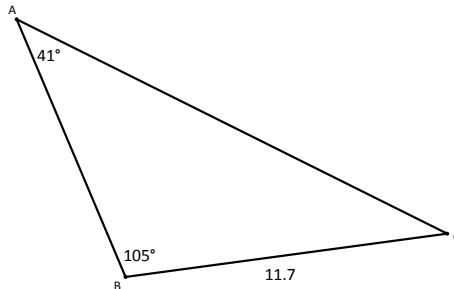


2. A triangle has sides with lengths 12.6 and 7.9. The angle opposite 7.9 is 37° . What are the possible values of the measure of the angle opposite 12.6?



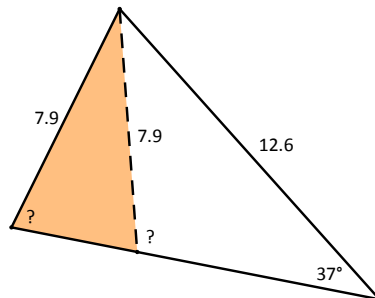
Exit Ticket Sample Solutions

1. Find the length of side \overline{AC} in the triangle below.



We have $\frac{AC}{\sin(105^\circ)} = \frac{11.7}{\sin(41^\circ)}$, which gives $AC = \frac{11.7 \cdot \sin(105^\circ)}{\sin(41^\circ)} \approx 17.2$.

2. A triangle has sides with lengths 12.6 and 7.9. The angle opposite 7.9 is 37° . What are the possible values of the measure of the angle opposite 12.6?



We have $\frac{\sin(x)}{12.6} = \frac{\sin(37^\circ)}{7.9}$, which means that $\sin(x) = \frac{12.6 \cdot \sin(37^\circ)}{7.9}$. The values of x that satisfy this equation are $x \approx 73.7^\circ$ and $x \approx 106.3^\circ$.

Problem Set Sample Solutions

1. Let $\triangle ABC$ be a triangle with the given lengths and angle measurements. Find all possible missing measurements using the law of sines.

a. $a = 5, m\angle A = 43^\circ, m\angle B = 80^\circ$.

$b \approx 7.22, c \approx 6.15, m\angle C = 57^\circ$

b. $a = 3.2, m\angle A = 110^\circ, m\angle B = 35^\circ$.

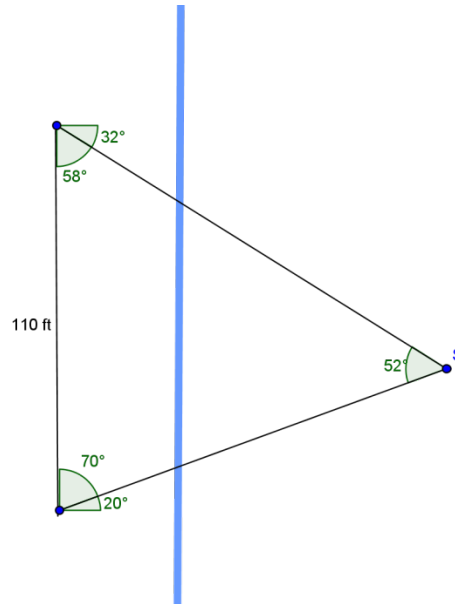
$b \approx 1.95, c \approx 1.95, m\angle C = 35^\circ$

c. $a = 9.1, m\angle A = 70^\circ, m\angle B = 95^\circ$.

$b \approx 9.65, c \approx 2.51, m\angle C = 15^\circ$

- d. $a = 3.2, m\angle B = 30^\circ, m\angle C = 45^\circ$.
 $m\angle A = 105^\circ, b \approx 1.66, c \approx 2.34$
- e. $a = 12, m\angle B = 29^\circ, m\angle C = 31^\circ$.
 $m\angle A = 120^\circ, b \approx 6.72, c \approx 7.14$
- f. $a = 4.7, m\angle B = 18.8^\circ, m\angle C = 72^\circ$.
 $m\angle A = 89.2^\circ, b \approx 1.51, c \approx 4.47$
- g. $a = 6, b = 3, m\angle A = 91^\circ$.
 $m\angle B \approx 29.99^\circ, m\angle C \approx 59.01^\circ, c \approx 5.14$
- h. $a = 7.1, b = 7, m\angle A = 70^\circ$.
 $m\angle B = 67.89^\circ, m\angle C \approx 42.11^\circ, c = 5.07$
- i. $a = 8, b = 5, m\angle A = 45^\circ$.
 $m\angle B = 26.23^\circ, m\angle C = 108.77^\circ, c = 10.71$
- j. $a = 3.5, b = 3.6, m\angle A = 37^\circ$.
 $m\angle B = 38.24^\circ, m\angle C = 104.76^\circ, c = 5.62$ or $m\angle B = 141.76^\circ, m\angle C = 1.24^\circ, c = 0.13$
- k. $a = 9, b = 10.1, m\angle A = 61^\circ$.
 $m\angle B = 78.97^\circ, m\angle C = 40.03^\circ, c = 6.62$ or $m\angle B = 101.03^\circ, m\angle C = 17.97^\circ, c = 3.17$
- l. $a = 6, b = 8, m\angle A = 41.5^\circ$.
 $m\angle B = 62.07^\circ, m\angle C = 76.43^\circ, c = 8.8$ or $m\angle B = 117.93^\circ, m\angle C = 20.57^\circ, c = 3.18$

2. A surveyor is working at a river that flows north to south. From her starting point, she sees a location across the river that is 20° north of east from her current position, she labels the position S . She moves 110 feet north and measures the angle to S from her new position, seeing that it is 32° south of east.
- a. Draw a picture representing this situation.



- b. Find the distance from her starting position to S .

$$\frac{\sin(52^\circ)}{110} = \frac{\sin(58^\circ)}{x}$$

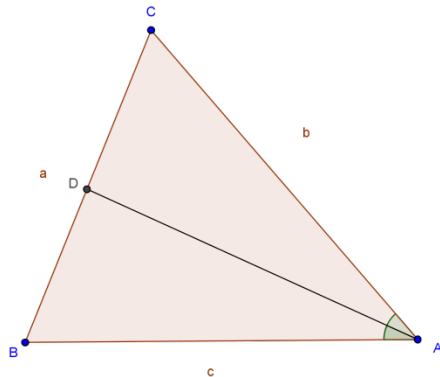
$$x \approx 118.4$$

The distance is about 118 ft.

- c. Explain how you can use the procedure the surveyor used in this problem (called triangulation) to calculate the distance to another object.

Calculate the angle from the starting point to the object. Travel a distance from the starting point, and again calculate the angle to the object. Create a triangle connecting the starting point, the object, and the new location. Use that distance traveled and the angles to find the distance to the object.

3. Consider the triangle pictured below.



Use the law of sines to prove the generalized angle bisector theorem, that is, $\frac{BD}{DC} = \frac{c \sin(m\angle BAD)}{b \sin(m\angle CAD)}$. (Although this is called the generalized angle bisector theorem, we do not assume that the bisector of $\angle BAC$ intersects side \overline{BC} at D . In the case that \overline{AD} is an angle bisector, then the formula simplifies to $\frac{BD}{DC} = \frac{c}{b}$.)

- a. Use the triangles ABD and ACD to express $\frac{c}{BD}$ and $\frac{b}{DC}$ as a ratio of sines.

$$\frac{c}{BD} = \frac{\sin(m\angle BDA)}{\sin(m\angle BAD)}$$

$$\frac{b}{DC} = \frac{\sin(m\angle CDA)}{\sin(m\angle CAD)}$$

- b. Note that angles BDA and ADC form a linear pair. What does this tell you about the value of the sines of these angles?

Since the angles are supplementary, the sines of these values are equal.

- c. Solve each equation in part (a) to be equal to the sine of either $\angle BDA$ or $\angle ADC$.

$$\sin(m\angle BAD) \cdot \frac{c}{BD} = \sin(m\angle BDA)$$

$$\sin(m\angle CAD) \cdot \frac{b}{DC} = \sin(m\angle CDA)$$

- d. What do your answers to parts (b) and (c) tell you?

The answers tell me that the two equations written in part (c) are equal to each other.

- e. Prove the generalized angle bisector theorem.

From part (d), we have

$$\sin(m\angle BAD) \cdot \frac{c}{BD} = \sin(m\angle CAD) \cdot \frac{b}{DC}$$

Dividing both sides by $\sin(m\angle CAD) \cdot b$, and multiplying by BD , we get

$$\frac{BD}{DC} = \frac{c \sin(m\angle BAD)}{b \sin(m\angle CAD)}$$

4. As an experiment, Carrie wants to independently confirm the distance to Alpha Centauri. She knows that if she measures the angle of Alpha Centauri and waits 6 months and measures again, then she will have formed a massive triangle with two angles and the side between them being 2 AU long.

- a. Carrie measures the first angle at $82^\circ 8' 24.5''$ and the second at $97^\circ 51' 34''$. How far away is Alpha Centauri according to Carrie's measurements?

The third angle would be $1.5''$.

$$\frac{\sin\left(82 + \frac{8}{60} + \frac{24.5}{3600}\right)}{a} = \frac{\sin\left(\frac{1.5}{3600}\right)}{2}$$

$$a \approx 272\,436$$

About 272,436 AU.

- b. Today, astronomers use the same triangulation method on a much larger scale by finding the distance between different spacecraft using radio signals and then measuring the angles to stars. Voyager 1 is about 122 AU away from Earth. What fraction of the distance from Earth to Alpha Centauri is this? Do you think that measurements found in this manner are very precise?

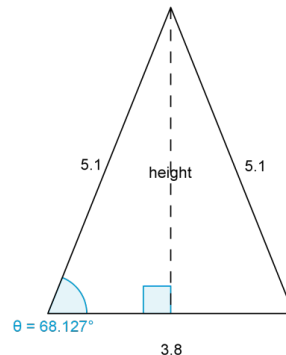
Voyager 1 is about $\frac{122}{276\,364}$, or 0.0004 the distance of Earth to Alpha Centauri. Depending on how far away the object being measured is, the distances are fairly precise on an astronomical scale. One AU is almost 93 million miles, which is not very precise.

5. A triangular room has sides of length 3.8 m, 5.1 m, and 5.1 m. What is the area of the room?

Since the room is isosceles, the height bisects the side of length 3.8 at a right angle. We get $\cos(\theta) = \frac{1.9}{5.1}$, therefore, $\theta \approx 68.127^\circ$.

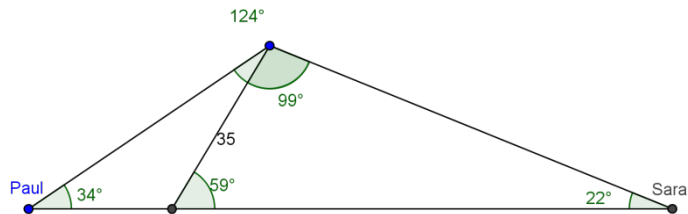
$$\frac{1}{2} \cdot 3.8 \cdot 5.1 \cdot \sin(68.127^\circ) \approx 8.99$$

The area of the room is approximately 8.99 m^2 .



6. Sara and Paul are on opposite sides of a building that a telephone pole fell on. The pole is leaning away from Paul at an angle of 59° and towards Sara. Sara measures the angle of elevation to the top of the telephone pole to be 22° , and Paul measures the angle of elevation to be 34° . Knowing that the telephone pole is about 35 ft. tall, answer the following questions.

- a. Draw a diagram of the situation.



- b. How far apart are Sara and Paul?

We can use law of sines to find the shared side between the two triangles and then again with the larger triangle to find the distance:

$$\frac{\sin(22^\circ)}{35} = \frac{\sin(59^\circ)}{r}$$

$$r \approx 80.086$$

Then we use this value in the larger triangle:

$$\frac{\sin(124^\circ)}{x} = \frac{\sin(34^\circ)}{80.086}$$

$$x \approx 118.733$$

They are about 118.7 ft. away from each other.

- c. If we assume the building is still standing, how tall is the building?

$$\sin(59^\circ) = \frac{y}{35}$$

$$y \approx 30.001$$

The building is about 30 ft. tall.



Lesson 9: Law of Cosines

Student Outcomes

- Students prove the law of cosines and use it to solve problems (**G-SRT.D.10**).

Lesson Notes

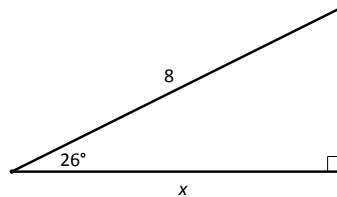
In this lesson, students continue the study of oblique triangles. In the previous lesson, students learned the law of sines. In this lesson, students add the law of cosines to their repertoire of tools that can be used to analyze the measurements of triangles. To prove the law of cosines, students build squares on the sides of an oblique triangle and then analyze their areas. In this way, students build on their prior knowledge of the Pythagorean theorem.

Classwork

Exercises 1–2 (2 minutes)

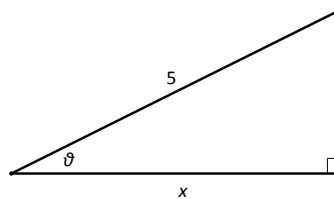
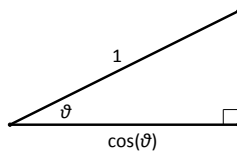
Exercises

- Find the value of x in the triangle below.



We have $\cos(26^\circ) = \frac{x}{8}$. This means $x = 8 \cdot \cos(26^\circ) \approx 7.2$.

- Explain how the figures below are related. Then, describe x in terms of ϑ .

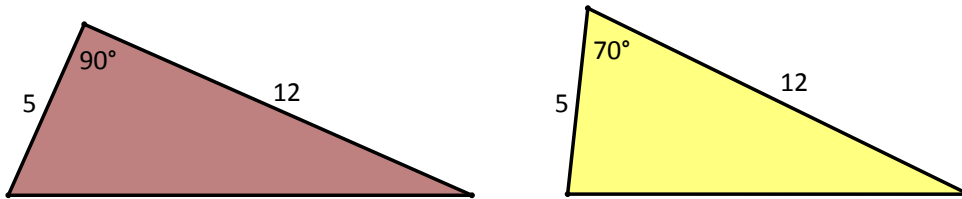


The triangles above are similar. We can produce the triangle at the right by dilating the triangle at the left using a scale factor of 5. This means that $x = 5 \cdot \cos(\vartheta)$.

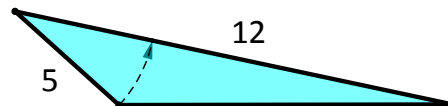
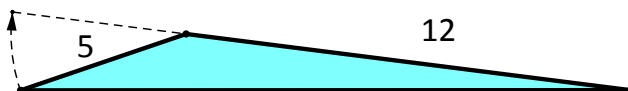
Exploratory Challenge (9 minutes): Exploring the SAS Case

Scaffolding:

- If students need support with this task, prompt them to consider extreme cases and to represent them with drawings.
- What happens if you open up the included angle so that it is as large as possible? What happens when you close the included angle as much as possible?



- Look at the triangles in the diagram above. What is the length of the third side of the triangle on the left? How did you determine your answer? Can you use the same strategy to find the third side of the triangle on the right?
 - *The triangle on the left has a right angle, so we can apply the Pythagorean theorem. The third side satisfies $c^2 = 5^2 + 12^2$, which means that $c = 13$. The triangle on the right does not have a right angle, so we cannot use this strategy.*
- Despite the fact that we can't use the Pythagorean theorem on the triangle at the right, do you think you should be able to determine the remaining measurements in this triangle? Why or why not? Think about this for a moment, then explain your reasoning to a neighbor.
 - *Yes, the SAS criterion for triangle congruence says that the three given measurements determine the remaining measurements in the figure.*
- Let's try to get some idea of how large the third side of the triangle can be. If two sides of a triangle are 5 and 12, how big can the third side be? Think about this for a moment, then explain your reasoning to a neighbor.
 - *If we open up the angle between the two given sides so that it approaches a straight angle, the third side of the triangle approaches a line segment with length $5 + 12 = 17$.*
 - *If we close the angle so that it approaches the zero angle, the third side of the triangle approaches a line segment with length $12 - 5 = 7$.*

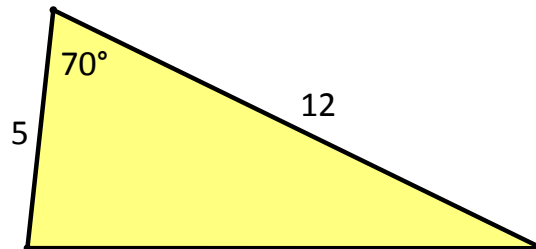


- So it appears that the third side takes on values between 7 and 17 and that the length of the third side is a function of the angle between the two given sides. There is a formula called the law of cosines that makes this relationship precise. We develop this formula together during the course of the lesson.

MP.3

MP.7

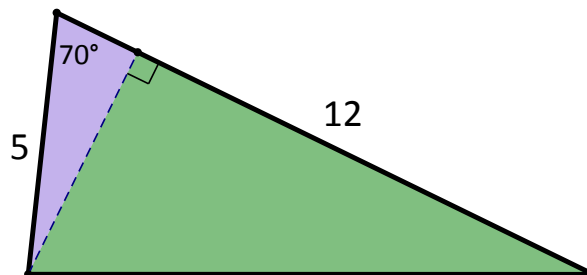
- Take a few minutes to explore the triangle below. Can you find the length of the third side? Work together with a partner, and see how much progress you can make.
- Can we use the law of sines? Why or why not?
 - No, we do not know the length of the side opposite a known angle.



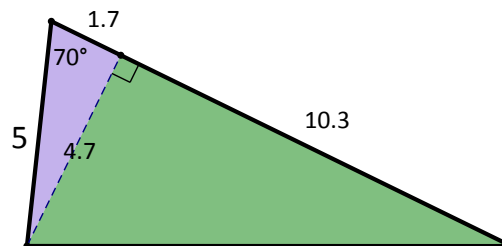
Scaffolding:

- Where could you add an auxiliary line to this figure?
- Add an auxiliary line that would allow you to capitalize on your knowledge of right triangles.

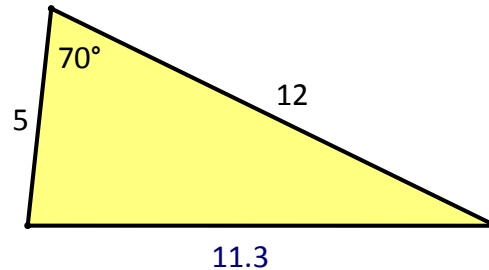
- We can draw an altitude to form two right triangles:



- Next, we can use trigonometry to describe the sides of the upper triangle. In particular, the small leg is $5 \cos(70^\circ) \approx 1.7$, and the longer leg is $5 \sin(70^\circ) \approx 4.7$.
- Since the original triangle has a side of length 12, we can also figure out the longer leg of the lower triangle, namely $12 - 1.7 = 10.3$.



- Finally, we can apply the Pythagorean theorem to the lower triangle. The hypotenuse of this triangle is a number c which satisfies $4.7^2 + 10.3^2 = c^2$, so we must have $c \approx 11.3$. So the third side of the original triangle is about 11.3 units long.

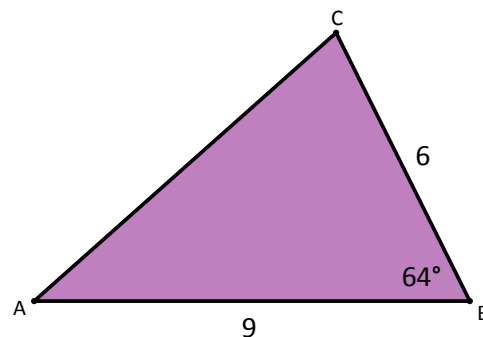


- Let's do another example of this kind, and then we'll look for a way to generalize our work.

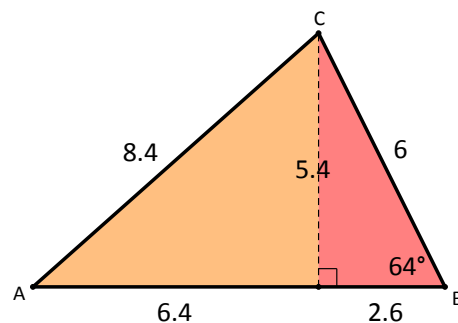
Exercise 3 (5 minutes)

Instruct students to solve the following problems, and then have them compare their work with a partner. Select two students to write their work on the board to share with the class.

3. Find the length of side \overline{AC} in the triangle below.

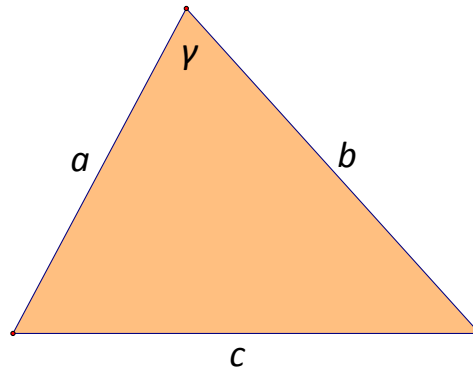


We have $6 \cdot \cos(64^\circ) \approx 2.6$ and $6 \cdot \sin(64^\circ) \approx 5.4$. Then we have $9 - 2.6 = 6.4$. Finally, we have $\sqrt{6.4^2 + 5.4^2} \approx 8.4$.

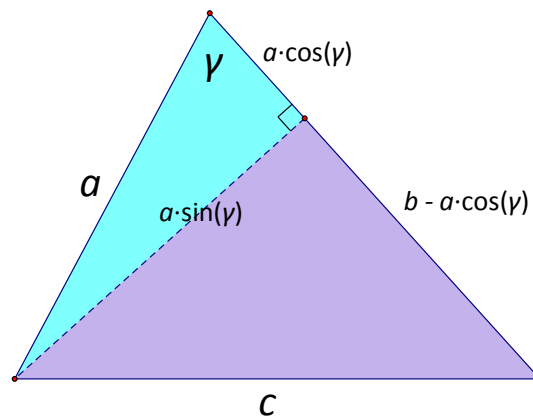


Discussion (5 minutes): Proving the Law of Cosines

- In the examples on the previous page, what information was given to you? What information were you able to find?
 - *We were given two sides and the included angle. We determined the length of the third side.*
- Let's try to develop a general formula that handles problems of this type. In the diagram below, find a way to describe the side c in terms of the other three measurements in the figure. Take a few minutes to work together with the students around you on this task.



- *First, we draw in an altitude to create right triangles, and then we describe the segments as before.*

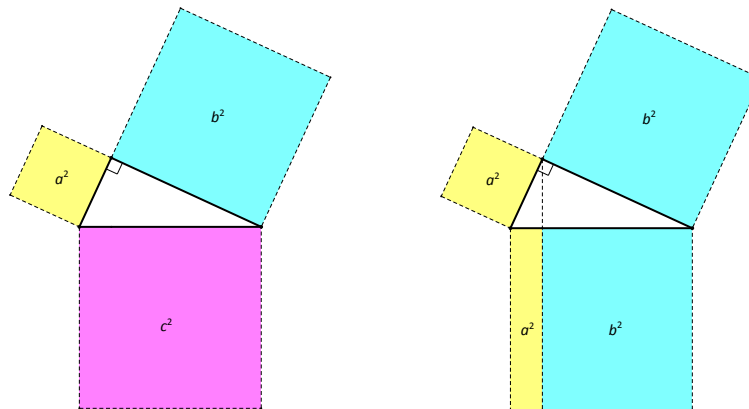


- *The third side c must satisfy $c^2 = (a \cdot \sin(\gamma))^2 + (b - a \cdot \cos(\gamma))^2$.*
- *Let's do some algebra: We have $c^2 = (a \cdot \sin(\gamma))^2 + b^2 - 2ab \cdot \cos(\gamma) + (a \cdot \cos(\gamma))^2$.*
- *We also see in the upper triangle that $(a \cdot \sin(\gamma))^2 + (a \cdot \cos(\gamma))^2 = a^2$, so we can rearrange and replace terms as*
 - $c^2 = [(a \cdot \sin(\gamma))^2 + (a \cdot \cos(\gamma))^2] + b^2 - 2ab \cdot \cos(\gamma)$
 - $c^2 = a^2 + b^2 - 2ab \cdot \cos(\gamma)$
- *We can use this relation to find c if we are given a , b , and γ .*
- Since this relation features the cosine function, it is known as the **law of cosines**.

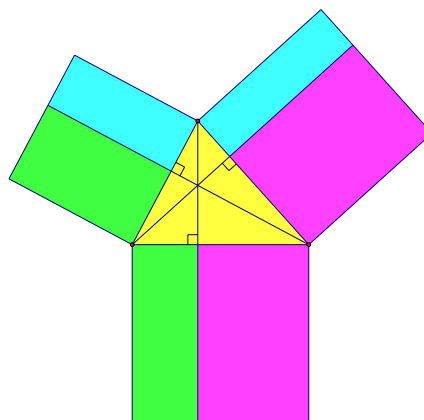
Discussion (13 minutes): Geometric Interpretation

MP.2

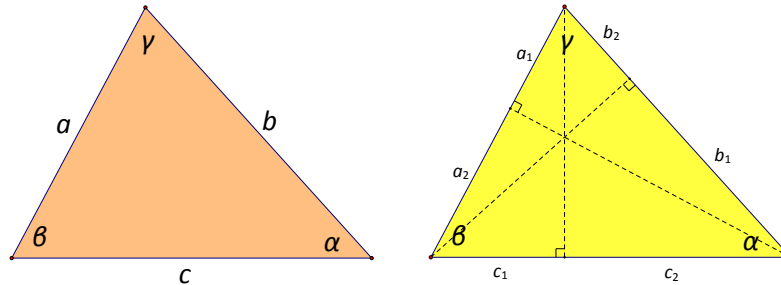
- Now that we have the law of cosines, let's explore its geometric meaning. How do all of the symbols in the formula $c^2 = a^2 + b^2 - 2ab \cdot \cos(\gamma)$ relate to the parts of the triangle? This is a complex question! Let's begin by reviewing what we know about right triangles.
- Do you recall the geometric interpretation of the Pythagorean theorem? In particular, what do the quantities a^2 , b^2 , and c^2 represent, and how are these three quantities related?
 - The symbols a , b , and c represent the sides of a right triangle, with c being the hypotenuse. The quantities a^2 , b^2 , and c^2 represent the areas of the squares that are built on the sides of the triangle, as shown in the figure below:



- The equation $a^2 + b^2 = c^2$ tells us that the area of the large square is equal to the sum of the areas of the two smaller squares.
- The picture on the right even suggests a proof of this fact and is closely related to the work you'll do below. But our main interest today is in the study of oblique triangles, so let's turn our attention back to those.
- Altitudes played a key role in our work above. In particular, they allowed us to split the figure into right triangles, which allowed us to utilize our knowledge of right triangle trigonometry. The key to understanding the law of cosines from a geometric point of view is to draw squares on the sides of the triangle and then to divide those squares by drawing in the altitudes as shown below:



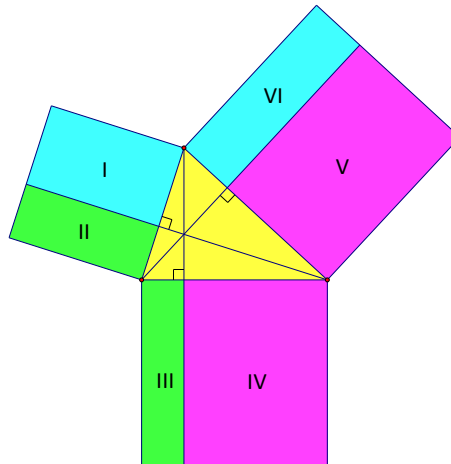
- Can you find a way to describe each of the rectangles in the figure? Take several minutes to work with the students around you on this task.
 - Let's begin by choosing labels for each of the components of an oblique triangle.



- Now we will attempt to describe each of the small segments in the figure.

a_1	a_2	b_1	b_2	c_1	c_2
$b \cdot \cos(\gamma)$	$c \cdot \cos(\beta)$	$c \cdot \cos(\alpha)$	$a \cdot \cos(\gamma)$	$a \cdot \cos(\beta)$	$b \cdot \cos(\alpha)$

- We can use the information in the table above to analyze the areas in the figure.



I	II	III	IV	V	VI
$a \cdot a_1$	$a \cdot a_2$	$c \cdot c_1$	$c \cdot c_2$	$b \cdot b_1$	$b \cdot b_2$
$ab \cdot \cos(\gamma)$	$ac \cdot \cos(\beta)$	$ca \cdot \cos(\beta)$	$cb \cdot \cos(\alpha)$	$bc \cdot \cos(\alpha)$	$ba \cdot \cos(\gamma)$

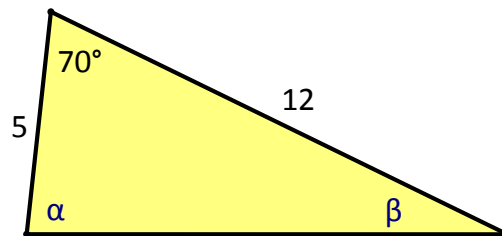
- From this table, it's clear that the pink regions have equal areas, as do the blue and green regions. So the equation $c^2 = a^2 + b^2 - 2ab \cdot \cos(\gamma)$ says that the area of the lower square is equal to the sum of the areas of the upper squares minus the area of the two blue rectangles.
- Fascinating, isn't it? The law of cosines truly is a generalization of the Pythagorean theorem.

- Note that if we shift our focus to a different square in the picture, we get a different version of this formula. Here are all three versions:
 - $a^2 + b^2 = c^2 + 2ab \cdot \cos(\gamma)$
 - $b^2 + c^2 = a^2 + 2bc \cdot \cos(\alpha)$
 - $c^2 + a^2 = b^2 + 2ca \cdot \cos(\beta)$
- Let's do a few examples to put all of our hard work in this lesson to use!

Example (3 minutes)

In this example, students use the law of cosines to find an unknown side of a triangle, and then they use the law of cosines to find an unknown angle in a triangle.

- Let's revisit the triangle you encountered at the start of the lesson. This time, find the missing side length by directly applying the law of cosines.



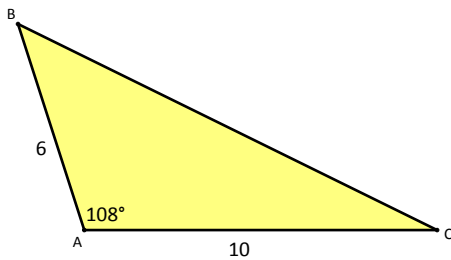
- $5^2 + 12^2 = c^2 + 2(5)(12) \cdot \cos(70^\circ)$
 $25 + 144 = c^2 + 120 \cdot \cos(70^\circ)$
 $169 = c^2 + 120 \cdot \cos(70^\circ)$
 $c^2 = 169 - 120 \cdot \cos(70^\circ)$
 $c = \sqrt{169 - 120 \cdot \cos(70^\circ)}$
 $c \approx 11.3$
- The law of cosines can also be used to find the missing angles in the triangle. Give it a try!
 - $5^2 + 11.3^2 = 12^2 + 2(5)(11.3) \cdot \cos(\alpha)$
 $25 + 127.69 = 144 + 113 \cdot \cos(\alpha)$
 $152.69 = 144 + 113 \cdot \cos(\alpha)$
 $8.69 = 113 \cdot \cos(\alpha)$
 $\cos(\alpha) = \frac{8.69}{113}$
 $m\angle\alpha \approx 86^\circ$

Now that we know two of the angles, we can easily find the third: $\beta = 180^\circ - 70^\circ - 86^\circ = 24^\circ$.

Exercises 4–5 (3 minutes)

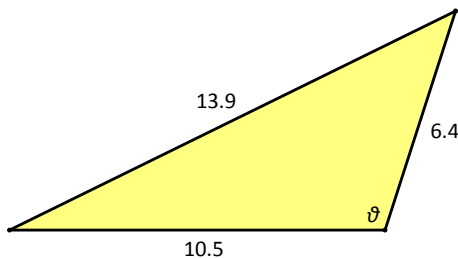
Instruct students to solve the problems below and then to compare their answers with a partner.

4. Points B and C are located at the edges of a large body of water. Point A is 6 km from point B and 10 km from point C . The angle formed between \overline{BA} and \overline{AC} is 108° . How far apart are points B and C ?



$$\begin{aligned} 6^2 + 10^2 &= BC^2 + 2(6)(10) \cdot \cos(108^\circ) \\ 136 &= BC^2 + 120 \cdot \cos(108^\circ) \\ BC^2 &= 136 - 120 \cdot \cos(108^\circ) \\ BC &\approx 13.2 \end{aligned}$$

5. Use the law of cosines to find the value of ϑ in the triangle below.

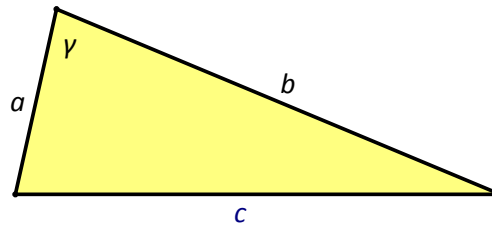
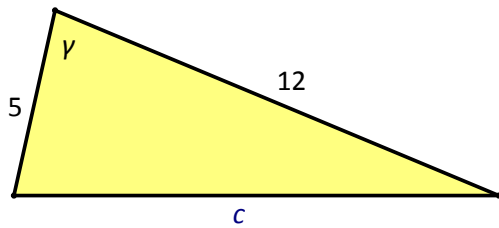


$$\begin{aligned} 10.5^2 + 6.4^2 &= 13.9^2 + 2(10.5)(6.4) \cdot \cos(\vartheta) \\ -42 &= 134.4 \cdot \cos(\vartheta) \\ m\angle \vartheta &\approx 108^\circ \end{aligned}$$

Scaffolding:

- Be aware that it is common for students to make order-of-operations errors in this context.
- For example, students may write $121 - 96 \cdot \cos(\vartheta) = 25 \cdot \cos(\vartheta)$.
- To address this, ask students to solve the equation $10 = 16 - 2x$ and then check to be sure that their solution is correct. If they write $10 = 14x$, they will find that their answer does not satisfy the equation.

Lesson Extension (optional)



- At the start of the lesson, we observed that $7 < c < 17$. More generally, the **triangle inequality** tells us that $b - a < c < b + a$. Does the law of cosines support these constraints on the value of c ?
- What does the law of cosines have to say about the triangle on the right? Write an equation.
 - $a^2 + b^2 = c^2 + 2ab \cdot \cos(\gamma)$
- Find the minimum and maximum values of c using what you know about the cosine function.
 - The cosine function has a maximum value of 1. In this case, we get the following:

$$\begin{aligned} a^2 + b^2 &= c^2 + 2ab \\ a^2 - 2ab + b^2 &= c^2 \\ (a - b)^2 &= c^2 \\ c &= a - b \end{aligned}$$

The cosine function reaches a value of 1 when the input is 0, so this corresponds to a zero angle between the two sides of the triangle (i.e., they form a straight line).

- We also know that the cosine function has a minimum value of -1 . This time our analysis is as follows:

$$\begin{aligned} a^2 + b^2 &= c^2 - 2ab \\ a^2 + 2ab + b^2 &= c^2 \\ (a + b)^2 &= c^2 \\ a + b &= c \end{aligned}$$

The cosine function reaches a value of -1 when the input is 180, so this corresponds to a straight angle between the two sides of the triangle (i.e., they form a straight line here as well).

- Let's try another special case. What does the law of cosines have to say when the angle involved is a right angle?

- The cosine of 90° is 0, so in this case we have the following:

$$\begin{aligned} a^2 + b^2 &= c^2 + 2ab \cdot \cos(90^\circ) \\ a^2 + b^2 &= c^2 + 2ab \cdot 0 \\ a^2 + b^2 &= c^2 + 0 \\ a^2 + b^2 &= c^2 \end{aligned}$$

- So the law of cosines is truly a generalization of the Pythagorean theorem. It's nice to know that this formula does what it is supposed to even in these special cases!

Closing (2 minutes)

- State the law of cosines.
 - $a^2 + b^2 = c^2 + 2ab \cdot \cos(\gamma)$
- Interpret each quantity in the formula geometrically.
 - *The expressions a^2 , b^2 , and c^2 represent the areas of squares.*
 - *The expression $2ab \cdot \cos(\gamma)$ represents the combined area of two rectangles with equal areas.*
- What is the law of cosines used for?
 - *You can use the law of cosines to find unknown parts of an oblique triangle.*

Exit Ticket (3 minutes)

Name _____

Date _____

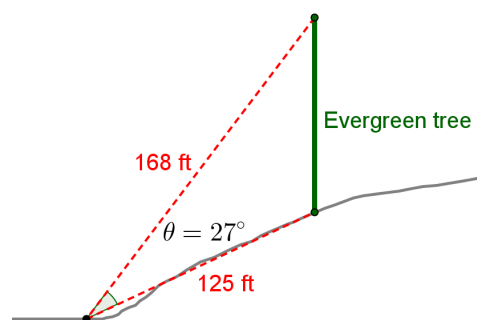
Lesson 9: Law of Cosines

Exit Ticket

Use the law of cosines to solve the following problems.

1. Josie wishes to install a new television that will take up 15° of her vertical field of view. Using the angle she wants, she uses a laser measure and finds the distances from the wall to her couch are 8 ft. and 12 ft., but she does not have any way to mark the spots on the wall. How tall is the television that she wants?

2. Given the figure shown, find the height of the evergreen tree. Round your answers to the nearest thousandths.



Exit Ticket Sample Solutions

Use the law of cosines to solve the following problems.

1. Josie wishes to install a new television that will take up 15° of her vertical field of view. Using the angle she wants, she uses a laser measure and finds the distances from the wall to her couch are 8 ft. and 12 ft., but she does not have any way to mark the spots on the wall. How tall is the television that she wants?

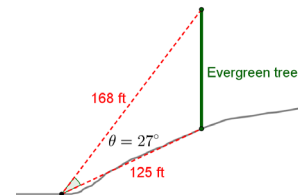
$$\sqrt{8^2 + 12^2 - 2 \cdot 8 \cdot 12 \cdot \cos(15^\circ)} \approx 4.748$$

Josie's television will be more than 4.748 ft. tall.

2. Given the figure shown, find the height of the evergreen tree. Round your answers to the nearest thousandths.

$$\sqrt{125^2 + 168^2 - 2 \cdot 125 \cdot 168 \cdot \cos(27^\circ)} \approx 80.167$$

The height of the tree is approximately 80.167 ft.



Problem Set Sample Solutions

The first few problems reframe the optional lesson extension as problems for the students to explore. If there was not enough time to complete the lesson extension in class, then use these problems as an alternative. If there was time to cover the material in class, then students may want to start with Problem 5, or Problems 1–4 can be used to emphasize the extension.

1. Consider the case of a triangle with sides 5, 12, and the angle between them 90° .

- a. What is the easiest method to find the missing side?

Pythagorean theorem

- b. What is the easiest method to find the missing angles?

Right triangle trigonometry

- c. Can you use the law of cosines to find the missing side? If so, perform the calculations. If not, show why not.

Yes, this is an example of SAS. The missing side is 13.

- d. Can you use the law of cosines to find the missing angles? If so, perform the calculations. If not, show why not.

Yes. The measures of the missing angles are 22.62° and 67.38° .

- e. Consider a triangle with sides a , b , and the angle between them 90° . Use the law of cosines to prove a well-known theorem. State the theorem.

$$c^2 = a^2 + b^2 - 2ab \cos(90^\circ)$$

$$c^2 = a^2 + b^2 - 2ab \cdot 0$$

$$c^2 = a^2 + b^2$$

The Pythagorean theorem

- f. Summarize what you have learned in parts (a) through (e).

The law of cosines appears to be a more general form of the Pythagorean theorem, one that can work with any angle between two sides of a triangle, not just a right angle.

2. Consider the case of \overline{CA} and \overline{CB} of lengths 5 and 12, respectively, with $m\angle C = 180^\circ$.

- a. Is ABC a triangle?

\overline{AB} is a line segment containing C . $\triangle ABC$ does not exist.

- b. What is the easiest method to find the distance between A and B ?

Add the lengths of the two line segments partitioning \overline{AB} : $5 + 12 = 17$.

- c. Can you use the law of cosines to find the distance between A and B ? If so, perform the calculations. If not, show why not.

Yes. $5^2 + 12^2 - 2 \cdot 5 \cdot 12 \cos(180^\circ) = 25 + 144 + 120 = 289 = 17^2$

The distance between A and B is 17 units.

- d. Summarize what you have learned in parts (a) through (c).

The law of cosines can work even when the line segments are a linear pair and do not form a triangle.

3. Consider the case of \overline{CA} and \overline{CB} of lengths 5 and 12, respectively, with $m\angle C = 0^\circ$.

- a. Is ABC a triangle?

\overline{CB} is a line segment containing A . $\triangle ABC$ does not exist.

- b. What is the easiest method to find the distance between A and B ?

Subtract the two line segments: $12 - 5 = 7$.

- c. Can you use the law of cosines to find the distance between A and B ? If so, perform the calculations. If not, show why not.

Yes, the law of cosines still applies. $5^2 + 12^2 - 2 \cdot 5 \cdot 12 \cos(0^\circ) = 25 + 144 - 120 = 49 = 7^2$

The distance between A and B is 7 units.

- d. Summarize what you have learned in parts (a) through (c).

The law of cosines applies even when the line segments have the same starting point and one lies on top of the other.

4. Consider the case of \overline{CA} and \overline{CB} of lengths 5 and 12, respectively, with $m\angle C > 180^\circ$.

- a. Is the law of cosines consistent in being able to calculate the length of \overline{AB} even using an angle this large? Try it for $m\angle C = 200^\circ$, and compare your results to the triangle with $m\angle C = 160^\circ$. Explain your findings.

Yes, the law of cosines is still able to calculate the length of \overline{AB} :

$$c^2 = 5^2 + 12^2 - 2 \cdot 5 \cdot 12 \cos(200^\circ) \\ \approx 281.76$$

Thus, $c \approx 16.786$, which represents the length of line segment AB . We get the same result for $\cos(160^\circ)$, which makes sense since it is true that $\cos(\theta) = \cos(360^\circ - \theta)$.

- b. Consider what you have learned in Problems 1–4. If you were designing a computer program to be able to measure sides and angles of triangles created from different line segments and angles, would it make sense to use the law of cosines or several different techniques depending on the shape? Would a computer program created from the law of cosines have any errors based on different inputs for the line segments and angle between them?

The benefits of using the law of cosines would be that there would be no need for logic involving different cases to be programmed, and there would be no exceptions to the formula. The law of cosines would work even when the shape formed was not a triangle or if the shape was formed using an angle greater than 180° . The triangle inequality theorem would need to be used to verify whether the side lengths could represent those of a triangle. There would be no errors for two line segments and the angle between them.

5. Consider triangles with the following measurements. If two sides are given, use the law of cosines to find the measure of the third side. If three sides are given, use the law of cosines to find the measure of the angle between a and b .

- a. $a = 4, b = 6, m\angle C = 35^\circ$
 $c \approx 3.56$

- b. $a = 2, b = 3, m\angle C = 110^\circ$
 $c \approx 4.14$

- c. $a = 5, b = 5, m\angle C = 36^\circ$
 $c \approx 3.09$

- d. $a = 7.5, b = 10, m\angle C = 90^\circ$
 $c \approx 12.5$

- e. $a = 4.4, b = 6.2, m\angle C = 9^\circ$
 $c \approx 1.98$

- f. $a = 12, b = 5, m\angle C = 45^\circ$
 $c \approx 9.17$

- g. $a = 3, b = 6, m\angle C = 60^\circ$
 $c \approx 5.2$

h. $a = 4, b = 5, c = 6$

$$m\angle C \approx 82.82^\circ$$

i. $a = 1, b = 1, c = 1$

$$m\angle C = 60^\circ$$

j. $a = 7, b = 8, c = 3$

$$m\angle C \approx 21.79^\circ$$

k. $a = 6, b = 5.5, c = 6.5$

$$m\angle C \approx 68.68^\circ$$

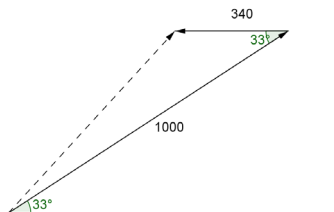
l. $a = 8, b = 5, c = 12$

$$m\angle C \approx 133.43^\circ$$

m. $a = 4.6, b = 9, c = 11.9$

$$m\angle C \approx 118.45^\circ$$

6. A trebuchet launches a boulder at an angle of elevation of 33° at a force of 1,000 N. A strong gale wind is blowing against the boulder parallel to the ground at a force of 340 N. The figure is shown below.



- a. What is the force in the direction of the boulder's path?

Since the wind is blowing in the opposite direction, finding the sum of the two vectors is similar to finding the difference between two vectors if the wind is in the same direction. Thus, we are finding the third side of a triangle (the diagonal of the parallelogram).

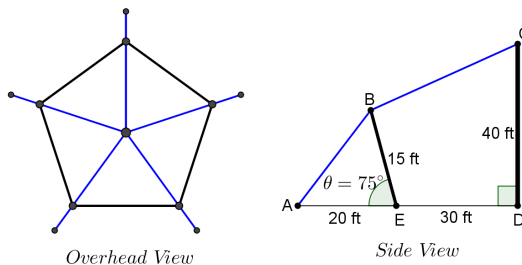
$$\begin{aligned} c^2 &= 1000^2 + 340^2 - 2 \cdot 1000 \cdot 340 \cos(33^\circ) \\ &\approx 738.45 \end{aligned}$$

The boulder is traveling with an initial force of 738.45 N.

- b. What is the angle of elevation of the boulder after the wind has influenced its path?

The angle between the original trajectory and the new trajectory is 14.52° , so the new angle of elevation is 47.52° .

7. Cliff wants to build a tent for his son's graduation party. The tent is a regular pentagon, as illustrated below. How much guy-wire (show in blue) does Cliff need to purchase to build this tent? Round your answers to the nearest thousandths.



$$AB = \sqrt{20^2 + 15^2 - 2 \cdot 20 \cdot 15 \cdot \cos(75^\circ)} \approx 21.673$$

$$EC = \sqrt{40^2 + 30^2} = 50$$

$$m\angle CED = \arctan\left(\frac{40}{30}\right) = 55.130^\circ$$

$$m\angle BEC = 180^\circ - 75^\circ - 55.130^\circ = 49.870^\circ$$

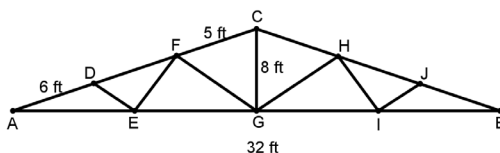
$$BC = \sqrt{15^2 + 50^2 - 2 \cdot 15 \cdot 50 \cdot \cos(49.870^\circ)} \approx 41.931$$

$$AB + BC = 63.604$$

$$(5)(63.604) = 318.020$$

Total guy-wire is 318.020 feet long.

8. A roofing contractor needs to build roof trusses for a house. The side view of the truss is shown below. Given that G is the midpoint of \overline{AB} , E is the midpoint of \overline{AG} , I is the midpoint of \overline{GB} , $AB = 32$ ft., $AD = 6$ ft., $FC = 5$ ft., and $m\angle AGC = 90^\circ$. Find DE , EF , and FG . Round your answers to the nearest thousandths.



$$m\angle CAG = \arctan\left(\frac{8}{16}\right) = 26.565^\circ$$

$$DE = \sqrt{6^2 + 8^2 - 2 \cdot 6 \cdot 8 \cdot \cos(26.565^\circ)} \approx 3.760$$

$$m\angle ACG = 180^\circ - 90^\circ - 26.565^\circ = 63.435^\circ$$

$$FG = \sqrt{5^2 + 8^2 - 2 \cdot 5 \cdot 8 \cdot \cos(63.435^\circ)} \approx 7.295$$

$$m\angle FGC = \arccos\left(\frac{8^2 + 7.295^2 - 5^2}{2 \cdot 8 \cdot 7.295}\right) \approx 37.808^\circ$$

$$m\angle FGE = 90^\circ - 37.808^\circ = 52.192^\circ$$

$$EF = \sqrt{8^2 + 7.295^2 - 2 \cdot 8 \cdot 7.295 \cdot \cos(52.192^\circ)} \approx 6.758$$

DE is approximately 3.760 feet long, FG is approximately 7.295 feet long, and EF is approximately 6.758 feet long.



Lesson 10: Putting the Law of Cosines and the Law of Sines to Use

Student Outcomes

- Students apply the law of sines or the law of cosines to determine missing measurements in real-world situations that can be modeled using non-right triangles, including situations that involve navigation, surveying, and resultant forces.

Lesson Notes

The Opening Exercise asks students to categorize which triangles can be solved by using the law of sines versus using the law of cosines. Then the bulk of the lesson presents different real-world scenarios using the law of sines or the law of cosines to compute missing measurements in situations involving non-right triangles. Depending on the size of your class, groups should be assigned one or two problems to present to the class. Groups that finish early can work on additional problems in the set.

As groups work through their assigned problems, they engage in the modeling cycle by making sense of the problem and then formulating a model to compute the required missing measurements (MP.1 and MP.4). They interpret and validate their responses and then report their results to the class.

Classwork

Opening (2 minutes)

Use these questions as a quick introduction to this lesson. Have students discuss their answers with a partner before having a few students share their responses.

- What is the law of sines? The law of cosines?
 - Given a triangle with sides a , b , and c and angles opposite those sides measuring A , B , and C , respectively, the law of sines is
$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c},$$
and the law of cosines is
$$a^2 = b^2 + c^2 - 2bc \cos(A).$$
- How many measurements are needed to determine the rest of the measurements in a non-right triangle? Explain your reasoning.
 - You need at least three measurements, and then you can write an equation to solve for the fourth unknown measurement.

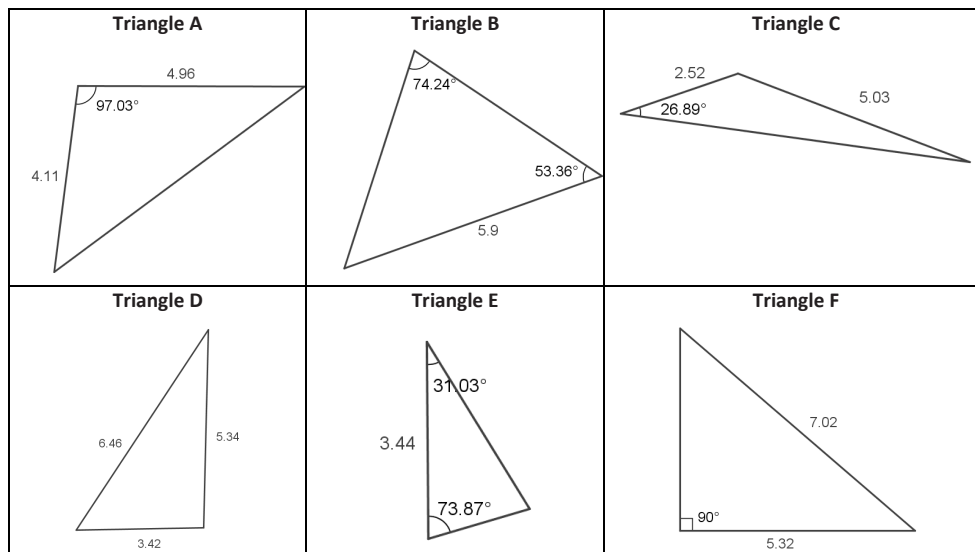
- Look at Triangle A below, how would you know whether to use the law of sines or the law of cosines to find its missing measurements?
 - *You are not provided with a side opposite the given angle, so you would need to use the law of cosines.*

Opening Exercise (5 minutes)

There are six problems presented below. Have students work with a partner to decide which formula, the law of sines or the law of cosines, would be required to find the missing measurements. One of the triangles is a right triangle, so as students work, be sure to point out that it is not necessary to use the law of sines or the law of cosines to find missing measures in these types of triangles.

Opening Exercise

- a. For each triangle shown below, decide whether you should use the law of sines, the law of cosines, or neither to begin finding the missing measurements. Explain how you know.



Triangle A is solved using the law of cosines because we are given two sides and the included angle.

Triangle B is solved using the law of sines because we are given two angles and one side opposite one of the angles.

Triangle C is solved using the law of sines because we are given two sides and one angle with the angle being opposite one side.

Triangle D is solved using the law of cosines because three sides are given.

Triangle E is solved using the law of sines because two angles are given. We can easily find the third using the triangle sum theorem, and then we will have an angle and opposite side pairing.

Triangle F does not require the law of sines or the law of cosines because it is a right triangle. We can find the missing sides or angles using the Pythagorean theorem and right triangle trigonometry functions.

- b. What types of given information will help you to decide which formula to use to determine missing measurements? Summarize your ideas in the table shown below:

Determining Missing Measurements

Given Measurements	Formulas to Use
Right Triangle <i>Two side measurements</i> <i>One angle and one side measurement</i>	Trigonometry Functions $\sin(\theta) = \frac{O}{H}$ $\cos(\theta) = \frac{A}{H}$ $\tan(\theta) = \frac{O}{A}$ <p>Where O is the leg opposite θ, A is the leg adjacent to θ, and H is the hypotenuse</p> Pythagorean Theorem $a^2 + b^2 = c^2$ <p>Where a and b are legs of a right triangle and c is the hypotenuse</p>
Non-Right Triangle <i>Any two angles and one side</i> <i>Two sides and the angle opposite one of them</i>	Law of Sines $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$ <p>Where a is opposite angle A, b is opposite angle B, and c is opposite angle C</p>
Non-Right Triangle <i>Three sides because this formula relates all three sides of a triangle</i> <i>Two sides and the angle between them because the law of sines requires an angle and the opposite side</i>	Law of Cosines $a^2 = b^2 + c^2 - 2bc \cos(A)$ <p>Where A is the measure of the angle opposite side a</p>

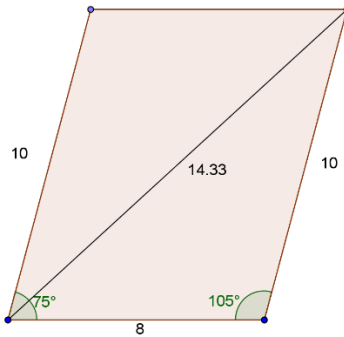
Exercises 1–7 (17 minutes)

MP.4

Students should work in groups of 2–4 to apply the law of sines or the law of cosines to solve these problems. Students engage in the modeling cycle (MP.4) as they formulate the problem by drawing and labeling a diagram to determine the requested measurement(s). They decide on an appropriate formula and compute the missing measurements. Once they have computed measurements, they must interpret and validate their results in terms of the given situation. Their solutions should include a diagram that illustrates the problem description and worked solutions to find the measurements required to solve the problem. Consider posting the numerical solution to each problem on the board so groups can check and validate their answers as they work. If time is running short, consider assigning some of these problems as homework exercises. Be sure to allow enough time to consider the two navigation problems at the end of this set of problems. You can invite different groups to present their solutions as this portion of the lesson comes to a close.

Exercises 1–7

1. A landscape architect is given a survey of a parcel of land that is shaped like a parallelogram. On the scale drawing the sides of the parcel of land are 8 in. and 10 in., and the angle between these sides measures 75° . The architect is planning to build a fence along the longest diagonal. If the scale on the survey is 1 in. = 120 ft., how long will the fence be?



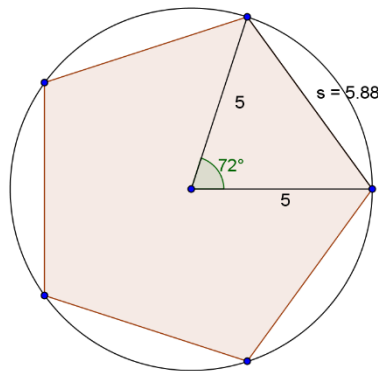
Let d be the measure of the longest diagonal.

$$d^2 = 10^2 + 8^2 - 2(10)(8)\cos(105^\circ)$$

$$d \approx 14.33$$

On the survey, this diagonal is approximately 14.33 in. The actual length of the fence will be 1,719.6 ft.

2. A regular pentagon is inscribed in a circle with a radius of 5 cm. What is the perimeter of the pentagon?



Let s be the measure of one side of the regular pentagon.

$$s^2 = 5^2 + 5^2 - 2(5)(5)\cos(72^\circ)$$

$$s \approx 5.88$$

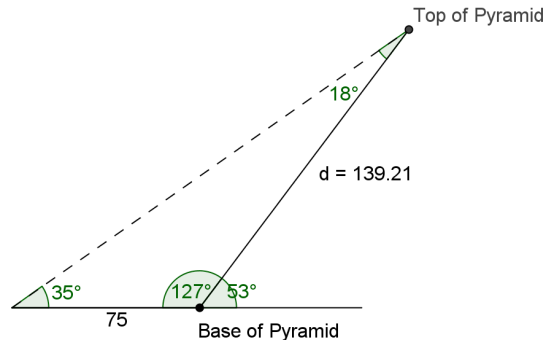
Since there are five sides: $5 \cdot 5.88 = 29.4$

Thus, the perimeter is 29.4 cm.

Scaffolding:

- For English language learners, provide additional support on the word problems in Exercises 1–7 by providing diagrams that illustrate the verbal descriptions in the problems.
- Ask students, “Where is the triangle in this situation?” and “What measurements in the triangle do we know?”
- Discuss viewpoint, and have students visualize flying above the problem description or viewing it from a distance (as if watching a video).
- On some technology applications, such as GeoGebra, you can import a photo of a situation similar those described in the Exercises 1–7 and superimpose a triangle on the photo.

3. At the base of a pyramid, a surveyor determines that the angle of elevation to the top is 53° . At a point 75 meters from the base, the angle of elevation to the top is 35° . What is the distance from the base of the pyramid up the slanted face to the top?



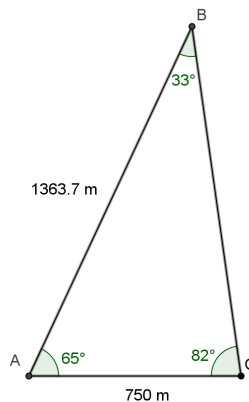
Let d be the distance from the base to the top of the pyramid.

$$\frac{\sin(18^\circ)}{75} = \frac{\sin(35^\circ)}{d}$$

$$d \approx 139.21$$

The distance is approximately 139 meters.

4. A surveyor needs to determine the distance across a lake between an existing ferry dock at point A and a second dock across the lake at point B . He locates a point C along the shore from the dock at point A that is 750 meters away. He measures the angle at A between the sight lines to points B and C to be 65° and the angle at C between the sight lines to points A and B to be 82° . How far is it from the dock at A and the dock at B ?



To find $m\angle B$: $180^\circ - (65^\circ + 82^\circ) = 33^\circ$.

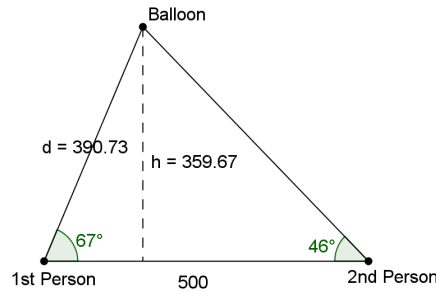
Let d be the distance from the dock at A and the dock at B .

$$\frac{\sin(33^\circ)}{750} = \frac{\sin(82^\circ)}{d}$$

$$d \approx 1363.7$$

The distance between the two docks across the lake is approximately 1,363.7 meters.

5. Two people located 500 yards apart have spotted a hot air balloon. The angle of elevation from one person to the balloon is 67° . From the second person to the balloon the angle of elevation is 46° . How high is the balloon when it is spotted?



Let d be the distance between the first person and the balloon. Let h be the height of the balloon in the air.

By the law of sines,

$$\frac{\sin(46^\circ)}{d} = \frac{\sin(67^\circ)}{500}$$

$$d \approx 390.73$$

Then,

$$\sin(67^\circ) = \frac{h}{390.73}$$

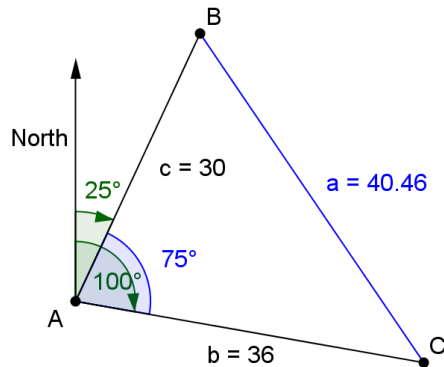
$$h \approx 359.67$$

The balloon is approximately 360 yards in the air when it is spotted.

Take time to discuss the concept of a bearing as it applies to navigation. Often, when measuring distance on water, we use nautical miles rather than statute miles, which we use to measure distance on land. These problems have been greatly simplified to provide an introduction to students on how the law of sines and law of cosines can be applied to navigation problems. Wind, currents, and elevation are not being taken into account in these situations. Students may wish to research this topic further on the Internet as well. If needed, lead a whole-class discussion modeling how to draw the diagram for Exercise 7. Determining the angle measurements in Exercise 8 is challenging, as students may need to draw several auxiliary lines to help them. One suggestion is to draw a ray representing north at each point where you are given a bearing and sketch in the angle. It may also help to draw in a line perpendicular to the north-facing ray indicating east and west directions so angle measurements can be quickly calculated as needed.

When applying mathematics to navigation, direction is often given as a bearing. The **bearing** of an object is the degrees rotated clockwise from north that indicates the direction of travel or motion. The next exercises apply the law of cosines and the law of sines to navigation problems.

6. Two fishing boats start from a port. One travels 15 nautical miles per hour on a bearing of 25° and the other travels 18 nautical miles per hour on a bearing of 100° . Assuming each maintains its course and speed, how far apart will the fishing boats be after two hours?



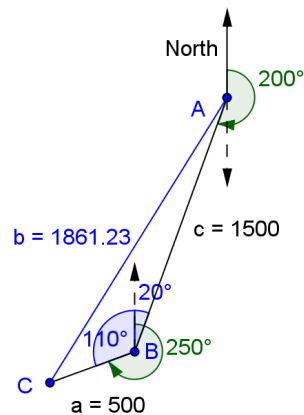
Let point A be the starting location of the two fishing boats at the port. After two hours one ship will have traveled 30 nautical miles from A to B. The other ship will have traveled 36 nautical miles from A to C. The law of cosines can be used to find the distance between the ships, a .

$$a^2 = 30^2 + 36^2 - 2(30)(36)\cos(75^\circ)$$

$$a \approx 40.46$$

The ships will be approximately 40.5 nautical miles apart after two hours.

7. An airplane travels on a bearing of 200° for 1500 miles and then changes to a bearing of 250° and travels an additional 500 miles. How far is the airplane from its starting point?



The measure of $\angle ABC$ is $110^\circ + 20^\circ$, or 130° . It is the sum of a 20° angle that is congruent to the angle formed by side c and the south-facing direction line from point A, and the difference between 250° and a full rotation of 360° about point B.

By the law of cosines, side b , which represents the distance from the starting point at A and the final point at C is given by

$$b^2 = 1500^2 + 500^2 - 2(1500)(500)\cos(130^\circ)$$

$$b \approx 1861.23$$

Thus, the airplane is approximately 1,861 miles from where it started. Note: This solution takes neither the elevation of the airplane nor the curvature of the Earth into account.

Discussion (2 minutes)

After students report their solutions to the class, lead a short discussion asking them to generate a list of tips for setting up and solving modeling problems that can be represented using non-right triangles.

- What advice would you give to students who are solving similar types of problems?
 - *Make sense of the problem, and begin to formulate the solution by reading the problem and drawing a diagram that contains a triangle. Represent unknown measurements with variable symbols.*
 - *Decide how to compute the answer using either the law of sines or the law of cosines. Remember that you must know at least three measurements in the triangle to use one of these formulas.*
 - *Re-read the problem to help you interpret and validate your solution and to be sure you have determined the required information to solve the problem.*
 - *Report your solution by providing the requested measurements and information.*

Example (5 minutes): Revisiting Vectors and Resultant Forces

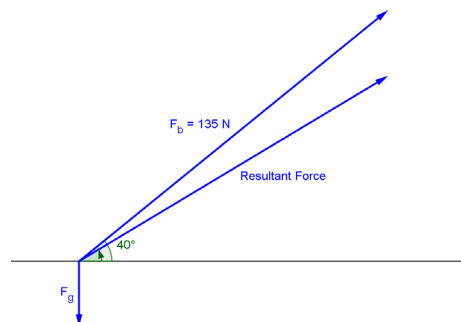
As you start this example, you may wish to remind students that a vector is a quantity that has a magnitude and direction. In physics, the resultant of two forces (which can be represented with vectors) acting on an object is the sum of the individual forces (vectors). Students need to recall the parallelogram rule for adding two vectors and then determine the magnitude and direction using the law of cosines and the law of sines. In the example, the force of the kick is given in newtons, a metric system (SI) unit of force based on the formula $\text{force} = \text{mass} \cdot \text{acceleration}$. The SI base units of a newton are $\text{kg} \cdot \frac{\text{m}}{\text{s}^2}$. The force due to gravity is the product of the mass of the ball measured in kilograms and the gravitational constant $9.8 \frac{\text{m}}{\text{s}^2}$. This model assumes that the only two forces acting on the ball are the force of the throw and the force due to gravity. In the second example, the speeds given would result from the force of each player's kick. The model is simplified to include only the forces of each player's kick acting on the ball, and we assume the kicks send the ball traveling along the ground and not up into the air. You can discuss the set-up of this problem and then let students work on it in their small groups or provide a more direct approach and work through the problem together as a whole class.

Example: Revisiting Vectors and Resultant Forces

The goalie on the soccer team kicks a ball with an initial force of 135 Newtons at a 40° angle with the ground. The mass of a soccer ball is 0.45 kg. Assume the acceleration due to gravity is $9.8 \frac{\text{m}}{\text{s}^2}$.

- a. Draw a picture representing the force vectors acting on the ball and the resultant force vector.

In the diagram below, F_b represents the force of the kick and F_g represents the force due to gravity.



MP.4

- b. What is the magnitude of the resultant force vector?

The force due to gravity is the product of the mass of the ball and the acceleration due to gravity.

$$F_g = 0.45 \cdot 9.8 = 4.41$$

Translating the gravitational force vector to the terminal point of the ball's force vector and using the law of cosines gives the magnitude of the resultant force. The angle between the two vectors is $40^\circ + 90^\circ = 130^\circ$. This would make the angle in the triangle that we are using for law of cosines 50° . Let b represent the magnitude of the resultant force vector.

$$\begin{aligned} b^2 &= 135^2 + 4.41^2 - 2(135)(4.41) \cos(50^\circ) \\ b &\approx 132.21 \end{aligned}$$

The magnitude of the resultant force is approximately 132 newtons.

- c. What are the horizontal and vertical components of this vector?

The components of the initial force on the ball are $\langle 135 \cos(40^\circ), 135 \sin(40^\circ) \rangle$, and the components of the gravitational force vector are $\langle 0, -4.41 \rangle$. Adding the vector components gives the resultant force in component form.

$$\langle 135 \cos(40^\circ) + 0, 135 \sin(40^\circ) - 4.41 \rangle = \langle 103.416, 82.366 \rangle$$

- d. What is the angle of elevation of the resulting vector?

Using right triangle trigonometry ratios, we can compute the angle of elevation.

$$\begin{aligned} \tan(\theta) &= \frac{82.366}{103.416} \\ \theta &= \arctan\left(\frac{82.366}{103.416}\right) \\ \theta &\approx 38.5^\circ \end{aligned}$$

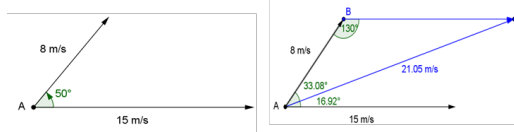
MP.4

Exercises 8–10 (7 minutes)

Have students work these final exercises in their groups. Present one or two solutions after groups have had time to do the exercises together.

Exercises 8–10

8. Suppose a soccer player runs up to a moving soccer ball located at A and kicks the ball into the air. The diagram below shows the initial velocity of the ball along the ground and the initial velocity and direction of the kick. What is the resultant velocity and angle of elevation of the soccer ball immediately after it is kicked?



If we translate the vector with magnitude $15 \frac{\text{m}}{\text{s}}$ to point B , then the angle at B will be 130° . Then the sum of the two vectors is the vector with tail at the origin at C . Let b be the magnitude of this vector. By the law of cosines,

$$b = 8^2 + 15^2 - 2(8)(15) \cos(130^\circ)$$

$$b \approx 21.05$$

The direction can be found using the law of sines. Let θ be the angle between the $8 \frac{\text{m}}{\text{s}}$ vector and b .

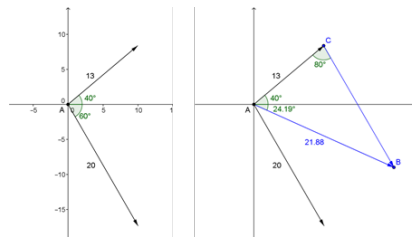
$$\frac{\sin(\theta)}{15} = \frac{\sin(130^\circ)}{21.05}$$

$$\theta = \arcsin\left(\frac{15 \sin(130^\circ)}{21.05}\right)$$

$$\theta \approx 33.08^\circ$$

Thus, the direction of the ball would be $50^\circ - 33.08^\circ = 16.92^\circ$.

9. A 13 lb. force and a 20 lb. force are applied to an object located at A as shown in the diagram below. What is the resulting force and direction being applied to the object at A ?



The resulting force is the sum of the two forces, which can be represented as vectors. The parallelogram rule gives us the resulting force vector. Using the law of cosines, we can determine the magnitude, and using the law of sines, we can determine the direction.

Let c be the distance between point A and B .

$$c^2 = 13^2 + 20^2 - 2(13)(20) \cos(80^\circ)$$

$$c \approx 21.88$$

The measure of $\angle CAB$ can be found using the law of sines. Let θ be the measure of $\angle CAB$.

$$\frac{\sin(80^\circ)}{21.88} = \frac{\sin(\theta)}{20}$$

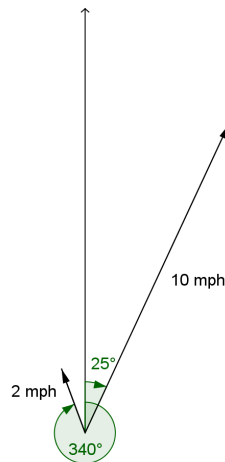
$$\theta = \arcsin\left(\frac{20 \sin(80^\circ)}{21.88}\right)$$

$$\theta \approx 64.19^\circ$$

The resulting force of 21.88 lb. would be in a direction of 24.19° clockwise from the horizontal axis.

10. A motorboat travels across a lake at a speed of 10 mph at a bearing of 25° . The current of the lake due to the wind is a steady 2 mph at a bearing of 340° .

- a. Draw a diagram that shows the two velocities that are affecting the boat's motion across the lake.



- b. What is the resulting speed and direction of the boat?

The resulting speed and direction is the sum of these two velocity vectors. Translating the current vector to the tip of the boat's speed vector allows us to quickly draw the resulting vector. Its magnitude and direction can be determined using the law of cosines and the law of sines.

Let a be the distance between points C and B . By the law of cosines,

$$a^2 = 10^2 + 2^2 - 2(10)(2) \cos(135^\circ)$$

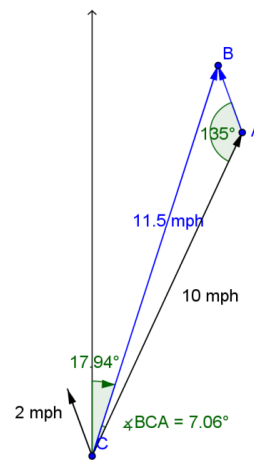
$$a \approx 11.5$$

Let θ be the measure of $\angle BCA$. Then, using the law of sines,

$$\frac{\sin(135^\circ)}{11.5} = \frac{\sin(\theta)}{2}$$

$$\theta \approx 7.06^\circ$$

Then, bearing is $25^\circ - 7.06^\circ = 17.94^\circ$, and the speed is 11.5 mph.



Closing (2 minutes)

Have students answer the following questions either individually in writing or with a partner.

- When do you use the law of sines to find missing measurements?
 - *When you are given two angles plus one side that is opposite a known angle or two sides and the non-included angle so long as the measurements can actually make a triangle.*
- When do you use the law of cosines to find missing measurements?
 - *When you are given three sides or two sides plus the included angle.*
- How do the law of cosines and the law of sines apply when working with vectors?
 - *We can use the magnitude of the vectors and the direction angles to form triangles whose missing measurements can be calculated. The sum and difference of two vectors is the third side of a triangle formed by the two vectors when positioned with the same terminal point or when positioned end-to-end.*

Lesson Summary

The law of sines and the law of cosines can be used to solve problems that can be represented with triangles with three known measurements.

The law of sines and the law of cosines can be used to find the magnitude and direction of the resultant sum of two vectors, which can represent velocities, distances, or forces.

Exit Ticket (5 minutes)

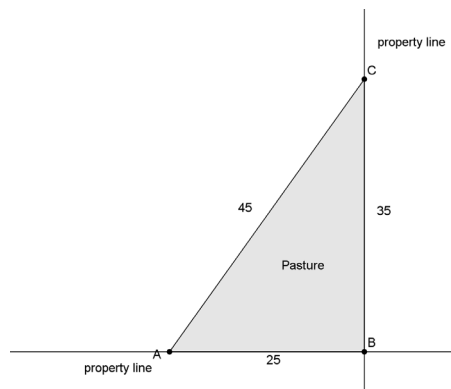
Name _____

Date _____

Lesson 10: Putting the Law of Cosines and the Law of Sines to Use

Exit Ticket

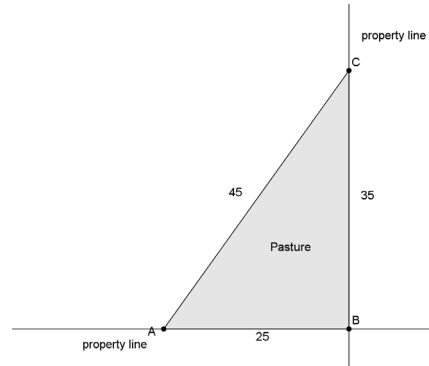
A triangular pasture is enclosed by fencing measuring 25, 35, and 45 yards at the corner of a farmer's property.



- According to the fencing specifications, what is the measure of $\angle ABC$?
- A survey of the land indicates that the property lines form a right angle at B . Explain why a portion of the pasture is actually on the neighboring property.
- Where does the 45-yard section of the fence cross the vertical property line?

Exit Ticket Sample Solutions

A triangular pasture is enclosed by fencing measuring 25, 35, and 45 yards at the corner of a farmer's property.



- a. According to the fencing specifications, what is the measure of $\angle ABC$?

$$\begin{aligned} 45^2 &= 25^2 + 35^2 - 2 \cdot 25 \cdot 35 \cdot \cos(B) \\ B &= \arccos\left(\frac{45^2 - 25^2 - 35^2}{-2 \cdot 25 \cdot 35}\right) \\ &= \arccos(-0.1) \\ &\approx 95.739 \end{aligned}$$

The measure of $\angle ABC$ is 95.739° .

- b. A survey of the land indicates that the property lines form a right angle at B . Explain why a portion of the pasture is actually on the neighboring property.

If the fencing went along the actual property lines, then the angle at B would be 90° . Since it is larger than 90° , a portion of the pasture must lie in the adjacent property.

- c. Where does the 45-yard section of the fence cross the vertical property line?

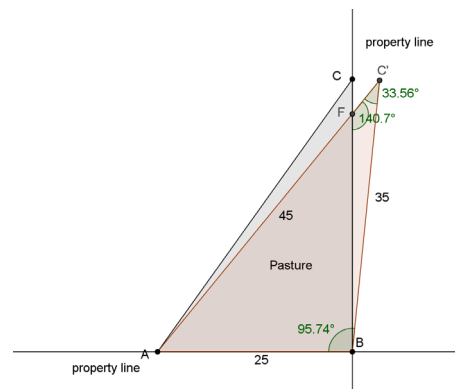
$$C' = \arcsin\left(25 \cdot \frac{\sin(95.74^\circ)}{45}\right) \approx 33.56^\circ$$

The measure of $\angle FBC'$ is 5.74° , and the measure of $\angle C'FB$ is 140.7° . Let c' be the measure of segment BF .

$$\frac{\sin(33.56^\circ)}{c'} = \frac{\sin(140.7^\circ)}{35}$$

$$c' \approx 30.55$$

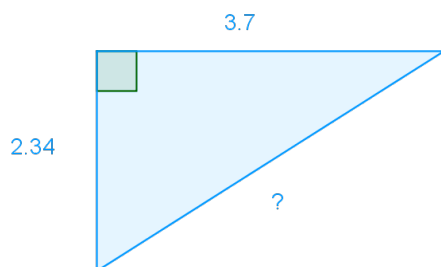
The 45-yard section of fence crosses the vertical property line approximately 30.55 yards from point B .



Problem Set Sample Solutions

1. For each of the situations below, determine whether to use the Pythagorean theorem, right triangle trigonometry, law of sines, law of cosines, or some other method.

a.

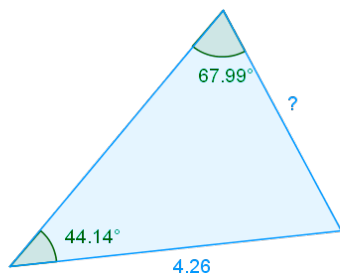


Pythagorean theorem

- b. Know one side and an angle of a right triangle and want to find any other side.

Right triangle trigonometry

c.

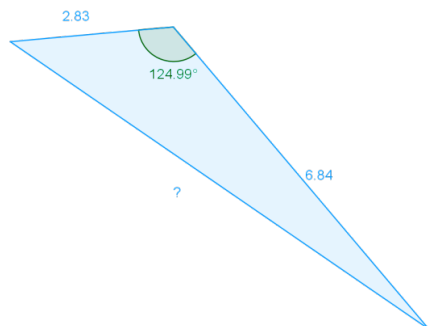


Law of sines

- d. Know two angles of a triangle and want to find the third.

Find the sum of the measures of the two known angles, and subtract the result from 180°.

e.

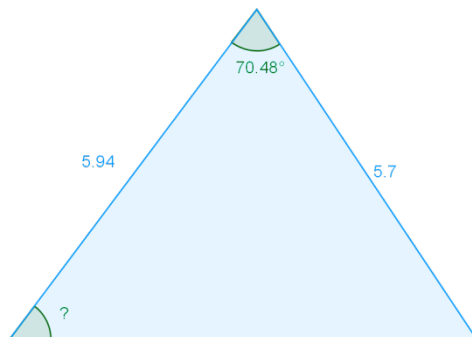


Law of cosines

- f. Know three sides of a triangle and want to find an angle.

Law of cosines

- g.

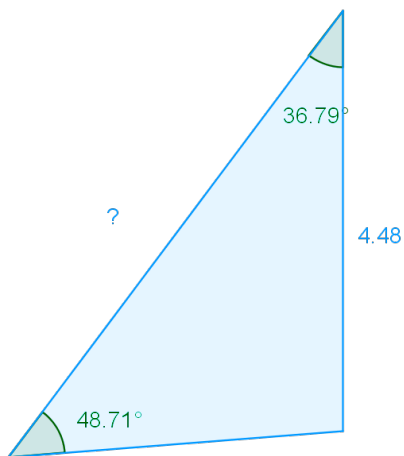


Either law of cosines twice or a combination of law of cosines and law of sines

- h. Know a side and two angles and want to find the third angle.

Find the sum of the measures of the two known angles, and subtract the result from 180° .

- i.



Law of sines

2. Mrs. Lane's trigonometry class has been asked to judge the annual unmanned hot-air balloon contest, which has a prize for highest flying balloon.

- a. Sarah thinks that the class needs to set up two stations to sight each balloon as it passes between them. Construct a formula that Mrs. Lane's class can use to find the height of the balloon by plugging the two angles of elevation so that they can program their calculators to automatically output the height of the balloon. Use 500 ft. for the distance between the stations and α and β for the angles of elevation.

$$b = 500 \frac{\sin(\beta)}{\sin(180^\circ - \alpha - \beta)}$$

$$h = b \sin(\alpha)$$

$$h = \frac{500 \sin(\alpha) \sin(\beta)}{\sin(180^\circ - \alpha - \beta)}$$

- b. The students expect the balloons to travel no higher than 500 ft. What distance between the stations would you recommend? Explain.

Answers may vary. Depending on how close the balloons pass to the stations, students may be sighting the balloon at near vertical angles. More accurate measurements can probably be obtained the closer the balloons are to 45° from the stations, so a distance of greater than 500 ft. is probably better suited.

- c. Find the heights of balloons sighted with the following angles of elevation to the nearest ten feet. Assume a distance of 500 ft. between stations.

- i. $5^\circ, 15^\circ$

$$32.977 \text{ ft.} \approx 30 \text{ ft.}$$

- ii. $38^\circ, 72^\circ$

$$311.553 \text{ ft.} \approx 310 \text{ ft.}$$

- iii. $45^\circ, 45^\circ$

$$250 \text{ ft.}$$

- iv. $45^\circ, 59^\circ$

$$312.33 \text{ ft.} \approx 310 \text{ ft.}$$

- v. $28^\circ, 44^\circ$

$$171.45 \text{ ft.} \approx 170 \text{ ft.}$$

- vi. $50^\circ, 66^\circ$

$$389 \text{ ft.} \approx 390 \text{ ft.}$$

- vii. $17^\circ, 40^\circ$

$$112 \text{ ft.} \approx 110 \text{ ft.}$$

- d. Based on your results in part (c), which balloon won the contest?

The student with the balloon that went 390 ft.

- e. The balloons were released several hundred feet away but directly in the middle of the two stations. If the first angle represents the West station and the second angle represents the East station, what can you say about the weather conditions during the contest?

It appears as though a wind was blowing the balloons to the east.

- f. Are there any improvements to Mrs. Lane's class's methods that you would suggest? Explain.

Answers may vary. Students could suggest higher degrees of accuracy by adjusting the distance between the students. Multiple angles of elevation could be taken from different spots, or additional students could help measure to minimize human error.

3. Bearings on ships are often given as a clockwise angle from the direction the ship is heading (0° represents something in the path of the boat, and 180° represents something behind the boat). Two ships leave port at the same time. The first ship travels at a constant speed of 30 kn. After $2\frac{1}{2}$ hours, the ship sights the second at a bearing of 110° and 58 nautical miles away.

- a. How far is the second ship from the port where it started?

109 nautical miles from port

- b. How fast is the second ship traveling on average?

$\frac{109}{2.5} = 43.6$; The second ship is traveling 43.6 kn.

4. A paintball is fired from a gun with a force of 59 N at an angle of elevation of 1° . If the force due to gravity on the paintball is 0.0294 N, then answer the following:

- a. Is this angle of elevation enough to overcome the initial force due to gravity and still have an angle of elevation greater than 0.5° ?

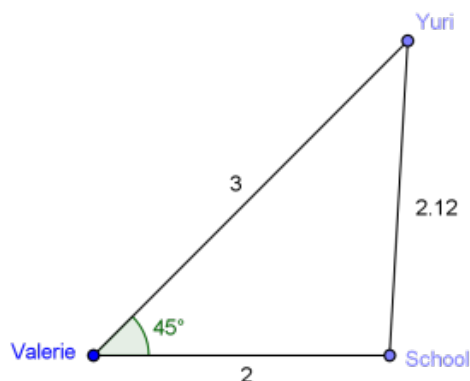
Yes. The force due to gravity is so small that there is effectively no difference initially. The third side has a magnitude of 58.999. The angle of elevation is reduced by less than 0.029° .

- b. What is the resultant magnitude of the vector in the direction of the paintball?

58.999 N

5. Valerie lives 2 miles west of her school and her friend Yuri lives 3 miles directly northeast of her.

- a. Draw a diagram representing this situation.



- b. How far does Yuri live from school?

$$\begin{aligned} c &= \sqrt{2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cdot \cos(45)} \\ &= \sqrt{13 - 6\sqrt{2}} \\ &\approx 2.125 \end{aligned}$$

Yuri lives approximately 2.125 mi. from school.

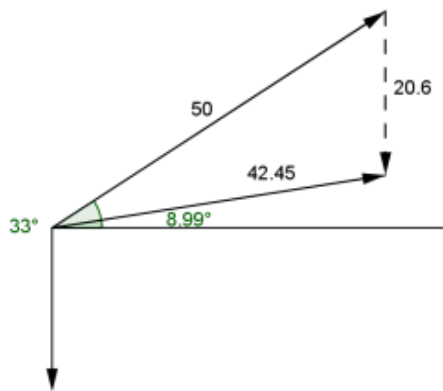
- c. What is the bearing of the school to Yuri's house?

$$\arccos\left(\frac{9 - 2.125^2 - 4}{-8.499}\right) \approx 93.27$$

Yuri's house is 93.27° N of W from the school, so the school is on a bearing of -176.73 from N. This can be worded different ways, for instance, 86.73° S of W, 93.27° S of E.

6. A 2.1 kg rocket is launched at an angle of 33° with an initial force of 50 N. Assume the acceleration due to gravity is $9.81 \frac{\text{m}}{\text{s}^2}$.

- a. Draw a picture representing the force vectors and their resultant vector.



- b. What is the magnitude of the resultant vector?

The force due to gravity: $F = 2.1 \cdot 9.81 = 20.601$

$$\begin{aligned} c &= \sqrt{50^2 + 20.601^2 - 2 \cdot 50 \cdot 20.601 \cdot \cos(57^\circ)} \\ &\approx 42.455 \end{aligned}$$

The resultant vector is about 42.455 N.

- c. What are the horizontal and vertical components of the resultant vector?

The initial force of the rocket can be expressed by the vector $\langle 50 \cos(33^\circ), 50 \sin(33^\circ) \rangle$. The force due to gravity can be expressed by $\langle 0, 20.601 \rangle$. The resultant vector is $\langle 41.934, 6.631 \rangle$.

- d. What is the angle of elevation of the resultant vector?

$$\arctan\left(\frac{6.631}{41.934}\right) \approx 8.986$$

The angle of elevation is about 8.986° .

7. Use the distance formula to find c , the distance between A and B for $\triangle ABC$, with $A = (b \cos(\gamma), b \sin(\gamma))$, $B = (a, 0)$, and $C = (0, 0)$. After simplifying, what formula have you proven?

$$AB = \sqrt{(b \cos(\gamma) - a)^2 + (b \sin(\gamma))^2}$$

Multiplying out on the inside of the square root, we get

$$b^2(\cos(\gamma))^2 - 2ab \cos(\gamma) + a^2 + b^2(\sin(\gamma))^2.$$

Factoring out b^2 , we get

$$b^2((\cos(\gamma))^2 + (\sin(\gamma))^2) - 2ab \cos(\gamma) + a^2 = b^2 + a^2 - 2ab \cos(\gamma).$$

So we have

$$AB = \sqrt{a^2 + b^2 - 2ab \cos(\gamma)}.$$

Since $\triangle ABC$ could be any triangle translated to the origin and rotated so that one side lays on the x -axis, we have proven the law of cosines.

8. For isosceles triangles with $a = b$, show the law of cosines can be written as $\cos(\gamma) = 1 - \frac{c^2}{2a^2}$.

$$c^2 = a^2 + b^2 - 2ab \cos(\gamma)$$

$$c^2 = 2a^2 - 2a^2 \cos(\gamma)$$

$$c^2 = 2a^2(1 - \cos(\gamma))$$

$$\frac{c^2}{2a^2} = 1 - \cos(\gamma)$$

$$\cos(\gamma) = 1 - \frac{c^2}{2a^2}$$



Topic C

Inverse Trigonometric Functions

F-TF.B.6, F-TF.B.7

Focus Standards:	F-TF.B.6	(+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
	F-TF.B.7	(+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.*
Instructional Days:	4	
Lesson 11:	Revisiting the Graphs of the Trigonometric Functions (E) ¹	
Lesson 12:	Inverse Trigonometric Functions (P)	
Lessons 13–14:	Modeling with Inverse Trigonometric Functions (P, P)	

Topic C returns student focus to the graphs of trigonometric functions. Students visualize these graphs with the aid of appropriate software and recall how changing the parameters of a trigonometric function affects its graph. Students extend their knowledge of inverse functions to trigonometric functions, restricting their domains to create inverses (**F-TF.B.6**). Students use inverses to solve trigonometric equations that arise in modeling contexts (**F-TF.B.7**).

In Module 2 of Algebra II, students graphed the sine and cosine functions and explored trigonometric identities that could be discovered from their graphs. Students also graphed functions in the form $f(x) = A \sin(\omega(x - h)) + k$ and modeled periodic data using a sinusoidal function. The graph of the tangent function was introduced but not covered extensively. The purpose of Lesson 11 is to review the graphs of the sine, cosine, and tangent functions as a lead-in to understanding the inverse trigonometric functions which are studied in Lesson 12. The graphs reinforce properties covered in Topic A, such as symmetry and periodicity. In Lesson 12, students extend their knowledge of inverse functions to trigonometric functions; students reason to develop their own domain restrictions before learning the conventional restrictions for each trigonometric function. Students apply inverse trigonometric functions to solve problems arriving in modeling contexts in Lesson 13 (**F-TF.B.6**). Specifically, this lesson focuses on using inverse trigonometric functions to determine the best horizontal viewing distance that an observer should stand from an object being viewed. Students use technology to solve the problems and engage in several parts of the modeling cycle (creating, computing, and interpreting with models) (**F-TF.B.7**). In Lesson 14,

¹Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

students continue applying inverse functions in modeling contexts as they design ramps and determine calendar dates based on the motion of the sun.

Mathematical practices MP.4 and MP.5 are highlighted in Topic C. Students model mathematics (MP.4) with trigonometric and inverse trigonometric functions designing a wheelchair ramp and a ramp into a parking garage and predicting rabbit populations. Students use technology (MP.5) to visualize graphs and explore the effects of the parameters A , ω , h , and k on the graph of the function $f(x) = A \sin(\omega(x - h)) + k$.



Lesson 11: Revisiting the Graphs of the Trigonometric Functions

Student Outcomes

- Students graph the sine, cosine, and tangent functions and analyze characteristics of their graphs.

Lesson Notes

In Module 2 of Algebra II, students graphed the sine and cosine functions and explored trigonometric identities that could be discovered from their graphs. Students also graphed functions in the form $f(x) = A \sin(\omega(x - h)) + k$ and modeled periodic data using a sinusoidal function. The graph of the tangent function was introduced but not covered extensively. The purpose of this lesson is to review the graphs of the sine, cosine, and tangent functions (F-IF.C.7e) as a lead-in to the inverse trigonometric functions that are studied in Lesson 12. The graphs are also used as a way to reinforce properties covered in Topic A such as symmetry and periodicity (F-TF.A.4). This lesson is an opportunity to highlight MP.5: Use appropriate tools strategically. Students should have access to technology throughout this lesson. It should be left to the individual student to decide how to effectively use technology such as a computer or graphing calculator. If computers are available, consider using Desmos (www.desmos.com), which is a free, on-line graphing utility.

Classwork

Opening Exercise (7 minutes)

Give students time to work on the Opening Exercise either individually or in pairs. If time is an issue, you could assign Exercise 1 to some students and Exercise 2 to other students. Then, share results as a class.

Scaffolding:

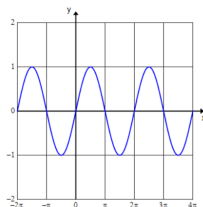
For students below grade level, consider using an anchor chart that illustrates the vocabulary necessary for this lesson such as amplitude, period, and midline. See Algebra II Module 2 Lesson 11 for these definitions.

Opening Exercise

Graph each of the following functions on the interval $-2\pi \leq x \leq 4\pi$ by making a table of values. The graph should show all key features (intercepts, asymptotes, relative maxima and minima).

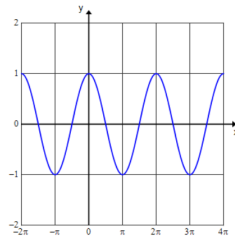
a. $f(x) = \sin(x)$

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$\sin(x)$	0	1	0	-1	0	1	0	-1	0	1	0	-1	0



b. $f(x) = \cos(x)$

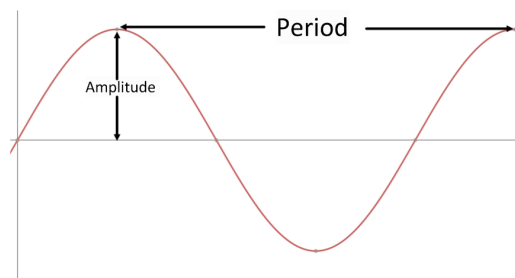
x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$\cos(x)$	1	0	-1	0	1	0	-1	0	1	0	-1	0	1



- What values of x did you use to graph the sine and cosine functions?
 - Values that result in a rotation that lies on the x -axis or y -axis of the unit circle
- Why?
 - These rotations represent the points of the unit circle that result in a maximum value, minimum value, or zero of the sine or cosine function.
- What are the domain and range of the sine and cosine functions?
 - D : the set of all real numbers R : $[-1, 1]$
- What is the amplitude of the sine and cosine functions?
 - The amplitude is 1.
- What is the period of the sine and cosine functions?
 - The period is 2π .

You may need to remind students of the definitions of amplitude and period. Have them label these characteristics on the graphs.

- A function f whose domain is a subset of the real numbers is said to be *periodic with period* $P > 0$ if the domain of f contains $x + P$ whenever it contains x , and if $f(x + P) = f(x)$ for all real numbers x in its domain.
- If a least positive number P exists that satisfies this equation, it is called the *fundamental period* or, if the context is clear, just the *period* of the function.
- The *amplitude* of the sine or cosine is the distance between a maximal point of the graph of the function and the midline.



Exploratory Challenge/Exercises 1–7 (30 minutes)

Allow students to work through the exercises either in pairs or in small groups. Stop to debrief periodically. Use a projector to display the graphs when discussing. Make sure that students have the correct graph for $f(x) = \tan(x)$ before they move on to Exercise 4.

Exploratory Challenge/Exercises 1–7

1. Consider the trigonometric function $f(x) = \tan(x)$.

- a. Rewrite $\tan(x)$ as a quotient of trigonometric functions. Then, state the domain of the tangent function.

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

D: The set of all real numbers such that $x \neq \frac{\pi}{2} + k\pi$ where k is an integer.

- b. Why is this the domain of the function?

Since $(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$, the domain cannot include a value of x that would make the $\cos(x) = 0$.

- c. Complete the table.

x	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
$\tan(x)$	0	und	0	und	0	und	0	und	0	und	0	und	0

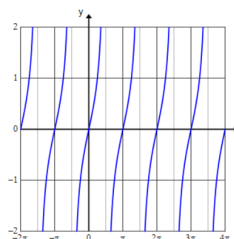
- d. What will happen on the graph of $f(x) = \tan(x)$ at the values of x for which the tangent function is undefined?

The graph will have vertical asymptotes.

- e. Expand the table to include angles that have a reference angle of $\frac{\pi}{4}$.

x	$-\frac{7\pi}{4}$	$-\frac{5\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$	$\frac{11\pi}{4}$	$\frac{13\pi}{4}$	$\frac{15\pi}{4}$
$\tan(x)$	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1

- f. Sketch the graph of $f(x) = \tan(x)$ on the interval $-2\pi \leq x \leq 4\pi$. Verify by using a graphing utility.



Scaffolding:

Early finishers could be challenged to explore the graphs of the other three trigonometric functions: secant, cosecant, and cotangent. These were not graphed in Algebra II.

- Why did we need additional points to graph the tangent function?
 - Using the rotations that lie on the x -axis or y -axis provided us with x -intercepts and vertical asymptotes. We needed some additional points to show the behavior of the graph between vertical asymptotes.
- As $x \rightarrow \frac{\pi}{2}$, what does $\tan(x)$ approach?
 - Looking at the graph, as x approaches $\frac{\pi}{2}$ moving to the right, y approaches $+\infty$ (for odd multiples of $\frac{\pi}{2}$). As x approaches $\frac{\pi}{2}$ moving to the left, y approaches $-\infty$ (for odd multiples of $\frac{\pi}{2}$).
- What is the range of the tangent function?
 - R : the set of all real numbers

2. Use the graphs of the sine, cosine, and tangent functions to answer each of the following.

- a. How do the graphs of the sine and cosine functions support the following identities for all real numbers x ?

$$\sin(-x) = -\sin(x)$$

$$\cos(-x) = \cos(x)$$

The graph of $f(x) = \sin(x)$ is symmetric with respect to the origin, which means that it is an odd function. Therefore, $\sin(-x) = -\sin(x)$. The graph of $f(x) = \cos(x)$ is symmetric with respect to the y -axis, which means that it is an even function. Therefore, $\cos(-x) = \cos(x)$.

- b. Use the symmetry of the graph of the tangent function to write an identity. Explain your answer.

$$\tan(-x) = -\tan(x), x \neq \frac{\pi}{2} + k\pi \text{ where } k \text{ is an integer}$$

This is because tangent is an odd function, which can be seen by the fact that the graph of $f(x) = \tan(x)$ is symmetric with respect to the origin.

- c. How do the graphs of the sine and cosine functions support the following identities for all real numbers x ?

$$\sin(x + 2\pi) = \sin(x)$$

$$\cos(x + 2\pi) = \cos(x)$$

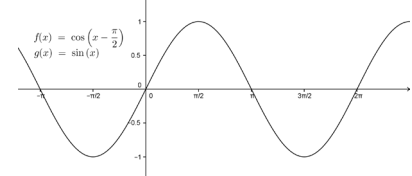
Both the sine and cosine functions are periodic and have a period of 2π . Using the definition of a periodic function, which says that $f(x + P) = f(x)$, it follows that $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$ for all real numbers x .

- d. Use the periodicity of the tangent function to write an identity. Explain your answer.

The tangent function has a period of π . Therefore, $\tan(x + \pi) = \tan(x)$ for all $x \neq \frac{\pi}{2} + k\pi$ where k is an integer.

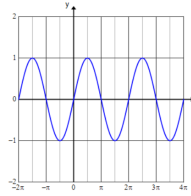
Scaffolding:

If students are struggling, allow them to use technology to confirm that the graphs of $f(x) = \cos\left(x - \frac{\pi}{2}\right)$ and $g(x) = \sin(x)$ coincide.



3. Consider the function $f(x) = \cos\left(x - \frac{\pi}{2}\right)$.

a. Graph $y = f(x)$ by using transformations of functions.



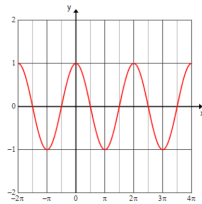
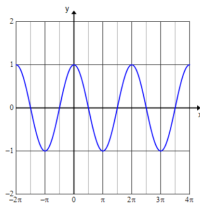
b. Based on your graph, write an identity.

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$

4. Verify the identity $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$ for all real numbers x by using a graph.

$$y = \sin\left(x + \frac{\pi}{2}\right):$$

$$y = \cos(x):$$



The graphs coincide, which illustrates that $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$ for all real numbers x .

5. Use a graphing utility to explore the graphs of the family of functions in the form $f(x) = A \sin(\omega(x - h)) + k$. Write a summary of the effect that changing each parameter has on the graph of the sine function.

a. A

The parameter A is a vertical scaling of the graph and affects the amplitude of the wave. The amplitude of the wave is equal to $|A|$. When A is negative, the graph is reflected across the x -axis.

b. ω

The parameter ω is a horizontal scaling of the graph and affects the period of the wave. For positive values of ω , as ω increases, the period decreases, which causes the wave to repeat more often. The graph is compressed horizontally. For positive values of ω , as ω decreases, the period increases, which causes the wave to repeat less often. The graph is stretched horizontally. The period is equal to $\frac{2\pi}{\omega}$.

c. h

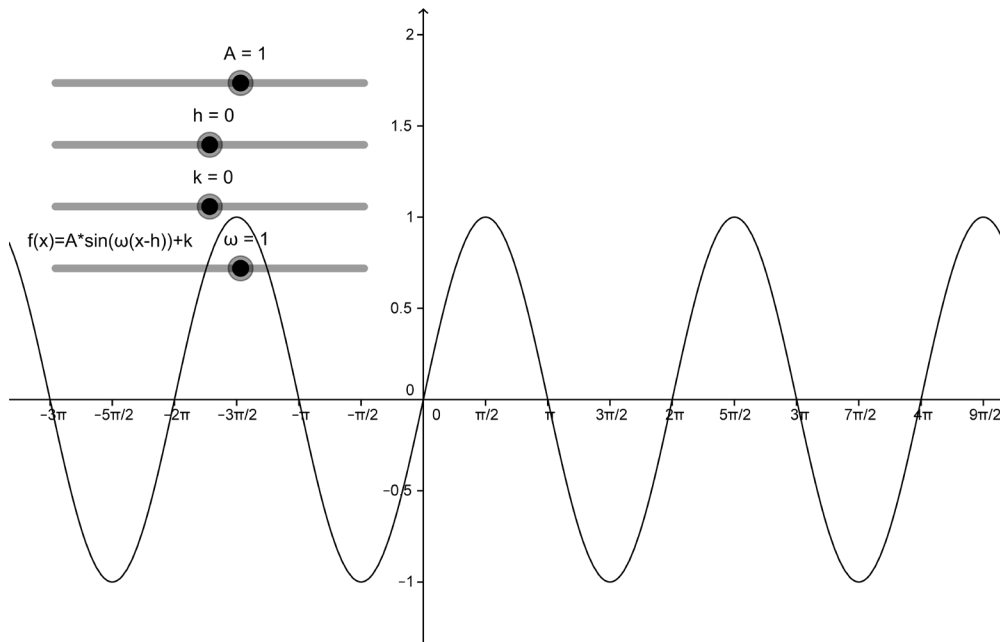
The parameter h is the horizontal shift of the wave, also called the phase shift. For $h > 0$, the graph will shift right h units. For $h < 0$, the graph will shift left h units.

d. k

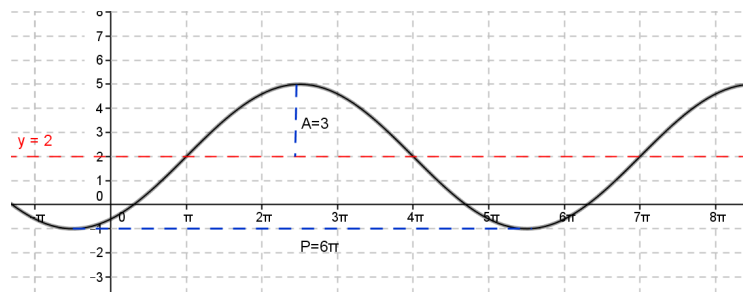
The parameter k is the vertical shift of the wave. For $k > 0$, the graph will shift up k units. For $k < 0$, the graph will shift down k units. The midline of the wave is given by the equation $y = k$.

MP.5

- In what ways did you use technology to assist you with Exercise 7?
 - We graphed various sine functions on the graphing calculator and changed one parameter at a time in order to explore the effect of each parameter.
 - We graphed the sine function on Desmos and created a slider for each parameter. We used the sliders to examine the effect of changing each parameter on the graph.



6. Graph at least one full period of the function $f(x) = 3 \sin\left(\frac{1}{3}(x - \pi)\right) + 2$. Label the amplitude, period, and midline on the graph.



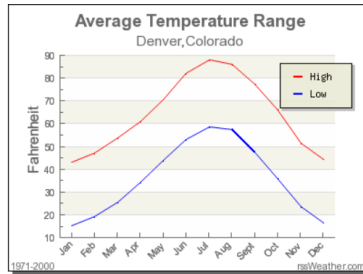
Amplitude = 3

Period = 6π

Midline: $y = 2$

7. The graph and table below show the average monthly high and low temperature for Denver, Colorado.
(source: <http://www.rssweather.com/climate/Colorado/Denver/>)

Average Temperatures for Denver



Month	Low	High
Jan	15.2°F	43.2°F
Feb	19.1°F	47.2°F
Mar	25.4°F	53.7°F
Apr	34.2°F	60.9°F
May	43.8°F	70.5°F
Jun	53.0°F	82.1°F
Jul	58.7°F	88.0°F
Aug	57.4°F	86.0°F
Sept	47.3°F	77.4°F
Oct	35.9°F	66.0°F
Nov	23.5°F	51.5°F
Dec	16.4°F	44.1°F

- a. Why would a sinusoidal function be appropriate to model this data?

The data is periodic, repeating every twelve months.

- b. Write a function to model the average monthly high temperature as a function of the month.

$T(t) = 22.4 \sin\left(\frac{\pi}{6}(t - 4)\right) + 65.6$ where $T(t)$ represents the average monthly high temperature and t represents the month (e.g., January would be represented by $t = 1$).

- c. What does the midline represent within the context of the problem?

The midline represents the mean average high temperature in Denver throughout the year.

- d. What does the amplitude represent within the context of the problem?

The amplitude represents how far from the mean the average highest and lowest temperatures vary throughout the year.

- e. Name a city whose temperature graphs would have a smaller amplitude. Explain your reasoning.

San Diego, CA would have a temperature graph with a smaller amplitude because the temperature does not vary much from the median throughout the year. The temperature from month to month stays relatively constant.

- f. Name a city whose temperature graphs would have a larger vertical shift. Explain your reasoning.

Miami, FL would have a temperature graph with a larger vertical shift because the average monthly high and low temperatures would be larger than in Denver. The median temperature would be greater, resulting in a larger vertical shift.

- What is the period of the function you used to model the data in Exercise 9? Why?
 - *The period is 12 because the data repeats every year.*
- Did you use technology to assist you on this exercise?
 - *No. We estimated the values using the graph and table given.*
 - *Yes. We created a scatterplot on the graphing calculator and used the regression feature to find a model. Our model was $T(t) = 21.794 \sin(0.554x - 2.396) + 65.371$ or $T(t) = 21.794 \sin(0.554(x - 4.325)) + 65.371$.*

Closing (3 minutes)

Have students record the key characteristics of the graphs of sine, cosine, and tangent and then share with the class.

- What are the key characteristics of the graph of $f(x) = \sin(x)$?
 - *D: all real numbers; R: $[-1,1]$; x-intercepts: $x = k\pi$ where k is an integer; relative maximum points: $x = \frac{\pi}{2} + 2\pi k$ where k is an integer; relative minimum points: $x = \frac{3\pi}{2} + 2\pi k$ where k is an integer; Amplitude = 1; Period = 2π*
- What are the key characteristics of the graph of $f(x) = \cos(x)$?
 - *D: all real numbers; R: $[-1,1]$; x-intercepts: $x = \frac{\pi}{2} + k\pi$ where k is an integer; relative maximum points: $x = 2\pi k$ where k is an integer; relative minimum points: $x = \pi + 2\pi k$ where k is an integer; Amplitude = 1; Period = 2π*
- What are the key characteristics of the graph of $f(x) = \tan(x)$?
 - *D: all real numbers $x \neq \frac{\pi}{2} + k\pi$ where k is an integer; R: all real numbers; x-intercepts: $x = k\pi$ where k is an integer; Period = π .*

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 11: Revisiting the Graphs of the Trigonometric Functions

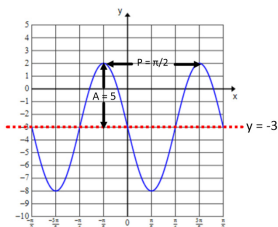
Exit Ticket

Consider a sinusoidal function whose graph has an amplitude of 5, a period of $\frac{\pi}{2}$, a phase shift of $\frac{\pi}{4}$ units to the left, and a midline of $y = -3$. Write the function in the form $f(x) = A \sin(\omega(x - h)) + k$ for positive A, ω, h, k . Then, graph at least one full period of the function. Label the midline, amplitude, and period on the graph.

Exit Ticket Sample Solutions

Consider a sinusoidal function whose graph has an amplitude of 5, a period of $\frac{\pi}{2}$, a phase shift of $\frac{\pi}{4}$ units to the left, and a midline of $y = -3$. Write the function in the form $f(x) = A \sin(\omega(x - h)) + k$ for positive A, ω, h, k . Then, graph at least one full period of the function. Label the midline, amplitude, and period on the graph.

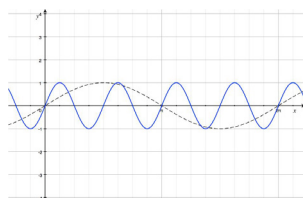
$$f(x) = 5 \sin\left(4\left(x + \frac{\pi}{4}\right)\right) - 3$$



Problem Set Sample Solutions

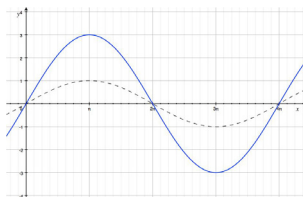
1. Sketch the graph of $y = \sin(x)$ on the same set of axes as the function $f(x) = \sin(4x)$. Explain the similarities and differences between the two graphs.

The solid curve shown is the graph of $f(x) = \sin(4x)$, and the dashed curve is the graph of $y = \sin(x)$. The graph of f is a horizontal scaling of the graph of the sine function by a factor of $\frac{1}{4}$. The graph of $f(x) = \sin(4x)$ has a different period and frequency than the sine function, which changes the x -intercepts, the maximum and minimum points, and the increasing and decreasing intervals for the function. The amplitudes of the two graphs are the same, with $|A| = 1$.



2. Sketch the graph of $y = \sin\left(\frac{x}{2}\right)$ on the same set of axes as the function $g(x) = 3\sin\left(\frac{x}{2}\right)$. Explain the similarities and differences between the two graphs.

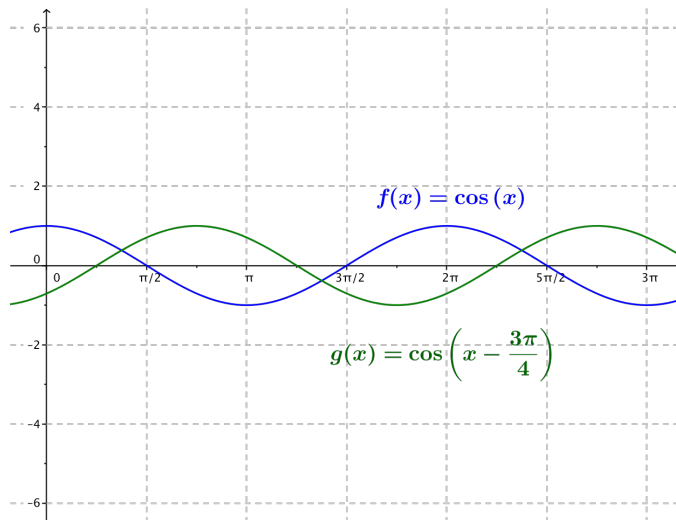
The dashed curve is the graph of $y = \sin\left(\frac{x}{2}\right)$, and the curve shown in blue is the graph of $g(x) = 3\sin\left(\frac{x}{2}\right)$. The graphs have different amplitudes. The graph of g is a vertical scaling of the graph of the sine function by a factor of 3. The y -coordinates of the maximum and minimum points are different for these two graphs. The two graphs have the same x -intercepts because the period of each function is 4π .



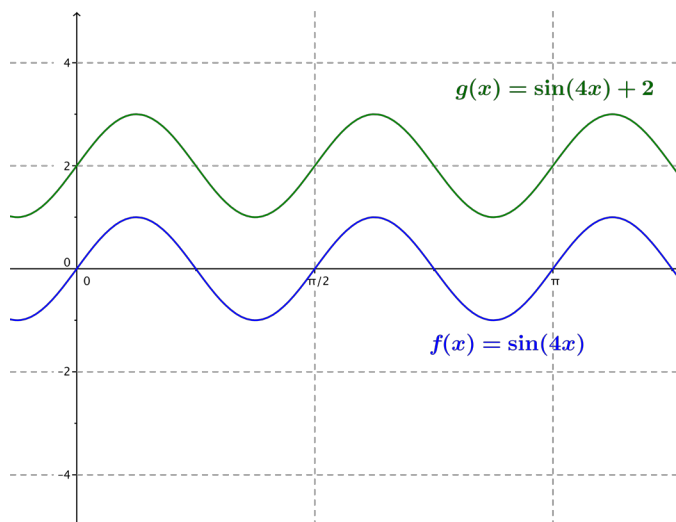
3. Indicate the amplitude, frequency, period, phase shift, horizontal and vertical translations, and equation of the midline. Graph the function on the same axes as the graph of the cosine function $f(x) = \cos(x)$. Graph at least one full period of each function.

$$g(x) = \cos\left(x - \frac{3\pi}{4}\right)$$

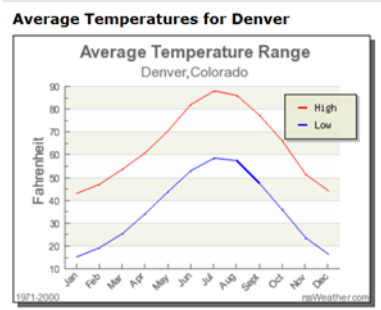
The amplitude is 1, the frequency is $\frac{1}{2\pi}$, the period is 2π , and the phase shift is $\frac{3\pi}{4}$. The horizontal translation is $\frac{\pi}{3}$ units to the right, there is no vertical translation, and the equation of the midline is $y = 0$.



4. Sketch the graph of the pairs of functions on the same set of axes: $f(x) = \sin(4x)$, $g(x) = \sin(4x) + 2$.



5. The graph and table below show the average monthly high and low temperature for Denver, Colorado.
(source: <http://www.rssweather.com/climate/Colorado/Denver/>)



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Aug	57.4°F	86.0°F
Sept	47.3°F	77.4°F
Oct	35.9°F	66.0°F
Nov	23.5°F	51.5°F
Dec	16.4°F	44.1°F

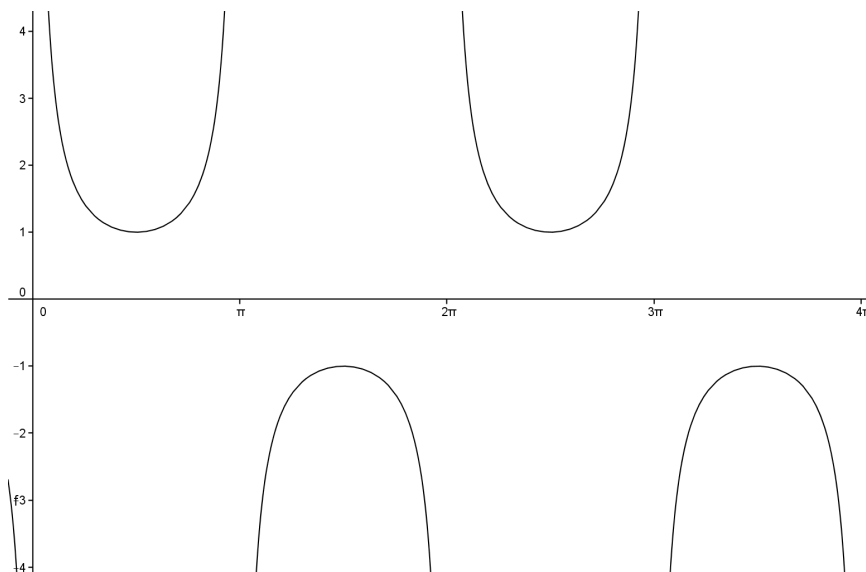
Write a function to model the average monthly low temperature as a function of the month.

$$T(t) = 21.75 \sin\left(\frac{\pi}{6}(t - 4.5)\right) + 36.95$$

Extension:

6. Consider the cosecant function.

- a. Use technology to help you sketch $y = \csc(x)$ for $0 \leq x \leq 4\pi$, $-4 \leq y \leq 4$.

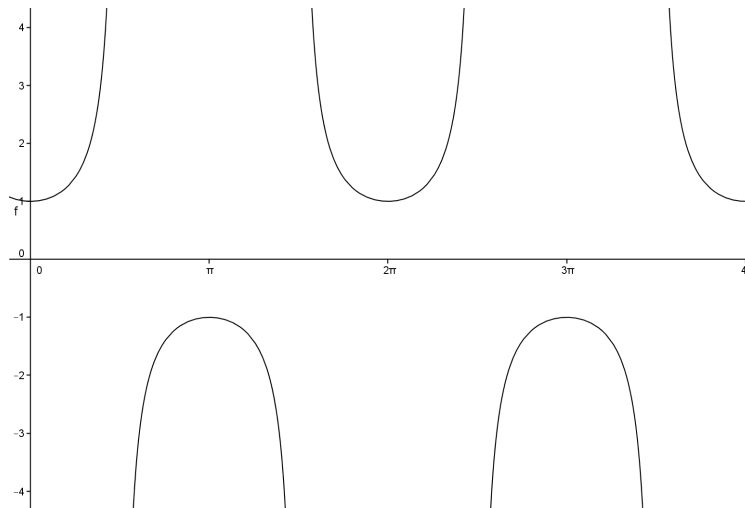


- b. What do you notice about the graph of the function? Compare this to your knowledge of the graph of $y = \sin(x)$.

The graph of the function touches the graph of the sine function whenever $y = \sin(x)$ has a minimum or maximum point. There appear to be infinitely many asymptotes where $y = \sin(x) = 0$. The range is all real numbers y such that $|y| \geq 1$, which makes sense as the reciprocal function of sine.

7. Consider the secant function.

- a. Use technology to help you sketch
- $y = \sec(x)$
- for
- $0 \leq x \leq 4\pi$
- ,
- $-4 \leq y \leq 4$
- .

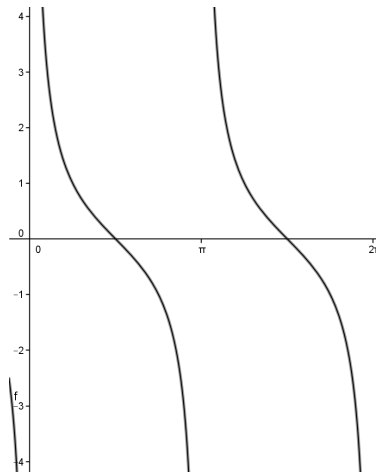


- b. What do you notice about the graph of the function? Compare this to your knowledge of the graph of
- $y = \cos(x)$
- .

The graph of the function touches the graph of the cosine function whenever $y = \cos(x)$ has a minimum or maximum point. There appear to be infinitely many asymptotes where $y = \cos(x) = 0$. The range is all real numbers y such that $|y| \geq 1$, which makes sense as the reciprocal function of cosine.

8. Consider the cotangent function.

- a. Use technology to help you sketch
- $y = \cot(x)$
- for
- $0 \leq x \leq 2\pi$
- ,
- $-4 \leq y \leq 4$
- .



- b. What do you notice about the graph of the function? Compare this to your knowledge of the graph of
- $y = \tan(x)$
- .

There appear to be infinitely many asymptotes where $y = \sin(x) = 0$ while the tangent function had asymptotes where $\cos(x) = 0$. The range is all real numbers, which makes sense as the reciprocal function of tangent.



Lesson 12: Inverse Trigonometric Functions

Student Outcomes

- Students understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
- Students use inverse functions to solve trigonometric equations.

Lesson Notes

Students studied inverse functions in Module 3 and came to the realization that not every function has an inverse that is also a function. Students considered how to restrict the domain of a function to produce an invertible function (**F-BF.B.4d**). This lesson builds on that understanding of inverse functions by restricting the domains of the trigonometric functions in order to develop the inverse trigonometric functions (**F-TF.B.6**). In Geometry, students used arcsine, arccosine, and arctangent to find missing angles, but they did not understand inverse functions and, therefore, did not use the terminology or notation for inverse trigonometric functions. Students define the inverse trigonometric functions in this lesson. Then they use the notation $\sin^{-1}(x)$ rather than $\arcsin(x)$. The focus shifts to using the inverse trigonometric functions to solve trigonometric equations (**F-TF.B.7**).

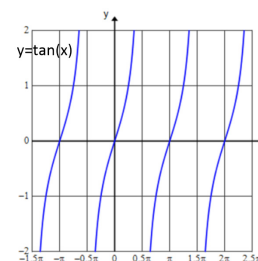
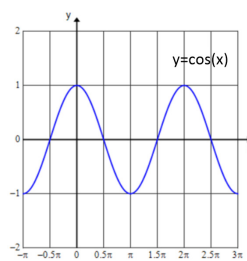
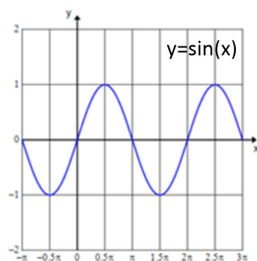
Classwork

Opening Exercise (5 minutes)

Give students time to work on the Opening Exercise independently. Then have them compare answers with a partner before sharing as a class.

Opening Exercise

Use the graphs of the sine, cosine, and tangent functions to answer each of the following questions.



- a. State the domain of each function.

The domain of the sine and cosine functions is the set of all real numbers. The domain of the tangent function is the set of all real numbers $x \neq \frac{\pi}{2} + k\pi$ for all integers k .

MP.7

- b. Would the inverse of the sine, cosine, or tangent functions also be functions? Explain.

None of these functions are invertible. Multiple elements of the domain are paired with a single range element. When the domain and range are exchanged to form the inverse, the result does not satisfy the definition of a function.

- c. For each function, select a suitable domain that will make the function invertible.

Answers will vary, so share a variety of responses. Any answer is suitable as long as the restricted domain leaves an interval of the graph that is either always increasing or always decreasing.

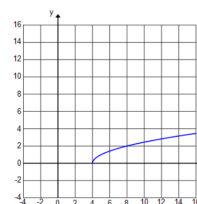
Sample response:

$$y = \sin(x), D: \left[0, \frac{\pi}{2}\right] \quad y = \cos(x), D: [0, \pi] \quad y = \tan(x), D: \left[0, \frac{\pi}{2}\right]$$

Scaffolding:

If students need a review of inverse functions, use this exercise:

Consider the function $f(x) = \sqrt{x - 4}$, which is graphed below. Graph f^{-1} . Find the equation of the inverse and its domain.



- Are any of the trigonometric functions invertible?
 - No. *The inverses of the trigonometric functions are no longer functions.*

If necessary, remind students of the definition of an invertible function.

INVERTIBLE FUNCTION: The domain of a function f can be restricted to make it *invertible* so that its inverse is also a function. A function is said to be *invertible* if its inverse is also a function.

- Was there only one way to restrict the domain to make each function invertible?
 - No. *There are an infinite number of ways in which we could restrict the domain of each function. We just need to erase enough of the graph so where the function is either only increasing or only decreasing.*
- How much of the graph should we keep?
 - *We want to choose the largest subset of the domain of the function as we can (such as $f(x) = \sin(x)$) and still have the function be continuously increasing or continuously decreasing on that interval.*

Ask students to share the domain restriction they chose for each of the three functions. Then, point out that while there is more than one way to do this, the convention is to use an interval that contains zero.

- Based on this, the convention is to restrict the domain of $f(x) = \sin(x)$ to be $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Does this satisfy all of our requirements?
 - Yes. *The graph is entirely increasing. We kept as much of the domain as possible, and we included zero in the domain.*
- Would this same restriction work for $f(x) = \cos(x)$?
 - No. *The graph would contain an interval of increasing and an interval of decreasing and still would not be invertible.*
- If we want to include zero and keep the largest subset of the domain possible, what would be a logical way to restrict the domain of $f(x) = \cos(x)$?
 - *Either $0 \leq x \leq \pi$ or $-\pi \leq x \leq 0$*
- The convention is to restrict the domain of $f(x) = \cos(x)$ to $0 \leq x \leq \pi$.

- If we want to include zero and keep the largest subset of the domain possible, what would be a logical way to restrict the domain of $f(x) = \tan(x)$?
 - *We could restrict the domain to values from 0 to π , but we would have to exclude $\frac{\pi}{2}$ from the domain. If we want to keep one branch of the graph and include 0, we should restrict the domain to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.*
- The convention is to restrict the domain of $f(x) = \tan(x)$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Example 1 (6 minutes)

Allow students time to read through the example and answer part (a). Then discuss part (b) as a class.

- How can we find the equation of the inverse sine?

Write the following on the board.

$$x = \sin(y)$$

- Now what? We need a function that denotes that it is the inverse of the sine function. The inverse sine function is usually written as $y = \sin^{-1}(x)$. Why does this notation make sense for an inverse function?
 - *The notation $f^{-1}(x)$ means the inverse function of x , so it makes sense that $\sin^{-1}(x)$ means the inverse of sine.*
- What is the value of $\sin\left(\frac{\pi}{6}\right)$? What about $\sin\left(\frac{5\pi}{6}\right)$?
 - *Both equal $\frac{1}{2}$.*
- What is the value of $\sin^{-1}\left(\frac{1}{2}\right)$?
 - $\frac{\pi}{6}$
- Why $\frac{\pi}{6}$ and not $\frac{5\pi}{6}$?
 - *The range of the inverse sine function is restricted to $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, which means that while there are an infinite number of possible answers, there is only one answer that lies within this restricted interval.*
- What is the value of $\sin\left(\frac{11\pi}{6}\right)$?
 - $-\frac{1}{2}$
- What is the value of $\sin^{-1}\left(-\frac{1}{2}\right)$?
 - $-\frac{\pi}{6}$
- Would it be acceptable to give the answer as $\frac{11\pi}{6}$?
 - *No. $\frac{11\pi}{6}$ is greater than $\frac{\pi}{2}$.*

- Why is it important for the inverse of sine to be a function?
 - Otherwise there would be an infinite number of possible values of $\sin^{-1}\left(\frac{1}{2}\right)$. $\sin^{-1}\left(\frac{1}{2}\right)$ could be any value y such that $\sin(y) = \frac{1}{2}$. Within the restricted range, there is only one value of y that satisfies the equation.

Example 1

Consider the function $(x) = \sin(x)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

- a. State the domain and range of this function.

$$D: -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$R: -1 \leq y \leq 1$$

- b. Find the equation of the inverse function.

$$x = \sin(y)$$

$$y = \sin^{-1}(x)$$

- c. State the domain and range of the inverse.

$$D: -1 \leq x \leq 1$$

$$R: -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Scaffolding:

If students need additional practice, consider using a rapid whiteboard exchange where you present a problem such as the examples listed and students hold up their answer on a small whiteboard. In this way, you can quickly assess student understanding.

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

$$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

$$\sin^{-1}(0) = 0$$

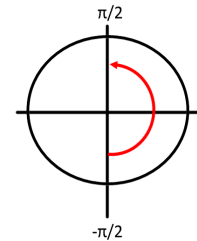
$$\cos^{-1}(0) = \frac{\pi}{2}$$

$$\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

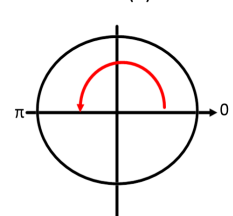
$$\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$$

If students are struggling, use a unit circle diagram to assist them in evaluating these expressions.

$\sin^{-1}(x)$ and $\tan^{-1}(x)$



$\cos^{-1}(x)$

**Exercises 1–3 (8 minutes)**

In these exercises, students are familiarizing themselves with the inverse trigonometric functions. Give students time to work through the exercises either individually or in pairs before sharing answers as a class.

Exercises 1–3

1. Write an equation for the inverse cosine function, and state its domain and range.

$$y = \cos^{-1}(x)$$

$$D: -1 \leq x \leq 1$$

$$R: 0 \leq y \leq \pi$$

2. Write an equation for the inverse tangent function, and state its domain and range.

$$y = \tan^{-1}(x)$$

$$D: \text{set of all real numbers}$$

$$R: -\frac{\pi}{2} < y < \frac{\pi}{2}$$

3. Evaluate each of the following expressions without using a calculator. Use radian measures.

a. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
 $\frac{\pi}{3}$

b. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
 $-\frac{\pi}{3}$

c. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$
 $\frac{\pi}{6}$

d. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
 $\frac{5\pi}{6}$

e. $\sin^{-1}(1)$
 $\frac{\pi}{2}$

f. $\sin^{-1}(-1)$
 $-\frac{\pi}{2}$

g. $\cos^{-1}(1)$
 0

h. $\cos^{-1}(-1)$
 π

i. $\tan^{-1}(1)$
 $\frac{\pi}{4}$

j. $\tan^{-1}(-1)$
 $-\frac{\pi}{4}$

- Why is the domain of the inverse cosine function restricted to values from -1 to 1 ?
 - *The domain is so restricted because the input is the value of cosine, and the values of cosine range from -1 to 1 .*
- Why is the range of the inverse cosine function restricted to values from 0 to π ?
 - *The range is so restricted because we restricted the domain of the cosine function to only the values from 0 to π in order to make it an invertible function. The domain of the cosine function became the range of the inverse cosine function.*
- What does this restriction mean in terms of evaluating an inverse trigonometric function?
 - *The answer must lie within the restricted values of the range.*

Scaffolding:

Pose this question to students who like a challenge:

Does $\sin(\sin^{-1}(x)) = x$ for all values of x ?

Yes, for all values in the domain of $\sin^{-1}(x)$ ($-1 \leq x \leq 1$).

Does $\sin^{-1}(\sin(x)) = x$ for all values of x ?

No, only for values of x such that $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Example 2 (6 minutes)

Work through the examples as a class.

- What is the difference between solving the equation $\cos(x) = \frac{1}{2}$ and evaluating the expression $\cos^{-1}\left(\frac{1}{2}\right)$?
 - *When solving the equation $\cos(x) = \frac{1}{2}$ we are looking for all values of x within the interval $0 \leq x \leq 2\pi$ such that $\cos(x) = \frac{1}{2}$. When evaluating $\cos^{-1}\left(\frac{1}{2}\right)$, we are looking for the one value within the interval $0 \leq y \leq \pi$ such that $\cos(y) = \frac{1}{2}$.*

- In part (b), why do we find the inverse sine of $\frac{2}{3}$ instead of $-\frac{2}{3}$?
 - We are looking for the reference angle, which is a positive, acute measure in order to find the other solutions.
- When do we need a calculator to find the reference angle?
 - We need a calculator when we are dealing with a value that is not a multiple of $\frac{\pi}{6}$, $\frac{\pi}{4}$, or $\frac{\pi}{3}$ or on the x - or y -axis.

Example 2

Solve each trigonometric equation such that $0 \leq x \leq 2\pi$. Round to three decimal places when necessary.

a. $2 \cos(x) - 1 = 0$

$$\cos(x) = \frac{1}{2}$$

Reference angle: $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

The cosine function is positive in Quadrants I and IV.

$$x = \frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$

b. $3 \sin(x) + 2 = 0$

$$\sin(x) = -\frac{2}{3}$$

Reference angle: $\sin^{-1}\left(\frac{2}{3}\right) = 0.730$

The sine function is negative in Quadrants III and IV.

$$x = \pi + 0.730 = 3.871 \text{ and } x = 2\pi - 0.730 = 5.553$$

Exercises 4–8 (12 minutes)

Give students time to work through the exercises either individually or in pairs. Circulate the room to ensure students understand the process of solving a trigonometric equation. For Exercises 7–8, consider using a graphing utility to either solve the equations or to check solutions calculated manually.

Exercises 4–8

4. Solve each trigonometric equation such that $0 \leq x \leq 2\pi$. Give answers in exact form.

a. $\sqrt{2}\cos(x) + 1 = 0$

$$x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

b. $\tan(x) - \sqrt{3} = 0$

$$x = \frac{\pi}{3}, \frac{4\pi}{3}$$

c. $\sin^2(x) - 1 = 0$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

5. Solve each trigonometric equation such that $0 \leq x \leq 2\pi$. Round answers to three decimal places.

a. $5 \cos(x) - 3 = 0$

$$x = 0.927, 5.356$$

b. $3 \cos(x) + 5 = 0$

There are no solutions to this equation within the domain of the function.

c. $3 \sin(x) - 1 = 0$

$$x = 0.340, 2.802$$

d. $\tan(x) = -0.115$

$$x = 3.027, 6.169$$

6. A particle is moving along a straight line for $0 \leq t \leq 18$. The velocity of the particle at time t (in seconds) is given by the function $v(t) = \cos\left(\frac{\pi}{5}t\right)$. Find the time(s) on the interval $0 \leq t \leq 18$ where the particle is at rest ($v(t) = 0$).

The particle is at rest at $t = 2.5$ seconds, 7.5 seconds, 12.5 seconds, and 17.5 seconds.

7. In an amusement park, there is a small Ferris wheel, called a kiddie wheel, for toddlers. The formula $H(t) = 10 \sin\left(2\pi\left(t - \frac{1}{4}\right)\right) + 15$ models the height H (in feet) of the bottom-most car t minutes after the wheel begins to rotate. Once the ride starts, it lasts 4 minutes.

- a. What is the initial height of the car?

5 ft.

- b. How long does it take for the wheel to make one full rotation?

1 minute

- c. What is the maximum height of the car?

25 ft.

- d. Find the time(s) on the interval $0 \leq t \leq 4$ when the car is at its maximum height.

The car is at its maximum height when $\sin\left(2\pi\left(t - \frac{1}{4}\right)\right) = 1$, which is at $t = 0.5, 1.5, 2.5$, and 3.5 minutes.

8. Many animal populations fluctuate periodically. Suppose that a wolf population over an 8-year period is given by the function $W(t) = 800 \sin\left(\frac{\pi}{4}t\right) + 2200$, where t represents the number of years since the initial population counts were made.

- a. Find the time(s) on the interval $0 \leq t \leq 8$ such that the wolf population equals 2,500.

$$t = 0.489, 3.511$$

The wolf population equals 2,500 after approximately 0.5 years and again after 3.5 years.

- b. On what time interval during the 8-year period is the population below 2,000?

$$W(t) = 2000 \text{ at } t = 4.332 \text{ and } 7.678$$

The wolf population is below 2,000 on the time interval (4.332, 7.678).

- c. Why would an animal population be an example of a periodic phenomenon?

An animal population might increase while their food source is plentiful. Then, when the population becomes too large, there is less food and the population begins to decrease. At a certain point, there are few enough animals that there is plenty of food for the entire population at which point the population begins to increase again.

Closing (3 minutes)

Use the following questions to summarize the lesson and check for student understanding.

- What does $y = \sin^{-1}(x)$ mean?
 - *It means find the value y on the interval $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ such that $\sin(y) = x$.*
- Is cosecant the same as inverse sine?
 - *No. Cosecant is the reciprocal of sine not the inverse of sine.*
- Suzanne says that $\tan^{-1}(-\sqrt{3})$ is $\frac{5\pi}{3}$. When Rosanne says that it is $-\frac{\pi}{3}$, Suzanne says either answer is fine because the two rotations lie on the same spot on the unit circle. What is wrong with Suzanne's thinking?
 - *$\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ and cannot equal $\frac{5\pi}{3}$ because $\frac{5\pi}{3}$ is outside of the restricted range. Because inverse tangent is a function, there can only be one output value. That value must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.*

MP.3

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 12: Inverse Trigonometric Functions

Exit Ticket

1. State the domain and range for $f(x) = \sin^{-1}(x)$, $g(x) = \cos^{-1}(x)$, and $h(x) = \tan^{-1}(x)$.

2. Solve each trigonometric equation such that $0 \leq x \leq 2\pi$. Give answers in exact form.

a. $2 \sin(x) + \sqrt{3} = 0$

b. $\tan^2(x) - 1 = 0$

3. Solve the trigonometric equation such that $0 \leq x \leq 2\pi$. Round to three decimal places.

$$\sqrt{5} \cos(x) - 2 = 0$$

Exit Ticket Sample Solutions

1. State the domain and range for $f(x) = \sin^{-1}(x)$, $g(x) = \cos^{-1}(x)$, and $h(x) = \tan^{-1}(x)$.

For f , the domain is all real numbers x , such that $-1 \leq x \leq 1$, and the range is all real numbers y , such that $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

For g , the domain is all real numbers x , such that $-1 \leq x \leq 1$, and the range is all real numbers y , such that $0 \leq y \leq \pi$.

For h , the domain is all real numbers x , and the range is all real numbers y , such that $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

2. Solve each trigonometric equation such that $0 \leq x \leq 2\pi$. Give answers in exact form.

a. $2 \sin(x) + \sqrt{3} = 0$

$$x = \frac{4\pi}{3}, \frac{5\pi}{3}$$

b. $\tan^2(x) - 1 = 0$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

3. Solve the trigonometric equation such that $0 \leq x \leq 2\pi$. Round to three decimal places.

$$\sqrt{5} \cos(x) - 2 = 0$$

$$x = 0.464, 5.819$$

Problem Set Sample Solutions

1. Solve the following equations. Approximate values of the inverse trigonometric functions to the thousandths place, where x refers to an angle measured in radians.

a. $5 = 6 \cos(x)$

$$2\pi k \pm 0.586$$

b. $-\frac{1}{2} = 2 \cos\left(x - \frac{\pi}{4}\right) + 1$

$$2\pi k + \frac{5\pi}{4} - 0.723$$

$$2\pi k - \frac{3\pi}{4} + 0.723$$

c. $1 = \cos(3(x - 1))$

$$\frac{2\pi k}{3} + 1$$

d. $1.2 = -0.5 \cos(\pi x) + 0.9$

$$\frac{-0.927 + \pi + 2\pi k}{\pi}$$

$$\frac{0.927 - \pi + 2\pi k}{\pi}$$

e. $7 = -9 \cos(x) - 4$

No solutions

f. $2 = 3 \sin(x)$

$$0.730 + 2\pi k$$

$$\pi - 0.730 + 2\pi k$$

g. $-1 = \sin\left(\frac{\pi(x-1)}{4}\right) - 1$

$$4k + 1$$

h. $\pi = 3 \sin(5x + 2) + 2$

$$\frac{0.390 - 2 + 2\pi k}{5}$$

$$\frac{\pi - 0.390 - 2 + 2\pi k}{5}$$

i. $\frac{1}{9} = \frac{\sin(x)}{4}$

$$0.461 + 2\pi k$$

$$\pi - 0.461 + 2\pi k$$

j. $\cos(x) = \sin(x)$

$$1 = \frac{\sin(x)}{\cos(x)} = \tan(x)$$

$$\frac{\pi}{4} + \pi k \text{ (or } 0.785 + \pi k)$$

k. $\sin^{-1}(\cos(x)) = \frac{\pi}{3}$

$$2\pi k \pm \frac{\pi}{6}$$

l. $\tan(x) = 3$

$$1.249 + \pi k$$

m. $-1 = 2 \tan(5x + 2) - 3$

$$\frac{\frac{\pi}{4} - 2 + \pi k}{5}$$

Alternatively, $\frac{0.785 - 2 + \pi k}{5}$.

n. $5 = -1.5 \tan(-x) - 3$

$$1.385 + \pi k$$

2. Fill out the following tables.

x	$\sin^{-1}(x)$	$\cos^{-1}(x)$
-1	$-\frac{\pi}{2}$	π
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$	$\frac{5\pi}{6}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$	$\frac{3\pi}{4}$
$-\frac{1}{2}$	$-\frac{\pi}{6}$	$\frac{2\pi}{3}$

x	$\sin^{-1}(x)$	$\cos^{-1}(x)$
0	0	$\frac{\pi}{2}$
$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$
1	$\frac{\pi}{2}$	0

3. Let the velocity v in miles per second of a particle in a particle accelerator after t seconds be modeled by the function $v = \tan\left(\frac{\pi t}{6000} - \frac{\pi}{2}\right)$ on an unknown domain.

a. What is the t -value of the first vertical asymptote to the right of the y -axis?

$$t = 6000$$

b. If the particle accelerates to 99% of the speed of light before stopping, then what is the domain?
Note: $c \approx 186,000$. Round your solution to the ten-thousandths place.

$$0.99 \cdot 186\,000 = 184\,140$$

$$184\,140 = \tan\left(\frac{\pi t}{6000} - \frac{\pi}{2}\right)$$

$$\tan^{-1}(184\,140) = \frac{\pi t}{6000} - \frac{\pi}{2}$$

$$\frac{6000}{\pi} \cdot \left(\tan^{-1}(184\,140) + \frac{\pi}{2}\right) = t$$

$$t \approx 5999.9896$$

So the domain is $0 < t \leq 5999.9896$.

c. How close does the domain get to the vertical asymptote of the function?

Very close. They are only different at the hundredths place.

- d. How long does it take for the particle to reach the velocity of Earth around the sun (about 18.5 miles per second)?

$$18.5 = \tan\left(\frac{\pi t}{6000} - \frac{\pi}{2}\right)$$

$$\tan^{-1}(18.5) = \frac{\pi t}{6000} - \frac{\pi}{2}$$

$$\frac{6000}{\pi} \cdot \left(\tan^{-1}(18.5) + \frac{\pi}{2}\right) = t$$

$$t \approx 5896.864$$

It takes approximately 5,896.864 seconds to reach the velocity of Earth around the sun.

- e. What does it imply that v is negative up until $t = 3000$?

The particle is traveling in the opposite direction.



Lesson 13: Modeling with Inverse Trigonometric Functions

Student Outcomes

- Students use inverse trigonometric functions to solve real-world problems addressing best viewing angles.

Lesson Notes

Students apply inverse trigonometric functions to solve problems in modeling contexts. The lesson focuses on using inverse trigonometric functions to determine the best horizontal viewing distance that an observer should stand from an object being viewed. Students use technology to solve the problems and interpret their answers in context (**F-TF.B.7**).

Classwork

Opening (5 minutes)

Display this image of the Statue of Liberty:



“The Statue of Liberty, taken from the Ellis Island ferry.” by Iolaire,
is licensed under CC BY-SA 3.0

<http://creativecommons.org/licenses/by-sa/3.0/deed.en>

Explain to students that the Statue of Liberty is 151 feet tall and stands on a pedestal of 154 feet, which goes from the ground to the statue’s feet. Students should each create a sketch of the statue and pedestal with appropriate heights labeled.

Have students imagine that they have traveled to Liberty Island in New York to see the Statue of Liberty. Once on the island, they stand in front of the statue, trying to get the best possible view. Have students represent themselves on their sketch with an estimated best viewing distance. Then, students should reflect on this prompt, sharing their ideas with a partner after a minute:

- Describe the ideal distance you should stand from the statue that would provide the best view.

Several students should be selected to share their ideas, which might include suggestions such as:

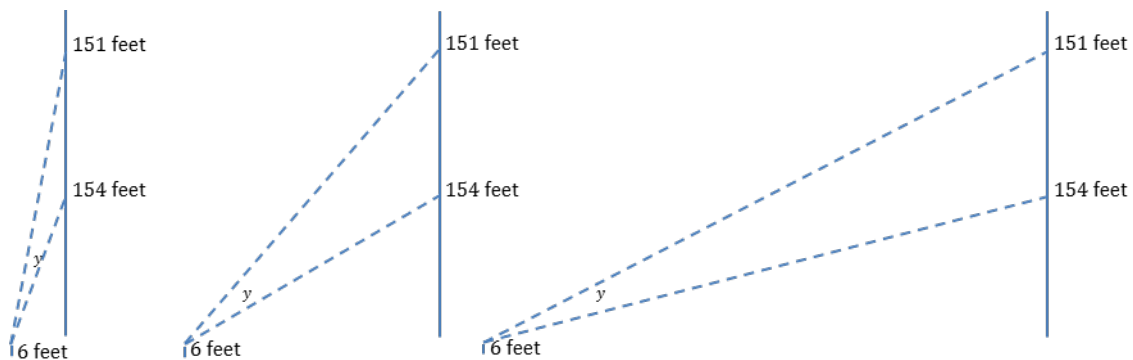
- I should not stand within 50 feet of the statue because then I would not be able to see the entire statue at once from this distance.*
- I should not stand several hundred feet away because then I would not be able to see the details of the statue.*
- The distance with the best view is the closest distance I can stand and see the whole statue without moving my head.*

Briefly explain to the class that Johannes Müller, a fifteenth-century German scholar, was interested in the distance where objects are “best viewed.” In his studies, he focused on maximizing the viewing angle, which is the angle created by the line of sight to the base of an object and the line of sight to the top of the object. Explain that this definition will be applied to help us determine the best viewing distance for the Statue of Liberty and later for other objects.

Discussion (3 minutes): Exploring Best Viewing Distance

This discussion should help students to think critically about why the distance where the viewing angle is maximized represents the best viewing distance.

- Look at the three viewing distances illustrated in the diagrams. In each diagram, the vertical line segments labeled 6 feet represent the height of the eyes of the viewer, and y is the viewing angle. Which would represent the best viewing distance among these three? Justify your selection.



- Answers will vary, but students will probably select the middle view and suggest that the first view would require them to look up at an uncomfortable angle to see the statue, while the third view is far enough away that it might be hard to see the features of the statue.*
- Based on the diagrams, form a conjecture about the best viewing distance and the measure of y , the viewing angle between the line of sight to the base of the statue and the line of sight with the top of the statue.
 - Conjectures will vary, but students should recognize that at the ideal viewing distance, y is maximized.*
- How might we be able to determine the value of y in diagrams like these? Share your ideas with a partner.
 - Answers will vary but are likely to address applying right triangle trigonometry and perhaps applying inverse trigonometric functions to find the angle measure.*

MP.3

Example (7 minutes)

This example demonstrates how inverse trigonometric functions can be applied in real-world modeling contexts. Students apply the inverse tangent function to determine the optimal distance a viewer should stand in front of the Statue of Liberty so that the viewing angle, as defined in the Opening, is maximized. Students apply a similar process to explore how to maximize viewing angles for objects in general. This example should be completed as part of a teacher-led discussion. It may be beneficial to display the sketch from part (a) and then have students add any missing components to their sketches from the Opening. Graphing calculators or other graphing software are needed to complete the problem.

- What components do we need to model in the diagram of the situation in order to determine the best viewing distance?
 - *Answers will vary but should include the viewing angle, the horizontal distance between the viewer and the statue that maximizes the viewing angle, the height of the pedestal, the height of the statue on the pedestal, and the height of the observer.*
- Why might it be helpful to include in our sketch a line segment parallel to the ground from our observer's line of sight to the statue?
 - *Answers will vary but might address that this line segment will create right triangles in our diagram.*
- Why would creating right triangles help us in determining the maximum viewing angle y ?
 - *Answers will vary but should address using trigonometric ratios and solving for y .*
- How could we use trigonometric ratios to find the maximum value of y when the triangle that contains y is oblique?
 - *Answers will vary but should address making use of the fact that $y = (y + w) - w$.*
- Why would the tangent function be most effective to use in trying to determine the values of $(y + w)$ and w ?
 - *Our known lengths represent opposite and adjacent side lengths in the right triangles we have sketched, and the tangent function applied to an angle is defined as the ratio of the length of the side opposite the angle to the length of the side adjacent to the angle.*
- How can we represent $(y + w)$ and w using the tangent function?
 - $\tan(y + w) = \frac{b-h}{x}$ and $\tan(w) = \frac{a-h}{x}$
- How can we isolate $(y + w)$ in this equation? Explain.
 - *We can apply the inverse tangent function to both sides of the equation. Applying the inverse tangent function will undo the tangent operation, resulting in $(y + w)$.*
- What does this look like mathematically?
 - $\tan^{-1}(\tan(y + w)) = \tan^{-1}\left(\frac{b-h}{x}\right)$, which simplifies to $(y + w) = \tan^{-1}\left(\frac{b-h}{x}\right)$.
- If we apply this procedure to find w , what do we get?
 - $\tan^{-1}(\tan(w)) = \tan^{-1}\left(\frac{a-h}{x}\right)$, which simplifies to $w = \tan^{-1}\left(\frac{a-h}{x}\right)$.
- Which means that y is equivalent to what?
 - $y = (y + w) - w = \tan^{-1}\left(\frac{b-h}{x}\right) - \tan^{-1}\left(\frac{a-h}{x}\right)$

MP.2

- How do we use this equation to find the maximum value of y ?
 - *Substitute the known values for a , b , and h into the equation. Graph the equation $y = \tan^{-1}\left(\frac{299}{x}\right) - \tan^{-1}\left(\frac{148}{x}\right)$, and identify its maximum.*
- What mode should we use to graph this function?
 - *Since we are modeling a viewing angle using triangles, degree mode is probably best to use.*
- Look at the graph to determine the coordinates of its maximum.
 - $(210.4, 19.7)$
- What do these values mean in context?
 - *The ideal viewing distance is 210.4 feet in front of the statue, which results in a viewing angle of approximately 19.7° .*
- What aspects of our model, if any, were inaccurate?
 - *Answers may vary but might address that our observer's known height is to the top of his or her head, not to eye level, but we used this value as the height to eye-level in the model. This means the actual height we desire is probably at least a few inches shorter than 6 feet.*
- How much do you think the true ideal viewing distance was affected by the fact that we used the observer's height and not the distance to his or her eye level? Justify your response.
 - *Not much. Small differences in height did not change the ideal viewing distances much in part (c), which indicates that the discrepancy in our model would not have a large effect on our answer.*
- Based on our findings, what advice would you give a friend who was going to visit the Statue of Liberty soon and wanted to know where to stand to get the best view from outside the statue?
 - *Answers will vary. An example of an appropriate response is shown: You should stand approximately 210 feet in front of the statue to get the best view.*

Example

The Statue of Liberty is 151 feet tall and sits on a pedestal that is 154 feet above the ground. An observer who is 6 feet tall wants to stand at the ideal viewing distance in front of the statue.

- a. Sketch the statue and observer. Label all appropriate measurements on the sketch, and define them in context.

h : height of the observer at eye level

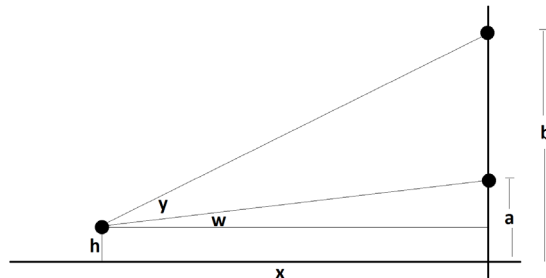
a : height of pedestal

b : height of the pedestal and statue

x : horizontal distance between viewer and base of the statue

y : viewing angle

w : angle formed between line of sight from viewer to the base of the statue and a line segment parallel to x from the viewer's eye level to the statue



MP.2

- b. How far back from the statue should the observer stand so that his or her viewing angle (from the feet of the statue to the tip of the torch) is largest? What is the value of the largest viewing angle?

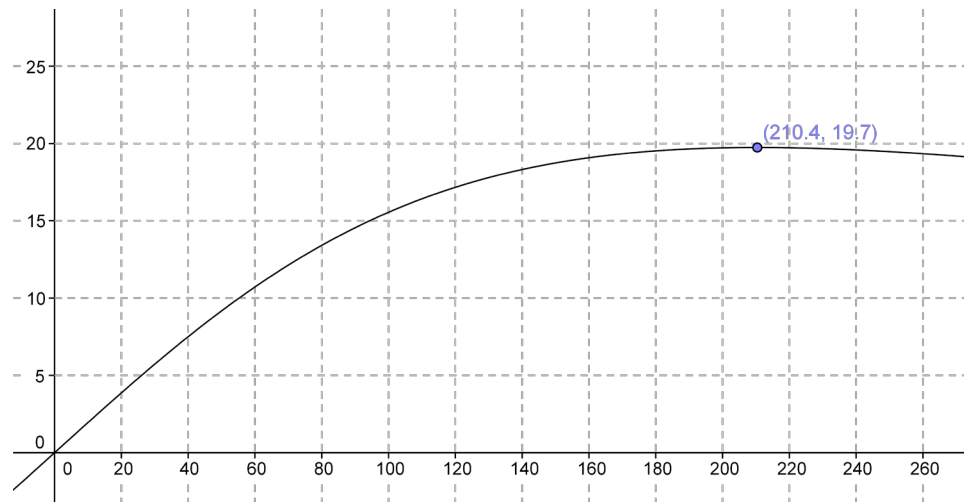
$$h = 6$$

$$a = 151$$

$$b = 154 + 151 = 305$$

$$y = \tan^{-1}\left(\frac{b-h}{x}\right) - \tan^{-1}\left(\frac{a-h}{x}\right)$$

$$y = \tan^{-1}\left(\frac{299}{x}\right) - \tan^{-1}\left(\frac{148}{x}\right)$$



The ideal viewing distance is approximately 210.4 feet from the base of the statue, which produces a maximum viewing angle of approximately 19.7° .

- c. What would be your best viewing distance from the statue?

Answers will vary. For example, for a viewer whose height is 5.5 feet:

$y = \tan^{-1}\left(\frac{299.5}{x}\right) - \tan^{-1}\left(\frac{148.5}{x}\right)$, which has a maximum y value of 19.7° when $x = 210.9$. Therefore, the best viewing distance is 210.9 feet from the base of the statue.

- d. If there are 66 meters of dry land in front of the statue, is the viewer still on dry land at the best viewing distance?

Since 66 meters is approximately 216.5 feet, the observer would be on land at our best viewing angle with about 6 feet between him or her and the water.

MP.2

Discussion (2 minutes)

- So we have determined the ideal viewing distance for the Statue of Liberty. How can we use our model to help us determine the best viewing distances for other objects?
 - Answers will vary but might address using similar sketches and equations to determine the maximum viewing angle and corresponding ideal viewing distance.

- Based on our results, what generalizations can be made about the ideal viewing distance for objects?
 - Answers will vary but could address that it is difficult to make generalizations based on a single, specific example; students might try to form generalizations based on the given measurements (e.g., the viewing distance was greater than the height of the base but less than the height of the torch).*
- What are some limitations in how our model can be applied to determining ideal viewing distances for other objects?
 - Answers will vary but might address that the Statue of Liberty is a specific height and that each object's ideal viewing distance will vary based on its base and top heights.*
- Let's see if we can apply what we learned in our example to a more general setting.

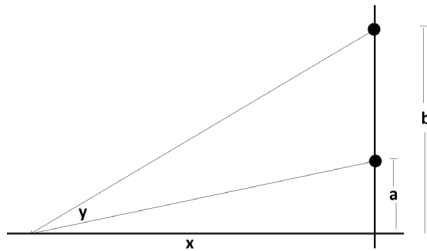
Exercise (7 minutes)

Students should complete the exercise in pairs. After a few minutes, responses to parts (a) and (b) should be discussed in a whole-class setting. Then each pair should be assigned specific values for the height of the base of the picture (a inches above eye level in the sketch shown) and the height of the top of the picture (b inches above eye level in the sketch shown). A list of possible values is provided in the table that follows. Each pair should use graphing software to compute the ideal viewing distance and maximum viewing angle for the measures assigned to them. These values can be combined in a class chart, which students analyze to form conjectures about ideal viewing distances. Students could take turns using graphing software if it is not accessible for each pair of students (e.g., a single website could be used and the parameters changed for each pair to determine the maximum viewing angle for specified values of a and b).

Exercise

Hanging on a museum wall is a picture with base a inches above a viewer's eye level and top b inches above the viewer's eye level.

- a. Model the situation with a diagram.



- b. Determine an expression that could be used to find the ideal viewing distance x that maximizes the viewing angle y .

$$y = \tan^{-1}\left(\frac{b}{x}\right) - \tan^{-1}\left(\frac{a}{x}\right)$$

Scaffolding:

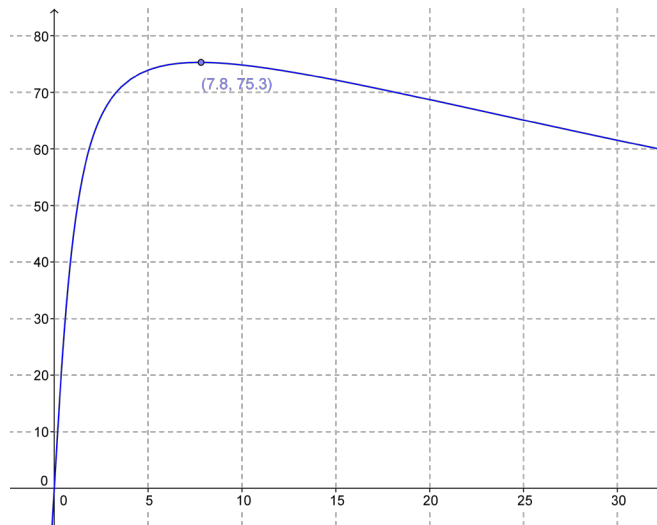
- Have advanced students determine a general expression for x without further prompting.
- Cue students to compute x^2 as they analyze the class data for patterns.

MP.3

- c. Find the ideal viewing distance, given the a and b values assigned to you. Calculate the maximum viewing angle in degrees.

Answers will vary. An example of an appropriate response is shown.

For $a = 1$ and $b = 60$, $y = \tan^{-1}\left(\frac{60}{x}\right) - \tan^{-1}\left(\frac{1}{x}\right)$



The ideal viewing distance is approximately 7.8 inches, with a maximized viewing angle of 75.3°.

- d. Complete the table using class data, which indicates the ideal values for x given different assigned values of a and b . Note any patterns you see in the data.

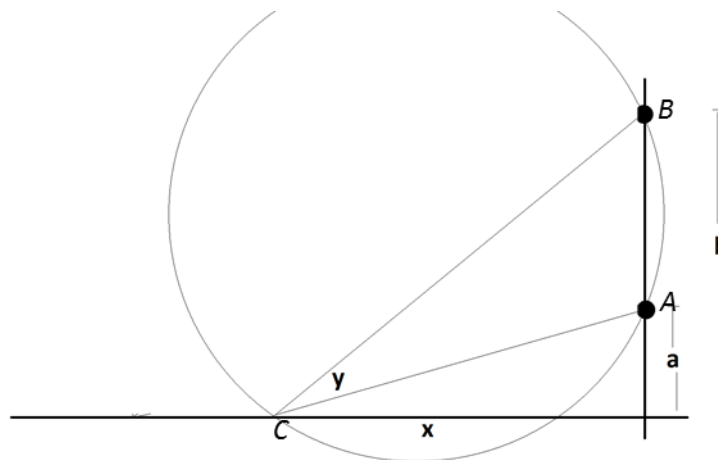
a (inches)	b (inches)	x at max (inches)	y max (degrees)
1	60	7.8	75.3
6	54	18	53.1
12	72	29.4	45.6
12	132	39.8	56.4
18	72	36	36.9
18	90	40.2	41.8
24	96	48	36.9
36	96	58.8	27
36	144	72	36.9
48	108	72	22.6
48	192	96	36.9

It appears that x is the geometric mean of a and b .

Discussion (13 minutes): Generalizing Ideal Viewing Distance

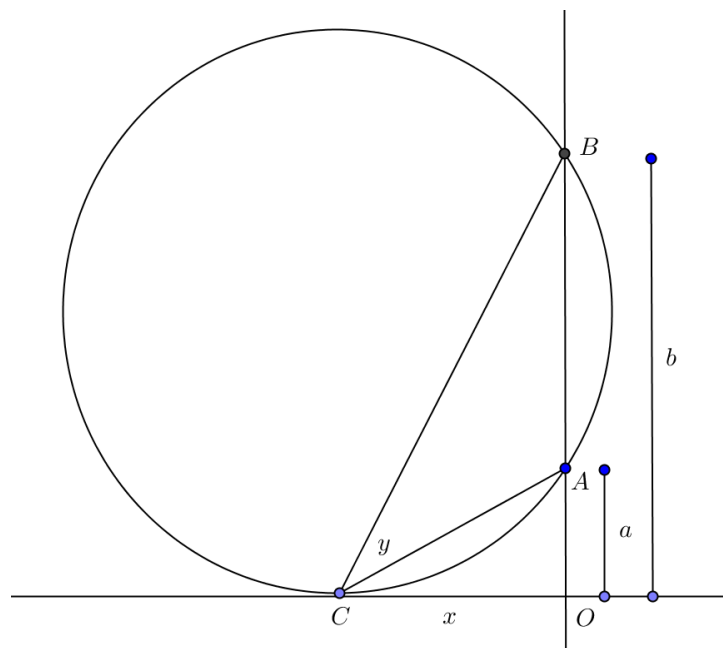
Students discuss conjectures about the ideal viewing distance based on their data from Exercise 1, arriving at the conclusion that the ideal viewing distance can be represented as $x = \sqrt{ab}$ where a represents the height of the base of the object above the viewer's eye level and b represents the height of the top of the object above the viewer's eye level. Students verify this finding using geometry.

- What generalizations can you make about the ideal viewing distance based on the data in the table?
 - *Answers will vary but might include that the ideal viewing distance is between the values a and b ; maximum viewing angles are greater for taller paintings than for shorter ones.*
- Do you notice any patterns in the data that might help you form a conjecture about how to calculate the ideal viewing distance?
 - *It appears to be the geometric mean of a and b .*
- What would be a formula relating the ideal viewing distance x , height from eye level to the base of an object a , and height from eye level to the top of the object b ?
 - $x = \sqrt{ab}$
- Is this true for our Statue of Liberty problem?
 - *Yes: $210.4 \approx \sqrt{299(148)}$*
- Let's explore this relationship between viewing distance and object height using geometry. First, let's draw a circle through the points A , B , and C where:
 - A is a point on a vertical line at the height of the base of the viewed object,
 - B is a point on the same vertical line at the height of the top of the viewed object,
 - C is a point on a horizontal line at the ideal viewing distance at the viewer's eye level.



- How do we know that only one circle can be drawn through these three points?
 - *Answers will vary but should address that given any three coplanar points, one circle can be drawn using these points.*

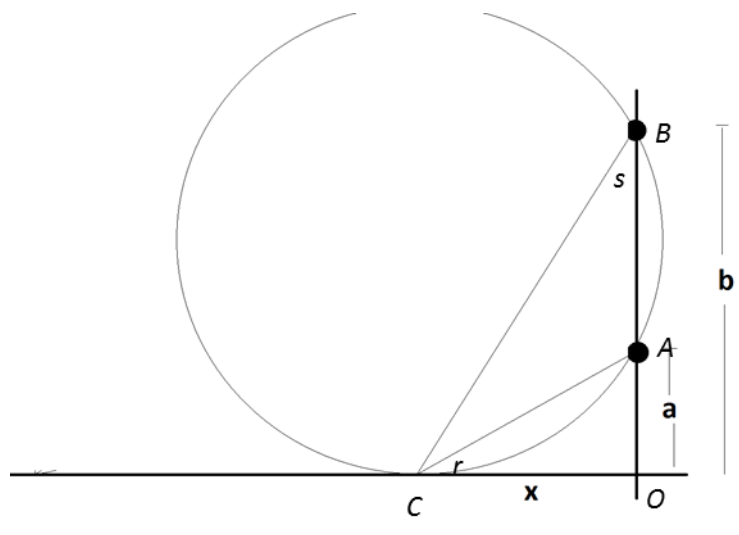
- Now we want to maximize y , the measure of our viewing angle BCA . If we move C farther from the segment containing A and B , how will that affect the circle drawn through A , B , and C ?
 - *The radius of the circle increases.*
- And how will that affect the central angle that intersects the circle at A and B ?
 - *The measure of the central angle decreases.*
- What does this imply about the effect of an increasing radius on y ? Explain.
 - *The value of y decreases with an increasing radius. The angle BCA is an inscribed angle whose measure is half that of the central angle that intersects points A and B . If the measure of the central angle decreases with an increasing radius, the measure of inscribed angle BCA also decreases with an increasing radius.*
- What type of circle would be ideal if we want to maximize y ?
 - *An ideal circle would intersect points A , B , and C and have the smallest possible radius.*
- What would characterize this circle?
 - *It would be tangent to the line segment containing C .*
- Our ideal circle would look like this:



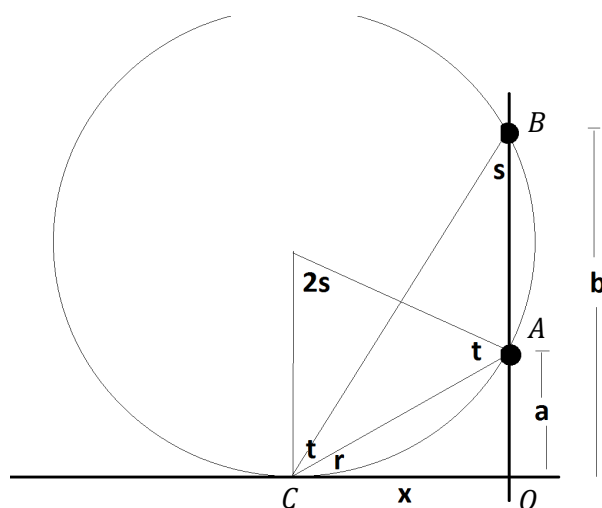
- What do you notice about the triangles formed in our diagram that might help us determine an expression for the ideal viewing distance x ?
 - *There are two right triangles, each of which has a leg length x : triangle AOC and triangle COB .*

MP.7

- Let's see if we can establish a relationship between the right triangles. We can start by labeling the angles opposite the shortest side of each triangle. We can call the measure of this angle in the larger triangle s and the measure of this angle in the smaller right triangle r . How can we characterize angle s ?



- *It is an inscribed angle that intersects points A and C .*
- What does that imply about the central angle that intersects points A and C ?
 - *It has a measure $2s$.*
- Let's sketch this central angle. Notice, the central angle forms a triangle with chord \overline{AC} . What do we know about the angles of this triangle besides the central angle?
 - *They are congruent.*
- Why is that?
 - *The triangle formed is isosceles because two of the legs are radii of the circle, which means that the base angles are congruent.*
- To make our discussion of the angle measures simpler, let's all label these two angles t . Our sketch now contains the angles and segments shown:



- How can we use this diagram to verify that the two right triangles are similar?
 - *Answers will vary but should address the fact that the triangle angle sum requires that $2s + 2t = 180^\circ$, which means that $s + t = 90^\circ$. The radius that intersects the circle at point C is perpendicular to the circle's tangent line through C , which means that this radius and the horizontal tangent line form a right angle, so $t + r = 90^\circ$. Using substitution and subtraction, we can determine that $s = r$, which means that triangle $AOC \sim$ triangle COB by the AA similarity theorem.*
- How can we use the fact that these triangles are similar to write an expression for x in terms of a and b ?
 - *Because the triangles are similar, their corresponding side lengths are proportionate. This means that $\frac{a}{x} = \frac{x}{b}$, so $x^2 = ab$ and $x = \sqrt{ab}$.*
- This supports our conjecture based on the data from Exercise 1.

Closing (3 minutes)

Students should summarize in writing how to determine the optimal viewing distance for an object whose base is a units above an observer's eye level and whose top is b units above eye level. This summary should include the role of the inverse tangent function in establishing the optimal viewing distance. As time permits, students should share their responses with a partner.

- The optimal viewing distance, defined as the horizontal distance between the observer and the viewed object, is $x = \sqrt{ab}$, where a is the vertical distance from an observer's eye level to the base of the object, and b is the vertical distance from the observer's eye level to the top of the object.
- The optimal viewing distance was defined as that which maximized y , the viewing angle formed by the line of sight from the viewer to the base of the viewed object and the line of sight from the viewer to the top of the object.
- The tangent function was used to represent the ratio of the heights a and b with x , and the inverse tangent function was used to isolate the y by undoing the operation performed by the tangent function.

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 13: Modeling with Inverse Trigonometric Functions

Exit Ticket

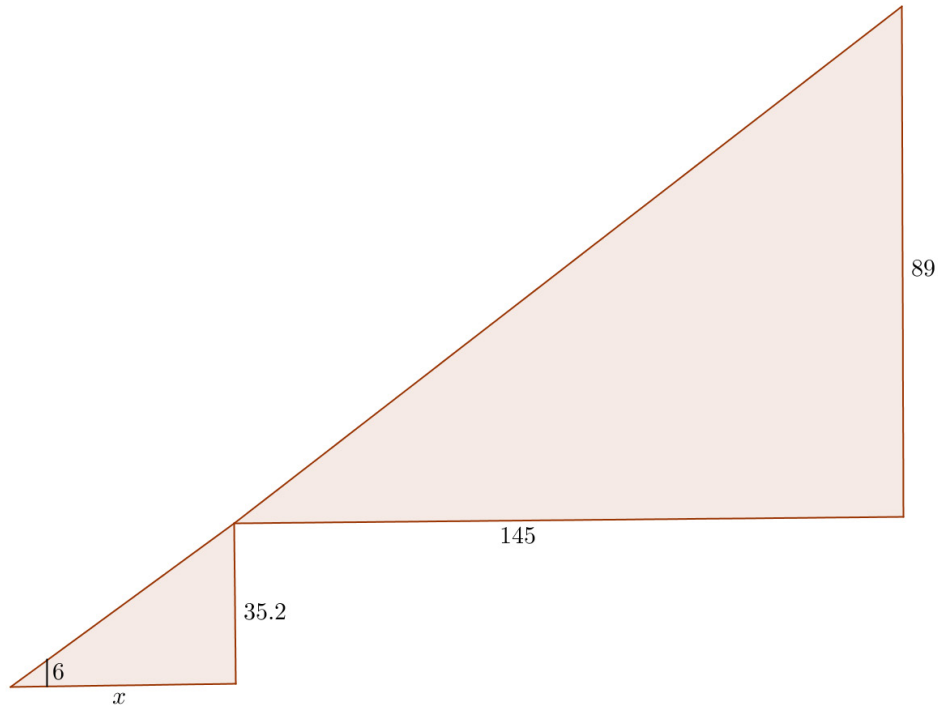
The pedestal that the Statue of Liberty sits on is 89 ft. tall with a foundation fashioned in the shape of an eleven-point star making up the rest of the height. The front point of the star juts out about 145 ft. from the front of the statue and stands about 35.2 ft. tall.

How far from the Statue of Liberty does someone whose eye-height is 6 ft. need to stand in order to see the base of the statue without being obscured by the foundation? Include a diagram and appropriate work to justify your answer.

Exit Ticket Sample Solutions

The pedestal that the Statue of Liberty sits on is 89 ft. tall with a foundation fashioned in the shape of an eleven-point star making up the rest of the height. The front point of the star juts out about 145 ft. from the front of the statue and stands about 35.2 ft. tall.

How far from the Statue of Liberty does someone whose eye-height is 6 ft. need to stand in order to see the base of the statue without being obscured by the foundation? Include a diagram and appropriate work to justify your answer.



Note that the larger triangle is similar to the smaller triangle, so the angles are equal. This tells us that

$$\begin{aligned}\tan^{-1}\left(\frac{89 + 35.2 - 6}{x + 145}\right) &= \tan^{-1}\left(\frac{35.2 - 6}{x}\right) \\ \frac{118.2}{x + 142} &= \frac{29.2}{x} \\ 118.2x &= 29.2x + 4234 \\ 89x &= 4234 \\ x &\approx 47.6\end{aligned}$$

The person will have to stand about 47.6 ft. away from the point of the star.

Problem Set Sample Solutions

1. Consider the situation of sitting down with eye level at 46 in. Find the missing distances and heights for the following:
 - a. The bottom of the picture is at 50 in. and the top is at 74 in. What is the optimal viewing distance?
 $\sqrt{4 \cdot 28} = 4\sqrt{7} \approx 10.583$
About 10.6 in. away
 - b. The bottom of the picture is at 52 in. and the top is at 60 in. What is the optimal viewing distance?
 $\sqrt{6 \cdot 14} = 2\sqrt{21} \approx 9.165$
About 9.2 in. away
 - c. The bottom of the picture is at 48 in. and the top is at 64 in. What is the optimal viewing distance?
 $\sqrt{2 \cdot 18} = 6$
6 in. away
 - d. What is the height of the picture if the optimal viewing distance is 1 ft. and the bottom of the picture is hung at 47 in.?

$$12 = \sqrt{1 \cdot x}$$

$$x = 144$$
12 ft. above eye level, so 15 ft. 10 in. tall
2. Consider the situation where you are looking at a painting a inches above your line of sight and b inches below your line of sight.
 - a. Find the optimal viewing distance if it exists.
It is not humanly possible to maximize the optimal viewing distance. The closer you get to the painting, the larger the angles become. Eventually, the picture fully encompasses the field of view of the person.
 - b. If the average standing eye height of Americans is 61.4 in., at what height should paintings and other works of art be hung?
The center of the artwork should be hung at eye level so that viewers can get the largest image of the picture they want by getting closer to the image without having to worry about finding the optimal viewing distance.

3. The amount of daylight per day is periodic with respect to the day of the year. The function $y = -3.016 \cos\left(\frac{2\pi x}{365}\right) + 12.25$ gives the number of hours of daylight in New York, y , as a function of the number of days since the winter solstice (December 22), which is represented by x .

- a. On what days will the following hours of sunlight occur?

- i. 15 hours, 15 minutes

$$\begin{aligned} 15.25 &= -3.016 \cos\left(\frac{2\pi x}{365}\right) + 12.25 \\ 3 &= -3.016 \cos\left(\frac{2\pi x}{365}\right) \\ -0.995 &= \cos\left(\frac{2\pi x}{365}\right) \\ \cos^{-1}(-0.995) &= \cos^{-1}\left(\cos\left(\frac{2\pi x}{365}\right)\right) \\ \left(\frac{2\pi x}{365}\right) &= 3.0416 \text{ and } 3.2416 \\ x &\approx 177 \text{ and } 188 \end{aligned}$$

On the 177th and 188th days

- ii. 12 hours

On the 86th and 279th days

- iii. 9 hours, 15 minutes

On the 6th and 359th days

- iv. 10 hours

On the 42nd and 323rd days

- v. 9 hours

This will never happen. The function does not go that low.

- b. Give a function that will give the day of the year from the solstice as a function of the hours of daylight.

$$y = \frac{365}{2\pi} \cdot \cos^{-1}\left(-\frac{x - 12.25}{3.016}\right)$$

- c. What is the domain of the function you gave in part (b)?

This function is only accurate for as long as the argument to the inverse cosine stays between -1 and 1 , so $|x - 12.25| \leq 3.016$, which tells us $9.234 \leq x \leq 15.266$.

- d. What does the domain tell you in the context of the problem?

The hours of sunlight for the year vary between about 9 hours, 14 minutes and 15 hours, 16 minutes.

- e. What is the range of the function? Does this make sense in the context of the problem? Explain.

The range of the function is 0 to 182.5. This only covers half of the year because if the entire year was covered, the inverse would not be a function. We can find other dates with the same amount of daylight by subtracting the number of the day from 365.

4. Ocean tides are an example of periodic behavior. At a particular harbor, data was collected over the course of 24 hours to create the following model: $y = 1.236 \sin\left(\frac{\pi}{3}x\right) + 1.798$, which gives the water level, y , in feet above the MLLW (mean lower low water) as a function of the time, x , in hours.

- a. How many periods are there each day?

Four

- b. Write a function that gives the time in hours as a function of the water level. How many other times per day will have the same water levels as those given by the function?

$$y = \frac{3}{\pi} \sin^{-1}\left(\frac{x - 1.798}{1.236}\right)$$

There are potentially eight times per day that have the same water level (with the exception of peaks and troughs, which only occur four times per day).



Lesson 14: Modeling with Inverse Trigonometric Functions

Student Outcomes

- Students model situations using trigonometric functions and apply inverse trigonometric functions to solve problems in modeling contexts.

Lesson Notes

In the previous lesson, students explored how inverse trigonometric functions could be used to solve best viewing problems. Throughout the lesson, students have opportunities to model real-world situations using trigonometric equations (MP.4) and solve the equations using inverse trigonometric functions. They apply inverse trigonometric functions to determine the angle of elevation for inclined surfaces and to solve problems involving periodic phenomena.

Classwork

Opening (5 minutes)

Ask students to imagine that they have been hired to construct a ramp from the ground to reach a door on a building that is 20 feet above the ground. To minimize construction costs, the slanted part of the ramp cannot be longer than 250 feet. Have students discuss in pairs how they could determine the minimum steepness of the ramp they have been commissioned to make. The pairs should produce a sketch to illustrate their thinking. After a few minutes, a few pairs could display their sketches and discuss their solutions, including the use of inverse trigonometric functions to determine the minimum steepness of the ramp. Students should be told that the methods they used to model the steepness of the ramp can be applied in a variety of modeling contexts, which are explored in this lesson.

Example 1 (10 minutes)

This example demonstrates how inverse trigonometric functions can be used to help architects design ramps to meet safety specifications. Part (a) should be completed in pairs, or if students are unfamiliar with force diagrams, it could be completed as part of a teacher-led discussion. If students complete the sketches in pairs, the sketches should be discussed in a whole-class setting or approved by the teacher to resolve any confusion before the rest of the problem is completed. Parts (b) and (c) should be completed in pairs.

Note: Students might need to be told that the unit newton is equivalent to $\frac{\text{kg} \cdot \text{m}}{\text{s}^2}$.

- What features should be included in our sketch of the scale-model ramp?
 - Answers should address that a right triangle could be used to represent the ramp, with the hypotenuse representing the inclined ramp surface and the angle of elevation representing the acute angle between the inclined surface and the ground; the model should also include a circle to represent an object rolling down the ramp and vectors to represent the force applied to the ball and the acceleration of the ball down the ramp.

- How can we represent the force of gravity applied to the ball on the ramp?
 - *It is a vector facing downward because the force of gravity is always directed perpendicular to the surface of the ground.*
- How can we represent the motion of the ball on the ramp?
 - *Answers may address drawing a vector parallel to the ramp pointed downward to represent the acceleration of the ball, which is $2.4 \frac{\text{m}}{\text{s}^2}$.*
- What is causing the ball to accelerate down the ramp?
 - *The force due to gravity*
- That's right. However, our force vector is directed downward, not in the direction of the ball's motion. How could we represent the force vector in a way that demonstrates its effect on the ball's motion?
 - *Answers will vary but should address that we could decompose the force vector into a component that is parallel to the ramp and facing downward and a component that is perpendicular to the ramp.*
- If we sketch the vector's components parallel and perpendicular to the ramp, these component vectors and the vector representing the gravitational force create a right triangle. How can we find the measures of the acute angles of this triangle?
 - *Answers will vary but should address that since the gravitational force vector is parallel to the height of the right triangle that represents the ramp, the angle formed between the height of the triangle and the hypotenuse is congruent to the angle formed by the gravitational force vector and the component parallel to the ramp. Because of the triangle angle sum theorem, the measure of the other acute angles (one in each triangle) are congruent with measure θ .*
- What does θ represent?
 - *The angle of elevation*
- How can we use our sketch to find the value of θ ?
 - *Answers should address applying the inverse sine function to the ratio of the force vector parallel to the ramp and the gravitational force vector.*
- How do we determine the magnitude of the force parallel to the ramp, which is causing the ball to roll down the ramp?
 - *We were told that the force is the product of the object's mass and its acceleration, so we can multiply the ball's mass, 0.1 kg, by its acceleration, $2.4 \frac{\text{m}}{\text{s}^2}$.*

Scaffolding:

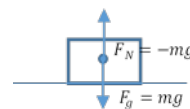
- Introduce students to the concept of the free-body diagram (FBD) as a means of illustrating the forces acting upon an object. Explain that the object of interest is drawn as a rectangle, and the forces acting upon the object are represented as vectors whose initial point is the center of the object. Demonstrate each of the following situations for students, discuss the forces acting on the objects, and have them represent the forces acting on the object using a free-body diagram.
- Drop a small object from waist height:

FBD:



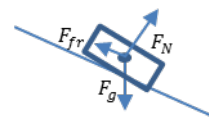
- Lay the same object on a flat surface:

FBD:



- Lay the object on an inclined surface so that it slides down:

FBD:



F_g : gravitational force, which points towards the center of Earth.

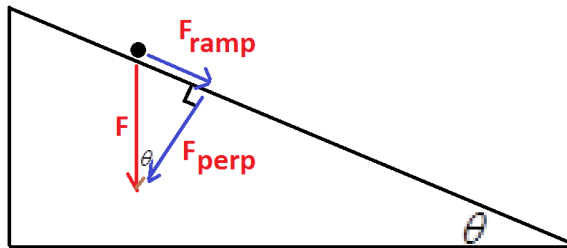
F_N : normal or contact force, which is perpendicular to the contact surface.

F_{fr} : frictional force, which is parallel and opposite in direction to the object's motion.

Example 1

A designer wants to test the safety of a wheelchair ramp she has designed for a building before constructing it, so she creates a scale model. To meet the city's safety requirements, an object that starts at a standstill from the top of the ramp and rolls down should not experience an acceleration exceeding $2.4 \frac{\text{m}}{\text{s}^2}$.

- a. A ball of mass 0.1 kg is used to represent an object that rolls down the ramp. As it is placed at the top of the ramp, the ball experiences a downward force due to gravity, which causes it to accelerate down the ramp. Knowing that the force applied to an object is the product of its mass and acceleration, create a sketch to model the ball as it accelerates down the ramp.



- b. If the ball rolls at the maximum allowable acceleration of $2.4 \frac{\text{m}}{\text{s}^2}$, what is the angle of elevation for the ramp?

$$F_{\text{ramp}} = m_{\text{ball}} \times a_{\text{ball}} = 0.1 \text{ kg} \times 2.4 \frac{\text{m}}{\text{s}^2} = 0.24 \text{ N parallel to the ramp and directed down the ramp}$$

$$F = m_{\text{ball}} \times a_{\text{gravity}} = 0.1 \text{ g} \times 9.8 \frac{\text{m}}{\text{s}^2} = 0.98 \text{ N directed down toward the ground}$$

$$\sin(\theta) = \frac{F_{\text{ramp}}}{F} = \frac{0.24 \text{ N}}{0.98 \text{ N}}$$

$$\sin^{-1}(\sin(\theta)) = \sin^{-1}\left(\frac{0.24 \text{ N}}{0.98 \text{ N}}\right) \approx 14.2^\circ$$

The angle of elevation for the ramp is approximately 14.2° .

Scaffolding:

- Students who are familiar with mechanics can create the sketch without being cued about the relationship between force, mass, and acceleration.
- Have advanced students calculate the maximum angle of elevation given a frictional force directed up the ramp.

MP.4

- c. If the designer wants to exceed the safety standards by ensuring the acceleration of the object does not exceed $2.0 \frac{\text{m}}{\text{s}^2}$, by how much will the maximum angle of elevation decrease?

$$F_{\text{ramp}} = m_{\text{ball}} \times a_{\text{ball}} = 0.1 \text{ kg} \times 2.0 \frac{\text{m}}{\text{s}^2} = 0.2 \text{ N parallel to the ramp and directed down the ramp}$$

$$F = m_{\text{ball}} \times a_{\text{gravity}} = 0.1 \text{ g} \times 9.8 \frac{\text{m}}{\text{s}^2} = 0.98 \text{ N directed down toward the ground}$$

$$\sin(\theta) = \frac{F_{\text{ramp}}}{F} = \frac{0.2 \text{ N}}{0.98 \text{ N}}$$

$$\sin^{-1}(\sin(\theta)) = \sin^{-1}\left(\frac{0.2 \text{ N}}{0.98 \text{ N}}\right) \approx 11.8^\circ$$

The maximum angle of elevation would be approximately 11.8° , which is 2.4 degrees less than the maximum angle permitted to meet the safety specification.

- d. How does the mass of the ball used in the scale model affect the value of θ ? Explain your response.

It doesn't. The mass of the ball is a common factor in the numerator and denominator in the ratio used to calculate θ .

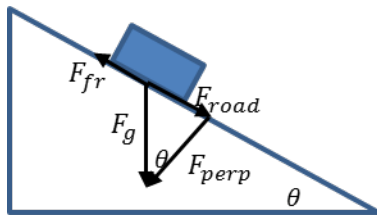
Exercise 1 (5 minutes)

This exercise should be completed in pairs. At an appropriate time, selected students should share their responses. Additional students should be allowed to suggest revisions to the model or suggest alternative approaches to solving the problem.

Exercise 1

A vehicle with a mass of 1,000 kg rolls down a slanted road with an acceleration of $0.07 \frac{\text{m}}{\text{s}^2}$. The frictional force between the wheels of the vehicle and the wet concrete road is 2,800 newtons.

- a. Sketch the situation.



- b. What is the angle of elevation of the road?

$$F_{\text{road}} = m_{\text{vehicle}} \times a_{\text{vehicle}} + F_{\text{fr}} = 1000 \text{ kg} \times 0.07 \frac{\text{m}}{\text{s}^2} = 70 \text{ N} + 2800 \text{ N} = 2870 \text{ N}$$

$$F_{\text{gravity}} = m_{\text{vehicle}} \times a_{\text{gravity}} = 1000 \text{ kg} \times 9.8 \frac{\text{m}}{\text{s}^2} = 9800 \text{ N}$$

$$\sin(\theta) = \frac{F_{\text{road}}}{F_{\text{gravity}}} = \frac{2870 \text{ N}}{9800 \text{ N}}$$

$$\sin^{-1}(\sin(\theta)) = \sin^{-1}\left(\frac{2870 \text{ N}}{9800 \text{ N}}\right) \approx 17^\circ$$

The angle of elevation for the road is approximately 17° .

Scaffolding:

Show the diagram, and have students explain how each part of the context is represented in the diagram by having them answer questions such as “What is F_{fr} ?” “What does F_{perp} represent?”

MP.4

- c. What is the maximum angle of elevation the road could have so that the vehicle described would not slide down the road?

If the vehicle does not slide, then the frictional force must be greater than or equal to the downward force parallel to the road. So, for the maximum angle θ , $F_{\text{road}} = F_{\text{friction}} = 2800 \text{ N}$.

$$\sin(\theta) = \frac{F_{\text{road}}}{F_{\text{gravity}}} = \frac{2800 \text{ N}}{9800 \text{ N}}$$

$$\sin^{-1}(\sin(\theta)) = \sin^{-1}\left(\frac{2800 \text{ N}}{9800 \text{ N}}\right) \approx 16.6^\circ$$

The angle of elevation for the road that would prohibit sliding is approximately 16.6° .

Example 2 (5 minutes)

This example demonstrates how inverse trigonometric functions can be used to solve problems addressing periodic phenomena. Students apply inverse trigonometric functions to create a function equation that enables them to predict the calendar dates that correspond to specific angles of solar declination. This example should be completed as part of a teacher-led discussion. Alternatively, students could complete the problem in pairs and share their responses in a whole-class setting.

- How can we determine the domain of the function modeled in the problem?
 - *Since N represents a calendar date, its values are counting numbers between 1, which represents January 1, and 365, which represents December 31 (excepting leap years).*
- What do we know about the range?
 - *We know that the cosine function has a range between -1 and 1 , inclusive. Therefore, the expression $-23.44^\circ \left(\cos \left(\left(\frac{360}{365} \right) (N + 10) \right) \right)$ has a maximum value of 23.44° when $\cos \left(\left(\frac{360}{365} \right) (N + 10) \right) = -1$ and a minimum value of -23.44° when $\cos \left(\left(\frac{360}{365} \right) (N + 10) \right) = 1$.*
- What first step should we take to write N as a function of δ ?
 - *Divide by -23.44° to isolate the cosine function.*
- What's next?
 - *Apply the inverse cosine function to both sides of the equation.*
- How does this help us to isolate N ?
 - *The inverse function undoes the operation applied by the cosine function.*
- Once we have isolated N , explain in context the meaning of this function.
 - *Answers will vary but should address that the function provides a calendar date output given a declination angle input.*
- What is the domain of the equation that represents N as a function of δ ? Explain.
 - $-23.44^\circ \leq \delta \leq 23.44^\circ$. *Any other values produce a ratio $\left| \frac{\delta}{-23.44^\circ} \right| > 1$, where the inverse cosine function is undefined.*
- The range is a bit trickier. What is the domain of the inverse cosine function?
 - $0^\circ \leq \theta \leq 180^\circ$

- Which means that the expression $-10 + \left(\frac{365}{360}\right) \cos^{-1}\left(\frac{\delta}{-23.44^\circ}\right)$ is between which two values?
 - -10 if $\cos^{-1}\left(\frac{\delta}{-23.44^\circ}\right) = 0^\circ$ and 172.5 if $\cos^{-1}\left(\frac{\delta}{-23.44^\circ}\right) = 180^\circ$
- These values account for about half the calendar dates. Clearly, solar declination exists for calendar dates greater than 172.5. How can we find appropriate dates in the second half of the calendar year that correspond to a given declination angle?
 - *Answers will vary but might address using the property that $\cos(360^\circ - x) = \cos(x)$.*
- How can we apply this to the declination angle 10° ?
 - *Answers will vary but might address that $x \approx 115.3^\circ$ and also $(360^\circ - 115.3^\circ) \approx 244.7^\circ$, so both of these values could be used to determine N .*
- If we evaluate the function for $\cos^{-1}\left(\frac{10^\circ}{-23.44^\circ}\right) \approx 115.3^\circ$, we get $N \approx 107$. Why can't we just subtract this N value from 365 to find the other value of N ?
 - *Answers will vary but should address that the original function was phase shifted 10 units to the left.*
- How can we determine the dates when the sun is directly overhead?
 - *It is when the declination is 0° .*
- When would you predict these dates to be?
 - *Answers will vary, but some students might predict that this occurs at the spring and autumn equinoxes.*
- Our model provided dates of March 22 and September 21 as outputs for a declination input of 0° . The dates of the equinoxes are known to be March 20 and September 22. What might account for the differences?
 - *Answers will vary but might address issues such as not accounting for leap days to possible simplifications in the original function for ease of use.*

Example 2

The declination of the sun is the path the sun takes overhead the earth throughout the year. When the sun passes directly overhead, the declination is defined as 0° , while a positive declination angle represents a northward deviation and a negative declination angle represents a southward deviation. Solar declination is periodic and can be roughly estimated using the equation

$\delta = -23.44^\circ \left(\cos \left(\left(\frac{360}{365} \right) (N + 10) \right) \right)$, where N represents a calendar date (e.g., $N = 1$ is January 1, and δ is the declination angle of the sun measured in degrees).

- a. Describe the domain and range of the function.

D: $1 \leq N \leq 365$ where N is a counting number

R: $-23.44^\circ \leq \delta \leq 23.44^\circ$

- b. Write an equation that represents N as a function of δ .

$$\begin{aligned}\delta &= -23.44^\circ \left(\cos \left(\left(\frac{360}{365} \right) (N + 10) \right) \right) \\ \frac{\delta}{-23.44^\circ} &= \left(\cos \left(\left(\frac{360}{365} \right) (N + 10) \right) \right) \\ \cos^{-1} \left(\frac{\delta}{-23.44^\circ} \right) &= \left(\frac{360}{365} \right) (N + 10) \\ -10 + \left(\frac{365}{360} \right) \cos^{-1} \left(\frac{\delta}{-23.44^\circ} \right) &= N\end{aligned}$$

- c. Determine the calendar date(s) for the given angles of declination:

- i. 10°

$$N = -10 + \left(\frac{365}{360} \right) \cos^{-1} \left(\frac{10^\circ}{-23.44^\circ} \right) \approx 107 \text{ and } 238$$

$N = 107$ corresponds to a calendar date of April 17, and $N = 238$ corresponds to August 26.

- ii. -5.2°

$$N = -10 + \left(\frac{365}{360} \right) \cos^{-1} \left(\frac{-5.2^\circ}{-23.44^\circ} \right) \approx 68 \text{ and } 277$$

$N = 68$ corresponds to a calendar date of March 9, and $N = 277$ corresponds to October 4.

- iii. 25°

No date will correspond to this angle because it lies outside of the domain of the function.

- d. When will the sun trace a direct path above the equator?

When the sun passes directly overhead, the declination is 0° . This means that

$$N = -10 + \left(\frac{365}{360} \right) \cos^{-1} \left(\frac{0^\circ}{-23.44^\circ} \right) \approx 81 \text{ and } 264, \text{ which correspond to the calendar dates March 22 and September 21.}$$

Exercises 2–3 (10 minutes)

These exercises should be completed in pairs. At an appropriate time, selected students should share their responses. If students struggle with the curve fitting in Exercise 3, this exercise could be completed as part of a teacher-led discussion. Graphing calculators, graphing software, or grid paper are needed for students to complete Exercise 3.

Exercises 2–3

2. The average monthly temperature in a coastal city in the United States is periodic and can be modeled with the equation $y = -8 \cos \left((x - 1) \left(\frac{\pi}{6} \right) \right) + 17.5$, where y represents the average temperature in degrees Celsius and x represents the month, with $x = 1$ representing January.

- a. Write an equation that represents x as a function of y .

$$x = 1 + \frac{6}{\pi} \cos^{-1} \left(\frac{y - 17.5}{-8} \right)$$

- b. A tourist wants to visit the city when the average temperature is closest to 25° Celsius. What recommendations would you make regarding when the tourist should travel? Justify your response.

$$x = 1 + \frac{6}{\pi} \cos^{-1} \left(\frac{25 - 17.5}{-8} \right) \approx 6.32$$

If the tourist wants to visit when the temperature is closest to 25° Celsius, she should travel about the second week in June.

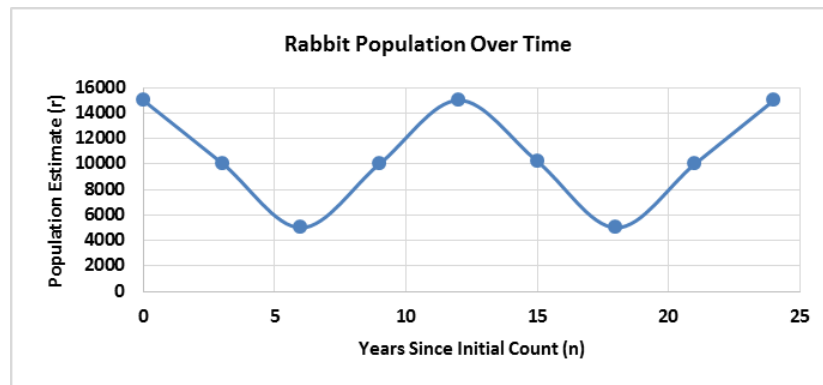
3. The estimated size for a population of rabbits and a population of coyotes in a desert habitat are shown in the table. The estimated population sizes were recorded as part of a long-term study related to the effect of commercial development on native animal species.

Years since initial count (n)	0	3	6	9	12	15	18	21	24
Estimated number of rabbits (r)	14,989	10,055	5,002	10,033	15,002	10,204	4,999	10,002	14,985
Estimated number of coyotes (c)	1,995	2,201	2,003	1,795	1,999	2,208	2,010	1,804	2,001

- a. Describe the relationship between sizes of the rabbit and coyote populations throughout the study.

The rabbit population started at approximately 15,000 rabbits and then decreased while the coyote population increased (perhaps because of the abundance of prey for the coyotes). Over time, both the rabbit and coyote populations declined until the coyote population was about 1,800, when the rabbit population increased again. Both species' population numbers appear to cycle, with the coyotes' values shifted about 3 years from the rabbit's values.

- b. Plot the relationship between the number of years since the initial count and the number of rabbits. Fit a curve to the data.

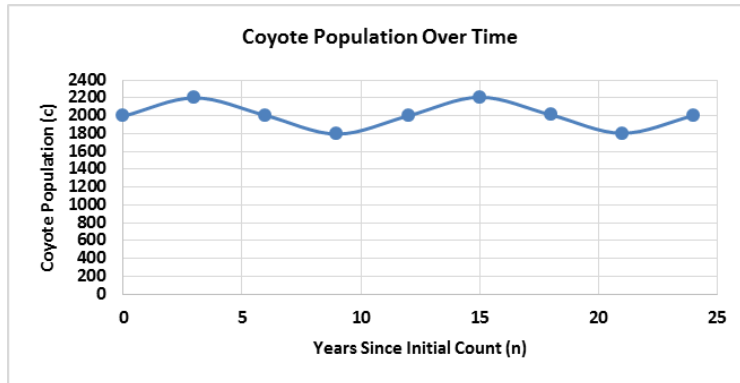


Equation of curve: $r = 5000 \cos \left(\frac{\pi n}{6} \right) + 10000$

MP.2

MP.4

- c. Repeat the procedure described in part (b) for the estimated number of coyotes over the course of the study.



Equation of curve: $c = 200 \sin\left(\frac{\pi n}{6}\right) + 2000$

- d. During the study, how many times was the rabbit population approximately 12,000? When were these times?

$$r = 5000 \cos\left(\frac{\pi n}{6}\right) + 10000$$

$$n = \frac{6}{\pi} \cos^{-1}\left(\frac{r - 10000}{5000}\right)$$

$$n = \frac{6}{\pi} \cos^{-1}\left(\frac{12000 - 10000}{5000}\right) \approx 2.2$$

Given that the function cycles every 12 years, $n = 0 + 2.2 = 2.2$; $n = (12 - 2.2) = 9.8$; $n = (12 + 2.2) = 14.2$; and $n = (24 - 2.2) = 21.8$. The rabbit population was approximately 12,000 four times at 2.2, 9.8, 14.2, and 21.8 years.

- e. During the study, when was the coyote population estimate below 2,100?

$$c = 200 \sin\left(\frac{\pi n}{6}\right) + 2000$$

$$n = \frac{6}{\pi} \sin^{-1}\left(\frac{c - 2000}{200}\right)$$

$$n = \frac{6}{\pi} \sin^{-1}\left(\frac{2100 - 2000}{200}\right) = 1, 5, 13, 17$$

By analyzing the graph of the coyote population estimates, the population of coyotes was less than 2,100 prior to $n = 1$, between $n = 5$ and $n = 13$, and from $n = 17$ to $n = 24$.

Scaffolding:

- Cue the students to use their graphs to verify their solutions.
- Remind the students of the periodicity and symmetry of the sine and cosine functions, which lead to multiple values for n within their domain.

MP.4

Closing (5 minutes)

Have students summarize, in writing, the different contexts they have encountered that could be modeled with trigonometric functions. The summaries should include a brief discussion of how inverse trigonometric functions were applied in solving problems in the modeling contexts. Students should share their summaries with a partner.

- *Right triangle trigonometry can be applied to model the forces applied to an object on an inclined surface.*
- *Inverse trigonometric functions can be applied to undo the effects of the trigonometric operation, which, in the inclined surface problem, was used to determine the angle of elevation.*
- *Trigonometric functions may be used to represent periodic phenomena, such as the declination of the sun.*
- *Inverse trigonometric functions can be applied to represent the relationship between variables so that the input is modeled as a function of the output. For instance, when we were presented an equation that modeled declination as a function of calendar date, inverse trigonometric functions allowed us to model calendar date as a function of the solar declination angle.*

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 14: Modeling with Inverse Trigonometric Functions

Exit Ticket

The minimum radius of the turn r needed for an aircraft traveling at true airspeed v is given by the following formula

$$r = \frac{v^2}{g \tan(\theta)}$$

where r is the radius in meters, g is the acceleration due to gravity, and θ is the banking angle of the aircraft. Use $g = 9.78 \frac{\text{m}}{\text{s}^2}$ instead of $9.81 \frac{\text{m}}{\text{s}^2}$ to model the acceleration of the airplane accurately at 30,000 ft.

- If an aircraft is traveling at $103 \frac{\text{m}}{\text{s}}$, what banking angle is needed to successfully turn within 1 km?
- Write the formula that gives the banking angle as a function of the radius of the turn available for a fixed airspeed v .
- For a variety of reasons, including motion sickness from fluctuating g -forces and the danger of losing lift, many airplanes have a maximum banking angle of around 60° . Does this maximum on the model affect the domain or range of the formula you gave in part (b)?

Exit Ticket Sample Solutions

The minimum radius of the turn r needed for an aircraft traveling at true airspeed v is given by the following formula

$$r = \frac{v^2}{g \tan(\theta)}$$

where r is the radius in meters, g is the acceleration due to gravity, and θ is the banking angle of the aircraft. Use $g = 9.78 \frac{\text{m}}{\text{s}^2}$ instead of $9.81 \frac{\text{m}}{\text{s}^2}$ to model the acceleration of the airplane accurately at 30,000 ft.

- a. If an aircraft is traveling at $103 \frac{\text{m}}{\text{s}}$, what banking angle is needed to successfully turn within 1 km?

$$\begin{aligned} 1000 &= \frac{103^2}{9.78 \cdot \tan(\theta)} \\ \theta &= \tan^{-1}\left(\frac{103^2}{1000 \cdot 9.78}\right) \\ &\approx 47.328 \end{aligned}$$

About 47.3°

- b. Write the formula that gives the banking angle as a function of the radius of the turn available for a fixed airspeed v .

$$\theta = \tan^{-1}\left(\frac{v^2}{r \cdot g}\right)$$

- c. For a variety of reasons, including motion sickness from fluctuating g -forces and the danger of losing lift, many airplanes have a maximum banking angle of around 60° . Does this maximum on the model affect the domain or range of the formula you gave in part (b)?

If the angle cannot be greater than 60° , then the range caps out at 60° instead of normally being able to go up to 90° .

Problem Set Sample Solutions

1. A particle is moving along a line at a velocity of $y = 3 \sin\left(\frac{2\pi x}{5}\right) + 2 \frac{\text{m}}{\text{s}}$ at location x meters from the starting point on the line for $0 \leq x \leq 20$.

- a. Find a formula that represents the location of the particle given its velocity.

$$y = \frac{5}{2\pi} \cdot \sin^{-1}\left(\frac{x-2}{3}\right)$$

- b. What is the domain and range of the function you found in part (a)?

The domain is $-1 \leq x \leq 5$, and the range is $-\frac{5}{4} \leq y \leq \frac{5}{4}$.

- c. Use your answer to part (a) to find where the particle is when it is traveling $5 \frac{\text{m}}{\text{s}}$ for the first time.

The particle will be located at $x = 1.2$ meters from the starting point on the line.

- d. How can you find the other locations the particle is traveling at this speed?

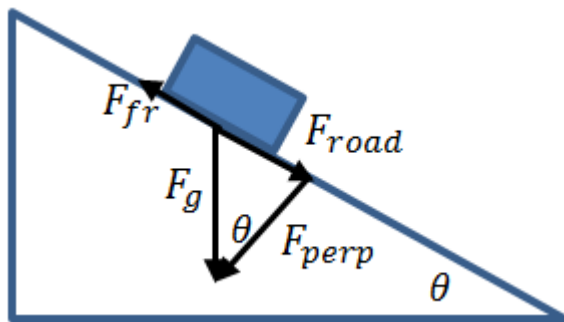
In this case, the velocity is a maximum, so it will only occur once every period. All other values can be found by adding multiples of 5 to the location. If it was not a maximum, we could subtract the location from $\frac{5}{2}$ to find another value within the same period and then add multiples of 5 to find analogous values in other periods.

2. In general, since the cosine function is merely the sine function under a phase shift, mathematicians and scientists regularly choose to use the sine function to model periodic phenomena instead of a mixture of the two. What behavior in data would prompt a scientist to use a tangent function instead of a sine function?

The tangent function has infinitely many vertical asymptotes and rapidly takes on extreme values. Since the tangent function is the ratio between the sine and cosine functions, it will probably show up when comparing the ratio of two sets of periodic data. Otherwise, the extreme values would be reasons to use the tangent function.

3. A vehicle with a mass of 500 kg rolls down a slanted road with an acceleration of $0.04 \frac{\text{m}}{\text{s}^2}$. The frictional force between the wheels of the vehicle and the road is 1,800 newtons.

- a. Sketch the situation.



- b. What is the angle of elevation of the road?

$$F_{\text{road}} = ma + F_{\text{fr}} = 500 \text{ kg} \times 0.04 \frac{\text{m}}{\text{s}^2} + 1800 \text{ N} = 20 \text{ N} + 1800 \text{ N} = 1820 \text{ N}$$

$$F_g = mg = 500 \text{ kg} \times 9.8 \frac{\text{m}}{\text{s}^2} = 4900 \text{ N}$$

$$\sin(\theta) = \frac{F_{\text{road}}}{F_g} = \frac{1820 \text{ N}}{4900 \text{ N}}$$

$$\sin^{-1}(\sin(\theta)) = \sin^{-1}\left(\frac{1820 \text{ N}}{4900 \text{ N}}\right) \approx 21.8^\circ$$

The angle of elevation for the road is approximately 21.8° .

- c. The steepness of a road is frequently measured as grade, which expresses the slope of a hill as a ratio of the change in height to the change in the horizontal distance. What is the grade of the hill described in this problem?

We need to find the perpendicular force. We get

$$\cos(21.8) = \frac{F_{\text{perp}}}{4900}$$

$$F_{\text{perp}} \approx 4549.46.$$

$$\text{So the grade of the hill is } \frac{1820}{4549.46} \approx 40\%.$$

4. Canton Avenue in Pittsburgh, PA is considered to be one of the steepest roads in the world with a grade of 37%.

- a. Assuming no friction on a particularly icy day, what would be the acceleration of a 1,000 kg car with only gravity acting on it?

$$\tan(\theta) = 0.37$$

$$\theta \approx 20.30$$

Since the acceleration due to gravity is the only acceleration on the car, the acceleration due to gravity is being transferred into an acceleration as the car goes down the hill. We can envision the force due to gravity as two separate forces, one that is parallel to the road and the second that is perpendicular to the road. The force parallel to the road is $F_{\parallel} = m \cdot g \cdot \sin(\theta)$, and the acceleration parallel to the road is

$$a = g \cdot \sin(\theta)$$

$$a = 9.8 \sin(20.3)$$

$$a \approx 3.4$$

$$\text{The acceleration is } 3.4 \frac{\text{m}}{\text{s}^2}.$$

- b. The force due to friction is equal to the product of the force perpendicular to the road and the coefficient of friction μ . For icy roads and a non-moving vehicle, assume the coefficient of friction is $\mu = 0.3$. Find the force due to friction for the car above. If the car is in park, will it begin sliding down Canton Avenue if the road is this icy?

$$\text{The force perpendicular to the road is } 9191.3 \text{ N. } F_{\perp} = m \cdot g \cdot \cos(\theta) = 1000 \cdot 9.8 \cdot \cos(20.3) \approx 9191.3$$

$$\text{Thus, the force due to friction is approximately } 2757.4 \text{ N. } \mu F_{\perp} \approx 2757.4.$$

$$\text{The force parallel to the road is } 3400.0 \text{ N. } F_{\parallel} = 1000 \cdot 9.8 \cdot \sin(20.3) \approx 3400.0$$

Because the force parallel to the road is greater than the force due to friction, the car will slide down the road once Canton Avenue gets this icy.

- c. Assume the coefficient of friction for moving cars on icy roads is $\mu = 0.2$. What is the maximum angle of road that the car will be able to stop on?

The car will be able to slow (and eventually stop) when the force due to friction is greater than the force parallel to the road, so we need to solve,

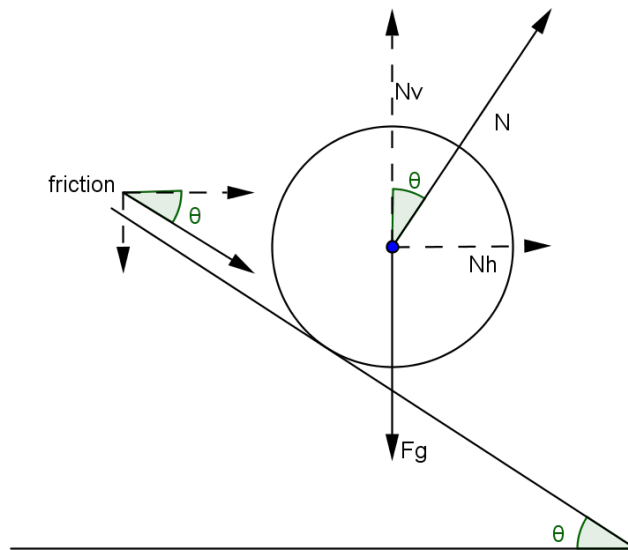
$$9800 \cdot \sin(\theta) = 0.2 \cdot 9800 \cos(\theta)$$

$$\frac{\sin(\theta)}{\cos(\theta)} = 0.2$$

$$\tan(\theta) = 0.2.$$

So the car will be able to slow on any hill with less than a 20% grade, which corresponds to an angle of 11.3° .

5. Talladega Superspeedway has some of the steepest turns in all of NASCAR. The main turns have a radius of about 305 m and are pitched at 33° . Let N be the perpendicular force on the car and N_v and N_h be the vertical and horizontal components of this force, respectively. See the diagram below.



- a. Let μ represent the coefficient of friction; recall that μN gives the force due to friction. To maintain the position of a vehicle traveling around the bank, the centripetal force must equal the horizontal force in the direction of the center of the track. Add the horizontal component of friction to the horizontal component of the perpendicular force on the car to find the centripetal force. Set your expression equal to $\frac{mv^2}{r}$, the centripetal force.

The force due to friction is μN , and the horizontal component then is $\mu N \cos(\theta)$.

The horizontal perpendicular force would be $N \sin(\theta)$.

We get, $\frac{mv^2}{r} = N \sin(\theta) + \mu N \cos(\theta)$.

- b. Add the vertical component of friction to the force due to gravity, and set this equal to the vertical component of the perpendicular force.

$N \cos(\theta)$ is the vertical component of the perpendicular force.

The force due to gravity is mg , and the vertical force of friction is $\mu N \sin(\theta)$.

We get, $N \cos(\theta) = mg + \mu N \sin(\theta)$.

- c. Solve one of your equations in part (a) or part (b) for m , and use this with the other equation to solve for v .

$$m = \frac{N \cos(\theta) - \mu N \sin(\theta)}{g}$$

$$\frac{N \cos(\theta) - \mu N \sin(\theta)}{g} \cdot \frac{v^2}{r} = N \sin(\theta) + \mu N \cos(\theta)$$

$$\frac{v^2}{gr} = \frac{N \sin(\theta) + \mu N \cos(\theta)}{N \cos(\theta) - \mu N \sin(\theta)}$$

We can factor out the N and cancel the common factor from here onward.

$$v^2 = gr \cdot \frac{\sin(\theta) + \mu \cos(\theta)}{\cos(\theta) - \mu \sin(\theta)}$$

$$v = \sqrt{gr \cdot \frac{\sin(\theta) + \mu \cos(\theta)}{\cos(\theta) - \mu \sin(\theta)}}$$

- d. Assume $\mu = 0.75$, the standard coefficient of friction for rubber on asphalt. For the Talladega Superspeedway, what is the maximum velocity on the main turns? Is this about how fast you might expect NASCAR stock cars to travel? Explain why you think NASCAR takes steps to limit the maximum speeds of the stock cars.

$$v = \sqrt{gr \cdot \frac{\sin(\theta) + \mu \cos(\theta)}{\cos(\theta) - \mu \sin(\theta)}}$$

$$= \sqrt{9.8 \cdot 305 \cdot \frac{\sin(33) + 0.75 \cos(33)}{\cos(33) - 0.75 \sin(33)}}$$

$$\approx 90.3$$

$90.3 \frac{\text{m}}{\text{s}}$ is about 202 mph, which is the speed most people associate with NASCAR stock cars (although they usually go slower than this). This means that at Talladega, the race cars are able to go at their maximum speeds through the main turns.

If the cars go any faster than this, then they would drift up toward the wall, which may prompt them to oversteer and possibly spin out of control. NASCAR may limit the speeds of the cars because the racetracks themselves are not designed for cars that can go faster.

- e. Does the friction component allow the cars to travel faster on the curve or force them to drive slower? What is the maximum velocity if the friction coefficient is zero on the Talladega roadway?

The friction component is in the direction toward the center of the racetrack (away from the walls), so it allows them to travel faster more safely. If $\mu = 0$, the equation becomes $v = \sqrt{gr \tan(\theta)}$. Therefore, the maximum velocity is about $44.1 \frac{\text{m}}{\text{s}}$ or 99 mph.

- f. Do cars need to travel slower on a flat roadway making a turn than on a banked roadway? What is the maximum velocity of a car traveling on a 305 m turn with no bank?

They need to travel much slower on a flat roadway making a turn than when traveling on a banked roadway. The normal force does not prevent the car from spinning out of control the way it does on a banked turn. If $\theta = 0$, then the equation becomes $v = \sqrt{gr\mu}$. Therefore, the maximum velocity is about $47.3 \frac{\text{m}}{\text{s}}$ or 106 mph, which is much slower than the velocity that cars can travel on the banked turns (202 mph).

6. At a particular harbor over the course of 24 hours, the following data on peak water levels was collected (measurements are in feet above the MLLW):

Time	1:30	7:30	14:30	20:30
Water Level	-0.211	8.21	-0.619	7.518

- a. What appears to be the average period of the water level?

It takes 13 hours to get from the first low-point to the second, and 13 hours to get from the first high-point to the second, so the average period is $\frac{13 + 13}{2} = 13$.

- b. What appears to be the average amplitude of the water level?

There are three areas we can examine to get amplitudes, from 1:30 to 7:30, 7:30 to 14:30, and 14:30 to 20:30. We get amplitudes of $\frac{8.21 - (-0.211)}{2} = 4.2105$, $\frac{8.21 - (-0.619)}{2} = 4.4145$, and $\frac{7.518 - (-0.619)}{2} = 4.0685$. We get 4.231 as the average amplitude.

- c. What appears to be the average midline for the water level?

3.748

- d. Fit a curve of the form $y = A \sin(\omega(x - h)) + k$ or $y = A \cos(\omega(x - h)) + k$ modeling the water level in feet as a function of the time.

Since it would make our curve more inaccurate to guess at what point the water levels will cross the midline, we can either start at 1:30 or 7:30 and use the cosine function. For 7:30, we get

$$y = 4.231 \cos\left(\frac{2\pi}{13}(x - 7.5)\right) + 3.748.$$

- e. According to your function, how many times per day will the water level reach its maximum?

It should reach its maximum levels twice a day usually, but there is the rare possibility that it will reach its maximum only once.

- f. How can you find other time values for a particular water level after finding one value from your function?

The values repeat every 13 hours, so immediately once a value is found, adding any multiple of 13 will give other values that work. Additionally, if you have the inverse cosine value for a particular water level (but have not solved for x yet), then take the opposite, solve normally, and you will have another time to which you can add multiples of 13.

- g. Find the inverse function associated with the function in part (d). What is the domain and range of this function? What type of values does this function output?

$$y = \frac{13}{2\pi} \cos^{-1}\left(\frac{x - 3.748}{4.231}\right) + 7.5$$

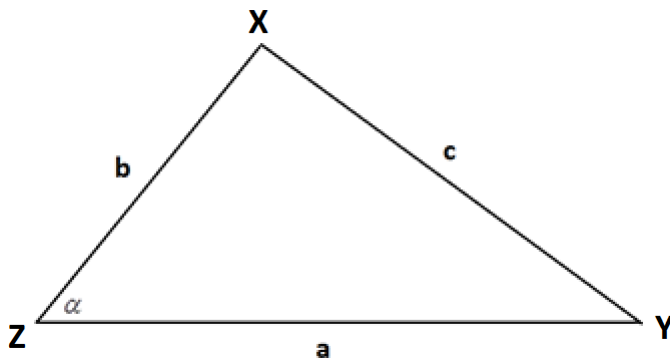
The domain is all real numbers x , such that $-0.483 \leq x \leq 7.979$, and the range is all real numbers y such that $7.5 \leq y \leq 14$.

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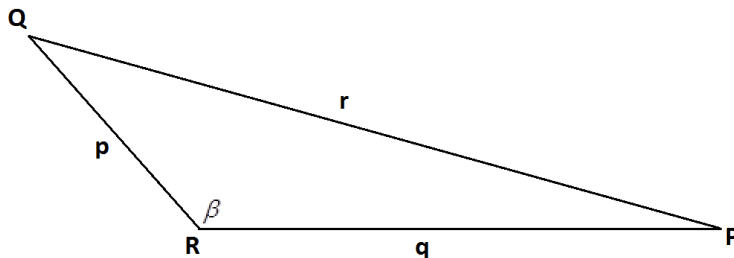
Date _____

1.

- a. In the following diagram, triangle XYZ has side lengths a , b , and c as shown. The angle α indicated is acute. Show that the area A of the triangle is given by $A = \frac{1}{2}ab \sin(\alpha)$.

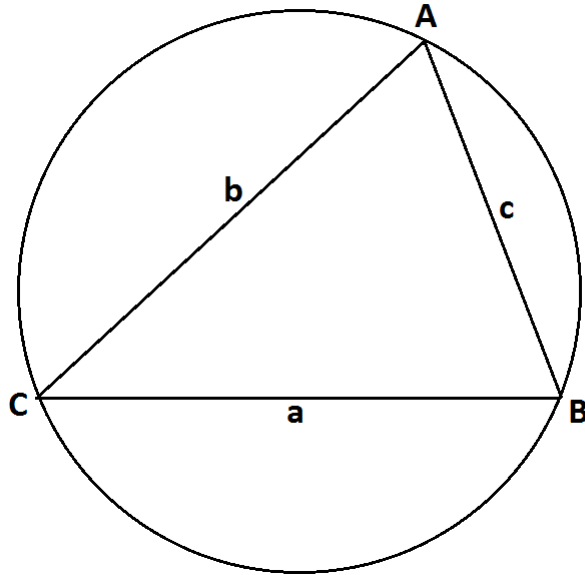


- b. In the following diagram, triangle PQR has side lengths p , q , and r as shown. The angle β indicated is obtuse. Show that the area A of the triangle is given by $A = \frac{1}{2}pq \sin(\beta)$.



- c. To one decimal place, what is the area of the triangle with sides of lengths 10 cm, 17 cm, and 21 cm? Explain how you obtain your answer.

2. Triangle ABC with side lengths a , b , and c as shown is circumscribed by a circle with diameter d .



- a. Show that $\frac{a}{\sin(A)} = d$.

- b. The law of sines states that $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ for any triangle ABC with side lengths a , b , and c (with the side of length a opposite vertex A , the side of length b opposite vertex B , and the side of length c opposite vertex C). Explain why the law of sines holds for all triangles.

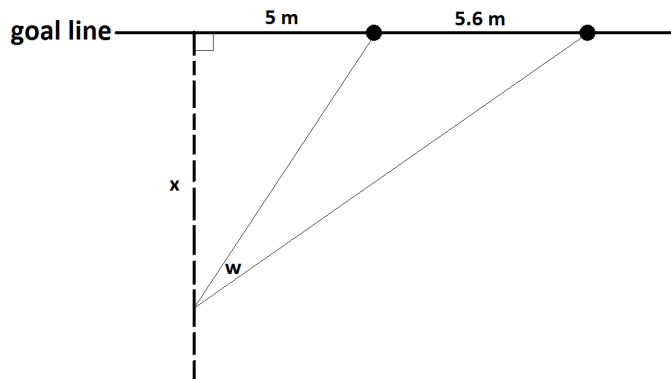
- c. Prove that $c^2 = a^2 + b^2 - 2ab \cos(C)$ for the triangle shown in the original diagram.

3. Beatrice is standing 20 meters directly east of Ari, and Cece is standing 15 meters directly northeast of Beatrice.
- a. To one decimal place, what is the distance between Ari and Cece?
- b. To one decimal place, what is the measure of the smallest angle in the triangle formed by Ari, Beatrice, and Cece?

- 4.
- a. Is it possible to construct an inverse to the sine function if the domain of the sine function is restricted to the set of real values between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$? If so, what is the value of $\sin^{-1}\left(\frac{1}{2}\right)$ for this inverse function? Explain how you reach your conclusions.
- b. Is it possible to construct an inverse to the cosine function if the domain of the cosine function is restricted to the set of real values between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$? If so, what is the value of $\cos^{-1}\left(-\frac{1}{2}\right)$ for this inverse function? Explain how you reach your conclusions.

- c. Is it possible to construct an inverse to the tangent function if the domain of the tangent function is restricted to the set of real values between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$? If so, what is the value of $\tan^{-1}(-1)$ for this inverse function? Explain how you reach your conclusions.

5. The diagram shows part of a rugby union football field. The goal line (marked) passes through two goal posts (marked as black circles) set 5.6 meters apart.

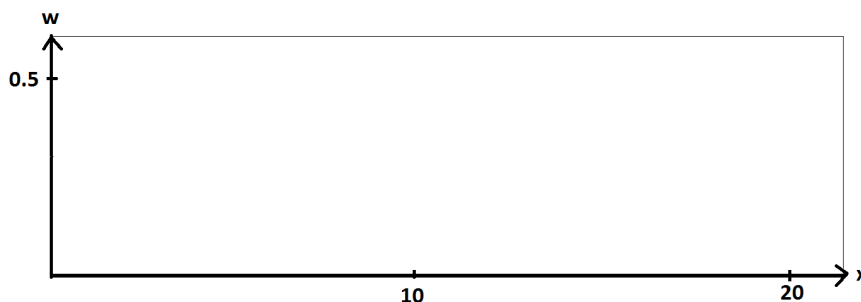


According to the rules of the game, an attempt at a conversion must be taken at a point on a line through the point of touchdown and perpendicular to the goal line. If a touchdown occurred 5 meters to one side of a goal post on the goal line, for example, the dashed line in the diagram indicates the line on which the conversion must be attempted.

Suppose the conversion is attempted at a distance of x meters from the goal line. Let w be the angle (measured in radians) indicated subtended by the goal posts.

- a. Using inverse trigonometric functions, write an expression for w in terms of the distance x .

- b. Using a graphing calculator or mathematics software, sketch a copy of the graph of the angle measure w as a function of x on the axes below. Indicate on your sketch the value of x that maximizes w . What is that maximal angle measure? (Give all your answers to two decimal places.)



- c. In the original diagram, we see that the angle of measure w is one of three angles in an obtuse triangle. To two decimal places, what is the measure of the obtuse angle in that triangle when w has its maximal possible measure?

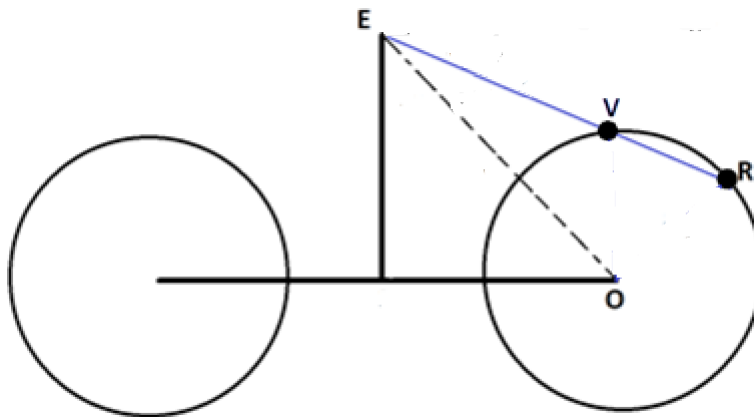
6. While riding her bicycle, Anu looks down for an instant to notice a reflector attached to the front wheel near its rim. As the bicycle moves, the wheel rotates and the position of the reflector relative to the frame of the bicycle changes. Consequently, the angle down from the horizontal that Anu needs to look in order to see the reflector changes with time.

Anu also notices the air valve on the rim of the front wheel tire and observes that the valve and the reflector mark off about one-sixth of the perimeter of the front wheel.

As Anu rides along a straight path, she knows that there will be a moment in time when the reflector, the valve, and her eye will be in line. She wonders what the angle between the horizontal from her eye and the line from her eye to the reflector passing through the valve is at this special moment.

She estimates that the reflector and the valve are each 1.5 feet from the center of the front wheel, that her eye is 6 feet away from the center of that wheel, and that the line between her eye and the wheel center is 45° down from the horizontal.

According to these estimates, what is the measure, to one decimal place in radians, of the angle Anu seeks?



In this diagram, O represents the center of the front wheel, E the location of Anu's eye, and R and V the positions of the reflector and valve, respectively, at the instant R , V , and E are collinear.

A Progression Toward Mastery

Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem, OR an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	a G-SRT.D.9	Student shows little or no understanding of the area of a triangle using trigonometry.	Student writes the altitude in terms of sine but does not verify the area formula.	Student writes the altitude in terms of sine but makes a mistake when verifying the area formula.	Student writes the altitude in terms of sine and verifies the area formula.
	b G-SRT.D.9	Student shows little or no understanding of the area of a triangle using trigonometry.	Student writes the altitude in terms of sine but does not verify the area formula.	Student writes the altitude in terms of sine but makes a mistake when verifying the area formula.	Student writes the altitude in terms of sine and verifies the area formula.
	c G-SRT.D.9	Student shows little or no understanding of the area of a triangle using trigonometry.	Student writes the area formula but does not use it correctly.	Student finds the area with supporting work but does not explain the answer.	Student finds and explains the area correctly.
2	a G-SRT.D.10	Student shows little or no understanding of the concept.	Student attempts an explanation but makes a major mathematical error.	Student uses Thales' theorem and identifies a right angle inscribed in a diameter but the explanation is incomplete.	Student uses Thales' theorem and identifies a right angle inscribed in a diameter leading to a correct explanation.
	b G-SRT.D.10	Student shows little or no understanding of the law of sines.	Student attempts to explain why the law of sines holds but makes a major mathematical mistake.	Student shows knowledge of the law of sines but the explanation is incomplete.	Student completely and correctly explains why the law of sines holds for all triangles.

	c G-SRT.D.10	Student shows little or no understanding of the concept.	Student attempts proof but makes a major mathematical error.	Student shows knowledge of proof but makes minor error in proof.	Student correctly proves the statement.
3	a G-SRT.D.11	Student shows little or no understanding of the law of cosines.	Student uses the law of cosines but makes mathematical mistakes leading to an incorrect answer.	Student uses the law of cosines correctly but does not round properly.	Student uses the law of cosines correctly and rounds answer correctly.
	b G-SRT.D.11	Student shows little or no understanding of the law of sines.	Student knows the shortest angle is A but cannot find A .	Student uses the law of sines to find A correctly but makes a mathematical mistake leading to an incorrect answer.	Student uses the law of sines to find A correctly.
4	a F-TF.B.6	Student shows little or no understanding of inverse trigonometric functions.	Student knows that there is an inverse function on the restricted domain but does not explain why or calculate value.	Student either finds the correct value OR explains the restricted domain correctly.	Student correctly explains the restricted domain of the inverse function and calculates the value.
	b F-TF.B.6	Student shows little or no understanding of inverse trigonometric functions.	Student knows that there is not an inverse function on the restricted domain but does not explain why or calculate the value.	Student explains that there is not an inverse function on the restricted domain but explanation is incomplete.	Student completely explains that there is not an inverse function on the restricted domain.
	c F-TF.B.6	Student shows little or no understanding of inverse trigonometric functions.	Student knows that there is an inverse function on the restricted domain but does not explain why or calculate the value.	Student either finds the correct value OR explains the restricted domain correctly.	Student correctly explains the restricted domain of the inverse function and calculates the value.
5	a F-TF.B.7	Student shows little or no knowledge of inverse trigonometric functions.	Student attempts to write w in terms of x but makes major mathematical mistakes.	Student writes w in terms of x with a minor mathematical mistake.	Student correctly writes w in terms of x .
	b F-TF.B.7	Student shows little or no knowledge of trigonometric functions.	Student sketches the graph but does not identify maximal angle measure.	Student sketches the graph and identifies the maximal angle measure but does not round correctly.	Student sketches the graph and identifies the maximal angle measure rounding correctly.
	c F-TF.B.7	Student shows little or no knowledge of trigonometric functions.	Student attempts to find the obtuse angle but makes major mathematical errors.	Student finds the obtuse angle but a minor mathematical error leads to an incorrect answer.	Student correctly finds the measure of the obtuse angle.

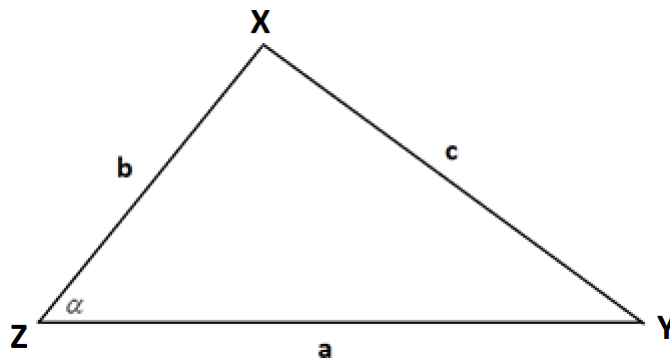
6	G-SRT.D.11	Student shows little or no knowledge of the law of sines.	Student attempts to use the law of sines but makes mathematical errors leading to an incorrect answer.	Student uses the law of sines and finds the measure of the angle but does not round correctly.	Student uses the law of sines, finds the measure of the angle, and rounds correctly.
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Name _____

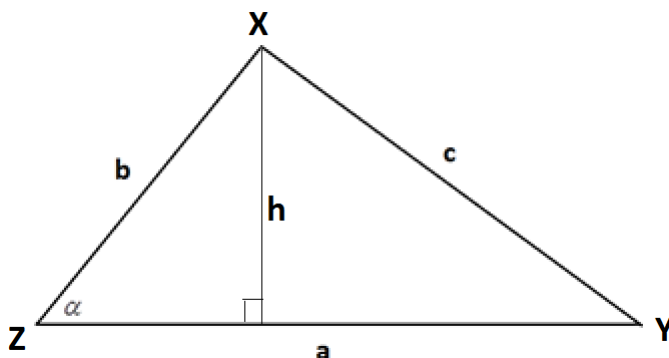
Date _____

1.

- a. In the following diagram, triangle XYZ has side lengths a , b , and c as shown. The angle α indicated is acute. Show that the area A of the triangle is given by $A = \frac{1}{2}ab \sin(\alpha)$.



Draw an altitude as shown. Call its length h .

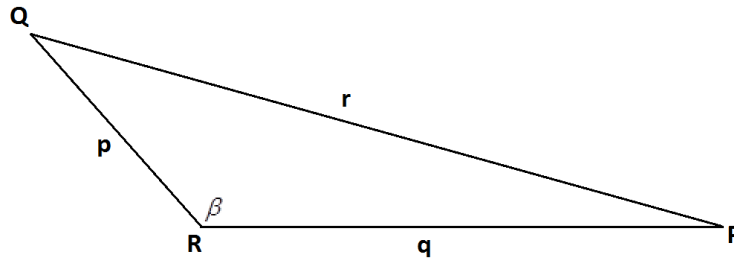


We have $\sin(\alpha) = \frac{h}{b}$, so $h = b \sin(\alpha)$.

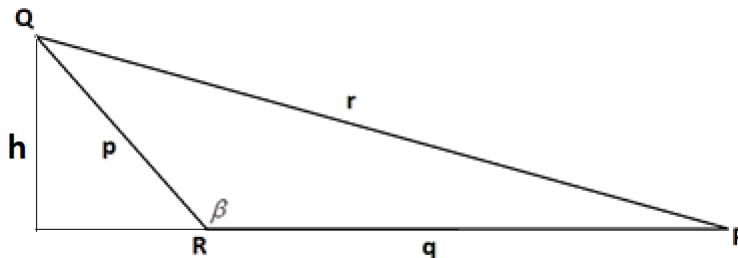
The area A of the triangle is given as “half base times height.” So

$$A = \frac{1}{2} \times a \times b \sin(\alpha) = \frac{1}{2}ab \sin(\alpha).$$

- b. In the following diagram, triangle PQR has side lengths p , q , and r as shown. The angle β indicated is obtuse. Show that the area A of the triangle is given by $A = \frac{1}{2}pq \sin(\beta)$.



Draw in an altitude as shown. Call its length h .



We have $\sin(\pi - \beta) = \frac{h}{p}$, so $h = p \sin(\pi - \beta)$. Since $\sin(\pi - \beta) = \sin(\beta)$, this can be rewritten $h = p \sin(\beta)$.

The area A of the triangle is thus $\frac{1}{2} \times q \times p \sin(\beta) = \frac{1}{2}pq \sin(\beta)$.

- c. To one decimal place, what is the area of the triangle with sides of lengths 10 cm, 17 cm, and 21 cm? Explain how you obtain your answer.

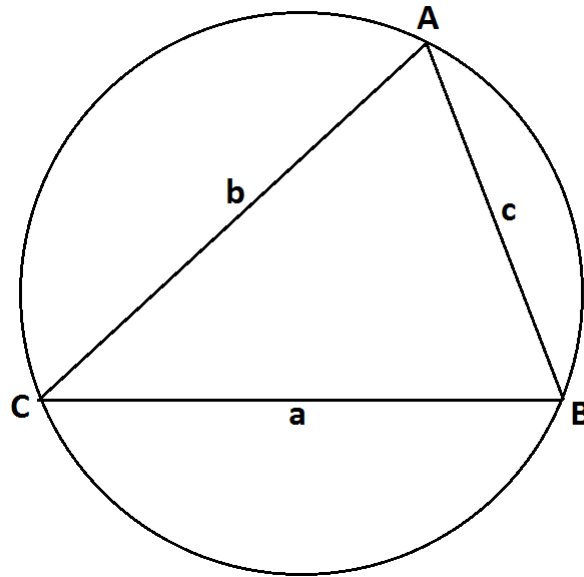
Let θ be the measure of the angle between the sides of lengths 10 cm and 17 cm. By the law of cosines, we have $21^2 = 10^2 + 17^2 - 2 \cdot 10 \cdot 17 \cos(\theta)$. This gives

$$\cos(\theta) = \frac{52}{340} = \frac{13}{85}, \text{ and so } \theta = \cos^{-1}\left(\frac{13}{85}\right) \approx 1.42 \text{ radians.}$$

$$\frac{1}{2} \cdot 10 \cdot 17 \sin(\theta) \approx 85 \sin(1.42) \approx 84.0$$

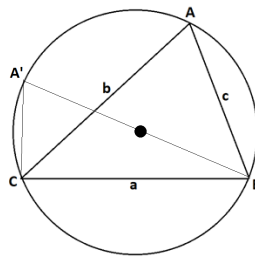
The area of the triangle is 84.0 square centimeters.

2. Triangle ABC with side lengths a , b , and c as shown is circumscribed by a circle with diameter d .



- a. Show that $\frac{a}{\sin(A)} = d$.

Consider the point A' on the circle with $\overline{A'B}$ a diameter of the circle.



By Thales' theorem (an inscribed angle that intercepts a semi-circle is a right angle), $\angle A'CB$ is a right angle. Thus, $\sin(A') = \frac{a}{d}$.

But by the inscribed angle theorem (angles intercepting the same arc are congruent), the inscribed angle at A' has the same measure as the inscribed angle at A . So, $\sin(A') = \sin(A)$, and our equation reads $\sin(A) = \frac{a}{d}$. Rearranging gives $\frac{a}{\sin(A)} = d$.

- b. The law of sines states that $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ for any triangle ABC with side lengths a , b , and c (with the side of length a opposite vertex A , the side of length b opposite vertex B , and the side of length c opposite vertex C). Explain why the law of sines holds for all triangles.

The relationship between a , $\sin(A)$, and d holds for any side of the triangle. So we also have $\frac{b}{\sin(B)} = d$ and $\frac{c}{\sin(C)} = d$. This shows that $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ for a triangle circumscribed by a circle.

As every triangle can be circumscribed by a circle, the law of sines,

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}, \text{ thus holds for all triangles.}$$

- c. Prove that $c^2 = a^2 + b^2 - 2ab \cos(C)$ for the triangle shown in the original diagram.

Draw an altitude as shown for the triangle, and identify the three lengths x , y , and h as shown.

Now $x = b \cos(C)$, $h = b \sin(C)$, and $y = a - x = a - b \cos(C)$.

Applying the Pythagorean theorem to the right triangle on the right, we have

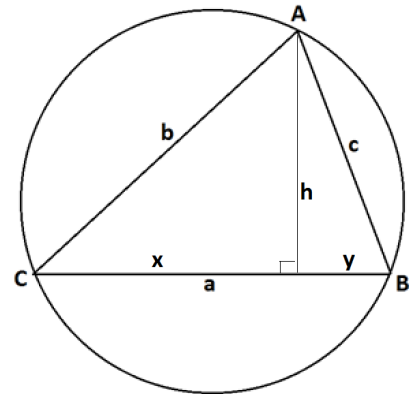
$$\begin{aligned} y^2 + h^2 &= c^2 \\ (a - b \cos(C))^2 + (b \sin(C))^2 &= c^2 \\ a^2 - 2ab \cos(C) + b^2 \cos^2(C) + b^2 \sin^2(C) &= c^2 \end{aligned}$$

Using $\cos^2(C) + \sin^2(C) = 1$ this reads

$$a^2 - 2ab \cos(C) + b^2 = c^2$$

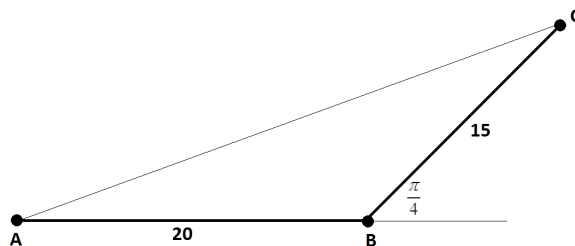
or

$$c^2 = a^2 + b^2 - 2ab \cos(C).$$



3. Beatrice is standing 20 meters directly east of Ari, and Cece is standing 15 meters directly northeast of Beatrice.
- a. To one decimal place, what is the distance between Ari and Cece?

The following diagram depicts the situation described:



By the law of cosines

$$|AC|^2 = 20^2 + 15^2 - 2 \cdot 15 \cdot 20 \cdot \cos\left(\frac{3\pi}{4}\right)$$

$$|AC|^2 = 400 + 225 - 600\left(-\frac{1}{\sqrt{2}}\right)$$

$$|AC|^2 = 625 + 300\sqrt{2}$$

$$|AC| = \sqrt{625 + 300\sqrt{2}} \approx 32.4$$

Thus, the distance between Ari and Cece is approximately 32.4 meters.

- b. To one decimal place, what is the measure of the smallest angle in the triangle formed by Ari, Beatrice, and Cece?

The angle of the smallest measure in a triangle lies opposite the shortest side of the triangle. Thus, we seek the measure of the angle at Ari's position.

By the law of sines

$$\frac{\sin(A)}{15} = \frac{\sin\left(\frac{3\pi}{4}\right)}{|AC|}$$

giving

$$\sin(A) = \frac{15}{\sqrt{2}|AC|} \approx \frac{15}{\sqrt{2} \cdot 32.4} \approx 0.33.$$

Thus, $m\angle A \approx \sin^{-1}(0.33) \approx 0.33$ radian. (This is about 19° .)

4.

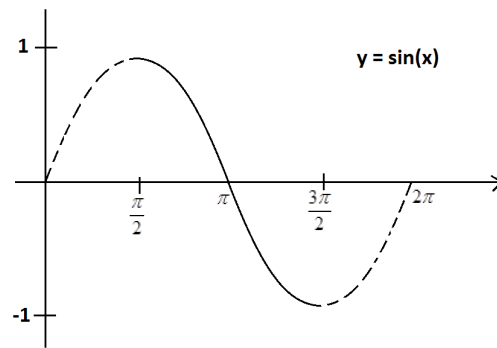
- a. Is it possible to construct an inverse to the sine function if the domain of the sine function is restricted to the set of real values between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$? If so, what is the value of $\sin^{-1}\left(\frac{1}{2}\right)$ for this inverse function? Explain how you reach your conclusions.

We see, when restricted to inputs between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, the graph of $y = \sin(x)$ is strictly decreasing.

Thus for each value between -1 and 1 , there is an input value x within this range for which $\sin(x)$ has this value. That is, there is indeed an inverse function for sine in this restricted domain.

We know that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. So it follows that $\sin\left(\pi + \frac{\pi}{6}\right) = -\frac{1}{2}$. For our inverse function we have

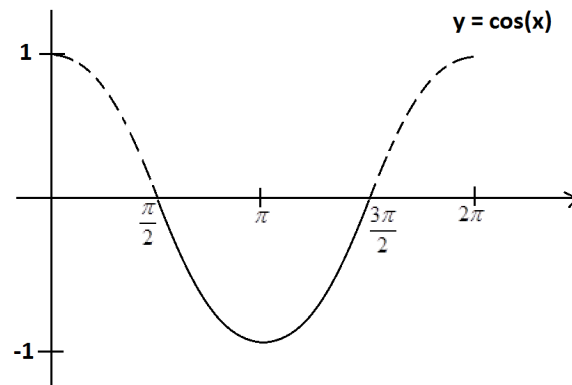
$$\sin^{-1}\left(-\frac{1}{2}\right) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}.$$



- b. Is it possible to construct an inverse to the cosine function if the domain of the cosine function is restricted to the set of real values between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$? If so, what is the value of $\cos^{-1}\left(-\frac{1}{2}\right)$ for this inverse function? Explain how you reach your conclusions.

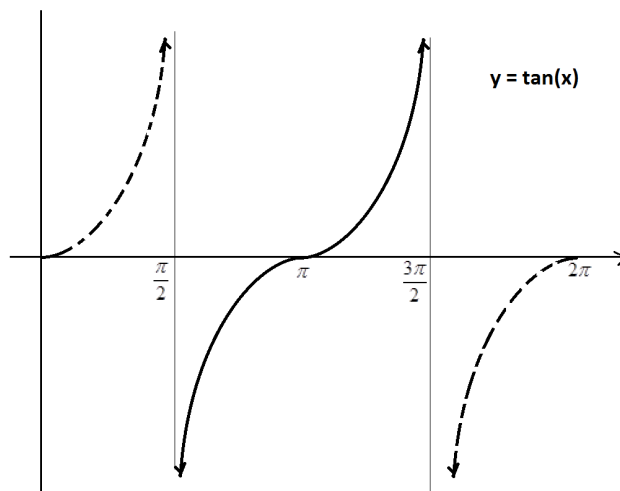
The value of the cosine function is neither strictly increasing nor strictly decreasing on that restricted domain.

There are distinct inputs from the restricted domain that give the same cosine values, and so it is not possible to construct an inverse to the cosine for this domain.



- c. Is it possible to construct an inverse to the tangent function if the domain of the tangent function is restricted to the set of real values between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$? If so, what is the value of $\tan^{-1}(-1)$ for this inverse function? Explain how you reach your conclusions.

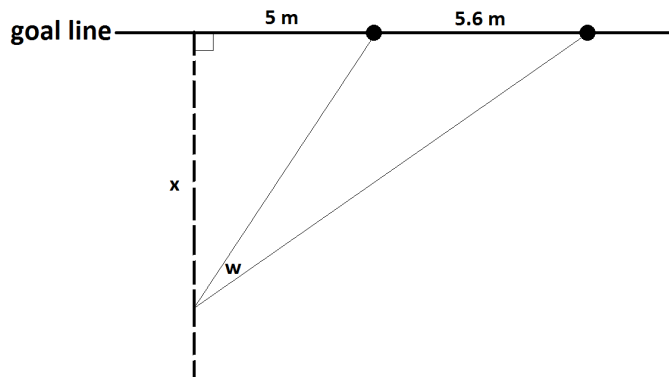
The graph of the tangent function is strictly increasing on the restricted domain:



For each real number y , there is indeed a unique real number x in this restricted domain with $\tan(x) = y$. We can thus construct an inverse function.

We know that $\tan\left(\frac{\pi}{4}\right) = 1$, and so we see that $\tan\left(\pi + \frac{\pi}{4}\right) = -1$. Thus for our inverse function, $\tan^{-1}(-1) = \frac{5\pi}{4}$.

5. The diagram shows part of a rugby union football field. The goal line (marked) passes through two goal posts (marked as black circles) set 5.6 meters apart.

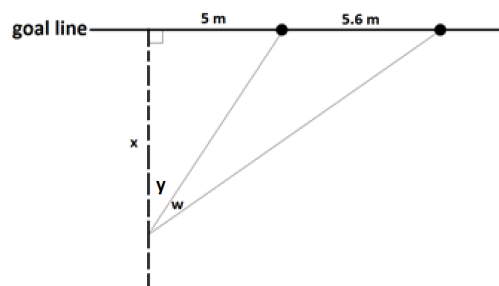


According to the rules of the game, an attempt at a conversion must be taken at a point on a line through the point of touchdown and perpendicular to the goal line. If a touchdown occurred 5 meters to one side of a goal post on the goal line, for example, the dashed line in the diagram indicates the line on which the conversion must be attempted.

Suppose the conversion is attempted at a distance of x meters from the goal line. Let w be the angle (measured in radians) indicated subtended by the goal posts.

- a. Using inverse trigonometric functions, write an expression for w in terms of the distance.

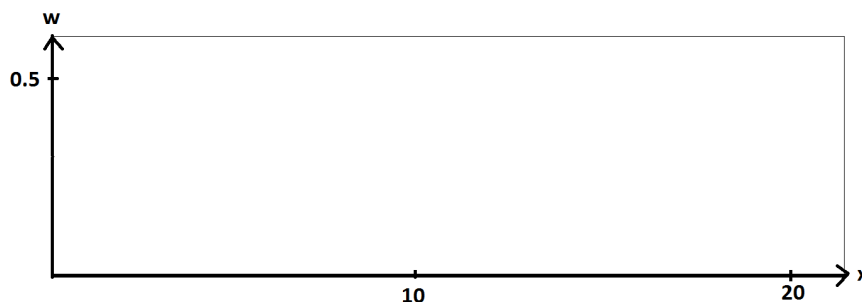
Label the angle y as shown:



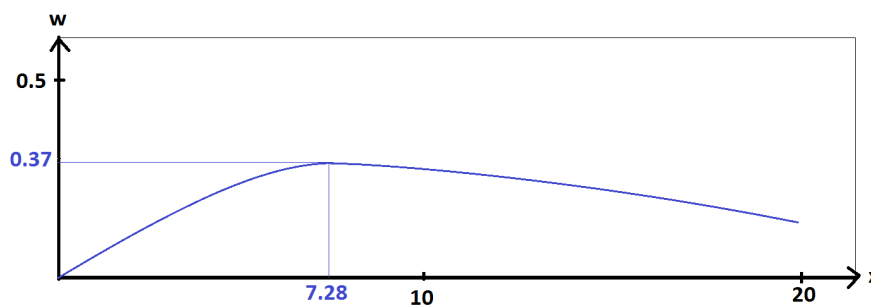
We have $\tan(y) = \frac{5}{x}$ and $\tan(y + w) = \frac{10.6}{x}$ and so

$$w = (y + w) - y = \tan^{-1}\left(\frac{10.6}{x}\right) - \tan^{-1}\left(\frac{5}{x}\right).$$

- b. Using a graphing calculator or mathematics software, sketch a copy of the graph of the angle measure w as a function of x on the axes below. Indicate on your sketch the value of x that maximizes w . What is that maximal angle measure? (Give all your answers to two decimal places.)



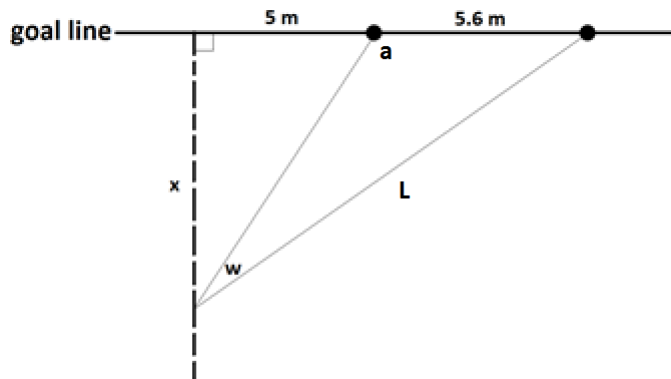
We see



At $x = 7.28$, the angle w has a measure of 0.37 radian. (This is about 21° .)

- c. In the original diagram, we see that the angle of measure w is one of three angles in an obtuse triangle. To two decimal places, what is the measure of the obtuse angle in that triangle when w has its maximal possible measure?

Label the obtuse angle a and the length L as shown.



For $x = 7.28$ and $w = 0.37$, we have

$$L = \sqrt{10.6^2 + 7.28^2} \approx 12.86, \text{ in meters.}$$

By the law of sines,

$$\frac{\sin(a)}{L} = \frac{\sin(w)}{5.6}.$$

$$\text{So } \sin(a) = \frac{L \sin(w)}{5.6} \approx \frac{12.86 \times \sin(0.37)}{5.6} \approx 0.83.$$

$$\text{Thus, } a = \sin^{-1}(0.83) = 0.98 \text{ or } \pi - 0.98.$$

Since we are working with an obtuse angle, we must have $a = \pi - 0.98 \approx 2.16$ radians.
(This is about 124° .)

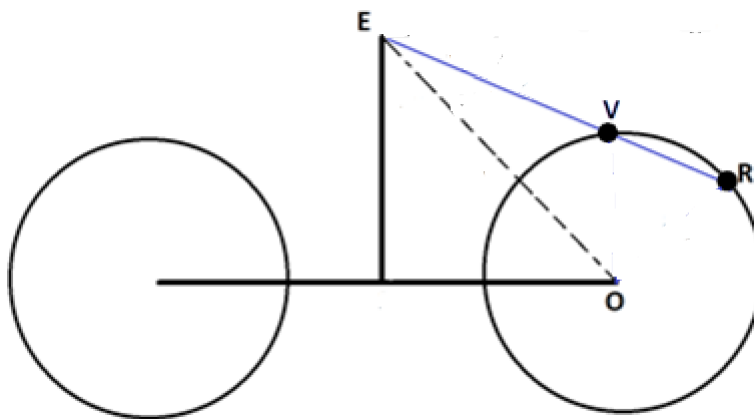
6. While riding her bicycle, Anu looks down for an instant to notice a reflector attached to the front wheel near its rim. As the bicycle moves, the wheel rotates and the position of the reflector relative to the frame of the bicycle changes. Consequently, the angle down from the horizontal that Anu needs to look in order to see the reflector changes with time.

Anu also notices the air valve on the rim of the front wheel tire and observes that the valve and the reflector mark off about one-sixth of the perimeter of the front wheel.

As Anu rides along a straight path, she knows that there will be a moment in time when the reflector, the valve, and her eye will be in line. She wonders what the angle between the horizontal from her eye and the line from her eye to the reflector passing through the valve is at this special moment.

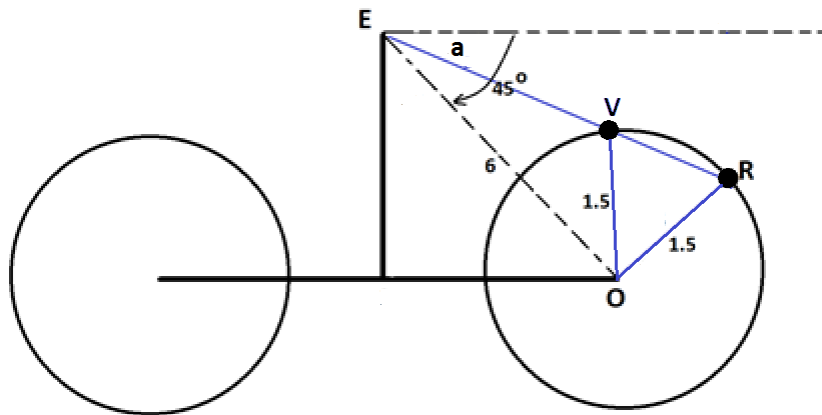
She estimates that the reflector and the valve are each 1.5 feet from the center of the front wheel, that her eye is 6 feet away from the center of that wheel, and that the line between her eye and the wheel center is 45° down from the horizontal.

According to these estimates, what is the measure, to one decimal place in radians, of the angle Anu seeks?



In this diagram, O represents the center of the front wheel, E the location of Anu's eye, and R and V the positions of the reflector and valve, respectively, at the instant R , V , and E are collinear.

The following is a schematic diagram of Anu on her bicycle. The point O is the center of the front wheel, the point E is the location of Anu's eye, and the points R and V are the locations of the reflector and the valve, respectively, on the rim of the front wheel and the instant those two points and E are collinear. We have $|EO| = 6$ feet and $|OR| = |OV| = 1.5$ feet, and we seek the measure of angle a shown.



We are told that the length of the arc between V and R is one-sixth of the perimeter of the wheel. Thus $m\angle VOR = \frac{1}{6} \cdot 2\pi = \frac{\pi}{3}$ radian. Consequently, $m\angle EVO = \frac{2\pi}{3}$.

Looking at triangle EVO , the law of sines gives $\frac{6}{\sin(\frac{2\pi}{3})} = \frac{1.5}{\sin(m\angle VEO)}$, and so

$\sin(m\angle VEO) = \frac{\sin(\frac{2\pi}{3})}{4} = \frac{\sqrt{3}}{8}$. Thus, $m\angle VEO = \sin^{-1}(\frac{\sqrt{3}}{8})$ radians. Since $\angle EVO$ is obtuse, $\angle VEO$ is acute, and we must work with the value of $\sin^{-1}(\frac{\sqrt{3}}{8})$ that corresponds to the measure of an acute angle.

It follows that $m\angle a = \frac{\pi}{4} - \sin^{-1}(\frac{\sqrt{3}}{8}) \approx 0.6$ radians (which is about 34°).