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Towards a measure of vulnerability, tenacity of a Graph

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ABSTRACT

If we think of the graph as modeling a network, the vulnerability measure the resistance of the network to disruption of operation after the failure of certain stations or communication links. Many graph theoretical parameters have been used to describe the vulnerability of communication networks, including connectivity, integrity, toughness, binding number, and tenacity.
In this paper we discuss tenacity and its properties in vulnerability calculation.

Keyword: connectivity, integrity, toughness, binding number, and tenacity.

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Introduction

We consider only finite undirected graphs without loops and multiple edges. Let G be a graph. We denote by V(G), E(G) and $\mid V(G) \mid$ the set of vertices, the set of edges and the order of a graph G, respectively. For a subset S of V(G), let G[S] denote the subgraph of G induced by S. The degree of a vertex v in a graph G is denoted by $d_G(v)$. The *end vertex* v of a graph G is a vertex of degree 1 in G, that is $d_G(v) = 1$.
A k-tree of a connected graph is a spanning tree with maximum degree k. Of course, for k=2, this notion reduces to that of a hamiltonian path.
The concept of tenacity of a graph G was introduced in [2], as a useful measure of the "vulnerability" of G. In [6], we compared integrity, connectivity, binding number, toughness, and tenacity for several classes of graphs. The results suggest that tenacity is a

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most suitable measure of stability or vulnerability in that for many graphs it is best able to distinguish between graphs that intuitively should have different levels of vulnerability. In [3,4,7,8,9,10], they studied more about this new invariant. The tenacity of a graph G, T(G), is defined by
$T(G) = min\{\frac{|S|+\tau(G-S)}{\omega(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G. We define $\tau(G-S)$ to be the number of the vertices in the largest component of the graph G - S, and $\omega(G-S)$ be the number of components of G - S. A connected graph G is called T-tenacious if $\mid S \mid + \tau(G-S) \geq T\omega(G-S)$ holds for any subset S of vertices of G with $\omega(G-S) > 1$. If G is not complete, then there is a largest T such that G is T-tenacious ; this T is the tenacity of G. On the other hand, a complete graph contains no vertex cutset and so it is T-tenacious for every T. Accordingly, we define $T(K_p) = \infty$ for every p $(p \geq 1)$. A set $S \subseteq V(G)$ is said to be a T-set of G if $T(G) = \frac{|S|+\tau(G-S)}{\omega(G-S)}$.
Any undefined terms can be found in the standard references on graph theory,including Bondy and Murty [1].

Theorem 1: If

$$T(G) \geq \frac{\tau(G-S)}{\omega(G-S)} + \frac{1}{k-2}, \qquad \text{with } k \geq 3, \qquad (1)$$

for any vertex cutset S of G, then G has a k-tree.
Lemma 1. The degree of a vertex of A in any k-tree of H is k.
Lemma 2. Suppose that, for some m and n with $m \neq n$, there is a vertex x_m of Kt_m adjacent in H to a vertex x_n in Kt_n. Then at least one of x_m and x_n, say x_m has the following property: if u is an end vertex in a k-tree of $G[V(Kt_m) \cup \{u\}]$, then in that k-tree, x_m has degree k and so also $d_{Kt}(x_m) = k$.

From Lemma 2, we can conclude that there is a k-tree, Kt of H and subsets P and Q of V(H), with P non-empty.
We may assume G_n^m is labeled by 0,1,2,$\cdots$,m. Let n be even, $n = 2r$ and two vertices i,j are adjacent, if $i - r \leq j \leq r$ (where addition is taken modulo m). Now let n and m be odd, $(n > 1)$. Let $n = 2r + 1$, $(r > 0)$. Then G_{2r+1}^m is constructed by first drawing G_{2r}^m, and adding edges joining vertex i to veretex $i + \frac{m+1}{2}$ for $1 \leq i \leq \frac{m-1}{2}$. Note that vertex 0 is adjacent to both vertices $\frac{m-1}{2}$ and $\frac{m+1}{2}$..
Harary in [5] proved the following theorem:
Theorem 2: Graph G_n^m is n-connected.
Through the rest of this paper we will let $n = 2r$ or $n = 2r + 1$ and $m = k(r + 1) + s$ for $0 \leq s \leq r + 1$. So we can indicate that $m \cong s \quad mod(r + 1)$ and $k = \lfloor \frac{m}{r+1} \rfloor$. Also we assume that the graph G_n^m is not complete, $n + 1 < m$.
In [6] we calculate the tenacity of G_{2r}^m by using the following theorem:
Theorem 3: $T(G_{2r}^m) = r + \frac{1+\lceil \frac{s}{k} \rceil}{k}$.
Lemma 3: Let G_n^m be the graph with m and n both odd, $n = 2r + 1$ and $r > 0$. Then $m \cong 1 \quad mod(n + 1)$ if and only if $s = 1$ and k is even.

Lemma 4: Let G_n^m be the graph with m and n both odd, $n = 2r + 1$ and $r > 0$. Then

$$\alpha(G_n^m) = \begin{cases} k & if \quad m \neq 1 \quad mod(n+1) \\ k-1 & if \quad m \cong 1 \quad mod(n+1) \end{cases}$$

Theorem 4: Let G_n^m be the graph with m and n odd, $n = 2r + 1$, then

$$r + \frac{1 + \lceil \frac{s}{k} \rceil}{k} \leq T(G_n^m) \leq \begin{cases} r + \frac{s+1}{k} & ifm \neq 1 \quad mod9n+1) \\ \frac{kr+s+2}{k-1} & ifm \cong 1 \quad mod(n+1) \end{cases}$$

Proof: Let $H = G_n^m$. We proved in [2] that $T(H) \leq \frac{m-\alpha(H)+1}{\alpha(H)}$. Thus by Lemma 4, if $m \neq 1 \quad mod(n+1)$, then
$T(H) = \frac{m-k-1}{k} = \frac{k(r+1)+s-k+1}{k} = r + \frac{s+1}{k}$, and if $m \cong 1 \quad mod(n+1)$, then $T(H) \leq \frac{m-(k-1)+1}{k-1} = \frac{kr+s+2}{k-1}$.
Since $V(G_{2r}^m = V(H)$ and $E(G_{2r}^m) \subseteq E(H)$, then G_{2r}^m is a spanning subgraph of H. We have in [2], $T(G_{2r}^m) \leq T(H)$. Thus by Theorem 3, we have

$$r + \frac{1 + \lceil \frac{s}{k} \rceil}{k} \leq T(G_n^m)$$

Lemma 5: Let G_n^m be the graph with m odd, $n = 2r+1$, $r \geq 2$, $1 < s < r = 1$, and k is even. Then there is an cutset A with kr+1 elements such that number of components is $\omega(G_n^m - A) = k$,and the largest component is $\tau(G_n^m - A) = 2$.
Proof: We may assume G_n^m is labeled by $0, 1, 2, \cdots, m-1$. Let $s < k$, then $s = k - l$ for some l. Since m is odd and k is even, then s is odd. Hence $l = 2t + 1$ and $k = 2q$, for some t and q. Thus $s = k - l = 2q - 2t - 1$, $q > t + 1$. Thus $m = s(r + 2) + l(r + 1)$. Choose the sets

$$D = \{1, 2, \cdots, \frac{m-1}{2}\} = \{1, 2, \cdots, qr + 2q - t - 1\},$$

$$F = \{\frac{m+1}{2}, \cdots, m-1\} = \{\frac{m+1}{2}, \cdots, \frac{m+1}{2} + qr + 2q - t - 2\}.$$

Hence $\mid D \mid = \mid F \mid$. Consider the cutset $A = C \cup Y$, such that C is the union of the set,

$$\begin{gathered} \{1, 2, \cdots, r\}, \{r+3, \cdots, 2r+2\}, \cdots, \\ \{(q-t-2)r + 2(q-t-1)r + 2(q-t-1) - 1, \cdots, (q-t-1)r + 2(q-t-1) - 2\}, \\ \{(q-t-1)r + 2(q-t-1) + 1, \cdots, (q-t)r + 2(q-t-1)\}, \\ \{(q-t)r + 2(q-t), \cdots, (q-t+1) + 2(q-t) - 1\}, \cdots, \\ \{(q-1)r + 2q - t - 1, \cdots, qr + 2q - t - 2\}, \end{gathered}$$

and $Y = \{\frac{m+1)}{2}, \cdots, \frac{m+1}{2} + r - 1\} \cup Y_1 \cup Y_2 \cup \{0\}$ where

$$Y_1 = \cup\{\{\frac{m+1}{2} + cr + 2c - 1, \cdots, \frac{m+1}{2} + (c+1)r + 2c - 2\} | 1 \leq c \leq q - t - 1\}$$
$$Y_2 = \cup\{\{\frac{m+1}{2} + hr + h + q - t - 1, \cdots, \frac{m+1}{2} + (h+1)r + h + q - t - 2\} | q - t \leq h \leq q - 1\}$$

Hence the cutset C is a subset of D and Y is a subset of B. Now consider W =D-C. Thus $W = W' \cup \{qr + 2q - t - 1\}$ wheren W'is the union of the sets

$$\{r+1, r+2\}, \{2r+3, 2r+4\}, \cdots,$$
$$\{(q-t-1)r + 2(q-t-1) - 1, (q-t-1)r + 2(q-t-1)\},$$
$$\{(q-t)r + (q-t) + q - t - 1\}, \cdots, \{(q-1)r + (q-1) + q - t - 1\}.$$

Thus $C = D - W$ is the union of the q set of r consecutive vertices. The doubleton sets in W' are of the form $\{dr + 2d - 1, dr + 2d\}$ where $1 \leq d \leq q - t - 1$.The singlton sets in W' are of the form $\{fr + f + q - t - 1\}$ where $q - t \leq f \leq q - 1$. We want to prove that if $x \in W$, then $x + \frac{m+1}{2} \in Y$.If x is a first element of doubleton set in W' then x is in the form $x = dr = 2d - 1$ where $1 \leq d \leq q - t - 1$. Hence $x + \frac{m+1}{2} = \frac{m+1}{2} + dr + 2d - 1 \in Y_1$. Similarly $x + 1 + \frac{m+1}{2} \in Y_1$, since $r \geq 2$. If x is in a singleton set, then x is of the form $x = fr + f + q - t - 1$ where $q - t \leq f \leq q - 1$. Hence $x + \frac{m+1}{2} + fr + f + q - t - 1 \in Y_2$. If $x = qr + 2q - t - 1$, then

$$x + \tfrac{m+1}{2} = qr + 2q - t - 1 + (qr + 2q - t\}$$
$$= 2q(r+1) + (2q - 2t) - 1$$
$$= k(r+1) + k - 1 = k(r+1) + s = m + 0 \in Y.$$

Thus each element in component set $W \subseteq D$ has adjacent element in cutset $Y \subseteq B$. Therefore the cardinality of cutset A is equal to $rk + 1$ and number of components, $\omega(G_n^m - A) = k$ and the largest component $\tau(G_n^m - A) = 2$. $\square$

Theorem 5: Let G_n^m be the graph with $n = 2r + 1$,m odd, k even, $k > 2$, $1 < s < r + 1$ and $s < k$. Then

$$r + \tfrac{2}{k} \leq T(G_n^m) \leq r + \tfrac{3}{k}.$$

Proof: By Theorem 4, we have $r + \frac{2}{k} \leq T(G_n^m)$. By Lemma 5, A is a cutset with kr+1 elements, $\mid A \mid= kr + 1$, $\tau(G_n^m - A) = 2$, and $\omega(G_n^m - A) = k$. Hence $\frac{|A| + \tau(G_n^m - A)}{\omega(G_n^m - A)} = \frac{kr+1+2}{k} = r + \frac{3}{k}$. Therefore $r + \frac{2}{k} \leq T(G_n^m) \leq r + \frac{3}{k}$. $\square$

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