Tensor Calculus

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Preface

These notes are the second part of the tensor calculus documents which started with the previous set of introductory notes [11]. In the present text, we continue the discussion of selected topics of the subject at a higher level expanding, when necessary, some topics and developing further concepts and techniques. The purpose of the present text is to solidify, generalize, fill the gaps and make more rigorous what have been presented in the previous set of notes and to prepare the ground for the next set of notes. Unlike the previous notes which are largely based on a Cartesian approach, the present notes are essentially based on assuming an underlying general curvilinear coordinate system. We also provide a small sample of proofs to familiarize the reader with the tensor techniques inline with the tutorial nature of the present text; however, due to the limited objectives of the present text we do not provide comprehensive proofs and complete theoretical foundations for the provided materials.

We generally follow the same conventions and notations used in the previous set of notes with the following amendments:

• We use capital Gamma, Γ^{i}_{jk}, for the Christoffel symbols of the second kind which is more elegant and readable than the curly bracket notation $\left\{ {i \atop jk} \right\}$ that we used in the previous notes insisting that, despite the suggestive appearance of the Gamma notation, the Christoffel symbols are not tensors in general.

• Due to the restriction of using real (non-complex) quantities, as stated in the previous notes, all arguments of real-valued functions, like square roots and logarithmic functions, are assumed to be non-negative by taking the absolute value, if necessary, without using the absolute value symbol, as done by some authors. This is to simplify the notation and avoid confusion with the determinant notation.

• We generalize the partial derivative notation so that ∂_i can symbolize the partial derivative with respect to the u^i coordinate of general curvilinear systems and not just for

Cartesian coordinates which are usually denoted by x^i. The type of coordinates, being Cartesian or general or otherwise, will be determined by the context which should be obvious in all cases.

• The summation symbol (i.e. $\sum$) is used in most cases when a summation is needed but the summation convention conditions do not apply or there is an ambiguity about it, e.g. when an index is repeated more than twice or a twice-repeated index is in an upper or lower state in both positions or a summation index is not repeated visually because it is part of a squared symbol.

• "Tensor" and "Matrix" are not the same; however for ease of expression they are used sometimes interchangeably and hence some tensors may be referred to as matrices meaning the matrix representing the tensor.

• In the present text, all coordinate transformations are assumed to be continuous, single valued and invertible.

Contents

1 Coordinate Systems, Spaces and Transformations

• The focus of this section is coordinate systems, their types and transformations as well as some general properties of spaces which are needed for the development of the concepts and techniques of tensor calculus in the present and forthcoming notes.

1.1 Coordinate Systems

• In simple terms, a coordinate system is a mathematical device, essentially of geometric nature, used by an observer to identify the location of points and objects and describe events in generalized space which may include space-time.

• The coordinates of a system can have the same or different physical dimensions. An example of the first is the Cartesian system where all the coordinates have the dimension of length, while examples of the second include the cylindrical and spherical systems where some coordinates have the dimension of length while others are dimensionless.

• Generally, the physical dimensions of the components and basis vectors of the covariant and contravariant forms of a tensor are different.

1.2 Spaces

• A Riemannian space is a manifold characterized by the existing of a symmetric rank-2 tensor called the metric tensor. The components of this tensor, which can be in covariant (g_{ij}) or contravariant (g^{ij}) forms, are in general continuous variable functions of coordinates, i.e. $g_{ij} = g_{ij}(u^1, u^2, \ldots, u^n)$ and $g^{ij} = g^{ij}(u^1, u^2, \ldots, u^n)$ where u^i symbolize general coordinates. This tensor facilitates, among other things, the generalization of lengths and distances in general coordinates where the length of an element of arc, ds, is defined by:

$$(ds)^2 = g_{ij} du^i du^j \qquad (1)$$

In the special case of a Euclidean space coordinated by a rectangular system, the metric becomes the identity tensor, that is:

$$g_{ij} = g^{ij} = g^i_j = \delta_{ij} = \delta^{ij} = \delta^i_j \tag{2}$$

• The metric of a Riemannian space may be called the Riemannian metric. Similarly, the geometry of the space may be described as a Riemannian geometry.

• All spaces dealt with in the present notes are Riemannian with well-defined metrics.

• A manifold or space is dubbed "flat" when it is possible to find a coordinate system for the space with a diagonal metric tensor whose all diagonal elements are ± 1; the space is called "curved" otherwise. Examples of flat space are the 3D Euclidean space coordinated by a rectangular Cartesian system whose metric tensor is diagonal with all the diagonal elements being $+1$, and the 4D Minkowski space-time whose metric is diagonal with elements of ± 1. Examples of curved space is the 4D space-time of general relativity in the presence of matter and energy.

• When all the diagonal elements of the metric tensor of a flat space are $+1$, the space and the coordinate system may be described as homogeneous.

• An nD manifold is Euclidean *iff* $R_{ijkl} = 0$ where R_{ijkl} is the Riemann tensor (see § 5.1); otherwise the manifold is curved to which the general Riemannian geometry applies.

• A "field" is a function of the position vector over a region of space. Scalars, vectors and tensors may be defined on a single point of the space or over an extended region of the space; in the latter case we have scalar fields, vector fields and tensor fields, e.g. temperature field, velocity field and stress field respectively.

• In metric spaces, the physical quantities are independent of the form of description, being covariant or contravariant, as the metric tensor facilitates the transformation between the different forms; hence making the description objective.

1.3 Transformations

• In general terms, a transformation from an nD space to another nD space is a correlation that maps a point from the first space (original) to a point in the second space (transformed) where each point in the original and transformed spaces is identified by n independent variables or coordinates. To distinguish between the two sets of coordinates in the two spaces, the coordinates of the points in the transformed space may be notated with barred symbols, e.g. $(\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^n)$ or $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$ where the superscripts and subscripts are indices, while the coordinates of the points in the original space are notated with unbarred similar symbols, e.g. $(u^1, u^2, \ldots, u^n)$ or $(u_1, u_2, \ldots, u_n)$. Under certain conditions, which will be clarified later, such a transformation is unique and hence an inverse transformation from the transformed space to the original space is also defined. Mathematically, each one of the direct and inverse transformations can be regarded as a correlation expressed by a set of equations in which each coordinate in one space is considered as a function of the coordinates in the other space. Hence the transformations between the two sets of coordinates in the two spaces can by expressed mathematically by the following two sets of independent relations:

$$\bar{u}^i = \bar{u}^i(u^1, u^2, \ldots, u^n) \qquad \& \qquad u^i = u^i(\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^n) \tag{3}$$

where $i = 1, 2, \ldots, n$ with n being the space dimension. The independence of the above relations is guaranteed *iff* the Jacobian of the transformation does not vanish on any point in the space (see about Jacobian the forthcoming points). An alternative to viewing the transformation as a mapping between two different spaces is to view it as a correlation of the same point in the same space but observed from two different coordinate frames of reference which are subject to a similar transformation. The following points will be largely based on the latter view.

• As far as the notation is concerned, there is no fundamental difference between the barred and unbarred systems and hence the notation can be interchanged.

• An injective transformation maps any two distinct points of the original space, $\mathbf{r}_1$ and $\mathbf{r}_2$, onto two distinct points of the transformed space, $\bar{\mathbf{r}}_1$ and $\bar{\mathbf{r}}_2$. The image of an injective transformation, $\bar{\mathbf{r}}$, is regarded as coordinates for the point, and the collection of all such coordinates of the space points may be considered as a representation of a coordinate system for the space. If the mapping from an original rectangular system is linear, the coordinate system obtained from such a transformation is called "affine". Coordinate systems which are not affine are described as "curvilinear" such as cylindrical and spherical systems.

• The following $n \times n$ matrix of n^2 partial derivatives of the barred coordinates with respect to the unbarred coordinates is called the "Jacobian matrix" of the transformation between the barred and unbarred systems:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \bar{u}^1}{\partial u^1} & \frac{\partial \bar{u}^1}{\partial u^2} & \cdots & \frac{\partial \bar{u}^1}{\partial u^n} \\ \frac{\partial \bar{u}^2}{\partial u^1} & \frac{\partial \bar{u}^2}{\partial u^2} & \cdots & \frac{\partial \bar{u}^2}{\partial u^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \bar{u}^n}{\partial u^1} & \frac{\partial \bar{u}^n}{\partial u^2} & \cdots & \frac{\partial \bar{u}^n}{\partial u^n} \end{bmatrix} \tag{4}$$

while its determinant:

$$J = \det(\mathbf{J}) \tag{5}$$

is called the "Jacobian" of the transformation.

• Barred and unbarred in the definition of Jacobian should be understood in a general sense not just as two labels since the Jacobian is not restricted to transformations between two systems of the same type but labeled as barred and unbarred. In fact the two coordinate systems can be fundamentally different in nature and not of the same type such as Cartesian and general curvilinear. The Jacobian matrix and determinant represent any transformation by the above partial derivative matrix system between two coordinates defined by two different sets of coordinate variables not necessarily as barred and unbarred. The objective of defining the Jacobian as between barred and unbarred systems is generality and clarity.

• The transformation from the unbarred coordinate system to the barred coordinate system is bijective[1] *iff* $J \neq 0$ on any point in the transformed region of the space. In this case the inverse transformation from the barred to the unbarred system is also defined and bijective and is represented by the inverse of the Jacobian matrix:[2]

$$\bar{\mathbf{J}} = \mathbf{J}^{-1} \tag{6}$$

Consequently, the Jacobian of the inverse transformation, being the determinant of the inverse Jacobian matrix, is the reciprocal of the Jacobian of the original transformation:

$$\bar{J} = \frac{1}{J} \tag{7}$$

• A coordinate transformation is admissible *iff* the transformation is bijective with non-vanishing Jacobian and the transformation function is of class C^2.[3]

• "Affine tensors" are tensors that correspond to admissible linear coordinate transformations from an original rectangular system of coordinates.

• Coordinate transformations are described as "proper" when they preserve the handed-

[1] Bijective transformation means injective (one-to-one) and surjective (onto) mapping.

[2] Notationally, there is no fundamental difference between the barred and unbarred systems and hence the labeling is rather arbitrary and can be interchanged. Therefore, the Jacobian may be notated as barred over unbarred or the other way around. Yes, in a specific context when one of these is labeled as the Jacobian, the other one should be labeled as the inverse Jacobian to distinguish between the two opposite Jacobians and their corresponding transformations.

[3] C^n continuity condition means that the function and all its first n partial derivatives do exist and are continuous in their domain of definition. Also, some authors impose a weaker condition of being of class C^1.

ness (right- or left-handed) of the coordinate system and "improper" when they reverse the handedness. Improper transformations involve an odd number of coordinate axes inversions through the origin.

• Inversion of axes may be called improper rotation while ordinary rotation is described as proper rotation.

• Transformations of coordinates can be active, when they change the state of the observed object such as rotating the object in the space, or passive when they are based on keeping the state of the object and changing the state of the coordinate system which the object is observed from. In brief, the subject of an active transformation is the object while the subject of a passive transformation is the coordinate system.

• An object that does not change by admissible coordinate transformations is described as "invariant" such as the value of a true scalar and the length of a vector.

• As there are essentially two different types of basis vectors, namely tangent vectors of covariant nature and gradient vectors of contravariant nature, there are two main types of non-scalar tensors: contravariant and covariant tensors which are based on the type of the employed basis vectors. Tensors of mixed type employ in their definition mixed basis vectors of the opposite type to the corresponding indices of their components. As indicated earlier, the transformation between these different types is facilitated by the metric tensor.

• A product or composition of coordinate transformations is a succession of transformations where the output of one transformation is taken as the input to the next transformation. In such cases, the Jacobian of the product is the product of the Jacobians of the individual transformations of which the product is made.

• The collection of all admissible coordinate transformations with non-vanishing Jacobian form a group, that is they satisfy the properties of closure, associativity, identity and inverse. Hence, any convenient admissible coordinate system can be chosen as the point of entry since other systems can be reached, if needed, through the set of admissible transformations. This is the cornerstone of building covariant physical theories which are independent of the subjective choice of coordinate systems and reference frames.

• Transformation of coordinates is not a commutative operation.

1.4 Coordinate Surfaces and Curves

• The surfaces of constant coordinates at a certain point of the space meet to form curves (i.e. curves of intersection of these surfaces in pairs). The coordinate curves are these curves of mutual intersection of the surfaces of constant coordinates of the curvilinear system.

• The above transformation equations (Eq. 3) are used to define the set of surfaces of constant coordinates and coordinate curves of mutual intersection of these surfaces. These coordinate surfaces and curves play a crucial role in the formulation and development of this subject.

• The coordinate axes of a coordinate system can be rectilinear, and hence the coordinate curves are straight lines and the surfaces of constant coordinates are planes, as in the case of rectangular Cartesian systems, or curvilinear, and hence the coordinate curves are generalized curved paths and the surfaces of constant coordinates are generalized curved surfaces, as in the case of cylindrical and spherical systems.

• In curvilinear coordinate systems, some or all of the coordinate surfaces are not planes and some or all of the coordinate lines are not straight lines.

• Orthogonal coordinate systems are those for which the vectors tangent to the coordinate curves, as well as the vectors normal to the surfaces of constant coordinates, are mutually perpendicular at all points of the space. Consequently, in orthogonal coordinates, the coordinate surfaces are mutually perpendicular and the coordinate lines are also perpendicular at the point of intersection.

• In orthogonal coordinate systems, the corresponding covariant and contravariant basis vectors at any given point in the space are in the same direction, i.e. the tangent vector to a particular coordinate curve u^i at a certain point and the gradient vector normal to the surface of constant u^i at the same point have the same direction although they may be of different length.

• A necessary and sufficient condition for a coordinate system to be orthogonal is that its metric tensor is diagonal.

• An admissible coordinate transformation from a Cartesian system defines another Cartesian system if the transformation is linear, and defines a curvilinear system if the transformation is nonlinear.

1.5 Scale Factors

• Scale factors (usually symbolized with $h_1, h_2, \ldots h_n$) of a coordinate system are those factors which are required to multiply the coordinate differentials to obtain distances traversed during a change in the coordinate of that magnitude, e.g. ρ in the plane polar coordinate system which multiplies the differential of the polar angle $d\phi$ to obtain the distance L traversed by a change of magnitude $d\phi$ in the polar angle which is $L = \rho\, d\phi$. They are also used to normalize the basis vectors (refer to the previous notes [11] and forthcoming notes).

• The scale factors for the Cartesian, cylindrical and spherical coordinate systems in 3D spaces are given in Table 1.

• The scale factors are also used in the expressions for the differential elements of arc, surface and volume in general orthogonal coordinates, as described in § 2.3.

Table 1: The scale factors for the three most commonly used orthogonal coordinate systems in 3D spaces. The squares of these entries and the reciprocals of these squares give the diagonal elements of the covariant and contravariant metric tensors g_{ij} and g^{ij} respectively of these systems.

	Cartesian (x, y, z)	Cylindrical (ρ, ϕ, z)	Spherical (r, θ, ϕ)
h_1	1	1	1
h_2	1	ρ	r
h_3	1	1	$r \sin\theta$

1.6 Basis Vectors in General Curvilinear Systems

• The vectors providing the basis set for a coordinate system, which are not necessarily of unit length or mutually orthogonal, are of covariant type when they are tangent to the coordinate curves, and of contravariant type when they are perpendicular to the local surfaces of constant coordinates. Formally, the covariant and contravariant basis vectors are defined respectively by:

$$\mathbf{E}_i = \frac{\partial \mathbf{r}}{\partial u^i} \qquad \& \qquad \mathbf{E}^i = \nabla u^i \tag{8}$$

where $\mathbf{r}$ is the position vector in Cartesian coordinates $(x^1, x^2, \ldots)$, and u^i are generalized curvilinear coordinates.

• In general curvilinear coordinate systems, the covariant and contravariant basis sets, $\mathbf{E}_i$ and $\mathbf{E}^i$, are functions of coordinates, i.e.

$$\mathbf{E}_i = \mathbf{E}_i\left(u^1, \ldots, u^n\right) \qquad \& \qquad \mathbf{E}^i = \mathbf{E}^i\left(u^1, \ldots, u^n\right) \tag{9}$$

• Like other vectors, the covariant and contravariant basis vectors are related to each other through the metric tensor, that is:

$$\mathbf{E}_i = g_{ij}\mathbf{E}^j \qquad \& \qquad \mathbf{E}^i = g^{ij}\mathbf{E}_j \tag{10}$$

• The covariant and contravariant basis vectors are reciprocal basis systems, and hence in a 3D space with a right-handed coordinate system (u_1, u_2, u_3) they are linked by the following relations:

$$\mathbf{E}^1 = \frac{\mathbf{E}_2 \times \mathbf{E}_3}{\mathbf{E}_1 \cdot (\mathbf{E}_2 \times \mathbf{E}_3)}, \qquad \mathbf{E}^2 = \frac{\mathbf{E}_3 \times \mathbf{E}_1}{\mathbf{E}_1 \cdot (\mathbf{E}_2 \times \mathbf{E}_3)}, \qquad \mathbf{E}^3 = \frac{\mathbf{E}_1 \times \mathbf{E}_2}{\mathbf{E}_1 \cdot (\mathbf{E}_2 \times \mathbf{E}_3)} \tag{11}$$

$$\mathbf{E}_1 = \frac{\mathbf{E}^2 \times \mathbf{E}^3}{\mathbf{E}^1 \cdot (\mathbf{E}^2 \times \mathbf{E}^3)}, \qquad \mathbf{E}_2 = \frac{\mathbf{E}^3 \times \mathbf{E}^1}{\mathbf{E}^1 \cdot (\mathbf{E}^2 \times \mathbf{E}^3)}, \qquad \mathbf{E}_3 = \frac{\mathbf{E}^1 \times \mathbf{E}^2}{\mathbf{E}^1 \cdot (\mathbf{E}^2 \times \mathbf{E}^3)} \tag{12}$$

• The relations in the last point may be expressed in a more compact form as follow:

$$\mathbf{E}^i = \frac{\mathbf{E}_j \times \mathbf{E}_k}{\mathbf{E}_i \cdot (\mathbf{E}_j \times \mathbf{E}_k)} \qquad \& \qquad \mathbf{E}_i = \frac{\mathbf{E}^j \times \mathbf{E}^k}{\mathbf{E}^i \cdot (\mathbf{E}^j \times \mathbf{E}^k)} \tag{13}$$

where i, j, k take respectively the values $1, 2, 3$ and the other two cyclic permutations (i.e. $2, 3, 1$ and $3, 1, 2$).

• The magnitude of the scalar triple product $\mathbf{E}_i \cdot (\mathbf{E}_j \times \mathbf{E}_k)$ represents the volume of the parallelepiped formed by $\mathbf{E}_i$, $\mathbf{E}_j$ and $\mathbf{E}_k$.

• The magnitudes of the basis vectors in general orthogonal coordinates are given by:

$$|\mathbf{E}_i| = h_i \qquad \& \qquad \left|\mathbf{E}^i\right| = \frac{1}{h_i} \tag{14}$$

where h_i is the scale factor for the i^{th} coordinate.

• The base vectors in the barred and unbarred general curvilinear coordinate systems are related by the following transformation rules:

$$\begin{aligned} \mathbf{E}_i &= \frac{\partial \bar{u}^j}{\partial u^i} \bar{\mathbf{E}}_j \qquad & \& \qquad \bar{\mathbf{E}}_i &= \frac{\partial u^j}{\partial \bar{u}^i} \mathbf{E}_j \\ \mathbf{E}^i &= \frac{\partial u^i}{\partial \bar{u}^j} \bar{\mathbf{E}}^j \qquad & \& \qquad \bar{\mathbf{E}}^i &= \frac{\partial \bar{u}^i}{\partial u^j} \mathbf{E}^j \end{aligned} \tag{15}$$

where the indexed u and $\bar{u}$ represent the coordinates in the unbarred and barred systems respectively. The transformation rules for the components can be straightforwardly concluded from the above rules; for example for a vector $\mathbf{A}$ which can be represented covariantly and contravariantly in the unbarred and barred systems as:

$$\begin{aligned} \mathbf{A} &= \mathbf{E}^i A_i = \bar{\mathbf{E}}^i \bar{A}_i \\ \mathbf{A} &= \mathbf{E}_i A^i = \bar{\mathbf{E}}_i \bar{A}^i \end{aligned} \tag{16}$$

the transformation equations of its components between the two systems are given respectively by:

$$\begin{aligned} A_i &= \frac{\partial \bar{u}^j}{\partial u^i} \bar{A}_j \qquad & \& \qquad \bar{A}_i &= \frac{\partial u^j}{\partial \bar{u}^i} A_j \\ A^i &= \frac{\partial u^i}{\partial \bar{u}^j} \bar{A}^j \qquad & \& \qquad \bar{A}^i &= \frac{\partial \bar{u}^i}{\partial u^j} A^j \end{aligned} \tag{17}$$

These transformation rules can be easily extended to higher rank tensors of different variance types, as detailed in the introductory notes [11].

• For a 3D manifold with a right-handed curvilinear coordinate system, we have:

$$\mathbf{E}_1 \cdot (\mathbf{E}_2 \times \mathbf{E}_3) = \sqrt{g} \qquad \& \qquad \mathbf{E}^1 \cdot \left(\mathbf{E}^2 \times \mathbf{E}^3\right) = \frac{1}{\sqrt{g}} \tag{18}$$

where g is the determinant of the covariant metric tensor, i.e.

$$g = \det(g_{ij}) = |g_{ij}| \tag{19}$$

• Because $\mathbf{E}_i \cdot \mathbf{E}_j = g_{ij}$ (Eq. 51) we have:

$$\mathbf{J}^T \mathbf{J} = [g_{ij}] \tag{20}$$

where $\mathbf{J}$ is the Jacobian matrix transforming between Cartesian and generalized coordinates, the superscript T represents matrix transposition, $[g_{ij}]$ is the matrix representing the covariant metric tensor and the product on the left is a matrix product as defined in linear algebra which is equivalent to a dot product in tensor algebra.

• Considering Eq. 20, the relation between the determinant of the metric tensor and the Jacobian is given by:

$$g = J^2 \tag{21}$$

where J ($= \left|\frac{\partial x}{\partial u}\right|$ with x for Cartesian and u for generalized coordinates) is the Jacobian of the transformation.

• As explained earlier, in orthogonal coordinate systems the covariant and contravariant basis vectors, $\mathbf{E}_i$ and $\mathbf{E}^i$, at any specific point of the space are in the same direction, and hence the normalization of each one of these basis sets, by dividing each basis vector by its magnitude, produces identical orthonormal basis sets.[4] This, however, is not true in general curvilinear coordinates where each normalized basis set is different in general from the other.

• When the covariant basis vectors $\mathbf{E}_i$ are mutually orthogonal at all points of the space, we have:

(A) the contravariant basis vectors $\mathbf{E}^i$ are mutually orthogonal as well,

(B) the covariant and contravariant metric tensors, g_{ij} and g^{ij}, are diagonal with non-vanishing diagonal elements, i.e.

$$g_{ij} = g^{ij} = 0 \qquad\qquad (i \neq j) \tag{22}$$

$$g_{ii} \neq 0 \qquad \& \qquad g^{ii} \neq 0 \qquad\qquad (\text{no sum on } i) \tag{23}$$

(C) the diagonal elements of the covariant and contravariant metric tensors are reciprocals,

[4] Consequently, there is no difference between the covariant and contravariant components of tensors with respect to such contravariant and covariant orthonormal basis sets.

i.e.[5]

$$g^{ii} = \frac{1}{g_{ii}} \qquad \text{(no summation)} \tag{24}$$

(D) the magnitude of the contravariant and covariant basis vectors are reciprocals, i.e.

$$\left|\mathbf{E}^i\right| = \frac{1}{|\mathbf{E}_i|} \tag{25}$$

1.7 Covariant, Contravariant and Physical Representations

• So far we are familiar with the covariant and contravariant (including mixed) representations of tensors. There is still another type of representation, that is the physical representation which is the common one in the applications of tensor calculus such as fluid and continuum mechanics.

• The covariant and contravariant basis vectors, as well as the covariant and contravariant components of a vector, do not in general have the same physical dimensions as indicated earlier; moreover, the basis vectors may not have the same magnitude. This motivates the introduction of a more standard form of vectors by using physical components (which have the same dimensions) with normalized basis vectors (which are dimensionless with unit magnitude) where the metric tensor and the scale factors are employed to facilitate this process. The normalization of the basis vectors is done by dividing each vector by its magnitude. For example, the normalized covariant basis vectors of a general coordinate system, $\hat{\mathbf{E}}_i$, are given by:

$$\hat{\mathbf{E}}_i = \frac{\mathbf{E}_i}{|\mathbf{E}_i|} \qquad \text{(no sum on } i\text{)} \tag{26}$$

[5] The comment "no summation" may not be needed in this type of expressions since both indices are of the same variance type in a generally non-Cartesian system.

which for an orthogonal coordinate system becomes:

$$\hat{\mathbf{E}}_i = \frac{\mathbf{E}_i}{\sqrt{g_{ii}}} = \frac{\mathbf{E}_i}{h_i} \qquad \text{(no sum on } i\text{)} \tag{27}$$

where g_{ii} is the i^{th} diagonal element of the covariant metric tensor and h_i is the scale factor of the i^{th} coordinate as described previously. Consequently if the physical components of a vector are notated with a hat, then for an orthogonal system we have:

$$\mathbf{A} = A^i \mathbf{E}_i = \hat{A}^i \hat{\mathbf{E}}_i = \hat{A}^i \frac{\mathbf{E}_i}{\sqrt{g_{ii}}} \qquad \Longrightarrow \qquad \hat{A}^i = \sqrt{g_{ii}} A^i = h_i A^i \qquad \text{(no sum)} \tag{28}$$

Similarly for the contravariant basis vectors we have:

$$\mathbf{A} = A_i \mathbf{E}^i = \hat{A}_i \hat{\mathbf{E}}^i = \hat{A}_i \frac{\mathbf{E}^i}{\sqrt{g^{ii}}} \qquad \Longrightarrow \qquad \hat{A}_i = \sqrt{g^{ii}} A_i = \frac{A_i}{\sqrt{g_{ii}}} = \frac{A_i}{h_i} \qquad \text{(no sum)} \tag{29}$$

where g^{ii} is the i^{th} diagonal element of the contravariant metric tensor. These definitions and processes can be easily extended to tensors of higher ranks.

• The physical components of higher rank tensors are similarly defined as for rank-1 tensors by considering the basis vectors of the coordinated space where similar simplifications apply to orthogonal systems with mutually-perpendicular basis vectors. For example, for a rank-2 tensor $\mathbf{A}$ with an orthogonal coordinate system, the physical components can be represented by:

$$\begin{aligned}
\hat{A}_{ij} &= \frac{A_{ij}}{h_i h_j} && \text{(no sum on } i \text{ or } j\text{, with basis } \hat{\mathbf{E}}^i \hat{\mathbf{E}}^j\text{)} \\
\hat{A}^{ij} &= h_i h_j A^{ij} && \text{(no sum on } i \text{ or } j\text{, with basis } \hat{\mathbf{E}}_i \hat{\mathbf{E}}_j\text{)} \\
\hat{A}^i_j &= \frac{h_i A^i_j}{h_j} && \text{(no sum on } i \text{ or } j\text{, with basis } \hat{\mathbf{E}}_i \hat{\mathbf{E}}^j\text{)}
\end{aligned} \tag{30}$$

• On generalizing the above pattern, the physical components of a tensor of type (m, n)

in a general orthogonal coordinate system are given by:

$$\hat{A}^{a_1 \ldots a_m}_{b_1 \ldots b_n} = \frac{h_{a_1} \ldots h_{a_m}}{h_{b_1} \ldots h_{b_n}} A^{a_1 \ldots a_m}_{b_1 \ldots b_n} \tag{31}$$

• As a consequence of the last points, in a space with a well defined metric any tensor can be expressed in covariant or contravariant (including mixed) or physical forms using different sets of basis vectors. Moreover, these forms can be transformed from each other using the raising and lowering operators and scale factors. As before, for the Cartesian rectangular systems the covariant, contravariant and physical components are the same where the Kronecker delta is the metric tensor.

• For orthogonal coordinate systems, the two sets of normalized covariant and contravariant basis vectors are identical as established earlier, and hence the physical components related to the covariant and contravariant components are identical as well. Consequently, for orthogonal systems with orthonormal basis vectors, the covariant, contravariant and physical components are identical.

• The physical components of a tensor may be represented by the symbol of the tensor with subscripts denoting the coordinates of the employed coordinate system. For instance, if $\mathbf{A}$ is a vector in a 3D space with contravariant components A^i or covariant components A_i, its physical components in Cartesian, cylindrical, spherical and general curvilinear systems may be denoted by (A_x, A_y, A_z), (A_ρ, A_ϕ, A_z), (A_r, A_θ, A_ϕ) and (A_u, A_v, A_w) respectively.

• For consistency and dimensional homogeneity, the tensors in scientific applications are normally represented by their physical components with a set of normalized unit base vectors. The invariance of the tensor form then guarantees that the same tensor formulation is valid regardless of any particular coordinate system where standard tensor transformations can be used to convert from one form to another without affecting the validity and invariance of the formulation.

2 Special Tensors

• The subject of investigation of this section is those tensors that form an essential part of the tensor calculus theory, namely the Kronecker, the permutation and the metric tensors.

2.1 Kronecker Tensor

• This is a rank-2 symmetric, constant, isotropic tensor in all dimensions.

• It is defined as:

$$\delta_{ij} = \delta^{ij} = \delta^{i}{}_{j} = \delta_{i}{}^{j} \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{32}$$

• The generalized Kronecker delta is defined as:

$$\delta^{i_1 \dots i_n}_{j_1 \dots j_n} = \begin{cases} 1 & [(j_1 \dots j_n) \text{ is even permutation of } (i_1 \dots i_n)] \\ -1 & [(j_1 \dots j_n) \text{ is odd permutation of } (i_1 \dots i_n)] \\ 0 & [\text{repeated } j\text{'s}] \end{cases} \tag{33}$$

It can also be defined by the following $n \times n$ determinant:

$$\delta^{i_1 \dots i_n}_{j_1 \dots j_n} = \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_n} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_n}_{j_1} & \delta^{i_n}_{j_2} & \cdots & \delta^{i_n}_{j_n} \end{vmatrix} \tag{34}$$

where the δ^i_j entries in the determinant are the normal Kronecker deltas as defined by Eq. 32.

• The relation between the rank-n permutation tensor and the generalized Kronecker delta in an nD space is given by:

$$\epsilon_{i_1 i_2 \dots i_n} = \delta^{1\,2 \dots n}_{i_1 i_2 \dots i_n} \qquad \& \qquad \epsilon^{i_1 i_2 \dots i_n} = \delta^{i_1 i_2 \dots i_n}_{1\,2 \dots n} \tag{35}$$

Hence, the permutation tensor ϵ may be considered as a special case of the generalized Kronecker delta. Consequently the permutation tensor can be written as an $n \times n$ determinant consisting of the normal Kronecker deltas.

• If we define

$$\delta^{ij}_{lm} = \delta^{ijk}_{lmk} \tag{36}$$

then the well known $\epsilon - \delta$ relation (Eq. 45) will take the following form:

$$\delta^{ij}_{lm} = \delta^i_l \delta^j_m - \delta^i_m \delta^j_l \tag{37}$$

Other identities involving δ and ϵ can also be formulated in terms of the generalized Kronecker delta.

2.2 Permutation Tensor

• This tensor has a rank equal to the number of dimensions of the space. Hence, a rank-n permutation tensor has n^n components.

• It is a relative tensor of weight -1 for its covariant form and $+1$ for its contravariant form.

• It is isotropic and totally anti-symmetric in each pair of its indices, i.e. it changes sign on swapping any two of its indices.

• It is a pseudo tensor since it acquires a minus sign under improper orthogonal transformation of coordinates.

• The rank-n permutation tensor is defined as:

$$\epsilon^{i_1 i_2 \ldots i_n} = \epsilon_{i_1 i_2 \ldots i_n} = \begin{cases} 1 & [(i_1, i_2, \ldots, i_n) \text{ is even permutation of } (1, 2, \ldots, n)] \\ -1 & [(i_1, i_2, \ldots, i_n) \text{ is odd permutation of } (1, 2, \ldots, n)] \\ 0 & [\text{repeated index}] \end{cases} \tag{38}$$

• For the rank-n permutation tensor we have:[6]

$$\epsilon_{a_1 a_2 \cdots a_n} = \prod_{i=1}^{n-1} \left[\frac{1}{i!} \prod_{j=i+1}^{n} (a_j - a_i) \right] = \frac{1}{S(n-1)} \prod_{1 \le i < j \le n} (a_j - a_i) \tag{39}$$

where $S(n-1)$ is the super-factorial function of $(n-1)$ which is defined by:

$$S(k) = \prod_{i=1}^{k} i! = 1! \cdot 2! \cdot \ldots \cdot k! \tag{40}$$

• A simpler formula for the rank-n permutation tensor can be obtained from the previous one by ignoring the magnitude of the multiplication factors and taking their signs only, that is:

$$\epsilon_{a_1 a_2 \cdots a_n} = \prod_{1 \le i < j \le n} \sigma(a_j - a_i) \tag{41}$$

where

$$\sigma(k) = \begin{cases} +1 & (k > 0) \\ -1 & (k < 0) \\ 0 & (k = 0) \end{cases} \tag{42}$$

• The sign function in the previous point can be expressed in a more direct form by dividing each argument of the multiplicative factors in Eq. 41 by its absolute value, noting that

[6] The following formulae also apply to the contravariant form.

none of these factors is zero, and hence Eq. 41 becomes:

$$\epsilon_{a_1 a_2 \cdots a_n} = \prod_{1 \le i < j \le n} \frac{(a_j - a_i)}{|a_j - a_i|} \tag{43}$$

• For the rank-3 permutation tensor we have:

$$\epsilon^{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta^i_l & \delta^i_m & \delta^i_n \\ \delta^j_l & \delta^j_m & \delta^j_n \\ \delta^k_l & \delta^k_m & \delta^k_n \end{vmatrix} \tag{44}$$

$$\epsilon^{ijk}\epsilon_{lmk} = \delta^i_l \delta^j_m - \delta^i_m \delta^j_l \tag{45}$$

• For the rank-n permutation tensor we have:

$$\epsilon^{i_1 i_2 \cdots i_n}\, \epsilon_{i_1 i_2 \cdots i_n} = n! \tag{46}$$

• For the rank-n permutation tensor we have:

$$\epsilon^{i_1 i_2 \cdots i_n}\, \epsilon_{j_1 j_2 \cdots j_n} = \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_n} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_n}_{j_1} & \delta^{i_n}_{j_2} & \cdots & \delta^{i_n}_{j_n} \end{vmatrix} \tag{47}$$

• On comparing Eqs. 34 and 47 we obtain the following identity:

$$\delta^{i_1 \ldots i_n}_{j_1 \ldots j_n} = \epsilon^{i_1 \ldots i_n}\, \epsilon_{j_1 \ldots j_n} \tag{48}$$

• Based on the previous point, the generalized Kronecker delta is the result of multiplying two relative tensors one of weight $w = +1$ and the other of weight $w = -1$ and hence the generalized Kronecker delta has a weight of $w = 0$; therefore the generalized Kronecker delta is an absolute tensor.[7]

• As has been stated previously, $\epsilon^{i_1 \ldots i_n}$ and $\epsilon_{i_1 \ldots i_n}$ are relative tensors of weight $+1$ and -1 respectively. It is desirable to define absolute covariant and contravariant forms of the permutation tensor, marked with underline, by the following relations:[8]

$$\underline{\epsilon}_{i_1 \ldots i_n} = \sqrt{g}\,\epsilon_{i_1 \ldots i_n} \qquad \& \qquad \underline{\epsilon}^{i_1 \ldots i_n} = \frac{1}{\sqrt{g}}\epsilon^{i_1 \ldots i_n} \tag{49}$$

where g is the determinant of the covariant metric tensor g_{pq}.

• The $\epsilon - \delta$ identity (Eqs. 37 and 45) can be generalized by employing the metric tensor with the absolute permutation tensor:

$$g^{ij}\underline{\epsilon}_{ikl}\underline{\epsilon}_{jmn} = g_{km}g_{ln} - g_{kn}g_{lm} \tag{50}$$

2.3 Metric Tensor

• One of the main objectives of the metric, which is a rank-2 symmetric absolute non-singular[9] tensor, is to generalize the concept of distance to general curvilinear coordinate frames and hence maintain the invariance of distance in different coordinate systems. This tensor is also used to raise and lower indices and thus facilitate the transformation between the covariant and contravariant types.

• In general, the coordinate system and the space metric are independent entities. Yes, some coordinate systems may be defined by having a specific metric in which case the two

[7] The multiplication of relative tensors produces a tensor whose weight is the sum of the weights of the original tensors.

[8] The contravariant form requires a sign function with details out of scope of the present text (see [13]); however, for the rank-3 permutation tensor which is the one used mostly in the forthcoming notes the above expression stands as it is.

[9] We mean that the matrix representing the tensor is invertible and hence its determinant does not vanish at any point of the space.

are correlated. This is the case in the Cartesian coordinate systems which are based in their definition on presuming an underlying Euclidean metric.

• The components of the metric tensor are given by:

$$g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j \qquad \& \qquad g^{ij} = \mathbf{E}^i \cdot \mathbf{E}^j \tag{51}$$

where the indexed $\mathbf{E}$ are the covariant and contravariant basis vectors as defined previously in § 1.6. Because of these relations, the vectors $\mathbf{E}_i$ and $\mathbf{E}^i$ may be denoted by $\mathbf{g}_i$ and $\mathbf{g}^i$ respectively which is more suggestive of their relation to the metric tensor.

• As a consequence of the last point, the covariant metric tensor can also be defined as:

$$g_{ij} = \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial u^j} \tag{52}$$

where

$$x^k = x^k \left(u^1, \ldots, u^n \right) \qquad (k = 1, \ldots, n) \tag{53}$$

are independent coordinates in an nD space with a rectangular Cartesian system, and u^i $(i = 1, \ldots, n)$ are independent generalized curvilinear coordinates. Similarly for the contravariant metric tensor we have:

$$g^{ij} = \frac{\partial u^i}{\partial x^k} \frac{\partial u^j}{\partial x^k} \tag{54}$$

• The coefficients of the metric tensor may also be considered as the components of the unit tensor in its two variance forms, that is:

$$\boldsymbol{\delta} = g_{ij} \mathbf{E}^i \mathbf{E}^j = g^{ij} \mathbf{E}_i \mathbf{E}_j \tag{55}$$

• As stated already, the basis vectors, whether covariant or contravariant, in general coordinate systems are not necessarily mutually orthogonal and hence the metric tensor is not diagonal in general since the dot products given in Eqs. 51, 52 and 54 are not necessarily zero when $i \neq j$. Moreover, since those basis vectors are not necessarily of unit length, the entries of the metric tensor are not of unit magnitude in general. However, since the dot product of vectors is a commutative operation, the metric tensor is necessarily symmetric.

• The entries of the metric tensor, including the diagonal elements, can be positive or negative.

• The covariant and contravariant forms of the metric tensor are inverses of each other and hence:

$$g^{ik}g_{kj} = \delta^i_{\ j} \qquad \& \qquad g_{ik}g^{kj} = \delta_i^{\ j} \tag{56}$$

where these equations can be seen as a matrix multiplication (row×column).

• A result from the previous points is that:

$$\begin{aligned} \left(\mathbf{E}^i \cdot \mathbf{E}^j\right)\left(\mathbf{E}_j \cdot \mathbf{E}_k\right) &= g^{ij}g_{jk} = \delta^i_{\ k} \\ \left(\mathbf{E}_i \cdot \mathbf{E}_j\right)\left(\mathbf{E}^j \cdot \mathbf{E}^k\right) &= g_{ij}g^{jk} = \delta_i^{\ k} \end{aligned} \tag{57}$$

• As the metric tensor has an inverse, it should not be singular and hence its determinant, which is in general a function of coordinates like the metric tensor itself, should not vanish at any point in the space, that is:

$$g(u^1, \ldots, u^n) = \det\left(g_{ij}\right) \neq 0 \tag{58}$$

• The mixed type metric tensor is given by:

$$g^i_{\ j} = \mathbf{E}^i \cdot \mathbf{E}_j = \delta^i_{\ j} \qquad \& \qquad g_i^{\ j} = \mathbf{E}_i \cdot \mathbf{E}^j = \delta_i^{\ j} \tag{59}$$

and hence it is the identity tensor. These equations represent the fact that the covariant and contravariant basis vectors are reciprocal sets.

• From the previous points, it can be concluded that the metric tensor is in fact a transformation of the Kronecker delta in its different variance types from a rectangular system to a general curvilinear system, that is:

$$\begin{aligned} g_{ij} &= \frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}\delta_{kl} = \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j} = \mathbf{E}_i \cdot \mathbf{E}_j && \text{(covariant)} \\ g^{ij} &= \frac{\partial u^i}{\partial x^k}\frac{\partial u^j}{\partial x^l}\delta^{kl} = \frac{\partial u^i}{\partial x^k}\frac{\partial u^j}{\partial x^k} = \mathbf{E}^i \cdot \mathbf{E}^j && \text{(contravariant)} \\ g^i_j &= \frac{\partial u^i}{\partial x^k}\frac{\partial x^l}{\partial u^j}\delta^k_l = \frac{\partial u^i}{\partial x^k}\frac{\partial x^k}{\partial u^j} = \mathbf{E}^i \cdot \mathbf{E}_j && \text{(mixed)} \end{aligned} \tag{60}$$

• Because of the relations:

$$\begin{aligned} A^i &= \mathbf{A} \cdot \mathbf{E}^i = A_j \mathbf{E}^j \cdot \mathbf{E}^i = A_j g^{ji} \\ A_i &= \mathbf{A} \cdot \mathbf{E}_i = A^j \mathbf{E}_j \cdot \mathbf{E}_i = A^j g_{ji} \end{aligned} \tag{61}$$

the metric tensor is used as an operator for raising and lowering indices and hence facilitating the transformation between the covariant and contravariant types of vectors. By a similar argument, the above can be easily generalized where the contravariant metric tensor is used for raising covariant indices and the covariant metric tensor is used for lowering contravariant indices of tensors of any rank, e.g.

$$A^i{}_k = g^{ij} A_{jk} \qquad \& \qquad A_i^{\ kl} = g_{ij} A^{jkl} \tag{62}$$

Consequently, any tensor in a Riemannian space with well-defined metric can be cast into covariant or contravariant or mixed forms.[10]

• In the raising and lowering of index operations the metric tensor acts, like a Kronecker delta, as an index replacement operator as well as shifting the index position.

[10] For mixed form the rank should be > 1.

• In general, the order of the raised and lowered indices is important and hence

$$g^{ik}A_{jk} = A_j{}^i \qquad \text{and} \qquad g^{ik}A_{kj} = A^i{}_j \tag{63}$$

are different unless the tensor is symmetric in its two indices, i.e. $A_{jk} = A_{kj}$. A dot may be used to indicate the original position of the shifted index and hence the order of the indices is recorded, e.g. $A_{j\cdot}^{\ i}$ and $A^{i}_{\cdot j}$ for the above examples respectively, although this is redundant in the case of symmetry.[11]

• Raising and lowering of indices is a reversible process; hence keeping a record of the original position of the shifted indices will facilitate the reversal.

• For a space with a coordinate system in which the metric tensor can be cast into a diagonal form with all the diagonal entries being of unity magnitude (i.e. ± 1) the metric is called flat.

• If g and $\bar{g}$ are the determinants of the covariant metric tensor in the unbarred and barred systems respectively, i.e. $g = \det(g_{ij})$ and $\bar{g} = \det(\bar{g}_{ij})$, then

$$\bar{g} = J^2 g \qquad \& \qquad \sqrt{\bar{g}} = J\sqrt{g} \tag{64}$$

where J $(= \left|\frac{\partial u}{\partial \bar{u}}\right|)$ is the Jacobian of the transformation between the unbarred and barred systems. Consequently, the determinant of the covariant metric and its square root are relative scalar invariants of weight $+2$ and $+1$ respectively.

• A "conjugate" or "associated" tensor of a tensor in a metric space is a tensor obtained by inner product multiplication, once or more, of the original tensor by the covariant or contravariant forms of the metric tensor.

• All tensors associated with a particular tensor through the metric tensor represent the

[11] Dots may also be inserted in the tensor symbols to remove any ambiguity about the order of the indices even without the action of the raising and lowering operators.

same tensor but in different reference frames since the association is no more than raising or lowering indices by the metric tensor which is equivalent to a representation of the components of the tensor relative to different basis sets.

• A sufficient and necessary condition for the components of the metric tensor to be constants in a given coordinate system is that the Christoffel symbols of the first or second kind vanish identically (refer to 3.1).

• The metric tensor behaves as a constant with respect to covariant and absolute differentiation (see § 3.2 and § 3.3). Hence, in all coordinate systems the covariant and absolute derivatives of the metric tensor are zero; moreover, the covariant and absolute derivative operators bypass the metric tensor in differentiating inner and outer products of tensors involving the metric tensor.

• In general orthogonal coordinate systems in nD spaces the metric tensor and its inverse are diagonal, that is:

$$g_{ij} = g^{ij} = 0 \qquad (i \neq j) \tag{65}$$

moreover, we have:

$$g_{ii} = (h_i)^2 = \frac{1}{g^{ii}} \qquad \text{(no sum on } i\text{)} \tag{66}$$

$$\det(g_{ij}) = g = g_{11}g_{22}\dots g_{nn} = \prod_i (h_i)^2 \tag{67}$$

$$\det\left(g^{ij}\right) = \frac{1}{g} = \frac{1}{g_{11}g_{22}\cdots g_{nn}} = \left[\prod_i (h_i)^2\right]^{-1} \tag{68}$$

where h_i $(= |\mathbf{E}_i|)$ are the scale factors, as described previously.

• A Riemannian metric, g_{ij}, in a particular coordinate system is a Euclidean metric if it can be transformed to the identity tensor, δ_{ij}, by a permissible coordinate transformation.

• The Minkowski metric, which is the metric tensor of special relativity, is given by one of the following two forms:

$$[g_{ij}] = \left[g^{ij}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad [g_{ij}] = \left[g^{ij}\right] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{69}$$

Consequently, the line element ds can be imaginary.

• The partial derivatives of the covariant and contravariant metric tensors satisfy the following identities:

$$\begin{aligned} \partial_k g_{ij} &= -g_{mj} g_{ni} \partial_k g^{nm} \\ \partial_k g^{ij} &= -g^{mj} g^{in} \partial_k g_{nm} \end{aligned} \tag{70}$$

• In the following subsections, we investigate a number of mathematical objects whose definitions and applications are dependent on the metric tensor.

2.3.1 Dot Product

• The dot product of two basis vectors in general curvilinear coordinates was given earlier in this section. This will be used in the following points to develop expressions for the dot product of vectors and tensors in general.

• The dot product of two vectors, **A** and **B**, in general curvilinear coordinates using their covariant and contravariant forms, as well as opposite forms, is given by:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= A_i \mathbf{E}^i \cdot B_j \mathbf{E}^j = A_i B_j \mathbf{E}^i \cdot \mathbf{E}^j = g^{ij} A_i B_j = A^j B_j = A_i B^i \\ \mathbf{A} \cdot \mathbf{B} &= A^i \mathbf{E}_i \cdot B^j \mathbf{E}_j = A^i B^j \mathbf{E}_i \cdot \mathbf{E}_j = g_{ij} A^i B^j = A_j B^j = A^i B_i \\ \mathbf{A} \cdot \mathbf{B} &= A_i \mathbf{E}^i \cdot B^j \mathbf{E}_j = A_i B^j \mathbf{E}^i \cdot \mathbf{E}_j = \delta^i_j A_i B^j = A_j B^j \\ \mathbf{A} \cdot \mathbf{B} &= A^i \mathbf{E}_i \cdot B_j \mathbf{E}^j = A^i B_j \mathbf{E}_i \cdot \mathbf{E}^j = \delta_i^j A^i B_j = A^i B_i \end{aligned} \tag{71}$$

In brief, the dot product of two vectors is the dot product of their two basis vectors multiplied algebraically by the algebraic product of their components. Because the dot product of basis vectors is a metric tensor, the metric tensor will act on the components by raising or lowering the index of one component or by replacing the index of a component.

• The dot product operations outlined in the previous point can be easily extended to tensors of higher ranks where the covariant and contravariant forms of the components and basis vectors are treated in a similar manner to the above to obtain the dot product. For instance, the dot product of a rank-2 tensor of contravariant components A^{ij} and a vector of covariant components B_k is given by:

$$\mathbf{A} \cdot \mathbf{B} = \left(A^{ij}\mathbf{E}_i\mathbf{E}_j\right) \cdot \left(B_k\mathbf{E}^k\right) = A^{ij}B_k\left(\mathbf{E}_i\mathbf{E}_j \cdot \mathbf{E}^k\right) = A^{ij}B_k\mathbf{E}_i\delta_j^k = A^{ij}B_j\mathbf{E}_i \tag{72}$$

that is, the i^{th} component of this product, which is a contravariant vector, is:

$$[\mathbf{A} \cdot \mathbf{B}]^i = A^{ij}B_j \tag{73}$$

• From the previous points, the dot product in general curvilinear coordinates occurs between two vectors of opposite variance type. Therefore, to obtain the dot product of two vectors of the same variance type, one of the vectors should be converted to the opposite type by the raising/lowering operator, followed by the inner product operation. This can be generalized to the dot product of higher-rank tensors where the two contracted indices of the dot product should be of opposite variance type and hence the index-shifting operator in the form of the metric tensor should be used, if necessary, to achieve this.

• The generalized dot product of two tensors is an invariant under permissible coordinate transformations.

2.3.2 Cross Product

• The cross product of two covariant basis vectors in general curvilinear coordinates is given by:

$$\mathbf{E}_i \times \mathbf{E}_j = \frac{\partial x^l}{\partial u^i}\mathbf{e}_l \times \frac{\partial x^m}{\partial u^j}\mathbf{e}_m = \frac{\partial x^l}{\partial u^i}\frac{\partial x^m}{\partial u^j}\mathbf{e}_l \times \mathbf{e}_m = \frac{\partial x^l}{\partial u^i}\frac{\partial x^m}{\partial u^j}\epsilon_{lmn}\mathbf{e}_n \tag{74}$$

where the indexed x and u are the coordinates of Cartesian and general curvilinear systems respectively, the indexed $\mathbf{e}$ are the Cartesian base vectors[12] and $\epsilon_{lmn} = \epsilon^{lmn}$ is the permutation relative tensor as defined in Eq. 38. Now since $\mathbf{e}_n = \mathbf{e}^n = \frac{\partial x^n}{\partial u^k}\mathbf{E}^k$, the last equation becomes:

$$\mathbf{E}_i \times \mathbf{E}_j = \frac{\partial x^l}{\partial u^i}\frac{\partial x^m}{\partial u^j}\frac{\partial x^n}{\partial u^k}\epsilon_{lmn}\mathbf{E}^k = \underline{\epsilon}_{ijk}\mathbf{E}^k \tag{75}$$

where the underlined absolute covariant permutation tensor is defined as:

$$\underline{\epsilon}_{ijk} = \frac{\partial x^l}{\partial u^i}\frac{\partial x^m}{\partial u^j}\frac{\partial x^n}{\partial u^k}\epsilon_{lmn} \tag{76}$$

So the final result is:

$$\mathbf{E}_i \times \mathbf{E}_j = \underline{\epsilon}_{ijk}\mathbf{E}^k \tag{77}$$

By a similar reasoning, we obtain the following expression for the cross product of two contravariant basis vectors in general curvilinear coordinates:

$$\mathbf{E}^i \times \mathbf{E}^j = \underline{\epsilon}^{ijk}\mathbf{E}_k \tag{78}$$

where the absolute contravariant permutation tensor is defined by:

$$\underline{\epsilon}^{ijk} = \frac{\partial u^i}{\partial x^l}\frac{\partial u^j}{\partial x^m}\frac{\partial u^k}{\partial x^n}\epsilon^{lmn} \tag{79}$$

[12] For Cartesian systems, there is no difference between covariant and contravariant tensors and hence $\mathbf{e}_i = \mathbf{e}^i$. We also note that for Cartesian systems $g = 1$.

• Considering Eq. 49, the above equations can also be expressed as:

$$\mathbf{E}_i \times \mathbf{E}_j = \underline{\epsilon}_{ijk}\mathbf{E}^k = \sqrt{g}\epsilon_{ijk}\mathbf{E}^k \tag{80}$$

$$\mathbf{E}^i \times \mathbf{E}^j = \underline{\epsilon}^{ijk}\mathbf{E}_k = \frac{\epsilon^{ijk}}{\sqrt{g}}\mathbf{E}_k \tag{81}$$

where $\epsilon_{ijk} = \epsilon^{ijk}$ are as defined previously (Eq. 38).

• The cross product of non-basis vectors follows similar rules to those outlined above for the basis vectors; the only difference is that the algebraic product of the components is used as a scale factor for the cross product of their basis vectors. For example, the cross product of two contravariant vectors, A^i and B^j, is given by:

$$\mathbf{A} \times \mathbf{B} = \left(A^i\mathbf{E}_i\right) \times \left(B^j\mathbf{E}_j\right) = A^iB^j\left(\mathbf{E}_i \times \mathbf{E}_j\right) = \underline{\epsilon}_{ijk}A^iB^j\mathbf{E}^k \tag{82}$$

that is, the k^{th} component of this product, which is a vector with covariant components, is:

$$[\mathbf{A} \times \mathbf{B}]_k = \underline{\epsilon}_{ijk}A^iB^j \tag{83}$$

Similarly, the cross product of two covariant vectors, A_i and B_j, is given by:

$$\mathbf{A} \times \mathbf{B} = \left(A_i\mathbf{E}^i\right) \times \left(B_j\mathbf{E}^j\right) = A_iB_j\left(\mathbf{E}^i \times \mathbf{E}^j\right) = \underline{\epsilon}^{ijk}A_iB_j\mathbf{E}_k \tag{84}$$

with the k^{th} contravariant component being given by:

$$[\mathbf{A} \times \mathbf{B}]^k = \underline{\epsilon}^{ijk}A_iB_j \tag{85}$$

2.3.3 Line Element

• The displacement differential vector in general curvilinear coordinate systems is given by:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^i} du^i = \mathbf{E}_i du^i = \sum_i |\mathbf{E}_i| \frac{\mathbf{E}_i}{|\mathbf{E}_i|} du^i = \sum_i |\mathbf{E}_i| \hat{\mathbf{E}}_i du^i \tag{86}$$

where $\mathbf{r}$ is the position vector as defined previously.

• The line element ds, which may also be called the differential of arc length, in general curvilinear coordinate systems is given by:

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = \mathbf{E}_i du^i \cdot \mathbf{E}_j du^j = (\mathbf{E}_i \cdot \mathbf{E}_j)\, du^i du^j = g_{ij} du^i du^j \tag{87}$$

where g_{ij} is the covariant metric tensor.

• For orthogonal coordinate systems, the metric tensor is given by:

$$g_{ij} = \begin{cases} 0 & (i \neq j) \\ (h_i)^2 & (i = j) \end{cases} \tag{88}$$

where h_i is the scale factor of the respective coordinate u^i. Hence, the last part of Eq. 87 becomes:

$$(ds)^2 = \sum_i (h_i)^2 \, du^i du^i \tag{89}$$

with no cross terms (i.e. terms of products involving more than one coordinate like $du^i du^j$ where $i \neq j$) which are generally present in the case of non-orthogonal curvilinear systems.

• On conducting a transformation from one coordinate system to another coordinate system, marked with barred coordinates, $\bar{u}$, the line element will be expressed in the new system as:

$$(ds)^2 = \bar{g}_{ij} d\bar{u}^i d\bar{u}^j \tag{90}$$

Since the line element is an invariant quantity, the same symbol $(ds)^2$ is used in both Eqs. 87 and 90.

2.3.4 Surface Element

• In general curvilinear coordinates of a 3D space, an infinitesimal element of area on the surface $u^1 = c_1$, where c_1 is a constant, is obtained by taking the magnitude of the cross product of the displacement vectors in the directions of the other two coordinates on that surface. Hence, the generalized differential of area element on the surface $u^1 = c_1$ is given by:

$$\begin{aligned} dA(u^1 = c_1) &= |d\mathbf{r}_2 \times d\mathbf{r}_3| \\ &= |\mathbf{E}_2 \times \mathbf{E}_3|\, du^2 du^3 \\ &= \left|\underline{\epsilon}_{231}\mathbf{E}^1\right| du^2 du^3 && \text{(Eq. 77)} \\ &= |\underline{\epsilon}_{231}| \left|\mathbf{E}^1\right| du^2 du^3 \\ &= \sqrt{g}\sqrt{\mathbf{E}^1 \cdot \mathbf{E}^1}\, du^2 du^3 && \text{(Eqs. 49 \& 98)} \\ &= \sqrt{g}\,\sqrt{g^{11}}\, du^2 du^3 && \text{(Eq. 51)} \\ &= \sqrt{gg^{11}}\, du^2 du^3 \end{aligned} \tag{91}$$

• On generalizing the above argument, the differential area element in a 3D space on the surface $u^i = c_i$ $(i = 1, 2, 3)$ where c_i is a constant is given by:

$$dA(u^i = c_i) = \sqrt{gg^{ii}} du^j du^k \qquad (i \neq j \neq k, \text{ no sum on } i) \tag{92}$$

• In general orthogonal coordinates in a 3D space we have:

$$\sqrt{gg^{ii}} = \sqrt{(h_i)^2 (h_j)^2 (h_k)^2 \frac{1}{(h_i)^2}} = h_j h_k \qquad (i \neq j \neq k, \text{ no sum on any index}) \tag{93}$$

and hence Eq. 92 becomes:

$$dA(u^i = c_i) = h_j h_k du^j du^k \qquad (i \neq j \neq k, \text{ no sum on any index}) \tag{94}$$

The last formula represents the area of a surface differential with sides $h_j du^j$ and $h_k du^k$ (no sum on j, k).

2.3.5 Volume Element

• In general curvilinear coordinates of a 3D space, an infinitesimal element of volume, represented by a parallelepiped spanned by the three displacement vectors $d\mathbf{r}_i$ $(i = 1, 2, 3)$, is obtained by taking the magnitude of the scalar triple product of these vectors. Hence, the generalized differential volume element is given by:

$$\begin{aligned}
dV &= |d\mathbf{r}_1 \cdot (d\mathbf{r}_2 \times d\mathbf{r}_3)| \\
&= |\mathbf{E}_1 \cdot (\mathbf{E}_2 \times \mathbf{E}_3)|\, du^1 du^2 du^3 \\
&= \left|\mathbf{E}_1 \cdot \underline{\epsilon}_{231} \mathbf{E}^1\right| du^1 du^2 du^3 && \text{(Eq. 77)} \\
&= \left|\mathbf{E}_1 \cdot \mathbf{E}^1\right| |\underline{\epsilon}_{231}|\, du^1 du^2 du^3 \\
&= \left|\delta_1^1\right| |\underline{\epsilon}_{231}|\, du^1 du^2 du^3 && \text{(Eq. 59)} \\
&= \sqrt{g}\, du^1 du^2 du^3 && \text{(Eq. 49)} \\
&= J\, du^1 du^2 du^3 && \text{(Eq. 21)}
\end{aligned} \tag{95}$$

where g is the determinant of the covariant metric tensor g_{ij}, and J is the Jacobian[13] of the transformation as defined previously. The last line in the last equation is particularly relevant to the case of change of variables in multivariate integrals where the Jacobian facilitates the transformation.

[13]Due to the freedom of choice in the order of the variables, which is related to the choice of the system handedness hence affecting the sign of the determinant Jacobian, the sign of the determinant should be adjusted if necessary to have a proper sign for the volume element.

• The formulae in the last point for a 3D space can be extended to the differential of generalized volume element[14] in general curvilinear coordinates in an nD space as follow:

$$dV = \sqrt{g} du^1 \dots du^n = J \, du^1 \dots du^n \tag{96}$$

• In general orthogonal coordinate systems in a 3D space, the above formulae become:

$$dV = h_1 h_2 h_3 \, du^1 du^2 du^3 \tag{97}$$

where h_1, h_2 and h_3 are the scale factors. The last formula represents the volume of a parallelepiped with edges $h_1 du^1$, $h_2 du^2$ and $h_3 du^3$.

2.3.6 Magnitude of Vector

• The magnitude of a contravariant vector $\mathbf{A}$ is given by:

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{(\mathbf{E}_i \cdot \mathbf{E}_j) \, A^i A^j} = \sqrt{g_{ij} A^i A^j} = \sqrt{A_j A^j} = \sqrt{A^i A_i} \tag{98}$$

A similar expression can be obtained for the covariant form of the vector, that is:

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{(\mathbf{E}^i \cdot \mathbf{E}^j) \, A_i A_j} = \sqrt{g^{ij} A_i A_j} = \sqrt{A^j A_j} = \sqrt{A_i A^i} \tag{99}$$

The magnitude of a vector can also be obtained more directly from the dot product of the covariant and contravariant forms of the vector:

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{(\mathbf{E}^i \cdot \mathbf{E}_j) \, A_i A^j} = \sqrt{\delta^i_j A_i A^j} = \sqrt{A_i A^i} = \sqrt{A_j A^j} \tag{100}$$

[14] Generalized volume elements are used, for instance, to represent the change of variables in multi-variable integrations.

2.3.7 Angle Between Vectors

• The angle θ between two contravariant or two covariant vectors $\mathbf{A}$ and $\mathbf{B}$ is given respectively by:

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|} = \frac{g_{ij}A^iB^j}{\sqrt{g_{kl}A^kA^l}\sqrt{g_{mn}B^mB^n}} = \frac{g^{ij}A_iB_j}{\sqrt{g^{kl}A_kA_l}\sqrt{g^{mn}B_mB_n}} \tag{101}$$

For two vectors of opposite variance type we have:

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|} = \frac{A^iB_i}{\sqrt{g_{kl}A^kA^l}\sqrt{g^{mn}B_mB_n}} = \frac{A_iB^i}{\sqrt{g^{kl}A_kA_l}\sqrt{g_{mn}B^mB^n}} \tag{102}$$

2.3.8 Length of Curve

• In general curvilinear coordinates, the length of a t-parameterized space curve $\mathbf{r}(t)$ defined by $u^i = u^i(t)$, which represents the distance traversed along the curve on moving between its start point S and end point E, is given by:[15]

$$L = \int_S^E \sqrt{g_{ij}du^idu^j} = \int_{t_1}^{t_2} \sqrt{g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\,dt \tag{103}$$

where t is a scalar variable parameter, and t_1 and t_2 are the values of t corresponding to the start and end points respectively.

• The length of curve is used to define the geodesic which is the path of the shortest distance connecting two points in a Riemannian space. Although the geodesic is a straight line in a Euclidean space, it is a generalized curved path in a general Riemannian space.

[15] Some authors add a sign indicator to ensure that the argument of the square root is positive. However, as indicated in the Preface, such a condition is assumed when needed since we deal with non-complex values only.

3 Covariant and Absolute Differentiation

• The focus of this section is the investigation of covariant and absolute differentiation operations which are closely linked. These operations represent generalization of tensor differentiation in general curvilinear coordinate systems. Briefly, the differential change of a tensor in general curvilinear coordinate systems is the result of a change in the base vectors and a change in the tensor components. Hence, covariant and absolute differentiation, in place of the normal differentiation, are defined and employed to account for both of these changes. Since Christoffel symbols are crucial in the formulation and application of covariant and absolute differentiation, the first subsection of the present section will be dedicated to these symbols and their properties.

3.1 Christoffel Symbols

• We start by investigating the main properties of the Christoffel symbols which play crucial roles in tensor calculus in general and are needed for the subsequent development of the present and forthcoming sections as well as the future notes.

• Christoffel symbols are classified as those of the first kind and those of the second kind. These two kinds are linked through the index raising and lowering operators. Both kinds of Christoffel symbols are variable functions of coordinates in general.

• Christoffel symbols of the first and second kind are not tensors in general although they are affine tensors of rank-3.

• As a consequence of the last point, if all the Christoffel symbols of either kind vanished in a particular coordinate system they will not necessarily vanish in other systems; for instance they all vanish in Cartesian systems but not in cylindrical or spherical systems, as has been established previously [11] and will be investigated further in the forthcoming points.

• Christoffel symbols of the first kind are given by:

$$[ij,l] = \frac{1}{2}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right) \tag{104}$$

where the indexed g is the covariant form of the metric tensor.

• Christoffel symbols of the second kind are obtained by raising the third index of the Christoffel symbols of the first kind, that is:

$$\Gamma^k_{ij} = g^{kl}\,[ij,l] = \frac{g^{kl}}{2}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right) \tag{105}$$

where the indexed g is the metric tensor in its contravariant and covariant forms with implied summation over l.

• Similarly, the Christoffel symbols of the first kind can be obtained from the Christoffel symbols of the second kind by reversing the above process through lowering the upper index, that is:

$$g_{km}\Gamma^k_{ij} = g_{km}g^{kl}\,[ij,l] = \delta^l_m\,[ij,l] = [ij,m] \tag{106}$$

• For an nD space with n covariant basis vectors $(\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n)$ spanning the space, the derivative $\partial_j \mathbf{E}_i$ for any given i is a vector within the space and hence it is in general a linear combination of all the basis vectors. The Christoffel symbols of the second kind are the components of this linear combination, that is:

$$\partial_j \mathbf{E}_i = \Gamma^k_{ij}\mathbf{E}_k \tag{107}$$

Similarly, for the contravariant basis vectors we have:

$$\partial_j \mathbf{E}^i = -\Gamma^i_{kj}\mathbf{E}^k \tag{108}$$

• By inner product multiplication of the previous relations with the basis vectors we obtain:

$$\mathbf{E}^i \cdot \partial_k \mathbf{E}_j = \Gamma^i_{jk} \qquad \& \qquad \mathbf{E}_i \cdot \partial_k \mathbf{E}^j = -\Gamma^j_{ik} \tag{109}$$

Similarly:

$$\mathbf{E}_k \cdot \partial_j \mathbf{E}_i = g_{mk} \mathbf{E}^m \cdot \partial_j \mathbf{E}_i = g_{mk} \Gamma^m_{ij} = [ij, k] \tag{110}$$

• Christoffel symbols of the first and second kind are symmetric in their paired indices, that is:

$$[ij, k] = [ji, k] \qquad \& \qquad \Gamma^k_{ij} = \Gamma^k_{ji} \tag{111}$$

• The partial derivative of the components of the covariant metric tensor and the Christoffel symbols of the first kind satisfy the following identity, which is essentially based on the forthcoming Ricci Theorem:

$$\partial_j g_{il} = [ij, l] + [jl, i] \tag{112}$$

This relation can also be written in terms of the Christoffel symbols of the second kind using the index shifting operator:

$$\partial_j g_{il} = g_{kl} \Gamma^k_{ij} + g_{ki} \Gamma^k_{jl} \tag{113}$$

A related formula for the partial derivative of the components of the contravariant metric tensor, which can be obtained by partial differentiation of the relation $g_{im} g^{mj} = \delta^j_i$ with respect to the k^{th} coordinate, is given by:

$$g_{im} \partial_k g^{mj} = -g^{mj} \partial_k g_{im} \tag{114}$$

• Christoffel symbols of the second kind with two identical indices of opposite variance type satisfy the following relations:

$$\Gamma^j_{ji} = \Gamma^j_{ij} = \frac{1}{2g}\partial_i g = \frac{1}{2}\partial_i \left(\ln\, g\right) = \partial_i \left(\ln \sqrt{g}\right) = \frac{1}{\sqrt{g}}\partial_i \sqrt{g} \tag{115}$$

where the main relation can be derived as follow:

$$\begin{aligned}
\Gamma^j_{ij} &= \frac{g^{jl}}{2}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right) && \text{(Eq. 105 with } k = j\text{)} \\
&= \frac{g^{jl}}{2}\left(\partial_l g_{ij} + \partial_i g_{jl} - \partial_l g_{ij}\right) && \text{(relabeling dummy } j, l \text{ in } 1^{st} \text{ term \& } g^{jl} = g^{lj}\text{)} \\
&= \frac{1}{2} g^{jl} \partial_i g_{jl} \\
&= \frac{1}{2g} g g^{jl} \partial_i g_{jl} \\
&= \frac{1}{2g}\partial_i g && \text{(derivative of determinant)}
\end{aligned} \tag{116}$$

• In orthogonal coordinate systems, the Christoffel symbols of the first kind are given by:

$$\begin{aligned}
[ij, i] = [ji, i] &= \frac{1}{2}\partial_j g_{ii} && \text{(no sum on } i\text{)} \\
[ii, j] &= -\frac{1}{2}\partial_j g_{ii} && (i \neq j, \text{ no sum on } i) \\
[ij, k] &= 0 && (i \neq j \neq k)
\end{aligned} \tag{117}$$

The first relation is a special case of Eq. 112 with $l = i$ taking into account that the Christoffel symbols are symmetric in their paired indices; moreover, the relation includes the case of $i = j$, i.e. when all the three indices are identical.

• In orthogonal coordinate systems, the Christoffel symbols of the second kind are given by:

$$\Gamma^i_{jk} = \frac{[jk, i]}{g_{ii}} = g^{ii}\,[jk, i] \qquad \text{(no sum on } i\text{)} \tag{118}$$

and hence from the results of the previous point we have:

$$\begin{aligned}
\Gamma^i_{ij} &= \Gamma^i_{ji} = \frac{1}{2g_{ii}}\partial_j g_{ii} = \frac{g^{ii}}{2}\partial_j g_{ii} = \frac{1}{2}\partial_j \ln g_{ii} && \text{(no sum on } i\text{)} \\
\Gamma^i_{jj} &= -\frac{1}{2g_{ii}}\partial_i g_{jj} = -\frac{g^{ii}}{2}\partial_i g_{jj} && \text{(no sum on } i \text{ or } j\text{, and } i \neq j\text{)} \\
\Gamma^i_{jk} &= 0 && (i \neq j \neq k)
\end{aligned} \tag{119}$$

As for the first kind in the last point, the first relation includes the case of $i = j$, i.e. when all the three indices are identical.

• In general orthogonal coordinate systems in a 3D space, the Christoffel symbols of the first kind vanish when the indices are all different, as shown earlier; moreover, the non-identically vanishing symbols of the first kind are given by:

$$\begin{array}{lll}
[11,1] = +h_1h_{1,1} & [11,2] = -h_1h_{1,2} & [11,3] = -h_1h_{1,3} \\
[12,1] = +h_1h_{1,2} & [12,2] = +h_2h_{2,1} & [13,1] = +h_1h_{1,3} \\
[13,3] = +h_3h_{3,1} & [22,1] = -h_2h_{2,1} & [22,2] = +h_2h_{2,2} \\
[22,3] = -h_2h_{2,3} & [23,2] = +h_2h_{2,3} & [23,3] = +h_3h_{3,2} \\
[33,1] = -h_3h_{3,1} & [33,2] = -h_3h_{3,2} & [33,3] = +h_3h_{3,3}
\end{array} \tag{120}$$

where $(1,2,3)$ stand for (u^1,u^2,u^3) respectively, h_1, h_2, h_3 are the scale factors as defined previously, and the comma indicates, as always, partial derivative; for example in cylindrical coordinates given by (ρ, ϕ, z), $h_{2,1}$ means the partial derivative of h_2 with respect to the first coordinate and hence $h_{2,1} = \partial_\rho \rho = 1$ since $h_2 = \rho$ and the first coordinate is ρ (refer to Table 1). Because the Christoffel symbols of the first kind are symmetric in their first two indices, the $[21,1]$ symbol for instance can be obtained from the value of the $[12,1]$ symbol.

• In general orthogonal coordinate systems in a 3D space, the Christoffel symbols of the second kind vanish when the indices are all different, as shown earlier; moreover, the non-identically vanishing symbols of the second kind are given by:

$$\begin{array}{lll} \Gamma^1_{11} = +\frac{h_{1,1}}{h_1} & \Gamma^2_{11} = -\frac{h_1 h_{1,2}}{(h_2)^2} & \Gamma^3_{11} = -\frac{h_1 h_{1,3}}{(h_3)^2} \\ \Gamma^1_{12} = +\frac{h_{1,2}}{h_1} & \Gamma^2_{12} = +\frac{h_{2,1}}{h_2} & \Gamma^1_{13} = +\frac{h_{1,3}}{h_1} \\ \Gamma^3_{13} = +\frac{h_{3,1}}{h_3} & \Gamma^1_{22} = -\frac{h_2 h_{2,1}}{(h_1)^2} & \Gamma^2_{22} = +\frac{h_{2,2}}{h_2} \\ \Gamma^3_{22} = -\frac{h_2 h_{2,3}}{(h_3)^2} & \Gamma^2_{23} = +\frac{h_{2,3}}{h_2} & \Gamma^3_{23} = +\frac{h_{3,2}}{h_3} \\ \Gamma^1_{33} = -\frac{h_3 h_{3,1}}{(h_1)^2} & \Gamma^2_{33} = -\frac{h_3 h_{3,2}}{(h_2)^2} & \Gamma^3_{33} = +\frac{h_{3,3}}{h_3} \end{array} \tag{121}$$

where $(1, 2, 3)$ stand for (u^1,u^2,u^3) respectively. Again, since the Christoffel symbols of the second kind are symmetric in their lower indices, the missing non-vanishing entries can be obtained from the given entries by permuting the lower indices.

• In Cartesian coordinate systems (x, y, z), all the Christoffel symbols of the first and second kind are identically zero.

• In cylindrical coordinate systems (ρ, ϕ, z), the non-zero Christoffel symbols of the first kind are:

$$\begin{aligned} [22, 1] &= -\rho \\ [12, 2] &= [21, 2] = \rho \end{aligned} \tag{122}$$

where $(1, 2, 3)$ stand for (ρ, ϕ, z) respectively.

• In cylindrical coordinate systems (ρ, ϕ, z), the non-zero Christoffel symbols of the second kind are:

$$\begin{aligned} \Gamma^1_{22} &= -\rho \\ \Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{\rho} \end{aligned} \tag{123}$$

where $(1, 2, 3)$ stand for (ρ, ϕ, z) respectively.

• In spherical coordinate systems (r, θ, ϕ), the non-zero Christoffel symbols of the first kind are:

$$\begin{aligned} [22,1] &= -r \\ [33,1] &= -r\sin^2\theta \\ [12,2] &= [21,2] = r \\ [33,2] &= -r^2\sin\theta\cos\theta \\ [13,3] &= [31,3] = r\sin^2\theta \\ [23,3] &= [32,3] = r^2\sin\theta\cos\theta \end{aligned} \tag{124}$$

where $(1, 2, 3)$ stand for (r, θ, ϕ) respectively.

• In spherical coordinate systems (r, θ, ϕ), the non-zero Christoffel symbols of the second kind are:

$$
\begin{aligned}
\Gamma_{22}^{1} &= -r \\
\Gamma_{33}^{1} &= -r\sin^{2}\theta \\
\Gamma_{12}^{2} &= \Gamma_{21}^{2} = \frac{1}{r} \\
\Gamma_{33}^{2} &= -\sin\theta\cos\theta \\
\Gamma_{13}^{3} &= \Gamma_{31}^{3} = \frac{1}{r} \\
\Gamma_{23}^{3} &= \Gamma_{32}^{3} = \cot\theta
\end{aligned}
\tag{125}
$$

where $(1,2,3)$ stand for (r,θ,ϕ) respectively.

• Because there is an element of arbitrariness in the choice of the coordinates order and hence their indices, the Christoffel symbols may be given in terms of coordinate symbols rather than their indices to be more explicit and avoid ambiguity; for instance in the above examples of cylindrical and spherical coordinate systems we have: $[22,1] \equiv [\phi\phi,\rho]$, $\Gamma_{12}^{2} \equiv \Gamma_{\rho\phi}^{\phi}$ for cylindrical, and $[22,1] \equiv [\theta\theta,r]$, $\Gamma_{13}^{3} \equiv \Gamma_{r\phi}^{\phi}$ for spherical.

• The Christoffel symbols may be subscripted by the symbol of the metric tensor for the given space to reveal the metric which they are based upon.

• In any coordinate system, all the Christoffel symbols of the first and second kind vanish identically *iff* all the components of the metric tensor in the given coordinate system are constants.

• In affine coordinates, all the components of the metric tensor are constants and hence all the Christoffel symbols of both kinds vanish identically.

• The number of independent Christoffel symbols of each kind (first and second) in general curvilinear coordinates is given by:

$$
N_{\mathrm{CI}} = \frac{n^{2}(n+1)}{2}
\tag{126}
$$

where n is the space dimension. The reason is that, due to the symmetry of the metric tensor there are $\frac{n(n+1)}{2}$ independent metric components, g_{ij}, and for each independent component there are n distinct Christoffel symbols.

• The following relations are useful in the manipulation of tensor expressions involving Christoffel symbols:[16]

$$\frac{1}{\sqrt{g}}\partial_j\left(\sqrt{g}g^{ij}\right) + g^{kl}\Gamma^i_{kl} = 0 \tag{127}$$

$$\partial_j\left[ik, l\right] = g_{la}\partial_j\Gamma^a_{ik} + \Gamma^a_{ik}\left[lj, a\right] + \Gamma^a_{ik}\left[aj, l\right] \tag{128}$$

3.2 Covariant Derivative

• The basis vectors in general curvilinear coordinate systems undergo changes in magnitude and direction as they move around in their own space, and hence they are functions of position. These changes should be accounted for when calculating the derivatives of tensors in such general systems. Therefore, terms based on using Christoffel symbols are added to the ordinary derivative terms to correct for these changes and this more comprehensive form of derivative is called the covariant derivative.

• Since in rectilinear coordinate systems the basis vectors are constants, the Christoffel symbol terms vanish identically and hence the covariant derivative reduces to the ordinary derivative, but in the other coordinate systems these terms are present in general.

• As a consequence of the last point, the ordinary derivative of a non-scalar tensor is a tensor *iff* the coordinate transformations are linear.

• It has been stated that the "covariant" label is an indication that the differentiation operator, $\nabla_{;i}$, is in the covariant position. However, it may also be true that "covariant" means "invariant" as pointed out earlier in the previous set of notes.

[16] The first relation is a special case of the relation: $A^{ij}_{;j} = \frac{1}{\sqrt{g}}\partial_j\left(\sqrt{g}A^{ij}\right) + A^{kl}\Gamma^i_{kl}$ noting that the covariant derivative of the metric tensor is identically zero according to the Ricci Theorem.

• Contravariant differentiation ($\nabla^{;j}$) can also be defined for covariant and contravariant tensors by raising the differentiation index using the index raising operator, e.g.

$$A_i^{\ ;j} = g^{jk} A_{i;k} \qquad \& \qquad A^{i;j} = g^{jk} A^i_{\ ;k} \tag{129}$$

However, practically such operations are rarely used.[17]

• As an example of how to obtain the covariant derivative of a tensor, let have a vector represented by contravariant components: $\mathbf{A} = A^i \mathbf{E}_i$ in general coordinates. We differentiate this vector following the normal rules of differentiation and taking account of the fact that the basis vectors in general coordinates are differentiable functions of position and hence they, unlike their Cartesian counterparts, are subject to differentiation using the product rule, that is:

$$\begin{aligned}
\partial_j \mathbf{A} &= \mathbf{E}_i \partial_j A^i + A^i \partial_j \mathbf{E}_i && \text{(product rule)} \\
&= \mathbf{E}_i \partial_j A^i + A^i \Gamma^k_{\ ij} \mathbf{E}_k && \text{(Eq. 107)} \\
&= \mathbf{E}_i \partial_j A^i + A^k \Gamma^i_{\ kj} \mathbf{E}_i && \text{(relabeling dummy indices } i \text{ \& } k\text{)} \\
&= \left(\partial_j A^i + A^k \Gamma^i_{\ kj} \right) \mathbf{E}_i \\
&= A^i_{\ ;j} \mathbf{E}_i
\end{aligned} \tag{130}$$

where $A^i_{\ ;j}$, which is a rank-2 mixed tensor, is labeled the "covariant derivative" of A^i. Similarly, for a vector represented by covariant components: $\mathbf{A} = A_i \mathbf{E}^i$ in general curvilinear coordinates we have:

$$\partial_j \mathbf{A} = A_{i;j} \mathbf{E}^i \tag{131}$$

• Following the method and techniques outlined in the previous point, to obtain the covariant derivative of a tensor in general, we start with an ordinary partial derivative

[17] An example of contravariant differentiation is in the definition of the Laplacian in general curvilinear coordinates (refer to § 4.1.4).

term of the given tensor. Then for each tensor index an extra Christoffel symbol term is added, positive for contravariant indices and negative for covariant indices, where the differentiation index is one of the lower indices in the Christoffel symbol. Hence, for a general differentiable rank-n tensor $\mathbf{A}$ the covariant derivative is given by:

$$\begin{aligned} A^{ij\ldots k}_{lm\ldots p;q} &= \partial_q A^{ij\ldots k}_{lm\ldots p} + \Gamma^i_{aq} A^{aj\ldots k}_{lm\ldots p} + \Gamma^j_{aq} A^{ia\ldots k}_{lm\ldots p} + \cdots + \Gamma^k_{aq} A^{ij\ldots a}_{lm\ldots p} \\ &\quad - \Gamma^a_{lq} A^{ij\ldots k}_{am\ldots p} - \Gamma^a_{mq} A^{ij\ldots k}_{la\ldots p} - \cdots - \Gamma^a_{pq} A^{ij\ldots k}_{lm\ldots a} \end{aligned} \tag{132}$$

• Practically, there is only one possibility for the arrangement of the indices in the Christoffel symbol terms if the following rules are observed:

(A) the second subscript index of the Christoffel symbol is the differentiation index,

(B) the concerned tensor index in the Christoffel symbol term is contracted with one of the indices of the Christoffel symbol and hence they are opposite in their lower/upper position,

(C) the contracted index is transferred from the tensor to the Christoffel symbol keeping its lower/upper position, and

(D) all the other indices of the tensor keep their names and position.

• The ordinary partial derivative term in the above covariant derivative expression (Eq. 132) represents the rate of change of the tensor components with change of position as a result of moving along the coordinate curve of the differentiated index, while the Christoffel symbol terms represent the change experienced by the local basis vectors as a result of the same movement. This can be seen from the development of Eq. 130.

• From the above discussion it is obvious that to obtain the covariant derivative, the Christoffel symbols are required and these symbols are dependent on the metric tensor; hence the covariant derivative is dependent on having the space metric.

• In all coordinate systems, the covariant derivative of a differentiable scalar function of position, f, is the same as the ordinary partial derivative, that is:

$$f_{;i} = f_{,i} = \partial_i f \tag{133}$$

This is justified by the fact that the covariant derivative is different from the ordinary partial derivative because the basis vectors in general coordinate systems are dependent on their spatial position, and since a scalar is independent of the basis vectors the covariant derivative and partial derivative are identical. This can also be concluded from the covariant derivative rule as stated in the previous points and formulated in Eq. 132.

• Several rules of normal differentiation are naturally extended to covariant differentiation. For example, covariant differentiation is a linear operation with respect to algebraic sums of tensor terms and hence the covariant derivative of a sum is the sum of the covariant derivatives of the terms:

$$(a\mathbf{A} \pm b\mathbf{B})_{;i} = a\,(\mathbf{A})_{;i} \pm b\,(\mathbf{B})_{;i} \tag{134}$$

where a and b are scalar constants and $\mathbf{A}$ and $\mathbf{B}$ are differentiable tensors. The product rule of ordinary differentiation also applies to covariant differentiation of inner and outer products of tensors:

$$(\mathbf{A} \circ \mathbf{B})_{;i} = (\mathbf{A})_{;i} \circ \mathbf{B} + \mathbf{A} \circ (\mathbf{B})_{;i} \tag{135}$$

where the symbol $\circ$ denotes an inner or outer product operator.

• According to the "Ricci Theorem", the covariant derivative of the covariant and contravariant metric tensor is zero. This has nothing to do with the metric tensor being a constant function of coordinates, which is true only for the rectilinear systems, but this arises from the fact that the covariant derivative quantifies the change with position of the basis vectors in magnitude and direction as well as the change in components, and these contributions in the case of the metric tensor cancel each other resulting in a total null effect. As a result, the metric tensor behaves as a constant with respect to the covariant derivative operation:

$$g_{ij;k} = 0 \qquad \& \qquad g^{ij}_{;k} = 0 \tag{136}$$

for all values of the indices, and hence the covariant derivative operator bypasses the metric tensor:

$$(\mathbf{g} \circ \mathbf{A})_{;k} = \mathbf{g} \circ (\mathbf{A})_{;k} \tag{137}$$

where $\mathbf{A}$ is a general tensor, $\mathbf{g}$ is the metric tensor in its covariant or contravariant form and $\circ$ denotes an inner or outer tensor product.[18]

• As a result of the Ricci Theorem, the covariant derivative operator and the index shifting operator are commutative, e.g.

$$\left(g_{ik}A^k\right)_{;j} = g_{ik}A^k_{;j} = A_{i;j} = (A_i)_{;j} = \left(g_{ik}A^k\right)_{;j} \tag{138}$$

$$(g_{mi}g_{nj}A^{mn})_{;k} = g_{mi}g_{nj}A^{mn}_{;k} = A_{ij;k} = (A_{ij})_{;k} = (g_{mi}g_{nj}A^{mn})_{;k} \tag{139}$$

• Like the metric tensor, the Kronecker delta is constant with regard to the covariant differentiation and hence the covariant derivative of the Kronecker delta is identically zero:

$$\delta^i_{j;k} = \partial_k \delta^i_j + \delta^a_j \Gamma^i_{ak} - \delta^i_a \Gamma^a_{jk} = 0 + \Gamma^i_{jk} - \Gamma^i_{jk} = 0 \tag{140}$$

The rule of the Kronecker delta may be regarded as an instance of the rule of the metric tensor, as stated by the Ricci Theorem, since the Kronecker delta is a metric tensor. Likewise, the covariant differentiation operator bypasses the Kronecker delta which is

[18] Although the metric tensor is normally used in inner product operations for raising and lowering of indices, the possibility of its involvement in outer product operations should not be ruled out.

involved in inner and outer tensor products:[19]

$$(\boldsymbol{\delta} \circ \mathbf{A})_{;k} = \boldsymbol{\delta} \circ (\mathbf{A})_{;k} \tag{141}$$

• Like the ordinary Kronecker delta, the covariant derivative of the generalized Kronecker delta is identically zero.

• For a differentiable function $f(x, y)$ of class C^2 (i.e. all the second order partial derivatives of the function do exist and are continuous), the mixed partial derivatives are equal, that is:

$$\partial_x \partial_y f = \partial_y \partial_x f \tag{142}$$

However, even if the components of a tensor satisfy this condition (i.e. being of class C^2), this is not sufficient for the equality of the mixed covariant derivatives. What is required for the mixed covariant derivatives to be equal is the vanishing of the Riemann Tensor (see § 5.1).

• Higher order covariant derivatives are similarly defined as derivatives of derivatives by successive repetition of the process of covariant differentiation; however the order of differentiation should be respected as stated in the previous point. For example, the second order mixed jk covariant derivative of a contravariant vector $\mathbf{A}$ is given by:

$$A^i_{;jk} = \partial_k \partial_j A^i + \Gamma^i_{ka} \partial_j A^a - \Gamma^a_{jk} \partial_a A^i + \Gamma^i_{ja} \partial_k A^a + A^a \left(\partial_j \Gamma^i_{ka} - \Gamma^b_{jk} \Gamma^i_{ba} + \Gamma^i_{jb} \Gamma^b_{ka} \right) \tag{143}$$

while the second order mixed jk covariant derivative of a covariant vector $\mathbf{A}$ is given by:

$$A_{i;jk} = \partial_k \partial_j A_i - \Gamma^a_{ij} \partial_k A_a - \Gamma^a_{ik} \partial_j A_a - \Gamma^a_{jk} \partial_a A_i - A_a \left(\partial_k \Gamma^a_{ij} - \Gamma^a_{ib} \Gamma^b_{jk} - \Gamma^b_{ik} \Gamma^a_{bj} \right) \tag{144}$$

[19] Like the metric tensor, the Kronecker delta is normally used in inner product operations for replacement of indices; however the possibility of its involvement in outer product operations should not be ruled out.

• The second order mixed kj covariant derivative of contravariant and covariant vectors can be obtained from the equations in the last point by interchanging the j and k indices and hence the inequality of the jk and kj mixed derivatives in general can be verified (refer to § 5.1).

• The covariant derivative of a tensor is a tensor whose covariant rank is higher than the covariant rank of the original tensor by one. Hence, the covariant derivative of a rank-n tensor of type (r, s) is a rank-$(n + 1)$ tensor of type $(r, s + 1)$.

• Covariant differentiation and contraction of index operations commute with each other, e.g.

$$\left(A^{ij}_{k}\right)_{;l}\delta^{k}_{j} = \left(A^{ij}_{k}\delta^{k}_{j}\right)_{;l} = \left(A^{ik}_{k}\right)_{;l} \tag{145}$$

• Since the Christoffel symbols vanish when the components of the metric tensor in a given coordinate system are constants, the covariant derivative is reduced to the ordinary derivative in such systems. This is particularly true in Euclidean spaces coordinated by rectilinear systems.

• The covariant derivatives of relative tensors, which are also relative tensors of the same weight as the original tensors, are obtained by adding a weight term to the normal formulae of covariant derivative. Hence, the covariant derivative of a relative scalar with weight w is given by:

$$f_{;i} = f_{,i} - wf\Gamma^{j}_{ji} \tag{146}$$

while the covariant derivative of relative tensors of higher ranks with weight w is obtained by adding the following term to the right hand side of Eq. 132:

$$-wA^{ij\ldots k}_{lm\ldots p}\Gamma^{a}_{aq} \tag{147}$$

• Unlike ordinary differentiation, the covariant derivative of a non-scalar tensor with constant components is not zero in general due to the presence of the Christoffel symbols in the definition of the covariant derivative, as given by Eq. 132.

• In rectilinear coordinates, the Christoffel symbols are identically zero because the basis vectors are constants, and hence the covariant derivative is the same as the normal partial derivative for all tensor ranks. As a result, when the components of the metric tensor, g_{ij}, are constants as in the case of rectangular coordinate systems, the covariant derivative becomes the ordinary partial derivative.

• For a differentiable covariant vector $\mathbf{A}$ which is a gradient of a scalar we have:

$$A_{i;j} = A_{j;i} \tag{148}$$

• The covariant derivative of the basis vectors of the covariant and contravariant types is identically zero:

$$\begin{aligned} \mathbf{E}_{i;j} &= \partial_j \mathbf{E}_i - \Gamma^k_{ij} \mathbf{E}_k = \Gamma^k_{ij} \mathbf{E}_k - \Gamma^k_{ij} \mathbf{E}_k = \mathbf{0} \\ \mathbf{E}^i_{;j} &= \partial_j \mathbf{E}^i + \Gamma^i_{kj} \mathbf{E}^k = -\Gamma^i_{kj} \mathbf{E}^k + \Gamma^i_{kj} \mathbf{E}^k = \mathbf{0} \end{aligned} \tag{149}$$

3.3 Absolute Derivative

• The absolute derivative of a tensor along a t-parameterized curve in an nD space with respect to the parameter t is the inner product of the covariant derivative of the tensor and the tangent vector to the curve. In brief, the absolute derivative is a covariant derivative of a tensor along a curve.

• For a tensor A^i, the inner product of $A^i_{;j}$, which is a tensor, with another tensor is a tensor. Now, if the other tensor is $\frac{du^i}{dt}$, which is the tangent vector to a t-parameterized curve L given by the equations $u^i = u^i(t)$, then the inner product:

$$A^i_{;r} \frac{du^r}{dt} \tag{150}$$

is a tensor of the same rank and type as the tensor A^i. The tensor given by the expression

150 is called the "absolute" or "intrinsic" or "absolute covariant" derivative of the tensor A^i along the curve L and is symbolized by:

$$\frac{\delta A^i}{\delta t} \tag{151}$$

• For a differentiable scalar f, the absolute derivative, like the covariant derivative, is the same as the ordinary derivative, that is:

$$\frac{\delta f}{\delta t} = \frac{df}{dt} \tag{152}$$

• The absolute derivative of a differentiable contravariant vector A^k with respect to the parameter t is given by:

$$\frac{\delta A^k}{\delta t} = \frac{dA^k}{dt} + \Gamma^k_{ij} A^i \frac{du^j}{dt} \tag{153}$$

Similarly for a differentiable covariant vector A_k we have:

$$\frac{\delta A_k}{\delta t} = \frac{dA_k}{dt} - \Gamma^i_{kj} A_i \frac{du^j}{dt} \tag{154}$$

• Absolute differentiation can be easily extended to higher rank (> 1) differentiable tensors of type (m, n) along parameterized curves. For instance, the absolute derivative of a mixed tensor of type $(1, 2)$ A^i_{jk} along a t-parameterized curve L is given by:

$$\frac{\delta A^i_{jk}}{\delta t} = A^i_{jk;b} \frac{du^b}{dt} = \frac{dA^i_{jk}}{dt} + \Gamma^i_{ab} A^a_{jk} \frac{du^b}{dt} - \Gamma^a_{jb} A^i_{ak} \frac{du^b}{dt} - \Gamma^a_{bk} A^i_{ja} \frac{du^b}{dt} \tag{155}$$

• As the absolute derivative is given generically by:

$$\frac{\delta \mathbf{A}}{\delta t} = (\mathbf{A})_{;k} \frac{du^k}{dt} \tag{156}$$

it can be seen as an instance of the chain rule of differentiation where the two contracted indices represent the in-between coordinate differential.

• Because the absolute derivative along a curve is just an inner product of the covariant derivative with the vector tangent to the curve, the well known rules of ordinary differentiation of sums and products also apply to absolute differentiation, as for covariant differentiation, that is:

$$\frac{\delta}{\delta t}(a\mathbf{A} + b\mathbf{B}) = a\frac{\delta \mathbf{A}}{\delta t} + b\frac{\delta \mathbf{B}}{\delta t} \tag{157}$$

$$\frac{\delta}{\delta t}(\mathbf{A} \circ \mathbf{B}) = \left(\frac{\delta \mathbf{A}}{\delta t} \circ \mathbf{B}\right) + \left(\mathbf{A} \circ \frac{\delta \mathbf{B}}{\delta t}\right) \tag{158}$$

where a and b are constant scalars, $\mathbf{A}$ and $\mathbf{B}$ are differentiable tensors and the symbol $\circ$ denotes an inner or outer product of tensors.

• The covariant and contravariant metric tensors are in lieu of constants with respect to absolute differentiation, that is:

$$\frac{\delta g_{ij}}{\delta t} = 0 \qquad \& \qquad \frac{\delta g^{ij}}{\delta t} = 0 \tag{159}$$

and hence they pass through the absolute derivative operator:

$$\frac{\delta\left(g_{ij}A^j\right)}{\delta t} = g_{ij}\frac{\delta A^j}{\delta t} \qquad \& \qquad \frac{\delta\left(g^{ij}A_j\right)}{\delta t} = g^{ij}\frac{\delta A_j}{\delta t} \tag{160}$$

• For coordinate systems in which all the components of the metric tensor are constants, the absolute derivative is the same as the ordinary derivative. This is the case in the rectilinear coordinate systems.

• The absolute derivative of a tensor along a given curve is unique, and hence the ordinary derivative of the tensor along that curve in a rectangular coordinate system is the same as the absolute derivative of the tensor along that curve in any other system.

4 Differential Operations

• In this section we generalize and expand what have been given in the previous notes [11] about the main differential operations which are based on the nabla operator ∇. The section will investigate these operations in general curvilinear coordinate systems and in general orthogonal coordinate systems which are a special case of the general curvilinear systems. We also investigate the two most important and widely used non-Cartesian orthogonal coordinate systems, namely the cylindrical and spherical systems, due to their particular importance.

4.1 General Curvilinear Coordinate System

• Here, we investigate the differential operations and operators in general curvilinear coordinate systems, whether orthogonal or not.

• The previous definitions of the differential operations, as given in the first set of notes, are essentially valid in general non-Cartesian coordinate systems if the operations are extended to include the basis vectors as well as the components.

• The analytical expressions of the differential operations can be obtained directly if the expression for the nabla operator ∇ and the spatial derivatives of the basis vectors in the general curvilinear coordinate system are known.

4.1.1 Gradient

• The nabla operator ∇ in general curvilinear coordinate systems is defined as follow:

$$\nabla = \mathbf{E}^i \partial_i \tag{161}$$

Hence, the gradient of a differentiable scalar function of position, f, is given by:

$$\nabla f = \mathbf{E}^i \partial_i f = \mathbf{E}^i f_{,i} \tag{162}$$

The components of this expression represent the covariant form of a rank-1 tensor, i.e. $[\nabla f]_i = f_{,i}$, as it should be since the gradient operation increases the covariant rank of a tensor by one. Since this expression consists of a contravariant basis vector and a covariant component, the gradient in general curvilinear systems is invariant under admissible transformations of coordinates.

• The contravariant form of the gradient of a scalar f can be obtained by using the index raising operator, that is:

$$[\nabla f]^i = \partial^i f = g^{ij} \partial_j f = g^{ij} f_{,j} = f^{,i} \tag{163}$$

• The gradient of a differentiable covariant vector $\mathbf{A}$ can similarly be defined as follow:

$$\begin{aligned}
\nabla \mathbf{A} &= \mathbf{E}^i \partial_i \left(A_j \mathbf{E}^j \right) \\
&= \mathbf{E}^i \mathbf{E}^j \partial_i A_j + \mathbf{E}^i A_j \partial_i \mathbf{E}^j && \text{(product rule)} \\
&= \mathbf{E}^i \mathbf{E}^j \partial_i A_j + \mathbf{E}^i A_j \left(-\Gamma^j_{ki} \mathbf{E}^k \right) && \text{(Eq. 108)} \\
&= \mathbf{E}^i \mathbf{E}^j \partial_i A_j - \mathbf{E}^i \mathbf{E}^j \Gamma^k_{ji} A_k && \text{(relabeling dummy indices } j \text{ \& } k\text{)} \\
&= \mathbf{E}^i \mathbf{E}^j \left(\partial_i A_j - \Gamma^k_{ji} A_k \right) && \text{(taking common factor } \mathbf{E}^i \mathbf{E}^j\text{)} \\
&= \mathbf{E}^i \mathbf{E}^j A_{j;i} && \text{(definition of covariant derivative)}
\end{aligned} \tag{164}$$

Similarly, for a differentiable contravariant vector $\mathbf{A}$ the gradient is given by:

$$\begin{aligned}
\nabla\mathbf{A} &= \mathbf{E}^i\partial_i\left(A^j\mathbf{E}_j\right) \\
&= \mathbf{E}^i\mathbf{E}_j\partial_i A^j + \mathbf{E}^i A^j\partial_i\mathbf{E}_j && \text{(product rule)} \\
&= \mathbf{E}^i\mathbf{E}_j\partial_i A^j + \mathbf{E}^i A^j\left(\Gamma^k_{ji}\mathbf{E}_k\right) && \text{(Eq. 107)} \\
&= \mathbf{E}^i\mathbf{E}_j\partial_i A^j + \mathbf{E}^i\mathbf{E}_j\Gamma^j_{ki}A^k && \text{(relabeling dummy indices } j \text{ \& } k\text{)} \\
&= \mathbf{E}^i\mathbf{E}_j\left(\partial_i A^j + \Gamma^j_{ki}A^k\right) && \text{(taking common factor } \mathbf{E}^i\mathbf{E}_j\text{)} \\
&= \mathbf{E}^i\mathbf{E}_j A^j{}_{;i} && \text{(definition of covariant derivative)}
\end{aligned} \tag{165}$$

The components of the gradients of covariant and contravariant vectors represent, respectively, the covariant and mixed forms of a rank-2 tensor, as they should be since the gradient operation increases the covariant rank of a tensor by one.

• The gradient of higher rank tensors is similarly defined. For example, the gradient of a rank-2 tensor is given by:

$$\nabla\mathbf{A} = \mathbf{E}^i\mathbf{E}^j\mathbf{E}^k\left(\partial_i A_{jk} - \Gamma^l_{ji}A_{lk} - \Gamma^l_{ki}A_{jl}\right) = \mathbf{E}^i\mathbf{E}^j\mathbf{E}^k A_{jk;i} \qquad \text{(covariant)} \tag{166}$$

$$\nabla\mathbf{A} = \mathbf{E}^i\mathbf{E}_j\mathbf{E}_k\left(\partial_i A^{jk} + \Gamma^j_{li}A^{lk} + \Gamma^k_{li}A^{jl}\right) = \mathbf{E}^i\mathbf{E}_j\mathbf{E}_k A^{jk}_{;i} \qquad \text{(contravariant)} \tag{167}$$

$$\nabla\mathbf{A} = \mathbf{E}^i\mathbf{E}^j\mathbf{E}_k\left(\partial_i A_j^{\ k} - \Gamma^l_{ji}A_l^{\ k} + \Gamma^k_{li}A_j^{\ l}\right) = \mathbf{E}^i\mathbf{E}^j\mathbf{E}_k A_{j;i}^{\ k} \qquad \text{(mixed)} \tag{168}$$

$$\nabla\mathbf{A} = \mathbf{E}^i\mathbf{E}_j\mathbf{E}^k\left(\partial_i A^j_{\ k} + \Gamma^j_{li}A^l_{\ k} - \Gamma^l_{ki}A^j_{\ l}\right) = \mathbf{E}^i\mathbf{E}_j\mathbf{E}^k A^j_{\ k;i} \qquad \text{(mixed)} \tag{169}$$

4.1.2 Divergence

• Generically, the divergence of a differentiable contravariant vector $\mathbf{A}$ is defined as follow:

$$\nabla\cdot\mathbf{A} = \mathbf{E}^i\partial_i\cdot\left(A^j\mathbf{E}_j\right) = \mathbf{E}^i\cdot\partial_i\left(A^j\mathbf{E}_j\right) = \mathbf{E}^i\cdot\left(A^j_{;i}\mathbf{E}_j\right) = \left(\mathbf{E}^i\cdot\mathbf{E}_j\right)A^j_{;i} = \delta^i_j A^j_{;i} = A^i_{;i} \tag{170}$$

In more details, the divergence of a differentiable contravariant vector A^i is a scalar obtained by contracting the covariant derivative index with the contravariant index of the vector, and hence:

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= A^i_{;i} \\
&= \partial_i A^i + \Gamma^i_{ji} A^j && \text{(definition of covariant derivative)} \\
&= \partial_i A^i + A^j \frac{1}{\sqrt{g}} \partial_j \left(\sqrt{g}\right) && \text{(Eq. 115)} \\
&= \partial_i A^i + A^i \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g}\right) && \text{(renaming dummy index } j\text{)} \\
&= \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} A^i\right) && \text{(product rule)}
\end{aligned} \tag{171}$$

where g is the determinant of the covariant metric tensor g_{ij}. The last equality may be called the Voss-Weyl formula.

• The divergence can also be obtained by raising the first index of the covariant derivative of a covariant vector using a contracting contravariant metric tensor:

$$\begin{aligned}
g^{ji} A_{j;i} &= \left(g^{ji} A_j\right)_{;i} && \text{(Eq. 137)} \\
&= \left(A^i\right)_{;i} && \text{(raising index)} \\
&= A^i_{;i} \\
&= \nabla \cdot \mathbf{A} && \text{(Eq. 171)}
\end{aligned} \tag{172}$$

as before.

• Based on the previous point, the divergence of a covariant vector A_j is obtained by using the raising operator, that is

$$A^i_{;i} = g^{ij} A_{j;i} \tag{173}$$

• For a rank-2 contravariant tensor $\mathbf{A}$, the divergence is generically defined by:

$$\nabla \cdot \mathbf{A} = \mathbf{E}^i \partial_i \cdot \left(A^{jk}\mathbf{E}_j\mathbf{E}_k\right) = \left(\mathbf{E}^i \cdot \mathbf{E}_j\right)\mathbf{E}_k A^{jk}_{\ \ ;i} = \delta^i_j \mathbf{E}_k A^{jk}_{\ \ ;i} = \mathbf{E}_k A^{ik}_{\ \ ;i} \tag{174}$$

The components of this expression represent a contravariant vector, as it should be since the divergence operation reduces the contravariant rank of a tensor by one.

• More generally, considering the tensor components, the divergence of a differentiable rank-2 contravariant tensor A^{ij} is a contravariant vector obtained by contracting the covariant derivative index with one of the contravariant indices, e.g.

$$[\nabla \cdot \mathbf{A}]^j = A^{ij}_{\ \ ;i} \tag{175}$$

And for a rank-2 mixed tensor A^i_j we have:

$$[\nabla \cdot \mathbf{A}]_j = A^i_{j;i} \tag{176}$$

• Similarly, for a general tensor of type (m, n): $\mathbf{A} = A^{i_1 i_2 \cdots i_k \cdots i_m}_{j_1 j_2 \cdots\cdots\cdots j_n}$, the divergence with respect to its k^{th} contravariant index is defined by:

$$[\nabla \cdot \mathbf{A}]^{i_1 i_2 \cdots\ \cdots i_m}_{j_1 j_2 \cdots\cdots\cdots j_n} = \left(A^{i_1 i_2 \cdots s \cdots i_m}_{j_1 j_2 \cdots\cdots\cdots j_n}\right)_{;s} \tag{177}$$

with the absence of the contracted contravariant index i_k on the left hand side.

4.1.3 Curl

• The curl of a differentiable vector is the cross product of the nabla operator ∇ with the vector, e.g. the curl of a vector $\mathbf{A}$ represented by covariant components is given by:

$$\begin{aligned}
\operatorname{curl}\mathbf{A} &= \nabla \times \mathbf{A} \\
&= \mathbf{E}^i \partial_i \times A_j \mathbf{E}^j \\
&= \mathbf{E}^i \times \partial_i \left(A_j \mathbf{E}^j\right) \\
&= \mathbf{E}^i \times \left(A_{j;i} \mathbf{E}^j\right) && \text{(Eq. 131)} \\
&= A_{j;i} \left(\mathbf{E}^i \times \mathbf{E}^j\right) \\
&= A_{j;i}\, \underline{\epsilon}^{ijk} \mathbf{E}_k && \text{(Eq. 78)} \\
&= \underline{\epsilon}^{ijk} A_{j;i} \mathbf{E}_k \\
&= \frac{\epsilon^{ijk}}{\sqrt{g}} \left(\partial_i A_j - \Gamma^l_{ji} A_l\right) \mathbf{E}_k && \text{(Eq. 49 \& definition of covariant derivative)}
\end{aligned} \tag{178}$$

and hence the k^{th} contravariant component of $\operatorname{curl}\mathbf{A}$ is:

$$[\nabla \times \mathbf{A}]^k = \frac{\epsilon^{ijk}}{\sqrt{g}} \left(\partial_i A_j - \Gamma^l_{ji} A_l\right) \tag{179}$$

On expanding the last equation for the three components of a 3D space, considering that the terms of the Christoffel symbols cancel out due to their symmetry in the two lower indices,[20] we obtain:

$$[\nabla \times \mathbf{A}]^1 = \frac{1}{\sqrt{g}} \left(\partial_2 A_3 - \partial_3 A_2\right) \tag{180}$$

$$[\nabla \times \mathbf{A}]^2 = \frac{1}{\sqrt{g}} \left(\partial_3 A_1 - \partial_1 A_3\right) \tag{181}$$

$$[\nabla \times \mathbf{A}]^3 = \frac{1}{\sqrt{g}} \left(\partial_1 A_2 - \partial_2 A_1\right) \tag{182}$$

[20] That is: $A_{i;j} - A_{j;i} = \partial_j A_i - A_k \Gamma^k_{ij} - \partial_i A_j + A_k \Gamma^k_{ji} = \partial_j A_i - A_k \Gamma^k_{ij} - \partial_i A_j + A_k \Gamma^k_{ij} = \partial_j A_i - \partial_i A_j = A_{i,j} - A_{j,i}$.

Hence, Eq. 179 will reduce to:

$$[\nabla \times \mathbf{A}]^k = \frac{\epsilon^{ijk}}{\sqrt{g}} \partial_i A_j \tag{183}$$

4.1.4 Laplacian

• Generically, the Laplacian of a differentiable scalar function of position, f, is defined as follow:

$$\nabla^2 f = \operatorname{div}(\operatorname{grad} f) = \nabla \cdot (\nabla f) \tag{184}$$

Hence the simplest approach for obtaining the Laplacian in general coordinates is to insert the expression for the gradient, ∇f, into the expression for the divergence. However, because in general curvilinear coordinates the divergence is defined only for contravariant tensors whereas the gradient of a scalar is a covariant tensor, the index of the gradient should be raised first before applying the divergence operation, that is:

$$[\nabla f]^i = \partial^i f = g^{ij} \partial_j f \tag{185}$$

Now, according to Eq. 171 the divergence is given by:

$$\nabla \cdot \mathbf{A} = A^i{}_{;i} = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} A^i \right) \tag{186}$$

On defining $\mathbf{A} \equiv \nabla f = \mathbf{E}_i \partial^i f$ and replacing A^i in the last equation with $\partial^i f$ we obtain:

$$\nabla^2 f = \nabla \cdot \left(\mathbf{E}_i \partial^i f \right) = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f \right) \tag{187}$$

which is the expression for the Laplacian of a scalar function f in general curvilinear coordinates.

• Another approach for developing the Laplacian expression in general coordinates is to apply the first principles by using the definitions and basic properties of the operations involved, that is:

$$\begin{aligned}
\nabla^2 f &= \nabla \cdot (\nabla f) \\
&= \mathbf{E}^i \partial_i \cdot \left(\mathbf{E}^j \partial_j f\right) \\
&= \mathbf{E}^i \cdot \partial_i \left(\mathbf{E}^j \partial_j f\right) \\
&= \mathbf{E}^i \cdot \partial_i \left(\mathbf{E}^j f_{,j}\right) \\
&= \mathbf{E}^i \cdot \left(\mathbf{E}^j f_{,j;i}\right) && \text{(Eq. 131)} \\
&= \left(\mathbf{E}^i \cdot \mathbf{E}^j\right) f_{,j;i} \\
&= g^{ij} f_{,j;i} && \text{(Eq. 51)} \\
&= \left(g^{ij} f_{,j}\right)_{;i} && \text{(Eq. 137)} \\
&= \left(g^{ij} \partial_j f\right)_{;i} \\
&= \partial_i \left(g^{ij} \partial_j f\right) + \left(g^{kj} \partial_j f\right) \Gamma^i_{ki} && \text{(Eq. 132)} \\
&= \partial_i \left(g^{ij} \partial_j f\right) + \left(g^{ij} \partial_j f\right) \Gamma^k_{ik} && \text{(renaming dummy indices } i \text{ \& } k\text{)} \\
&= \partial_i \left(g^{ij} \partial_j f\right) + \left(g^{ij} \partial_j f\right) \frac{1}{\sqrt{g}} \left(\partial_i \sqrt{g}\right) && \text{(Eq. 115)} \\
&= \frac{1}{\sqrt{g}} \left[\sqrt{g} \partial_i \left(g^{ij} \partial_j f\right) + \left(g^{ij} \partial_j f\right) \partial_i \sqrt{g}\right] && \text{(taking } \tfrac{1}{\sqrt{g}} \text{ factor)} \\
&= \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f\right) && \text{(product rule)}
\end{aligned} \tag{188}$$

as before.

• The Laplacian of a scalar f may also be shorthand notated with:

$$\nabla^2 f = g^{ij} f_{,ij} \tag{189}$$

• The Laplacian of non-scalar tensors can be similarly defined. For example, the Laplacian of a vector $\mathbf{B}$ is a vector $\mathbf{A}$ (i.e. $\mathbf{A} = \nabla^2 \mathbf{B}$) which may be defined in general coordinates as:

$$A^i = g^{jk} B^i{}_{;jk} \qquad \text{and} \qquad A_i = g^{jk} B_{i;jk} \tag{190}$$

• The Laplacian of a tensor is a tensor of the same rank and variance type.

4.2 General Orthogonal Coordinate System

• In this section we state the main differential operations in general orthogonal coordinate systems. These operations are special cases of the operations in general curvilinear systems which were derived in the previous section. However, due to the wide spread use of orthogonal systems, it is worth to state the most important of these operations although they can be easily obtained from the formulae of general curvilinear systems.

• General orthogonal coordinate systems are identified in the following notes by the coordinates $(u^1, \ldots, u^n)$ with unit basis vectors $(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ and scale factors $(h_1, \ldots, h_n)$ where:[21]

$$\mathbf{u}_i = \sum_j \frac{1}{h_i} \frac{\partial x^j}{\partial u^i} \mathbf{e}_j = \sum_j h_i \frac{\partial u^i}{\partial x^j} \mathbf{e}_j \qquad \text{(no sum on } i\text{)} \tag{191}$$

$$h_i = |\mathbf{E}_i| = \left| \frac{\partial \mathbf{r}}{\partial u_i} \right| = \left[\sum_j \left(\frac{\partial x^j}{\partial u^i} \right)^2 \right]^{1/2} = \left[\sum_j \left(\frac{\partial u^i}{\partial x^j} \right)^2 \right]^{-1/2} \tag{192}$$

In the last equations, x^j and $\mathbf{e}_j$ are respectively the coordinates and unit basis vectors in the Cartesian rectangular system, and $\mathbf{r}$ is the position vector in that system.

4.2.1 Gradient

• The nabla operator in general orthogonal coordinates is given by:

$$\nabla = \sum_i \frac{\mathbf{u}_i}{h_i} \frac{\partial}{\partial u^i} \tag{193}$$

[21] In orthogonal coordinates, the covariant and contravariant normalized basis vectors are identical, as established previously in § 1.7, and hence $\mathbf{u}^i = \mathbf{u}_i$ and $\mathbf{e}^j = \mathbf{e}_j$.

Hence, the gradient of a differentiable scalar f in orthogonal coordinates is given by:

$$\nabla f = \sum_i \frac{\mathbf{u}_i}{h_i} \frac{\partial f}{\partial u^i} \tag{194}$$

4.2.2 Divergence

• The divergence in orthogonal coordinates can be obtained from Eq. 171. Since for orthogonal coordinate systems the metric tensor is diagonal with $\sqrt{g} = h_1 h_2 h_3$ in a 3D space and $h_i A^i = \hat{A}^i$ (component-wise with no summation), the last line of Eq. 171 becomes:

$$\nabla \cdot \mathbf{A} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left(\sqrt{g} A^i \right) = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial u^i} \left(\frac{h_1 h_2 h_3}{h_i} \hat{A}^i \right) \tag{195}$$

where $\mathbf{A}$ is a contravariant differentiable vector and $\hat{A}^i$ represents the physical components (refer to Eq. 28).[22] This equation is the divergence of a vector in general orthogonal coordinates as defined in vector calculus.

4.2.3 Curl

• The curl of a differentiable vector $\mathbf{A}$ in orthogonal coordinate systems in 3D spaces is given by:

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{u}_1 & h_2 \mathbf{u}_2 & h_3 \mathbf{u}_3 \\ \frac{\partial}{\partial u^1} & \frac{\partial}{\partial u^2} & \frac{\partial}{\partial u^3} \\ h_1 \hat{A}_1 & h_2 \hat{A}_2 & h_3 \hat{A}_3 \end{vmatrix} \tag{196}$$

where the hat indicates a physical component. The last equation may also be given in a more compact form as:

$$[\nabla \times \mathbf{A}]_i = \sum_{k=1}^{3} \frac{\epsilon_{ijk} h_i}{h_1 h_2 h_3} \frac{\partial (h_k \hat{A}_k)}{\partial u^j} \tag{197}$$

[22] In orthogonal coordinate systems the physical components are the same for covariant and contravariant forms, as established before in § 1.7, and hence $\hat{A}_i = \hat{A}^i$.

4.2.4 Laplacian

• For general orthogonal coordinate systems in 3D spaces we have:

$$\sqrt{g} = h_1 h_2 h_3 \qquad \& \qquad g^{ij} = \frac{\delta^{ij}}{h_i h_j} \quad \text{(no sum on } i \text{ or } j\text{)} \tag{198}$$

and hence Eq. 187 becomes:

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial u^i} \left(\frac{h_1 h_2 h_3}{h_i^2} \frac{\partial f}{\partial u^i} \right) \tag{199}$$

which is the Laplacian of a scalar function of position, f, in orthogonal coordinates as defined in vector calculus.

4.3 Cylindrical Coordinate System

• For cylindrical systems identified by the coordinates (ρ, ϕ, z), the orthonormal basis vectors are $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$.[23] We use for brevity $\mathbf{e}_{\rho\phi}$ as a shorthand notation for the unit dyad $\mathbf{e}_\rho \mathbf{e}_\phi$ and similar notations for the other dyads.

4.3.1 Gradient

• The gradient of a differentiable scalar f is:

$$\nabla f = \mathbf{e}_\rho \partial_\rho f + \mathbf{e}_\phi \frac{1}{\rho} \partial_\phi f + \mathbf{e}_z \partial_z f \tag{200}$$

[23]Hence the given components (i.e. A_ρ, A_ϕ and A_z) are physical (see § 1.7). Despite that, we do not use hats since the components are suffixed with coordinate symbols (refer to § 1.7).

• The gradient of a differentiable vector $\mathbf{A}$ is:

$$\begin{aligned}\nabla\mathbf{A} &= \mathbf{e}_{\rho\rho}A_{\rho,\rho}+\mathbf{e}_{\rho\phi}A_{\phi,\rho}+\mathbf{e}_{\rho z}A_{z,\rho}+\\ &\quad \mathbf{e}_{\phi\rho}\left(\frac{1}{\rho}A_{\rho,\phi}-\frac{A_{\phi}}{\rho}\right)+\mathbf{e}_{\phi\phi}\left(\frac{1}{\rho}A_{\phi,\phi}+\frac{A_{\rho}}{\rho}\right)+\mathbf{e}_{\phi z}\frac{1}{\rho}A_{z,\phi}+\\ &\quad \mathbf{e}_{z\rho}A_{\rho,z}+\mathbf{e}_{z\phi}A_{\phi,z}+\mathbf{e}_{zz}A_{z,z}\end{aligned} \tag{201}$$

4.3.2 Divergence

• The divergence of a differentiable vector $\mathbf{A}$ is:

$$\nabla\cdot\mathbf{A}=\frac{1}{\rho}\left[\partial_{\rho}\left(\rho A_{\rho}\right)+\partial_{\phi}A_{\phi}+\rho\partial_{z}A_{z}\right] \tag{202}$$

• The divergence of a differentiable rank-2 tensor $\mathbf{A}$ is a vector given by:[24]

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \mathbf{e}_{\rho}\left(A_{\rho\rho,\rho}+\frac{A_{\rho\rho}-A_{\phi\phi}}{\rho}+\frac{1}{\rho}A_{\phi\rho,\phi}+A_{z\rho,z}\right)+\\ &\quad \mathbf{e}_{\phi}\left(A_{\rho\phi,\rho}+\frac{2A_{\rho\phi}}{\rho}+\frac{1}{\rho}A_{\phi\phi,\phi}+A_{z\phi,z}+\frac{A_{\phi\rho}-A_{\rho\phi}}{\rho}\right)+\\ &\quad \mathbf{e}_{z}\left(A_{\rho z,\rho}+\frac{A_{\rho z}}{\rho}+\frac{1}{\rho}A_{\phi z,\phi}+A_{zz,z}\right)\end{aligned} \tag{203}$$

4.3.3 Curl

• The curl of a differentiable vector $\mathbf{A}$ is:

$$\nabla\times\mathbf{A}=\frac{1}{\rho}\begin{vmatrix}\mathbf{e}_{\rho} & \rho\mathbf{e}_{\phi} & \mathbf{e}_{z}\\ \partial_{\rho} & \partial_{\phi} & \partial_{z}\\ A_{\rho} & \rho A_{\phi} & A_{z}\end{vmatrix} \tag{204}$$

[24] It should be understood that $\nabla\cdot\mathbf{A}$ is lower than the original tensor $\mathbf{A}$ by just one contravariant index and hence, unlike the common use of this notation, it is not scalar in general.

4.3.4 Laplacian

• The Laplacian of a differentiable scalar f is:

$$\nabla^2 f = \partial_{\rho\rho} f + \frac{1}{\rho}\partial_\rho f + \frac{1}{\rho^2}\partial_{\phi\phi} f + \partial_{zz} f \tag{205}$$

• The Laplacian of a differentiable vector $\mathbf{A}$ is:

$$\begin{aligned} \nabla^2 \mathbf{A} \quad = \quad & \mathbf{e}_\rho \left[\partial_\rho \left(\frac{1}{\rho}\partial_\rho \left(\rho A_\rho\right)\right) + \frac{1}{\rho^2}\partial_{\phi\phi} A_\rho + \partial_{zz} A_\rho - \frac{2}{\rho^2}\partial_\phi A_\phi\right] + \\ & \mathbf{e}_\phi \left[\partial_\rho \left(\frac{1}{\rho}\partial_\rho \left(\rho A_\phi\right)\right) + \frac{1}{\rho^2}\partial_{\phi\phi} A_\phi + \partial_{zz} A_\phi + \frac{2}{\rho^2}\partial_\phi A_\rho\right] + \\ & \mathbf{e}_z \left[\frac{1}{\rho}\partial_\rho \left(\rho \partial_\rho A_z\right) + \frac{1}{\rho^2}\partial_{\phi\phi} A_z + \partial_{zz} A_z\right] \end{aligned} \tag{206}$$

4.4 Spherical Coordinate System

• For spherical coordinate systems identified by the coordinates (r, θ, ϕ), the orthonormal basis vectors are $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.[25] We use for brevity $\mathbf{e}_{r\theta}$ as a shorthand notation for the dyad $\mathbf{e}_r\mathbf{e}_\theta$ and similar notations for the other unit dyads.

4.4.1 Gradient

• The gradient of a differentiable scalar f is:

$$\nabla f = \mathbf{e}_r \partial_r f + \mathbf{e}_\theta \frac{1}{r}\partial_\theta f + \mathbf{e}_\phi \frac{1}{r\sin\theta}\partial_\phi f \tag{207}$$

[25] Again, the components are physical and we do not use hats.

• The gradient of a differentiable vector $\mathbf{A}$ is:

$$\begin{aligned}\nabla\mathbf{A} &= \mathbf{e}_{rr}A_{r,r} + \mathbf{e}_{r\theta}A_{\theta,r} + \mathbf{e}_{r\phi}A_{\phi,r} + \\ &\quad \mathbf{e}_{\theta r}\left(\frac{A_{r,\theta}}{r} - \frac{A_\theta}{r}\right) + \mathbf{e}_{\theta\theta}\left(\frac{A_{\theta,\theta}}{r} + \frac{A_r}{r}\right) + \mathbf{e}_{\theta\phi}\frac{A_{\phi,\theta}}{r} + \\ &\quad \mathbf{e}_{\phi r}\left(\frac{A_{r,\phi}}{r\sin\theta} - \frac{A_\phi}{r}\right) + \mathbf{e}_{\phi\theta}\left(\frac{A_{\theta,\phi}}{r\sin\theta} - \frac{A_\phi\cot\theta}{r}\right) + \mathbf{e}_{\phi\phi}\left(\frac{A_{\phi,\phi}}{r\sin\theta} + \frac{A_r}{r} + \frac{A_\theta\cot\theta}{r}\right)\end{aligned} \tag{208}$$

4.4.2 Divergence

• The divergence of a differentiable vector $\mathbf{A}$ is:

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial\left(r^2A_r\right)}{\partial r} + r\frac{\partial\left(\sin\theta A_\theta\right)}{\partial\theta} + r\frac{\partial A_\phi}{\partial\phi}\right] \tag{209}$$

• The divergence of a differentiable rank-2 tensor $\mathbf{A}$ is a vector given by:

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \mathbf{e}_r\left(\frac{\partial_r\left(r^2A_{rr}\right)}{r^2} + \frac{\partial_\theta\left(A_{\theta r}\sin\theta\right)}{r\sin\theta} + \frac{\partial_\phi A_{\phi r}}{r\sin\theta} - \frac{A_{\theta\theta}+A_{\phi\phi}}{r}\right) + \\ &\quad \mathbf{e}_\theta\left(\frac{\partial_r\left(r^3A_{r\theta}\right)}{r^3} + \frac{\partial_\theta\left(A_{\theta\theta}\sin\theta\right)}{r\sin\theta} + \frac{\partial_\phi A_{\phi\theta}}{r\sin\theta} + \frac{A_{\theta r}-A_{r\theta}-A_{\phi\phi}\cot\theta}{r}\right) + \\ &\quad \mathbf{e}_\phi\left(\frac{\partial_r\left(r^3A_{r\phi}\right)}{r^3} + \frac{\partial_\theta\left(A_{\theta\phi}\sin\theta\right)}{r\sin\theta} + \frac{\partial_\phi A_{\phi\phi}}{r\sin\theta} + \frac{A_{\phi r}-A_{r\phi}+A_{\phi\theta}\cot\theta}{r}\right)\end{aligned} \tag{210}$$

4.4.3 Curl

• The curl of a differentiable vector $\mathbf{A}$ is:

$$\nabla\times\mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix} \tag{211}$$

4.4.4 Laplacian

- The Laplacian of a differentiable scalar f is:

$$\nabla^2 f = \partial_{rr} f + \frac{2}{r}\partial_r f + \frac{1}{r^2}\partial_{\theta\theta} f + \frac{\cos\theta}{r^2 \sin\theta}\partial_\theta f + \frac{1}{r^2 \sin^2\theta}\partial_{\phi\phi} f \tag{212}$$

- The Laplacian of a differentiable vector $\mathbf{A}$ is:

$$\begin{aligned}
\nabla^2 \mathbf{A} \quad = \quad & \mathbf{e}_r \left[\partial_r \left(\frac{\partial_r \left(r^2 A_r\right)}{r^2}\right) + \frac{\partial_\theta \left(\sin\theta \partial_\theta A_r\right)}{r^2 \sin\theta} + \frac{\partial_{\phi\phi} A_r}{r^2 \sin^2\theta} - \frac{2\partial_\theta \left(A_\theta \sin\theta\right)}{r^2 \sin\theta} - \frac{2\partial_\phi A_\phi}{r^2 \sin\theta}\right] + \\
& \mathbf{e}_\theta \left[\frac{\partial_r \left(r^2 \partial_r A_\theta\right)}{r^2} + \frac{1}{r^2}\partial_\theta \left(\frac{\partial_\theta \left(A_\theta \sin\theta\right)}{\sin\theta}\right) + \frac{\partial_{\phi\phi} A_\theta}{r^2 \sin^2\theta} + \frac{2\partial_\theta A_r}{r^2} - \frac{2\cot\theta}{r^2 \sin\theta}\partial_\phi A_\phi\right] + \\
& \mathbf{e}_\phi \left[\frac{\partial_r \left(r^2 \partial_r A_\phi\right)}{r^2} + \frac{1}{r^2}\partial_\theta \left(\frac{\partial_\theta \left(A_\phi \sin\theta\right)}{\sin\theta}\right) + \frac{\partial_{\phi\phi} A_\phi}{r^2 \sin^2\theta} + \frac{2\partial_\phi A_r}{r^2 \sin\theta} + \frac{2\cot\theta}{r^2 \sin\theta}\partial_\phi A_\theta\right]
\end{aligned} \tag{213}$$

5 Tensors in Applications

• In this section we conduct a preliminary investigation of some commonly-used tensors in physical and mathematical applications of tensor calculus in anticipation of the forthcoming set of notes. Most of these tensors come from differential geometry, fluid, continuum and relativistic mechanics, since these disciplines are intimately linked to tensor calculus as large parts of the subject were developed, and are still developing, within these disciplines. These tensors also form the building blocks of several physical and mathematical theories. We would like to insist that these are just a few partially representative examples for the use of tensors in scientific and mathematical applications to have more familiarity with tensor language and techniques and hence they are not meant to provide a comprehensive view in any way. We should also indicate that some tensors are defined differently in different disciplines and hence the given definitions and properties may not be thorough or general.

5.1 Riemann Tensor

• This rank-4 tensor, which is also called Riemann curvature tensor and Riemann-Christoffel tensor, is a property of the space. It characterizes important properties of spaces and surfaces and hence it plays an important role in geometry in general and in non-Euclidean geometries in particular.

• The covariant differentiation operators in mixed derivatives are not commutative and hence for a covariant vector **A** we have:

$$A_{j;kl} - A_{j;lk} = R^{i}_{jkl} A_i \tag{214}$$

where R^i_{jkl} is the Riemann tensor of the second kind which is given by:

$$R^i_{\ jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^r_{jl}\Gamma^i_{rk} - \Gamma^r_{jk}\Gamma^i_{rl} \tag{215}$$

• The last equation can be put into the following mnemonic determinantal form:

$$R^i_{\ jkl} = \begin{vmatrix} \partial_k & \partial_l \\ \Gamma^i_{jk} & \Gamma^i_{jl} \end{vmatrix} + \begin{vmatrix} \Gamma^r_{jl} & \Gamma^r_{jk} \\ \Gamma^i_{rl} & \Gamma^i_{rk} \end{vmatrix} \tag{216}$$

• The Riemann-Christoffel tensor of the second kind is also called the mixed Riemann-Christoffel tensor.

• From Eq. 214, it is obvious that the mixed second order covariant derivatives are equal *iff* the Riemann tensor of the second kind vanishes identically.

• On lowering the contravariant index of the Riemann tensor of the second kind, the Riemann tensor of the first kind is obtained:

$$R_{ijkl} = g_{ia} R^a_{\ jkl} \tag{217}$$

• Alternatively, the Riemann tensor of the first kind can be established independently as:

$$\begin{aligned} R_{ijkl} &= \partial_k [jl,i] - \partial_l [jk,i] + [il,r]\,\Gamma^r_{jk} - [ik,r]\,\Gamma^r_{jl} \\ &= \frac{1}{2}\left(\partial_j\partial_k g_{il} + \partial_i\partial_l g_{jk} - \partial_j\partial_l g_{ik} - \partial_i\partial_k g_{jl}\right) + [il,r]\,\Gamma^r_{jk} - [ik,r]\,\Gamma^r_{jl} \\ &= \frac{1}{2}\left(\partial_j\partial_k g_{il} + \partial_i\partial_l g_{jk} - \partial_j\partial_l g_{ik} - \partial_i\partial_k g_{jl}\right) + g^{rs}\left([il,r]\,[jk,s] - [ik,r]\,[jl,s]\right) \end{aligned} \tag{218}$$

• The first line of the last equation can be cast in the following mnemonic determinantal form:

$$R_{ijkl} = \begin{vmatrix} \partial_k & \partial_l \\ [jk,i] & [jl,i] \end{vmatrix} + \begin{vmatrix} \Gamma^r_{jk} & \Gamma^r_{jl} \\ [ik,r] & [il,r] \end{vmatrix} \tag{219}$$

• Similarly, the Riemann-Christoffel tensor of the second kind can be obtained by raising the first covariant index of the Riemann-Christoffel tensor of the first kind:

$$R^{i}{}_{jkl} = g^{ia} R_{ajkl} \tag{220}$$

• The Riemann-Christoffel tensor of the first kind is also called the covariant (or totally covariant) Riemann-Christoffel tensor.

• The covariant differentiation operators become commutative when the metric makes the Riemann tensor of either kind vanish.

• For the mixed second order covariant derivatives of a contravariant vector **A** we have:

$$A^{j}_{;kl} - A^{j}_{;lk} = R^{j}{}_{ilk} A^{i} \tag{221}$$

which is similar to Eq. 214 for a covariant vector **A**.

• The Riemann-Christoffel tensor vanishes identically *iff* the space is globally flat. Hence, the Riemann tensor is zero in Euclidean spaces, and consequently the mixed second order covariant derivatives, which become ordinary derivatives, are equal when the C^2 continuity condition is satisfied.

• The Riemann curvature tensor depends only on the metric which, in general curvilinear coordinates, is a function of position and hence the Riemann tensor follows this dependency on position. Yes, for affine coordinates the metric tensor is constant and hence the Riemann tensor vanishes identically.

• The totally covariant Riemann tensor satisfies the following symmetric and skew-symmetric relations in its four indices:

$$\begin{aligned} R_{ijkl} &= R_{klij} && \text{(block symmetry)} \\ &= -R_{jikl} && \text{(anti-symmetry in the first two indices)} \\ &= -R_{ijlk} && \text{(anti-symmetry in the last two indices)} \end{aligned} \tag{222}$$

• The skew-symmetric property of the covariant Riemann tensor with respect to the last two indices also applies to the mixed Riemann tensor:

$$R^{i}{}_{jkl} = -R^{i}{}_{jlk} \tag{223}$$

• As a consequence of the first and second anti-symmetric properties of the covariant Riemann tensor, the entries of the Riemann tensor with identical values of the first two indices or/and the last two indices are zero.

• As a consequence of the skew-symmetric properties of the Riemann tensor, all entries of the tensor with identical values of more than two indices (e.g. R_{iiji}) are zero.

• In an nD space, the Riemann tensor has n^4 components.

• As a consequence of the symmetric and anti-symmetric properties of the Riemann tensor, in an nD space there are three types of distinct non-vanishing entries:

A. Entries with only two distinct indices (type R_{ijij}) which count:

$$N_1 = \frac{n\,(n-1)}{2} \tag{224}$$

B. Entries with only three distinct indices (type R_{ijik}) which count:

$$N_2 = \frac{n\,(n-1)\,(n-2)}{2} \tag{225}$$

C. Entries with four distinct indices (type R_{ijkl}) which count:

$$N_3 = \frac{n(n-1)(n-2)(n-3)}{12} \tag{226}$$

• By adding the numbers of the three types of non-zero distinct entries, as given in the last point, it can be shown that the Riemann tensor in an nD space has a total of

$$N_{\mathrm{RI}} = N_1 + N_2 + N_3 = \frac{n^2(n^2-1)}{12} \tag{227}$$

independent components which do not vanish identically. For example, in a 2D Riemannian space the Riemann tensor has $2^4 = 16$ components; however there is only one independent component (with the principal suffix 1212) which is not identically zero represented by the following four dependent components:

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112} \tag{228}$$

Similarly, in a 3D Riemannian space the Riemann tensor has $3^4 = 81$ components but only six of these are distinct non-zero entries which are the ones with the following principal suffixes:

$$1212,\ 1313,\ 1213,\ 2123,\ 3132,\ 2323 \tag{229}$$

where the permutations of indices in each of these suffixes are subject to the symmetric and anti-symmetric properties of the four indices of the Riemann tensor, as in the case of a 2D space in the above example, and hence these permutations do not produce independent entries.

• Following the pattern in the last point, in a 4D Riemannian space the Riemann tensor has $4^4 = 256$ components but only 20 of these are independent non-zero entries, while in a 5D Riemannian space the Riemann tensor has $5^4 = 625$ components but only 50 are independent non-zero entries.

• The Riemann tensor satisfies the following identity:

$$R_{ijkl;s} + R_{iljk;s} = R_{iksl;j} + R_{ikjs;l} \tag{230}$$

• A necessary and sufficient condition that a manifold for which there is a coordinate system with all the components of the metric tensor being constants[26] is that:

$$R_{ijkl} = 0 \tag{231}$$

• On contracting the first covariant index with the contravariant index of the Riemann tensor of the second kind we obtain:

$$\begin{aligned}
R^{i}_{ikl} &= \partial_k \Gamma^{i}_{il} - \partial_l \Gamma^{i}_{ik} + \Gamma^{r}_{il}\Gamma^{i}_{rk} - \Gamma^{r}_{ik}\Gamma^{i}_{rl} && (j = i \text{ in Eq. } 215) \\
&= \partial_k \Gamma^{i}_{il} - \partial_l \Gamma^{i}_{ik} + \Gamma^{r}_{il}\Gamma^{i}_{rk} - \Gamma^{i}_{rk}\Gamma^{r}_{il} && (\text{relabeling dummy } i, r \text{ in last term}) \\
&= \partial_k \Gamma^{i}_{il} - \partial_l \Gamma^{i}_{ik} \\
&= \partial_k \left[\partial_l \left(\ln \sqrt{g}\right)\right] - \partial_l \left[\partial_k \left(\ln \sqrt{g}\right)\right] && (\text{Eq. } 115) \\
&= \partial_k \partial_l \left(\ln \sqrt{g}\right) - \partial_l \partial_k \left(\ln \sqrt{g}\right) \\
&= \partial_k \partial_l \left(\ln \sqrt{g}\right) - \partial_k \partial_l \left(\ln \sqrt{g}\right) && (C^2 \text{ condition is assumed}) \\
&= 0
\end{aligned} \tag{232}$$

That is:

$$R^{i}_{ikl} = 0 \tag{233}$$

[26] This may be called flat pseudo Riemannian manifold.

5.1.1 Bianchi identities

• The Riemann tensor of the first and second kind satisfies a number of identities called the Bianchi identities.

• The first Bianchi identity is:

$$\begin{aligned} R_{ijkl} + R_{iljk} + R_{iklj} &= 0 \qquad \text{(first kind)} \\ R^{i}{}_{jkl} + R^{i}{}_{ljk} + R^{i}{}_{klj} &= 0 \qquad \text{(second kind)} \end{aligned} \tag{234}$$

These two forms of the first identity can be obtained from each other by the raising and lowering operators.

• The above first Bianchi identity is an instance of the fact that by fixing the position of one of the four indices and permuting the other three indices cyclically, the algebraic sum of these three permuting forms is zero, that is:

$$\begin{aligned} R_{ijkl} + R_{iljk} + R_{iklj} &= 0 \qquad (i \text{ fixed}) \\ R_{ijkl} + R_{ljik} + R_{kjli} &= 0 \qquad (j \text{ fixed}) \\ R_{ijkl} + R_{likj} + R_{jlki} &= 0 \qquad (k \text{ fixed}) \\ R_{ijkl} + R_{kijl} + R_{jkil} &= 0 \qquad (l \text{ fixed}) \end{aligned} \tag{235}$$

• Another one of the Bianchi identities is:

$$\begin{aligned} R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} &= 0 \qquad \text{(first kind)} \\ R^{i}{}_{jkl;m} + R^{i}{}_{jlm;k} + R^{i}{}_{jmk;l} &= 0 \qquad \text{(second kind)} \end{aligned} \tag{236}$$

Again, these two forms can be obtained from each other by the raising and lowering operators.

• The Bianchi identities are valid regardless of the metric.

5.2 Ricci Tensor

• The Ricci tensor of the first kind is obtained by contracting the contravariant index with the last covariant index of the Riemann tensor of the second kind, that is:

$$R_{ij} = R^a_{ija} = \partial_j \Gamma^a_{ia} - \partial_a \Gamma^a_{ij} + \Gamma^a_{bj}\Gamma^b_{ia} - \Gamma^a_{ba}\Gamma^b_{ij} \tag{237}$$

and hence it is a rank-2 tensor.

• The Ricci tensor, as given by the last equation, can be written in the following mnemonic determinantal form:

$$R_{ij} = \begin{vmatrix} \partial_j & \partial_a \\ \Gamma^a_{ij} & \Gamma^a_{ia} \end{vmatrix} + \begin{vmatrix} \Gamma^a_{bj} & \Gamma^a_{ba} \\ \Gamma^b_{ij} & \Gamma^b_{ia} \end{vmatrix} \tag{238}$$

• Because of Eq. 115 (i.e. $\Gamma^j_{ij} = \partial_i \left(\ln \sqrt{g}\right)$), the Ricci tensor can also be written in the following forms as well as several other forms:

$$\begin{aligned} R_{ij} &= \partial_j \partial_i (\ln \sqrt{g}) - \partial_a \Gamma^a_{ij} + \Gamma^a_{bj}\Gamma^b_{ia} - \Gamma^b_{ij}\partial_b (\ln \sqrt{g}) \\ &= \partial_j \partial_i (\ln \sqrt{g}) + \Gamma^a_{bj}\Gamma^b_{ia} - \partial_a \Gamma^a_{ij} - \Gamma^b_{ij}\partial_b (\ln \sqrt{g}) \\ &= \partial_j \partial_i (\ln \sqrt{g}) + \Gamma^a_{bj}\Gamma^b_{ia} - \frac{1}{\sqrt{g}}\partial_a \left(\sqrt{g}\Gamma^a_{ij}\right) \end{aligned} \tag{239}$$

where g is the determinant of the covariant metric tensor.

• The Ricci tensor of the first kind is symmetric, that is:

$$R_{ij} = R_{ji} \tag{240}$$

• On raising the first index of the Ricci tensor of the first kind, the Ricci tensor of the second kind is obtained:

$$R^i{}_j = g^{ik} R_{kj} \tag{241}$$

• The Ricci scalar, which is also called the curvature scalar and the curvature invariant, is the result of contracting the indices of the Ricci tensor of the second kind, that is:

$$R = R^i_i \tag{242}$$

• Since the Ricci scalar is obtained by raising a subscript index of the Ricci tensor of the first kind using the raising operator followed by contracting the two indices, it can be written as:

$$R = g^{ij} R_{ij} = g^{ij} \left[\partial_j \partial_i \left(\ln \sqrt{g} \right) + \Gamma^a_{bj} \Gamma^b_{ia} - \frac{1}{\sqrt{g}} \partial_a \left(\sqrt{g} \Gamma^a_{ij} \right) \right] \tag{243}$$

where the expression in the square brackets is obtained from the last line of Eq. 239; similar expressions can be obtained from the other lines of that equation.

• In an nD space, the Ricci tensor has n^2 entries. However, because of its symmetry it possesses a maximum of

$$N_{\text{RD}} = \frac{n(n+1)}{2} \tag{244}$$

distinct entries. As an example, in the 4D manifold of general relativity $n = 4$, and hence the Ricci tensor has $4^2 = 16$ components. However, due to the symmetry of the Ricci tensor there are only ten independent entries according to Eq. 244. The gravitational field equations in a free space are obtained by setting the Ricci tensor components equal to zero, and hence there are ten partial differential equations describing the gravitational field in this space according to the general relativistic mechanics.

5.3 Einstein Tensor

• The Einstein tensor **G** is a rank-2 tensor defined in terms of the Ricci tensor **R** and the Ricci curvature scalar R as follow:[27]

$$\begin{aligned} G_{mn} &= R_{mn} - \frac{1}{2} g_{mn} R & &\text{(covariant)} \\ G^{mn} &= R^{mn} - \frac{1}{2} g^{mn} R & &\text{(contravariant)} \\ G^{m}_{n} &= R^{m}_{n} - \frac{1}{2} \delta^{m}_{n} R & &\text{(mixed)} \end{aligned} \tag{245}$$

• Since both the Ricci tensor and the metric tensor are symmetric, the Einstein tensor is symmetric as well.

• The divergence of the Einstein tensor vanishes at all points of the space for any Riemannian metric.

• On contracting the Bianchi identity twice with using the anti-symmetric properties of the Riemann tensor we obtain:

$$G^{mn}_{\ \ ;n} = 0 \tag{246}$$

which is inline with the above statement. The following form can also be derived based on the Bianchi identity:

$$g^{jk} G_{ki;j} = 0 \tag{247}$$

5.4 Infinitesimal Strain Tensor

• This is a rank-2 tensor which describes the state of strain in a continuum medium and hence it is used in continuum and fluid mechanics.

[27] We notate this tensor with **G** rather than **E**, which is more natural, to avoid potential confusion with the first displacement gradient tensor (see § 5.6).

• The infinitesimal strain tensor $\boldsymbol{\gamma}$ is defined by:[28]

$$\boldsymbol{\gamma} = \frac{\nabla \mathbf{d} + \nabla \mathbf{d}^T}{2} \tag{248}$$

where $\mathbf{d}$ is the displacement vector and the superscript T represents matrix transposition. The displacement vector $\mathbf{d}$ represents the change in distance and direction which an infinitesimal element of the medium experiences as a consequence of the applied stress.

• In Cartesian coordinates with tensor notation, the last equation is given as:

$$\gamma_{ij} = \frac{\partial_i d_j + \partial_j d_i}{2} \tag{249}$$

5.5 Stress Tensor

• The stress tensor, which is also called Cauchy stress tensor, is a rank-2 symmetric[29] tensor used for *transforming* a normal vector to a surface *to* a traction vector acting on that surface, that is:

$$\mathbf{T} = \boldsymbol{\sigma}\mathbf{n} \tag{250}$$

where $\mathbf{T}$ is the traction vector, $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{n}$ is the normal vector. This is usually represented in tensor notation using Cartesian coordinates as:

$$T_i = \sigma_{ij} n_j \tag{251}$$

• The diagonal components of the stress tensor represent normal stresses while the off-diagonal components represent shear stresses.

[28]Some authors do not include the factor $\frac{1}{2}$ in the definition of $\boldsymbol{\gamma}$.

[29]In fact it is symmetric in many applications (e.g. in the flow of Newtonian fluids) but not all, as it can be asymmetric in some cases. We also choose to define it within the context of Cauchy stress law which is more relevant to the continuum mechanics; however it can be defined differently in other disciplines and in a more general form.

• Because the stress tensor is symmetric, in an nD space it possesses $\frac{n(n+1)}{2}$ independent components instead of n^2. Hence in a 3D space (which is the ordinary space for this tensor) it has six independent components.

• In fluid dynamics, the stress tensor (or total stress tensor) is decomposed into two main parts: a viscous contribution part and a pressure contribution part. The viscous part may then be split into a normal stress and a shear stress while the pressure part may be split into a hydrostatic pressure and an extra pressure.

5.6 Displacement Gradient Tensors

• These are rank-2 tensors which are denoted by **E** and **Δ**. They are defined in Cartesian coordinates using tensor notation as:

$$E_{ij} = \frac{\partial x_i}{\partial x'_j} \qquad \& \qquad \Delta_{ij} = \frac{\partial x'_i}{\partial x_j} \tag{252}$$

where x and x' represent the Cartesian coordinates of an observed continuum particle at the present and past times respectively. These tensors may also be called deformation gradient tensors.

• **E** quantifies the displacement of the particle at the present time relative to its position at the past time, while **Δ** quantifies its displacement at the past time relative to its position at the present time.

• From their definitions, it is obvious that **E** and **Δ** are inverses of each other and hence:

$$E_{ik}\Delta_{kj} = \delta_{ij} \tag{253}$$

5.7 Finger Strain Tensor

• This, which may also be called the left Cauchy-Green deformation tensor, is a rank-2 tensor used in the fluid and continuum mechanics to describe the strain in a continuum object, e.g. fluid, in a series of time frames. It is defined as:

$$\mathbf{B} = \mathbf{E} \cdot \mathbf{E}^T \tag{254}$$

which in Cartesian coordinates with tensor notation becomes:

$$B_{ij} = \frac{\partial x_i}{\partial x'_k} \frac{\partial x_j}{\partial x'_k} \tag{255}$$

where $\mathbf{E}$ is the first displacement gradient tensor as defined in § 5.6, the superscript T represents matrix transposition, and the indexed x and x' represent the Cartesian coordinates of an element of the continuum at the present and past times respectively.

5.8 Cauchy Strain Tensor

• This, which may also be called the right Cauchy-Green deformation tensor, is the inverse of the Finger strain tensor and hence it is denoted by $\mathbf{B}^{-1}$. Consequently, it is defined as:

$$\mathbf{B}^{-1} = \mathbf{\Delta}^T \cdot \mathbf{\Delta} \tag{256}$$

which in Cartesian coordinates with tensor notation becomes:

$$B_{ij}^{-1} = \frac{\partial x'_k}{\partial x_i} \frac{\partial x'_k}{\partial x_j} \tag{257}$$

where $\mathbf{\Delta}$ is the second displacement gradient tensor as defined in § 5.6.

• The Finger and Cauchy strain tensors may be labeled as "finite strain tensors" as opposite to infinitesimal strain tensors. They are symmetric positive definite tensors; moreover they become the unity tensor when the change in the state of the object from the past to the present times consists of rotation and translation with no deformation.

5.9 Velocity Gradient Tensor

• This is a rank-2 tensor which is often used in fluid dynamics and rheology. As its name suggests, it is the gradient of the velocity vector $\mathbf{v}$ and hence it is given in Cartesian coordinates by:

$$[\nabla \mathbf{v}]_{ij} = \partial_i v_j \tag{258}$$

• The velocity gradient tensor in other coordinate systems can be obtained from the expressions of the gradient of vectors in these systems, as given, for instance, in § 4.3 and § 4.4 for cylindrical and spherical coordinates.

• The term "velocity gradient tensor" my also be used for the transpose of this tensor, i.e. $(\nabla \mathbf{v})^T$.

• The velocity gradient tensor is usually decomposed into a symmetric part which is the rate of strain tensor $\mathbf{S}$ (see § 5.10), and an anti-symmetric part which is the vorticity tensor $\bar{\mathbf{S}}$ (see § 5.11), that is:

$$\nabla \mathbf{v} = \mathbf{S} + \bar{\mathbf{S}} \tag{259}$$

5.10 Rate of Strain Tensor

• This tensor, which is also called the rate of deformation tensor, is the symmetric part of the velocity gradient tensor and hence is given by:[30]

$$\mathbf{S} = \frac{\nabla \mathbf{v} + (\nabla \mathbf{v})^T}{2} \tag{260}$$

[30]Some authors do not include the factor $\frac{1}{2}$ in the definition of $\mathbf{S}$ and $\bar{\mathbf{S}}$ and hence this factor is moved to the definition of $\nabla \mathbf{v}$. Also these tensors are commonly denoted by $\dot{\boldsymbol{\gamma}}$ and $\boldsymbol{\omega}$ respectively.

which, in tensor notation with Cartesian coordinates, is given by:

$$S_{ij} = \frac{\partial_i v_j + \partial_j v_i}{2} \tag{261}$$

• The rate of strain tensor is a quantitative measure of the local rate at which neighboring material elements of a deforming continuum move with respect to each other.

• As a rank-2 symmetric tensor, it has $\frac{n(n+1)}{2}$ independent components which is six in a 3D space.

• The rate of strain tensor is related to the infinitesimal strain tensor (refer to § 5.4) by:

$$\mathbf{S} = \frac{\partial \boldsymbol{\gamma}}{\partial t} \tag{262}$$

where t is time. Hence, the rate of strain tensor is normally denoted by $\dot{\boldsymbol{\gamma}}$ where the dot represents the temporal rate of change.

5.11 Vorticity Tensor

• This is the anti-symmetric part of the velocity gradient tensor and hence is given by:

$$\bar{\mathbf{S}} = \frac{\nabla \mathbf{v} - (\nabla \mathbf{v})^T}{2} \tag{263}$$

which, in tensor notation with Cartesian coordinates, is given by:

$$\bar{S}_{ij} = \frac{\partial_i v_j - \partial_j v_i}{2} \tag{264}$$

• The vorticity tensor quantifies the local rate of rotation of a deforming continuum medium.

• As a rank-2 anti-symmetric tensor, it has $\frac{n(n-1)}{2}$ independent components which is three in a 3D space. These three components added to the six components of the rate of strain tensor give nine independent components which is the total number of independent components of their parent tensor $\nabla \mathbf{v}$.

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