Math 3228 - Week 9

- The Riemann Zeta function
- Extension to the whole complex plane
- Connection with prime numbers
- The prime number theorem
- The functional equation

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The Riemann zeta function

- The *Riemann zeta function* $\zeta(z)$ is defined on the half-plane $\{z \in \mathbf{C} : \mathrm{Re}(z) > 1\}$ by the formula

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \dots,$$

thus for instance

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$$

(see Assignment). Notice that if $z = x + iy$, then

$$\zeta(x+iy) = \sum_{n=1}^{\infty} \frac{1}{n^x n^{iy}} = \sum_{n=1}^{\infty} \frac{e^{-iy \log n}}{n^x}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^x}$ is convergent for $x > 1$ by the integral test, we thus see that the sum defining $\zeta(z)$ is absolutely convergent. In fact we have

$$|\zeta(z)| \leq \sum_{n=1}^{\infty} \frac{1}{n^x} \leq 1 + \int_1^{\infty} \frac{1}{a^x}\, da = 1 + \frac{1}{x-1}.$$

- This function is analytic on the half-plane. Indeed by Morera's theorem it suffices to show that $\int_\gamma \zeta(z)\ dz = 0$ for all closed contours γ on the half-plane. We can write this as

$$\int_\gamma \zeta(z)\ dz = \int_\gamma \sum_{n=1}^\infty \frac{1}{n^z}\ dz.$$

Because γ is a closed contour in the open half-plane, there exists an $\varepsilon > 0$ such that $\mathrm{Re}(z) > 1 + \varepsilon$ for all $z \in \gamma$. Thus $|\frac{1}{n^z}| \leq \frac{1}{n^{1+\varepsilon}}$ for all $z \in \gamma$. Since $\frac{1}{n^{1+\varepsilon}}$ is uniformly convergent, we may use the Weierstrass M-test to swap the sum and integral to write this as

$$\sum_{n=1}^\infty \int_\gamma \frac{1}{n^z}\ dz.$$

But each integral is zero by Cauchy's theorem (note that $\frac{1}{n^z} = e^{-z\log n}$ is clearly analytic in z for each $n \geq 1$) so we have $\int_\gamma \zeta(z)\ dz = 0$ as desired.

- The Riemann zeta function is connected to the Gamma function as follows. Recall that for all $\mathrm{Re}(z) > 1$ we have

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\ dt.$$

Now let $n \geq 1$ be an integer. Making the change of variables $t = na$, we obtain

$$\Gamma(z) = \int_0^\infty (na)^{z-1}e^{-na}\ nda$$

or in other words

$$\int_0^\infty a^{z-1}e^{-na}\ da = \frac{1}{n^z}\Gamma(z).$$

Now we sum in n to obtain

$$\sum_{n=0}^\infty \int_0^\infty a^{z-1}e^{-na}\ da = \zeta(z)\Gamma(z).$$

Now we have to swap the sum and integral again. Writing $z = x + iy$ we see that $a^{z-1}e^{-na}$ has magnitude $a^{x-1}e^{-na}$. Summing this over all n would give $a^{x-1}\frac{e^{-a}}{1-e^{-a}} = \frac{a^{x-1}}{e^a-1}$ by the geometric series formula. This is absolutely integrable in a (for $0 \leq a \leq 1$ we use the bound $e^a - 1 \geq a$ and the observation that a^{x-2} is absolutely integrable on $[0, 1]$ when $x > 1$; when $a > 1$ we use the bound $e^a - 1 > e^a/2$ and observe that $a^{x-1}e^{-a}$ is absolutely integrable on $[1, \infty)$ for any x), so we may use the Lebesgue dominated convergence theorem to justify the swapping of the sum and integral:

$$\int_0^\infty a^{z-1} \sum_{n=0}^\infty e^{-na}\ da = \zeta(z)\Gamma(z).$$

Using the geometric series formula as above, we thus have

$$\zeta(z)\Gamma(z) = \int_0^\infty \frac{a^{z-1}}{e^a - 1}\ da.$$

Now fix z and let $f(w)$ be the function

$$f(w) := \frac{e^{(z-1)\mathrm{Log}_0(w)}}{e^w - 1}$$

where $\mathrm{Log}_0(w)$ is the branch of the logarithm whose argument ranges between 0 and 2π. So for w just above the positive real axis $w = a + \varepsilon i$, we have $f(a + \varepsilon i) \approx \frac{a^{z-1}}{e^a-1}$; for w just below the positive real axis $w = a - \varepsilon i$, we have $f(a + \varepsilon i) \approx e^{2\pi i(z-1)}\frac{a^{z-1}}{e^a-1}$. Thus if we let $\gamma_{r,\varepsilon}$ be the clockwise contour consisting of the leftward half-infinite ray from $+\infty - \varepsilon i$ to $r - \varepsilon i$, the (nearly full) circle from $r - \varepsilon i$ to $r + \varepsilon i$ transversed once clockwise, and then the half-infinite ray from εi to $+\infty + \varepsilon i$, we see that

$$\lim_{\varepsilon \to 0} \int_{\gamma_{r,\varepsilon}} f(w)\ dw = (1 - e^{2\pi i(z-1)}) \int_r^\infty \frac{a^{z-1}}{e^a - 1}\ da - \int_{|w|=r} f(w)\ dw.$$

Let us get rid of the integral on the circle $|w| = r$. Since $\frac{1}{e^w-1}$ has a simple pole at the origin we see that $|\frac{1}{e^w-1}| \leq \frac{C}{r}$ on this circle for some constant $C > 0$. Also writing $w = re^{it}$, $0 \leq t < 2\pi$ we see that

$w^{z-1} = r^{z-1}e^{it(z-1)}$ and hence $|w^{z-1}| \leq r^{x-1}e^{2\pi|z-1|}$. Thus $|f(w)| \leq Ce^{2\pi|z-1|}r^{x-2}$ on the circle, and hence

$$|\int_{|w|=r} f(w)\ dw| \leq 2\pi r C e^{2\pi|z-1|}r^{x-2}$$

which goes to zero as $r \to 0$ because $x > 1$. Thus we have

$$\lim_{r,\varepsilon\to 0}\int_{\gamma_{r,\varepsilon}} f(w)\ dw = (1-e^{2\pi i(z-1)})\int_0^\infty \frac{a^{z-1}}{e^a-1}\ da = (1-e^{2\pi i(z-1)})\zeta(z)\Gamma(z).$$

Actually, since $f(w)$ has no poles except on the integers (and a discontinuity on the positive real axis) we see that the integral $\gamma_{r,\varepsilon} f(w)\ dw$ does not actually depend on r or ε as long as they are small. So we can write

$$\int_{\gamma_{r,\varepsilon}} f(w)\ dw = (1-e^{2\pi i(z-1)})\zeta(z)\Gamma(z)$$

or equivalently (using the previous formula)

$$\zeta(z) = \frac{1}{\Gamma(z)(1-e^{2\pi i(z-1)})}((1-e^{2\pi i(z-1)})\int_r^\infty \frac{a^{z-1}}{e^a-1}\ da - \int_{|w|=r} f(w)\ dw).$$

These integrals is actually convergent for all z in the complex plane (not just those with $\mathrm{Re}(z) > 1$); the point being is that we have moved away from the origin $w = 0$ where singularities occur. Thus this gives an analytic definition of the ζ function for all z in $\mathbf{C}$, except possibly when z is an integer, in which case $1 - e^{2\pi i(z-1)}$ has a simple zero (the analyticity can be proven by yet another tedious application of Morera's theorem, which we omit). But we know that Γ has a simple pole for every negative integer z (or 0), so in fact ζ has no poles except at 1 (we already know ζ has no poles for the positive integers).

- The above formula tells us that at 1, $\zeta(z)$ has at most a simple pole. Indeed we can work out its residue here, which is equal to $\lim_{\alpha\to 0^+}\alpha\zeta(1+\alpha)$. Observe from the integral test that

$$\zeta(1+\alpha) = \sum_{n=1}^\infty \frac{1}{n^{1+\alpha}} \leq \int_1^\infty \frac{1}{a^{1+\alpha}}\ da = \frac{1}{\alpha}$$

and

$$\zeta(1+\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}} \geq 1 + \int_1^{\infty} \frac{1}{a^{1+\alpha}}\, da = 1 + \frac{1}{\alpha}$$

and hence by the squeeze test

$$\lim_{\alpha \to 0^+} \alpha\zeta(1+\alpha) = 1.$$

Thus ζ has a simple pole with residue 1 at 1, and no other poles.

- Thus the poles of ζ are very well understood. The *zeroes* of ζ, on the other hand, are much more difficult, and are closely related to the distribution of the prime numbers.

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Connection with prime numbers

- Now we connect the Riemann zeta function $\zeta(z)$ with the prime numbers $p = 2, 3, 5, 7, \ldots$. Let $\mathrm{Re} z > 1$. Observe that the series

$$\sum_{n \text{ is a power of } 2} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{(2^2)^z} + \ldots$$

is absolutely convergent and is equal to $\frac{1}{1-2^{-z}}$ by the geometric series formula. Similarly

$$\sum_{n \text{ is a power of } 3} \frac{1}{n^z} = 1 + \frac{1}{3^z} + \frac{1}{3^{2z}} + \ldots$$

is absolutely convergent and equal to $\frac{1}{1-3^{-z}}$. Multiplying the two together and collecting all the terms, we see that

$$\sum_{n \text{ is a power of } 2 \text{ times a power of } 3} \frac{1}{n^z} = \sum_{j,k \geq 0} \frac{1}{(2^j 3^k)^z}$$

is absolutely convergent and is equal to $\frac{1}{1-2^{-z}}\frac{1}{1-3^{-z}}$. Continuing in this fashion, we see that

$$\sum_{n \text{ is a power of } 2 \text{ times a power of } 3 \text{ times a power of } 5} \frac{1}{n^z}$$

converges to $\frac{1}{1-2^{-z}}\frac{1}{1-3^{-z}}\frac{1}{1-5^{-z}}$. As you can see, these series continue to increase monotonically. Using the unique factorization theorem (every positive integer can be written as the product of powers of distinct primes in a unique way) (and the monotone convergence theorem) we thus see that

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \text{ prime}} \frac{1}{1-p^{-z}}$$

whenever $\mathrm{Re}(z) > 1$. This is known as *Euler's formula*. In particular, we can take absolute values and conclude that

$$|\zeta(z)| = \prod_{p \text{ prime}} \frac{1}{|1-p^{-z}|} \geq \prod_{p \text{ prime}} \frac{1}{1+p^{-x}}.$$

Now use the estimate $1+y < e^y$ for $|y| \leq 1/2$ to conclude that

$$|\zeta(z)| \geq \prod_{p \text{ prime}} e^{-p^{-x}} = \exp(-\sum_{p \text{ prime}} p^{-x}) \geq \exp(-\sum_{n=1}^{\infty} n^{-x}).$$

But this sum is absolutely convergent by the integral test, and thus $|\zeta(z)| > 0$ for $\mathrm{Re}(z) > 1$; i.e. ζ has no zeroes to the right of 1.

- We can also take logarithms, and define the log-ζ function for $\mathrm{Re}(z) > 1$ to be

$$\log \zeta(z) := \sum_{p \text{ prime}} -\mathrm{Log}(1-p^{-z})$$

where Log is the principal branch of the logarithm (note that $1-p^{-z}$ stays on the right half-plane if $\mathrm{Re} z > 1$). This is clearly a branch of the logarithm of ζ. Differentiating this (one can use the generalized Cauchy integral formulae to make this rigorous), we thus see that

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{p \text{ prime}} \frac{p^{-z}\log p}{1-p^{-z}} = -\sum_{p \text{ prime}} \sum_{m \geq 1} \log p p^{-mz}.$$

- We have just proven that ζ has no zeroes to the right of 1. We now show that ζ also has no zeroes on the line $\mathrm{Re}(z) = 1$.

- **Lemma.** Suppose $z = 1 + it$ for some $t \neq 0$. Then $\zeta(z)$ does not have a zero at $1 + it$. (For $t = 0$ we already know that ζ has a simple pole).

- **Proof.** Suppose for contradiction that ζ had a zero of order m at $1+it$ for some $m \geq 1$, then $\frac{\zeta'}{\zeta}$ would have a simple pole with residue $m \geq 1$ at $1 + it$. In particular we have

$$\frac{\zeta'}{\zeta}(1 + \varepsilon + it) = \frac{m}{\varepsilon} + O(1).$$

Suppose that ζ also had a zero of order n at $1 + 2it$ (n could be zero), so

$$\frac{\zeta'}{\zeta}(1 + \varepsilon + 2it) = \frac{n}{\varepsilon} + O(1).$$

Meanwhile $\frac{\zeta'}{\zeta}$ has a simple pole of residue 1 at 1, so

$$\frac{\zeta'}{\zeta}(1 + \varepsilon) = \frac{-1}{\varepsilon} + O(1).$$

We combine these three facts as

$$3\frac{\zeta'}{\zeta}(1 + \varepsilon) + 4\frac{\zeta'}{\zeta}(1 + \varepsilon + it) + \frac{\zeta'}{\zeta}(1 + \varepsilon + 2it) = \frac{4m + n - 3}{\varepsilon} + O(1).$$

In particular, we see that the left-hand side has positive real part for ε large enough. But we can rewrite the left-hand side as

$$-\sum_{p \text{ prime}} \sum_{m \geq 1} \frac{\log m}{p^{m(1+\varepsilon)}}(3 + 4p^{-itm} + p^{-2itm})$$

which has real part

$$-\sum_{p \text{ prime}} \sum_{m \geq 1} \frac{\log p}{p^{m(1+\varepsilon)}}(3 + 4\cos(mt \log p) + \cos(2mt \log p)).$$

But $3 + 4\cos x + \cos(2x) = 2 + 4\cos x + 2\cos x^2 = 2(1 + \cos x)^2 \geq 0$, so this is negative; a contradiction. □

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The prime number theorem

- For any integer $N > 1$, let $\pi(N)$ denote the number of primes less than or equal to N. We can now prove (glossing over a few details, because the theorem is quite long and messy) the famous *prime number theorem*, which asserts that $\pi(N)$ is approximately $N/\log N$:

- **Prime number theorem.** $\lim_{N\to+\infty} \frac{\pi(N)}{N/\log N} = 1$.

- This theorem was first conjectured by Legendre and Gauss in around 1800. It was one of the outstanding mathematical problems of the nineteenth century, and was finally proven by Hadamard and de la Vallée Poussin in 1986.

- To prove this we have to introduce a new object, the *Möbius function* $\mu(n)$. This is defined as $\mu(n) = (-1)^k$ if n is the product of k distinct primes (thus $\mu(1) = 1$, $\mu(2) = \mu(3) = -1$, $\mu(6) = +1$), and $\mu(n) = 0$ otherwise (i.e. of n has some repeated primes, e.g. $\mu(12) = 0$). Thus μ oscillates between -1, 0, and +1. Its relationship to the Riemann zeta function can be seen from the formula

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \prod_{p \text{ prime}} (1 - \frac{1}{p^z})$$

  for $\text{Re}(z) > 1$, as can be seen by multiplying out the right-hand side and examining all the terms which appear (one can easily verify that the collection of such terms is absolutely convergent when $\text{Re}(z) > 1$ and so there is no problem justifying the expansion of the infinite product). In other words,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \frac{1}{\zeta(z)}$$

  for $\text{Re}(z) > 1$.

- We now prove a key result concerning the Möbius function.

- **Theorem.** The sum $\sum_{n=1}^{\infty} \frac{\mu(n)}{n}$ is (conditionally) convergent to zero.

- **Proof.** Since $\zeta(z)$ has a simple pole at 1, $\frac{1}{\zeta(z)}$ has a simple zero at 1 (if we remove the singularity). In principle, this finishes the proof because all we need to do is substitute $z = 1$ in the above formula.

Unfortunately the above formula is only rigorously derived for $\mathrm{Re}(z) > 1$, so we need to do a fair bit more work to conclude.

- The function $\frac{1}{\zeta(z)}$ is analytic on all of $\mathrm{Re}(z) \geq 1$ (if we remove the singularity at 1) because we already showed that ζ has no zeroes for $\mathrm{Re}(z) \geq 1$. We now let $R > 0$ be a large number, and consider the semicircular contour $\gamma_R(t) := 1 + Re^{it}$, $-\pi/2 < t < \pi$.

- **Lemma 1.** We have $0 = \lim_{N\to\infty} \int_{\gamma_R} \frac{N^{z-1}}{\zeta(z)}(\frac{1}{z-1} + \frac{z-1}{R^2})\ dz$.

- **Proof.** We know that $\frac{1}{\zeta(z)}$ is analytic on the line segment from $1 - Ri$ to $1 + Ri$, hence it is also analytic a little bit to the left of this line segment (analyticity is an open property). Thus we can find a contour γ'_R from $1 + Ri$ to $1 - Ri$ which lives entirely in the half-space where $\mathrm{Re} z < 1$. Then by the Cauchy theorem

$$0 = \int_{\gamma_R+\gamma'_R} \frac{N^{z-1}}{\zeta(z)}(\frac{1}{z-1} + \frac{z-1}{R^2})\ dz,$$

since the integrand on the right-hand side is analytic inside $\gamma_R + \gamma'_R$ except for 1, where it has a removable singularity (recall $\frac{1}{\zeta(z)}$ has a simple zero here to cancel out the $\frac{1}{z-1}$ pole). □

- We now split $\frac{1}{\zeta(z)} = f_{\leq N} + f_{>N}$, where

$$f_{\leq N}(z) := \sum_{n\leq N} \frac{\mu(n)}{n^z}$$

and

$$f_{>N}(z) = \sum_{n>N} \frac{\mu(n)}{n^z}.$$

- **Lemma 2.** We have $|\int_{\gamma_R} N^{z-1} f_{>N}(z)(\frac{1}{z-1} + \frac{z-1}{R^2})\ dz| \leq 2\pi/R$.

- **Proof.** The left-hand side is less than or equal to

$$\pi R \sup_{z\in\gamma_R} |N^{z-1}||f_{>N}(z)||\frac{1}{z-1} + \frac{z-1}{R^2}|.$$

Write $z = x + iy$. Observe that if $z \in \gamma_R$, then $\frac{1}{z-1} = \frac{\overline{z-1}}{R^2}$, and hence

$$|\frac{1}{z-1} + \frac{z-1}{R^2}| = \frac{2(x-1)}{R^2}.$$

Also we have $|N^{z-1}| = N^{x-1}$, while (since $|\mu(n)| \leq 1$ for all n)

$$|f_{>N}(z)| \leq \sum_{n>N} \frac{1}{n^x} \leq \int_N^\infty \frac{da}{a^x} = \frac{N^{x-1}}{x-1}.$$

Putting this all together, we obtain the result. □

- **Lemma 3.** We have

$$|\int_{\gamma_R} N^{z-1} f_{\leq N}(z)(\frac{1}{z-1} + \frac{z-1}{R^2})\ dz - 2\pi i \sum_{n=1}^N \frac{\mu(n)}{n}| \leq 2\pi/R + 2\pi/N.$$

- **Proof.** We introduce the contour γ_R^- which is the other half of the circle traversed by γ_R anticlockwise, i.e. $\gamma_R^-(t) := 1 + Re^{it}$, $\pi/2 \leq t \leq 3\pi/2$. Observe that $f_{\leq N}(z)$ is analytic everywhere, and so by the Cauchy integral formula (or residue theorem)

$$\int_{\gamma_R + \gamma_R^-} N^{z-1} f_{\leq N}(z)(\frac{1}{z-1} + \frac{z-1}{R^2})\ dz = 2\pi i f_{\leq N}(1) = 2\pi i \sum_{n=1}^N \frac{\mu(n)}{n}.$$

So to prove the lemma it will suffice to show that

$$|\int_{\gamma_R^-} N^{z-1} f_{\leq N}(z)(\frac{1}{z-1} + \frac{z-1}{R^2})\ dz| \leq 2\pi/R + 2\pi/N.$$

Once again, we estimate the left-hand side by

$$\pi R \sup_{z \in \gamma_R^-} |N^{z-1}||f_{\leq N}(z)||\frac{1}{z-1} + \frac{z-1}{R^2}|.$$

As before we have $|N^{z-1}| = N^{x-1}$ and $|\frac{1}{z-1} + \frac{z-1}{R^2}| = \frac{2|x-1|}{R^2}$, while

$$|f_{\leq N}(z)| \leq \sum_{n=1}^N \frac{1}{n^x}.$$

If $0 \leq x \leq 1$, then the integral test gives

$$\sum_{n=1}^N \frac{1}{n^x} \leq 1 + \int_1^N \frac{da}{a^x} \leq \frac{N^{1-x}}{|x-1|},$$

while if $x < 0$ then the integral test gives

$$\sum_{n=1}^N \frac{1}{n^x} \leq \int_1^N \frac{da}{a^x} + N^{-x} \leq \frac{N^{1-x}}{|x-1|} + N^{-x}.$$

Putting this all together we obtain

$$\sup_{z \in \gamma_R^-} |N^{z-1}||f_{\leq N}(z)||\frac{1}{z-1} + \frac{z-1}{R^2}| \leq \frac{2}{R^2} + \frac{2}{RN}$$

and the claim follows. □

- Combining Lemma 2 and Lemma 3 we see that

$$|\int_{\gamma_R} N^{z-1} f(z)(\frac{1}{z-1} + \frac{z-1}{R^2})\ dz - 2\pi i \sum_{n=1}^N \frac{\mu(n)}{n}| \leq 4\pi/R + 2\pi/N.$$

Taking limit suprema as $N \to \infty$ and using Lemma 1 we obtain

$$\limsup_{N \to \infty} |2\pi i \sum_{n=1}^N \frac{\mu(n)}{n}| \leq 4\pi/R.$$

Then letting $R \to \infty$ gives the result. □

- We now convert the above statement about $\mu(n)$ into a statement about $\pi(n)$. This now leaves the realm of complex analysis and into the world of number theory. We use the notation $O(A)$ to denote anything which is bounded by CA for some constant $C > 0$, and $o(A(N))$ to denote any quantity $B(N)$ such that $\lim_{N \to \infty} B(N)/A(N) = 0$. Thus the prime number theorem states that $\pi(N) = \frac{N}{\log N}(1 + o(1))$.

- We first need to estimate some partial sums.

- **Lemma 3.** We have $\sum_{n=1}^N \mu(n) = o(N)$.

- **Proof.** Let $\varepsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} \mu(n)/n$ is conditionally convergent, we see that $\sum_{n=(1-\varepsilon)N}^{N} \mu(n)/n = o(1)$. But $1/n = 1/N + O(\varepsilon/N)$, and hence

$$\sum_{n=(1-\varepsilon)N}^{N} \mu(n)/N = o(1) + \sum_{n=(1-\varepsilon)N}^{N} O(\varepsilon/N) = o(1) + O(\varepsilon^2).$$

Subdividing the interval from $N/2$ to N into about $1/\varepsilon$ intervals and doing the above estimate on each interval, we see upon summing that

$$\sum_{n=N/2}^{N} \mu(n) = o(N/\varepsilon) + O(\varepsilon N).$$

In particular, for N sufficiently large (say $N > N_0$, where N_0 depends on ε) we have

$$\sum_{n=N/2}^{N} \mu(n) = O(\varepsilon N).$$

Applying this with N replaced by $N/2$, $N/4$, etc. until one reaches N_0 and then summing the telescoping series, we obtain

$$\sum_{n=N_0}^{N} \mu(n) = O(\varepsilon N) + O(N_0)$$

and thus if N is large enough

$$\sum_{n=1}^{N} \mu(n) = O(\varepsilon N)$$

Since ε was arbitrary, the claim follows. □

- **Lemma 4.** We have $\sum_{n=1}^{N} \log n = N \log N - N + O(N^{1/2})$.

- **Proof.** From the integral test we have

$$\int_1^N \log a \; da = \sum_{n=1}^{N} \log n \leq \int_1^N \log a \; da + \log N.$$

Since $\int_1^N \log a \; da = N \log N - N$, the claim follows ($\log N$ is smaller than $N^{1/2}$ for N large). □

- Let $\tau(n)$ denote the number of divisors of a positive integer n, thus for instance $\tau(6) = 4$.
- **Lemma 5.** We have $\sum_{n=1}^N \tau(n) = N \log N - N + 2\gamma N + O(N^{1/2})$, where γ is a fixed number (called Euler's constant).
- **Proof.** We can write $\tau(n) = \sum_{d|n} 1$, where $d|n$ means "d divides n". Thus

$$\sum_{n=1}^N \tau(n) = \sum_{n=1}^N \sum_{d|n} 1.$$

Writing $n = dd'$, we obtain

$$\sum_{n=1}^N \tau(n) = \sum_{d,d':dd' \leq N} 1.$$

If $dd' \leq N$, then at least one of d, d' is less than or equal to $\sqrt{N}$. Thus we can write the previous as

$$\sum_{d,d':d \leq \sqrt{N}, dd' \leq N} 1 + \sum_{d,d':d' \leq \sqrt{N}, dd' \leq N} 1 - \sum_{d,d':d,d' \leq \sqrt{N}, dd' \leq N} 1.$$

The third sum is just $(\sqrt{N} + O(1))^2 = N + O(N^{1/2})$ (note that the condition $de \leq N$ is irrelevant here). The first two are symmetric, so it will suffice to show that

$$\sum_{d,d':d \leq \sqrt{N}, dd' \leq N} 1 = \frac{1}{2} N \log N + \gamma N + O(N^{1/2}).$$

Writing $dd' \leq N$ as $d' \leq N/d$ we see that for fixed d, the number of availale d' is $N/d + O(1)$, thus the left-hand side is

$$\sum_{d \leq \sqrt{N}} (\frac{N}{d} + O(1))$$

and it will suffice to show that

$$\sum_{d \leq \sqrt{N}} \frac{1}{d} = \frac{1}{2} \log N + \gamma + O(N^{-1/2}).$$

But we can write $\frac{1}{2}\log N = \int_1^{\sqrt{N}} \frac{da}{a}$, and the claim will then follow from the integral test if we choose γ to be the discrepancy between the infinite integral and infinite sum. $\square$

- **Lemma 6.** We have $\sum_{d,d':d,d' \leq N} \mu(d)(\log d' - \tau(d') + 2\gamma) = o(N)$.

- **Proof.** Let $M > 0$ be a large number. If N is large enough (depending on M), we can split the left-hand side as

$$\sum_{d,d':d,d' \leq N; d' < M} \mu(d)(\log d' - \tau(d') + 2\gamma) + \sum_{d,d':d,d' \leq N; d < N/M} \mu(d)(\log d' - \tau(d') + 2\gamma) - \sum_{d,d':d,d' \leq N; d < N/M}$$

The first sum can be rewritten

$$\sum_{d' < M} (\log d' - \tau(d') + 2\gamma)(\sum_{d \leq N/d'} \mu(d))$$

and this is $o(N)$ if N is large enough depending on M thanks to Lemma 3. In particular we can make this less than $O(M^{-1/2}N)$ if N is large enouh. The third sum can similarly be made less than $O(M^{-1/2}N)$. As for the second sum, we write it as

$$\sum_{d \leq N/M} \mu(d) \sum_{d' \leq N/d} \log d' - \tau(d') + 2\gamma$$

which by Lemmas 4, 5 is equal to

$$\sum_{d \leq N/M} O((N/d)^{1/2}) = N^{1/2} O(\int_1^{N/M} d^{-1/2}) = N^{1/2} O((N/M)^{1/2}) = O(M^{-1/2}N).$$

Thus we have estimated $\sum_{d,d':d,d' \leq N} \mu(d)(\log d' - \tau(d') + 2\gamma) = O(M^{-1/2}N)$; since M was arbitrary, the claim follows. $\square$

- **Lemma 7** Let $n > 1$. Then we have $\sum_{d|n} \mu(d)(\log \frac{n}{d} - \tau(\frac{n}{d}) + 2\gamma) = \log p - 1$ if n is equal to a power of a prime number p, and equal to -1 otherwise.

- **Proof.** Recall that for Re$z > 1$

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

and hence (differentiating in z)

$$\zeta'(z) = \sum_{n=1}^{\infty} \frac{-\log n}{n^z}$$

or by squaring and collecting terms

$$\zeta^2(z) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^z}$$

and thus

$$-\zeta'(z) - \zeta^2(z) + 2\gamma\zeta(z) = \sum_{n=1}^{\infty} \frac{\log n - \tau(n) + 2\gamma}{n^z}.$$

On the other hand, we have

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}.$$

Multiplying the two, we obtain

$$-\frac{\zeta'(z)}{\zeta(z)} - \zeta(z) + 2\gamma = \sum_{d=1}^{\infty}\sum_{m=1}^{\infty} \frac{\mu(d)}{d^z}\frac{\log m - \tau(m) + 2\gamma)}{m^z}.$$

Making the substitution $n = md$, this becomes

$$-\frac{\zeta'(z)}{\zeta(z)} - \zeta(z) + 2\gamma = \sum_{n=1}^{\infty} \frac{\sum_{d|n} \mu(d)(\log \frac{n}{d} - \tau(\frac{n}{d}) + 2\gamma)}{n^z}.$$

On the other hand, we have from before that

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{p \text{ prime}} \sum_{m \geq 1} \log p p^{-mz}.$$

The claim then follows by comparing coefficients of n^{-z} for all n. $\square$

- From Lemma 6 and making the substitution $n = dd'$, discarding the $n = 1$ case (which is $O(1)$), and then using Lemma 7, we obtain

$$\sum_{n<N; n \text{ is the power of a prime } p} \log p - \sum_{n<N} 1 = o(N).$$

There are at most $O(N^{1/2})$ numbers less than N which are squares of primes; at most $O(N^{1/2})$ numbers which are cubes of primes, and so on up to $\log N^{th}$ powers of primes (after that, it's impossible to stay less than N). Each such number contributes at most $\log N$ to the first sum, for a net contribution of $O(N^{1/2}\log^2 N) = o(N)$. Thus we have

$$\sum_{p<N; p \text{ prime}} \log p = N + o(N).$$

Now let ε be arbitrary. Applying this with N replaced by $1 - \varepsilon$, and then subtracting, we obtain

$$\sum_{(1-\varepsilon)N \leq p < N; p \text{ prime}} \log p = \varepsilon N + o(N).$$

In this interval we have $\log p = \log N - \log N/p = \log N + O(\varepsilon) = \log N(1 + O(\varepsilon/\log N))$ and thus

$$\log N(1 + O(\varepsilon/\log N)) \sum_{(1-\varepsilon)N \leq p < N; p \text{ prime}} 1 = \varepsilon N + o(N).$$

If N is large enough, we can write $\varepsilon N + o(N) = \varepsilon N(1 + o(1))$ and $1 + O(\varepsilon/\log N) = 1 + o(1)$, while the sum is just $\pi(N) - \pi((1-\varepsilon)N)$. So we obtain

$$\log N\pi(N) - \log N\pi((1-\varepsilon)N) = \varepsilon N(1 + o(1)).$$

In particular, if N is large enough (say $N > N_0$, we have

$$\log N\pi(N) - \log N\pi((1-\varepsilon)N) = \varepsilon N(1 + O(\varepsilon)).$$

If we define $f(N) := \pi(N) - \frac{N}{\log N}$, then we see that

$$f(N) - f((1-\varepsilon)N) \leq O(\varepsilon^2 \frac{N}{\log N})$$

(note that $\log(1-\varepsilon)N = \log N + \log 1 - \varepsilon = \log N + O(\varepsilon)$). More generally, we see that

$$f((1-\varepsilon)^k N) - f((1-\varepsilon)^{k+1} N) \leq O(\varepsilon^2 (1-\varepsilon)^{k/2} \frac{N}{\log N})$$

whenever $(1-\varepsilon)^k N > N_0$ (because $N/\log N$ grows faster than $\sqrt{N}$). Summing this telescoping series and noting that $f(N) = O(N_0)$ when $N = O(N_0)$, we obtain

$$f(N) - O(N_0) = O(\varepsilon \frac{N}{\log N});$$

when N is large enough we can absorb the N_0 error into the $\varepsilon N/\log N$ error, and thus

$$\frac{\pi(N)}{N/\log N} - 1 = O(\varepsilon).$$

Since ε is arbitrary, the prime number theorem follows.

* * * * *

The functional equation

- The proof of the prime number theorem was quite complicated, however one thing that can be seen was that it relied quite substantially on there being no zeroes of ζ for $\mathrm{Re} z \geq 1$. The famous *Riemann hypothesis* asserts that in fact there are no zeroes of ζ for $\mathrm{Re} z > 1/2$, and in fact they all lie on the line $\mathrm{Re} z = 1/2$ (except for the so called "trivial zeroes", which we discuss shortly) This is a major unsolved problem in mathematics. What is known is that the zeta function has infinitely many zeroes, and the first five hundred billion (!) zeroes are on this so-called *critical line* $\mathrm{Re} z = 1/2$. (There is in fact a distributed computing project going on right now which is computing about a billion new zeroes of the Zeta function each day). This result has many implications, for instance it will allow one to improve the prime number theorem estimate from $\pi(N) = N/\log N + o(N/\log N)$ (which is what we just proved) to what is basically $\pi(N) = N/\log N + O(N^{1/2} \log N)$. (Actually, this is a slight oversimplification, one has to replace $N/\log N$ by the variant quantity $\int_0^N \frac{dt}{\log t}$). The main reason is that knowing that there are no zeroes in the strip $\text{Re}z > 1/2$ allows one to push the region where ζ'/ζ is analytic back quite a bit, allowing one to improve Lemma 1.

- The Riemann hypothesis is still extremely far from resolution; it will probably require not just all the existing tricks and machinery known for these problems, but some totally new ones besides. However, there is one thing that is known about the zeroes of the Riemann zeta function, which is that they are *symmetric* about the critical strip. More precisely, we have

- **Functional equation** For any z (not an integer) we have

$$\zeta(1-z) = \frac{2}{(2\pi)^z}\cos(\frac{z\pi}{2})\Gamma(z)\zeta(z).$$

- **Proof.** By uniqueness of analytic continuation it suffices to prove this when $\text{Re}(z) < -10$ (say). In which case the left-hand side is just $\sum_{n=1}^{\infty} n^{z-1}$. Recall that

$$\Gamma(z)\zeta(z) = \frac{1}{(1-e^{2\pi i(z-1)})}\int_{\gamma_{\varepsilon,r}} \frac{e^{(z-1)\text{Log}_0 w}}{e^w - 1}\,dw$$

where $\gamma_{\varepsilon,r}$ is a clockwise contour going around the positive real axis. Writing $e^{2\pi i(z-1)}$ as $e^{2\pi iz}$, it thus suffices to show that

$$\int_{\gamma_{\varepsilon,r}} \frac{e^{(z-1)\text{Log}_0 w}}{e^w - 1}\,dw = \frac{(2\pi)^z(1-e^{2\pi iz})}{2\cos(\frac{z\pi}{2})}\sum_{n=1}^{\infty} n^{z-1}.$$

We use the residue theorem. The function $\frac{e^{(z-1)\text{Log}_0 w}}{e^w-1}$ is not analytic on the positive real axis, but is meromorphic everywhere else, with simple poles at integer multiples of $2\pi i$, and a residue of $e^{(z-1)\text{Log}_0 2\pi in} = (2\pi n)^{z-1}e^{\pi i(z-1)/2}$ when $w = 2\pi in$ for some positive n, and a residue of $e^{(z-1)\text{Log}_0 -2\pi in} = (2\pi n)^{z-1}e^{3\pi i(z-1)/2}$ when $w = -2\pi in$. For any $N > 0$, let γ_N be the contour consisting of the horizontal ray from $+\infty - (2N+1)\pi i$ to $-(2N+1)\pi - (2N+1)\pi i$, the vertical line segent from $-(2N+1)\pi - (2N+1)\pi i$ to $-(2N+1)\pi + (2N+1)\pi i$, and the horizontal ray from $-(2N+1)\pi + (2N+1)\pi i$ to $+\infty + (2N+1)\pi i$. The region of space between γ_N and $\gamma_{\varepsilon,r}$ contains the poles $2\pi i n$ for $-N \leq n \leq N$ (excluding $n = 0$), and so by the residue theorem

$$\int_{\gamma_{\varepsilon,r}} \frac{e^{(z-1)\mathrm{Log}_0 w}}{e^w - 1}\, dw = \int_{\gamma_N} \frac{e^{(z-1)\mathrm{Log}_0 w}}{e^w - 1}\, dw + 2\pi i \sum_{n=1}^{N} (2\pi n)^{z-1} e^{\pi i(z-1)/2} + (2\pi n)^{z-1} e^{3\pi i(z-1)/2}.$$

We now claim that the γ_N integral goes to zero as N goes to infinity. On the vertical line segment of γ_N, the point is that e^w is very small and so $\frac{1}{e^w-1}$ is bounded, while $e^{(z-1)\mathrm{Log}_0 w} = |w|^{z-1} e^{i(z-1)\mathrm{Arg}_0(w)} = O(N^{-11})$ since $\mathrm{Re}(z) < -10$. On either of the two horizontal rays, e^w is negative and so $\frac{1}{e^w-1}$ is again bounded, and again we can argue to show that these integrals decay very quickly in N. Taking limits we thus have

$$\int_{\gamma_{\varepsilon,r}} \frac{e^{(z-1)\mathrm{Log}_0 w}}{e^w - 1}\, dw = 2\pi i \sum_{n=1}^{\infty} (2\pi n)^{z-1} e^{\pi i(z-1)/2} + (2\pi n)^{z-1} e^{3\pi i(z-1)/2}.$$

We can the cancel the powers of 2π, and reduce to showing that

$$i(e^{\pi i(z-1)/2} + e^{3\pi i(z-1)/2}) \sum_{n=1}^{\infty} n^{z-1} = \frac{(1 - e^{2\pi i z})}{2\cos(\frac{z\pi}{2})} \sum_{n=1}^{\infty} n^{z-1}.$$

We can cancel the summation and reduce to verifying that

$$i(e^{\pi i(z-1)/2} + e^{3\pi i(z-1)/2}) = \frac{1 - e^{2\pi i z}}{2\cos(\frac{z\pi}{2})}.$$

But the left-hand side is $2ie^{\pi i(z-1)} \cos(\pi(z-1)/2) = -2ie^{\pi i z}\sin(\pi z/2)$ while the right-hand side is $e^{\pi i z} \frac{2i \sin(\pi z)}{2\cos(\pi z/2)}$, and the claim then follows from the double angle formula for sine. □

- From the functional equation we see in particular (since Γ has no zeroes, and $\cos(z\pi/2)$ only has zeroes when z is an odd integer) that if z is a zero of ζ that is not an integer, then $1 - z$ is also a zero of ζ; thus the zeroes of ζ are symmetric around the point $1/2$. It is also easy to show that if z is a zero of ζ, then so is $\overline{z}$ (this is in fact similar to one of the questions in the assignment), so they are also symmetric around the real axis, and thus around the critical line.

- It remains to check what happens when z is an integer. We already know that $\zeta(z)$ has a simple pole at $z = 1$ and is non-zero for larger integers. The functional equation then tells us that $\zeta(z)$ has no zero at 0, has zeroes at $-1, -3, -5, \ldots$, and is non-zero at the other integers. These are the so called *trivial zeroes* of the Riemann zeta function.