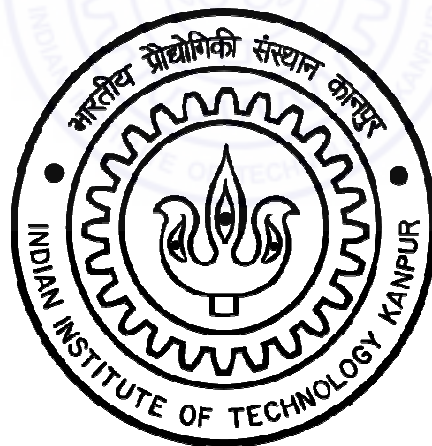


**STUDIES AND VALIDATION OF SOLUTIONS  
TO THE FORWARD PROBLEM OF  
ELECTRICAL IMPEDANCE TOMOGRAPHY**

*A Thesis submitted  
in partial fulfillment of the requirements  
for the degree of  
MASTER OF TECHNOLOGY*

*by*  
**RAJARSHI PAL CHOWDHURY**



*to the*  
**NUCLEAR ENGINEERING AND TECHNOLOGY PROGRAMME  
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR  
August 2013**



**CERTIFICATE:**

It is certified that the work done in this thesis entitled "STUDIES AND VALIDATION OF SOLUTION TO THE FORWARD PROBLEM OF ELECTRICAL IMPEDANCE TOMOGRAPHY" by Mr. RAJARSHI PAL CHOWDHURY has been carried out under our supervision and this work has not been done or submitted elsewhere for a degree.

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## **Abstract:**

Electrical Impedance Tomography (EIT) is a popular area of research for non-destructive evaluation. EIT has applications in the fields of bio-medical engineering as well as nuclear engineering and sciences.

The basic concept behind EIT is that excitation in the form of current or voltage is inserted to the object to be imaged. The output which is obtained in the form of voltage or current is measured at the boundary of the object. Based on the input excitation and output response an approximate conductivity or resistivity map of the object is formed. The whole EIT operation can be divided into two parts. First one is the forward problem and the second one is the inverse problem.

The present work concentrates on the validation of the forward problem. Forward problem in EIT refers to solution of the partial differential equation of EIT and implementation of the solution through coding. The forward problem solution provides the simulated results of measurements made at the boundary of the object, when a known excitation has been given. The present work provides a solution of the forward problem. The boundary conditions needed to solve the equation are established. The forward problem is then solved with the help of finite element method. The solution of forward problem has been done for both 2 dimensional and 3-dimensional EIT. The results of the solution and corresponding simulations are then validated against standard software tools for EIT (EIDORS version 3.6) in MATLAB (v. 2009b) and analytical solutions of the EIT partial differential equation in circular and spherical domain. The Jacobian or sensitivity matrix for both types of problems have been calculated and validated.

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**Date:**

**Rajarshi Pal Chowdhury**

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## **Table of Contents:**

<i>Certificate</i>	<i>ii</i>
<i>Abstract</i>	<i>iii</i>
<i>Acknowledgement</i>	<i>iv</i>
<i>List of figures</i>	<i>vii</i>
<i>List of tables</i>	<i>vii</i>
<i>List of abbreviations</i>	<i>viii</i>

## **Chapter 1: Introduction**

1.1 Overview	1
1.2 Electrical impedance tomography	2
1.3 Literature review	3-4

## **Chapter 2: EIT Forward Problem**

2.1 Mathematical formulation of EIT Forward problem	5
2.2 Boundary Conditions	6
2.3 Electrode Models	7
2.4 Mathematical Formulation of boundary conditions	8
2.5 Forward Model Mathematical Representation	9
2.6 Current Injection and Measurement pattern	9-10

### **Chapter 3: FEM SOLUTION OF FORWARD PROBLEM IN EIT**

3.1 Finite Element Method	11-12
3.2 Mathematical Formulation of FEM in 2D and 3D Forward Problem	13-14
3.3 Implementation of FEM	14-19
3.4 Implementation of CEM through FEM	19-22
3.5 Sensitivity Analysis of Forward Problem	22-24

### **Chapter 4: Simulated 2D and 3D forward problem of EIT**

4.1 The simulated 2D forward problem.	25-27
4.2 The simulated 3D forward problem.	27

### **Chapter 5: Results and Discussions**

28-35

### **Chapter 6: Conclusions**

36

### **References:**

37-38

### **Appendix A:**

39-40

### **Appendix B:**

41-43

## **List of figures:**

1.1 EIT basic Block diagram	2
3.1 Mesh analysis for 2D and 3D elements	11
3.2 Connectivity analysis between local and global elements	12
4.1 Simulation set up for 2D EIT	26
5.1 The 2D FEM Meshing of a circular geometry	28
5.2 The 3D Elements dividing a spherical geometry	29
5.3 (a) Comparison of Nodal Voltages with EIDORS solution for mesh size=0.08	30
5.3 (b) Comparison of Nodal Voltages with EIDORS solution for mesh size=0.07	30
5.3 (c) Comparison of Nodal Voltages with EIDORS solution for mesh size=0.06	31
5.4 (a) Nodal voltage distribution for mesh size=0.08	31
5.4 (b) Nodal voltage distribution for mesh size=0.07	32
5.4 (c) Nodal voltage distribution for mesh size=0.06	32
5.5 Comparison between 2D FEM EIT solution to Analytic solution.	34
5.6 3D Nodes and elements of a spherical geometry	35
5.7 Comparison between 3D FEM EIT solution to Analytic solution.	36
B.1 Analytic Solution for 2D EIT in circular geometry	43
B.2 Analytical Solution of 3D EIT in spherical geometry.	44

## **List of Abbreviations:**

1D:	One dimensional
2D:	Two dimensional
3D:	Three dimensional
EIT:	Electrical Impedance Tomography
FEM:	Finite Element Method
DUT:	Domain under test
N:	No of nodes
K:	No of elements
L:	No of electrodes
M:	No of measurements in a data frame
$\sigma$ :	Conductivity
$\varphi$ :	Electric potential
u:	Nodal Voltage
J:	Current Density
$\Omega$ :	Domain to be imaged.
$\Gamma$ :	Domain Boundary
$\vartheta$ :	Unit normal outward to the domain.
$V_l$ :	Electrode Voltage
$I_l$	Electrode Current
$E_l$ :	Area of the boundary under electrodes.
$A_e$ :	Electrode Area
$S_e$ :	Area of an element
$V_e$	Volume of an element



**List of Table:**

5.1 Comparison between different Mesh sizes for 2D FEM

32



# Chapter 1

## **INTRODUCTION:**

### **1.1 Overview:**

Electrical Impedance Tomography is a non-linear and ill-posed form of tomography. It is an imaging modality which approximates electrical properties like conductivity or resistivity of an object by measurements done on its boundary. Typically currents are injected from the boundary of the object to be imaged, through electrodes that are attached on the periphery of the body. Voltages are measured on the electrodes which are not injecting currents. Using different current patterns and voltage measurement schemes the approximate spatial distribution of conductivity or resistivity distribution within the object can be formed. It is also possible to inject voltage and measure current along the boundary but the former process is widely used and considered here in this present work.

### **1.2 Electrical Impedance Tomography:**

Electrical Impedance Tomography has several applications in the field of science and engineering. These are:

#### **Applications:**

**a)Bio-Medical:** It can be used in the field of bio-medical engineering, namely to measure cardiac and pulmonary functions, detection of bone fracture, and internal hemorrhage, breast and thorax cancer and tumor detection etc.

**b)Nuclear Engineering:** These applications relates to imaging of fluid flow through a pipe in a nuclear reactor, detection of cracks within the structure of a reactor etc.

Electrical impedance tomography operation can be divided into two parts. First one is the forward problem and 2<sup>nd</sup> one is the inverse problem. In this thesis only 2D and 3D forward problem has been discussed and established. The object to be imaged or DUT (Domain under test) is first divided into several 2D or 3D sub-domains in the forms of triangles or tetrahedrons respectively. This process is called meshing. An approximate conductivity of the DUT has been assumed and based on the conductivity a voltage distribution within the DUT has been obtained when the right boundary conditions are applied. The voltage obtained through the forward simulation is obtained on the vertices of the triangles or tetrahedrons. These voltages are called the nodal voltage. The objective of forward problem is to solve the basic differential equation of electrical impedance tomography through finite element method and simulate the nodal voltages. The electrode consists of a single node or several single nodes. The simulated nodal voltage on the nodes under an electrode gives the approximate value of the measured voltage at that electrode.

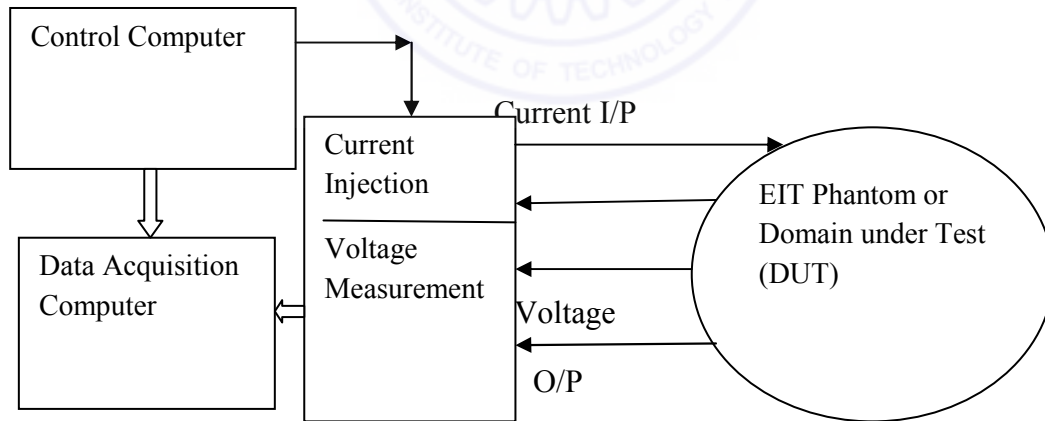


Figure 1.1: Basic Block Diagram of EIT Operation.

### 1.3: Literature review:

Study of electrical impedance tomography is a prime research aspect in the field of non-invasive imaging due to its wide application area and relatively cheap instrumentation. EIT is an imaging modality that produces images by computing electrical conductivity within the material. Currents are applied to the DUT (Domain under test) using electrodes and the resulting voltages are measured. The internal conductivity map of the body is constructed based on the data measures at boundary. Many applications of this technology are used in medical and industrial uses [1].

The domain under test is surrounded by electrodes. The currents are injected through two electrodes and differential voltages are measured between rest of the electrodes [2]. Based on the simulations done by the 2DFEM an approximate voltage distribution has been simulated [3].

It is assumed that the current stays on the 2 dimensions in case of 2D EIT, but when current passes through the object it spreads into all 3 dimensions analysis so analysis of 3D EIT becomes important. For 3D EIT calculations, the DUT has been divided into 3D elements and the corresponding nodal voltages have been simulated through 3D FEM analysis [4-7].

The simulated results have been compared to the standard software tool (EIDORS 2D and 3D) in MATLAB and a comparative study has been shown. [6-8].

The sensitivity analysis for both the 2D and the 3D EIT Forward problem which is required for the reconstruction procedure has been described in the papers [5-7]. Analytical solution methodology for different geometry has been analyzed in the papers [9] and [10].

In the present work the comparison 2D and 3D nodal voltage solutions have been done and validated against the standard software solution and corresponding analytic solution. The Jacobian or sensitivity analysis has been done and compared with the standard solution.



## Chapter 2

### **EIT FORWARD PROBLEM:**

#### **2.1 Mathematical formulation of EIT forward problem:**

The EIT forward problem statement can be established from the basic Maxwell's equations of electro-magnetics.

The object to be imaged is denoted by the symbol  $\Omega$ . The energy inserted to the medium in the form of current through the boundary of the medium  $\Gamma$ , which creates a voltage distribution within the DUT (Domain under test) denoted as  $u$ . The conductivity of the DUT is assumed to be  $\sigma$ .

From Maxwell's electromagnetic equations and Kirchhoff's law of charge conservation the basic equation of EIT can be formed. [Appendix A.1]

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{on } \Omega \quad (2.1)$$

This is an example of elliptic partial differential equation.

The main aim of this thesis is to solve this partial differential equation numerically with the help of finite element method and solve the forward problem.

## 2.2:Boundary Conditions:

Solution of the EIT forward problem requires forming proper boundary conditions. The boundary conditions depend on the way the electrodes are located around the DUT and the current injection pattern. There are two main boundary conditions in the EIT Forward problem solution.

### a) Dirichlet Boundary Condition:

Dirichlet boundary condition or forced boundary condition suggests that the voltage on the electrode is the voltage measured on that electrode.

$$u = V_l \text{ on } E_l \quad l = 1, 2, 3 \dots L \quad (2.2)$$

$E_l$  denotes the area under the  $l^{th}$  electrode.

### b) Neumann Boundary Condition:

Neumann boundary condition or mixed boundary condition states that:

$$\frac{\sigma \partial u}{\partial \nu} = J_l \text{ on } E_l$$

$$= 0 \quad \text{on } \Gamma / \bigcup_{l=1}^L E_l \quad (2.3)$$

Here  $E_l$  denotes the space covered by the  $l^{th}$  electrode and  $J_l$  denotes the current density on the  $l^{th}$  electrode. The symbol  $\Omega$  represents the space of the DUT and  $\bigcup_{l=1}^L E_l$  indicates the total space acquired by the electrodes. The notation  $\Gamma / \bigcup_{l=1}^L E_l$  means the boundary of DUT which is not covered by the electrodes.

From first statement of Neumann boundary condition the electrode current can be expressed as:

$$I_t = \int_{E_t} \sigma \partial u / \partial \vartheta \cdot d\Gamma \quad (2.4)$$

## 2.3 Electrode Models:

Electrode models are set of boundary conditions that need to be followed for a particular arrangement of current injection and electrode placement. There are two main electrode models that are generally followed. They are:

### 2.3.1 Point Electrode Model:

The point electrode model is the most basic boundary condition in EIT. This is mainly used in case of solution of 2D EIT forward problem. Here, electrodes are considered to be single nodes. One node can be used as current source while another can be used as current sink.

### 2.3.2 Complete Electrode Model:

Complete electrode model is the most popular model in case of EIT forward problem solution. This model can be used for both 2D and 3D EIT forward problem. In this model electrodes are considered to be formed with 3 or more nodes. Contact impedance is considered to be formed between the electrode and the DUT. The formation of contact impedance across the electrode creates a voltage drop. The complete electrode model introduces more variables to solve the EIT forward problem.



## 2.4 Mathematical formulation of boundary conditions:

The complete electrode model boundary conditions have been shown below:

$$u + Z_l \frac{\sigma \partial u}{\partial \vartheta} = V_l \text{ on } E_l, \quad \text{where } l = 1, 2, \dots, L \quad (2.5)$$

$$\int_{E^\ell} \frac{\sigma \partial u}{\partial \vartheta} \cdot d\Omega = I_l \text{ where } l = 1, 2, \dots, L \quad (2.6)$$

$$\sigma \partial u / \partial \vartheta = 0 \text{ on } \Gamma / \bigcup_{l=1}^L E_l \text{ where } l = 1, 2, \dots, L \quad (2.7)$$

Equation (2.4) accounts for the electrode contact impedance which is denoted as  $Z_l$ . The contact impedance originates from the layer that forms between the electrode and the object to which it is attached. This also shows that the voltage formed across electrode consists of voltage on the boundary plus the voltage drop across the electrode.

Equation (2.5) shows that the integral of the current density across the electrode equals the total current flowing through the electrode. Equation (2.6) implies no current entering or leaving through the Domain where there is no electrode.

There are also the conditions of conservation of charges:

$$\sum_{l=1}^L I_l = 0 \quad (2.8)$$

An arbitrary choice of a ground gives the condition:

$$\sum_{l=1}^L V_l = 0 \quad (2.9)$$

## 2.5 Forward Model mathematical representation:

The CEM model of equations (2.4-2.7) can be formed in linear equations and solved numerically as:

$$AV_C = I_c \quad (2.10)$$

$A$  = the global conductance or stiffness matrix (Refer to chapter 3.2).

$I_c$  = Injected current vector.

$V_C$  = Vector of simulated voltages at FEM nodes (Refer to chapter 3.1) and electrodes.

## 2.6 Current injection and measurement pattern:

The number of measurements of voltage depends on the current injection and measurement pattern. There are mainly two kinds of current injection methods. One is pair drive injection pattern where a single source is used and another is multiple drive where more than one current sources are used simultaneously. In case of pair drive a current source has been put between two electrodes and voltages are measured from the remaining electrodes. The current source is then switched and placed between two new electrodes and voltages are measured again. For multiple drive mechanism one current source is placed between more than one pair of electrodes. There are mainly 2 types of current injection patterns which are used in the EIT. They are:

**a) Adjacent Pattern:** Here currents are injected between two adjacent electrodes and differential voltages are measured between remaining electrodes. If the total number of electrodes is  $L$  then the number of independent measurements is equal to  $L(L-3)/2$ . This number

is also called the degree of freedom of measurement. In the Figure 2.1 the adjacent pattern of current injection has been illustrated. Currents are injected from two adjacent electrodes and differential voltages are measured from other electrodes. After one set of measurements the source and sink electrodes are changed and then another set of measurements are taken. If the number of electrodes is 16 then the number of measurements is 208. The number of independent measurements is 104.

**b) Cross or Opposite pattern:**

Currents are injected between two diametrically opposite (by  $180^\circ$ ) located electrodes in this pattern of current injection. Voltages are measured from the rest of the electrodes differentially. The total number of measurements, in this case, is also  $L(L-3)/2$ .

The currents are injected through the electrodes that are  $180^\circ$  apart and differential voltages are measured from rest of the electrodes.

## Chapter 3

### **FINITE ELEMENT FORMULATION OF FORWARD PROBLEM**

#### **IN EIT:**

##### **3.1 Finite Element Method:**

Finite element method is a mathematical tool to solve partial differential equations numerically. Finite element method is used in EIT to solve the elliptic partial differential equation (2.1). The whole domain under test is divided into several small sub-domains called elements. This process of dividing a large domain into several small sub-domains is called meshing. The domain is subdivided into triangles or rectangles in case of 2D EIT, and in case of 3D EIT the domain is divided into 3 dimensional elements such as tetrahedrons. The domain is divided into elements the equation is solved on the values on the vertices of the elements called the nodes. The voltages on the nodes called as nodal voltages are assumed as an interpolation function of the vertices or nodes. The nodal voltages can be represented as a weighted function of the values of vertices. Solving the system of linear equations by Cramer's rule, the value of the weights can be found. These values when placed in the original interpolation function, gives the approximate values of the nodal voltages. The meshing of 2D DUT is expressed in the Figure 3.1.

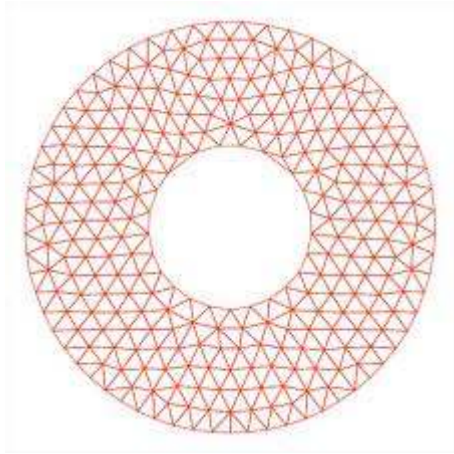


Figure 3.1: Triangular meshes dividing a hollow circular geometry

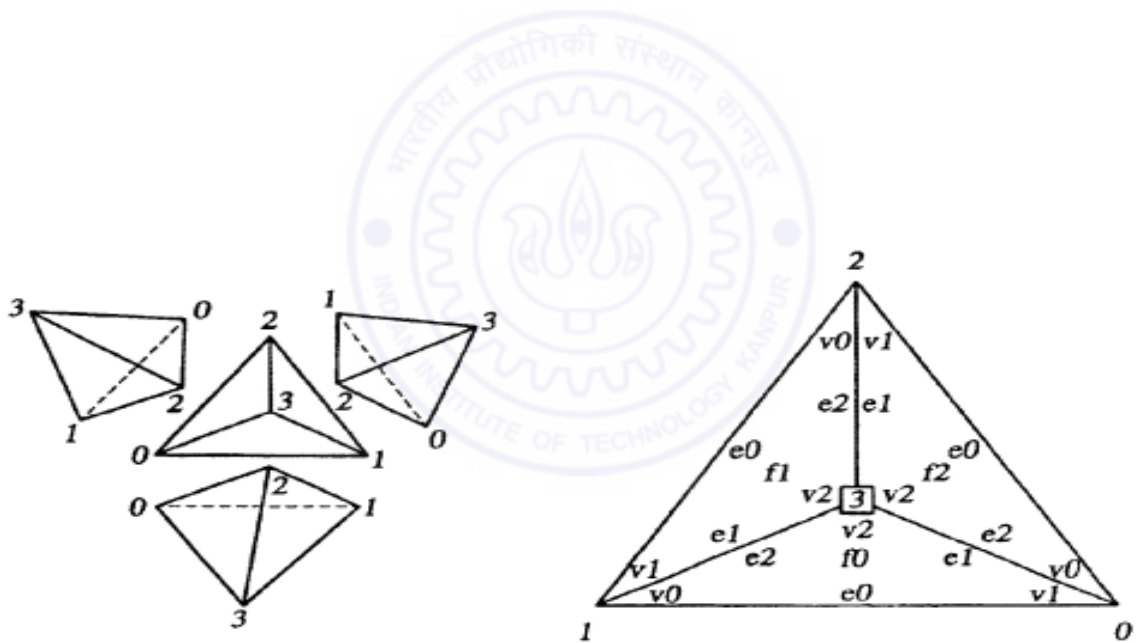


Figure 3.2: The 3D tetrahedron elements and combination of 4 elements to make a global element.

### 3.2 Mathematical formulation for FEM in 2D and 3D EIT:

The DUT is meshed in 2D finite elements that are triangles. Each element has 3 vertices called nodes.

The nodal voltages on each element can be approximated as:

$$u_1 = a_1x_1 + b_1y_1 + c_1 \quad (3.1)$$

$$u_2 = a_2x_2 + b_2y_2 + c_2 \quad (3.2)$$

$$u_3 = a_3x_3 + b_3y_3 + c_3 \quad (3.3)$$

$u_1, u_2$  and  $u_3$  represent the nodal voltages of vertices 1, 2 and 3 of a single element.

$a_i, b_i$  and  $c_i$  are the weights of interpolation function for  $i=1,2,3$ .

The voltage  $u$  can be approximated based on the nodal voltages ( $u_1, u_2, u_3$ ) as

$$u(x, y) = \frac{1}{D} * \left( \begin{bmatrix} u_1 & x_1 & y_1 \\ u_2 & x_2 & y_2 \\ u_3 & x_3 & y_3 \end{bmatrix} + \begin{bmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{bmatrix} \cdot x + \begin{bmatrix} 1 & x_1 & u_1 \\ 1 & x_2 & u_2 \\ 1 & x_3 & u_3 \end{bmatrix} \cdot y \right) \quad (3.4)$$

where,  $D = 2 * S_e$

$$S_e = \text{Area of the element (triangle)} = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

The total no of nodes = N

And total no of elements = K

The potential of a linear 2D element can be expressed as:

$$u(x, y) = \sum_{i=1}^3 (a_i \cdot x + b_i \cdot y + c_i) \cdot u_i \quad (3.5)$$

The potential inside of a 3D element can also be written as:

$$u(x, y, z) = \sum_{i=1}^4 (a_i \cdot x + b_i \cdot y + c_i \cdot z + d_i) \cdot u_i \quad (3.6)$$

The values of  $a_i, b_i, c_i$  and  $d_i$  for a 3D element are given in appendix A.2

So the potential inside an element can also be expressed as:

$$u = \sum_i N_i \cdot u_i \quad (3.7)$$

$N_i$ 's are called the shape function or interpolation function of the elements.

The shape functions or potential functions are chosen in that way so that

$$N_i(x_i, y_j) = 1 \text{ when } i = j$$

$$= 0 \text{ when } i \neq j$$

The sum of an interpolation function over an element is equal to zero.

$$\sum_i N_i = 0$$

### 3.3 Finite Element implementation in solving equations:

#### Method of weighted residuals:

The method of weighted residuals is one of the finite element methods that has been used in EIT to solve the partial differential equation. The fundamental equation of EIT is:

$$\nabla \cdot (\sigma \nabla u) = 0$$

In the MWR procedure we form a weight  $W$  such that the weighted integration of the EIT equation becomes zero.

Let us consider, that the approximate solution of the FEM method is  $u_x$  and the exact solution is  $u$ . From equation (2.1) it can be expressed that

$$\nabla \cdot (\sigma \nabla u) = 0 \text{ and}$$

$$\nabla \cdot (\sigma \nabla u_x) \approx 0$$

It is assumed that

$$\nabla \cdot (\sigma \nabla u_x) = R \quad (3.8)$$

The term  $R$  is called the residual. The method of weighted residuals refers that the weighted sum or the integral of the residuals over all the elements can be zero if proper weight can be found. The basic principle of MWR suggests that for an arbitrary ' $W$ ' the integral of the weighted residual will be zero on the medium.

$$\int_{\Omega} WR d\Omega = 0 \quad (3.9)$$

From Equations (2.1) and (3.8) it can be written that:

$$\nabla \cdot (\sigma \nabla u_x) - \nabla \cdot (\sigma \nabla u) = R \quad (3.10)$$

Where  $W$  is a weight function which can be approximated as:



$$W = \sum_{j=1}^D N_j \cdot v_j \quad (3.11)$$

Here  $N_i$  is interpolation or shape function for weights and  $v_i$ 's are the coefficients of the weight function.  $D$  represents the dimension of the element.  $D=3$  if the element is 2D and  $D=4$  if the element is 3D. Using Equations (3.10) and (3.11) it can be written that:

$$\int W \nabla \cdot (\sigma \nabla u_x) d\Omega = 0 \quad (3.12)$$

From the basic equations of vector calculus it can be written that (Appendix: A)

$$\int [\nabla \cdot (W \sigma \nabla u_x)] d\Omega = \int [\sigma \nabla u_x \cdot \nabla W] d\Omega \quad (3.13)$$

From Gauss's theorem it can be concluded that

$$\int W \sigma \nabla u_x \cdot \vartheta d\Gamma = \int [\sigma \nabla u_x \cdot \nabla W] d\Omega \quad (3.14)$$

This equation refers to the relationship the integral of dot product of two fields with the given boundary condition. As it can be conferred that  $\nabla u_x \cdot \vartheta = \partial u_x / \partial \vartheta$  the equation (3.14) can be written as:

$$\int [\sigma \nabla u_x \cdot \nabla W] d\Omega = \int W \sigma \frac{\partial u_x}{\partial \vartheta} d\Gamma \quad (3.15)$$

The LHS of equation (3.16) represents the whole mesh that is the domain  $\Omega$  and the RHS represents the boundary conditions. The equation (3.16) can be formed for each element inside the DUT. The integral will be over an element instead of the whole mesh. If the domain is divided into  $K$  elements then we have  $K$  integrals to be solved to find the nodal voltage. For each

element a set of linear equation are formed, and these are combined to form the system equation for the whole domain. From equations (3.8) and (3.12) the equation (3.16) can be rearranged for an element  $k$  ( $k = 1, 2, \dots, K$ ). The conductivity of each element is assumed to be constant along the element. This type of conductivity is called piecewise-constant conductivity. The integral over the area of the  $k^{th}$  element  $E_k$  whose conductivity is assumed as  $\sigma_k$  can be written as:

$$\int_{E_k} \sigma_k \nabla u \cdot \nabla W = \sigma_k \sum_{i=1}^3 u_i \sum_{j=1}^3 v_j S_{ij}^k \quad (3.16)$$

where  $S_{ij}^k$  is called the stiffness matrix of element ' $k$ ' and it is defined as:

$$S_{ij}^k = \int_{E_k} \nabla N_i^k \cdot \nabla N_j^k d\Omega \quad (3.17) N_i^k \text{ and } N_j^k \text{ are the}$$

shape functions of the FEM nodal voltage and weight function respectively for element  $k$ . From this point onwards the FEM nodal voltage  $u_x$  is considered as nodal voltage  $u$ .

Thus a 3x3 matrix can be formed to solve the equations of a 2D element. Similarly a 4x 4 matrix can be formed when 3D element is considered. The number of elements inside the DUT is  $K$ . To solve the whole system equation the total no of equations needed are:

$K \times 3 \times 3$  for 2D FEM EIT and  $K \times 4 \times 4$  for 3D FEM EIT.

Equation (3.18) represents the LHS of the equation (3.16). To form the equivalent for the RHS the terms are required to be expanded in the FEM equations. The RHS can be represented as

$$\int W \sigma \frac{\partial u_x}{\partial \vartheta} d\Gamma = \sigma_k \sum_{i=1}^3 u_i \sum_{j=1}^3 v_j \int_{\Gamma} N_i \cdot \nabla N_j \cdot \vartheta d\Gamma \quad (3.18)$$

So, for an individual element  $k$  the equation reduces to:

$$\sigma_k \sum_{i=1}^3 u_i \sum_{j=1}^3 S_{ij}^k = \sigma_k \sum_{i=1}^3 \sum_{j=1}^3 u_i D_{ij}^k \quad (3.19)$$

where  $D_{ij}^k = \int_{\Gamma} N_i^k \cdot \nabla N_j^k \cdot \vartheta d\Gamma$

The equation (3.20) is the final formulation of the 2D forward problem of EIT with the help of finite element method. From equation (3.20) the global system matrix  $S_g$  can be formed by combining the local stiffness matrix  $S^k$  for each element. This global stiffness matrix is then used to solve the forward problem for calculating the nodal voltages  $u$  from the injected current pattern  $I$  depending upon which boundary condition has been implemented. The system matrix or global stiffness matrix is also used to calculate the sensitivity or Jacobian matrix which is used for reconstruction. The forward problem can now be expressed as:

$$S_g U = I \quad (3.20)$$

Here  $U$  is the combined nodal voltage of all the nodes in the domain. Its dimension is  $N \times 1$ .

$S_g$  has the dimension  $N \times N$  and  $I$  be the injected current vector having dimension:  $N \times 1$ .

The approach for constructing a global matrix from  $K$  local matrices is to construct a connectivity matrix for the whole mesh based on the connectivity map of the elements. The forward problem can be expressed as:

$$C^T S_k \Sigma_k C = I \quad (3.21)$$

Here  $C$  is connectivity matrix.

$S_k \Sigma_k$  = The diagonal concatenation of the global matrix with each global matrices multiplied by its own assumed conductivity. The whole matrix  $S_k \Sigma_k$  can be represented as:

$$S_k \Sigma_k = \begin{bmatrix} \sigma_1 S_1 & \cdots & \\ \vdots & \ddots & \vdots \\ \cdots & \cdots & \sigma_K S_K \end{bmatrix} \quad (3.22)$$

$S_k \Sigma_k$  is a block diagonal matrix whose diagonal elements are  $[\sigma_k S_k]_{3 \times 3}$ . The dimension of the matrix  $S_k \Sigma_k$  is  $3K \times 3K$ .

The diagonal elements are a  $4 \times 4$  matrix in case of 3D forward model.

$I$  = Vector of injected currents.

The point electrode model has been considered to form the basic forward equation. Individual nodes are considered to be the electrodes in case of point electrode model. If the point electrode model is considered the dimension of the current injection vector  $I$  will be  $N \times 1$ . For the complete electrode model configuration the forward model is needed to be modified according to the boundary condition. The dimensions of the vectors will be changed accordingly.

### 3.4 Implementation of CEM through the FEM:

The complete electrode model has been implemented here with the help of FEM. From the equation (2.8) it can be written

$$\frac{\sigma \partial u}{\partial \vartheta} = \sigma \nabla u \cdot \vartheta = \frac{1}{Z_l} \cdot (V_l - u) \quad (3.23)$$

So, the RHS of the FEM EIT equation (3.16) can be written as:

$$\int W \nabla u \cdot \vartheta d\Gamma = \sum_{l=1}^L \int W \cdot 1/Z_l (V_l - u) \cdot d\Omega' \quad (3.24)$$

Here  $\Omega'$  denotes the areas which are under the electrodes. This integration is done on the area under each electrode from  $(l = 1, 2, \dots, L)$  and the summation has been taken. The RHS of the equation (3.22) can be expanded to the FEM terms in the following expression:

$$\sum_{l=1}^L V_l \int \frac{1}{Z_l} \left( \sum_{j=1}^3 N_j v_j \right) d\Gamma - \sum_{l=1}^L \int \frac{1}{Z_l} \left( \sum_{i=1}^3 N_i u_i \right) \left( \sum_{j=1}^3 N_j v_j \right) d\Omega$$

Rearranging the terms the full governing equation of FEM EIT with CEM boundary conditions for an element can be expressed as:

$$\begin{aligned} \sigma_k \sum_{i=1}^3 u_i \sum_{j=1}^3 v_j \left( \int \nabla N_i \cdot \nabla N_j d\Omega \right) = \\ \sum_{l=1}^L V_l \sum_{j=1}^3 v_j \int \frac{1}{Z_l} N_j d\Gamma - \sum_{l=1}^L \sum_{i=1}^3 u_i \sum_{j=1}^3 v_j \int \frac{1}{Z_l} (N_i \cdot N_j) d\Gamma \end{aligned} \quad (3.25)$$

Equation (3.24) is valid for every test function  $v$ . So eliminating  $v$  from both sides of the equation the final form of CEM equation is formed.

$$\begin{aligned} \sum_{k=1}^K \sigma_k \sum_{i=1}^3 u_i \sum_{j=1}^3 \left( \int \nabla N_i \cdot \nabla N_j d\Omega \right) = \\ \sum_{l=1}^L V_l \int_{E_l} 1/Z_l \cdot N_j d\Gamma - \sum_{l=1}^L \sum_{i=1}^3 u_i \sum_{j=1}^3 \int_{E_l} 1/Z_l \cdot (N_i \cdot N_j) d\Gamma \end{aligned} \quad (3.26)$$

The current on the  $l^{th}$  electrode is given by equation (2.3) as:

$$I_l = \int_{E_l} \nabla u \cdot \vartheta d\Gamma$$

It can also be expressed as:

$$I_l = \int_{E_l} 1/Z_l \cdot (V_l - u) d\Gamma = \int_{E_l} \frac{V_l}{Z_l} \cdot d\Gamma - \sum_{i=1}^3 u_i \int_{E_l} \frac{N_i}{Z_l} \cdot d\Gamma \quad (3.27)$$

The contact impedance  $Z_l$  is assumed to be constant for each electrode and so it can be taken out from the integral. The modified form of the electrode current can be expressed as:

$$I_l = \frac{1}{Z_l} \cdot A_{el} \cdot V_l - \frac{1}{Z_l} \cdot \sum_{i=1}^3 u_i \int_{E_l} N_i d\Gamma \quad (3.28)$$

$A_{el}$ =Area under the  $l^{th}$  electrode. In case of 2D it is the length of the electrode.

Combining equations (3.26) and (3.27) a new system matrix for 2D FEM EIT with CEM is formed

This system matrix can be represented as:

$$\begin{bmatrix} A_1 + A_2 & A_3 \\ A_3^T & A_4 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad (3.29)$$

$U = (u_1, u_2, u_3 \dots \dots \dots u_N)_{(N \times 1)}^T$  = Matrix of all nodal voltages

$V = (V_1, V_2, V_3 \dots \dots V_L)_{(L \times 1)}^T$  = Matrix of all voltages measured at the electrodes.

$I = (I_1, I_2, I_3 \dots \dots \dots I_L)_{(L \times 1)}$  = Matrix of the current injection vector.

$0$  = Null matrix with dimension= $N \times 1$

The matrices  $A_1, A_2, A_3, A_4$  are expressed as:

$$[A_1]_{ij} = \int \sigma \nabla N_i \cdot \nabla N_j d\Omega$$

$$[A_2]_{ij} = \sum_{l=1}^L \int \frac{1}{Z_l} \cdot N_i \cdot N_j d\Gamma, j=1,2,3,\dots,N$$

$$[A_3]_{ij} = - \int \frac{1}{Z_l} \cdot N_i d\Gamma \quad i=1,2,3,\dots,N. \quad j=1,2,3,\dots,L$$

$$[A_4]_{LL} = \text{diag}\left(\frac{1}{Z_l} \cdot A_{el}\right)$$

Matrix  $A_1$  denotes the global stiffness matrix multiplied by the elemental conductivity matrix. The matrix  $A_1$  is the system matrix if point electrode model has been used, but since the CEM has been used it has been modified with the addition of other matrices. Matrix  $A_1$  does not contain any information about boundary conditions. Matrix  $A_2$  refers to the effects of contact impedances on the system matrix. Matrix  $A_3$  represents the weights of voltages on the electrodes. Matrix  $A_4$  adds the effect of contact impedance to the nodes which are situated under the electrodes.

### 3.5 Calculation of Jacobian or Sensitivity matrix:

The reconstruction of the EIT images requires the calculation of the Jacobian or sensitivity matrix which denotes the change of output voltage with change of conductivity.

The reconstruction of the EIT images requires the calculation of the Jacobian or sensitivity matrix which denotes the change of output voltage with change of conductivity. The Jacobian or sensitivity analysis expressed the rate of change of output voltage with respect to the change of conductivity. This measure helps to determine whether a system is sensitive enough to reflect the

change of voltage when conductivity is changed even for a slightest amount. The expression for the Jacobian is given by

$$J = \partial V_i / \partial \sigma_j \quad (3.30)$$

That implies change in the  $j^{th}$  voltage measurement when the the conductivity of the  $i^{th}$  element has been changed.

The forward problem in EIT can be expressed as:

$$Y(\sigma, r)V = I \quad (3.31)$$

From the above equation the voltage matrix  $V$  can be expressed as:

$$V = Y(\sigma, r)^{-1} \cdot I \quad (3.32)$$

Here  $Y(\sigma, r)$  is the admittance matrix which is both function of conductivity and position.

Taking derivative on both sides of equation (3.32) it can be written as:

$$\frac{\partial V}{\partial \sigma} = Y(\sigma, r)^{-1} \cdot \frac{\partial Y(\sigma, r)}{\partial \sigma} \cdot Y(\sigma, r)^{-T} \cdot I \quad (3.33)$$

The term  $\frac{\partial V}{\partial \sigma}$  is referred to as the Jacobian matrix or sensitivity matrix.

### **Adjoint method of calculating the Jacobian:**

There is another method of calculating the Jacobian for the 2D and 3D EIT forward problem.

This method is called the adjoint method of calculation of Jacobian. The formula for Jacobian in the adjoint method is given by:



$$J(i, k) = \frac{\partial V_i}{\partial \sigma_k} = - \int_{\Omega_k} \nabla u(I^d) \cdot \nabla u(I^m) d\Omega \quad (3.34)$$

Here  $V(i)$  is the  $i^{th}$  voltage measurement.  $\sigma_k$  is the conductivity of the  $k^{th}$  element. The terms  $u(I^d)$  and  $u(I^m)$  denotes the voltage distribution inside the  $k^{th}$  element when the injected current pattern are the drive and measurement pattern. The integral is a surface integral when 2D EIT problem is considered and is a volume integral when 3D EIT is considered.

The Jacobian matrix of dimension  $K \times M$  where  $K$  is the number of elements and  $M$  is the number of measurement patterns. In this analysis both the 2D and 3D Jacobian matrix has been calculated and they are verified with the Jacobian obtained through the EIDORS version 3.6 Jacobian calculation module ‘Jacobian\_2d’ and ‘Jacobian\_3d’ when the same injection and measurement pattern has been analyzed.

## **Chapter 4**

### **SIMULATED 2D AND 3D FORWARD PROBLEM OF EIT:**

The forward problem of 2D and 3D EIT has been simulated here with the help of proper boundary conditions. The solutions in the form of nodal voltages that is obtained are verified with the solutions obtained from the standard software tool.

#### **4.1 2D EIT Forward problem simulations:**

The 2D EIT forward problem has been simulated on a circular geometry. The electrodes are attached to the boundary of this circular domain. The radius of the circular DUT is considered to be of 1unit.

The total number of electrodes attached = 16.

The electrodes are 2 dimensional so only their lengths are considered here.

The length of each electrode =0.1 unit.

The conductivity of the medium is considered to be homogenous and =1 unit.

Cross pattern of current injection has been done for the simulation.

$$\text{Inter-electrode gap} = \frac{2\pi \times 1 - 16 \times 0.1}{16} = 0.292 \text{ unit}$$

The currents that are injected are direct current in nature and the conductivity that is assumed to have only real part.

The currents that are injected through electrodes are of magnitude =0.01mA

Value of contact impedance for electrodes: 60 Ohm-meter.

Number of nodes used in the FEM: 737

Number of elements for the FEM: 1385

The picture in Figure 4.1 shows the 2D EIT simulation on a circular geometry:

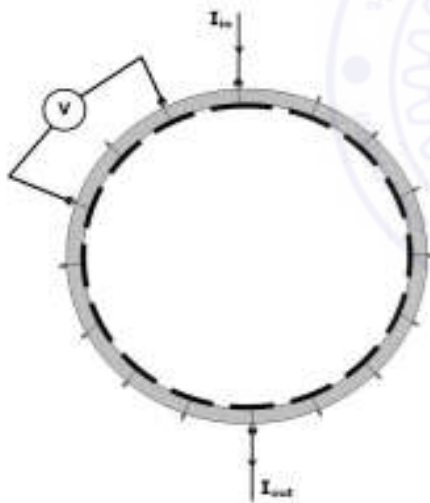


Figure 4.1: The 2D EIT forward problem simulation with the help of cross-pattern injection.

$I_{in}$  = The current injected through one electrode.

$I_{out}$  = The output or sink current through diametrically opposite electrode.

$V$  = The voltage measured differentially between two electrodes.

The number of measurements =208. The number of independent measurements=208/2=104.

Each electrode contains more than 2 nodes. The forward problem solution with point electrode model gives the nodal voltage on each node of the DUT. The nodes which are under the electrode are first determined. A weighted average based on the distance function is used to measure the electrode voltages.

The voltage simulated on each electrode is calculated as:

$$V_l = W_1V_1 + W_2V_2 + W_3V_3 + \dots \quad (4.1)$$

The voltages  $V_1, V_2, V_3$  are the nodal voltages of the nodes under the  $l^{th}$  electrode. The weights  $W_1, W_2, W_3$  are the weights based on the distances of the nodes in the electrode. For example, if there are three nodes under an electrode the node which is at middle will have higher weight than the nodes on the terminal.

#### **4.2 The 3D EIT Forward problem Simulation:**

The 3D EIT forward problem is done on a spherical domain. A sphere of radius unity is considered. The sphere is sub-divided into tetrahedron elements. Electrodes are placed on some particular positions of the sphere. Two electrodes are used here for the simulation purpose. The two electrodes are placed in the position  $(0, 0, 1)$  and  $(0, 0, -1)$ .

The nodal voltages are obtained by solving the 3D forward model. The conductivity is assumed to be constant throughout the material and equal to 10.

The solutions of the nodal voltages that are obtained from the forward model simulation are compared to the results obtained from the analytical solution of EIT equation on spherical domain.

## Chapter 5

### **RESULTS AND DISCUSSIONS:**

This chapter presents the comparative analysis between the nodal voltages obtained from the 2D and 3D FEM EIT to the solutions obtained from the standard software tool EIDORS and the analytical solution. This analysis proves the accuracy and closeness of solution between the finite element based method and other methods available. In this work two cases have been considered.

**Case I:** A circular geometry of unit radius is considered. Meshing is done on the geometry with varying mesh size.

Currents are injected from the boundary nodes. The magnitude of applied current is equal to 1 mili-Ampere. The analysis is done here on different mesh sizes and then the nodal voltages obtained are compared to that of analytic solution and EIDORS solution. Comparative analysis

on simulated voltage have been done for different mesh size and based on that calculation an optimal mesh size has been chosen. Background conductivity is chosen to be 10unit.

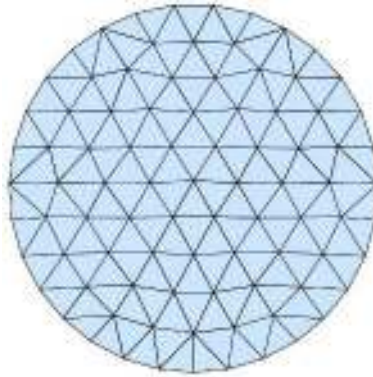


Figure 5.1: The 2D FEM Meshing of a circular geometry

**Case II:** The 3D FEM EIT equation is solved on a spherical domain. Two electrodes have been attached to that spherical domain. One is for injection of current and another is for sink. The radius of this spherical domain is 1 unit. The electrodes are attached in the vertices  $(0,0,1)$  and  $(0,0,-1)$  respectively. The mesh size used is 0.08. Number of nodes are 4385 and Number of elements are 23000. The solution is then compared with the analytic solution of 3D EIT equation for verification. Figure 5.2 describes the geometry of operation for 3D FEM EIT.

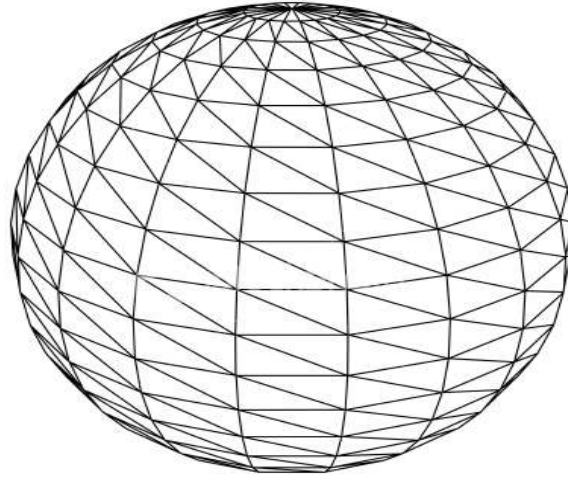
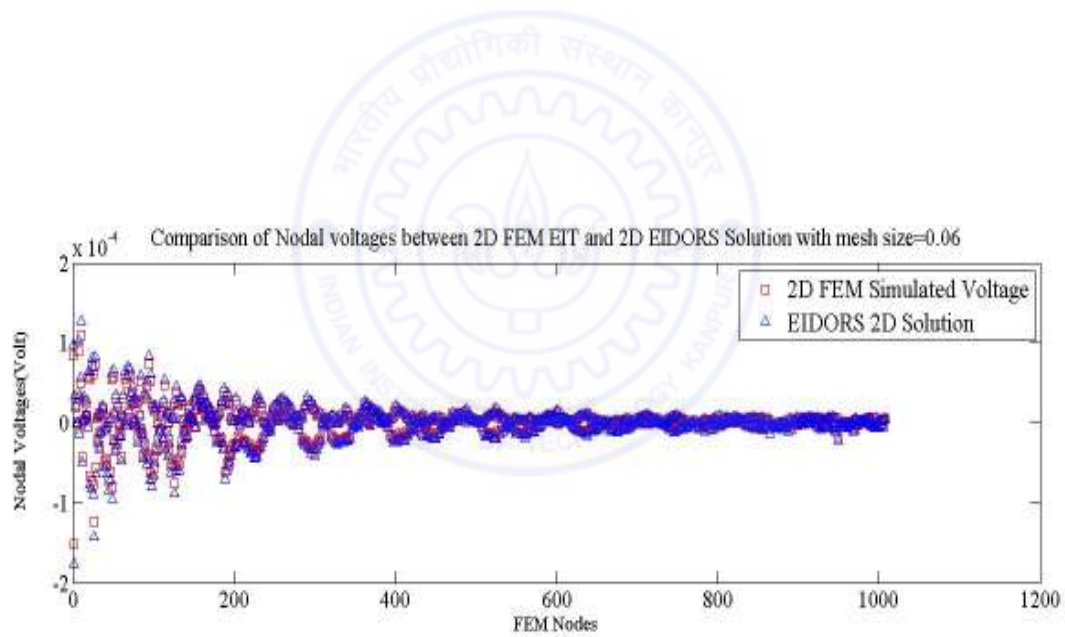
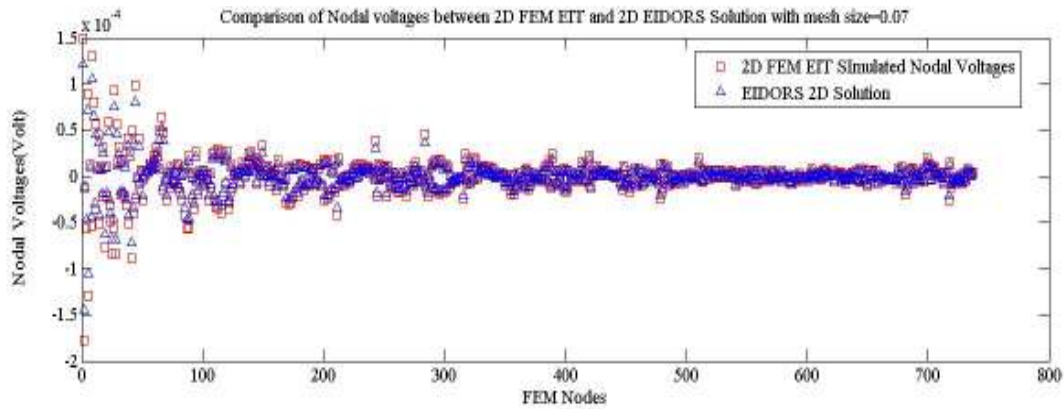


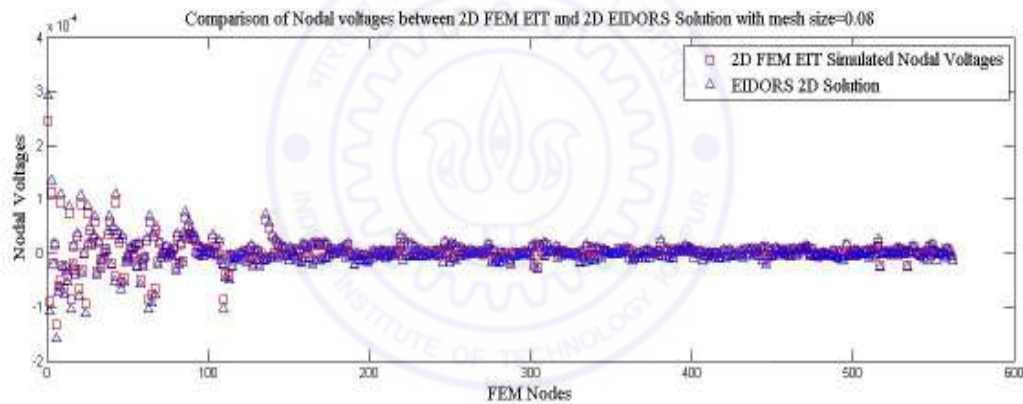
Figure 5.2: The 3D Elements dividing a spherical geometry with radius 1



(a) Mesh Size=0.06



(b) Mesh Size=0.07

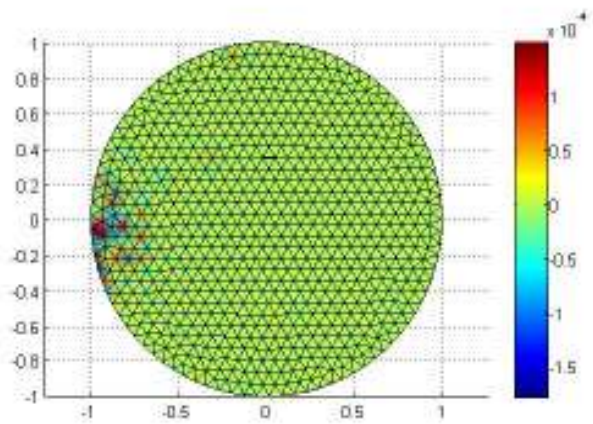


(c) Mesh Size=0.08

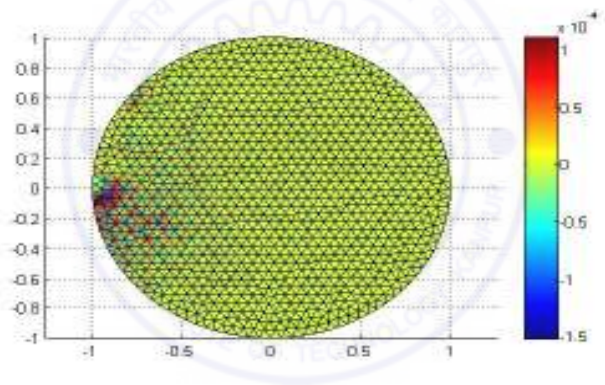
Figure 5.3: Comparison of Nodal Voltages with EIDORS solution for different mesh size (a) Mesh size=0.06 (b) Mesh size=0.07 and (c) Mesh size=0.08

Figure 5.3 shows the comparative analysis between Nodal Voltages obtained from 2D FEM EIT with different mesh resolution or size to standard software solution.

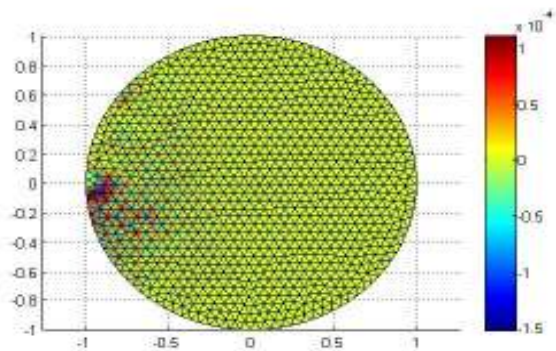




(a) Mesh Size=0.08



(b) Mesh Size=0.07



(c) Mesh Size=0.06

Figure 5.4: Nodal Voltage distribution within a circular geometry for different mesh sizes (a) 0.08 (b) 0.07 and (c) 0.06.

**Table 5.1: Comparison between different Mesh sizes for 2D FEM:**

S. No	Mesh Dimension	Average Error $[\frac{1}{N} \cdot (V_{FEM} - V_{EIDORS}) / (V_{EIDORS})]$	Number of Nodes	Number of Elements
1	0.06	0.0186	1008	1910
2	0.07	0.0297	737	1385
3	0.08	0.0359	562	1045

Table 5.1 shows the corresponding analysis of average error and number of nodes and elements for each mesh size. This comparison proves that with the decrease of mesh dimension and increase in the number of mesh nodes or elements the average error tends to decrease. Increase in the number of elements also suggests greater calculation complexity and greater run time. So, a trade-off between calculation complexity and accuracy has to be chosen. In this case of circular geometry with radius 1 unit the mesh dimension 0.07 seems to be a reasonable solution.

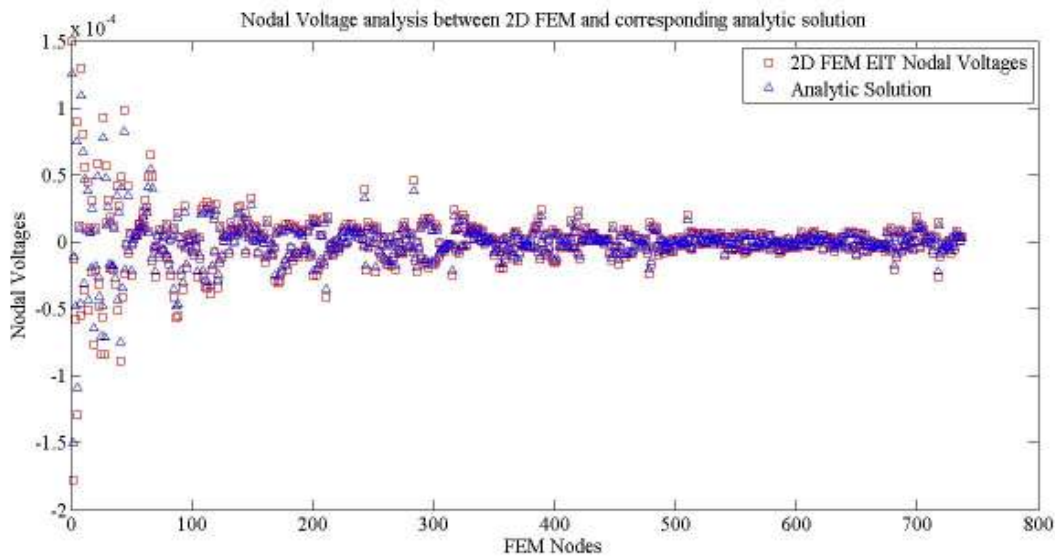


Figure 5.5: Comparison between 2D FEM EIT solutions of Nodal Voltages to that obtained from Analytic solution.

Figure 5.5 shows the comparative analysis between the 2D FEM EIT nodal voltages and the analytic solution. The mesh dimension of 0.07 has been used here for comparison. The average error in this case is:

Average Error= 0.0273.

This analysis proves the closeness of solutions between 2D FEM analysis, standard software tool analysis and analytic solution of 2D EIT equation.

The average error pattern for 2D and 3D EIT forward problem proves that the error margin is less in case of 3D analysis but the calculation complexity resources (memory) usage is very high.

## Case II: 3D FEM

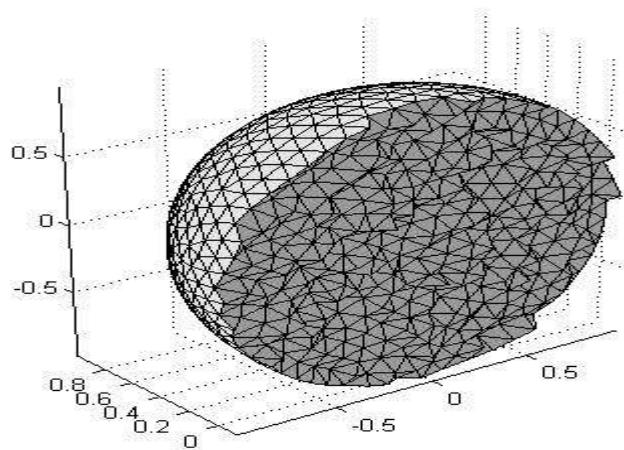


Figure 5.6: The 3D Nodes and elements of a spherical geometry

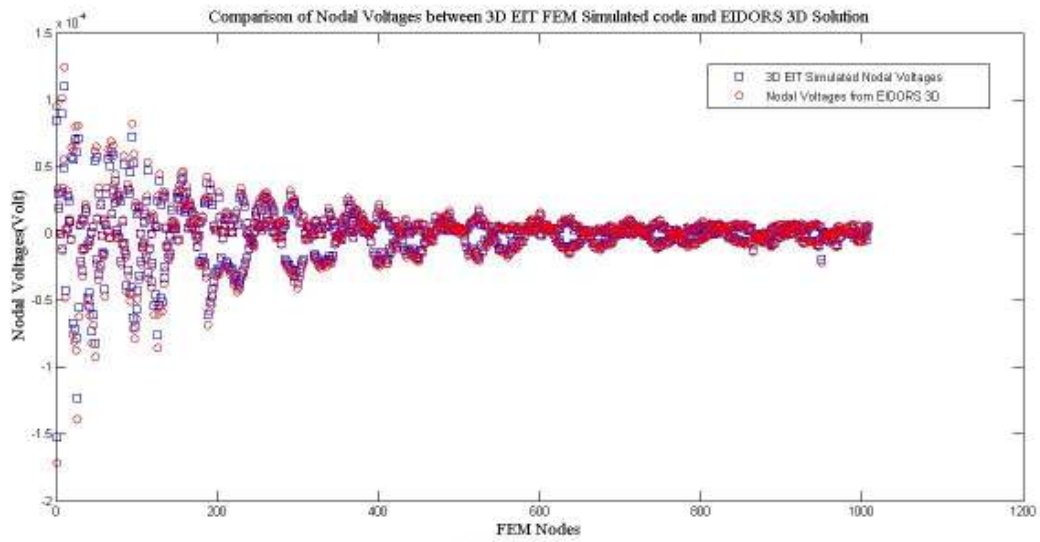


Figure 5.7: Comparison between Nodal voltages obtained by solving 3D EIT problem and analytical solution of 3D EIT equation on spherical domain.

For 3D FEM EIT solution the average error with the analytic solution and standard software solution are respectively:

**Average Error for 3D FEM EIT with Analytic solution= 0.0087**

**Average Error for 3D FEM EIT with EIDORS 3D solution=0.00793**

## Chapter 6

### **CONCLUSION:**

The forward problem of Electrical Impedance Tomography has been implemented and validated here. EIT forward problem analysis can lead to the following conclusions:

- 1) The present work in this thesis establishes and validates the 2D and 3D forward model of Electrical Impedance Tomography.
- 2) The results obtained in this thesis prove the correctness of the solutions of both analytic and that obtained from the EIDORS software tools [8].
- 3) The analytic solutions that has been done here solves the EIT partial differential equation in some particular geometry like circular in 2D and spherical in 3D [9 and 10].
- 4) The Jacobian for both 2D and 3D EIT has been calculated and compared to Jacobian obtained from EIDORS 2D and 3D. The results differ very small in the order of  $10^{-7}$ . This proves the accuracy of the calculation.

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## **APPENDIX A:**

### **A.1:**

$\nabla \cdot J = 0$  When there is no charge accumulated in the medium considered.

$$J = \sigma E$$

$$E = -\nabla u$$

These equations form the basic equation of EIT that is:

$$\nabla \cdot (\sigma \nabla u) = 0$$

### **A.2:**

#### **3D EIT Calculations:**

The voltage inside a 3D element in EIT can be expressed as

$$u(x, y, z) = a \cdot x + b \cdot y + c \cdot z + d$$

(a,b,c,d) are constants. (x ,y,z) the vertices of the point.

The vertices of a 3D element be  $= (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}^T \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$$

Solving the equation by Kramer's rule the values of a, b ,c ,d can be found.

The values of the co-efficients (a,b,c,d) can be expressed as:

$$a = (a_1 \cdot u_1 + a_2 \cdot u_2 + a_3 \cdot u_3 + a_4 \cdot u_4) / 6V_e$$

The values of  $a_1, a_2, a_3, a_4$  are obtained from solving the above equation.



Similarly the values of  $b$ ,  $c$  and  $d$  can be expressed as the same way:

The local stiffness matrix of  $k^{th}$  element is given by:

$$S^k = \frac{\sigma_e}{36V_e} \cdot \begin{bmatrix} u_1 x_1 y_1 z_1 \\ u_2 x_2 y_2 z_2 \\ u_3 x_3 y_3 z_3 \\ u_4 x_4 y_4 z_4 \end{bmatrix} / \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix}$$



## **APPENDIX B:**

### **Analytical Solution of 2D circular geometry:**

The analytical solution on the 2D circular geometry can be obtained by solving the EIT governing equation with boundary conditions on a circular geometry.

Two simple cases have been chosen and their corresponding solutions are discussed here. First one is the analytic solution when the conductivity of the DUT is homogenous. The solution of

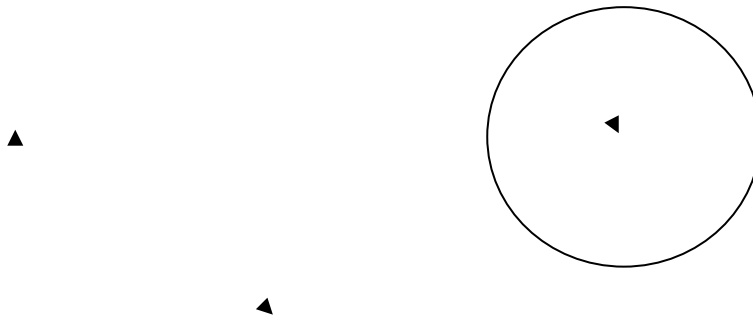
In the first case a circular geometry with radius =R has been considered (shown in Fig 4.1). To solve this problem the Fourier coefficients of the injected current density is to be determined. These values are then multiplied with the eigen-values of the resistivity to find the voltage distribution inside the DUT.

Fourier distribution of nth component of current density is given by:

$$J_n = \frac{1}{2\pi} \cdot \int J_\theta e^{-jn\theta} d\theta = \frac{1}{\pi} \cdot \sum_{l=1}^L (I_l / nA_e e^{-jn\theta_l} \sin \frac{nw}{2}) \quad (B.1)$$

Where w= Width of the electrode (In case of 2D).

$\theta$  is the angle which varies from 0 to  $2\pi$  as the DUT considered is circular.



The analytic value of voltage distribution within the DUT can be found as:

Case 1: When the DUT is homogenous.

$$V_1(\theta) = \sum_{n=-\infty}^{\infty} J_n \frac{e^{jn\theta}}{|n|\sigma} \quad (B.2)$$

Case 2: When the DUT is not homogenous

$$V_2(\theta) = \sum_{n=-\infty}^{\infty} J_n e^{jn\theta} \cdot \frac{1 - KR_1^{2n}}{1 + KR_1^{2n}} \quad (B.3)$$

Here  $K = (\sigma_1 - \sigma_0)/(\sigma_1 + \sigma_0)$

$R_1$ =Radius of the inhomogeneity.

## **B.2 Analytic Solution of 3D EIT in Spherical domain:**

In order to validate the 3D FEM code an analytic solution for 3D EIT has been done in the spherical domain. The values of voltages obtained on a spherical object can be obtained analytically by the following formula.

$$U(P) = \frac{I}{4\pi\sigma} \cdot \left[ \frac{2}{R_1} - \log(1 - P \cdot A + R_1) \right] - \frac{I}{4\pi\sigma} \left[ \frac{2}{R_2} - \log(1 - P \cdot B + R_2) \right] \quad (B.4)$$

Here  $\sigma$  is the conductivity of the uniform sphere (Shown in Fig B.2).  $P$  denotes the distance from the centre of the sphere to the point  $P$  where the voltage is being measured.  $A, B$  denote the distance of the current ( $I$ ) source and sink point.  $R_1$  is the distance between points  $A$  and  $P$ .

Similarly,  $R_2$  is the distance between point  $P$  and point  $B$ . The current value and conductivity value are kept the same to that of the FEM code so that they can be compared for validation.

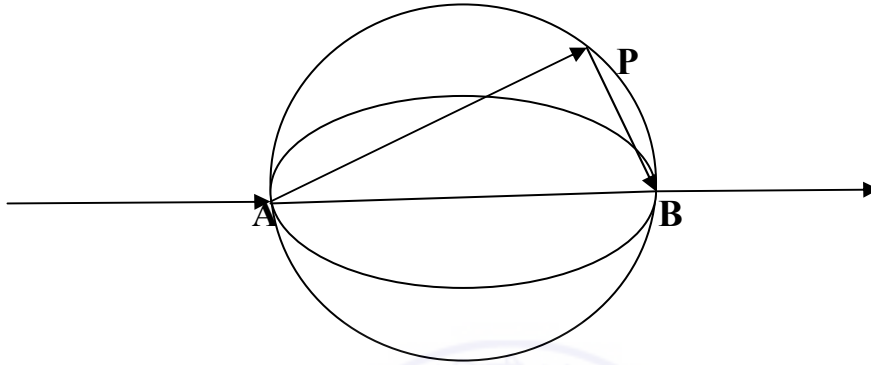


Figure B.2: Analytical solution of 3D FEM