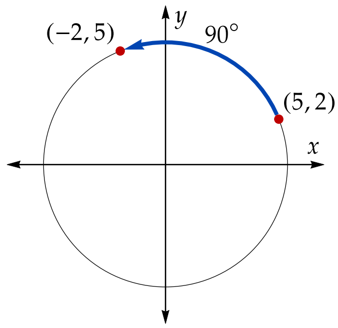
12.2 Linear Transformations



▲ **Figure 1:** A 90° counterclockwise rotation.

As we have seen, we can rotate any point in the plane 90° counterclockwise around the origin by switching the two coordinates and negating the first one:

$$(5, 2) \longmapsto (-2, 5).$$

This transformation is shown in Figure 1.

This transformation is equivalent to multiplying by the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

For example,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

and more generally

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

This is a simple example of a linear transformation.

> **Linear Transformations**
>
> A transformation of the plane is called a **linear transformation** if it corresponds to multiplying each point (x, y) by some 2×2 matrix A, i.e.
>
> $$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto A\begin{bmatrix} x \\ y \end{bmatrix}.$$

It turns out that many geometric transformations of the plane are linear transformations, including:

1. Rotation of the plane by any angle around the origin.
2. Reflection of the plane across any line that goes through the origin.

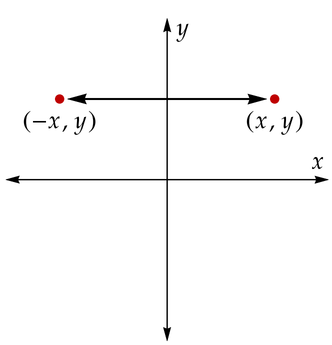
EXAMPLE 1

Describe the linear transformation of the plane corresponding to the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

SOLUTION We have

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

so this matrix negates the x-coordinate of each point of the plane. Geometrically, this corresponds to $\boxed{\text{reflection across the } y\text{-axis}}$, as shown in Figure 2.



▲ **Figure 2:** Reflection across the y-axis.

Finding the Matrix

There is a nice trick that can be used to find the matrix for a given transformation.

Column Trick

If A is a 2×2 matrix, then

$$A\begin{bmatrix}1\\0\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix}$$

are the first and second columns of A, respectively.

For example,

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

The following example shows how to use this trick to find the matrix for a linear transformation.

EXAMPLE 2

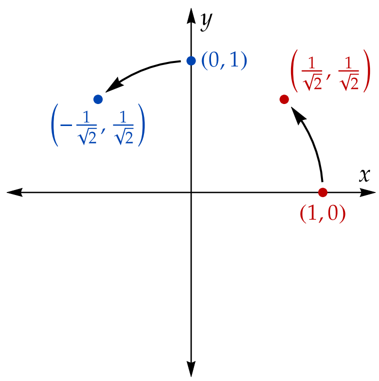
Find the matrix for a 45° counterclockwise rotation of the plane about the origin.

SOLUTION This transformation is shown in Figure 3. Note that $(1,0)$ maps to $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $(0,1)$ maps to $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. If A is the matrix for this transformation, it follows that

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

so these vectors are the columns of A. We conclude that

$$A = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$



▲ **Figure 3:** A 45° rotation of the plane.

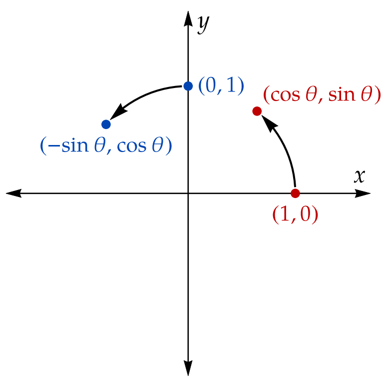
The previous example is a special case of a more general formula.

2 × 2 Rotation Matrices

The matrix

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

rotates the plane counterclockwise around the origin by an angle of θ.

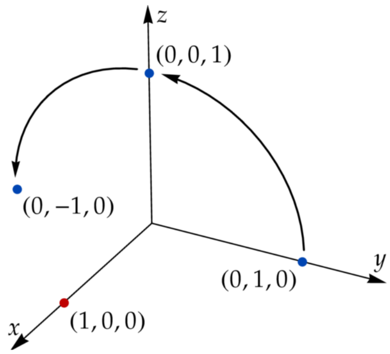


▲ **Figure 4:** A rotation of the plane by an angle of θ.

The justification for this formula is shown in Figure 4. If A is the matrix for this transformation, then

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

so these vectors are the columns of A.



▲ **Figure 5:** A 90° rotation around the x-axis.

A Closer Look — Transformations of $\mathbb{R}^3$

We can use 3×3 matrices to describe certain transformations in three dimensions, such as rotation around a line through the origin, or reflection across a plane through the origin. Such a transformation is called a **linear transformation of** $\mathbb{R}^3$.

For example, consider the 90° rotation of $\mathbb{R}^3$ about the x-axis shown in Figure 5. How can we find a 3×3 matrix A for this transformation? Well, it is obvious from the figure that

$$A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} \qquad A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\0\end{bmatrix}.$$

Then these three vectors must be the three columns of A. We conclude that

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}$$

EXERCISES

1–4 ■ Give a geometric description of the linear transformation corresponding to the given matrix.

1. $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$

2. $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

3–4 ■ Find the matrix for the reflection of $\mathbb{R}^2$ across the given line.

3. the line $y = x$

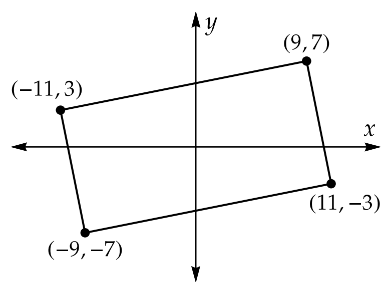
4. the x-axis

5–6 ■ Find the matrix for the given rotation of $\mathbb{R}^2$ around the origin.

5. 135° counterclockwise

6. 30° clockwise

7. The following figure shows a rectangle in the plane.



Find the new coordinates of the four vertices if this rectangle is rotated 45° counterclockwise around the origin.