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Interval Neutrosophic Sets with Applications in BCK/BCI-Algebra

Young Bae Jun [1,*], Seon Jeong Kim [2] and Florentin Smarandache [3]

1 Department of Mathematics Education Gyeongsang National University, Jinju 52828, Korea
2 Department of Mathematics, Natural Science of College, Gyeongsang National University, Jinju 52828, Korea; skim@gnu.ac.kr
3 Mathematics & Science Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; fsmarandache@gmail.com
* Correspondence: skywine@gmail.com

Received: 27 February 2018; Accepted: 06 April 2018; Published: 9 April 2018

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Abstract: For $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, the notion of $(T(i, j), I(k, l), F(m, n))$-interval neutrosophic subalgebra in BCK/BCI-algebra is introduced, and their properties and relations are investigated. The notion of interval neutrosophic length of an interval neutrosophic set is also introduced, and related properties are investigated.

Keywords: interval neutrosophic set; interval neutrosophic subalgebra; interval neutrosophic length

MSC: 06F35, 03G25, 03B52

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1. Introduction

Intuitionistic fuzzy set, which is introduced by Atanassov [1], is a generalization of Zadeh's fuzzy sets [2], and consider both truth-membership and falsity-membership. Since the sum of degree true, indeterminacy and false is one in intuitionistic fuzzy sets, incomplete information is handled in intuitionistic fuzzy sets. On the other hand, neutrosophic sets can handle the indeterminate information and inconsistent information that exist commonly in belief systems in a neutrosophic set since indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are independent, which is mentioned in [3]. As a formal framework that generalizes the concept of the classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set, interval valued intuitionistic fuzzy set and paraconsistent set, etc., the neutrosophic set is developed by Smarandache [4,5], which is applied to various parts, including algebra, topology, control theory, decision-making problems, medicines and in many real-life problems. The concept of interval neutrosophic sets is presented by Wang et al. [6], and it is more precise and more flexible than the single-valued neutrosophic set. The interval neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information, which exists in the real world. BCK-algebra is introduced by Imai and Iséki [7], and it has been applied to several branches of mathematics, such as group theory, functional analysis, probability theory and topology, etc. As a generalization of BCK-algebra, Iséki introduced the notion of BCI-algebra (see [8]).

In this article, we discuss interval neutrosophic sets in BCK/BCI-algebra. We introduce the notion of $(T(i, j), I(k, l), F(m, n))$-interval neutrosophic subalgebra in BCK/BCI-algebra for $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, and investigate their properties and relations. We also introduce the notion of interval neutrosophic length of an interval neutrosophic set, and investigate related properties.

2. Preliminaries

By a *BCI-algebra*, we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = y * x = 0 \Rightarrow x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is *BCK-algebra*.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

The collection of all BCK-algebra and all BCI-algebra are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. In addition, $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [9,10] for further information regarding BCK/BCI-algebra.

By a *fuzzy structure* over a nonempty set X, we mean an ordered pair (X, ρ) of X and a fuzzy set ρ on X.

Definition 1 ([11])**.** *For any $(X, *, 0) \in \mathcal{B}(X)$, a fuzzy structure (X, μ) over $(X, *, 0)$ is called a*

- *fuzzy subalgebra of $(X, *, 0)$ with type 1 (briefly, 1-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\, (\mu(x * y) \geq \min\{\mu(x), \mu(y)\})\,, \tag{1}$$

- *fuzzy subalgebra of $(X, *, 0)$ with type 2 (briefly, 2-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\, (\mu(x * y) \leq \min\{\mu(x), \mu(y)\})\,, \tag{2}$$

- *fuzzy subalgebra of $(X, *, 0)$ with type 3 (briefly, 3-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\, (\mu(x * y) \geq \max\{\mu(x), \mu(y)\})\,, \tag{3}$$

- *fuzzy subalgebra of $(X, *, 0)$ with type 4 (briefly, 4-fuzzy subalgebra of $(X, *, 0)$) if*

$$(\forall x, y \in X)\, (\mu(x * y) \leq \max\{\mu(x), \mu(y)\})\,. \tag{4}$$

Let X be a non-empty set. A neutrosophic set (NS) in X (see [4]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\},$$

where $A_T : X \to [0,1]$ is a truth-membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function.

An interval neutrosophic set (INS) A in X is characterized by truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0,1]$ (see [3,6]).

3. Interval Neutrosophic Subalgebra

In what follows, let $(X, *, 0) \in \mathcal{B}(X)$ and $\mathcal{P}^*([0,1])$ be the family of all subintervals of $[0,1]$ unless otherwise specified.

Definition 2 ([3,6])**.** *An interval neutrosophic set in a nonempty set X is a structure of the form:*

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\},$$

where

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),$$

which is called interval truth-membership function,

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),$$

which is called interval indeterminacy-membership function, and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),$$

which is called interval falsity-membership function.

For the sake of simplicity, we will use the notation $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ for the interval neutrosophic set

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}.$$

Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we consider the following functions:

$$\begin{aligned}
&\mathcal{I}[T]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[T](x)\},\\
&\mathcal{I}[I]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[I](x)\},\\
&\mathcal{I}[F]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[F](x)\},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{I}[T]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[T](x)\},\\
&\mathcal{I}[I]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[I](x)\},\\
&\mathcal{I}[F]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[F](x)\}.
\end{aligned}$$

Definition 3. *For any $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is called a $(T(i,j), I(k,l), F(m,n))$-interval neutrosophic subalgebra of X if the following assertions are valid.*

(1) *$(X, \mathcal{I}[T]_{\inf})$ is an i-fuzzy subalgebra of $(X, *, 0)$ and $(X, \mathcal{I}[T]_{\sup})$ is a j-fuzzy subalgebra of $(X, *, 0)$,*
(2) *$(X, \mathcal{I}[I]_{\inf})$ is a k-fuzzy subalgebra of $(X, *, 0)$ and $(X, \mathcal{I}[I]_{\sup})$ is an l-fuzzy subalgebra of $(X, *, 0)$,*
(3) *$(X, \mathcal{I}[F]_{\inf})$ is an m-fuzzy subalgebra of $(X, *, 0)$ and $(X, \mathcal{I}[F]_{\sup})$ is an n-fuzzy subalgebra of $(X, *, 0)$.*

Example 1. *Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the binary operation $*$, which is given in Table 1 (see [10]).*

Table 1. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**	**3**
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

(1) Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ for which $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1])\ \ x \mapsto \begin{cases} [0.4, 0.5) & \text{if } x = 0,\\ (0.3, 0.5] & \text{if } x = 1,\\ [0.2, 0.6) & \text{if } x = 2,\\ [0.1, 0.7] & \text{if } x = 3, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]) \ \ x \mapsto \begin{cases} [0.5, 0.8) & \text{if } x = 0, \\ (0.2, 0.7) & \text{if } x = 1, \\ [0.5, 0.6] & \text{if } x = 2, \\ [0.4, 0.8) & \text{if } x = 3, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \ \ x \mapsto \begin{cases} [0.4, 0.5) & \text{if } x = 0, \\ (0.2, 0.9) & \text{if } x = 1, \\ [0.1, 0.6] & \text{if } x = 2, \\ (0.4, 0.7] & \text{if } x = 3. \end{cases}$$

It is routine to verify that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(1,4), I(1,4), F(1,4))$*-interval neutrosophic subalgebra of* $(X, *, 0)$*.*

(2) Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *be an interval neutrosophic set in* $(X, *, 0)$ *for which* $\mathcal{I}[T]$*,* $\mathcal{I}[I]$ *and* $\mathcal{I}[F]$ *are given as follows:*

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]) \ \ x \mapsto \begin{cases} [0.1, 0.4) & \text{if } x = 0, \\ (0.3, 0.5) & \text{if } x = 1, \\ [0.2, 0.7] & \text{if } x = 2, \\ [0.4, 0.6) & \text{if } x = 3, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]) \ \ x \mapsto \begin{cases} (0.2, 0.5) & \text{if } x = 0, \\ [0.5, 0.8] & \text{if } x = 1, \\ (0.4, 0.5] & \text{if } x = 2, \\ [0.2, 0.6] & \text{if } x = 3, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \ \ x \mapsto \begin{cases} [0.3, 0.4) & \text{if } x = 0, \\ (0.4, 0.7) & \text{if } x = 1, \\ (0.6, 0.8) & \text{if } x = 2, \\ [0.4, 0.6] & \text{if } x = 3. \end{cases}$$

By routine calculations, we know that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,4), I(4,4), F(4,4))$*-interval neutrosophic subalgebra of* $(X, *, 0)$*.*

Example 2. *Consider a BCI-algebra* $X = \{0, a, b, c\}$ *with the binary operation* $*$*, which is given in Table 2 (see [10]).*

Table 2. Cayley table for the binary operation "$*$".

$*$	**0**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *be an interval neutrosophic set in* $(X, *, 0)$ *for which* $\mathcal{I}[T]$*,* $\mathcal{I}[I]$ *and* $\mathcal{I}[F]$ *are given as follows:*

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]) \ \ x \mapsto \begin{cases} [0.3, 0.9) & \text{if } x = 0, \\ (0.7, 0.9) & \text{if } x = a, \\ [0.7, 0.8) & \text{if } x = b, \\ (0.5, 0.8] & \text{if } x = c, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} [0.2, 0.65) & \text{if } x = 0, \\ [0.5, 0.55] & \text{if } x = a, \\ (0.6, 0.65) & \text{if } x = b, \\ [0.5, 0.55) & \text{if } x = c, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]) \;\; x \mapsto \begin{cases} (0.3, 0.6) & \text{if } x = 0, \\ [0.4, 0.6] & \text{if } x = a, \\ (0.4, 0.5] & \text{if } x = b, \\ [0.3, 0.5) & \text{if } x = c. \end{cases}$$

Routine calculations show that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(T(4,1), I(4,1), F(4,1))$*-interval neutrosophic subalgebra of* $(X, *, 0)$*. However, it is not a* $(T(2,1), I(2,1), F(2,1))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *since*

$$\mathcal{I}[T]_{\inf}(c * a) = \mathcal{I}[T]_{\inf}(b) = 0.7 > 0.5 = \min\{\mathcal{I}[T]_{\inf}(c), \mathcal{I}[T]_{\inf}(a)\}$$

and/or

$$\mathcal{I}[I]_{\inf}(a * c) = \mathcal{I}[I]_{\inf}(b) = 0.6 > 0.5 = \min\{\mathcal{I}[I]_{\inf}(a), \mathcal{I}[I]_{\inf}(c)\}.$$

In addition, it is not a $(T(4,3), I(4,3), F(4,3))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *since*

$$\mathcal{I}[T]_{\sup}(a * b) = \mathcal{I}[T]_{\sup}(c) = 0.8 < 0.9 = \max\{\mathcal{I}[T]_{\inf}(a), \mathcal{I}[T]_{\inf}(c)\}$$

and/or

$$\mathcal{I}[F]_{\sup}(a * b) = \mathcal{I}[F]_{\sup}(c) = 0.5 < 0.6 = \max\{\mathcal{I}[F]_{\inf}(a), \mathcal{I}[F]_{\inf}(c)\}.$$

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X. We consider the following sets:

$$\begin{aligned} U(\mathcal{I}[T]_{\inf}; \alpha_I) &:= \{x \in X \mid \mathcal{I}[T]_{\inf}(x) \geq \alpha_I\}, \\ L(\mathcal{I}[T]_{\sup}; \alpha_S) &:= \{x \in X \mid \mathcal{I}[T]_{\sup}(x) \leq \alpha_S\}, \\ U(\mathcal{I}[I]_{\inf}; \beta_I) &:= \{x \in X \mid \mathcal{I}[I]_{\inf}(x) \geq \beta_I\}, \\ L(\mathcal{I}[I]_{\sup}; \beta_S) &:= \{x \in X \mid \mathcal{I}[I]_{\sup}(x) \leq \beta_S\}, \end{aligned}$$

and

$$\begin{aligned} U(\mathcal{I}[F]_{\inf}; \gamma_I) &:= \{x \in X \mid \mathcal{I}[F]_{\inf}(x) \geq \gamma_I\}, \\ L(\mathcal{I}[F]_{\sup}; \gamma_S) &:= \{x \in X \mid \mathcal{I}[F]_{\sup}(x) \leq \gamma_S\}, \end{aligned}$$

where α_I, α_S, β_I, β_S, γ_I and γ_S are numbers in $[0,1]$.

Theorem 1. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,4), I(i,4), F(i,4))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{1,3\}$*, then* $U(\mathcal{I}[T]_{\inf}; \alpha_I)$*,* $L(\mathcal{I}[T]_{\sup}; \alpha_S)$*,* $U(\mathcal{I}[I]_{\inf}; \beta_I)$*,* $L(\mathcal{I}[I]_{\sup}; \beta_S)$*,* $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ *and* $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ *are either empty or subalgebra of* $(X, *, 0)$ *for all* α_I*,* α_S*,* β_I*,* β_S*,* γ_I*,* $\gamma_S \in [0,1]$*.*

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 1-fuzzy subalgebra of X; and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 4-fuzzy subalgebra of X. Let $\alpha_I, \alpha_S \in [0,1]$ be such that $U(\mathcal{I}[T]_{\inf}; \alpha_I)$ and $L(\mathcal{I}[T]_{\sup}; \alpha_S)$ are nonempty. For any $x, y \in X$, if $x, y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$, then $\mathcal{I}[T]_{\inf}(x) \geq \alpha_I$ and $\mathcal{I}[T]_{\inf}(y) \geq \alpha_I$, and so

$$\mathcal{I}[T]_{\inf}(x * y) \geq \min\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\} \geq \alpha_I,$$

that is, $x * y \in U(\mathcal{I}[T]_{\inf}; \alpha_I)$. If $x, y \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$, then $\mathcal{I}[T]_{\sup}(x) \leq \alpha_S$ and $\mathcal{I}[T]_{\sup}(y) \leq \alpha_S$, which imply that

$$\mathcal{I}[T]_{\sup}(x * y) \leq \max\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\} \leq \alpha_S,$$

that is, $x * y \in L(\mathcal{I}[T]_{\sup}; \alpha_S)$. Hence, $U(\mathcal{I}[T]_{\inf}; \alpha_I)$ and $L(\mathcal{I}[T]_{\sup}; \alpha_S)$ are subalgebra of $(X, *, 0)$ for all α_I, $\alpha_S \in [0, 1]$. Similarly, we can prove that $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all β_I, β_S, γ_I, $\gamma_S \in [0, 1]$. Suppose that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(3,4), I(3,4), F(3,4))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 3-fuzzy subalgebra of X; and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 4-fuzzy subalgebra of X. Let β_I and $\beta_S \in [0, 1]$ be such that $U(\mathcal{I}[I]_{\inf}; \beta_I)$ and $L(\mathcal{I}[I]_{\sup}; \beta_S)$ are nonempty. Let $x, y \in U(\mathcal{I}[I]_{\inf}; \beta_I)$. Then, $\mathcal{I}[I]_{\inf}(x) \geq \beta_I$ and $\mathcal{I}[I]_{\inf}(y) \geq \beta_I$. It follows that

$$\mathcal{I}[I]_{\inf}(x * y) \geq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\} \geq \beta_I$$

and so $x * y \in U(\mathcal{I}[I]_{\inf}; \beta_I)$. Thus, $U(\mathcal{I}[I]_{\inf}; \beta_I)$ is a subalgebra of $(X, *, 0)$. If $x, y \in L(\mathcal{I}[I]_{\inf}; \beta_S)$, then $\mathcal{I}[I]_{\inf}(x) \leq \beta_S$ and $\mathcal{I}[I]_{\inf}(y) \leq \beta_S$. Hence,

$$\mathcal{I}[I]_{\inf}(x * y) \leq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\} \leq \beta_S,$$

and so $x * y \in L(\mathcal{I}[I]_{\inf}; \beta_S)$. Thus, $L(\mathcal{I}[I]_{\inf}; \beta_S)$ is a subalgebra of $(X, *, 0)$. Similarly, we can show that $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all α_I, α_S, γ_I, $\gamma_S \in [0, 1]$. □

Since every 2-fuzzy subalgebra is a 4-fuzzy subalgebra, we have the following corollary.

Corollary 1. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(i,2), I(i,2), F(i,2))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $i \in \{1, 3\}$, then $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.*

By a similar way to the proof of Theorem 1, we have the following theorems.

Theorem 2. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(i,4), I(i,4), F(i,4))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $i \in \{2, 4\}$, then $L(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $L(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.*

Corollary 2. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(i,2), I(i,2), F(i,2))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $i \in \{2, 4\}$, then $L(\mathcal{I}[T]_{\inf}; \alpha_I)$, $L(\mathcal{I}[T]_{\sup}; \alpha_S)$, $L(\mathcal{I}[I]_{\inf}; \beta_I)$, $L(\mathcal{I}[I]_{\sup}; \beta_S)$, $L(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $L(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.*

Theorem 3. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(k,1), I(k,1), F(k,1))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $k \in \{1, 3\}$, then $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $U(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.*

Corollary 3. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(k,3), I(k,3), F(k,3))$-interval neutrosophic subalgebra of $(X, *, 0)$ for $k \in \{1, 3\}$, then $U(\mathcal{I}[T]_{\inf}; \alpha_I)$, $U(\mathcal{I}[T]_{\sup}; \alpha_S)$, $U(\mathcal{I}[I]_{\inf}; \beta_I)$, $U(\mathcal{I}[I]_{\sup}; \beta_S)$, $U(\mathcal{I}[F]_{\inf}; \gamma_I)$ and $U(\mathcal{I}[F]_{\sup}; \gamma_S)$ are either empty or subalgebra of $(X, *, 0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0, 1]$.*

Theorem 4. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(k,1), I(k,1), F(k,1))$-interval neutrosophic subalgebra of $(X,*,0)$ for $k \in \{2,4\}$, then $L(\mathcal{I}[T]_{\inf};\alpha_I)$, $U(\mathcal{I}[T]_{\sup};\alpha_S)$, $L(\mathcal{I}[I]_{\inf};\beta_I)$, $U(\mathcal{I}[I]_{\sup};\beta_S)$, $L(\mathcal{I}[F]_{\inf};\gamma_I)$ and $U(\mathcal{I}[F]_{\sup};\gamma_S)$ are either empty or subalgebra of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.*

Corollary 4. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(k,3), I(k,3), F(k,3))$-interval neutrosophic subalgebra of $(X,*,0)$ for $k \in \{2,4\}$, then $L(\mathcal{I}[T]_{\inf};\alpha_I)$, $U(\mathcal{I}[T]_{\sup};\alpha_S)$, $L(\mathcal{I}[I]_{\inf};\beta_I)$, $U(\mathcal{I}[I]_{\sup};\beta_S)$, $L(\mathcal{I}[F]_{\inf};\gamma_I)$ and $U(\mathcal{I}[F]_{\sup};\gamma_S)$ are either empty or subalgebra of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$.*

Theorem 5. *Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X in which $U(\mathcal{I}[T]_{\inf};\alpha_I)$, $L(\mathcal{I}[T]_{\sup};\alpha_S)$, $U(\mathcal{I}[I]_{\inf};\beta_I)$, $L(\mathcal{I}[I]_{\sup};\beta_S)$, $U(\mathcal{I}[F]_{\inf};\gamma_I)$ and $L(\mathcal{I}[F]_{\sup};\gamma_S)$ are nonempty subalgebra of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$. Then, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic subalgebra of $(X,*,0)$.*

Proof. Suppose that $(X, \mathcal{I}[T]_{\inf})$ is not a 1-fuzzy subalgebra of $(X,*,0)$. Then, there exists $x, y \in X$ such that

$$\mathcal{I}[T]_{\inf}(x*y) < \min\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\}.$$

If we take $\alpha_I = \min\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\}$, then $x, y \in U(\mathcal{I}[T]_{\inf};\alpha_I)$, but $x*y \notin U(\mathcal{I}[T]_{\inf};\alpha_I)$. This is a contradiction, and so $(X, \mathcal{I}[T]_{\inf})$ is a 1-fuzzy subalgebra of $(X,*,0)$. If $(X, \mathcal{I}[T]_{\sup})$ is not a 4-fuzzy subalgebra of $(X,*,0)$, then

$$\mathcal{I}[T]_{\sup}(a*b) > \max\{\mathcal{I}[T]_{\sup}(a), \mathcal{I}[T]_{\sup}(b)\}$$

for some $a, b \in X$, and so $a, b \in L(\mathcal{I}[T]_{\sup};\alpha_S)$ and $a*b \notin L(\mathcal{I}[T]_{\sup};\alpha_S)$ by taking

$$\alpha_S := \max\{\mathcal{I}[T]_{\sup}(a), \mathcal{I}[T]_{\sup}(b)\}.$$

This is a contradiction, and therefore $(X, \mathcal{I}[T]_{\sup})$ is a 4-fuzzy subalgebra of $(X,*,0)$. Similarly, we can verify that $(X, \mathcal{I}[I]_{\inf})$ is a 1-fuzzy subalgebra of $(X,*,0)$ and $(X, \mathcal{I}[I]_{\sup})$ is a 4-fuzzy subalgebra of $(X,*,0)$; and $(X, \mathcal{I}[F]_{\inf})$ is a 1-fuzzy subalgebra of $(X,*,0)$ and $(X, \mathcal{I}[F]_{\sup})$ is a 4-fuzzy subalgebra of $(X,*,0)$. Consequently, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,4), I(1,4), F(1,4))$-interval neutrosophic subalgebra of $(X,*,0)$. □

Using the similar method to the proof of Theorem 5, we get the following theorems.

Theorem 6. *Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X in which $L(\mathcal{I}[T]_{\inf};\alpha_I)$, $U(\mathcal{I}[T]_{\sup};\alpha_S)$, $L(\mathcal{I}[I]_{\inf};\beta_I)$, $U(\mathcal{I}[I]_{\sup};\beta_S)$, $L(\mathcal{I}[F]_{\inf};\gamma_I)$ and $U(\mathcal{I}[F]_{\sup};\gamma_S)$ are nonempty subalgebra of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$. Then, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,1), I(4,1), F(4,1))$-interval neutrosophic subalgebra of $(X,*,0)$.*

Theorem 7. *Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X in which $L(\mathcal{I}[T]_{\inf};\alpha_I)$, $L(\mathcal{I}[T]_{\sup};\alpha_S)$, $L(\mathcal{I}[I]_{\inf};\beta_I)$, $L(\mathcal{I}[I]_{\sup};\beta_S)$, $L(\mathcal{I}[F]_{\inf};\gamma_I)$ and $L(\mathcal{I}[F]_{\sup};\gamma_S)$ are nonempty subalgebra of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$. Then, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,4), I(4,4), F(4,4))$-interval neutrosophic subalgebra of $(X,*,0)$.*

Theorem 8. *Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in X in which $U(\mathcal{I}[T]_{\inf};\alpha_I)$, $U(\mathcal{I}[T]_{\sup};\alpha_S)$, $U(\mathcal{I}[I]_{\inf};\beta_I)$, $U(\mathcal{I}[I]_{\sup};\beta_S)$, $U(\mathcal{I}[F]_{\inf};\gamma_I)$ and $U(\mathcal{I}[F]_{\sup};\gamma_S)$ are nonempty subalgebra of $(X,*,0)$ for all α_I, α_S, β_I, β_S, γ_I, $\gamma_S \in [0,1]$. Then, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(1,1), I(1,1), F(1,1))$-interval neutrosophic subalgebra of $(X,*,0)$.*

4. Interval Neutrosophic Lengths

Definition 4. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X*, we define the interval neutrosophic length of* $\mathcal{I}$ *as an ordered triple* $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ *where*

$$\mathcal{I}[T]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x),$$
$$\mathcal{I}[I]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x),$$

and

$$\mathcal{I}[F]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x),$$

which are called interval neutrosophic T-length, interval neutrosophic I-length and interval neutrosophic F-length of $\mathcal{I}$*, respectively.*

Example 3. *Consider the interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X*, which is given in Example 2. Then, the interval neutrosophic length of* $\mathcal{I}$ *is given by Table 3.*

Table 3. Interval neutrosophic length of $\mathcal{I}$.

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.6	0.45	0.3
a	0.2	0.05	0.2
b	0.1	0.05	0.1
c	0.3	0.05	0.2

Theorem 9. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,3), I(i,3), F(i,3))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2,4\}$*, then* $(X, \mathcal{I}[T]_\ell)$*,* $(X, \mathcal{I}[I]_\ell)$ *and* $(X, \mathcal{I}[F]_\ell)$ *are 3-fuzzy subalgebra of* $(X, *, 0)$*.*

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(2,3), I(2,3), F(2,3))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 2-fuzzy subalgebra of X, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 3-fuzzy subalgebra of X. Thus,

$$\mathcal{I}[T]_{\inf}(x*y) \leq \min\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\},$$
$$\mathcal{I}[I]_{\inf}(x*y) \leq \min\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\},$$
$$\mathcal{I}[F]_{\inf}(x*y) \leq \min\{\mathcal{I}[F]_{\inf}(x), \mathcal{I}[F]_{\inf}(y)\},$$

and

$$\mathcal{I}[T]_{\sup}(x*y) \geq \max\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\},$$
$$\mathcal{I}[I]_{\sup}(x*y) \geq \max\{\mathcal{I}[I]_{\sup}(x), \mathcal{I}[I]_{\sup}(y)\},$$
$$\mathcal{I}[F]_{\sup}(x*y) \geq \max\{\mathcal{I}[F]_{\sup}(x), \mathcal{I}[F]_{\sup}(y)\},$$

for all $x, y \in X$. It follows that

$$\mathcal{I}[T]_\ell(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \geq \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_\ell(x),$$
$$\mathcal{I}[T]_\ell(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \geq \mathcal{I}[T]_{\sup}(y) - \mathcal{I}[T]_{\inf}(y) = \mathcal{I}[T]_\ell(y),$$
$$\mathcal{I}[I]_\ell(x*y) = \mathcal{I}[I]_{\sup}(x*y) - \mathcal{I}[I]_{\inf}(x*y) \geq \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_\ell(x),$$
$$\mathcal{I}[I]_\ell(x*y) = \mathcal{I}[I]_{\sup}(x*y) - \mathcal{I}[I]_{\inf}(x*y) \geq \mathcal{I}[I]_{\sup}(y) - \mathcal{I}[I]_{\inf}(y) = \mathcal{I}[I]_\ell(y),$$

and

$$\mathcal{I}[F]_\ell(x*y) = \mathcal{I}[F]_{\sup}(x*y) - \mathcal{I}[F]_{\inf}(x*y) \geq \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_\ell(x),$$
$$\mathcal{I}[F]_\ell(x*y) = \mathcal{I}[F]_{\sup}(x*y) - \mathcal{I}[F]_{\inf}(x*y) \geq \mathcal{I}[F]_{\sup}(y) - \mathcal{I}[F]_{\inf}(y) = \mathcal{I}[F]_\ell(y).$$

Hence,

$$\mathcal{I}[T]_\ell(x*y) \geq \max\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},$$
$$\mathcal{I}[I]_\ell(x*y) \geq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},$$

and

$$\mathcal{I}[F]_\ell(x*y) \geq \max\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\},$$

for all $x, y \in X$. Therefore, $(X, \mathcal{I}[T]_\ell)$, $(X, \mathcal{I}[I]_\ell)$ and $(X, \mathcal{I}[F]_\ell)$ are 3-fuzzy subalgebra of $(X, *, 0)$.

Suppose that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,3), I(4,3), F(4,3))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 4-fuzzy subalgebra of X, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 3-fuzzy subalgebra of X. Hence,

$$\begin{aligned}
&\mathcal{I}[T]_{\inf}(x*y) \leq \max\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\},\\
&\mathcal{I}[I]_{\inf}(x*y) \leq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\},\\
&\mathcal{I}[F]_{\inf}(x*y) \leq \max\{\mathcal{I}[F]_{\inf}(x), \mathcal{I}[F]_{\inf}(y)\},
\end{aligned} \tag{5}$$

and

$$\mathcal{I}[T]_{\sup}(x*y) \geq \max\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\},$$
$$\mathcal{I}[I]_{\sup}(x*y) \geq \max\{\mathcal{I}[I]_{\sup}(x), \mathcal{I}[I]_{\sup}(y)\},$$
$$\mathcal{I}[F]_{\sup}(x*y) \geq \max\{\mathcal{I}[F]_{\sup}(x), \mathcal{I}[F]_{\sup}(y)\},$$

for all $x, y \in X$. Label (5) implies that

$$\mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\inf}(x) \text{ or } \mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\inf}(y),$$
$$\mathcal{I}[I]_{\inf}(x*y) \leq \mathcal{I}[I]_{\inf}(x) \text{ or } \mathcal{I}[I]_{\inf}(x*y) \leq \mathcal{I}[I]_{\inf}(y),$$
$$\mathcal{I}[F]_{\inf}(x*y) \leq \mathcal{I}[F]_{\inf}(x) \text{ or } \mathcal{I}[F]_{\inf}(x*y) \leq \mathcal{I}[F]_{\inf}(y).$$

If $\mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\inf}(x)$, then

$$\mathcal{I}[T]_\ell(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \geq \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_\ell(x).$$

If $\mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\inf}(y)$, then

$$\mathcal{I}[T]_\ell(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \geq \mathcal{I}[T]_{\sup}(y) - \mathcal{I}[T]_{\inf}(y) = \mathcal{I}[T]_\ell(y).$$

It follows that $\mathcal{I}[T]_\ell(x*y) \geq \max\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\}$. Therefore, $(X, \mathcal{I}[T]_\ell)$ is a 3-fuzzy subalgebra of $(X, *, 0)$. Similarly, we can show that $(X, \mathcal{I}[I]_\ell)$ and $(X, \mathcal{I}[F]_\ell)$ are 3-fuzzy subalgebra of $(X, *, 0)$. □

Corollary 5. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,3), I(i,3), F(i,3))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2, 4\}$*, then* $(X, \mathcal{I}[T]_\ell)$*,* $(X, \mathcal{I}[I]_\ell)$ *and* $(X, \mathcal{I}[F]_\ell)$ *are* 1*-fuzzy subalgebra of* $(X, *, 0)$*.*

Theorem 10. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(3,4), I(3,4), F(3,4))$-interval neutrosophic subalgebra of $(X, *, 0)$, then $(X, \mathcal{I}[T]_\ell)$, $(X, \mathcal{I}[I]_\ell)$ and $(X, \mathcal{I}[F]_\ell)$ are 4-fuzzy subalgebra of $(X, *, 0)$.*

Proof. Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be a $(T(3,4), I(3,4), F(3,4))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 3-fuzzy subalgebra of X, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 4-fuzzy subalgebra of X. Thus,

$$\begin{aligned}
&\mathcal{I}[T]_{\inf}(x * y) \geq \max\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\},\\
&\mathcal{I}[I]_{\inf}(x * y) \geq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\},\\
&\mathcal{I}[F]_{\inf}(x * y) \geq \max\{\mathcal{I}[F]_{\inf}(x), \mathcal{I}[F]_{\inf}(y)\},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{I}[T]_{\sup}(x * y) \leq \max\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\},\\
&\mathcal{I}[I]_{\sup}(x * y) \leq \max\{\mathcal{I}[I]_{\sup}(x), \mathcal{I}[I]_{\sup}(y)\},\\
&\mathcal{I}[F]_{\sup}(x * y) \leq \max\{\mathcal{I}[F]_{\sup}(x), \mathcal{I}[F]_{\sup}(y)\},
\end{aligned} \tag{6}$$

for all $x, y \in X$. It follows from Label (6) that

$$\begin{aligned}
&\mathcal{I}[T]_{\sup}(x * y) \leq \mathcal{I}[T]_{\sup}(x) \text{ or } \mathcal{I}[T]_{\sup}(x * y) \leq \mathcal{I}[T]_{\sup}(y),\\
&\mathcal{I}[I]_{\sup}(x * y) \leq \mathcal{I}[I]_{\sup}(x) \text{ or } \mathcal{I}[I]_{\sup}(x * y) \leq \mathcal{I}[I]_{\sup}(y),\\
&\mathcal{I}[F]_{\sup}(x * y) \leq \mathcal{I}[F]_{\sup}(x) \text{ or } \mathcal{I}[F]_{\sup}(x * y) \leq \mathcal{I}[F]_{\sup}(y).
\end{aligned}$$

Assume that $\mathcal{I}[T]_{\sup}(x * y) \leq \mathcal{I}[T]_{\sup}(x)$. Then,

$$\mathcal{I}[T]_\ell(x * y) = \mathcal{I}[T]_{\sup}(x * y) - \mathcal{I}[T]_{\inf}(x * y) \leq \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_\ell(x).$$

If $\mathcal{I}[T]_{\sup}(x * y) \leq \mathcal{I}[T]_{\sup}(y)$, then

$$\mathcal{I}[T]_\ell(x * y) = \mathcal{I}[T]_{\sup}(x * y) - \mathcal{I}[T]_{\inf}(x * y) \leq \mathcal{I}[T]_{\sup}(y) - \mathcal{I}[T]_{\inf}(y) = \mathcal{I}[T]_\ell(y).$$

Hence, $\mathcal{I}[T]_\ell(x * y) \leq \max\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\}$ for all $x, y \in X$. By a similar way, we can prove that

$$\mathcal{I}[I]_\ell(x * y) \leq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\}$$

and

$$\mathcal{I}[F]_\ell(x * y) \leq \max\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}$$

for all $x, y \in X$. Therefore, $(X, \mathcal{I}[T]_\ell)$, $(X, \mathcal{I}[I]_\ell)$ and $(X, \mathcal{I}[F]_\ell)$ are 4-fuzzy subalgebra of $(X, *, 0)$. □

Theorem 11. *If an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is a $(T(3,2), I(3,2), F(3,2))$-interval neutrosophic subalgebra of $(X, *, 0)$, then $(X, \mathcal{I}[T]_\ell)$, $(X, \mathcal{I}[I]_\ell)$ and $(X, \mathcal{I}[F]_\ell)$ are 2-fuzzy subalgebra of $(X, *, 0)$.*

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(3,2), I(3,2), F(3,2))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$, $(X, \mathcal{I}[I]_{\inf})$ and $(X, \mathcal{I}[F]_{\inf})$ are 3-fuzzy subalgebra of X, and $(X, \mathcal{I}[T]_{\sup})$, $(X, \mathcal{I}[I]_{\sup})$ and $(X, \mathcal{I}[F]_{\sup})$ are 2-fuzzy subalgebra of X. Hence,

$$\begin{aligned}
&\mathcal{I}[T]_{\inf}(x * y) \geq \max\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\},\\
&\mathcal{I}[I]_{\inf}(x * y) \geq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\},\\
&\mathcal{I}[F]_{\inf}(x * y) \geq \max\{\mathcal{I}[F]_{\inf}(x), \mathcal{I}[F]_{\inf}(y)\},
\end{aligned}$$

and

$$\mathcal{I}[T]_{\sup}(x*y) \leq \min\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\},$$
$$\mathcal{I}[I]_{\sup}(x*y) \leq \min\{\mathcal{I}[I]_{\sup}(x), \mathcal{I}[I]_{\sup}(y)\},$$
$$\mathcal{I}[F]_{\sup}(x*y) \leq \min\{\mathcal{I}[F]_{\sup}(x), \mathcal{I}[F]_{\sup}(y)\},$$

for all $x, y \in X$, which imply that

$$\mathcal{I}[T]_{\ell}(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\ell}(x),$$
$$\mathcal{I}[T]_{\ell}(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\sup}(y) - \mathcal{I}[T]_{\inf}(y) = \mathcal{I}[T]_{\ell}(y),$$
$$\mathcal{I}[I]_{\ell}(x*y) = \mathcal{I}[I]_{\sup}(x*y) - \mathcal{I}[I]_{\inf}(x*y) \leq \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\ell}(x),$$
$$\mathcal{I}[I]_{\ell}(x*y) = \mathcal{I}[I]_{\sup}(x*y) - \mathcal{I}[I]_{\inf}(x*y) \leq \mathcal{I}[I]_{\sup}(y) - \mathcal{I}[I]_{\inf}(y) = \mathcal{I}[I]_{\ell}(y),$$

and

$$\mathcal{I}[F]_{\ell}(x*y) = \mathcal{I}[F]_{\sup}(x*y) - \mathcal{I}[F]_{\inf}(x*y) \leq \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\ell}(x),$$
$$\mathcal{I}[F]_{\ell}(x*y) = \mathcal{I}[F]_{\sup}(x*y) - \mathcal{I}[F]_{\inf}(x*y) \leq \mathcal{I}[F]_{\sup}(y) - \mathcal{I}[F]_{\inf}(y) = \mathcal{I}[F]_{\ell}(y).$$

It follows that

$$\mathcal{I}[T]_{\ell}(x*y) \leq \min\{\mathcal{I}[T]_{\ell}(x), \mathcal{I}[T]_{\ell}(y)\},$$
$$\mathcal{I}[I]_{\ell}(x*y) \leq \min\{\mathcal{I}[I]_{\ell}(x), \mathcal{I}[I]_{\ell}(y)\},$$

and

$$\mathcal{I}[F]_{\ell}(x*y) \leq \min\{\mathcal{I}[F]_{\ell}(x), \mathcal{I}[F]_{\ell}(y)\},$$

for all $x, y \in X$. Hence, $(X, \mathcal{I}[T]_{\ell})$, $(X, \mathcal{I}[I]_{\ell})$ and $(X, \mathcal{I}[F]_{\ell})$ are 2-fuzzy subalgebra of $(X, *, 0)$. □

Corollary 6. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(3,2), I(3,2), F(3,2))$*-interval neutrosophic subalgebra of* $(X, *, 0)$*, then* $(X, \mathcal{I}[T]_{\ell})$*,* $(X, \mathcal{I}[I]_{\ell})$ *and* $(X, \mathcal{I}[F]_{\ell})$ *are* 4*-fuzzy subalgebra of* $(X, *, 0)$*.*

Theorem 12. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,3), I(3,4), F(3,2))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2, 4\}$*, then*

(1) $(X, \mathcal{I}[T]_{\ell})$ *is a* 3*-fuzzy subalgebra of* $(X, *, 0)$*.*
(2) $(X, \mathcal{I}[I]_{\ell})$ *is a* 4*-fuzzy subalgebra of* $(X, *, 0)$*.*
(3) $(X, \mathcal{I}[F]_{\ell})$ *is a* 2*-fuzzy subalgebra of* $(X, *, 0)$*.*

Proof. Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(T(4,3), I(3,4), F(3,2))$-interval neutrosophic subalgebra of $(X, *, 0)$. Then, $(X, \mathcal{I}[T]_{\inf})$ is a 4-fuzzy subalgebra of X, $(X, \mathcal{I}[T]_{\sup})$ is a 3-fuzzy subalgebra of X, $(X, \mathcal{I}[I]_{\inf})$ is a 3-fuzzy subalgebra of X, $(X, \mathcal{I}[I]_{\sup})$ is a 4-fuzzy subalgebra of X, $(X, \mathcal{I}[F]_{\inf})$ is a 3-fuzzy subalgebra of X, and $(X, \mathcal{I}[F]_{\sup})$ is a 2-fuzzy subalgebra of X. Hence,

$$\mathcal{I}[T]_{\inf}(x*y) \leq \max\{\mathcal{I}[T]_{\inf}(x), \mathcal{I}[T]_{\inf}(y)\}, \quad (7)$$

$$\mathcal{I}[T]_{\sup}(x*y) \geq \max\{\mathcal{I}[T]_{\sup}(x), \mathcal{I}[T]_{\sup}(y)\}, \quad (8)$$

$$\mathcal{I}[I]_{\inf}(x*y) \geq \max\{\mathcal{I}[I]_{\inf}(x), \mathcal{I}[I]_{\inf}(y)\}, \quad (9)$$

$$\mathcal{I}[I]_{\sup}(x*y) \leq \max\{\mathcal{I}[I]_{\sup}(x), \mathcal{I}[I]_{\sup}(y)\}, \quad (10)$$

$$\mathcal{I}[F]_{\inf}(x*y) \geq \max\{\mathcal{I}[F]_{\inf}(x), \mathcal{I}[F]_{\inf}(y)\}, \quad (11)$$

and

$$\mathcal{I}[F]_{\sup}(x*y) \leq \min\{\mathcal{I}[F]_{\sup}(x), \mathcal{I}[F]_{\sup}(y)\}, \tag{12}$$

for all $x, y \in X$. Then,

$$\mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\inf}(x) \text{ or } \mathcal{I}[T]_{\inf}(x*y) \leq \mathcal{I}[T]_{\inf}(y)$$

by Label (7). It follows from Label (8) that

$$\mathcal{I}[T]_{\ell}(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \geq \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \mathcal{I}[T]_{\ell}(x)$$

or

$$\mathcal{I}[T]_{\ell}(x*y) = \mathcal{I}[T]_{\sup}(x*y) - \mathcal{I}[T]_{\inf}(x*y) \geq \mathcal{I}[T]_{\sup}(y) - \mathcal{I}[T]_{\inf}(y) = \mathcal{I}[T]_{\ell}(y),$$

and so that $\mathcal{I}[T]_{\ell}(x*y) \geq \max\{\mathcal{I}[T]_{\ell}(x), \mathcal{I}[T]_{\ell}(y)\}$ for all $x, y \in X$. Thus, $(X, \mathcal{I}[T]_{\ell})$ is a 3-fuzzy subalgebra of $(X, *, 0)$. The condition (10) implies that

$$\mathcal{I}[I]_{\sup}(x*y) \leq \mathcal{I}[I]_{\sup}(x) \text{ or } \mathcal{I}[I]_{\sup}(x*y) \leq \mathcal{I}[I]_{\sup}(y). \tag{13}$$

Combining Labels (9) and (13), we have

$$\mathcal{I}[I]_{\ell}(x*y) = \mathcal{I}[I]_{\sup}(x*y) - \mathcal{I}[I]_{\inf}(x*y) \leq \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \mathcal{I}[I]_{\ell}(x)$$

or

$$\mathcal{I}[I]_{\ell}(x*y) = \mathcal{I}[I]_{\sup}(x*y) - \mathcal{I}[I]_{\inf}(x*y) \leq \mathcal{I}[I]_{\sup}(y) - \mathcal{I}[I]_{\inf}(y) = \mathcal{I}[I]_{\ell}(y).$$

It follows that $\mathcal{I}[I]_{\ell}(x*y) \leq \max\{\mathcal{I}[I]_{\ell}(x), \mathcal{I}[I]_{\ell}(y)\}$ for all $x, y \in X$. Thus, $(X, \mathcal{I}[I]_{\ell})$ is a 4-fuzzy subalgebra of $(X, *, 0)$. Using Labels (11) and (12), we have

$$\mathcal{I}[F]_{\ell}(x*y) = \mathcal{I}[F]_{\sup}(x*y) - \mathcal{I}[F]_{\inf}(x*y) \leq \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \mathcal{I}[F]_{\ell}(x)$$

and

$$\mathcal{I}[F]_{\ell}(x*y) = \mathcal{I}[F]_{\sup}(x*y) - \mathcal{I}[F]_{\inf}(x*y) \leq \mathcal{I}[F]_{\sup}(y) - \mathcal{I}[F]_{\inf}(y) = \mathcal{I}[F]_{\ell}(y),$$

and so $\mathcal{I}[F]_{\ell}(x*y) \leq \min\{\mathcal{I}[F]_{\ell}(x), \mathcal{I}[F]_{\ell}(y)\}$ for all $x, y \in X$. Therefore, $(X, \mathcal{I}[F]_{\ell})$ is a 2-fuzzy subalgebra of $(X, *, 0)$. Similarly, we can prove the desired results for $i = 2$. □

Corollary 7. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,3), I(3,4), F(3,2))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2, 4\}$*, then*

(1) $(X, \mathcal{I}[T]_{\ell})$ *is a* 1*-fuzzy subalgebra of* $(X, *, 0)$*.*
(2) $(X, \mathcal{I}[I]_{\ell})$ *and* $(X, \mathcal{I}[F]_{\ell})$ *are* 4*-fuzzy subalgebra of* $(X, *, 0)$*.*

By a similar way to the proof of Theorem 12, we have the following theorems.

Theorem 13. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,3), I(3,2), F(3,2))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2, 4\}$*, then*

(1) $(X, \mathcal{I}[T]_{\ell})$ *is a* 3*-fuzzy subalgebra of* $(X, *, 0)$*.*
(2) $(X, \mathcal{I}[I]_{\ell})$ *and* $(X, \mathcal{I}[F]_{\ell})$ *are* 2*-fuzzy subalgebra of* $(X, *, 0)$*.*

Corollary 8. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,3), I(3,2), F(3,2))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2, 4\}$*, then*

(1) $(X, \mathcal{I}[T]_{\ell})$ *is a* 1*-fuzzy subalgebra of* $(X, *, 0)$*.*
(2) $(X, \mathcal{I}[I]_{\ell})$ *and* $(X, \mathcal{I}[F]_{\ell})$ *are* 4*-fuzzy subalgebra of* $(X, *, 0)$*.*

Theorem 14. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,3), I(3,2), F(2,3))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2, 4\}$*, then*

(1) $(X, \mathcal{I}[T]_\ell)$ *and* $(X, \mathcal{I}[F]_\ell)$ *are* 3*-fuzzy subalgebra of* $(X, *, 0)$*.*
(2) $(X, \mathcal{I}[I]_\ell)$ *is a* 2*-fuzzy subalgebra of* $(X, *, 0)$*.*

Corollary 9. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* X *is a* $(T(i,3), I(3,2), F(2,3))$*-interval neutrosophic subalgebra of* $(X, *, 0)$ *for* $i \in \{2, 4\}$*, then*

(1) $(X, \mathcal{I}[T]_\ell)$ *and* $(X, \mathcal{I}[F]_\ell)$ *are* 1*-fuzzy subalgebra of* $(X, *, 0)$*.*
(2) $(X, \mathcal{I}[I]_\ell)$ *is a* 4*-fuzzy subalgebra of* $(X, *, 0)$*.*

Acknowledgments: The authors wish to thank the anonymous reviewers for their valuable suggestions.

Author Contributions: Young Bae Jun conceived and designed the main idea and wrote the paper, Seon Jeong Kim and Florentin Smarandache performed the idea, checking contents and finding examples. All authors have read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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