

# A generalization of the Cassini formula

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## 1. Introduction: Cassini formula

**Great astronomer Giovanni Domenico Cassini.** Cassini is the name of a famous dynasty of French astronomers. Giovanni Domenico Cassini (1625-1712) is the most famous of them and the founder of this dynasty. The following facts illustrate his outstanding contribution to astronomy. The name Cassini was given to numerous astronomical objects, the “Cassini Crater” on the Moon, the “Cassini Crater” on Mars, “Cassini Slot” in Saturn's ring, and “Cassini Laws” of the Moon’s movement.

However, the name of Cassini is widely known not only in astronomy, but also in mathematics. Cassini developed a theory of the remarkable geometrical figures known under the name of *Cassini ovals*. The mathematical identity that connects three adjacent Fibonacci numbers is well known under the name *Cassini formula*.

**Cassini formula for the Fibonacci numbers.** The history of science is silent as to why Cassini took such a great interest in Fibonacci numbers. Most likely it was simply a hobby of the Great astronomer. At that time many serious scientists took a great interest in Fibonacci numbers and the golden mean. These mathematical objects were also a hobby of Cassini’s contemporary, Kepler.

Now, let us consider the Fibonacci series: 1, 1, 2, 3, 5, 8, 13, 21, 34, ... . Take the Fibonacci number 5 and square it, that is,  $5^2=25$ . Multiply two Fibonacci numbers 3 and 8 that encircle the Fibonacci number 5, so we get  $3 \times 8=24$ . Then we can write:

$$5^2-3 \times 8=1$$

Note that the difference is equal to (+1).

Now, we follow the same process with the next Fibonacci number 8, that is, at first, we square it  $8^2=64$  and then multiply two Fibonacci numbers 5 and 13 that encircle the Fibonacci number 8,  $5 \times 13=65$ . After a comparison of the result  $5 \times 13=65$  with the square  $8^2=64$  we can write:

$$8^2-5 \times 13=-1$$

Note that the difference is equal to (-1).

Further we have:

$$13^2-8 \times 21=1,$$
$$21^2-13 \times 34=-1,$$

and so on.

We see, that the quadrate of any Fibonacci number  $F_n$  always differs from the product of the two adjacent Fibonacci numbers  $F_{n-1}$  and  $F_{n+1}$ , which encircle it, by 1. However, the sign of 1 depends on the index  $n$  of the Fibonacci number  $F_n$ . If the index  $n$  is even, then the number 1 is taken with minus, and if odd, with plus. The indicated property of Fibonacci numbers can be expressed by the following mathematical formula:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}. \quad (1)$$

This wonderful formula evokes a reverent thrill, if one recognizes that this formula is valid for any value of  $n$  (we remember that  $n$  can be some integer in limits from  $-\infty$  up to  $+\infty$ ). The alternation of +1 and -1 in the expression (1) at the successive passing of all Fibonacci numbers produces genuine aesthetic enjoyment and a feeling of rhythm and harmony.

For centuries, it was believed that the Fibonacci numbers are the only number sequence that has a unique mathematical property given by the Cassini formula (1).

In the late 20 th and early 21 th centuries, several researchers from different countries – the Argentinean mathematician Vera Spinadel [1], the French mathematician Midhat Ghazal [2], the American mathematician Jay Kappraff [3], the Russian engineer Alexander Tatarenko [4], the Armenian philosopher and physicist Hrant Arakelyan [5], the Russian researcher Victor Shenyagin [6], the Ukrainian physicist Nikolai Kosinov [7], Ukrainian-Canadian mathematician Alexey Stakhov [8,9], the Spanish mathematicians Falcon Sergio and Plaza Angel [10] and others independently began to study a new class of recursive numerical sequences, which are a generalization of the classical Fibonacci numbers. These numerical sequences led to the discovery of a new class of mathematical constants, called Vera Spinadel "metallic proportions". [1].

The interest of a large number of researchers from different countries (USA, Canada, Argentina, France, Spain, Russia, Armenia, Ukraine) can not be accidental. This means that the problem of the generalized Fibonacci numbers has matured in modern science.

As shown in the works [1-9], the most attention of the researchers was drawn to the "metallic means," while the properties of the generalized Fibonacci numbers  $s$  are studied not enough. This article is an attempt to draw attention to some of the unusual properties of generalized Fibonacci numbers, in particular, to the generalized Cassini formula. For the first time, these results have been published in the author's article [10], written in Russian.

## 2. The Fibonacci $\lambda$ -numbers

Let us give a real number  $\lambda > 0$  and consider the following recurrence relation:

$$F_\lambda(n+2) = \lambda F_\lambda(n+1) + F_\lambda(n); \quad F_\lambda(0) = 0, F_\lambda(1) = 1. \quad (2)$$

The recurrence relation (2) "generates" an infinite number of new numerical sequences, because every real number  $\lambda$  "generates" its own numerical sequence.

Let us consider partial cases of the recurrence relation (2). For the case  $\lambda = 1$  the recurrence relation (2) is reduced to the following recurrence relation:

$$F_1(n+2) = F_1(n+1) + F_1(n); F_1(0) = 0, F_1(1) = 1, \quad (3)$$

This recurrence relation “generates” the classical Fibonacci numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots \quad (4)$$

Based on this fact, we will name the numerical sequences, generated by the recurrence relation (2), the Fibonacci  $\lambda$ -numbers.

For the case  $\lambda=2$  the recurrence relation (2) is reduced to the recurrence relation

$$F_2(n+2) = 2F_2(n+1) + F_2(n); F_2(0) = 0, F_2(1) = 1, \quad (5)$$

which gives the so-called Pelly numbers

$$0, 1, 2, 5, 12, 29, 70, \dots \quad (6)$$

For the cases  $\lambda=3,4$  the recurrence relation (2) is reduced to the following recurrence relations:

$$F_3(n+2) = 3F_3(n+1) + F_3(n); F_3(0) = 0, F_3(1) = 1 \quad (7)$$

$$F_4(n+2) = 4F_4(n+1) + F_4(n); F_4(0) = 0, F_4(1) = 1. \quad (8)$$

The  $\lambda$ -Fibonacci numbers have many remarkable properties, similar to the properties of the classical Fibonacci numbers. It is proved that the Fibonacci  $\lambda$ -numbers, as well as the classical Fibonacci numbers can be "expanded" to the negative values of the discrete variable  $n$ .

Table 1 shows the four “expanded” sequences of the Fibonacci  $\lambda$ -numbers, corresponding to the values  $\lambda = 1, 2, 3, 4$ .

**Table 1.** The “expanded”  $\lambda$ - Fibonacci  $\lambda$ -numbers ( $\lambda = 1, 2, 3, 4$ )

$n$	0	1	2	3	4	5	6	7	8
$F_1(n)$	0	1	1	2	3	5	8	13	21
$F_1(-n)$	0	1	-1	2	-3	5	-8	13	-21
$F_2(n)$	0	1	2	5	12	29	70	169	408
$F_2(-n)$	0	1	-2	5	-12	29	-70	169	-408
$F_3(n)$	0	1	3	10	33	109	360	1189	3927
$F_3(-n)$	0	1	-3	10	-33	109	-360	1199	-3927
$F_4(n)$	0	1	4	17	72	305	1292	5473	23184
$F_4(-n)$	0	1	-4	17	-72	305	-1292	5473	-23184

### 3. Cassini formula for the Fibonacci $\lambda$ -numbers

Let us prove that the Cassini formula (1) can be generalized for the case of the Fibonacci  $\lambda$ -numbers. The generalized Cassini formula is of the following form:

$$F_\lambda^2(n) - F_\lambda(n-1)F_\lambda(n+1) = (-1)^{n+1} \quad (9)$$

Let us prove a validity of the formula (9) by using the induction by  $n$ . For the case  $n = 1$  the Fibonacci  $\lambda$ -numbers  $F_\lambda(n-1), F_\lambda(n), F_\lambda(n+1)$  in the formula (9), according to the recurrence relation (2), takes the following values:

$$F_\lambda(0) = 0, F_\lambda(1) = 1, F_\lambda(2) = \lambda,$$

what implies that the identity (9) for the case  $n = 1$  is equal:

$$(1)^2 - 0 \times \lambda = (1)^2. \quad (10)$$

The base of the induction is proved.

We formulate the following inductive assumption. Suppose that the identity (9) is valid for any given integer  $n$ , and prove the validity of the identity (9) for the case of  $n+1$ , that is, the identity

$$F_\lambda^2(n+1) - F_\lambda(n)F_\lambda(n+2) = (-1)^{n+2} \quad (11)$$

is valid too.

In order to prove the identity (11), we represent the left part of the identity (11) as follows:

$$\begin{aligned} F_\lambda^2(n+1) - F_\lambda(n)F_\lambda(n+2) &= F_\lambda^2(n+1) - F_\lambda(n)[F_\lambda(n) + \lambda F_\lambda(n+1)] = \\ &= F_\lambda^2(n+1) - F_\lambda^2(n) - \lambda F_\lambda(n)F_\lambda(n+1) \\ &= F_\lambda(n+1)[F_\lambda(n+1) - \lambda F_\lambda(n)] - F_\lambda^2(n) \\ &= F_\lambda(n+1)F_\lambda(n-1) - F_\lambda^2(n) \end{aligned} \quad (12)$$

On the other hand, by using (9), we can rewrite:

$$F_\lambda(n+1)F_\lambda(n-1) - F_\lambda^2(n) = -(-1)^{n+1}. \quad (13)$$

This means that the identity (12) can be rewritten as follows:

$$F_\lambda^2(n+1) - F_\lambda(n)F_\lambda(n+2) = -(-1)^{n+1} = (-1)^{n+2}$$

The identity (11) is proved.

#### 4. Numerical examples

Consider the examples of the validity of the identity (9) for the various sequences shown in Table 1. Let us consider the  $F_2(n)$ -sequence for the case of  $n=7$ . For this case we should consider the following triple of the Fibonacci 2-numbers  $F_2(n)$ :  $F_2(6) = 70, F_2(7) = 169, F_2(8) = 408$ . By performing calculations over them according to (9), we obtain the following result:

$$(169)^2 - 70 \times 408 = 28561 - 28560 = 1,$$

what corresponds to the identity (9), because for the case  $n=7$  we have:  
 $(-1)^{n+1} = (-1)^8 = 1$ .

Now let us consider the  $F_3(n)$ -sequence from Table 1 for the case of  $n=6$ . For this case we should chose the following triple of the Fibonacci 3-numbers  $F_3(n)$ :

$$F_3(5) = 109, F_3(6) = 360, F_3(7) = 1189.$$

By performing calculations over them according to (9), we obtain the following result:

$$(360)^2 - 109 \times 1189 = 129600 - 129601 = -1,$$

what corresponds to the identity (9), because for the case  $n=6$  we have  
 $(-1)^{n+1} = (-1)^7 = -1$ .

Finally, let us consider the  $F_4(-n)$ -sequence from Table 1 for the case of  $n=-5$ . For this case we should chose the following triple of the Fibonacci 4-numbers  $F_4(-n)$  ::

$$F_4(-4) = -72, F_4(-5) = 305, F_4(-6) = -1292.$$

By performing calculations over them according to (9), we obtain the following result:

$$(305)^2 - (-72) \times (-1292) = 93025 - 93024 = 1,$$

what corresponds to the identity (9), because for the case  $n=-5$  we have  
 $(-1)^{n+1} = (-1)^{-4} = 1$ .

Thus, by studying the generalized formula Cassini (9) for the Fibonacci  $\lambda$ -numbers, we came to the discovery of an infinite number of integer recurrence sequences in the range from  $+\infty$  to  $-\infty$ , with the following unique mathematical property, expressed by the generalized Cassini formula (9), which sounds as follows:

The quadrate of any Fibonacci  $\lambda$ -number  $F_\lambda(n)$  are always different from the product of the two adjacent Fibonacci  $\lambda$ -numbers  $F_\lambda(n-1)$  and  $F_\lambda(n+1)$ , which surround the initial Fibonacci  $\lambda$ -number  $F_\lambda(n)$ , by the number 1; herewith the sign of the difference of 1 depends on the parity of  $n$ : if  $n$  is even, then the difference of 1 is taken with the sign "minus", otherwise, with the sign "plus."

Until now, we have assumed that only the classical Fibonacci numbers are of the unusual property, given by the Cassini formula (1). However, as is shown above, a number of such numerical sequences is infinite. All the Fibonacci  $\lambda$ -numbers given by (2) are of a similar property, given by the generalized Cassini formula (9)!

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