Chapter 3

Trigonometric Identities and Equations - 2nd edition

3.1 Fundamental Identities

Introduction

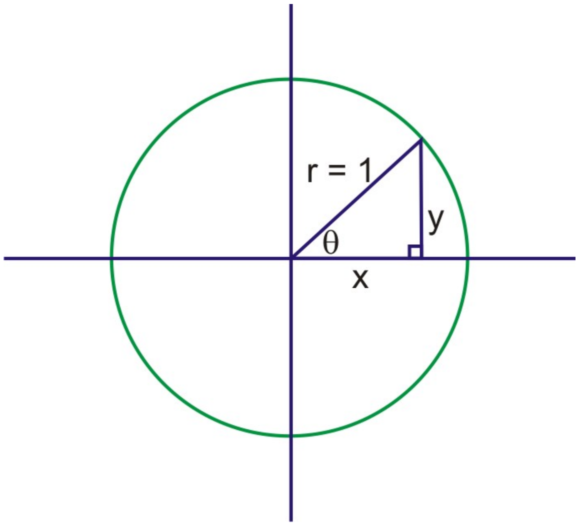
We now enter into the proof portion of trigonometry. Starting with the basic definitions of sine, cosine, and tangent, identities (or fundamental trigonometric equations) emerge. Students will learn how to prove certain identities, using other identities and definitions. Finally, students will be able solve trigonometric equations for theta, also using identities and definitions.

Learning Objectives

- use the fundamental identities to prove other identities.
- apply the fundamental identities to values of θ and show that they are true.

Quotient Identity

In Chapter 1, the three fundamental trigonometric functions sine, cosine and tangent were introduced. All three functions can be defined in terms of a right triangle or the unit circle.



$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r} = \frac{y}{1} = y$$
$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r} = \frac{x}{1} = x$$
$$\tan\theta = \frac{opposite}{adjacent} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

The Quotient Identity is $\tan\theta = \frac{\sin\theta}{\cos\theta}$. We see that this is true because tangent is equal to $\frac{y}{x}$, in the unit circle. We know that y is equal to the sine values of θ and x is equal to the cosine values of θ. Substituting $\sin\theta$ for y and $\cos\theta$ for x and we have a new identity.

Example 1: Prove $\tan\theta = \frac{\sin\theta}{\cos\theta}$ by using $\theta = 45^\circ$.

Solution: Plugging in 45°, we have: $\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ}$. Then, substitute each function with its actual value and simplify both sides.

$\frac{\sin 45^\circ}{\cos 45^\circ} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \cdot \frac{2}{\sqrt{2}} = 1$ and we know that $\tan 45^\circ = 1$, so this is true.

Example 2: Show that $\tan 90^\circ$ is undefined using the Quotient Identity.

Solution: $\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0}$, because you cannot divide by zero, the tangent at 90° is undefined.

Reciprocal Identities

Chapter 1 also introduced us to the idea that the three fundamental reciprocal trigonometric functions are cosecant (csc), secant (sec) and cotangent (cot) and are defined as:

$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

If we apply the Quotient Identity to the reciprocal of tangent, an additional quotient is created:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta}{\sin\theta}$$

Example 3: Prove $\tan\theta = \sin\theta\sec\theta$

Solution: First, you should change everything into sine and cosine. Feel free to work from either side, as long as the end result from both sides ends up being the same.

$$\begin{aligned}\tan\theta &= \sin\theta\ \sec\theta\\ &= \sin\theta\cdot\frac{1}{\cos\theta}\\ &= \frac{\sin\theta}{\cos\theta}\end{aligned}$$

Here, we end up with the Quotient Identity, which we know is true. Therefore, this identity is also true and we are finished.

Example 4: Given $\sin\theta = -\frac{\sqrt{6}}{5}$, find $\sec\theta$.

Solution: Secant is the reciprocal of cosine, so we need to find the adjacent side. We are given the opposite side, $\sqrt{6}$ and the hypotenuse, 5. Because θ is in the fourth quadrant, cosine will be positive. From the Pythagorean Theorem, the third side is:

$$\begin{aligned}\left(\sqrt{6}\right)^2 + b^2 &= 5^2\\ 6 + b^2 &= 25\\ b^2 &= 19\\ b &= \sqrt{19}\end{aligned}$$

from this we can now find $\cos\theta = \frac{\sqrt{19}}{5}$. Since secant is the reciprocal of cosine, $\sec\theta = \frac{5}{\sqrt{19}}$.

Pythagorean Identity

Using the fundamental trig functions, sine and cosine and some basic algebra can reveal some interesting trigonometric relationships. Note when a trig function such as $\sin\theta$ is multiplied by itself, the mathematical convention is to write it as $\sin^2\theta$. ($\sin\theta^2$ can be interpreted as the sine of the square of the angle, and is therefore avoided.)

$\sin^2\theta = \frac{y^2}{r^2}$ and $\cos^2\theta = \frac{x^2}{r^2}$ or $\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2+y^2}{r^r}$

Using the Pythagorean Theorem for the triangle above: $x^2 + y^2 = r^2$

Then, divide both sides by $r^2, \frac{x^2+y^2}{r^2} = \frac{r^2}{r^2} = 1$. So, because $\frac{x^2+y^2}{r^2} = 1, \sin^2\theta + \cos^2\theta$ also equals 1. This is known as the Trigonometric Pythagorean Theorem or the Pythagorean Identity and is written $\sin^2\theta + \cos^2\theta = 1$. Alternative forms of the Theorem are: $1 + \cot^2\theta = \csc^2\theta$ and $\tan^2\theta + 1 = \sec^2\theta$. The second form is found by taking the original form and dividing each of the terms by $\sin^2\theta$, while the third form is found by dividing all the terms of the first by $\cos^2\theta$.

Example 5: Use 30° to show that $\sin^2\theta + \cos^2\theta = 1$ holds true.

Solution: Plug in 30° and find the values of $\sin 30^\circ$ and $\cos 30^\circ$.

$$\sin^2 30^\circ + \cos^2 30^\circ$$
$$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$
$$\frac{1}{4} + \frac{3}{4} = 1$$

Even and Odd Identities

Functions are even or odd depending on how the end behavior of the graphical representation looks. For example, $y = x^2$ is considered an even function because the ends of the parabola both point in the same direction and the parabola is symmetric about the $y-$axis. $y = x^3$ is considered an odd function for the opposite reason. The ends of a cubic function point in opposite directions and therefore the parabola is not symmetric about the $y-$axis. What about the trig functions? They do not have exponents to give us the even or odd clue (when the degree is even, a function is even, when the degree is odd, a function is odd).

Even Function	Odd Function
$y = (-x)^2 = x^2$	$y = (-x)^3 = -x^3$

Let's consider sine. Start with $\sin(-x)$. Will it equal $\sin x$ or $-\sin x$? Plug in a couple of values to see.

$$\sin(-30^\circ) = \sin 330^\circ = -\frac{1}{2} = -\sin 30^\circ$$

$$\sin(-135^\circ) = \sin 225^\circ = -\frac{\sqrt{2}}{2} = -\sin 135^\circ$$

From this we see that sine is **odd**. Therefore, $\sin(-x) = -\sin x$, for any value of x. For cosine, we will plug in a couple of values to determine if it's even or odd.

$$\cos(-30^\circ) = \cos 330^\circ = \frac{\sqrt{3}}{2} = \cos 30^\circ$$

$$\cos(-135^\circ) = \cos 225^\circ = -\frac{\sqrt{2}}{2} = \cos 135^\circ$$

This tells us that the cosine is **even**. Therefore, $\cos(-x) = \cos x$, for any value of x. The other four trigonometric functions are as follows:

$$\tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x$$
$$\sec(-x) = \sec x$$
$$\cot(-x) = -\cot x$$

Notice that cosecant is odd like sine and secant is even like cosine.

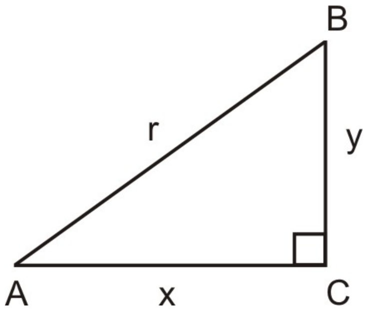
Example 6: If $\cos(-x) = \frac{3}{4}$ and $\tan(-x) = -\frac{\sqrt{7}}{3}$, find $\sin x$.

Solution: We know that sine is odd. Cosine is even, so $\cos x = \frac{3}{4}$. Tangent is odd, so $\tan x = \frac{\sqrt{7}}{3}$. Therefore, sine is positive and $\sin x = \frac{\sqrt{7}}{4}$.

Cofunction Identities

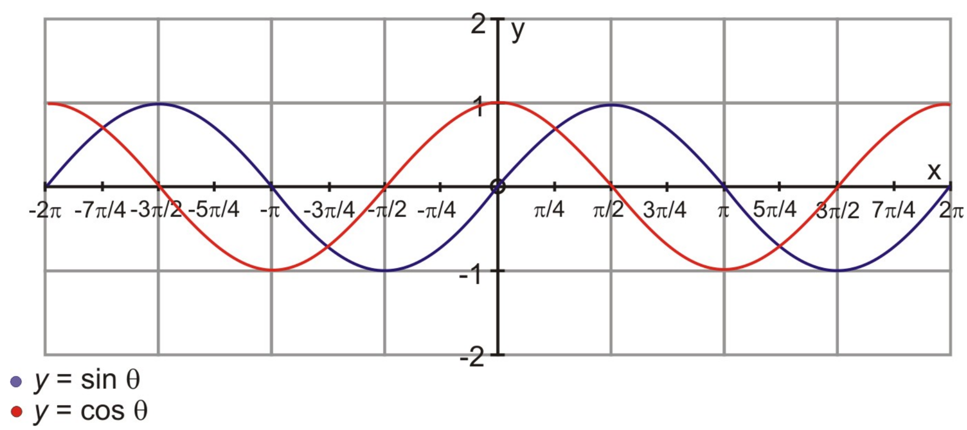
Recall that two angles are complementary if their sum is 90°. In every triangle, the sum of the interior angles is 180° and the right angle has a measure of 90°. Therefore, the two remaining acute angles of the triangle have a sum equal to 90°, and are complementary. Let's explore this concept to identify the relationship between a function of one angle and the function of its complement in any right triangle, or the cofunction identities. A cofunction is a pair of trigonometric functions that are equal when the variable in one function is the complement in the other.

In $\triangle ABC, \angle C$ is a right angle, $\angle A$ and $\angle B$ are complementary.



Chapter 1 introduced the cofunction identities (section 1.8) and because $\angle A$ and $\angle B$ are complementary, it was found that $\sin A = \cos B, \cos A = \sin B, \tan A = \cot B, \cot A = \tan B, \csc A = \sec B$ and $\sec A = \csc B$. For each of the above $\angle A = \frac{\pi}{2} - \angle B$. To generalize, $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ and $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ and $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta, \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$ and $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$.

The following graph represents two complete cycles of $y = \sin x$ and $y = \cos\theta$.



Notice that a phase shift of $\frac{\pi}{2}$ on $y = \cos x$, would make these graphs exactly the same. These cofunction identities hold true for all real numbers for which both sides of the equation are defined.

Example 7: Use the cofunction identities to evaluate each of the following expressions:

a. If $\tan\left(\frac{\pi}{2} - \theta\right) = -4.26$ determine $\cot\theta$

b. If $\sin\theta = 0.91$ determine $\cos\left(\frac{\pi}{2} - \theta\right)$.

Solution:

a. $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ therefore $\cot\theta = -4.26$

b. $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ therefore $\cos\left(\frac{\pi}{2} - \theta\right) = 0.91$

Example 8: Show $\sin\left(\frac{\pi}{2} - x\right) = \cos(-x)$ is true.

Solution: Using the identities we have derived in this section, $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, and we know that cosine is an even function so $\cos(-x) = \cos x$. Therefore, each side is equal to $\cos x$ and thus equal to each other.

Points to Consider

- Why do you think secant is even like cosine?
- How could you show that tangent is odd?

Review Questions

1. Use the Quotient Identity to show that the $\tan 270°$ is undefined.
2. If $\cos\left(\frac{\pi}{2} - x\right) = \frac{4}{5}$, find $\sin(-x)$.
3. If $\tan(-x) = -\frac{5}{12}$ and $\sin x = -\frac{5}{13}$, find $\cos x$.
4. Simplify $\sec x \cos\left(\frac{\pi}{2} - x\right)$.
5. Verify $\sin^2\theta + \cos^2\theta = 1$ using:
   (a) the sides $5, 12$, and 13 of a right triangle, in the first quadrant
   (b) the ratios from a $30 - 60 - 90$ triangle
6. Prove $1 + \tan^2\theta = \sec^2\theta$ using the Pythagorean Identity
7. If $\csc z = \frac{17}{8}$ and $\cos z = -\frac{15}{17}$, find $\cot z$.
8. Factor:
   (a) $\sin^2\theta - \cos^2\theta$
   (b) $\sin^2\theta + 6\sin\theta + 8$
9. Simplify $\frac{\sin^4\theta - \cos^4\theta}{\sin^2\theta - \cos^2\theta}$ using the trig identities
10. Rewrite $\frac{\cos x}{\sec x - 1}$ so that it is only in terms of cosine. Simplify completely.
11. Prove that tangent is an odd function.

Review Answers

1. $\tan 270° = \frac{\sin 270°}{\cos 270°} = \frac{-1}{0}$, you cannot divide by zero, therefore $\tan 270°$ is undefined.
2. If $\cos\left(\frac{\pi}{2} - x\right) = \frac{4}{5}$, then, by the cofunction identities, $\sin x = \frac{4}{5}$. Because sine is odd, $\sin(-x) = -\frac{4}{5}$.
3. If $\tan(-x) = -\frac{5}{12}$, then $\tan x = \frac{5}{12}$. Because $\sin x = -\frac{5}{13}$, cosine is also negative, so $\cos x = -\frac{12}{13}$.
4. Use the reciprocal and cofunction identities to simplify

$$\sec x \cos\left(\frac{\pi}{2} - x\right)$$
$$\frac{1}{\cos x} \cdot \sin x$$
$$\frac{\sin x}{\cos x}$$
$$\tan x$$

5. (a) Using the sides 5, 12, and 13 and in the first quadrant, it doesn't really matter which is cosine or sine. So, $\sin^2\theta + \cos^2\theta = 1$ becomes $\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = 1$. Simplifying, we get: $\frac{25}{169} + \frac{144}{169} = 1$, and finally $\frac{169}{169} = 1$.
   (b) $\sin^2\theta + \cos^2\theta = 1$ becomes $\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$. Simplifying we get: $\frac{1}{4} + \frac{3}{4} = 1$ and $\frac{4}{4} = 1$.
6. To prove $\tan^2\theta + 1 = \sec^2\theta$, first use $\frac{\sin\theta}{\cos\theta} = \tan\theta$ and change $\sec^2\theta = \frac{1}{\cos^2\theta}$.

$$\tan^2\theta + 1 = \sec^2\theta$$
$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$
$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$\sin^2\theta + \cos^2\theta = 1$$

7. If $\csc z = \frac{17}{8}$ and $\cos z = -\frac{15}{17}$, then $\sin z = \frac{8}{17}$ and $\tan z = -\frac{8}{15}$. Therefore $\cot z = -\frac{15}{8}$.
8. (a) Factor $\sin^2\theta - \cos^2\theta$ using the difference of squares.

$$\sin^2\theta - \cos^2\theta = (\sin\theta + \cos\theta)(\sin\theta - \cos\theta)$$

   (b) $\sin^2\theta + 6\ \sin\theta + 8 = (\sin\theta + 4)(\sin\theta + 2)$

9. You will need to factor and use the $\sin^2\theta + \cos^2\theta = 1$ identity.

$$\begin{aligned}
\frac{\sin^4\theta - \cos^4\theta}{\sin^2\theta - \cos^2\theta} \\
&= \frac{(\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)}{\sin^2\theta - \cos^2\theta} \\
&= \sin^2\theta + \cos^2\theta \\
&= 1
\end{aligned}$$

10. To rewrite $\frac{\cos x}{\sec x - 1}$ so it is only in terms of cosine, start with changing secant to cosine.

$$\frac{\cos x}{\sec x - 1} = \frac{\cos x}{\frac{1}{\cos x} - 1}$$

Now, simplify the denominator.

$$\frac{\cos x}{\frac{1}{\cos x} - 1} = \frac{\cos x}{\frac{1-\cos x}{\cos x}}$$

Multiply by the reciprocal $\frac{\cos x}{\frac{1-\cos x}{\cos x}} = \cos x \div \frac{1-\cos x}{\cos x} = \cos x \cdot \frac{\cos x}{1-\cos x} = \frac{\cos^2 x}{1-\cos x}$

11. The easiest way to prove that tangent is odd to break it down, using the Quotient Identity.

$$\begin{aligned}
\tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{from this statement, we need to show that } \tan(-x) = -\tan x \\
&= \frac{-\sin x}{\cos x} && \text{because } \sin(-x) = -\sin x \text{ and } \cos(-x) = \cos x \\
&= -\tan x
\end{aligned}$$

3.2 Proving Identities

Learning Objectives

- Prove identities using several techniques.

Working with Trigonometric Identities

During the course, you will see complex trigonometric expressions. Often, complex trigonometric expressions can be equivalent to less complex expressions. The process for showing two trigonometric expressions to be equivalent (regardless of the value of the angle) is known as validating or proving trigonometric identities.

There are several options a student can use when proving a trigonometric identity.

Option One: Often one of the steps for proving identities is to change each term into their sine and cosine equivalents:

Example 1: Prove the identity: $\csc\theta \times \tan\theta = \sec\theta$

Solution: Reducing each side separately. It might be helpful to put a line down, through the equals sign. Because we are proving this identity, we don't know if the two sides are equal, so wait until the end to include the equality.

$$\begin{array}{c|c}
\csc x \times \tan x & \sec x \\
\frac{1}{\sin x} \times \frac{\sin x}{\cos x} & \frac{1}{\cos x} \\
\frac{1}{\cancel{\sin x}} \times \frac{\cancel{\sin x}}{\cos x} & \frac{1}{\cos x} \\
\frac{1}{\cos x} & \frac{1}{\cos x}
\end{array}$$

At the end we ended up with the same thing, so we know that this is a valid identity.

Notice when working with identities, unlike equations, conversions and mathematical operations are performed only on one side of the identity. In more complex identities sometimes both sides of the identity are simplified or expanded. The thought process for establishing identities is to view each side of the identity separately, and at the end to show that both sides do in fact transform into identical mathematical statements.

Option Two: Use the Trigonometric Pythagorean Theorem and other Fundamental Identities.

Example 2: Prove the identity: $(1-\cos^2 x)(1+\cot^2 x) = 1$

Solution: Use the Pythagorean Identity and its alternate form. Manipulate $\sin^2\theta + \cos^2\theta = 1$ to be $\sin^2\theta = 1-\cos^2\theta$. Also substitute $\csc^2 x$ for $1+\cot^2 x$, then cross-cancel.

$$\begin{array}{c|l} (1-\cos^2 x)(1+\cot^2 x) & 1 \\ \sin^2 x\cdot\csc^2 x & 1 \\ \sin^2 x\cdot\frac{1}{\sin^2 x} & 1 \\ 1 & 1 \end{array}$$

Option Three: When working with identities where there are fractions- combine using algebraic techniques for adding expressions with unlike denominators:

Example 3: Prove the identity: $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\csc\theta$.

Solution: Combine the two fractions on the left side of the equation by finding the common denominator: $(1+\cos\theta)\times\sin\theta$, and the change the right side into terms of sine.

$$\begin{array}{c|l} \frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} & 2\csc\theta \\ \frac{\sin\theta}{\sin\theta}\cdot\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} & 2\csc\theta \\ \frac{\sin^2\theta+(1+\cos\theta)^2}{\sin\theta(1+\cos\theta)} & 2\csc\theta \end{array}$$

Now, we need to apply another algebraic technique, FOIL. Always leave the denominator factored, because you might be able to cancel something out at the end.

$$\begin{array}{c|l} \frac{\sin^2\theta+1+2\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta \end{array}$$

Using the second option, substitute $\sin^2\theta + \cos^2\theta = 1$ and simplify.

$$\begin{array}{c|l} \frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\ \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\ \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} & 2\csc\theta \\ \frac{2}{\sin\theta} & \frac{2}{\sin\theta} \end{array}$$

Option Four: If possible, factor trigonometric expressions. Actually procedure four was used in the above example: $\frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)} = 2\csc\theta$ can be *factored* to $\frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = 2\csc\theta$ and in this situation, the factors cancel each other.

Example 4: Prove the identity: $\frac{1+\tan\theta}{(1+\cot\theta)} = \tan\theta$.

Solution: Change $\cot\theta$ to $\frac{1}{\tan\theta}$ and find a common denominator.

$$\frac{1+\tan\theta}{\left(1+\frac{1}{\tan\theta}\right)} = \tan\theta$$

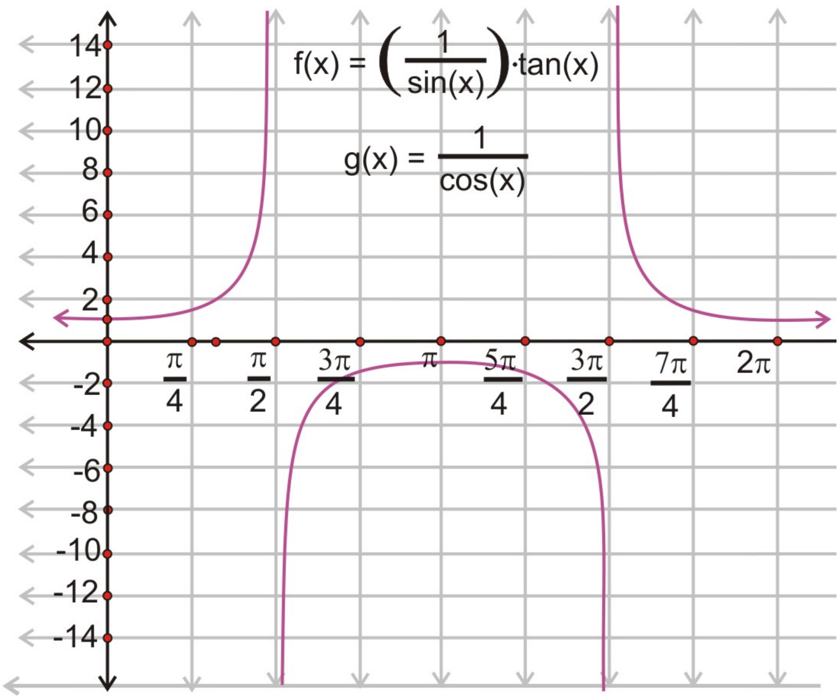
$$\frac{1+\tan\theta}{\left(\frac{\tan\theta}{\tan\theta}+\frac{1}{\tan\theta}\right)} = \tan\theta \qquad \text{or} \qquad \frac{1+\tan\theta}{\frac{\tan\theta+1}{\tan\theta}} = \tan\theta$$

Now invert the denominator and multiply.

$$\frac{\tan\theta(1+\tan\theta)}{\tan\theta+1} = \tan\theta$$
$$\tan\theta = \tan\theta$$

Technology Note

A graphing calculator can help provide the correctness of an identity. For example looking at: $\csc x \times \tan x = \sec x$, first graph $y = \csc x \times \tan x$, and then graph $y = \sec x$. Examining the viewing screen for each demonstrates that the results produce the same graph.



To summarize, when verifying a trigonometric identity, use the following tips:

1. Work on one side of the identity- usually the more complicated looking side.
2. Try rewriting all given expressions in terms of sine and cosine.
3. If there are fractions involved, combine them.
4. After combining fractions, if the resulting fraction can be reduced, reduce it.
5. The goal is to make one side look exactly like the other—so as you change one side of the identity, look at the other side for a potential hint to what to do next. If you are stumped, work with the other side. Don't limit yourself to working only on the left side, a problem might require you to work on the right.

Points to Consider

- Are there other techniques that you could use to prove identities?
- What else, besides what is listed in this section, do you think would be useful in proving identities?

Review Questions

Prove the following identities true:

1. $\sin x \tan x + \cos x = \sec x$
2. $\cos x - \cos x \sin^2 x = \cos^3 x$
3. $\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\csc x$
4. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$
5. $\frac{1}{1+\cos a} + \frac{1}{1-\cos a} = 2 + 2\cot^2 a$
6. $\cos^4 b - \sin^4 b = 1 - 2\sin^2 b$
7. $\frac{\sin y+\cos y}{\sin y} - \frac{\cos y-\sin y}{\cos y} = \sec y \csc y$
8. $(\sec x - \tan x)^2 = \frac{1-\sin x}{1+\sin x}$
9. Show that $2\sin x \cos x = \sin 2x$ is true using $\frac{5\pi}{6}$.
10. Use the trig identities to prove $\sec x \cot x = \csc x$

Review Answers

1. Step 1: Change everything into sine and cosine

$$\sin x \tan x + \cos x = \sec x$$
$$\sin x \cdot \frac{\sin x}{\cos x} + \cos x = \frac{1}{\cos x}$$

Step 2: Give everything a common denominator, $\cos x$.

$$\frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x} = \frac{1}{\cos x}$$

Step 3: Because the denominators are all the same, we can eliminate them.

$$\sin^2 x + \cos^2 x = 1$$

We know this is true because it is the Trig Pythagorean Theorem

2. Step 1: Pull out a $\cos x$

$$\cos x - \cos x \sin^2 x = \cos^3 x$$
$$\cos x(1 - \sin^2 x) = \cos^3 x$$

Step 2: We know $\sin^2 x + \cos^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$ is also true, therefore $\cos x(\cos^2 x) = \cos^3 x$. This, of course, is true, we are done!

3. Step 1: Change everything in to sine and cosine and find a common denominator for left hand side.

$$\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\csc x$$
$$\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = \frac{2}{\sin x} \leftarrow \text{LCD}: \ \sin x(1+\cos x)$$
$$\frac{\sin^2 x + (1+\cos x)^2}{\sin x(1+\cos x)}$$

Step 2: Working with the left side, FOIL and simplify.

$$\frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{\sin x(1+\cos x)} \quad \rightarrow \text{FOIL } (1+\cos x)^2$$

$$\frac{\sin^2 x + \cos^2 x + 1 + 2\cos x}{\sin x(1+\cos x)} \quad \rightarrow \text{move } \cos^2 x$$

$$\frac{1 + 1 + 2\cos x}{\sin x(1+\cos x)} \quad \rightarrow \sin^2 x + \cos^2 x = 1$$

$$\frac{2 + 2\cos x}{\sin x(1+\cos x)} \quad \rightarrow \text{add}$$

$$\frac{2(1+\cos x)}{\sin x(1+\cos x)} \quad \rightarrow \text{fator out 2}$$

$$\frac{2}{\sin x} \quad \rightarrow \text{cancel } (1+\cos x)$$

4. Step 1: Cross-multiply

$$\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$
$$\sin^2 x = (1+\cos x)(1-\cos x)$$

Step 2: Factor and simplify

$$\sin^2 x = 1 - \cos^2 x$$
$$\sin^2 x + \cos^2 x = 1$$

5. Step 1: Work with left hand side, find common denominator, FOIL and simplify, using $\sin^2 x + \cos^2 x = 1$.

$$\frac{1}{1+\cos x} + \frac{1}{1-\cos x} = 2 + 2\cot^2 x$$
$$\frac{1-\cos x + 1 + \cos x}{(1+\cos x)(1-\cos x)}$$
$$\frac{2}{1-\cos^2 x}$$
$$\frac{2}{\sin^2 x}$$

Step 2: Work with the right hand side, to hopefully end up with $\frac{2}{\sin^2 x}$.

$$= 2 + 2\cot^2 x$$

$$= 2 + 2\,\frac{\cos^2 x}{\sin^2 x}$$

$$= 2\left(1 + \frac{\cos^2 x}{\sin^2 x}\right) \quad \rightarrow \text{factor out the 2}$$

$$= 2\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right) \quad \rightarrow \text{common denominator}$$

$$= 2\left(\frac{1}{\sin^2 x}\right) \quad \rightarrow \text{trig pythagorean theorem}$$

$$= \frac{2}{\sin^2 x} \quad \rightarrow \text{simply/multiply}$$

Both sides match up, the identity is true.

6. Step 1: Factor left hand side

$$\begin{array}{r|l} \cos^4 b - \sin^4 b & 1 - 2\sin^2 b \\ (\cos^2 b + \sin^2 b)(\cos^2 b - \sin^2 b) & 1 - 2\sin^2 b \\ \cos^2 b - \sin^2 b & 1 - 2\sin^2 b \end{array}$$

Step 2: Substitute $1-\sin^2 b$ for $\cos^2 b$ because $\sin^2 x+\cos^2 x=1$.

$$\begin{array}{r|l} (1-\sin^2 b)-\sin^2 b & 1-2\sin^2 b \\ 1-\sin^2 b-\sin^2 b & 1-2\sin^2 b \\ 1-2\quad \sin^2 b & 1-2\sin^2 b \end{array}$$

7. Step 1: Find a common denominator for the left hand side and change right side in terms of sine and cosine.

$$\frac{\sin y+\cos y}{\sin y}-\frac{\cos y-\sin y}{\cos y}=\sec y\csc y$$
$$\frac{\cos y(\sin y+\cos y)-\sin y(\cos y-\sin y)}{\sin y\cos y}=\frac{1}{\sin y\cos y}$$

Step 2: Work with left side, simplify and distribute.

$$\frac{\sin y\cos y+\cos^2 y-\sin y\cos y+\sin^2 y}{\sin y\cos y}$$
$$\frac{\cos^2 y+\sin^2 y}{\sin y\cos y}$$
$$\frac{1}{\sin y\cos y}$$

8. Step 1: Work with left side, change everything into terms of sine and cosine.

$$(\sec x-\tan x)^2=\frac{1-\sin x}{1+\sin x}$$
$$\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)^2$$
$$\left(\frac{1-\sin x}{\cos x}\right)^2$$
$$\frac{(1-\sin x)^2}{\cos^2 x}$$

Step 2: Substitute $1-\sin^2 b$ for $\cos^2 b$ because $\sin^2 x+\cos^2 x=1$

$$\frac{(1-\sin x)^2}{1-\sin^2 x}\rightarrow \text{be careful, these are NOT the same!}$$

Step 3: Factor the denominator and cancel out like terms.

$$\frac{(1-\sin x)^2}{(1+\sin x)(1-\sin x)}$$
$$\frac{1-\sin x}{1+\sin x}$$

9. Plug in $\frac{5\pi}{6}$ for x into the formula and simplify.

$$2\sin x\cos x=\sin 2x$$
$$2\sin\frac{5\pi}{6}\cos\frac{5\pi}{6}=\sin 2\cdot\frac{5\pi}{6}$$
$$2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)=\sin\frac{5\pi}{3}$$

This is true because $\sin 300^\circ$ is $-\frac{\sqrt{3}}{2}$

10. Change everything into terms of sine and cosine and simplify.

$$\sec x\cot x=\csc x$$
$$\frac{1}{\cos x}\cdot\frac{\cos x}{\sin x}=\frac{1}{\sin x}$$
$$\frac{1}{\sin x}=\frac{1}{\sin x}$$

3.3 Solving Trigonometric Equations

Learning Objectives

- Use the fundamental identities to solve trigonometric equations.
- Express trigonometric expressions in simplest form.
- Solve trigonometric equations by factoring.
- Solve trigonometric equations by using the Quadratic Formula.

By now we have seen trigonometric functions represented in many ways: Ratios between the side lengths of right triangles, as functions of coordinates as one travels along the unit circle and as abstract functions with graphs. Now it is time to make use of the properties of the trigonometric functions to gain knowledge of the connections between the functions themselves. The patterns of these connections can be applied to simplify trigonometric expressions and to solve trigonometric equations.

Simplifying Trigonometric Expressions

Example 1: Simplify the following expressions using the basic trigonometric identities:

a. $\frac{1+\tan^2 x}{\csc^2 x}$

b. $\frac{\sin^2 x+\tan^2 x+\cos^2 x}{\sec x}$

c. $\cos x - \cos^3 x$

Solution:

a.

$$\frac{1+\tan^2 x}{\csc^2 x} \ldots (1+\tan^2 x = \sec^2 x)\text{Pythagorean Identity}$$

$$\frac{\sec^2 x}{\csc^2 x} \ldots (\sec^2 x = \frac{1}{\cos^2 x} \text{ and } \csc^2 x = \frac{1}{\sin^2 x})\text{Reciprocal Identity}$$

$$\frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} = \left(\frac{1}{\cos^2 x}\right) \div \left(\frac{1}{\sin^2 x}\right)$$

$$\left(\frac{1}{\cos^2 x}\right) \cdot \left(\frac{\sin^2 x}{1}\right) = \frac{\sin^2 x}{\cos^2 x}$$

$$= \tan^2 x \rightarrow \text{Quotient Identity}$$

b.

$$\frac{\sin^2 x+\tan^2 x+\cos^2 x}{\sec x} \ldots (\sin^2 x + \cos^2 x = 1)\text{Pythagorean Identity}$$

$$\frac{1+\tan^2 x}{\sec x} \ldots (1+\tan^2 x = \sec^2 x)\text{Pythagorean Identity}$$

$$\frac{\sec^2 x}{\sec x} = \sec x$$

c.

$$\begin{aligned}
&\cos x - \cos^3 x \\
&\cos x(1-\cos^2 x) \qquad \ldots\text{Factor out } \cos x \text{ and } \sin^2 x = 1-\cos^2 x \\
&\cos x(\sin^2 x)
\end{aligned}$$

In the above examples, the given expressions were simplified by applying the patterns of the basic trigonometric identities. We can also apply the fundamental identities to trigonometric equations to solve for x. When solving trig equations, restrictions on x (or θ) must be provided, or else there would be infinitely many possible answers (because of the periodicity of trig functions).

Solving Trigonometric Equations

Example 2: Without the use of technology, find all solutions $\tan^2(x) = 3$, such that $0 \le x \le 2\pi$.

Solution:

$$\begin{aligned}
\tan^2 x &= 3 \\
\sqrt{\tan^2 x} &= \sqrt{3} \\
\tan x &= \pm\sqrt{3}
\end{aligned}$$

This means that there are four answers for x, because tangent is positive in the first and third quadrants and negative in the second and fourth. Combine that with the values that we know would generate $\tan x = \sqrt{3}$ or $\tan x = -\sqrt{3}$, $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, and $\frac{5\pi}{3}$.

Example 3: Solve $2\cos x \sin x - \cos x = 0$ for all values of x between $[0, 2\pi]$.

Solution:

$$\cos x \ \ (2\sin x - 1) = 0 \rightarrow \text{set each factor equal to zero and solve them separately}$$

$$\begin{array}{ll}
\downarrow & \searrow \\
\cos x = 0 & 2\sin x = 1 \\
x = \frac{\pi}{2} \text{ and } x = \frac{3\pi}{2} & \sin x = \frac{1}{2} \\
& x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6}
\end{array}$$

In the above examples, exact values were obtained for the solutions of the equations. These solutions were within the domain that was specified.

Example 4: Solve $2\sin^2 x - \cos x - 1 = 0$ for all values of x.

Solution: The equation now has two functions – sine and cosine. Study the equation carefully and decide in which function to rewrite the equation. $\sin^2 x$ can be expressed in terms of cosine by manipulating the Pythagorean Identity, $\sin^2 x + \cos^2 x = 1$.

$$
\begin{aligned}
2\sin^2 x - \cos x - 1 &= 0 \\
2(1-\cos^2 x) - \cos x - 1 &= 0 \\
2 - 2\cos^2 x - \cos x - 1 &= 0 \\
-2\cos^2 x - \cos x + 1 &= 0 \\
2\cos^2 x + \cos x - 1 &= 0 \\
(2\cos x - 1)(\cos x + 1) &= 0
\end{aligned}
$$

$$
\begin{array}{lcl}
\swarrow & & \searrow \\
2\cos x - 1 = 0 & \text{or} & \cos x + 1 = 0 \\
\cos x = \frac{1}{2} & & \cos x = -1 \\
x = \frac{\pi}{3} + 2\pi k, k\epsilon Z & & x = \pi + 2\pi k, k\epsilon Z \\
x = \frac{5\pi}{3} + 2\pi k, k\epsilon Z & &
\end{array}
$$

Solving Trigonometric Equations Using Factoring

Algebraic skills like factoring and substitution that are used to solve various equations are very useful when solving trigonometric equations. As with algebraic expressions, one must be careful to avoid dividing by zero during these maneuvers.

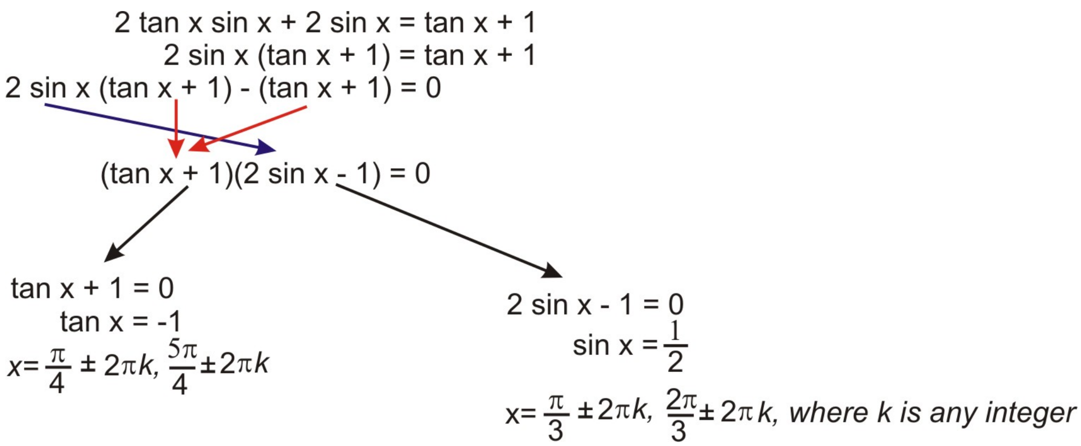
Example 5: Solve $2\sin^2 x - 3\sin x + 1 = 0$ for $0 < x \leq 2\pi$.

Solution:

$$
\begin{array}{lcl}
2\sin^2 x - 3\sin x + 1 = 0 & & \text{Factor this like a quadratic equation} \\
(2\sin x - 1)(\sin x - 1) = 0 & & \\
\downarrow & & \searrow \\
2\sin x - 1 = 0 & \text{or} & \sin x - 1 = 0 \\
2\sin x = 1 & & \sin x = 1 \\
\sin x = \frac{1}{2} & & x = \frac{\pi}{2} \\
x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6} & &
\end{array}
$$

Example 6: Solve $2\tan x \sin x + 2\sin x = \tan x + 1$ for all values of x.

Solution:



Pull out $\sin x$

There is a common factor of $(\tan x + 1)$

Think of the $-(\tan x + 1)$ as $(-1)(\tan x + 1)$, which is why there is a -1 behind the $2 \sin x$.

Example 7: Solve $2\sin^2 x + 3\sin x - 2 = 0$ for all x, $[0, \pi]$.

Solution:

$$2\sin^2 x + 3\sin x - 2 = 0 \rightarrow \text{Factor like a quadratic}$$
$$(2\sin x - 1)(\sin x + 2) = 0$$
$$\swarrow \qquad\qquad \searrow$$
$$2\sin x - 1 = 0 \qquad \sin x + 2 = 0$$
$$\sin x = \frac{1}{2} \qquad \sin x = -2$$
$$x = \frac{\pi}{6} \text{ and } x = \frac{5\pi}{6} \quad \text{There is no solution because the range of } \sin x \text{ is } [-1, 1].$$

Some trigonometric equations have no solutions. This means that there is no replacement for the variable that will result in a true expression.

Example 8: Solve $4\sin^3 x + 2\sin^2 x - 2\sin x - 1 = 0$ for x in the interval $[0, 2\pi]$.

Solution: Even though this does not look like a factoring problem, it is. We are going to use factoring by grouping, from Algebra II. First group together the first two terms and the last two terms. Then find the greatest common factor for each pair.

$$\underbrace{4\sin^3 x + 2\sin^2 x} \qquad \underbrace{-2\sin x - 1} = 0$$
$$2\sin^2 x(2\sin x + 1) - 1(2\sin x + 1)$$

Notice we have gone from four terms to two. These new two terms have a common factor of $2\sin x + 1$. We can pull this common factor out and reduce our number of terms from two to one, comprised of two factors.

$$2\sin^2 x(2\sin x + 1) - 1(2\sin x + 1) = 0$$
$$\searrow \qquad \swarrow$$
$$(2\sin x + 1)(2\sin^2 x \ - \ 1) = 0$$

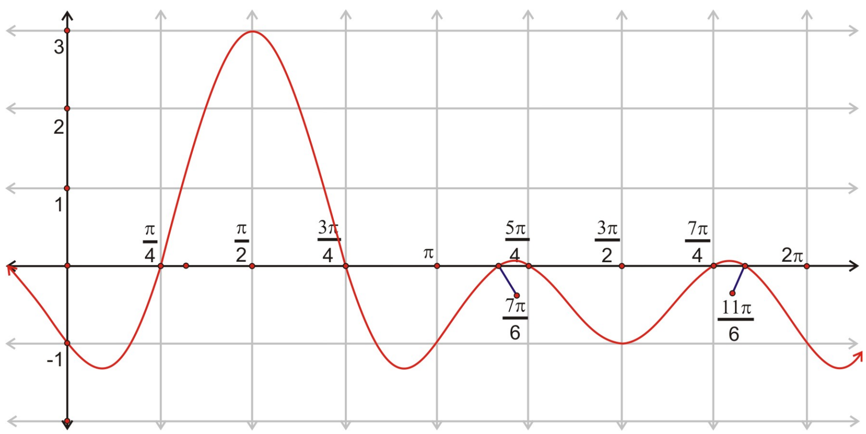
We can take this one step further because $2\sin^2 x - 1$ can factor again.

$$(2\sin x+1)\left(\sqrt{2}\sin x-1\right)\left(\sqrt{2}\sin x+1\right)=0$$

Set each factor equal to zero and solve.

$$\begin{aligned}2\sin x+1&=0\\2\sin x&=-1\\\sin x&=-\frac{1}{2}\\x&=\frac{7\pi}{6},\frac{11\pi}{6}\end{aligned}\qquad\text{or}\qquad\begin{aligned}\sqrt{2}\sin x+1&=0\\\sqrt{2}\sin x&=-1\\\sin x&=-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}\\x&=\frac{5\pi}{4},\frac{7\pi}{4}\end{aligned}\qquad\text{or}\qquad\begin{aligned}\sqrt{2}\sin x-1&=0\\\sqrt{2}\sin x&=1\\\sin x&=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}\\x&=\frac{\pi}{4},\frac{3\pi}{4}\end{aligned}$$

Notice there are six solutions for x. Graphing the original function would show that the equation crosses the $x-$axis six times in the interval $[0,2\pi]$.



Solving Trigonometric Equations Using the Quadratic Formula

When solving quadratic equations that do not factor, the quadratic formula is often used. The same can be applied when solving trigonometric equations that do not factor. The values for a is the numerical coefficient of the function's squared term, b is the numerical coefficient of the function term that is to the first power and c is a constant. The formula will result in two answers and both will have to be evaluated within the designated interval.

Example 8: Solve $3\ \cot^2 x-3\ \cot x=1$ for exact values of x over the interval $[0,2\pi]$.

Solution:

$$\begin{aligned}3\cot^2 x-3\cot x&=1\\3\cot^2 x-3\cot x-1&=0\end{aligned}$$

The equation will not factor. Use the quadratic formula for $\cot x$, $a=3,b=-3,c=-1$.

$$\cot x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\cot x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(3)(-1)}}{2(3)}$$
$$\cot x = \frac{3 \pm \sqrt{9 + 12}}{6}$$
$$\cot x = \frac{3 + \sqrt{21}}{6} \quad \text{or} \quad \cot x = \frac{3 - \sqrt{21}}{6}$$
$$\cot x = \frac{3 + 4.5826}{6} \qquad \cot x = \frac{3 - 4.5826}{6}$$
$$\cot x = 1.2638 \qquad \cot x = -0.2638$$
$$\tan x = \frac{1}{1.2638} \qquad \tan x = \frac{1}{-0.2638}$$
$$x = 0.6694, 3.81099 \qquad x = 1.8287, 4.9703$$

Example 2: Solve $-5\cos^2 x + 9\sin x + 3 = 0$ for values of x over the interval $[0, 2\pi]$.

Solution: Change $\cos^2 x$ to $1 - \sin^2 x$ from the Pythagorean Identity.

$$-5\cos^2 x + 9\sin x + 3 = 0$$
$$-5(1 - \sin^2 x) + 9\sin x + 3 = 0$$
$$-5 + 5\sin^2 x + 9\sin x + 3 = 0$$
$$5\sin^2 x + 9\sin x - 2 = 0$$

$$\sin x = \frac{-9 \pm \sqrt{9^2 - 4(5)(-2)}}{2(5)}$$
$$\sin x = \frac{-9 \pm \sqrt{81 + 40}}{10}$$
$$\sin x = \frac{-9 \pm \sqrt{121}}{10}$$
$$\sin x = \frac{-9 + 11}{10} \text{ and } \sin x = \frac{-9 - 11}{10}$$
$$\sin x = \frac{1}{5} \text{ and } -2$$
$$\sin^{-1}(0.2) \text{ and } \sin^{-1}(-2)$$

$x \approx .201$ *rad* and $\pi - .201 \approx 2.941$

This is the only solutions for x since -2 is not in the range of values.

To summarize, to solve a trigonometric equation, you can use the following techniques:

1. Simplify expressions with the fundamental identities.
2. Factor, pull out common factors, use factoring by grouping.
3. The Quadratic Formula.
4. Be aware of the intervals for x. Make sure your final answer is in the specified domain.

Points to Consider

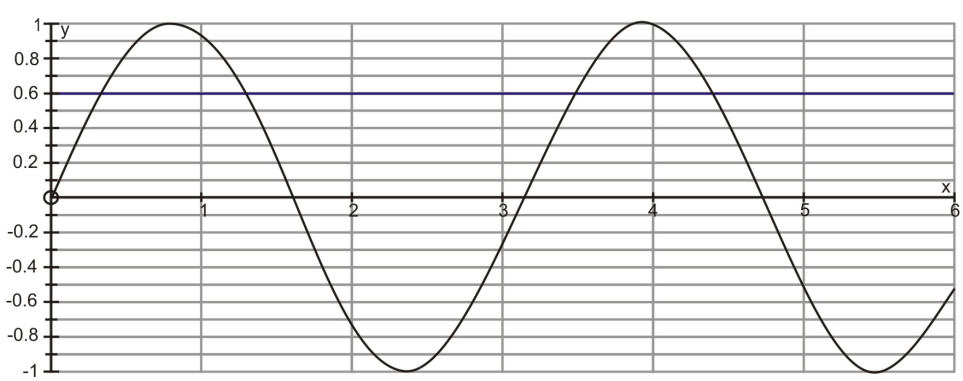
- Are there other methods for solving equations that can be adapted to solving trigonometric equations?
- Will any of the trigonometric equations involve solving quadratic equations?
- Is there a way to solve a trigonometric equation that will not factor?
- Is substitution of a function with an identity a feasible approach to solving a trigonometric equation?

Review Questions

1. Solve the equation $\sin 2\theta = 0.6$ for $0 \le \theta < 2\pi$.
2. Solve the equation $\cos^2 x = \frac{1}{16}$ over the interval $[0, 2\pi]$
3. Solve the trigonometric equation $\tan^2 x = 1$ for all values of θ such that $0 \le \theta \le 2\pi$
4. Solve the trigonometric equation $4 \sin x \cos x + 2 \cos x - 2 \sin x - 1 = 0$ such that $0 \le x < \pi$.
5. Solve $\sin^2 x - 2 \sin x - 3 = 0$ for x over $[0, \pi]$.
6. Solve $\tan^2 x = 3 \tan x$ for x over $[0, \pi]$.
7. Find all the solutions for the trigonometric equation $2 \sin^2 \frac{x}{4} - 3 \cos \frac{x}{4} = 0$ over the interval $[0, 2\pi)$.
8. Solve the trigonometric equation $3 - 3 \sin^2 x = 8 \sin x$ over the interval $[0, 2\pi]$.
9. Solve $2 \sin x \tan x = \tan x + \sec x$ for all values of $x \epsilon [0, 2\pi]$.
10. Solve the trigonometric equation $2 \cos^2 x + 3 \sin x - 3 = 0$ over the interval $[0, 2\pi]$.
11. Solve $\tan^2 x + \tan x + 2 = 0$ for values of x over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
12. Solve the trigonometric equation such that $5 \cos^2 \theta - 6 \sin \theta = 0$ over the interval $[0, 2\pi]$.

Review Answers

1. Because the problem deals with 2θ, the domain values must be doubled, making the domain $0 \le 2\theta < 4\pi$
   The reference angle is $\alpha = \sin^{-1} 0.6 = 0.6435$
   $2\theta = 0.6435, \pi - 0.6435, 2\pi + 0.6435, 3\pi - 0.6435$
   $2\theta = 0.6435, 2.2980, 6.9266, 8.7812$
   The values for θ are needed so the above values must be divided by 2.
   $\theta = 0.3218, 1.1490, 3.4633, 4.3906$
   The results can readily be checked by graphing the function. The four results are reasonable since they are the only results indicated on the graph that satisfy $\sin 2\theta = 0.6$.



2.

$$\cos^2 x = \frac{1}{16}$$
$$\sqrt{\cos^2 x} = \sqrt{\frac{1}{16}}$$
$$\cos x = \pm\frac{1}{4}$$

$$\text{Then } \cos x = \frac{1}{4} \qquad \text{or} \qquad \cos x = -\frac{1}{4}$$
$$\cos^{-1}\frac{1}{4} = x \qquad\qquad \cos^{-1} -\frac{1}{4} = x$$
$$x = 1.3181 \textit{ radians} \qquad\qquad x = 1.8235 \textit{ radians}$$

However, $\cos x$ is also positive in the fourth quadrant, so the other possible solution for $\cos x = \frac{1}{4}$ is $2\pi - 1.3181 = 4.9651$ *radians* and $\cos x$ is also negative in the third quadrant, so the other possible solution for $\cos x = -\frac{1}{4}$ is $2\pi - 1.8235 = 4.4597$ *radians*

3.

$$\tan^2 x = 1$$
$$\tan x = \pm\sqrt{1}$$
$$\tan x = \pm 1$$

so, $\tan x = 1$ or $\tan x = -1$. Therefore, x is all critical values corresponding with $\frac{\pi}{4}$ within the interval. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

4. Use factoring by grouping.

4 sin x cos x + 2 cos x - 2 sin x - 1= 0

2 cos x(2 sin x + 1)-1(2 sin x + 1)=0

(2 sin x + 1)(2 cos x - 1)=0

$$2\sin x + 1 = 0 \quad \text{or} \quad 2\cos x - 1 = 0$$
$$2\sin x = -1 \qquad 2\cos x = 1$$
$$\sin x = -\frac{1}{2} \qquad \cos x = \frac{1}{2}$$
$$x = \frac{7\pi}{6}, \frac{11\pi}{6} \qquad x = \frac{\pi}{x}, \frac{2\pi}{3}$$

5. You can factor this one like a quadratic.

$$\sin^2 x - 2\sin x - 3 = 0$$
$$(\sin x - 3)(\sin x + 1) = 0$$
$$\sin x - 3 = 0 \qquad\qquad \sin x + 1 = 0$$
$$\sin x = 3 \qquad \text{or} \qquad \sin x = -1$$
$$x = \sin^{-1}(3) \qquad\qquad x = \frac{3\pi}{2}$$

For this problem the only solution is $\frac{3\pi}{2}$ because sine cannot be 3 (it is not in the range).

6.

$$\tan^2 x = 3\tan x$$
$$\tan^2 x - 3\tan x = 0$$
$$\tan x(\tan x - 3) = 0$$
$$\tan x = 0 \qquad \text{or} \qquad \tan x = 3$$
$$x = 0, \pi \qquad\qquad x = 1.25, 4.39$$

7. $2\sin^2 \frac{x}{4} - 3\cos\frac{x}{4} = 0$

$$2\left(1-\cos^2\frac{x}{4}\right) - 3\cos\frac{x}{4} = 0$$
$$2 - 2\cos^2\frac{x}{4} - 3\cos\frac{x}{4} = 0$$
$$2\cos^2\frac{x}{4} + 3\cos\frac{x}{4} - 2 = 0$$
$$\left(2\cos\frac{x}{4} - 1\right)\left(\cos\frac{x}{4} + 2\right) = 0$$
$$\swarrow \qquad \searrow$$
$$2\cos\frac{x}{4} - 1 = 0 \quad \text{or} \quad \cos\frac{x}{4} + 2 = 0$$
$$2\cos\frac{x}{4} = 1 \qquad \cos\frac{x}{4} = -2$$
$$\cos\frac{x}{4} = \frac{1}{2}$$
$$\frac{x}{4} = \frac{\pi}{3} \quad \text{or} \quad \frac{5\pi}{3}$$
$$x = \frac{4\pi}{3} \quad \text{or} \quad \frac{20\pi}{3}$$

8. $\frac{20\pi}{3}$ is eliminated as a solution because it is outside of the range and $\cos\frac{x}{4} = -2$ will not generate any solutions because -2 is outside of the range of cosine. Therefore, the only solution is $\frac{4\pi}{3}$.

$$3 - 3\sin^2 x = 8\sin x$$
$$3 - 3\sin^2 x - 8\sin x = 0$$
$$3\sin^2 x + 8\sin x - 3 = 0$$
$$(3\sin x - 1)(\sin x + 3) = 0$$
$$3\sin x - 1 = 0 \quad \text{or} \quad \sin x + 3 = 0$$
$$3\sin x = 1$$
$$\sin x = \frac{1}{3} \qquad \sin x = -3$$
$$x = 0.3398 \textit{ radians} \quad \text{No solution exists}$$
$$x = \pi - 0.3398 = 2.8018 \textit{ radians}$$

9. $2\sin x\tan x = \tan x + \sec x$

$$2\sin x \cdot \frac{\sin x}{\cos x} = \frac{\sin x}{\cos x} + \frac{1}{\cos x}$$
$$\frac{2\sin^2 x}{\cos x} = \frac{\sin x + 1}{\cos}$$
$$2\sin^2 x = \sin x + 1$$
$$2\sin^2 x - \sin x - 1 = 0$$
$$(2\sin x + 1)(\sin x - 1) = 0$$
$$2\sin x + 1 = 0 \quad \text{or} \quad \sin x - 1 = 0$$
$$2\sin x = -1 \qquad \sin x = 1$$
$$\sin x = -\frac{1}{2}$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$$

10.

$$\begin{aligned}
2\cos^2 x + 3\sin x - 3 &= 0 \\
2(1-\sin^2 x) + 3\sin x - 3 &= 0 \text{ Pythagorean Identity} \\
2 - 2\sin^2 x + 3\sin x - 3 &= 0 \\
-2\sin^2 x + 3\sin x - 1 &= 0 \text{ Multiply by } -1 \\
2\sin^2 x - 3\sin x + 1 &= 0 \\
(2\sin x - 1)(\sin x - 1) &= 0
\end{aligned}$$

$$2\sin x - 1 = 0 \quad \text{or} \quad \sin x - 1 = 0$$

$$2\sin x = 1$$

$$\sin x = \frac{1}{2} \qquad\qquad \sin x = 1$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6} \qquad\qquad x = \frac{\pi}{2}$$

11. $\tan^2 x + \tan x - 2 = 0$

$$\begin{aligned}
\frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2} &= \tan x \\
\frac{-1 \pm \sqrt{1+8}}{2} &= \tan x \\
\frac{-1 \pm 3}{2} &= \tan x \\
\tan x = -2 \quad &\text{or} \quad 1
\end{aligned}$$

$\tan x = 1$ when $x = -\frac{3\pi}{4}$, in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\tan x = -2$ when $x = -4.249$ *rad*

12. $5\cos^2\theta - 6\sin\theta = 0$ over the interval $[0, 2\pi]$.

$$\begin{aligned}
5\left(1-\sin^2 x\right) - 6\sin x &= 0 \\
-5\sin^2 x - 6\sin x + 5 &= 0 \\
5\sin^2 x + 6\sin x - 5 &= 0 \\
\frac{-6 \pm \sqrt{6^2 - 4(5)(-5)}}{2(5)} &= \sin x \\
\frac{-6 \pm \sqrt{36+100}}{10} &= \sin x \\
\frac{-6 \pm \sqrt{136}}{10} &= \sin x \\
\frac{-6 \pm 2\sqrt{34}}{10} &= \sin x \\
\frac{-3 \pm \sqrt{34}}{5} &= \sin x
\end{aligned}$$

$x = \sin^{-1}\left(\frac{-3+\sqrt{34}}{5}\right)$ or $\sin^{-1}\left(\frac{-3-\sqrt{34}}{5}\right)$ $x = 0.6018$ *rad* or 2.5398 *rad* from the first expression, the second expression will not yield any answers because it is out the the range of sine.

3.4 Sum and Difference Identities

Learning Objectives

- Use and identify the sum and difference identities.

- Apply the sum and difference identities to solve trigonometric equations.
- Find the exact value of a trigonometric function for certain angles.

In this section we are going to explore $\cos(a \pm b), \sin(a \pm b)$, and $\tan(a \pm b)$. These identities have very useful expansions and can help to solve identities and equations.

Sum and Difference Formulas: cosine

Is $\cos 15^\circ = \cos(45^\circ - 30^\circ)$? Upon appearance, yes, it is. This section explores how to find an expression that would equal $\cos(45^\circ - 30^\circ)$. To simplify this, let the two given angles be a and b where $0 < b < a < 2\pi$.

Begin with the unit circle and place the angles a and b in standard position as shown in Figure A. Point Pt1 lies on the terminal side of b, so its coordinates are $(\cos b, \sin b)$ and Point Pt2 lies on the terminal side of a so its coordinates are $(\cos a, \sin a)$. Place the $a - b$ in standard position, as shown in Figure B. The point A has coordinates $(1, 0)$ and the Pt3 is on the terminal side of the angle $a - b$, so its coordinates are $(\cos[a - b], \sin[a - b])$.

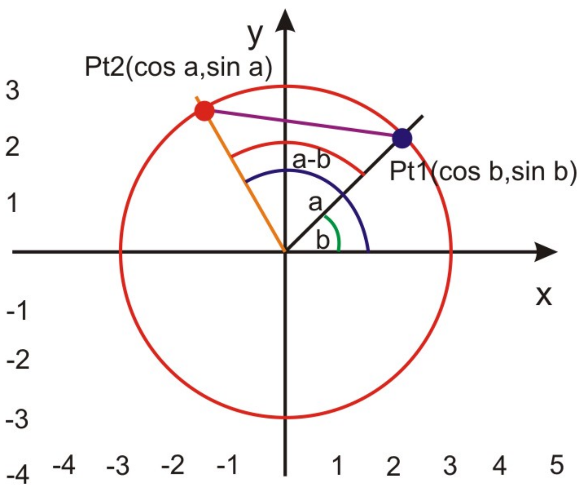


Figure A

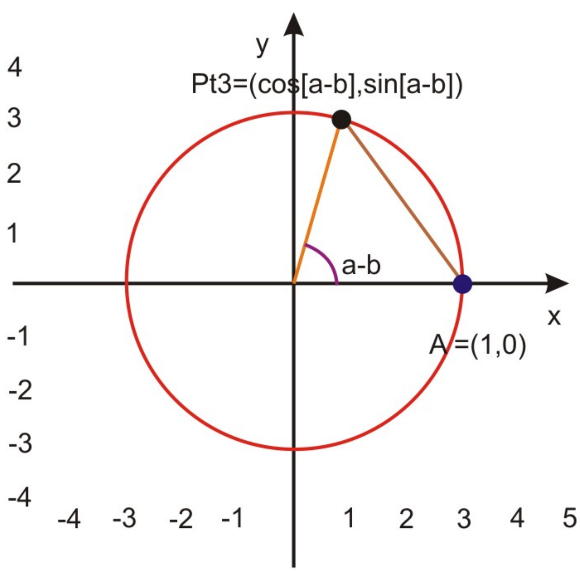


Figure B

Triangles OP_1P_2 in figure A and Triangle OAP_3 in figure B are congruent. (Two sides and the included angle, $a - b$, are equal). Therefore the unknown side of each triangle must also be equal. That is: $d\ (A, P_3) = d\ (P_1, P_2)$

Applying the distance formula to the triangles in Figures A and B and setting them equal to each other:

$$\sqrt{[\cos(a-b)-1]^2 + [\sin(a-b)-0]^2} = \sqrt{(\cos a - \cos b)^2 + (\sin a - \sin b)^2}$$

Square both sides to eliminate the square root.

$$[\cos(a-b)-1]^2 + [\sin(a-b)-0]^2 = (\cos a - \cos b)^2 + (\sin a - \sin b)^2$$

FOIL all four squared expressions and simplify.

$$\begin{aligned}
\cos^2(a-b) - 2\cos(a-b) + 1 + \sin^2(a-b) &= \cos^2 a - 2\cos a \cos b + \cos^2 b + \sin^2 a - 2\sin a \sin b + \sin^2 b \\
\underbrace{\sin^2(a-b) + \cos^2(a-b)} - 2\cos(a-b) + 1 &= \underbrace{\sin^2 a + \cos^2 a} - 2\cos a \cos b + \underbrace{\sin^2 b + \cos^2 b} - 2\sin a \sin b \\
1 - 2\cos(a-b) + 1 &= 1 - 2\cos a \cos b + 1 - 2\sin a \sin b \\
2 - 2\cos(a-b) &= 2 - 2\cos a \cos b - 2\sin a \sin b \\
-2\cos(a-b) &= -2\cos a \cos b - 2\sin a \sin b \\
\cos(a-b) &= \cos a \cos b + \sin a \sin b
\end{aligned}$$

In $\cos(a-b) = \cos a \cos b + \sin a \sin b$, the *difference* formula for cosine, you can substitute $a-(-b) = a+b$ to obtain: $\cos(a+b) = \cos[a-(-b)]$ or $\cos a \cos(-b) + \sin a \sin(-b)$. since $\cos(-b) = \cos b$ and $\sin(-b) = -\sin b$, then $\cos(a+b) = \cos a \cos b - \sin a \sin b$, which is the *sum* formula for cosine.

Using the Sum and Difference Identities of cosine

The sum/difference formulas for cosine can be used to establish other identities:

Example 1: Find an equivalent form of $\cos\left(\frac{\pi}{2} - \theta\right)$ using the cosine difference formula.

Solution:

$$\begin{aligned}
&\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\
&\cos\left(\frac{\pi}{2} - \theta\right) = 0 \times \cos\theta + 1 \times \sin\theta, \text{ substitute } \cos\frac{\pi}{2} = 0 \text{ and } \sin\frac{\pi}{2} = 1 \\
&\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta
\end{aligned}$$

We know that is a true identity because of our understanding of the sine and cosine curves, which are a phase shift of $\frac{\pi}{2}$ off from each other.

The cosine formulas can also be used to find exact values of cosine that we weren't able to find before, such as $15^\circ = (45^\circ - 30^\circ), 75^\circ = (45^\circ + 30^\circ)$, among others.

Example 2: Find the exact value of $\cos 15^\circ$

Solution: Use the difference formula where $a = 45^\circ$ and $b = 30^\circ$.

$$\begin{aligned}
\cos(45^\circ - 30^\circ) &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\
\cos 15^\circ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} \\
\cos 15^\circ &= \frac{\sqrt{6} + \sqrt{2}}{4}
\end{aligned}$$

Example 3: Find the exact value of $\cos 105^\circ$.

Solution: There may be more than one pair of key angles that can add up (or subtract to) 105°. Both pairs, $45^\circ + 60^\circ$ and $150^\circ - 45^\circ$, will yield the correct answer.

1.

$$\begin{aligned}\cos 105^\circ &= \cos(45^\circ + 60^\circ)\\ &= \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ, \text{ substitute in the known values}\\ &= \frac{\sqrt{2}}{2} \times \frac{1}{2} - \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2}\\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

2.

$$\begin{aligned}\cos 105^\circ &= \cos(150^\circ - 45^\circ)\\ &= \cos 150^\circ \cos 45^\circ + \sin 150^\circ \sin 45^\circ\\ &= -\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}\\ &= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

You do not need to do the problem multiple ways, just the one that seems easiest to you.

Example 4: Find the exact value of $\cos \frac{5\pi}{12}$, in radians.

Solution: $\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$, notice that $\frac{\pi}{4} = \frac{3\pi}{12}$ and $\frac{\pi}{6} = \frac{2\pi}{12}$

$$\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos \frac{\pi}{4} \cos \frac{\pi}{6} - \sin \frac{\pi}{4} \sin \frac{\pi}{6}$$

$$\begin{aligned}\cos \frac{\pi}{4} \cos \frac{\pi}{6} - \sin \frac{\pi}{4} \sin \frac{\pi}{6} &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Sum and Difference Identities: sine

To find $\sin(a + b)$, use Example 1, from above:

$$\begin{aligned}\sin(a + b) &= \cos\left[\frac{\pi}{2} - (a + b)\right] && \text{Set } \theta = a + b\\ &= \cos\left[\left(\frac{\pi}{2} - a\right) - b\right] && \text{Distribute the negative}\\ &= \cos\left(\frac{\pi}{2} - a\right)\cos b + \sin\left(\frac{\pi}{2} - a\right)\sin b && \text{Difference Formula for cosines}\\ &= \sin a \cos b + \cos a \sin b && \text{Co-function Identities}\end{aligned}$$

In conclusion, $\sin(a + b) = \sin a \cos b + \cos a \sin b$, which is the *sum* formula for sine.

To obtain the identity for $\sin(a - b)$:

$$\begin{aligned}
\sin(a-b) &= \sin[a+(-b)] \\
&= \sin a\cos(-b)+\cos a\sin(-b) && \text{Use the sine sum formula} \\
\sin(a-b) &= \sin a\cos b-\cos a\sin b && \text{Use } \cos(-b)=\cos b, \text{ and } \sin(-b)=-\sin b
\end{aligned}$$

In conclusion, $\sin(a-b)=\sin a\cos b-\cos a\sin b$, so, this is the *difference* formula for sine.

Example 5: Find the exact value of $\sin\frac{5\pi}{12}$

Solution: Recall that there are multiple angles that add or subtract to equal any angle. Choose whichever formula that you feel more comfortable with.

$$\begin{aligned}
\sin\frac{5\pi}{12} &= \sin\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right) \\
&= \sin\frac{3\pi}{12}\cos\frac{2\pi}{12}+\cos\frac{3\pi}{12}\sin\frac{2\pi}{12} \\
\sin\frac{5\pi}{12} &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\times\frac{1}{2} \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

Example 6: Given $\sin\alpha=\frac{12}{13}$, where α is in Quadrant II, and $\sin\beta=\frac{3}{5}$, where β is in Quadrant I, find the exact value of $\sin(\alpha+\beta)$.

Solution: To find the exact value of $\sin(\alpha+\beta)$, here we use $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. The values of $\sin\alpha$ and $\sin\beta$ are known, however the values of $\cos\alpha$ and $\cos\beta$ need to be found.

Use $\sin^2\alpha+\cos^2\alpha=1$, to find the values of each of the missing cosine values.

For $\cos a$: $\sin^2\alpha+\cos^2\alpha=1$, substituting $\sin\alpha=\frac{12}{13}$ transforms to $\left(\frac{12}{13}\right)^2+\cos^2\alpha=\frac{144}{169}+\cos^2\alpha=1$ or $\cos^2\alpha=\frac{25}{169}\cos\alpha=\pm\frac{5}{13}$, however, since α is in Quadrant II, the cosine is negative, $\cos\alpha=-\frac{5}{13}$.

For $\cos\beta$ use $\sin^2\beta+\cos^2\beta=1$ and substitute $\sin\beta=\frac{3}{5},\left(\frac{3}{5}\right)^2+\cos^2\beta=\frac{9}{25}+\cos^2\beta=1$ or $\cos^2\beta=\frac{16}{25}$ and $\cos\beta=\pm\frac{4}{5}$ and since β is in Quadrant I, $\cos\beta=\frac{4}{5}$

Now the sum formula for the sine of two angles can be found:

$$\begin{aligned}
\sin(\alpha+\beta) &= \frac{12}{13}\times\frac{4}{5}+\left(-\frac{5}{13}\right)\times\frac{3}{5} \text{ or } \frac{48}{65}-\frac{15}{65} \\
\sin(\alpha+\beta) &= \frac{33}{65}
\end{aligned}$$

Sum and Difference Identities: Tangent

To find the sum formula for tangent:

$$\begin{aligned}
\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} && \text{Using } \tan\theta = \frac{\sin\theta}{\cos\theta} \\
&= \frac{\sin a\cos b + \sin b\cos a}{\cos a\cos b - \sin a\sin b} && \text{Substituting the sum formulas for sine and cosine} \\
&= \frac{\frac{\sin a\cos b+\sin b\cos a}{\cos a\cos b}}{\frac{\cos a\cos b-\sin a\sin b}{\cos a\cos b}} && \text{Divide both the numerator and the denominator by } \cos a\cos b \\
&= \frac{\frac{\sin a\cos b}{\cos a\cos b} + \frac{\sin b\cos a}{\cos a\cos b}}{\frac{\cos a\cos b}{\cos a\cos b} - \frac{\sin a\sin b}{\cos a\cos b}} && \text{Reduce each of the fractions} \\
&= \frac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a\sin b}{\cos a\cos b}} && \text{Substitute } \frac{\sin\theta}{\cos\theta} = \tan\theta \\
\tan(a+b) &= \frac{\tan a + \tan b}{1 - \tan a\tan b} && \text{Sum formula for tangent}
\end{aligned}$$

In conclusion, $\tan(a+b) = \frac{\tan a+\tan b}{1-\tan a\tan b}$. Substituting $-b$ for b in the above results in the difference formula for tangent:

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\tan b}$$

Example 7: Find the exact value of $\tan 285^\circ$.

Solution: Use the difference formula for tangent, with $285^\circ = 330^\circ - 45^\circ$

$$\begin{aligned}
\tan(330^\circ - 45^\circ) &= \frac{\tan 330^\circ - \tan 45^\circ}{1 + \tan 330^\circ \tan 45^\circ} \\
&= \frac{-\frac{\sqrt{3}}{3} - 1}{1 - \frac{\sqrt{3}}{3}\cdot 1} = \frac{-3-\sqrt{3}}{3-\sqrt{3}} \\
&= \frac{-3-\sqrt{3}}{3-\sqrt{3}} \cdot \frac{3+\sqrt{3}}{3+\sqrt{3}} \\
&= \frac{-9-6\sqrt{3}-3}{9-3} \\
&= \frac{-12-6\sqrt{3}}{6} \\
&= -2-\sqrt{3}
\end{aligned}$$

To verify this on the calculator, $\tan 285^\circ = -3.732$ and $-2-\sqrt{3} = -3.732$.

Using the Sum and Difference Identities to Verify Other Identities

Example 8: Verify the identity $\frac{\cos(x-y)}{\sin x\sin y} = \cot x\cot y + 1$

$$\begin{aligned}
\cot x \cot y + 1 &= \frac{\cos(x-y)}{\sin x \sin y} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + \frac{\sin x \sin y}{\sin x \sin y} && \text{Expand using the cosine difference formula.} \\
&= \frac{\cos x \cos y}{\sin x \sin y} + 1 \\
\cot x \cot y + 1 &= \cot x \cot y + 1 && \text{cotangent equals cosine over sine}
\end{aligned}$$

Example 9: Show $\cos(a+b)\cos(a-b) = \cos^2 a - \sin^2 b$

Solution: First, expand left hand side using the sum and difference formulas:

$$\begin{aligned}
\cos(a+b)\cos(a-b) &= (\cos a \cos b - \sin a \sin b)(\cos a \cos b + \sin a \sin b) \\
&= \cos^2 a \cos^2 b - \sin^2 a \sin^2 b \rightarrow \text{FOIL, middle terms cancel out}
\end{aligned}$$

Substitute $(1-\sin^2 b)$ for $\cos^2 b$ and $(1-\cos^2 a)$ for $\sin^2 a$ and simplify.

$$\cos^2 a(1-\sin^2 b) - \sin^2 b(1-\cos^2 a)$$
$$\cos^2 a - \cos^2 a \sin^2 b - \sin^2 b + \cos^2 a \sin^2 b$$
$$\cos^2 a - \sin^2 b$$

Solving Equations with the Sum and Difference Formulas

Just like the section before, we can incorporate all of the sum and difference formulas into equations and solve for values of x. In general, you will apply the formula *before* solving for the variable. Typically, the goal will be to isolate $\sin x, \cos x$, or $\tan x$ and then apply the inverse. Remember, that you may have to use the identities in addition to the formulas seen in this section to solve an equation.

Example 10: Solve $3\sin(x-\pi) = 3$ in the interval $[0, 2\pi)$.

Solution: First, get $\sin(x-\pi)$ by itself, by dividing both sides by 3.

$$\begin{aligned}
\frac{3\sin(x-\pi)}{3} &= \frac{3}{3} \\
\sin(x-\pi) &= 1
\end{aligned}$$

Now, expand the left side using the sine difference formula.

$$\begin{aligned}
\sin x \cos \pi - \cos x \sin \pi &= 1 \\
\sin x(-1) - \cos x(0) &= 1 \\
-\sin x &= 1 \\
\sin x &= -1
\end{aligned}$$

The $\sin x = -1$ when x is $\frac{3\pi}{2}$.

Example 11: Find all the solutions for $2\cos^2\left(x+\frac{\pi}{2}\right) = 1$ in the interval $[0, 2\pi)$.

Solution: Get the $\cos^2\left(x+\frac{\pi}{2}\right)$ by itself and then take the square root.

$$2\cos^2\left(x+\frac{\pi}{2}\right)=1$$
$$\cos^2\left(x+\frac{\pi}{2}\right)=\frac{1}{2}$$
$$\cos\left(x+\frac{\pi}{2}\right)=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

Now, use the cosine sum formula to expand and solve.

$$\cos x\cos\frac{\pi}{2}-\sin x\sin\frac{\pi}{2}=\frac{\sqrt{2}}{2}$$
$$\cos x(0)-\sin x(1)=\frac{\sqrt{2}}{2}$$
$$-\sin x=\frac{\sqrt{2}}{2}$$
$$\sin x=-\frac{\sqrt{2}}{2}$$

The $\sin x=-\frac{\sqrt{2}}{2}$ is in Quadrants III and IV, so $x=\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.

Points to Consider

- What are the angles that have 15° and 75° as reference angles?
- Are the only angles that we can find the exact sine, cosine, or tangent values for, multiples of $\frac{\pi}{12}$? (Recall that $\frac{\pi}{2}$ would be $6\cdot\frac{\pi}{12}$, making it a multiple of $\frac{\pi}{12}$)

Review Questions

1. Find the exact value for:
   (a) $\cos\frac{5\pi}{12}$
   (b) $\cos\frac{7\pi}{12}$
   (c) $\sin 345^\circ$
   (d) $\tan 75^\circ$
   (e) $\cos 345^\circ$
   (f) $\sin\frac{17\pi}{12}$
2. If $\sin y=\frac{12}{13}$, y is in quad II, and $\sin z=\frac{3}{5}$, z is in quad I find $\cos(y-z)$
3. If $\sin y=-\frac{5}{13}$, y is in quad III, and $\sin z=\frac{4}{5}$, z is in quad II find $\sin(y+z)$
4. Simplify:
   (a) $\cos 80^\circ\cos 20^\circ+\sin 80^\circ\sin 20^\circ$
   (b) $\sin 25^\circ\cos 5^\circ+\cos 25^\circ\sin 5^\circ$
5. Prove the identity: $\frac{\cos(m-n)}{\sin m\cos n}=\cot m+\tan n$
6. Simplify $\cos(\pi+\theta)=-\cos\theta$
7. Verify the identity: $\sin(a+b)\sin(a-b)=\cos^2 b-\cos^2 a$
8. Simplify $\tan(\pi+\theta)$
9. Verify that $\sin\frac{\pi}{2}=1$, using the sine sum formula.
10. Reduce the following to a single term: $\cos(x+y)\cos y+\sin(x+y)\sin y$.

11. Prove $\frac{\cos(c+d)}{\cos(c-d)} = \frac{1-\tan c \tan d}{1+\tan c \tan d}$
12. Find all solutions to $2\cos^2\left(x+\frac{\pi}{2}\right) = 1$, when x is between $[0, 2\pi)$.
13. Solve for all values of x between $[0, 2\pi)$ for $2\tan^2\left(x+\frac{\pi}{6}\right) - 1 = 7$.
14. Find all solutions to $\sin\left(x+\frac{\pi}{6}\right) = \sin\left(x-\frac{\pi}{4}\right)$, when x is between $[0, 2\pi)$.

Review Answers

1. (a)

$$\cos\frac{5\pi}{12} = \cos\left(\frac{2\pi}{12}+\frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(b)

$$\cos\frac{7\pi}{12} = \cos\left(\frac{4\pi}{12}+\frac{3\pi}{12}\right) = \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

(c)

$$\sin 345^\circ = \sin(300^\circ + 45^\circ) = \sin 300^\circ \cos 45^\circ + \cos 300^\circ \sin 45^\circ$$
$$= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(d)

$$\tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$
$$= \frac{1+\frac{\sqrt{3}}{3}}{1-1\cdot\frac{\sqrt{3}}{3}} = \frac{\frac{3+\sqrt{3}}{3}}{\frac{3-\sqrt{3}}{3}} = \frac{3+\sqrt{3}}{3-\sqrt{3}}\cdot\frac{3+\sqrt{3}}{3+\sqrt{3}} = \frac{9+6\sqrt{3}+3}{9-3} = \frac{12+6\sqrt{3}}{6} = 2+\sqrt{3}$$

(e)

$$\cos 345^\circ = \cos(315^\circ + 30^\circ) = \cos 315^\circ \cos 30^\circ - \sin 315^\circ \sin 30^\circ$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(f)

$$\sin\frac{17\pi}{12} = \sin\left(\frac{9\pi}{12}+\frac{8\pi}{12}\right) = \sin\left(\frac{3\pi}{4}+\frac{2\pi}{3}\right) = \sin\frac{3\pi}{4}\cos\frac{2\pi}{3} + \cos\frac{3\pi}{4}\sin\frac{2\pi}{3}$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

2. If $\sin y = \frac{12}{13}$ and in Quadrant II, then by the Pythagorean Theorem $\cos y = -\frac{5}{13}(12^2 + b^2 = 13^2)$. And, if $\sin z = \frac{3}{5}$ and in Quadrant I, then by the Pythagorean Theorem $\cos z = \frac{4}{5}(a^2 + 3^2 = 5^2)$. So, to find $\cos(y - z) = \cos y \cos z + \sin y \sin z$ and $= -\frac{5}{13}\cdot\frac{4}{5} + \frac{12}{13}\cdot\frac{3}{5} = -\frac{20}{65} + \frac{36}{65} = \frac{16}{65}$
3. If $\sin y = -\frac{5}{13}$ and in Quadrant III, then cosine is also negative. By the Pythagorean Theorem, the second leg is $12(5^2 + b^2 = 13^2)$, so $\cos y = -\frac{12}{13}$. If the $\sin z = \frac{4}{5}$ and in Quadrant II, then the cosine is also negative. By the Pythagorean Theorem, the second leg is $3(4^2 + b^2 = 5^2)$, so $\cos = -\frac{3}{5}$. To find $\sin(y + z)$, plug this information into the sine sum formula.

$$\sin(y+z) = \sin y \cos z + \cos y \sin z$$
$$= -\frac{5}{13}\cdot -\frac{3}{5} + -\frac{12}{13}\cdot\frac{4}{5} = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}$$

4. (a) This is the cosine difference formula, so: $\cos 80^\circ \cos 20^\circ + \sin 80^\circ 20^\circ = \cos(80^\circ - 20^\circ) = \cos 60^\circ = \frac{1}{2}$

(b) This is the expanded sine sum formula, so: $\sin 25^\circ \cos 5^\circ + \cos 25^\circ \sin 5^\circ = \sin(25^\circ + 5^\circ) = \sin 30^\circ = \frac{1}{2}$

5. Step 1: Expand using the cosine sum formula and change everything into sine and cosine

$$\frac{\cos(m-n)}{\sin m \cos n} = \cot m + \tan n$$
$$\frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n} = \frac{\cos m}{\sin m} + \frac{\sin n}{\cos n}$$

Step 2: Find a common denominator for the right hand side.

$$= \frac{\cos m \cos n + \sin m \sin n}{\sin m \cos n}$$

The two sides are the same, thus they are equal to each other and the identity is true.

6. $\cos(\pi + \theta) = \cos \pi \cos \theta - \sin \pi \sin \theta = -\cos \theta$

7. Step 1: Expand $\sin(a+b)$ and $\sin(a-b)$ using the sine sum and difference formulas. $\sin(a+b)\sin(a-b) = \cos^2 b - \cos^2 a$ $(\sin a \cos b + \cos a \sin b)(\sin a \cos b - \cos a \sin b)$

Step 2: FOIL and simplify.

$$\sin^2 a \cos^2 b - \sin a \cos a \sin b \cos b + \sin a \sin b \cos a \cos b - \cos^2 a \sin^2 b \sin^2 a \cos^2 b - \cos a^2 \sin^2 b$$

Step 3: Substitute $(1 - \cos^2 a)$ for $\sin^2 a$ and $(1 - \cos^2 b)$ for $\sin^2 b$, distribute and simplify.

$$(1 - \cos^2 a)\cos^2 b - \cos a^2 (1 - \cos^2 b)$$
$$\cos^2 b - \cos^2 a \cos^2 b - \cos^2 a + \cos^2 a \cos^2 b$$
$$\cos^2 b - \cos^2 a$$

8. $\tan(\pi + \theta) = \frac{\tan \pi + \tan \theta}{1 - \tan \pi \tan \theta} = \frac{\tan \theta}{1} = \tan \theta$

9. $\sin \frac{\pi}{2} = \sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \sin \frac{\pi}{4} \cos \frac{\pi}{4} - \cos \frac{\pi}{4} \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{2}{4} - \frac{2}{4} = 0$ This could also be verified by using $60^\circ + 30^\circ$

10. Step 1: Expand using the cosine and sine sum formulas.

$$\cos(x+y)\cos y + \sin(x+y)\sin y = (\cos x \cos y - \sin x \sin y)\cos y + (\sin x \cos y + \cos x \sin y)\sin y$$

Step 2: Distribute $\cos y$ and $\sin y$ and simplify.

$$= \cos x \cos^2 y - \sin x \sin y \cos y + \sin x \sin y \cos y + \cos x \sin^2 y$$
$$= \cos x \cos^2 y + \cos x \sin^2 y$$
$$= \cos x \underbrace{(\cos^2 y + \sin^2 y)}_{1}$$
$$= \cos x$$

11. Step 1: Expand left hand side using the sum and difference formulas

$$\frac{\cos(c+d)}{\cos(c-d)} = \frac{1 - \tan c \tan d}{1 + \tan c \tan d}$$
$$\frac{\cos c \cos d - \sin c \sin d}{\cos c \cos d + \sin c \sin d} = \frac{1 - \tan c \tan d}{1 + \tan c \tan d}$$

Step 2: Divide each term on the left side by $\cos c \cos d$ and simplify

$$\frac{\frac{\cos c \cos d}{\cos c \cos d} - \frac{\sin c \sin d}{\cos c \cos d}}{\frac{\cos c \cos d}{\cos c \cos d} - \frac{\sin c \sin d}{\cos c \cos d}} = \frac{1 - \tan c \tan d}{1 + \tan c \tan d}$$
$$\frac{1 - \tan c \tan d}{1 + \tan c \tan d} = \frac{1 - \tan c \tan d}{1 + \tan c \tan d}$$

12. To find all the solutions, between $[0, 2\pi)$, we need to expand using the sum formula and isolate the $\cos x$.

$$\begin{aligned}
2\cos^2\left(x+\frac{\pi}{2}\right) &= 1\\
\cos^2\left(x+\frac{\pi}{2}\right) &= \frac{1}{2}\\
\cos\left(x+\frac{\pi}{2}\right) &= \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}\\
\cos x\cos\frac{\pi}{2} - \sin x\sin\frac{\pi}{2} &= \frac{\sqrt{2}}{2}\\
\cos x\cdot 0 - \sin x\cdot 1 &= \frac{\sqrt{2}}{2}\\
-\sin x &= \frac{\sqrt{2}}{2}\\
\sin x &= -\frac{\sqrt{2}}{2}
\end{aligned}$$

This is true when $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

13. First, solve for tan().

$$\begin{aligned}
2\tan^2\left(x+\frac{\pi}{6}\right)-1 &= 7\\
2\tan^2\left(x+\frac{\pi}{6}\right) &= 6\\
\tan^2\left(x+\frac{\pi}{6}\right) &= 3\\
\tan\left(x+\frac{\pi}{6}\right) &= \sqrt{3}
\end{aligned}$$

Now, use the tangent sum formula to expand.

$$\begin{aligned}
\frac{\tan x + \tan\frac{\pi}{6}}{1-\tan x\tan\frac{\pi}{6}} &= \sqrt{3}\\
\tan x + \tan\frac{\pi}{6} &= \sqrt{3}\left(1-\tan x\tan\frac{\pi}{6}\right)\\
\tan x + \frac{\sqrt{3}}{3} &= \sqrt{3} - \sqrt{3}\tan x\cdot\frac{\sqrt{3}}{3}\\
\tan x + \frac{\sqrt{3}}{3} &= \sqrt{3} - \tan x\\
2\tan x &= \frac{2\sqrt{3}}{3}\\
\tan x &= \frac{\sqrt{3}}{3}
\end{aligned}$$

This is true when $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$.

14. To solve, expand each side:

$$\begin{aligned}
\sin\left(x+\frac{\pi}{6}\right) &= \sin x\cos\frac{\pi}{6} + \cos x\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\\
\sin\left(x-\frac{\pi}{4}\right) &= \sin x\cos\frac{\pi}{4} - \cos x\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x
\end{aligned}$$

Set the two sides equal to each other:

$$\begin{aligned}
\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x &= \frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x \\
\sqrt{3}\sin x + \cos x &= \sqrt{2}\sin x - \sqrt{2}\cos x \\
\sqrt{3}\sin x - \sqrt{2}\sin x &= -\cos x - \sqrt{2}\cos x \\
\sin x\left(\sqrt{3} - \sqrt{2}\right) &= \cos x\left(-1 - \sqrt{2}\right) \\
\frac{\sin x}{\cos x} &= \frac{-1 - \sqrt{2}}{\sqrt{3} - \sqrt{2}} \\
\tan x &= \frac{-1 - \sqrt{2}}{\sqrt{3} - \sqrt{2}} \cdot \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}} \\
&= \frac{-\sqrt{3} - \sqrt{2} + \sqrt{6} - 2}{3 - 2} \\
&= -2 + \sqrt{6} - \sqrt{3} - \sqrt{2}
\end{aligned}$$

As a decimal, this is -2.69677, so $\tan^{-1}(-2.69677) = x, x = 290.35^\circ$ and 110.35°.

3.5 Double Angle Identities

Learning Objectives

- Use the double angle identities to solve other identities.
- Use the double angle identities to solve equations.

Deriving the Double Angle Identities

One of the formulas for calculating the sum of two angles is:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

If α and β are both the same angle in the above formula, then

$$\begin{aligned}
\sin(\alpha + \alpha) &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha
\end{aligned}$$

This is the double angle formula for the sine function. The same procedure can be used in the sum formula for cosine, start with the sum angle formula:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

If α and β are both the same angle in the above formula, then

$$\begin{aligned}
\cos(\alpha + \alpha) &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha
\end{aligned}$$

This is one of the double angle formulas for the cosine function. Two more formulas can be derived by using the Pythagorean Identity, $\sin^2\alpha + \cos^2\alpha = 1$.

$\sin^2\alpha = 1-\cos^2\alpha$ and likewise $\cos^2\alpha = 1-\sin^2\alpha$

Using $\sin^2\alpha = 1-\cos^2\alpha$:

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= \cos^2\alpha - (1-\cos^2\alpha) \\ &= \cos^2\alpha - 1 + \cos^2\alpha \\ &= 2\cos^2\alpha - 1\end{aligned}$$

Using $\cos^2\alpha = 1-\sin^2\alpha$:

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= (1-\sin^2\alpha) - \sin^2\alpha \\ &= 1 - \sin^2\alpha - \sin^2\alpha \\ &= 1 - 2\sin^2\alpha\end{aligned}$$

Therefore, the double angle formulas for $\cos 2a$ are:

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ \cos 2\alpha &= 2\cos^2\alpha - 1 \\ \cos 2\alpha &= 1 - 2\sin^2\alpha\end{aligned}$$

Finally, we can calculate the double angle formula for tangent, using the tangent sum formula:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

If α and β are both the same angle in the above formula, then

$$\begin{aligned}\tan(\alpha+\alpha) &= \frac{\tan\alpha + \tan\alpha}{1-\tan\alpha\tan\alpha} \\ \tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}\end{aligned}$$

Applying the Double Angle Identities

Example 1: If $\sin a = \frac{5}{13}$ and a is in Quadrant II, find $\sin 2a \cos 2a$, and $\tan 2a$.

Solution: To use $\sin 2a = 2\sin a \cos a$, the value of $\cos a$ must be found first.

$$\begin{aligned}&= \cos^2 a + \sin^2 a = 1 \\ &= \cos^2 a + \left(\frac{5}{13}\right)^2 = 1 \\ &= \cos^2 a + \frac{25}{169} = 1 \\ &= \cos^2 a = \frac{144}{169}, \cos a = \pm\frac{12}{13}\end{aligned}$$

.

However since a is in Quadrant II, $\cos a$ is negative or $\cos a = -\frac{12}{13}$.

$$\sin 2a = 2\sin a\cos a = 2\left(\frac{5}{13}\right)\times\left(-\frac{12}{13}\right) = \sin 2a = -\frac{120}{169}$$

For $\cos 2a$, use $\cos(2a) = \cos^2 a - \sin^2 a$

$$\begin{aligned}\cos(2a) &= \left(-\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 \text{ or } \frac{144-25}{169} \\ \cos(2a) &= \frac{119}{169}\end{aligned}$$

For $\tan 2a$, use $\tan 2a = \frac{2\tan a}{1-\tan^2 a}$. From above, $\tan a = \frac{\frac{5}{13}}{-\frac{12}{13}} = -\frac{5}{12}$.

$$\tan(2a) = \frac{2 \cdot \frac{-5}{12}}{1 - \left(\frac{-5}{12}\right)^2} = \frac{\frac{-5}{6}}{1 - \frac{25}{144}} = \frac{\frac{-5}{6}}{\frac{119}{144}} = -\frac{5}{6} \cdot \frac{144}{119} = -\frac{120}{119}$$

Example 2: Find $\cos 4\theta$.

Solution: Think of $\cos 4\theta$ as $\cos(2\theta + 2\theta)$.

$$\cos 4\theta = \cos(2\theta + 2\theta) = \cos 2\theta \cos 2\theta - \sin 2\theta \sin 2\theta = \cos^2 2\theta - \sin^2 2\theta$$

Now, use the double angle formulas for both sine and cosine. For cosine, you can pick which formula you would like to use. In general, because we are proving a cosine identity, stay with cosine.

$$\begin{aligned}
&= (2\cos^2\theta - 1)^2 - (2\sin\theta\cos\theta)^2 \\
&= 4\cos^4\theta - 4\cos^2\theta + 1 - 4\sin^2\theta\cos^2\theta \\
&= 4\cos^4\theta - 4\cos^2\theta + 1 - 4(1-\cos^2\theta)\cos^2\theta \\
&= 4\cos^4\theta - 4\cos^2\theta + 1 - 4\cos^2\theta + 4\cos^4\theta \\
&= 8\cos^4\theta - 8\cos^2\theta + 1
\end{aligned}$$

Example 3: If $\cot x = \frac{4}{3}$ and x is an acute angle, find the exact value of $\tan 2x$.

Solution: Cotangent and tangent are reciprocal functions, $\tan x = \frac{1}{\cot x}$ and $\tan x = \frac{3}{4}$.

$$\begin{aligned}
\tan 2x &= \frac{2\tan x}{1 - \tan^2 x} \\
&= \frac{2 \cdot \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} \\
&= \frac{\frac{3}{2}}{1 - \frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} \\
&= \frac{3}{2} \cdot \frac{16}{7} = \frac{24}{7}
\end{aligned}$$

Example 4: Given $\sin(2x) = \frac{2}{3}$ and x is in Quadrant I, find the value of $\sin x$.

Solution: Using the double angle formula, $\sin 2x = 2\sin x \cos x$. Because we do not know $\cos x$, we need to solve for $\cos x$ in the Pythagorean Identity, $\cos x = \sqrt{1 - \sin^2 x}$. Substitute this into our formula and solve for $\sin x$.

$$\begin{aligned}
\sin 2x &= 2\sin x \cos x \\
\frac{2}{3} &= 2\sin x \sqrt{1 - \sin^2 x} \\
\left(\frac{2}{3}\right)^2 &= \left(2\sin x \sqrt{1 - \sin^2 x}\right)^2 \\
\frac{4}{9} &= 4\sin^2 x(1 - \sin^2 x) \\
\frac{4}{9} &= 4\sin^2 x - 4\sin^4 x
\end{aligned}$$

At this point we need to get rid of the fraction, so multiply both sides by the reciprocal.

$$\frac{9}{4}\left(\frac{4}{9} = 4\sin^2 x - 4\sin^4 x\right)$$
$$1 = 9\sin^2 x - 9\sin^4 x$$
$$0 = 9\sin^4 x - 9\sin^2 x + 1$$

Now, this is in the form of a quadratic equation, even though it is a quartic. Set $a = \sin^2 x$, making the equation $9a^2 - 9a + 1 = 0$. Once we have solved for a, then we can substitute $\sin^2 x$ back in and solve for x. In the Quadratic Formula, $a = 9, b = -9, c = 1$.

$$\frac{9 \pm \sqrt{(-9)^2 - 4(9)(1)}}{2(9)} = \frac{9 \pm \sqrt{81 - 36}}{18} = \frac{9 \pm \sqrt{45}}{18} = \frac{9 \pm 3\sqrt{5}}{18} = \frac{3 \pm \sqrt{5}}{6}$$

So, $a = \frac{3+\sqrt{5}}{6} \approx 0.873$ or $\frac{3-\sqrt{5}}{6} \approx .1273$. This means that $\sin^2 x \approx 0.873$ or $.1273$ so $\sin x \approx 0.934$ or $\sin x \approx .357$.

Example 5: Prove $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta}$

Solution: Substitute in the double angle formulas. Use $\cos 2\theta = 1 - 2\sin^2\theta$, since it will produce only one term in the numerator.

$$\begin{aligned}\tan\theta &= \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}\\ &= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}\\ &= \frac{\sin\theta}{\cos\theta}\\ &= \tan\theta\end{aligned}$$

Solving Equations with Double Angle Identities

Much like the previous sections, these problems all involve similar steps to solve for the variable. Isolate the trigonometric function, using any of the identities and formulas you have accumulated thus far.

Example 6: Find all solutions to the equation $\sin 2x = \cos x$ in the interval $[0, 2\pi]$

Solution: Apply the double angle formula $\sin 2x = 2\sin x\cos x$

$$\begin{aligned}2\sin x\cos x &= \cos x\\ 2\sin x\cos x - \cos x &= \cos x - \cos x\\ 2\sin x\cos x - \cos x &= 0\\ \cos x(2\sin x - 1) &= 0 \text{ Factor out } \cos x\\ \text{Then } \cos x &= 0 \text{ or } 2\sin x - 1 = 0\\ \cos x &= 0 \text{ or } 2\sin x - 1 + 1 = 0 + 1\\ \frac{2}{2}\sin x &= \frac{1}{2}\\ \sin x &= \frac{1}{2}\end{aligned}$$

The values for $\cos x = 0$ in the interval $[0, 2\pi]$ are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ and the values for $\sin x = \frac{1}{2}$ in the interval $[0, 2\pi]$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Thus, there are four solutions.

Example 7: Solve the trigonometric equation $\sin 2x = \sin x$ such that $(-\pi \leq x < \pi)$

Solution: Using the sine double angle formula:

$$\sin 2x = \sin x$$
$$2\sin x\cos x \;= \sin x$$
$$2\sin x\cos x - \sin x = 0$$
$$\sin x(2\cos x - 1) = 0$$

$$\downarrow \qquad \searrow$$

$$2\cos x - 1 = 0$$
$$2\cos x = 1$$
$$\sin x = 0$$
$$x = 0, -\pi \qquad \cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, -\frac{\pi}{3}$$

Example 8: Find the exact value of $\cos 2x$ given $\cos x = -\frac{13}{14}$ if x is in the second quadrant.

Solution: Use the double-angle formula with cosine only.

$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 2\left(-\frac{13}{14}\right)^2 - 1$$
$$\cos 2x = 2\left(\frac{169}{196}\right) - 1$$
$$\cos 2x = \left(\frac{338}{196}\right) - 1$$
$$\cos 2x = \frac{338}{196} - \frac{196}{196}$$
$$\cos 2x = \frac{142}{196} = \frac{71}{98}$$

Example 9: Solve the trigonometric equation $4\sin\theta\cos\theta = \sqrt{3}$ over the interval $[0, 2\pi)$.

Solution: Pull out a 2 from the left-hand side and this is the formula for $\sin 2x$.

$$4\sin\theta\cos\theta = \sqrt{3}$$
$$2(2\sin\theta\cos\theta) = \sqrt{3}$$
$$2(2\sin\theta\cos\theta) = 2\sin 2\theta$$
$$2\sin 2\theta = \sqrt{3}$$
$$\sin 2\theta = \frac{\sqrt{3}}{2}$$

The solutions for 2θ are $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$, dividing each of these by 2, we get the solutions for θ, which are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{8\pi}{6}$.

Points to Consider

- Are there similar formulas that can be derived for other angles?
- Can technology be used to either solve these trigonometric equations or to confirm the solutions?

Review Questions

1. If $\sin x = \frac{4}{5}$ and x is in Quad II, find the exact values of $\cos 2x, \sin 2x$ and $\tan 2x$
2. Find the exact value of $\cos^2 15° - \sin^2 15°$
3. Verify the identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
4. Verify the identity: $\sin 2t - \tan t = \tan t \cos 2t$
5. If $\sin x = -\frac{9}{41}$ and x is in Quad III, find the exact values of $\cos 2x, \sin 2x$ and $\tan 2x$
6. Find all solutions to $\sin 2x + \sin x = 0$ if $0 \le x < 2\pi$
7. Find all solutions to $\cos^2 x - \cos 2x = 0$ if $0 \le x < 2\pi$
8. If $\tan x = \frac{3}{4}$ and $0° < x < 90°$, use the double angle formulas to determine each of the following:
   (a) $\tan 2x$
   (b) $\sin 2x$
   (c) $\cos 2x$
9. Use the double angle formulas to prove that the following equations are identities.
   (a) $2\csc 2x = \csc^2 x \tan x$
   (b) $\cos^4\theta - \sin^4\theta = \cos 2\theta$
   (c) $\frac{\sin 2x}{1+\cos 2x} = \tan x$
10. Solve the trigonometric equation $\cos 2x - 1 = \sin^2 x$ such that $[0, 2\pi)$
11. Solve the trigonometric equation $\cos 2x = \cos x$ such that $0 \le x < \pi$
12. Prove $2\csc 2x \tan x = \sec^2 x$.
13. Solve $\sin 2x - \cos 2x = 1$ for x in the interval $[0, 2\pi)$.
14. Solve the trigonometric equation $\sin^2 x - 2 = \cos 2x$ such that $0 \le x < 2\pi$

Review Answers

1. If $\sin x = \frac{4}{5}$ and in Quadrant II, then cosine and tangent are negative. Also, by the Pythagorean Theorem, the third side is $3(b = \sqrt{5^2 - 4^2})$. So, $\cos x = -\frac{3}{5}$ and $\tan x = -\frac{4}{3}$. Using this, we can find $\sin 2x, \cos 2x$, and $\tan 2x$.

$$\begin{aligned} \sin 2x &= 2\sin x \cos x \\ &= 2 \cdot \frac{4}{5} \cdot -\frac{3}{5} \\ &= -\frac{24}{25} \end{aligned}$$

$$\begin{aligned} \cos 2x &= 1 - \sin^2 x \\ &= 1 - 2 \cdot \left(\frac{4}{5}\right)^2 \\ &= 1 - 2 \cdot \frac{16}{25} \\ &= 1 - \frac{32}{25} \\ &= -\frac{7}{25} \end{aligned}$$

$$\begin{aligned} \tan 2x &= \frac{2\tan x}{1 - \tan^2 x} \\ &= \frac{2 \cdot -\frac{4}{3}}{1 - \left(-\frac{4}{3}\right)^2} \\ &= \frac{-\frac{8}{3}}{1 - \frac{16}{9}} = -\frac{8}{3} \div -\frac{7}{9} \\ &= -\frac{8}{3} \cdot -\frac{9}{7} \\ &= \frac{24}{7} \end{aligned}$$

2. This is one of the forms for $\cos 2x$.

$$\begin{aligned} \cos^2 15° - \sin^2 15° &= \cos(15° \cdot 2) \\ &= \cos 30° \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

3. Step 1: Use the cosine sum formula

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$$

Step 2: Use double angle formulas for $\cos 2\theta$ and $\sin 2\theta$

$$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$

Step 3: Distribute and simplify.

$$\begin{aligned}
&= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\
&= -\cos\theta(-2\cos^2\theta + 2\sin^2\theta + 1) \\
&= -\cos\theta[-2\cos^2\theta + 2(1-\cos^2\theta) + 1] \qquad \rightarrow \text{Substiture } 1-\cos^2\theta \text{ for } \sin^2\theta \\
&= -\cos\theta[-2\cos^2\theta + 2 - 2\cos^2\theta + 1] \\
&= -\cos\theta(-4\cos^2\theta + 3) \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}$$

4. Step 1: Expand $\sin 2t$ using the double angle formula.

$$\begin{aligned}
\sin 2t - \tan t &= \tan t\cos 2t \\
2\sin t\cos t - \tan t &= \tan t\cos 2t
\end{aligned}$$

Step 2: change $\tan t$ and find a common denominator.

$$2\sin t\cos t - \frac{\sin t}{\cos t}$$
$$\frac{2\sin t\cos^2 t - \sin t}{\cos t}$$
$$\frac{\sin t(2\cos^2 t - 1)}{\cos t}$$
$$\frac{\sin t}{\cos t}\cdot(2\cos^2 t - 1)$$
$$\tan t\cos 2t$$

5. If $\sin x = -\frac{9}{41}$ and in Quadrant III, then $\cos x = -\frac{40}{41}$ and $\tan x = \frac{9}{40}$ (Pythagorean Theorem, $b = \sqrt{41^2 - (-9)^2}$). So,

$$\begin{aligned}
\sin 2x &= 2\sin x\cos x \\
&= 2\cdot -\frac{9}{41}\cdot -\frac{40}{41} \\
&= \frac{720}{1681}
\end{aligned}$$

$$\begin{aligned}
\cos 2x &= 2\cos^2 x - 1 \\
&= 2\left(-\frac{40}{41}\right)^2 - 1 \\
&= \frac{3200}{1681} - \frac{1681}{1681} \\
&= \frac{1519}{1681}
\end{aligned}$$

$$\begin{aligned}
\tan 2x &= \frac{\sin 2x}{\cos 2x} \\
&= \frac{\frac{720}{1681}}{\frac{1519}{1681}} \\
&= \frac{720}{1519}
\end{aligned}$$

6. Step 1: Expand $\sin 2x$

$$\begin{aligned}
\sin 2x + \sin x &= 0 \\
2\sin x\cos x + \sin x &= 0 \\
\sin x(2\cos x + 1) &= 0
\end{aligned}$$

Step 2: Separate and solve each for x.

$$\sin x = 0 \qquad x = 0, \pi$$

or

$$2\cos x + 1 = 0 \qquad \cos x = -\frac{1}{2} \qquad x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

7. Expand $\cos 2x$ and simplify

$$\begin{aligned}
\cos^2 x - \cos 2x &= 0 \\
\cos^2 x - (2\cos^2 x - 1) &= 0 \\
-\cos^2 x + 1 &= 0 \\
\cos^2 x &= 1 \\
\cos x &= 1
\end{aligned}$$

$\cos x = 1$ when $x = 0, 2\pi$

8. (a) 3.429
   (b) 0.960
   (c) 0.280
9. (a)

$$\begin{aligned}
2\csc x\ 2x &= \frac{2}{\sin 2x} \\
2\csc x\ 2x &= \frac{2}{2\sin x \cos x} \\
2\csc x\ 2x &= \frac{1}{\sin x \cos x} \\
2\csc x\ 2x &= \left(\frac{\sin x}{\sin x}\right)\left(\frac{1}{\sin x \cos x}\right) \\
2\csc x\ 2x &= \frac{\sin x}{\sin^2 x \cos x} \\
2\csc x\ 2x &= \frac{1}{\sin^2 x} \cdot \frac{\sin x}{\cos x} \\
2\csc x\ 2x &= \csc^2 x \tan x
\end{aligned}$$

   (b)

$$\begin{aligned}
\cos^4 \theta - \sin^4 \theta &= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) \\
\cos^4 - \sin^4 \theta &= 1(\cos^2 \theta - \sin^2 \theta) \\
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
\therefore \cos^4 \theta - \sin^4 \theta &= \cos 2\theta
\end{aligned}$$

   (c)

$$\begin{aligned}
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x \cos x}{1+(1-2\sin^2 x)} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x \cos x}{2-2\sin^2 x} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x \cos x}{2(1-\sin^2 x)} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{2\sin x \cos x}{2\cos^2 x} \\
\frac{\sin 2x}{1+\cos 2x} &= \frac{\sin x}{\cos x} \\
\frac{\sin 2x}{1+\cos 2x} &= \tan x
\end{aligned}$$

10. $\cos 2x - 1 = \sin^2 x$

$$
\begin{aligned}
1-2\sin^2 x &= \sin^2 x\\
1 &= 3\sin^2 x\\
\frac{1}{3} &= \sin^2 x\\
\frac{\sqrt{3}}{3} &= \sin x\\
x &= 35.26^\circ, 144.74^\circ
\end{aligned}
$$

11.

$$
\begin{aligned}
\cos 2x &= \cos x\\
2\cos^2 x - 1 &= \cos x\\
2\cos^2 x - \cos x - 1 &= 0\\
(2\cos x + 1)(\cos x - 1) &= 0
\end{aligned}
$$

$$
\searrow \qquad\qquad \searrow
$$

$$
\begin{aligned}
2\cos x + 1 = 0 \quad &\text{or} \quad \cos x - 1 = 0\\
2\cos x = -1 \quad &\qquad\quad \cos x = 1\\
\cos x = -\frac{1}{2} &
\end{aligned}
$$

12. $\cos x = 1$ when $x = 0$ and $\cos x = -\frac{1}{2}$ when $x = \frac{5\pi}{6}$.

$$
\begin{aligned}
2\csc 2x \tan x &= \sec^2 x\\
\frac{2}{\sin 2x}\cdot\frac{\sin x}{\cos x} &= \frac{1}{\cos^2 x}\\
\frac{2}{2\sin x\cos x}\cdot\frac{\sin x}{\cos x} &= \frac{1}{\cos^2 x}\\
\frac{1}{\cos^2 x} &= \frac{1}{\cos^2 x}
\end{aligned}
$$

13. $\sin 2x - \cos 2x = 1$

$$2\sin x\cos x-(1-2\sin^2 x)=1$$
$$2\sin x\cos x-1+2\sin^2 x=1$$
$$2\sin x\cos x+2\sin^2 x=2$$
$$\sin x\cos x+\sin^2 x=1$$
$$\sin x\cos x=1-\sin^2 x$$
$$\sin x\cos x=\cos^2 x$$
$$\left(\sqrt{1-\cos^2 x}\right)\cos x=\cos^2 x$$
$$\left(1-\cos^2 x\right)\cos^2 x=\cos^4 x$$
$$\cos^2 x-\cos^4 x=\cos^4 x$$
$$\cos^2 x-2\cos^4 x=0$$
$$\cos^2 x(1-2\cos^2 x)=0$$
$$\swarrow \qquad \searrow$$
$$1-2\cos^2 x=0$$
$$\cos^2 x=0 \qquad -2\cos^2 x=-1$$
$$\cos x=0 \quad \text{or} \quad \cos^2 x=\frac{1}{2}$$
$$x=\frac{\pi}{2},\frac{3\pi}{2} \qquad \cos x=\frac{\sqrt{2}}{2}$$
$$x=\frac{\pi}{4},\frac{7\pi}{4}$$

14. Use the double angle identity for $\cos 2x$.

$$\sin^2 x-2=\cos 2x$$
$$\sin^2 x-2=\cos 2x$$
$$\sin^2 x-2=1-2\sin^2 x$$
$$3\sin^2 x=3$$
$$\sin^2 x=1$$
$$\sin x=\pm 1$$
$$x=\frac{\pi}{2},\frac{3\pi}{2}$$

3.6 Half-Angle Identities

Learning Objectives

- Apply the half angle identities to expressions, equations and other identities.
- Use the half angle identities to find the exact value of trigonometric functions for certain angles.

Just as there are double angle identities, there are also half angle identities. For example: $\sin\frac{1}{2}a$ can be found in terms of the angle "a". Recall that $\frac{1}{2}a$ and $\frac{a}{2}$ are the same thing and will be used interchangeably throughout this section.

Deriving the Half Angle Formulas

In the previous lesson, one of the formulas that was derived for the cosine of a double angle is: $\cos 2\theta = 1 - 2\sin^2\theta$. Set $\theta = \frac{\alpha}{2}$, so the equation above becomes $\cos 2\frac{\alpha}{2} = 1 - 2\sin^2\frac{\alpha}{2}$.

Solving this for $\sin\frac{\alpha}{2}$, we get:

$$\begin{aligned}
\cos 2\frac{\alpha}{2} &= 1 - 2\sin^2\frac{\alpha}{2} \\
\cos\alpha &= 1 - 2\sin^2\frac{\alpha}{2} \\
2\sin^2\frac{\alpha}{2} &= 1 - \cos\alpha \\
\sin^2\frac{\alpha}{2} &= \frac{1-\cos\alpha}{2} \\
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}}
\end{aligned}$$

$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or second quadrant.

$\sin\frac{\alpha}{2} = -\sqrt{\frac{1-\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in the third or fourth quadrant.

Example 1: Determine the exact value of $\sin 15^\circ$.

Solution: Using the half angle identity, $\alpha = 30^\circ$, and 15° is located in the first quadrant. Therefore, $\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$.

$$\begin{aligned}
\sin 15^\circ &= \sqrt{\frac{1-\cos 30^\circ}{2}} \\
&= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}}
\end{aligned}$$

Plugging this into a calculator, $\sqrt{\frac{2-\sqrt{3}}{4}} \approx 0.2588$. Using the sine function on your calculator will validate that this answer is correct.

Example 2: Use the half angle identity to find exact value of $\sin 112.5^\circ$

Solution: since $\sin\frac{225^\circ}{2} = \sin 112.5^\circ$, use the half angle formula for sine, where $\alpha = 225^\circ$. In this example, the angle 112.5° is a second quadrant angle, and the sin of a second quadrant angle is positive.

$$\begin{aligned}
\sin 112.5^\circ &= \sin\frac{225^\circ}{2} \\
&= \pm\sqrt{\frac{1-\cos 225^\circ}{2}} \\
&= +\sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}} \\
&= \sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}} \\
&= \sqrt{\frac{2+\sqrt{2}}{4}}
\end{aligned}$$

One of the other formulas that was derived for the cosine of a double angle is:

$\cos 2\theta = 2\cos^2\theta - 1$. Set $\theta = \frac{\alpha}{2}$, so the equation becomes $\cos 2\frac{\alpha}{2} = -1 + 2\cos^2\frac{\alpha}{2}$. Solving this for $\cos\frac{\alpha}{2}$, we get:

$$\begin{aligned}
\cos 2\frac{\alpha}{2} &= 2\cos^2\frac{\alpha}{2} - 1\\
\cos\alpha &= 2\cos^2\frac{\alpha}{2} - 1\\
2\cos^2\frac{\alpha}{2} &= 1 + \cos\alpha\\
\cos^2\frac{\alpha}{2} &= \frac{1+\cos\alpha}{2}\\
\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}}
\end{aligned}$$

$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the first or fourth quadrant.

$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$ if $\frac{\alpha}{2}$ is located in either the second or fourth quadrant.

Example 3: Given that the $\cos\theta = \frac{3}{4}$, and that θ is a fourth quadrant angle, find $\cos\frac{1}{2}\,\theta$

Solution: Because θ is in the fourth quadrant, the half angle would be in the second quadrant, making the cosine of the half angle negative.

$$\begin{aligned}
\cos\frac{\theta}{2} &= -\sqrt{\frac{1+\cos\theta}{2}}\\
&= -\sqrt{\frac{1+\frac{3}{4}}{2}}\\
&= -\sqrt{\frac{\frac{7}{4}}{2}}\\
&= -\sqrt{\frac{7}{8}} = -\frac{\sqrt{7}}{2\sqrt{2}} = -\frac{\sqrt{14}}{4}
\end{aligned}$$

Example 4: Use the half angle formula for the cosine function to prove that the following expression is an identity: $2\cos^2\frac{x}{2} - \cos x = 1$

Solution: Use the formula $\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$ and substitute it on the left-hand side of the expression.

$$\begin{aligned}
2\left(\sqrt{\frac{1+\cos\theta}{2}}\right)^2 - \cos\theta &= 1\\
2\left(\frac{1+\cos\theta}{2}\right) - \cos\theta &= 1\\
1 + \cos\theta - \cos\theta &= 1\\
1 &= 1
\end{aligned}$$

The half angle identity for the tangent function begins with the reciprocal identity for tangent.

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

The half angle formulas for sine and cosine are then substituted into the identity.

$$\tan\frac{\alpha}{2} = \frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}} = \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}$$

At this point, you can multiply by either $\frac{\sqrt{1-\cos\alpha}}{\sqrt{1-\cos\alpha}}$ or $\frac{\sqrt{1+\cos\alpha}}{\sqrt{1+\cos\alpha}}$. We will show both, because they produce different answers.

$$= \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} \cdot \frac{\sqrt{1-\cos\alpha}}{\sqrt{1-\cos\alpha}} = \frac{1-\cos\alpha}{\sqrt{1-\cos^2\alpha}} = \frac{1-\cos\alpha}{\sqrt{\sin^2\alpha}} = \frac{1-\cos\alpha}{\sin\alpha}$$

or

$$= \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}} \cdot \frac{\sqrt{1+\cos\alpha}}{\sqrt{1+\cos\alpha}} = \frac{\sqrt{1-\cos^2\alpha}}{1+\cos\alpha} = \frac{\sqrt{\sin^2\alpha}}{1+\cos\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

So, the two half angle identities for tangent are $\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$ and $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$.

Example 5: Use the half-angle identity for tangent to determine an exact value for $\tan\frac{7\pi}{12}$.

Solution:

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$
$$\tan\frac{7\pi}{12} = \frac{1-\cos\frac{7\pi}{6}}{\sin\frac{7\pi}{6}}$$
$$\tan\frac{7\pi}{12} = \frac{1+\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$$
$$\tan\frac{7\pi}{12} = -2-\sqrt{3}$$

Example 6: Prove the following identity: $\tan x = \frac{1-\cos 2x}{\sin 2x}$

Solution: Substitute the double angle formulas for $\cos 2x$ and $\sin 2x$.

$$\tan x = \frac{1-\cos 2x}{\sin 2x} = \frac{1-(1-2\sin^2 x)}{2\sin x\cos x} = \frac{1-1+2\sin^2 x}{2\sin x\cos x} = \frac{2\sin^2 x}{2\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x$$

Solving Trigonometric Equations Using Half Angle Formulas

Example 7: Solve the trigonometric equation $\sin^2\theta = 2\sin^2\frac{\theta}{2}$ over the interval $[0, 2\pi)$.

Solution:

$$\begin{aligned}
\sin^2\theta &= 2\sin^2\frac{\theta}{2} \\
\sin^2\theta &= 2\left(\frac{1-\cos\theta}{2}\right) && \text{Half angle identity} \\
1-\cos^2\theta &= 1-\cos\theta && \text{Pythagorean identity} \\
\cos\theta - \cos^2\theta &= 0 \\
\cos\theta(1-\cos\theta) &= 0
\end{aligned}$$

Then $\cos\theta = 0$ or $1-\cos\theta = 0$, which is $\cos\theta = 1$.

$\theta = 2\pi$ or $\theta = 0$.

Points to Consider

- Can you derive a third or fourth angle formula?
- How do $\frac{1}{2}\sin x$ and $\sin\frac{1}{2}x$ differ? Is there a formula for $\frac{1}{2}\sin x$?

Review Questions

1. Find the exact value of:
   (a) $\cos 112.5^\circ$
   (b) $\sin 105^\circ$
   (c) $\tan\frac{7\pi}{8}$
   (d) $\tan\frac{\pi}{8}$
   (e) $\sin 67.5^\circ$
   (f) $\tan 165^\circ$
2. If $\sin\theta = \frac{7}{25}$ and θ is in Quad II, find $\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, \tan\frac{\theta}{2}$
3. Prove the identity: $\tan\frac{b}{2} = \frac{\sec b}{\sec b\csc b + \csc b}$
4. Verify the identity: $\cot\frac{c}{2} = \frac{\sin c}{1-\cos c}$
5. Prove that $\sin x\tan\frac{\pi}{2} + 2\cos x = 2\cos^2\frac{\pi}{2}$
6. If $\sin u = -\frac{8}{13}$, find $\cos\frac{u}{2}$
7. Solve $2\cos^2\frac{x}{2} = 1$ for $0 \le x < 2\pi$
8. Solve $\tan\frac{a}{2} = 4$ for $0 \le x < 2\pi$
9. Solve the trigonometric equation $\cos\frac{x}{2} = 1 + \cos x$ such that $0 \le x < 2\pi$.
10. Prove $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$.

Review Answers

1. (a)

$$\begin{aligned}
\cos 112.5^\circ = \cos\frac{225^\circ}{2} &= -\sqrt{\frac{1+\cos 225^\circ}{2}} \\
&= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{2}}{4}} = -\frac{\sqrt{2-\sqrt{2}}}{2}
\end{aligned}$$

(b)

$$\sin 105^\circ = \sin \frac{210^\circ}{2} = \sqrt{\frac{1-\cos 210^\circ}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

(c)

$$\tan \frac{7\pi}{8} = \tan \frac{1}{2} \cdot \frac{7\pi}{4} = \frac{1-\cos \frac{7\pi}{4}}{\sin \frac{7\pi}{4}}$$

$$= \frac{1-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = \frac{\frac{2-\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -\frac{2-\sqrt{2}}{\sqrt{2}} = \frac{-2\sqrt{2}+2}{2} = -\sqrt{2}+1$$

(d) $\tan \frac{\pi}{8} = \tan \frac{1}{2} \cdot \frac{\pi}{4} = \frac{1-\cos \frac{\pi}{4}}{\sin \frac{\pi}{4}} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\frac{2-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}-2}{2} = \sqrt{2}-1$

(e) $\sin 67.5^\circ = \sin \frac{135^\circ}{2} = \sqrt{\frac{1-\cos 135^\circ}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

(f) $\tan 165^\circ = \tan \frac{330^\circ}{2} = \frac{1-\cos 330^\circ}{\sin 330^\circ} = \frac{1-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{\frac{2-\sqrt{3}}{2}}{-\frac{1}{2}} = -\left(2-\sqrt{3}\right) = -2+\sqrt{3}$

But, because 165° is in the second quadrant, tangent is negative, so the answer is $-\left(-2+\sqrt{3}\right) = 2-\sqrt{3}$.

2. If $\sin\theta = \frac{7}{25}$, then by the Pythagorean Theorem the third side is 24. Because θ is in the second quadrant, $\cos\theta = -\frac{24}{25}$.

$$\sin \frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1+\frac{24}{25}}{2}} = \sqrt{\frac{49}{50}} = \frac{7}{5\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{7\sqrt{2}}{10}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1-\frac{24}{25}}{2}} = \sqrt{\frac{1}{50}} = \frac{1}{5\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{10}$$

$$\tan \frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1+\frac{24}{25}}{1-\frac{24}{25}}} = \sqrt{\frac{49}{50}\cdot\frac{50}{1}} = \sqrt{49} = 7$$

3. Step 1: Change right side into sine and cosine.

$$\tan \frac{b}{2} = \frac{\sec b}{\sec b \csc b + \csc b}$$

$$= \frac{1}{\cos b} \div \csc b(\sec b + 1)$$

$$= \frac{1}{\cos b} \div \frac{1}{\sin b}\left(\frac{1}{\cos b}+1\right)$$

$$= \frac{1}{\cos b} \div \frac{1}{\sin b}\left(\frac{1+\cos b}{\cos b}\right)$$

$$= \frac{1}{\cos b} \div \frac{1+\cos b}{\sin b \cos b}$$

$$= \frac{1}{\cos b} \cdot \frac{\sin b \cos b}{1+\cos b}$$

$$= \frac{\sin b}{1+\cos b}$$

Step 2: At the last step above, we have simplified the right side as much as possible, now we simplify the left side, using the half angle formula.

$$\begin{aligned}
\sqrt{\frac{1-\cos b}{1+\cos b}} &= \frac{\sin b}{1+\cos b} \\
\frac{1-\cos b}{1+\cos b} &= \frac{\sin^2 b}{(1+\cos b)^2} \\
(1-\cos b)(1+\cos b)^2 &= \sin^2 b(1+\cos b) \\
(1-\cos b)(1+\cos b) &= \sin^2 b \\
1-\cos^2 b &= \sin^2 b
\end{aligned}$$

4. Step 1: change cotangent to cosine over sine, then cross-multiply.

$$\begin{aligned}
\cot\frac{c}{2} &= \frac{\sin c}{1-\cos c} \\
&= \frac{\cos\frac{c}{2}}{\sin\frac{c}{2}} = \sqrt{\frac{1+\cos c}{1-\cos c}} \\
\sqrt{\frac{1+\cos c}{1-\cos c}} &= \frac{\sin c}{1-\cos c} \\
\frac{1+\cos c}{1-\cos c} &= \frac{\sin^2 c}{(1-\cos c)^2} \\
(1+\cos c)(1-\cos c)^2 &= \sin^2 c(1-\cos c) \\
(1+\cos c)(1-\cos c) &= \sin^2 c \\
1-\cos^2 c &= \sin^2 c
\end{aligned}$$

5.

$$\begin{aligned}
\sin x\tan\frac{x}{2}+2\cos x &= \sin x\left(\frac{1-\cos x}{\sin x}\right)+2\cos x \\
\sin x\tan\frac{x}{2}+2\cos x &= 1-\cos x+2\cos x \\
\sin x\tan\frac{x}{2}+2\cos x &= 1+\cos x \\
\sin x\tan\frac{x}{2}+2\cos x &= 2\cos^2\frac{x}{2}
\end{aligned}$$

6. First, we need to find the third side. Using the Pythagorean Theorem, we find that the final side is $\sqrt{105}\left(b=\sqrt{13^2-(-8)^2}\right)$. Using this information, we find that $\cos u=\frac{\sqrt{105}}{13}$. Plugging this into the half angle formula, we get:

$$\begin{aligned}
\cos\frac{u}{2} &= -\sqrt{\frac{1+\sqrt{\frac{105}{13}}}{2}} \\
&= -\sqrt{\frac{\frac{13+\sqrt{105}}{13}}{2}} \\
&= -\sqrt{\frac{13+\sqrt{105}}{26}}
\end{aligned}$$

7. To solve $2\cos^2\frac{x}{2}=1$, first we need to isolate cosine, then use the half angle formula.

$$\begin{aligned}
2\cos^2\frac{x}{2} &= 1 \\
\cos^2\frac{x}{2} &= \frac{1}{2} \\
\frac{1+\cos x}{2} &= \frac{1}{2} \\
1+\cos x &= 1 \\
\cos x &= 0
\end{aligned}$$

$\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

8. To solve $\tan \frac{a}{2} = 4$, first isolate tangent, then use the half angle formula.

$$\begin{aligned}
\tan \frac{a}{2} &= 4 \\
\sqrt{\frac{1-\cos a}{1+\cos a}} &= 4 \\
\frac{1-\cos a}{1+\cos a} &= 16 \\
16 + 16\cos a &= 1 - \cos a \\
17\cos a &= -15 \\
\cos a &= -\frac{15}{17}
\end{aligned}$$

9. Using your graphing calculator, $\cos a = -\frac{15}{17}$ when $x = 152°, 208°$

$$\begin{aligned}
\cos \frac{x}{2} &= 1 + \cos x \\
\pm\sqrt{\frac{1+\cos x}{2}} &= 1 + \cos x && \text{Half angle identity} \\
\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2 &= (1+\cos x)^2 && \text{square both sides} \\
\frac{1+\cos x}{2} &= 1 + 2\cos x + \cos^2 x \\
2\left(\frac{1+\cos x}{2}\right) &= 2(1 + 2\cos x + \cos^2 x) \\
1 + \cos x &= 2 + 4\cos x + 2\cos^2 x \\
2\cos^2 x + 3\cos x + 1 &= 0 \\
(2\cos x + 1)(\cos x + 1) &= 0 \\
\text{Then} \quad 2\cos x + 1 &= 0 \\
\frac{2\cos x}{2} &= \frac{-1}{2} \\
x &= \frac{2\pi}{3}, \frac{4\pi}{3} \\
\text{Or} \quad \cos x + 1 &= 0 \\
\cos x &= -1 \\
x &= \pi
\end{aligned}$$

10. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$ This is the two formulas for $\tan \frac{x}{2}$. Cross-multiply.

$$\begin{aligned}
\frac{\sin x}{1+\cos x} &= \frac{1-\cos x}{\sin x} \\
(1-\cos x)(1+\cos x) &= \sin^2 x \\
1 + \cos x - \cos x - \cos^2 x &= \sin^2 x \\
1 - \cos^2 x &= \sin^2 x \\
1 &= \sin^2 x + \cos^2 x
\end{aligned}$$

3.7 Products, Sums, Linear Combinations, and Applications

Learning Objectives

- Use the transformation formulas to go from product to sum and sum to product.
- Derive multiple angle formulas.
- Use linear combinations to solve trigonometric equations.
- Apply trigonometric equations to real-life situations.

Sum to Product Formulas for Sine and Cosine

In some problems, the product of two trigonometric functions is more conveniently found by the sum of two trigonometric functions by use of identities such as this one:

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}$$

This can be verified by using the sum and difference formulas:

$$\begin{aligned}
2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} &= 2\left[\sin\left(\tfrac{\alpha}{2}+\tfrac{\beta}{2}\right)\cos\left(\tfrac{\alpha}{2}-\tfrac{\beta}{2}\right)\right]\\
&= 2\left[\left(\sin\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}+\cos\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\right)\left(\cos\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}+\sin\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\right)\right]\\
&= 2\left[\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\cos^2\tfrac{\beta}{2}+\sin^2\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}+\sin\tfrac{\beta}{2}\cos^2\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}+\sin\tfrac{\alpha}{2}\sin^2\tfrac{\beta}{2}\cos\tfrac{\alpha}{2}\right]\\
&= 2\left[\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\left(\sin^2\tfrac{\beta}{2}+\cos^2\tfrac{\beta}{2}\right)+\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}\left(\sin^2\tfrac{\alpha}{2}+\cos^2\tfrac{\alpha}{2}\right)\right]\\
&= 2\left[\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}+\sin\tfrac{\beta}{2}\cos\tfrac{\beta}{2}\right]\\
&= 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\\
&= \sin\left(2\cdot\frac{\alpha}{2}\right)+\sin\left(2\cdot\frac{\beta}{2}\right)\\
&= \sin\alpha+\sin\beta
\end{aligned}$$

The following variations can be derived similarly:

$$\begin{aligned}
\sin\alpha - \sin\beta &= 2\sin\frac{\alpha-\beta}{2} \times \cos\frac{\alpha+\beta}{2}\\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2} \times \cos\frac{\alpha-\beta}{2}\\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2} \times \sin\frac{\alpha-\beta}{2}
\end{aligned}$$

Example 1: Change $\sin 5x - \sin 9y$ into a product.

Solution: Use the formula $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2} \times \cos\frac{\alpha+\beta}{2}$.

$$\begin{aligned}
\sin 5x - \sin 9x &= 2\sin\frac{5x-9x}{2}\cos\frac{5x+9x}{2}\\
&= 2\sin(-2x)\cos 7x\\
&= -2\sin 2x\cos 7x
\end{aligned}$$

Example 2: Change $\cos(-3x) + \cos 8x$ into a product.

Solution: Use the formula $\cos \alpha + \cos \beta = 2 \cos \frac{\alpha+\beta}{2} \times \cos \frac{\alpha-\beta}{2}$.

$$\begin{aligned}
\cos(-3x) + \cos(8x) &= 2 \cos \frac{-3x+8x}{2} \cos \frac{-3x-8x}{2} \\
&= 2 \cos(2.5x) \cos(-5.5x) \\
&= 2 \cos(2.5) \cos(5.5x)
\end{aligned}$$

Example 3: Change $2 \sin 7x \cos 4x$ to a sum.

Solution: This is the reverse of what was done in the previous two examples. Looking at the four formulas above, take the one that has sine and cosine as a product, $\sin \alpha - \sin \beta = 2 \sin \frac{\alpha-\beta}{2} \times \cos \frac{\alpha+\beta}{2}$. Therefore, $7x = \frac{\alpha-\beta}{2}$ and $4x = \frac{\alpha+\beta}{2}$.

$$7\text{x} \ = \alpha - \beta \overline{\scriptstyle 24x=\frac{\alpha+\beta}{2}\text{and}14x=\alpha-\beta 8x=\alpha+\beta\alpha=14x+\beta 8x=[14x+\beta]+\beta\text{so}-6x=2\beta-3x=\beta\alpha=14x+(-3x)\alpha=11x}$$

So, this translates to $\sin(11x) + \sin(-3x)$ or $\sin(11x) - \sin(3x)$. A shortcut for this problem, would be to notice that the sum of $7x$ and $4x$ is $11x$ and the difference is $3x$.

Product to Sum Formulas for Sine and Cosine

There are two formulas for transforming a product of sine or cosine into a sum or difference. First, let's look at the product of the sine of two angles. To do this, start with cosine.

$$\begin{aligned}
\cos(a-b) = \cos a \cos b + \sin a \sin b &\text{ and } \cos(a+b) = \cos a \cos b - \sin a \sin b \\
\cos(a-b) - \cos(a+b) &= \cos a \cos b + \sin a \sin b - (\cos a \cos b - \sin a \sin b) \\
\cos(a-b) - \cos(a+b) &= \cos a \cos b + \sin a \sin b - \cos a \cos b + \sin a \sin b \\
\cos(a-b) - \cos(a+b) &= 2 \sin a \sin b \\
\frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] &= \sin a \sin b
\end{aligned}$$

The following product to sum formulas can be derived using the same method:

$$\begin{aligned}
\cos \alpha \cos \beta &= \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right] \\
\sin \alpha \cos \beta &= \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right] \\
\cos \alpha \sin \beta &= \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]
\end{aligned}$$

Example 4: Change $\cos 2x \cos 5y$ to a sum.

Solution: Use the formula $\cos \alpha \cos \beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$. Set $\alpha = 2x$ and $\beta = 5y$.

$$\cos 2x \cos 5y = \frac{1}{2}\left[\cos(2x-5y) + \cos(2x+5y)\right]$$

Example 5: Change $\frac{\sin 11z + \sin z}{2}$ to a product.

Solution: Use the formula $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$. Therefore, $\alpha+\beta = 11z$ and $\alpha-\beta = z$. Solve the second equation for α and plug that into the first.

$$\begin{aligned} \alpha = z+\beta \rightarrow (z+\beta)+\beta &= 11z \qquad\qquad \text{and} \quad \alpha = z+5z = 6z \\ z+2\beta &= 11z \\ 2\beta &= 10z \\ \beta &= 5z \end{aligned}$$

$\frac{\sin 11z+\sin z}{2} = \sin 6z \sin 5z$. Again, the sum of $6z$ and $5z$ is $11z$ and the difference is z.

Solving Equations with Product and Sum Formulas

Example 6: Solve $\sin 4x + \sin 2x = 0$.

Solution: Use the formula $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$.

$$\begin{aligned} \sin 4x + \sin 2x &= 0 & \text{So, } \sin 3x = 0 \ \text{ and } \ \cos x = 0 \rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \\ 2\sin 3x\cos x &= 0 & 3x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi \\ \sin 3x\cos x &= 0 & x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

Example 7: Solve $\cos 5x + \cos x = \cos 2x$.

Solution: Use the formula $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\times\cos\frac{\alpha-\beta}{2}$.

$$\begin{aligned} \cos 5x + \cos x &= \cos 2x \\ 2\cos 3x\cos 2x &= \cos 2x \\ 2\cos 3x\cos 2x - \cos 2x &= 0 \\ \cos 2x(2\cos 3x - 1) &= 0 \\ \swarrow \qquad\qquad & \searrow \\ \cos 2x = 0 \qquad\qquad & 2\cos 3x - 1 = 0 \\ & 2\cos 3x = 1 \\ 2x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{and} \quad & \cos 3x = \frac{1}{2} \\ x = \frac{\pi}{4}, \frac{3\pi}{4} \qquad\qquad & 3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3} \\ & x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9} \end{aligned}$$

Triple-Angle Formulas and Beyond

By combining the sum formula and the double angle formula, formulas for triple angles and more can be found.

Example 8: Find the formula for $\sin 3x$

Solution: Use both the double angle formula and the sum formula.

$$
\begin{aligned}
\sin 3x &= \sin(2x + x) \\
&= \sin(2x)\cos x + \cos(2x)\sin x \\
&= (2\sin x\cos x)\cos x + (\cos^2 x - \sin^2 x)\sin x \\
&= 2\sin x\cos^2 x + \cos^2 x\sin x - \sin^3 x \\
&= 3\sin x\cos^2 x - \sin^3 x \\
&= 3\sin x(1 - \sin^2 x) - \sin^3 x \\
&= 3\sin x - 4\sin^3 x
\end{aligned}
$$

Example 9: Find the formula for $\cos 4x$

Solution: Using the same method from the previous example, you can obtain this formula.

$$
\begin{aligned}
\cos 4x &= \cos(2x + 2x) \\
&= \cos^2 2x - \sin^2 2x \\
&= (\cos^2 x - \sin^2 x)^2 - (2\sin x\cos x)^2 \\
&= \cos^4 - 2\sin^2 x\cos^2 x + \sin^4 x - 4\sin^2 x\cos^2 x \\
&= \cos^4 - 6\sin^2 x\cos^2 x + \sin^4 x \\
&= \cos^4 - 6(1 - \cos^2 x)\cos^2 x + (1 - \cos^2 x)^2 \\
&= 1 - 8\cos^2 x + 8\cos^4 x
\end{aligned}
$$

Linear Combinations

Here, we take an equation which takes a linear combination of sine and cosine and converts it into a simpler cosine function.

$A\cos x + B\sin x = C\cos(x - D)$, where $C = \sqrt{A^2 + B^2}$, $\cos D = \frac{A}{C}$ and $\sin D = \frac{B}{C}$.

Example 10: Transform $3\cos 2x - 4\sin 2x$ into the form $C\cos(2x - D)$

Solution: $A = 3$ and $B = -4$, so $C = \sqrt{3^2 + (-4)^2} = 5$. Therefore $\cos D = \frac{3}{5}$ and $\sin D = -\frac{4}{5}$ which makes the reference angle is -53.1° or -0.927 radians. since cosine is positive and sine is negative, the angle must be a fourth quadrant angle. D must therefore be 306.9° or 5.36 radians.The final answer is $3\cos 2x - 4\sin 2x = 5\cos(2x - 5.36)$.

Example 11: Solve $5\cos x + 12\sin x = 6$.

Solution: First, transform the left-hand side into the form $C\cos(x - D)$. $A = 5$ and $B = 12$, so $C = \sqrt{5^2 + 12^2} = 13$. From this $\cos D = \frac{5}{13}$ and $\sin D = \frac{12}{13}$, which makes the angle in the first quadrant and 1.176 radians. Now, our equation looks like this: $13\cos(x - 1.176) = 6$ and we can solve for x.

$$
\begin{aligned}
\cos(x - 1.176) &= \frac{6}{13} \\
x - 1.176 &= \cos^{-1}\left(\frac{6}{13}\right) \\
x - 1.176 &= 1.09 \\
x &= 2.267 \text{ radians}
\end{aligned}
$$

Applications & Technology

Example 12: The range of a small rocket that is launched with an initial velocity v at an angle with θ the horizontal is given by $R(range) = \frac{v^2(velocity)}{g(9.8m/s^2)} \sin 2\theta$. If the rocket is launched with an initial velocity of 15 m/s, what angle is needed to reach a range of 20 m?

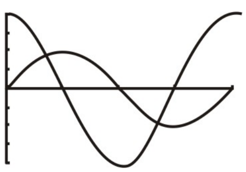
Solution: Plug in 15 m/s for v and 20 m for the range to solve for the angle.

$$\begin{aligned} 20 &= \frac{15^2}{9.8} \sin 2\theta \\ 20 &= 22.96 \sin 2\theta \\ 0.87\bar{1} &= \sin 2\theta \\ \sin^{-1}(0.87\bar{1}) &= 2\theta \\ 60.59^\circ, 119.41^\circ &= 2\theta \\ 30.3^\circ, 59.7^\circ &= \theta \end{aligned}$$

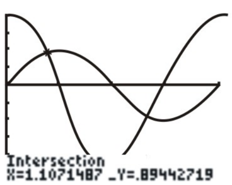
You can also use the TI-83 to solve trigonometric equations. It is sometimes easier than solving the equation algebraically. Just be careful with the directions and make sure your final answer is in the form that is called for. You calculator cannot put radians in terms of π.

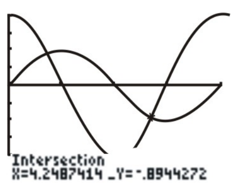
Example 13: Solve $\sin x = 2\cos x$ such that $0 \le x \le 2\pi$ using a graphing calculator.

Solution: In $y =$, graph $y1 = \sin x$ and $y2 = 2\cos x$.



Next, use **CALC** to find the intersection points of the graphs.





Review Questions

1. Express the sum as a product: $\sin 9x + \sin 5x$
2. Express the difference as a product: $\cos 4y - \cos 3y$
3. Verify the identity (using sum-to-product formula): $\frac{\cos 3a - \cos 5a}{\sin 3a + \sin 5a} = -\tan 4a$
4. Express the product as a sum: $\sin(6\theta)\sin(4\theta)$
5. Transform to the form $C\cos(x - D)$
   (a) $5\cos x - 5\sin x$
   (b) $-15\cos 3x - 8\sin 3x$
6. Solve $\sin 11x - \sin 5x = 0$ for all solutions $0 \le x < 2\pi$.
7. Solve $\cos 4x + \cos 2x = 0$ for all solutions $0 \le x < 2\pi$.
8. Solve $\sin 5x + \sin x = \sin 3x$ for all solutions $0 \le x < 2\pi$.
9. In the study of electronics, the function $f(t) = \sin(200t + \pi) + \sin(200t - \pi)$ is used to analyze frequency. Simplify this function using the sum-to-product formula.

10. Derive a formula for $\tan 4x$.
11. A spring is being moved up and down. Attached to the end of the spring is an object that undergoes a vertical displacement. The displacement is given by the equation $y = 3.50 \sin t + 1.20 \sin 2t$. Find the first two values of t (in seconds) for which $y = 0$.

Review Answers

1. Using the sum-to-product formula:

$$\sin 9x + \sin 5x$$
$$\frac{1}{2}\left(\sin\left(\frac{9x+5x}{2}\right)\cos\left(\frac{9x-5x}{2}\right)\right)$$
$$\frac{1}{2}\sin 7x \cos 2x$$

2. Using the difference-to-product formula:

$$\cos 4y - \cos 3y$$
$$-2\sin\left(\frac{4y+3y}{2}\right)\sin\left(\frac{4y-3y}{2}\right)$$
$$-2\sin\frac{7y}{2}\sin\frac{y}{2}$$

3. Using the difference-to-product formulas:

$$\frac{\cos 3a - \cos 5a}{\sin 3a - \sin 5a} = -\tan 4a$$
$$\frac{-2\sin\left(\frac{3a+5a}{2}\right)\sin\left(\frac{3a-5a}{2}\right)}{2\sin\left(\frac{3a-5a}{2}\right)\cos\left(\frac{3a+5a}{2}\right)}$$
$$-\frac{\sin 4a}{\cos 4a}$$
$$-\tan 4a$$

4. Using the product-to-sum formula:

$$\sin 6\theta \sin 4\theta$$
$$\frac{1}{2}(\cos(6\theta - 4\theta - \cos(6\theta + 4\theta))$$
$$\frac{1}{2}(\cos 2\theta - \cos 10\theta)$$

5. (a) If $5\cos x - 5\sin x$, then $A = 5$ and $B = -5$. By the Pythagorean Theorem, $C = 5\sqrt{2}$ and $\cos D = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So, because B is negative, D is in Quadrant IV. Therefore, $D = \frac{7\pi}{4}$. Our final answer is $5\sqrt{2}\cos\left(x - \frac{7\pi}{4}\right)$.
   (b) If $-15\cos 3x - 8\sin 3x$, then $A = -15$ and $B = -8$. By the Pythagorean Theorem, $C = 17$. Because A and B are both negative, D is in Quadrant III, which means $D = \cos^{-1}\left(-\frac{15}{17}\right) = 2.65$ *rad*. Our final answer is $17\cos 3(x - 2.65)$.
6. Using the sum-to-product formula:

$$\sin 11x - \sin 5x = 0$$
$$2\sin\frac{11x-5x}{2}\cos\frac{11x+5x}{2} = 0$$
$$2\sin 3x \cos 8x = 0$$
$$\sin 3x \cos 8x = 0$$

So,

$$\sin 3x = 0 \quad \text{or} \quad \cos 8x = 0$$
$$3x = 0, \pi \qquad 8x = \frac{\pi}{2}, \frac{3\pi}{2}$$
$$x = 0, \frac{\pi}{3} \qquad x = \frac{\pi}{16}, \frac{3\pi}{16}$$

7. Using the sum-to-product formula:

$$\begin{aligned}\cos 4x + \cos 2x &= 0\\ 2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2} &= 0\\ 2\cos 3x\cos x &= 0\\ \cos 3x \cos x &= 0\end{aligned}$$

So, either $\cos 3x = 0$ or $\cos x = 0$

$$\begin{aligned}3x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}\\ x &= \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\end{aligned}$$

8. Move $\sin 3x$ over to the other side and use the sum-to-product formula:

$$\begin{aligned}\sin 5x + \sin x &= \sin 3x\\ \sin 5x - \sin 3x + \sin x &= 0\\ 2\cos\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right) + \sin x &= 0\\ 2\cos 4x \sin x + \sin x &= 0\\ \sin x(2\cos 4x + 1) &= 0\end{aligned}$$

So $\sin x = 0$

$$\begin{aligned}x = 0, \pi \quad \text{or} \quad 2\cos 4x &= -1\\ \cos 4x &= -\frac{1}{2}\\ 4x &= \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}, \frac{20\pi}{3}, \frac{22\pi}{3}\\ &= \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}\\ x = 0, &= \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}\end{aligned}$$

9. Using the sum-to-product formula:

$$\begin{aligned}f(x) &= \sin(200x+\pi) + \sin(200x-\pi)\\ &= 2\sin\left(\frac{(200x+\pi)+(200t-\pi)}{2}\right)\cos\left(\frac{(200x+\pi)-(200x-\pi)}{2}\right)\\ &= 2\sin\left(\frac{400x}{2}\right)\cos\left(\frac{2\pi}{2}\right)\\ &= 2\sin 200x\cos\pi\\ &= 2\sin 200x(-1)\\ &= -2\sin 200x\end{aligned}$$

10. Derive a formula for $\tan 4x$.

$$\begin{aligned}
\tan 4x &= \tan(2x+2x)\\
&= \frac{\tan 2x + \tan 2x}{1-\tan 2x \tan 2x}\\
&= \frac{2\tan 2x}{1-\tan^2 2x}\\
&= \frac{2\cdot \frac{2\tan x}{1-\tan^2 x}}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2}\\
&= \frac{4\tan x}{1-\tan^2 x} \div \frac{(1-\tan^2 x)^2 - 4\tan^2 x}{(1-\tan^2 x)^2}\\
&= \frac{4\tan x}{1-\tan^2 x} \div \frac{1-2\tan^2 x + \tan^4 x - 4\tan^2 x}{(1-\tan^2 x)^2}\\
&= \frac{4\tan x}{1-\tan^2 x} \cdot \frac{(1-\tan^2 x)^2}{1-6\tan^2 x + \tan^4 x}\\
&= \frac{4\tan x - 4\tan^3 x}{1-6\tan^2 x + \tan^4 x}
\end{aligned}$$

11. Let $y = 0$.

$$\begin{aligned}
3.50\sin t + 1.20\sin 2t &= 0\\
3.50\sin t + 2.40\sin t\cos t &= 0, \text{ Double-Angle Identity}\\
\sin t(3.50 + 2.40\cos t) &= 0\\
\sin t = 0 \text{ or } 3.50 + 2.40\cos t &= 0\\
2.40\cos t &= -3.50\\
\cos t &= -1.46 \rightarrow \text{no solution because } -1 \le \cos t \le 1.\\
t &= 0, \pi
\end{aligned}$$

3.8 Chapter Review

Chapter Summary

Here are the identities studied in this chapter:

Quotient & Reciprocal Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \quad \sec\theta = \frac{1}{\cos\theta} \quad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta$$

Even & Odd Identities

$$\begin{aligned}
\sin(-x) &= -\sin x & \cos(-x) &= \cos x & \tan(-x) &= -\tan x\\
\csc(-x) &= -\csc x & \sec(-x) &= \sec x & \cot(-x) &= -\cot x
\end{aligned}$$

Co-Function Identities

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \qquad \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$$
$$\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta \qquad \sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$$

Sum and Difference Identities

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \qquad \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Double Angle Identities

$$\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha=2\cos^2\alpha-1=1-2\sin^2\alpha$$
$$\sin(2\alpha)=2\sin\alpha\cos\beta$$
$$\tan(2\alpha)=\frac{2\tan\alpha}{1-\tan^2\alpha}$$

Half Angle Identities

$$\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \tan\frac{\alpha}{2}=\frac{1-\cos\alpha}{\sin\alpha}=\frac{\sin\alpha}{1+\cos\alpha}$$

Product to Sum & Sum to Product Identities

$$\sin a+\sin b=2\sin\frac{a+b}{2}\cos\frac{a-b}{2} \qquad \sin a\sin b=\frac{1}{2}[\cos(a-b)-\cos(a+b)]$$
$$\sin a-\sin b=2\sin\frac{a-b}{2}\cos\frac{a+b}{2} \qquad \cos a\cos b=\frac{1}{2}[\cos(a-b)+\cos(a+b)]$$
$$\cos a+\cos b=2\cos\frac{a+b}{2}\cos\frac{a-b}{2} \qquad \sin a\cos b=\frac{1}{2}[\sin(a+b)+\sin(a-b)]$$
$$\cos a-\cos b=2-2\sin\frac{a+b}{2}\sin\frac{a-b}{2} \qquad \cos a\sin b=\frac{1}{2}[\sin(a+b)-\sin(a-b)]$$

Linear Combination Formula

$A\cos x+B\sin x=C\cos(x-D)$, where $C=\sqrt{A^2+B^2}, \cos D=\frac{A}{C}$ and $\sin D=\frac{B}{C}$

Review Questions

1. Find the sine, cosine, and tangent of an angle with terminal side on $(-8, 15)$.
2. If $\sin a=\frac{\sqrt{5}}{3}$ and $\tan a<0$, find $\sec a$.
3. Simplify: $\frac{\cos^4 x-\sin^4 x}{\cos^2-\sin^2 x}$.
4. Verify the identity: $\frac{1+\sin x}{\cos x\sin x}=\sec x(\csc x+1)$

For problems 5-8, find all the solutions in the interval $[0, 2\pi)$.

5. $\sec\left(x+\frac{\pi}{2}\right)+2=0$

6. $8\sin\left(\frac{x}{2}\right) - 8 = 0$
7. $2\sin^2 x + \sin 2x = 0$
8. $3\tan^2 2x = 1$
9. Solve the trigonometric equation $1 - \sin x = \sqrt{3}\sin x$ over the interval $[0, \pi]$.
10. Solve the trigonometric equation $2\cos 3x - 1 = 0$ over the interval $[0, 2\pi]$.
11. Solve the trigonometric equation $2\sec^2 x - \tan^4 x = -1$ for all real values of x.

Find the exact value of:

12. $\cos 157.5°$
13. $\sin \frac{13\pi}{12}$
14. Write as a product: $4(\cos 5x + \cos 9x)$
15. Simplify: $\cos(x - y)\cos y - \sin(x - y)\sin y$
16. Simplify: $\sin\left(\frac{4\pi}{3} - x\right) + \cos\left(x + \frac{5\pi}{6}\right)$
17. Derive a formula for $\sin 6x$.
18. If you solve $\cos 2x = 2\cos^2 x - 1$ for $\cos^2 x$, you would get $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$. This new formula is used to reduce powers of cosine by substituting in the right part of the equation for $\cos^2 x$. Try writing $\cos^4 x$ in terms of the first power of cosine.
19. If you solve $\cos 2x = 1 - 2\sin^2 x$ for $\sin^2 x$, you would get $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Similar to the new formula above, this one is used to reduce powers of sine. Try writing $\sin^4 x$ in terms of the first power of cosine.
20. Rewrite in terms of the first power of cosine:
    (a) $\sin^2 x \cos^2 2x$
    (b) $\tan^4 2x$

Review Answers

1. If the terminal side is on $(-8, 15)$, then the hypotenuse of this triangle would be 17 (by the Pythagorean Theorem, $c = \sqrt{(-8)^2 + 15^2}$). Therefore, $\sin x = \frac{15}{17}, \cos x = -\frac{8}{17}$, and $\tan x = -\frac{15}{8}$.
2. If $\sin a = \frac{\sqrt{5}}{3}$ and $\tan a < 0$, then a is in Quadrant II. Therefore $\sec a$ is negative. To find the third side, we need to do the Pythagorean Theorem.

$$\left(\sqrt{5}\right)^2 + b^2 = 3^2$$
$$5 + b^2 = 9 \text{ So, } \sec a = \frac{3}{2},$$
$$b^2 = 4$$
$$b = 2$$

3. Factor top, cancel like terms, and use the Pythagorean Theorem Identity.

$$\frac{\cos^4 x - \sin^4 x}{\cos^2 x - \sin^2 x}$$
$$\frac{(\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)}{\cos^2 x - \sin^2 x}$$
$$\frac{\cos^2 x + \sin^2 x}{1}$$

4. Change secant and cosecant into terms of sine and cosine, then find a common denominator.

$$\begin{aligned}\frac{1+\sin x}{\cos x \sin x} &= \sec x(\csc x + 1)\\ &= \frac{1}{\cos x}\left(\frac{1}{\sin x}+1\right)\\ &= \frac{1}{\cos x}\left(\frac{1+\sin x}{\sin x}\right)\\ &= \frac{1+\sin x}{\cos x \sin x}\end{aligned}$$

5.

$$\begin{aligned}\sec\left(x+\frac{\pi}{2}\right)+2 &= 0\\ \sec\left(x+\frac{\pi}{2}\right) &= -2\\ \cos\left(x+\frac{\pi}{2}\right) &= -\frac{1}{2}\\ x+\frac{\pi}{2} &= \frac{2\pi}{3}, \frac{4\pi}{3}\\ x &= \frac{2\pi}{3}-\frac{\pi}{2}, \frac{4\pi}{3}-\frac{\pi}{2}\\ x &= \frac{\pi}{6}, \frac{5\pi}{6}\end{aligned}$$

6.

$$\begin{aligned}8\sin\left(\frac{x}{2}\right)-8 &= 0\\ 8\sin\frac{x}{2} &= 8\\ \sin\frac{x}{2} &= 1\\ \frac{x}{2} &= \frac{x}{2}\\ x &= \pi\end{aligned}$$

7.

$$\begin{aligned}2\sin^2 x + \sin 2x &= 0\\ 2\sin^2 x + 2\sin x\cos x &= 0\\ 2\sin x(\sin x + \cos x) &= 0\\ \text{So, } 2\sin x = 0 \qquad &\text{or} \qquad \sin x + \cos x = 0\\ 2\sin x = 0 \qquad & \qquad \sin x + \cos x = 0\\ \sin x = 0 \qquad & \qquad \sin x = -\cos x\\ x = 0, \pi \qquad & \qquad x = \frac{3\pi}{4}, \frac{7\pi}{4}\end{aligned}$$

8.

$$\begin{aligned}\tan^2 2x &= \frac{1}{3}\\ \tan 2x &= \frac{\sqrt{3}}{3}\\ 2x &= \frac{\pi}{6}, \frac{7\pi}{6}\\ x &= \frac{\pi}{12}, \frac{7\pi}{12}\end{aligned}$$

9.

$$\begin{aligned}
1 - \sin x &= \sqrt{3} \sin x \\
1 &= \sin x + \sqrt{3} \sin x \\
1 &= \sin x \left(1 + \sqrt{3}\right) \\
\frac{1}{1 + \sqrt{3}} &= \sin x
\end{aligned}$$

$\sin^{-1}\left(\frac{1}{1+\sqrt{3}}\right) = x$ or $x = .3747$ radians and $x = 2.7669$ radians

10. Because this is $\cos 3x$, you will need to divide by 3 at the very end to get the final answer. This is why we went beyond the limit of 2π when finding $3x$.

$$\begin{aligned}
2 \cos 3x - 1 &= 0 \\
2 \cos 3x &= 1 \\
\cos 3x &= \frac{1}{2} \\
3x &= \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3} \\
x &= \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}
\end{aligned}$$

11. Rewrite the equation in terms of tan by using the Pythagorean identity, $1 + \tan^2\theta = \sec^2\theta$.

$$\begin{aligned}
2 \sec^2 x - \tan^4 x &= -1 \\
2(1 + \tan^2 x) - \tan^4 x &= -1 \\
2 + 2\tan^2 x - \tan^4 x &= -1 \\
\tan^4 x - 2\tan^2 x + 1 &= 0 \\
(\tan^2 x - 1)(\tan^2 x - 1) &= 0
\end{aligned}$$

Because these factors are the same, we only need to solve one for x.

$$\begin{aligned}
\tan^2 x - 1 &= 0 \\
\tan^2 x &= 1 \\
\tan x &= \pm 1 \\
x &= \frac{\pi}{4} + \pi k \text{ and } \frac{3\pi}{4} + \pi k
\end{aligned}$$

Where k is any integer.

12. Use the half angle formula with 315°.

$$\begin{aligned}
\cos 157.5^\circ &= \cos \frac{315^\circ}{2} \\
&= -\sqrt{\frac{1 + \cos 315^\circ}{2}} \\
&= -\sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} \\
&= -\sqrt{\frac{2 + \sqrt{2}}{4}} \\
&= -\frac{\sqrt{2 + \sqrt{2}}}{2}
\end{aligned}$$

13. Use the sine sum formula.

$$\begin{aligned}\sin\frac{13\pi}{12} &= \sin\left(\frac{10\pi}{12}+\frac{3\pi}{12}\right)\\ &= \sin\left(\frac{5\pi}{6}+\frac{\pi}{4}\right)\\ &= \sin\frac{5\pi}{6}\cos\frac{\pi}{4}+\cos\frac{5\pi}{6}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

14.

$$\begin{aligned}4(\cos 5x+\cos 9x) &= 4\left[2\cos\left(\frac{5x+9x}{2}\right)\cos\left(\frac{5x-9x}{2}\right)\right]\\ &= 8\cos 7x\cos(-2x)\\ &= 8\cos 7x\cos 2x\end{aligned}$$

15.

$$\begin{gathered}\cos(x-y)\cos y-\sin(x-y)\sin y\\ \cos y(\cos x\cos y+\sin x\sin y)-\sin y\ (\sin x\cos y-\cos x\sin y)\\ \cos x\cos^2 y+\sin x\sin y\cos y-\sin x\sin y\cos y+\cos x\sin^2 y\\ \cos x\cos^2 y+\cos x\sin^2 y\\ \cos x(\cos^2 y+\sin^2 y)\\ \cos x\end{gathered}$$

16. Use the sine and cosine sum formulas.

$$\begin{gathered}\sin\left(\frac{4\pi}{3}-x\right)+\cos\left(x+\frac{5\pi}{6}\right)\\ \sin\frac{4\pi}{3}\cos x-\cos\frac{4\pi}{3}\sin x+\cos x\cos\frac{5\pi}{6}-\sin x\sin\frac{5\pi}{6}\\ -\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x-\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x\\ -\sqrt{3}\cos x\end{gathered}$$

17. Use the sine sum formula as well as the double angle formula.

$$\begin{aligned}\sin 6x &= \sin(4x+2x)\\ &= \sin 4x\cos 2x+\cos 4x\sin 2x\\ &= \sin(2x+2x)\cos 2x+\cos(2x+2x)\sin 2x\\ &= \cos 2x\ (\sin 2x\cos 2x+\cos 2x\sin 2x)+\sin 2x(\cos 2x\cos 2x-\sin 2x\ \sin 2x)\\ &= 2\sin 2x\cos^2 2x+\sin 2x\cos^2 2x-\sin^3 2x\\ &= 3\sin 2x\ \cos^2 2x-\sin^3 2x\\ &= \sin 2x(3\cos^2 2x-\sin^2 2x)\\ &= 2\sin x\cos x[3(\cos^2 x-\sin^2 x)^2-(2\sin x\cos x)^2\\ &= 2\sin x\cos x[3(\cos^4 x-2\sin^2 x\cos^2 x+\sin^4 x)-4\ \sin^2 x\cos^2 x]\\ &= 2\sin x\cos x[3\cos^4 x-6\sin^2 x\cos^2 x+3\sin^4 x-4\ \sin^2 x\cos^2 x]\\ &= 2\sin x\cos x[3\cos^4 x+3\sin^4 x-10\sin^2 x\cos^2 x]\\ &= 6\sin x\cos^5 x+6\sin^5 x\cos x-20\sin^3 x\cos^3 x\end{aligned}$$

18. Using our new formula, $\cos^4 x = \left[\frac{1}{2}(\cos 2x + 1)\right]^2$ Now, our final answer needs to be in the first power of cosine, so we need to find a formula for $\cos^2 2x$. For this, we substitute $2x$ everywhere there is an x and the formula translates to $\cos^2 2x = \frac{1}{2}(\cos 4x + 1)$.
19. Using our new formula, $\sin^4 x = \left[\frac{1}{2}(1 - \cos 2x)\right]^2$ Now, our final answer needs to be in the first power of cosine, so we need to find a formula for $\cos^2 2x$. For this, we substitute $2x$ everywhere there is an x and the formula translates to $\cos^2 2x = \frac{1}{2}(\cos 4x + 1)$.
20. (a) First, we use both of our new formulas, then simplify:

$$\begin{aligned}\sin^2 x \cos^2 2x &= \frac{1}{2}(1 - \cos 2x)\frac{1}{2}(\cos 4x + 1)\\ &= \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\left(\frac{1}{2}\cos 4x + \frac{1}{2}\right)\\ &= \frac{1}{4}\cos 4x + \frac{1}{4} - \frac{1}{4}\cos 2x \cos 4x - \frac{1}{4}\cos 2x\\ &= \frac{1}{4}(1 - \cos 2x + \cos 4x - \cos 2x \cos 4x)\end{aligned}$$

(b) For tangent, we using the identity $\tan x = \frac{\sin x}{\cos x}$ and then substitute in our new formulas. $\tan^4 2x = \frac{\sin^4 2x}{\cos^4 2x} \rightarrow$ now, use the formulas we derived in #8 and 9.

Texas Instruments Resources

In the CK-12 Texas Instruments Trigonometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9701.

Chapter 4

Inverse Trigonometric Functions - 2nd edition

4.1 Basic Inverse Trigonometric Functions

Introduction

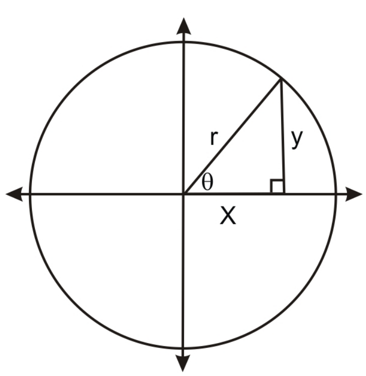
Recall that an inverse function is a reflection of the function over the line $y = x$. In order to find the inverse of a function, you must switch the x and y values and then solve for y. A function has an inverse if and only if it has exactly one output for every input and exactly one input for every output. All of the trig functions fit these criteria over a specific range. In this chapter, we will explore inverse trig functions and equations.

Learning Objectives

- Understand and evaluate inverse trigonometric functions.
- Extend the inverse trigonometric functions to include the $\csc^{-1}, \sec^{-1}$ and $\cot^{-1}$ functions.
- Apply inverse trigonometric functions to the critical values on the unit circle.

Defining the Inverse of the Trigonometric Ratios

Recall from Chapter 1, the ratios of the six trig functions and their inverses, with regard to the unit circle.

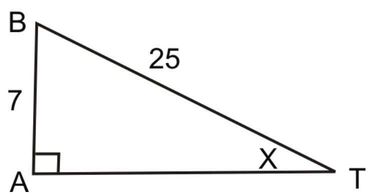


$$\sin\theta = \frac{y}{r} \to \sin^{-1}\frac{y}{r} = \theta \qquad \cos\theta = \frac{x}{r} \to \cos^{-1}\frac{x}{r} = \theta$$
$$\tan\theta = \frac{y}{x} \to \tan^{-1}\frac{y}{x} = \theta \qquad \cot\theta = \frac{x}{y} \to \cot^{-1}\frac{x}{y} = \theta$$
$$\csc\theta = \frac{r}{y} \to \csc^{-1}\frac{r}{y} = \theta \qquad \sec\theta = \frac{r}{x} \to \sec^{-1}\frac{r}{x} = \theta$$

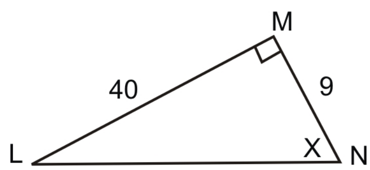
These ratios can be used to find any θ in standard position or in a triangle.

Example 1: Find the measure of the angles below.

a.



b.



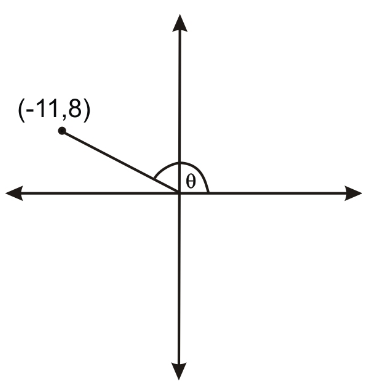
Solution: For part a, you need to use the sine function and part b utilizes the tangent function. Because both problems require you to solve for an angle, the inverse of each must be used.

a. $\sin x = \frac{7}{25} \to \sin^{-1}\frac{7}{25} = x \to x = 16.26^\circ$

b. $\tan x = \frac{40}{9} \to \tan^{-1}\frac{40}{9} = x \to x = 77.32^\circ$

The trigonometric value $\tan\theta = \frac{40}{9}$ of the angle is known, but not the angle. In this case the inverse of the trigonometric function must be used to determine the measure of the angle. (Directions for how to find inverse function values in the graphing calculator are in Chapter 1). The inverse of the tangent function is read "tangent inverse" and is also called the arctangent relation. The inverse of the cosine function is read "cosine inverse" and is also called the arccosine relation. The inverse of the sine function is read "sine inverse" and is also called the arcsine relation.

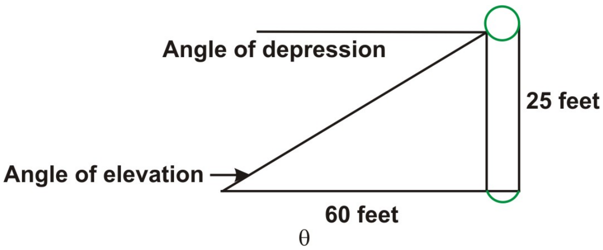
Example 2: Find the angle, θ, in standard position.



Solution: The $\tan\theta = \frac{y}{x}$ or, in this case, $\tan\theta = \frac{8}{-11}$. Using the inverse tangent, you get $\tan^{-1} -\frac{8}{11} = -36.03^\circ$. This is the reference angle and in the 4^{th} quadrant. This value of -36.03° is the angle you also see if you move clockwise from the -x axis. To find the corresponding angle in the second quadrant (which is the same as though you started at the +x axis and moved counterclockwise), subtract 36.03° from 180°, yielding 143.97°.

Recall that inverse trigonometric functions are also used to find the angle of depression or elevation.

Example 3: A new outdoor skating rink has just been installed outside a local community center. A light is mounted on a pole 25 feet above the ground. The light must be placed at an angle so that it will illuminate the end of the skating rink. If the end of the rink is 60 feet from the pole, at what angle of depression should the light be installed?



Solution: In this diagram, the angle of depression, which is located outside of the triangle, is not known. Recall, the angle of depression equals the angle of elevation. For the angle of elevation, the pole where the light is located is the opposite and is 25 feet high. The length of the rink is the adjacent side and is 60 feet in length. To calculate the measure of the angle of elevation the trigonometric ratio for tangent can be applied.

$$\begin{aligned}\tan\theta &= \frac{25}{60}\\ \tan\theta &= 0.4166\\ \tan^{-1}(\tan\theta) &= \tan^{-1}(0.4166)\\ \theta &= 22.6^\circ\end{aligned}$$

The angle of depression at which the light must be placed to light the rink is 22.6°

Exact Values for Inverse Sine, Cosine, and Tangent

Recall the unit circle and the critical values. With the inverse trigonometric functions, you can find the angle value (in either radians or degrees) when given the ratio and function. Make sure that you find all solutions within the given interval.

Example 4: Find the exact value of each expression without a calculator, in $[0, 2\pi)$.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

b. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

c. $\tan^{-1}\sqrt{3}$

Solution: These are all values from the special right triangles and the unit circle.

a. Recall that $-\frac{\sqrt{3}}{2}$ is from the $30-60-90$ triangle. The reference angle for sin and $\frac{\sqrt{3}}{2}$ would be 60°. Because this is sine and it is negative, it must be in the third or fourth quadrant. The answer is either $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

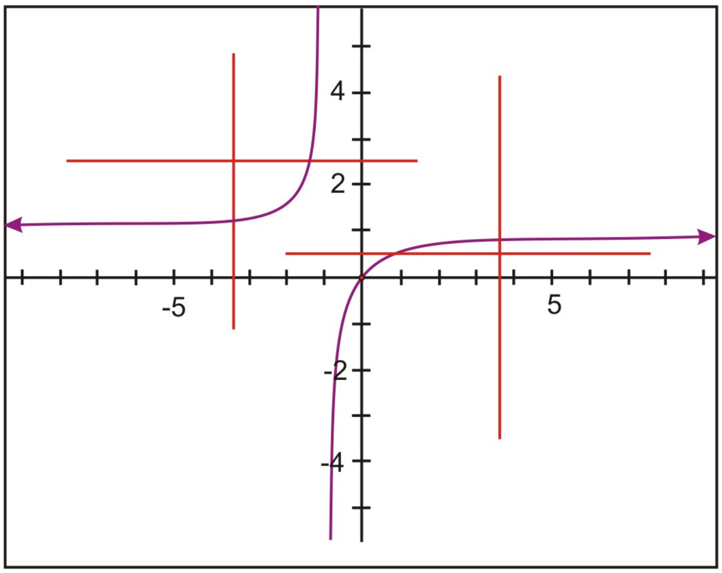
b. $-\frac{\sqrt{2}}{2}$ is from an isosceles right triangle. The reference angle is then 45°. Because this is cosine and negative, the angle must be in either the second or third quadrant. The answer is either $\frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

c. $\sqrt{3}$ is also from a $30-60-90$ triangle. Tangent is $\sqrt{3}$ for the reference angle 60°. Tangent is positive in the first and third quadrants, so the answer would be $\frac{\pi}{3}$ or $\frac{4\pi}{3}$.

Notice how each one of these examples yield two answers. This poses a problem when finding a singular inverse for each of the trig functions. Therefore, we need to restrict the domain in which the inverses can be found, which will be addressed in the next section. Unless otherwise stated, all angles are in radians.

Finding Inverses Algebraically

In the Prerequisite Chapter, you learned that each function has an inverse relation and that this inverse relation is a function only if the original function is one-to-one. A function is one-to-one when its graph passes both the vertical and the horizontal line test. This means that every vertical and horizontal line will intersect the graph in exactly one place.



This is the graph of $f(x) = \frac{x}{x+1}$. The graph suggests that f is one-to-one because it passes both the vertical and the horizontal line tests. To find the inverse of f, switch **the x and y and solve for y.**

First, switch x and y.

$$x = \frac{y}{y+1}$$

Next, multiply both sides by $(y+1)$.

$$\begin{aligned}(y+1)x &= \frac{y}{y+1}(y+1)\\ x(y+1) &= y\end{aligned}$$

Then, apply the distributive property and put all the y terms on one side so you can pull out the y.

$$\begin{aligned}xy + x &= y\\ xy - y &= -x\\ y(x-1) &= -x\end{aligned}$$

Divide by $(x-1)$ to get y by itself.

$$y = \frac{-x}{x-1}$$

Finally, multiply the right side by $\frac{-1}{-1}$.

$$y = \frac{x}{1-x}$$

Therefore the inverse of f is $f^{-1}(x) = \frac{x}{1-x}$.

The symbol f^{-1} is read "f inverse" and is not the reciprocal of f.

Example 5: Find the inverse of $f(x) = \frac{1}{x-5}$ algebraically.

Solution: To find the inverse algebraically, switch $f(x)$ to y and then switch x and y.

$$\begin{aligned} y &= \frac{1}{x-5} \\ x &= \frac{1}{y-5} \\ x(y-5) &= 1 \\ xy - 5x &= 1 \\ xy &= 5x + 1 \\ y &= \frac{5x+1}{x} \end{aligned}$$

Example 6: Find the inverse of $f(x) = 5\sin^{-1}\left(\frac{2}{x-3}\right)$

Solution:

a.

$$\begin{aligned} f(x) &= 5\sin^{-1}\left(\frac{2}{x-3}\right) \\ x &= 5\sin^{-1}\left(\frac{2}{y-3}\right) \\ \frac{x}{5} &= \sin^{-1}\left(\frac{2}{y-3}\right) \\ \sin\frac{x}{5} &= \left(\frac{2}{y-3}\right) \\ (y-3)\sin\frac{x}{5} &= 2 \\ (y-3) &= \frac{2}{\sin\frac{x}{5}} \\ y &= \frac{2}{\sin\frac{x}{5}} + 3 \end{aligned}$$

Example 7: Find the inverse of the trigonometric function $f(x) = 4\tan^{-1}(3x+4)$

Solution:

$$\begin{aligned} x &= 4\tan^{-1}(3y+4) \\ \frac{x}{4} &= \tan^{-1}(3y+4) \\ \tan\frac{x}{4} &= 3y+4 \\ \tan\frac{x}{4} - 4 &= 3y \\ \frac{\tan\frac{x}{4} - 4}{3} &= y \\ f^{-1}(x) &= \frac{\tan\frac{x}{4} - 4}{3} \end{aligned}$$

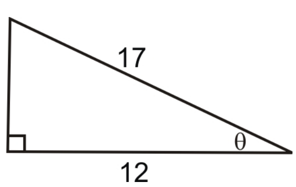
Points to Consider

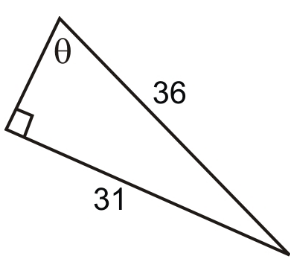
- What is the difference between an inverse and a reciprocal?
- Considering that most graphing calculators do not have csc, sec or cot buttons, how would you find the inverse of each of these?
- Besides algebraically, is there another way to find the inverse?

Review Questions

1. Use the special triangles or the unit circle to evaluate each of the following:
   (a) $\cos 120^\circ$
   (b) $\csc\frac{3\pi}{4}$
   (c) $\tan\frac{5\pi}{3}$
2. Find the exact value of each inverse function, without a calculator in $[0, 2\pi)$:
   (a) $\cos^{-1}(0)$
   (b) $\tan^{-1}\left(-\sqrt{3}\right)$
   (c) $\sin^{-1}\left(-\frac{1}{2}\right)$

Find the value of the missing angle.

3. 

4. 

5. What is the value of the angle with its terminal side passing through (-14, -23)?

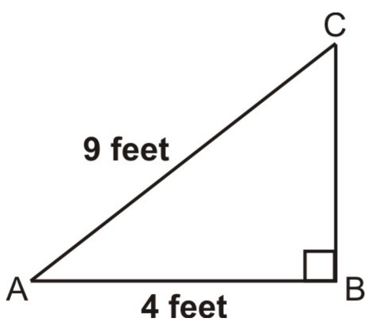
6. A 9-foot ladder is leaning against a wall. If the foot of the ladder is 4 feet from the base of the wall, what angle does the ladder make with the floor?

Find the inverse of the following functions.

7. $f(x) = 2x^3 - 5$
8. $y = \frac{1}{3}\tan^{-1}\left(\frac{3}{4}x - 5\right)$
9. $g(x) = 2\sin(x - 1) + 4$
10. $h(x) = 5 - \cos^{-1}(2x + 3)$

Review Answers

1. (a) $-\frac{1}{2}$
   (b) $\sqrt{2}$
   (c) $-\sqrt{3}$
2. (a) $\frac{\pi}{2}, \frac{3\pi}{2}$
   (b) $\frac{2\pi}{3}, \frac{5\pi}{3}$
   (c) $\frac{11\pi}{6}, \frac{7\pi}{6}$
3. $\cos\theta = \frac{12}{17} \to \cos^{-1}\frac{12}{17} = 45.1^\circ$
4. $\sin\theta = \frac{25}{36} \to \sin^{-1}\frac{31}{36} = 59.44^\circ$
5. This problem uses tangent inverse. $\tan x = \frac{-14}{-23} \to x = \tan^{-1}\frac{14}{23} = 31.33^\circ$ (value graphing calculator will produce). However, this is the reference angle. Our angle is in the third quadrant because both the x and y values are negative. The angle is $180^\circ + 31.33^\circ = 211.33^\circ$.
6.



$$\cos A = \frac{4}{9}$$
$$\cos^{-1}\frac{4}{9} = A$$
$$\angle A = 63.6^\circ$$

7.

$$f(x) = 2x^3 - 5$$
$$y = 2x^3 - 5$$
$$x = 2y^3 - 5$$
$$x + 5 = 2y^3$$
$$\frac{x+5}{2} = y^3$$
$$\sqrt[3]{\frac{x+5}{2}} = y$$

8.

$$\begin{aligned}
y &= \frac{1}{3}\tan^{-1}\left(\frac{3}{4}x-5\right)\\
x &= \frac{1}{3}\tan^{-1}\left(\frac{3}{4}y-5\right)\\
3x &= \tan^{-1}\left(\frac{3}{4}y-5\right)\\
\tan(3x) &= \frac{3}{4}y-5\\
\tan(3x)+5 &= \frac{3}{4}y\\
\frac{4(\tan(3x)+5)}{3} &= y
\end{aligned}$$

9.

$$\begin{aligned}
g(x) &= 2\sin(x-1)+4\\
y &= 2\sin(x-1)+4\\
x &= 2\sin(y-1)+4\\
x-4 &= 2\sin(y-1)\\
\frac{x-4}{2} &= \sin(y-1)\\
\sin^{-1}\left(\frac{x-4}{2}\right) &= y-1\\
1+\sin^{-1}\left(\frac{x-4}{2}\right) &= y
\end{aligned}$$

10.

$$\begin{aligned}
h(x) &= 5-\cos^{-1}(2x+3)\\
y &= 5-\cos^{-1}(2x+3)\\
x &= 5-\cos^{-1}(2y+3)\\
x-5 &= -\cos^{-1}(2y+3)\\
5-x &= \cos^{-1}(2y+3)\\
\cos(5-x) &= 2y+3\\
\cos(5-x)-3 &= 2y\\
\frac{\cos(5-x)-3}{2} &= y
\end{aligned}$$

4.2 Graphing Inverse Trigonometric Functions

Learning Objectives

- Understand the meaning of restricted domain as it applies to the inverses of the six trigonometric functions.
- Apply the domain, range and quadrants of the six inverse trigonometric functions to evaluate expressions.

Finding the Inverse by Mapping

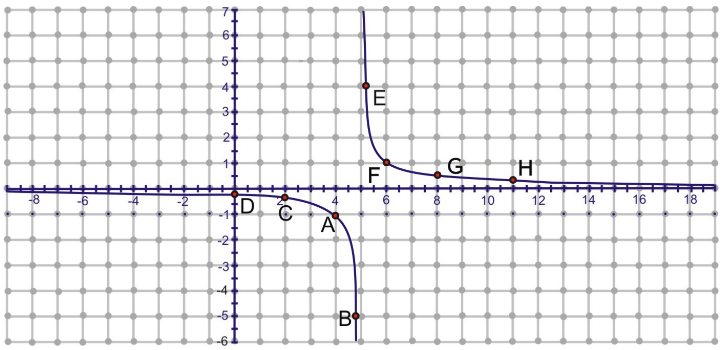
Determining an inverse function algebraically can be both involved and difficult, so it is useful to know how to map f to f^{-1}. The graph of f can be used to produce the graph of f^{-1} by applying the inverse reflection principle:

The points (a,b) and (b,a) in the coordinate plane are symmetric with respect to the line $y = x$.

The points (a,b) and (b,a) are reflections of each other across the line $y = x$.

Example 1: Find the inverse of $f(x) = \frac{1}{x-5}$ mapping.

Solution: From the last section, we know that the inverse of this function is $y = \frac{5x+1}{x}$. To find the inverse by mapping, pick several points on $f(x)$, reflect them using the reflection principle and plot.



A: (4, -1)

B: (4.8, -5)

C: (2, -0.3)

D: (0, -0.2)

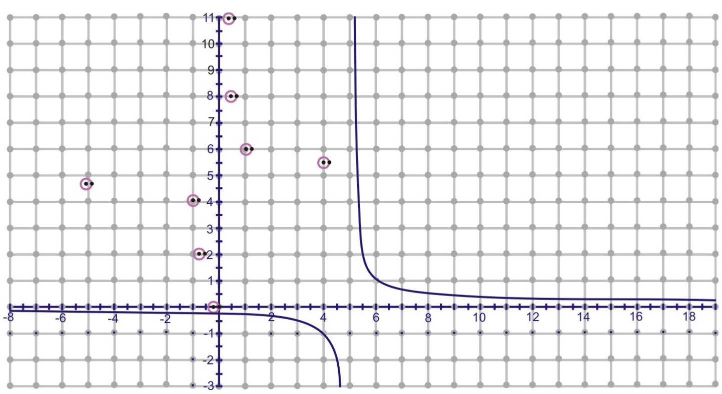
E: (5.3, 4)

F: (6, 1)

G: (8, 0.3)

H: (11, 0.2)

Now, take these eight points, switch the x and y and plot (y,x). Connect them to make the inverse function.



A^{-1} : $(-1,4)$

B^{-1} : $(-5,4.8)$

C^{-1} : $(-0.3,2)$

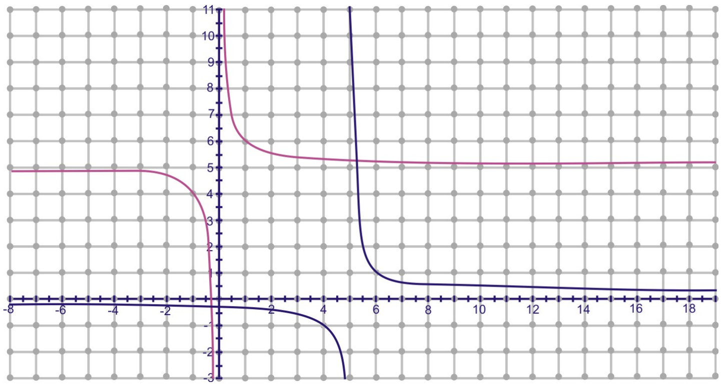
D^{-1} : $(-0.2,0)$

E^{-1} : $(4,5.3)$

F^{-1} : $(1,6)$

$G^{-1}: \ (0.3, 8)$

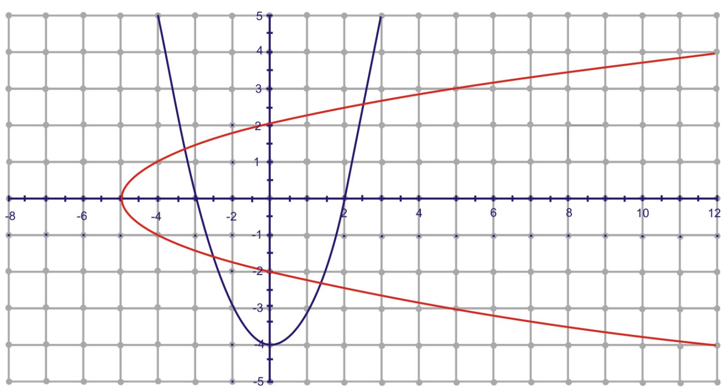
$H^{-1}: \ (0.2, 11)$



Not all functions have inverses that are one-to-one. However, the inverse can be modified to a one-to-one function if a "restricted domain" is applied to the inverse function.

Example 2: Find the inverse of $f(x) = x^2 - 4$.

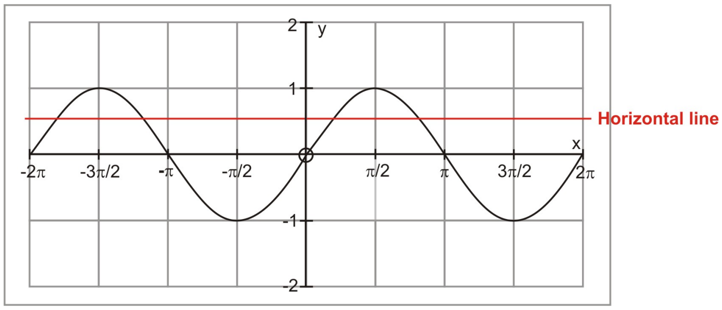
Solution: Let's use the graphic approach for this one. The function is graphed in blue and its inverse is red.



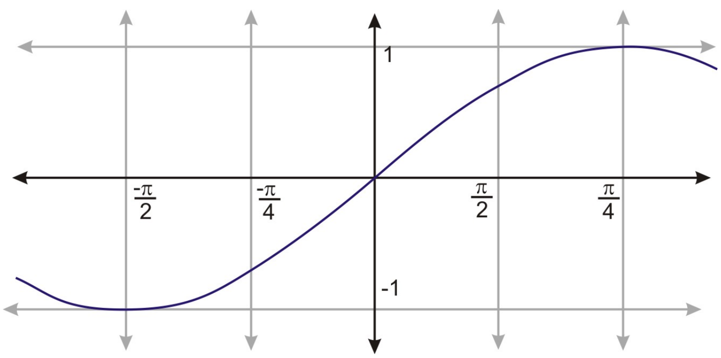
Clearly, the inverse relation is not a function because it does not pass the vertical line test. This is because all parabolas fail the horizontal line test. To "make" the inverse a function, we restrict the domain of the original function. For parabolas, this is fairly simple. To find the inverse of this function algebraically, we get $f^{-1}(x) = \sqrt{x+4}$. Technically, however, the inverse is $\pm\sqrt{x+4}$ because the square root of any number could be positive or negative. So, the inverse of $f(x) = x^2 - 4$ is both parts of the square root equation, $\sqrt{x+4}$ and $-\sqrt{x+4}$. $\sqrt{x+4}$ will yield the top portion of the horizontal parabola and $-\sqrt{x+4}$ will yield the bottom half. Be careful, because if you just graph $f^{-1}(x) = \sqrt{x+4}$ in your graphing calculator, it will only graph the top portion of the inverse.

This technique of sectioning the inverse is applied to finding the inverse of trigonometric functions because it is periodic.

Finding the Inverse of the Trigonometric Functions



In order to consider the inverse of this function, we need to restrict the domain so that we have a section of the graph that is one-to-one. If the domain of f is restricted to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ a new function $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$. is defined. This new function is one-to-one and takes on all the values that the function $f(x) = \sin x$ takes on. Since the restricted domain is smaller, $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ takes on all values once and only once.

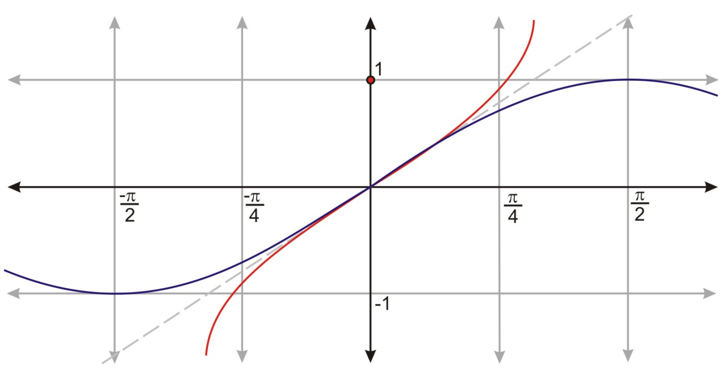


In the previous lesson the inverse of $f(x)$ was represented by the symbol $f^{-1}(x)$, and $y = f^{-1}(x) \Leftrightarrow f(y) = x$. The inverse of $\sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ will be written as $\sin^{-1} x$. or $\arcsin x$.

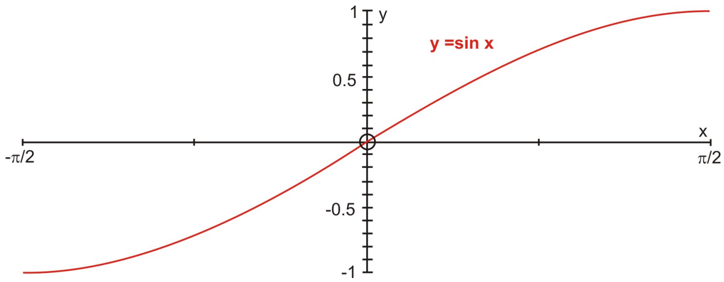
$$\left\{\begin{matrix} y = \sin^{-1} x \\ or \\ y = \arcsin x \end{matrix}\right\} \Leftrightarrow \sin y = x$$

In this lesson we will use both $\sin^{-1} x$ and $\arcsin x$ and both are read as "the inverse sine of x" or "the number between $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ whose sine is x."

The graph of $y = \sin^{-1} x$ is obtained by applying the inverse reflection principle and reflecting the graph of $y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ in the line $y = x$. The domain of $y = \sin x$ becomes the range of $y = \sin^{-1} x$, and hence the range of $y = \sin x$ becomes the domain of $y = \sin^{-1} x$.

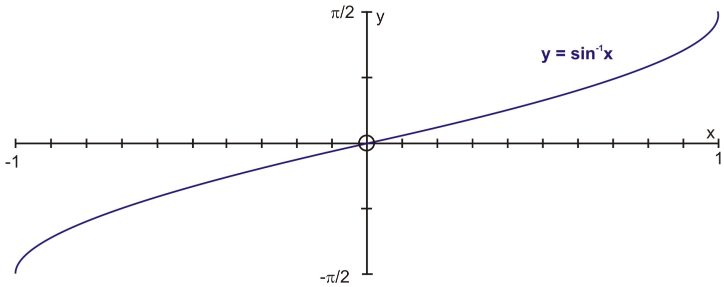


Another way to view these graphs is to construct them on separate grids. If the domain of $y = \sin x$ is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the result is a restricted one-to one function. The inverse sine function $y = \sin^{-1} x$ is the inverse of the restricted section of the sine function.



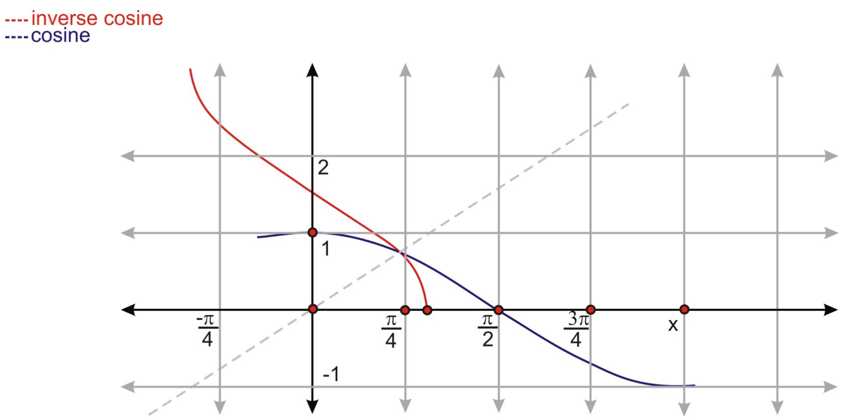
The domain of $y = \sin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **and the range is** [-1, 1].

The restriction of $y = \sin x$ is a one-to-one function and it has an inverse that is shown below.

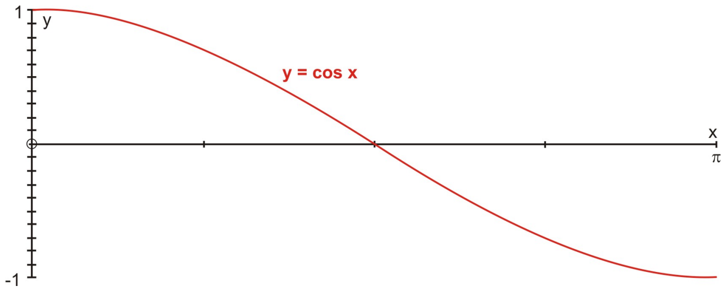


The domain of $y = \sin^{-1}$ **is** [-1, 1] **and the range is** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The inverse functions for cosine and tangent are defined by following the same process as was applied for the inverse sine function. However, in order to create one-to-one functions, different intervals are used. The cosine function is restricted to the interval $0 \le x \le \pi$ and the new function becomes $y = \cos x, 0 \le x \le \pi$. The inverse reflection principle is then applied to this graph as it is reflected in the line $y = x$ The result is the graph of $y = \cos^{-1} x$ (also expressed as $y = \arccos x$).

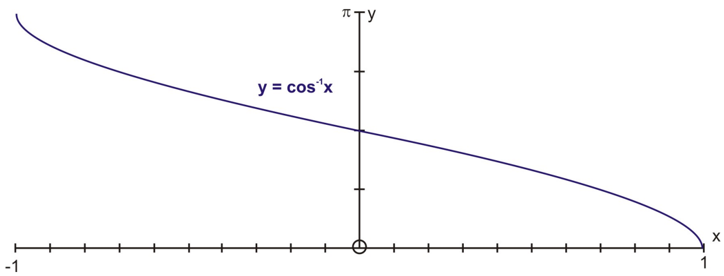


Again, construct these graphs on separate grids to determine the domain and range. If the domain of $y = \cos x$ is restricted to the interval $[0, \pi]$, the result is a restricted one-to one function. The inverse cosine function $y = \cos^{-1} x$ is the inverse of the restricted section of the cosine function.



The domain of $y = \cos x$ is $[0, \pi]$ **and the range is** [-1, 1].

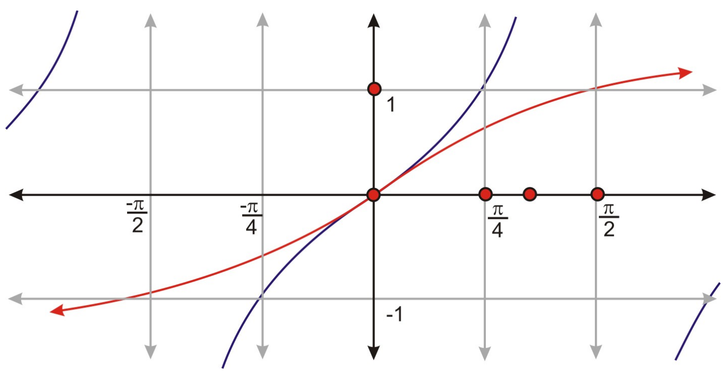
The restriction of $y = \cos x$ is a one-to-one function and it has an inverse that is shown below.



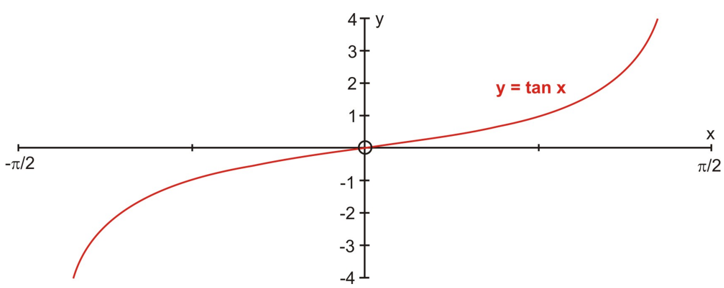
The statements $y = \cos x$ **and** $x = \cos y$ **are equivalent for** $y-$**values in the restricted domain** $[0, \pi]$ **and** $x-$**values between** -1 **and** 1.

The domain of $y = \cos^{-1} x$ is [-1, 1] **and the range is** $[0, \pi]$.

The tangent function is restricted to the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and the new function becomes $y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. The inverse reflection principle is then applied to this graph as it is reflected in the line $y = x$. The result is the graph of $y = \tan^{-1} x$ (also expressed as $y = \arctan x$).

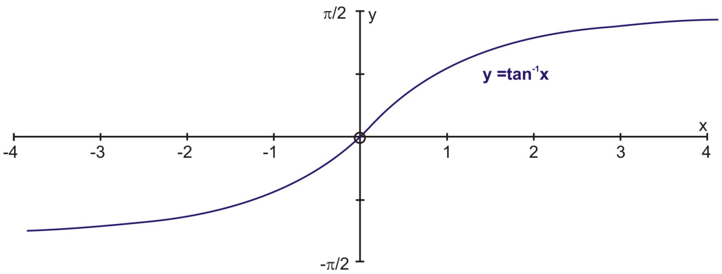


Graphing the two functions separately will help us to determine the domain and range. If the domain of $y = \tan x$ is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the result is a restricted one-to one function. The inverse tangent function $y = \tan^{-1} x$ is the inverse of the restricted section of the tangent function.



The domain of $y = \tan x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range is $[-\infty, \infty]$.

The restriction of $y = \tan x$ is a one-to-one function and it has an inverse that is shown below.



The statements $y = \tan x$ and $x = \tan y$ are equivalent for y–values in the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and x–values between -4 **and** +4.

The domain of $y = \tan^{-1} x$ is $[-\infty, \infty]$ **and the range is** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

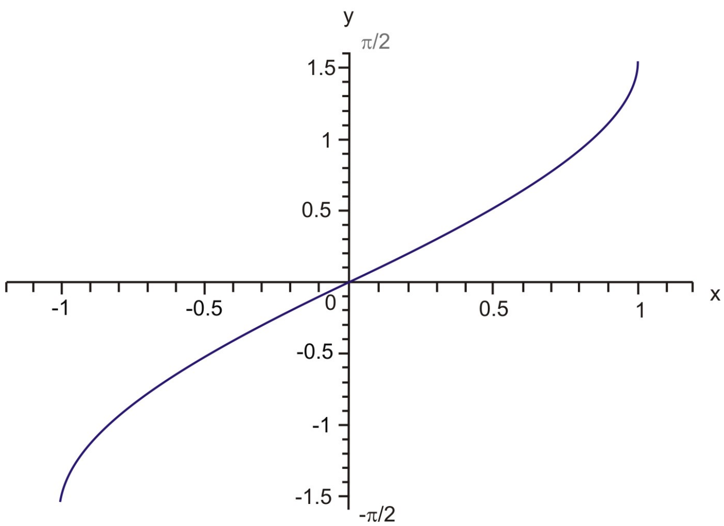
The above information can be readily used to evaluate inverse trigonometric functions without the use of a calculator. These calculations are done by applying the restricted domain functions to the unit circle. To summarize:

Table 4.1:

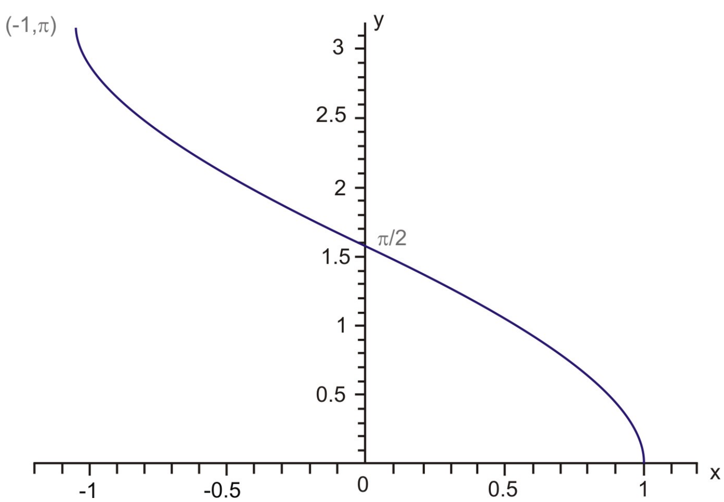
Restricted Domain Function	Inverse Trigonometric Function	Domain	Range	Quadrants
$y = \sin x$		$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	[-1, 1]	1 AND 4
	$y = \arcsin x$ $y = \sin^{-1} x$	[-1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	
$y = \cos x$		$[0, \pi]$	[-1, 1]	1 AND 2
	$y = \arccos x$ $y = \cos^{-1} x$	[-1, 1]	$[0, \pi]$	
$y = \tan x$		$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(-\infty, \infty)$	1 AND 4
	$y = \arctan x$ $y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	

Now that the three trigonometric functions and their inverses have been summarized, let's take a look at the graphs of these inverse trigonometric functions.

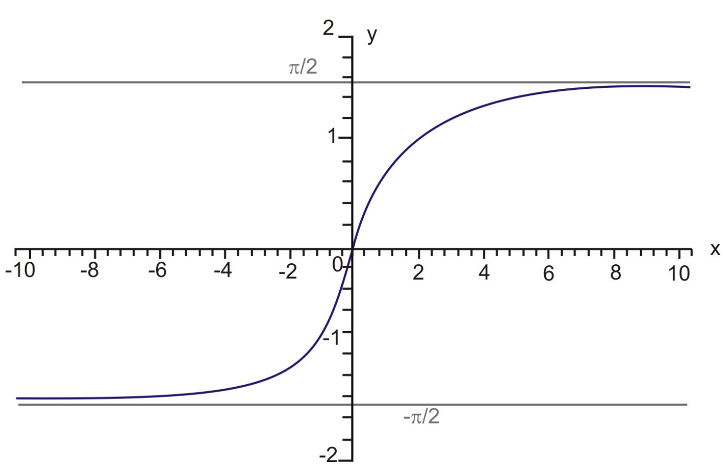
Graph of $y=\sin^{-1}x$



Graph of $y = \cos^{-1}x$



Graph of $y = \tan^{-1}x$



Points to Consider

- What are the restricted domains for the inverse relations of the trigonometric functions?
- Can the values of the special angles of the unit circle be applied to the inverse trigonometric functions?

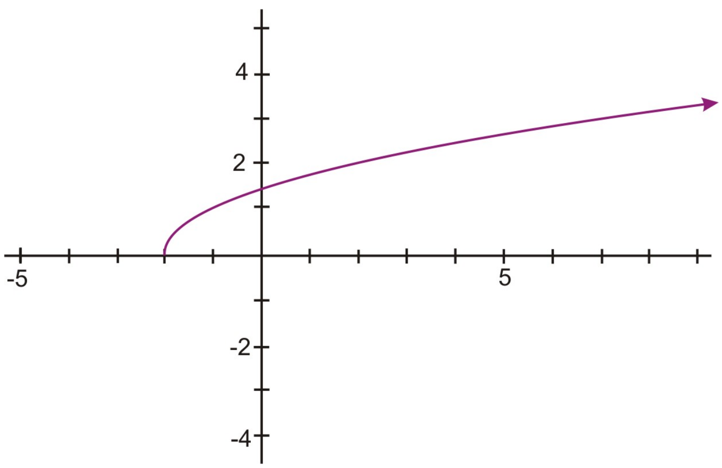
Review Questions

Study each of the following graphs and answer these questions:

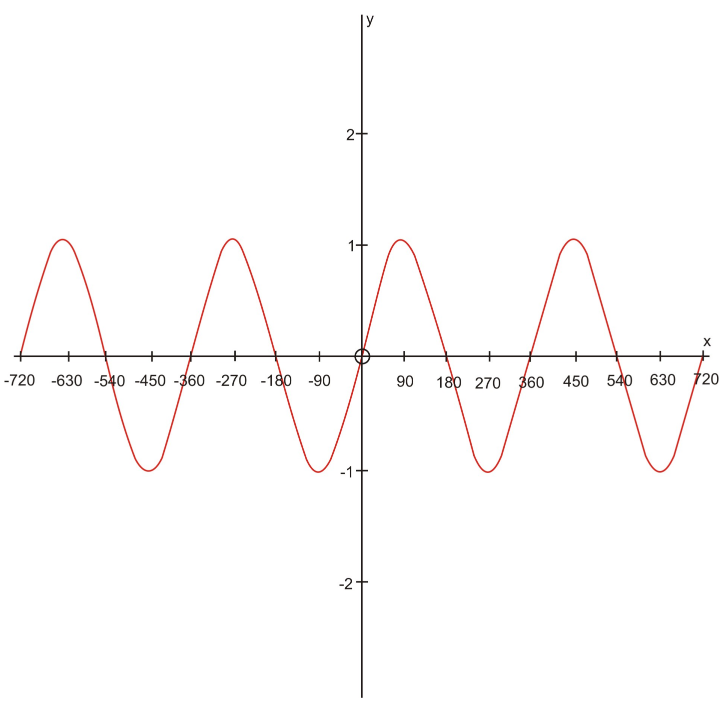
(a) Is the graphed relation a function?

(b) Does the relation have an inverse that is a function?

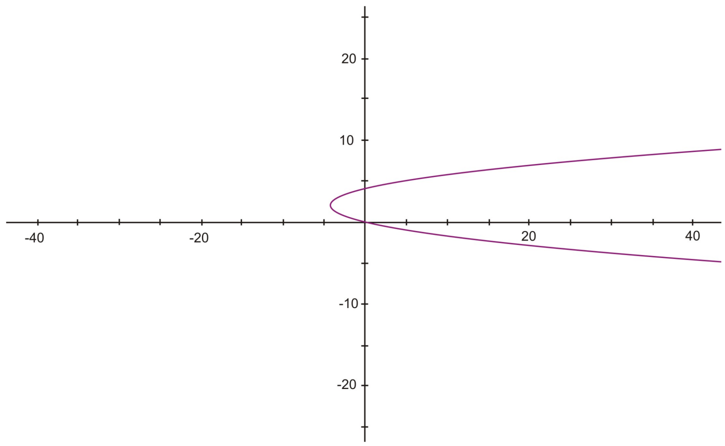
1.



2.



3.

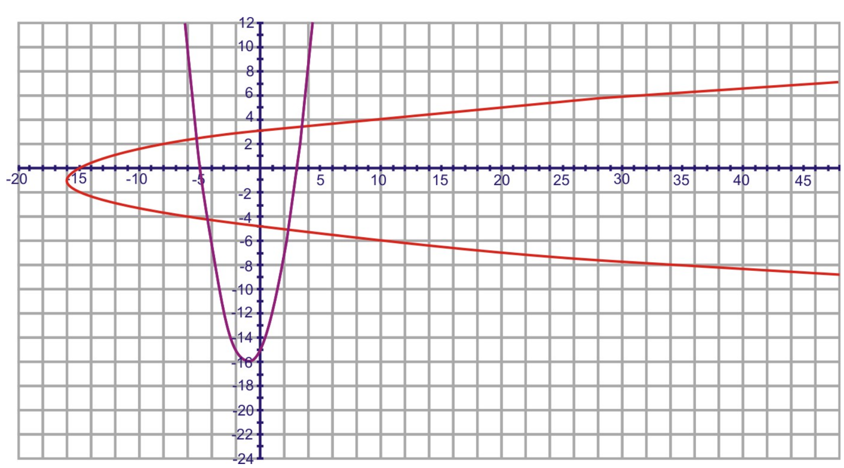


Find the inverse of the following functions using the mapping principle.

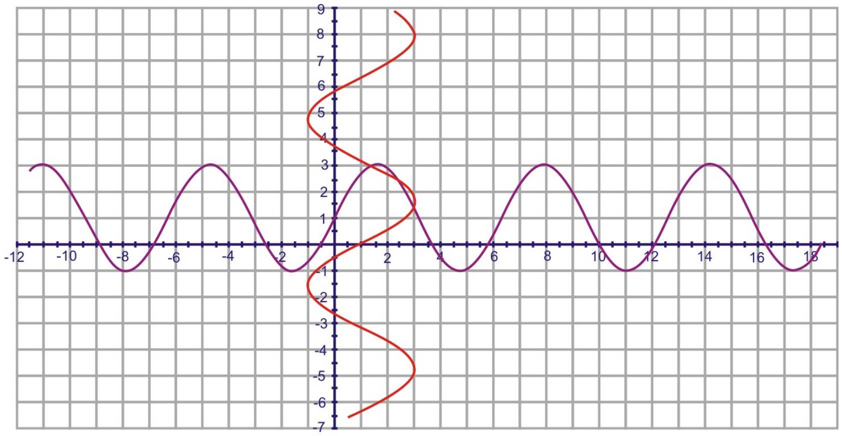
4. $f(x) = x^2 + 2x - 15$
5. $y = 1 + 2\sin x$
6. Sketch a graph of $y = \frac{1}{2}\cos^{-1}(3x+1)$. Sketch $y = \cos^{-1} x$ on the same set of axes and compare how the two differ.
7. Sketch a graph of $y = 3 - \tan^{-1}(x-2)$. Sketch $y = \tan^{-1} x$ on the same set of axes and compare how the two differ.
8. Graph $y = 2\sin^{-1}(2x-1) + 1$
9. Graph $y = 4 + \cos^{-1} \frac{1}{3}x$
10. Remember that sine and cosine are out of phase with each other, $\sin x = \cos\left(x - \frac{\pi}{2}\right)$. Find the inverse of $y = \cos\left(x - \frac{\pi}{2}\right)$. Is the inverse of $y = \left(\cos x - \frac{\pi}{2}\right)$ the same as $y = \sin^{-1} x$? Why or why not?

Review Answers

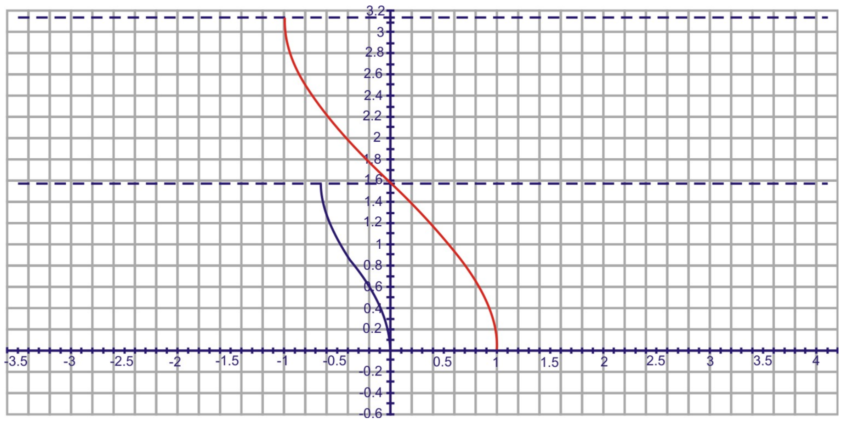
1. The graph represents a one-to-one function. It passes both a vertical and a horizontal line test. The inverse would be a function.
2. The graph represents a function, but is not one-to-one because it does not pass the horizontal line test. Therefore, it does not have an inverse that is a function.
3. The graph does not represent a one-to-one function. It fails a vertical line test. However, its inverse would be a function.
4. By selecting 4-5 points and switching the x and y values, you will get the red graph below.



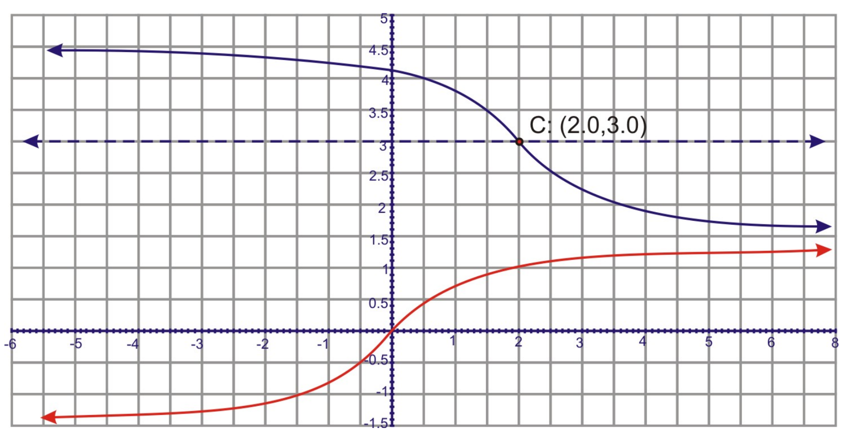
5. By selecting 4-5 points and switching the x and y values, you will get the red graph below.



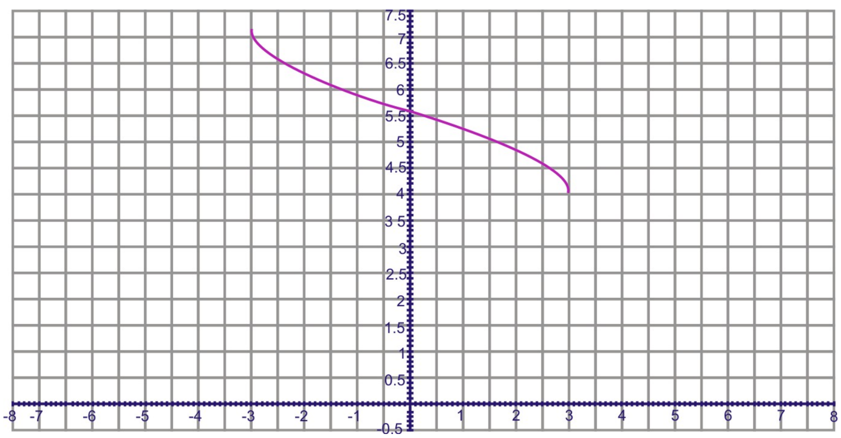
6. $y = \frac{1}{2} \cos^{-1}(3x + 1)$ is in blue and $y = \cos^{-1}(x)$ is in red. Notice that $y = \frac{1}{2} \cos^{-1}(3x + 1)$ has half the amplitude and is shifted over -1. The 3 seems to narrow the graph.



7. $y = 3 - \tan^{-1}(x - 2)$ is in blue and $y = \tan^{-1} x$ is in red. $y = 3 - \tan^{-1}(x - 2)$ is shifted up 3 and to the right 2 (as indicated by point C, the "center") and is flipped because of the $-\tan^{-1}$.



8. 

9. 

10.

$$
\begin{aligned}
y &= \cos\left(x - \frac{\pi}{2}\right) \\
x &= \cos\left(y - \frac{\pi}{2}\right) \\
\cos^{-1} x &= y - \frac{\pi}{2} \\
\frac{\pi}{2} + \cos^{-1} x &= y
\end{aligned}
$$

$\sin^{-1} x \neq \frac{\pi}{2} + \cos^{-1} x$, graphing the two equations will illustrate that the two are not the same. This is because of the restricted domain on the inverses. Since the functions are periodic, there is a phase shift of cosine that, when the inverse is found, is equal to sine inverse.

4.3 Inverse Trigonometric Properties

Learning Objectives

- Relate the concept of inverse functions to trigonometric functions.
- Reduce the composite function to an algebraic expression involving no trigonometric functions.
- Use the inverse reciprocal properties.
- Compose each of the six basic trigonometric functions with $\tan^{-1} x$.

Composing Trig Functions and their Inverses

In the Prerequisite Chapter, you learned that for a function $f(f^{-1}(x)) = x$ for all values of x for which $f^{-1}(x)$ is defined. If this property is applied to the trigonometric functions, the following equations will be true whenever they are defined:

$$\sin(\sin^{-1}(x)) = x \qquad \cos(\cos^{-1}(x)) = x \qquad \tan(\tan^{-1}(x)) = x$$

As well, you learned that $f^{-1}(f(x)) = x$ for all values of x for which $f(x)$ is defined. If this property is applied to the trigonometric functions, the following equations that deal with finding an inverse trig. function of a trig. function, will only be true for values of x within the restricted domains.

$$\sin^{-1}(\sin(x)) = x \qquad \cos^{-1}(\cos(x)) = x \qquad \tan^{-1}(\tan(x)) = x$$

These equations are better known as composite functions and are composed of one trigonometric function in conjunction with another different trigonometric function. The composite functions will become algebraic functions and will not display any trigonometry. Let's investigate this phenomenon.

Example 1: Find $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$.

Solution: We know that $\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$, within the defined restricted domain. Then, we need to find $\sin\frac{\pi}{4}$, which is $\frac{\sqrt{2}}{2}$. So, the above properties allow for a short cut. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$, think of it like the sine and sine inverse cancel each other out and all that is left is the $\frac{\sqrt{2}}{2}$.

Composing Trigonometric Functions

Besides composing trig functions with their own inverses, you can also compose any trig functions with any inverse. When solving these types of problems, start with the function that is composed inside of the other and work your way out. Use the following examples as a guideline.

Example 2: Without using technology, find the exact value of each of the following:

a. $\cos\left(\tan^{-1}\sqrt{3}\right)$

b. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

c. $\cos(\tan^{-1}(-1))$

d. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$

Solution: For all of these types of problems, the answer is restricted to the inverse functions' ranges.

a. $\cos\left(\tan^{-1}\sqrt{3}\right)$: First find $\tan^{-1}\sqrt{3}$, which is $\frac{\pi}{3}$. Then find $\cos\frac{\pi}{3}$. Your final answer is $\frac{1}{2}$. Therefore, $\cos\left(\tan^{-1}\sqrt{3}\right) = \frac{1}{2}$.

b. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

c. $\cos(\tan^{-1}(-1)) = \cos^{-1}\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

d. $\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Inverse Reciprocal Functions

We already know that the cosecant function is the reciprocal of the sine function. This will be used to derive the reciprocal of the inverse sine function.

$$\begin{aligned}
y &= \sin^{-1} x \\
x &= \sin y \\
\frac{1}{x} &= \csc y \\
\csc^{-1} \frac{1}{x} &= y \\
\csc^{-1} \frac{1}{x} &= \sin^{-1} x
\end{aligned}$$

Because cosecant and sect are inverses, $\sin^{-1} \frac{1}{x} = \csc^{-1} x$ is also true.

The inverse reciprocal identity for cosine and secant can be proven by using the same process as above. However, remember that these inverse functions are defined by using restricted domains and the reciprocals of these inverses must be defined with the intervals of domain and range on which the definitions are valid.

$$\sec^{-1} \frac{1}{x} = \cos^{-1} x \leftrightarrow \cos^{-1} \frac{1}{x} = \sec^{-1} x$$

Tangent and cotangent have a slightly different relationship. Recall that the graph of cotangent differs from tangent by a reflection over the $y-$axis and a shift of $\frac{\pi}{2}$. As an equation, this can be written as $\cot x = -\tan\left(x - \frac{\pi}{2}\right)$. Taking the inverse of this function will show the inverse reciprocal relationship between arccotangent and arctangent.

$$\begin{aligned}
y &= -\tan\left(x - \frac{\pi}{2}\right) \\
x &= -\tan\left(y - \frac{\pi}{2}\right) \\
-x &= \tan\left(y - \frac{\pi}{2}\right) \\
\tan^{-1}(-x) &= y - \frac{\pi}{2} \\
\frac{\pi}{2} + \tan^{-1}(-x) &= y \\
\frac{\pi}{2} - \tan^{-1} x &= y
\end{aligned}$$

Remember that tangent is an odd function, so that $\tan(-x) = -\tan(x)$. Because tangent is odd, its inverse is also odd. So, this tells us that $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$ and $\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x$. You will determine the domain and range of all of these functions when you graph them in the exercises for this section. To graph arcsecant, arccosecant, and arccotangent in your calculator you will use these conversion identities: $\sec^{-1} x = \cos^{-1} \frac{1}{x}, \csc^{-1} x = \sin^{-1} \frac{1}{x}, \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$.

Now, let's apply these identities to some problems that will give us an insight into how they work.

Example 3: Evaluate $\sec^{-1} \sqrt{2}$

Solution: Use the inverse reciprocal property. $\sec^{-1} x = \cos^{-1} \frac{1}{x} \rightarrow \sec^{-1} \sqrt{2} = \cos^{-1} \frac{1}{\sqrt{2}}$. Recall that $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. So, $\sec^{-1} \sqrt{2} = \cos^{-1} \frac{\sqrt{2}}{2}$, and we know that $\cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$. Therefore, $\sec^{-1} \sqrt{2} = \frac{\pi}{4}$.

Example 4: Find the exact value of each expression within the restricted domain, without a calculator.

a. $\sec^{-1}\sqrt{2}$

b. $\cot^{-1}\left(-\sqrt{3}\right)$

c. $\csc^{-1}\frac{2\sqrt{3}}{3}$

Solution: For each of these problems, first find the reciprocal and then determine the angle from that.

a. $\sec^{-1}\sqrt{2} = \cos^{-1}\frac{\sqrt{2}}{2}$ From the unit circle, we know that the answer is $\frac{\pi}{4}$.

b. $\cot^{-1}\left(-\sqrt{3}\right) = \frac{\pi}{2} - \tan^{-1}\left(-\sqrt{3}\right)$ From the unit circle, the answer is $\frac{5\pi}{6}$.

c. $\csc^{-1}\frac{2\sqrt{3}}{3} = \sin^{-1}\frac{\sqrt{3}}{2}$ Within our interval, there are is one answer, $\frac{\pi}{3}$.

Example 5: Using technology, find the value in radian measure, of each of the following:

a. $\arcsin 0.6384$

b. $\arccos(-0.8126)$

c. $\arctan(-1.9249)$

Solution:

a.
```
sin⁻¹(.6384)
         .69241775
```

b.
```
cos⁻¹(-.8126
     2.519395724
```

c.
```
tan⁻¹(-1.9249
    -1.091664781
```

Make sure that your calculator's MODE is RAD (radians).

Composing Inverse Reciprocal Trig Functions

In this subsection, we will combine what was learned in the previous two sections. Here are a few examples:

Example 6: Without a calculator, find $\cos\left(\cot^{-1}\sqrt{3}\right)$.

Solution: First, find $\cot^{-1}\sqrt{3}$, which is also $\tan^{-1}\frac{\sqrt{3}}{3}$. This is $\frac{\pi}{6}$. Now, find $\cos\frac{\pi}{6}$, which is $\frac{\sqrt{3}}{2}$. So, our answer is $\frac{\sqrt{3}}{2}$.

Example 7: Without a calculator, find $\sec^{-1}\left(\csc\frac{\pi}{3}\right)$.

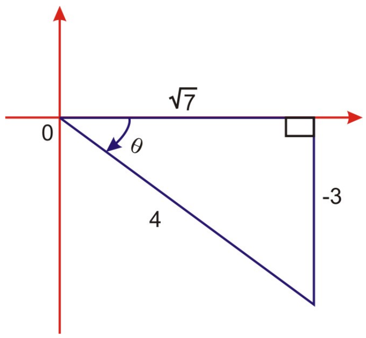
Solution: First, $\csc\frac{\pi}{3} = \frac{1}{\sin\frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. Then $\sec^{-1}\frac{2\sqrt{3}}{3} = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$.

Example 8: Evaluate $\cos\left(\sin^{-1}\frac{3}{5}\right)$.

Solution: Even though this problem is not a critical value, it can still be done without a calculator. Recall that sine is the opposite side over the hypotenuse of a triangle. So, 3 is the opposite side and 5 is the hypotenuse. This is a Pythagorean Triple, and thus, the adjacent side is 4. To continue, let $\theta = \sin^{-1}\frac{3}{5}$ or $\sin\theta = \frac{3}{5}$, which means θ is in the Quadrant 1 (from our restricted domain, it cannot also be in Quadrant II). Substituting in θ we get $\cos\left(\sin^{-1}\frac{3}{5}\right) = \cos\theta$ and $\cos\theta = \frac{4}{5}$.

Example 9: Evaluate $\tan\left(\sin^{-1}\left(-\frac{3}{4}\right)\right)$

Solution: Even though $\frac{3}{4}$ does not represent two lengths from a Pythagorean Triple, you can still use the Pythagorean Theorem to find the missing side. $(-3)^2 + b^2 = 4^2$, so $b = \sqrt{16-9} = \sqrt{7}$. From the restricted domain, sine inverse is negative in the 4^{th} Quadrant. To illustrate:

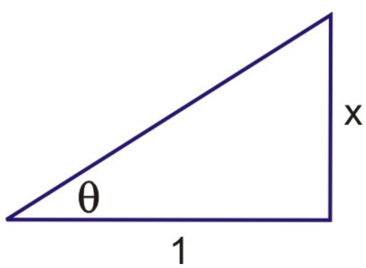


Let

$$\theta = \sin^{-1}\left(-\frac{3}{4}\right)$$
$$\sin\theta = -\frac{3}{4}$$
$$\tan\left(\sin^{-1}\left(-\frac{3}{4}\right)\right) = \tan\theta$$
$$\tan\theta = \frac{-3}{\sqrt{7}} \text{ or } \frac{-3\sqrt{7}}{7}$$

Trigonometry in Terms of Algebra

All of the trigonometric functions can be rewritten in terms of only x, when using one of the inverse trigonometric functions. Starting with tangent, we draw a triangle where the opposite side (from θ) is defined as x and the adjacent side is 1. The hypotenuse, from the Pythagorean Theorem would be $\sqrt{x^2+1}$. Substituting $\tan^{-1} x$ for θ, we get:



$$\tan\theta = \frac{x}{1}$$
$$\tan\theta = x \qquad \textit{hypotenuse} = \sqrt{x^2+1}$$
$$\theta = \tan^{-1} x$$

$$\sin(\tan^{-1} x) = \sin\theta = \frac{x}{\sqrt{x^2+1}} \qquad \csc(\tan^{-1} x) = \csc\theta = \frac{\sqrt{x^2+1}}{x}$$
$$\cos(\tan^{-1} x) = \cos\theta = \frac{1}{\sqrt{x^2+1}} \qquad \sec(\tan^{-1} x) = \sec\theta = \sqrt{x^2+1}$$
$$\tan(\tan^{-1} x) = \tan\theta = x \qquad \cot(\tan^{-1} x) = \cot\theta = \frac{1}{x}$$

Example 10: Find $\sin(\tan^{-1} 3x)$.

Solution: Instead of using x in the ratios above, use $3x$.

$$\sin(\tan^{-1} 3x) = \sin\theta = \frac{3x}{\sqrt{(3x)^2+1}} = \frac{3x}{\sqrt{9x^2+1}}$$

Example 11: Find $\sec^2(\tan^{-1} x)$.

Solution: This problem might be better written as $[\sec(\tan^{-1} x)]^2$. Therefore, all you need to do is square the ratio above.

$$[\sec(\tan^{-1} x)]^2 = \left(\sqrt{x^2+1}\right)^2 = x^2+1$$

You can also write the all of the trig functions in terms of arcsine and arccosine. However, for each inverse function, there is a different triangle. You will derive these formulas in the exercise for this section.

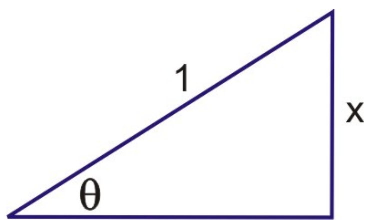
Points to Consider

- Is it possible to graph these composite functions? What happens when you graph $y = \sin(\cos^{-1} x)$ in your calculator?
- Do exact values of functions of inverse functions exist if any value is used?

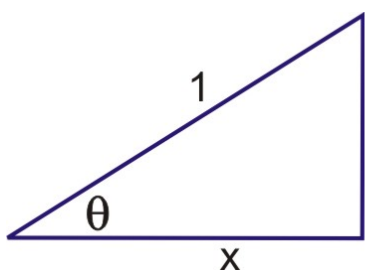
Review Questions

1. Evaluate each of the following:
   (a) $\cos^{-1} \frac{\sqrt{3}}{2}$
   (b) $\sec^{-1} \sqrt{2}$
   (c) $\sec^{-1}\left(-\sqrt{2}\right)$
   (d) $\sec^{-1}(-2)$
   (e) $\cot^{-1}(-1)$
   (f) $\csc^{-1}\left(\sqrt{2}\right)$
2. Use your calculator to find:
   (a) $\arccos(-0.923)$
   (b) $\arcsin 0.368$
   (c) $\arctan 5.698$
3. Find the exact value of the functions, without a calculator, over their restricted domains.
   (a) $\csc\left(\cos^{-1} \frac{\sqrt{3}}{2}\right)$
   (b) $\sec^{-1}(\tan(\cot^{-1} 1))$
   (c) $\tan^{-1}\left(\cos \frac{\pi}{2}\right)$
   (d) $\cot\left(\sec^{-1} \frac{2\sqrt{3}}{3}\right)$
4. Using your graphing calculator, graph $y = \sec^{-1} x$. Sketch this graph, determine the domain and range, $x-$ and/or $y-$intercepts. (Your calculator knows the restriction on this function, there is no need to input it into $Y =$.)
5. Using your graphing calculator, graph $y = \csc^{-1} x$. Sketch this graph, determine the domain and range, $x-$ and/or $y-$intercepts. (Your calculator knows the restriction on this function, there is no need to input it into $Y =$.)

6. Using your graphing calculator, graph $y = \cot^{-1} x$. Sketch this graph, determine the domain and range, $x-$ and/or $y-$intercepts. (Your calculator knows the restriction on this function, there is no need to input it into $Y =$.)
7. Evaluate:
   (a) $\sin\left(\cos^{-1} \frac{5}{13}\right)$
   (b) $\tan\left(\sin^{-1}\left(-\frac{6}{11}\right)\right)$
   (c) $\cos\left(\csc^{-1} \frac{25}{7}\right)$
8. Express each of the following functions as an algebraic expression involving no trigonometric functions.
   (a) $\cos^2(\tan^{-1} x)$
   (b) $\cot(\tan^{-1} x^2)$
9. To find trigonometric functions in terms of sine inverse, use the following triangle.



   (a) Determine the sine, cosine and tangent in terms of arcsine.
   (b) Find $\tan(\sin^{-1} 2x^3)$.
10. To find the trigonometric functions in terms of cosine inverse, use the following triangle.



   (a) Determine the sine, cosine and tangent in terms of arccosine.
   (b) Find $\sin^2\left(\cos^{-1} \frac{1}{2}x\right)$.

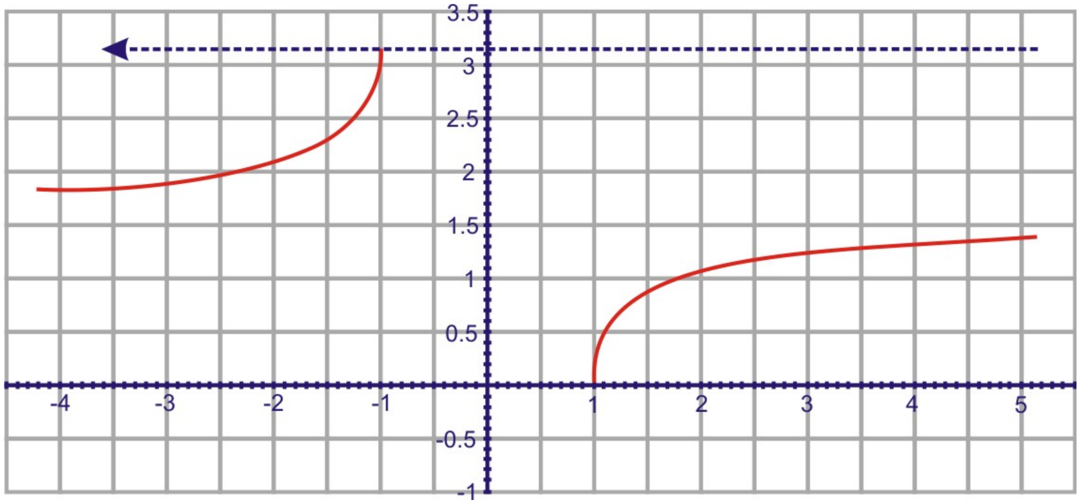
Review Answers

1. (a) $\frac{\pi}{6}$
   (b) $\frac{\pi}{4}$
   (c) $\frac{5\pi}{4}$
   (d) $\frac{2\pi}{3}, \frac{4\pi}{3}$
   (e) $\frac{3\pi}{4}, \frac{7\pi}{4}$
   (f) $\frac{\pi}{4}, \frac{3\pi}{4}$
2. (a) 2.747
   (b) 0.377
   (c) 1.397
3. (a) $\csc\left(\cos^{-1} \frac{\sqrt{3}}{2}\right) = \csc \frac{\pi}{6} = 2$
   (b) $\sec^{-1}(\tan(\cot^{-1} 1)) = \sec^{-1}\left(\tan \frac{\pi}{4}\right) = \sec^{-1} 1 = 0$

(c) $\tan^{-1}\left(\cos\frac{\pi}{2}\right) = \tan^{-1} 0 = 0$

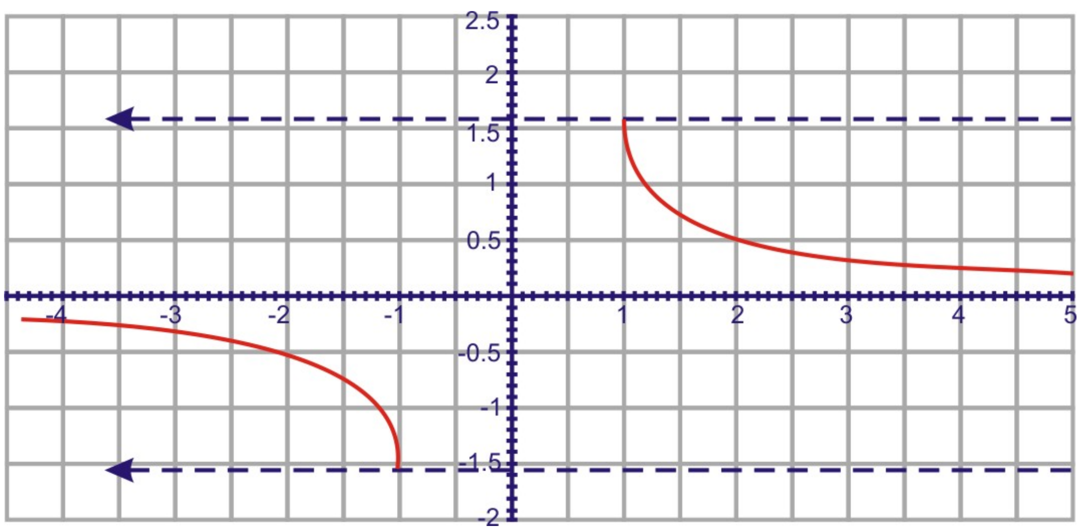
(d) $\cot\left(\sec^{-1}\frac{2\sqrt{3}}{3}\right) = \cot\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) = \cot\frac{\pi}{6} = \frac{1}{\tan\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{3}} = \sqrt{3}$

4. $y = \sec^{-1} x$ when plugged into your graphing calculator is $y = \cos^{-1}\frac{1}{x}$.



The domain is all reals, excluding the interval (-1, 1). The range is all reals in the interval $[0, \pi], y \neq \frac{\pi}{2}$. There are no y intercepts and the only x intercept is at 1.

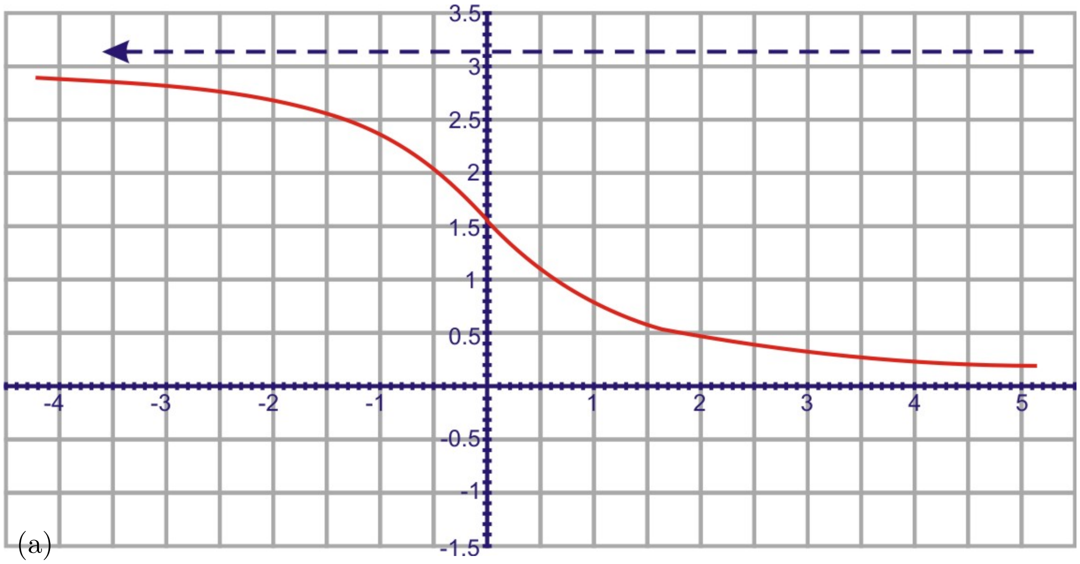
5. $y = \csc^{-1} x$ when plugged into your graphing calculator is $y = \sin^{-1}\frac{1}{x}$.



The domain is all reals, excluding the interval (-1, 1). The range is all reals in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}], y \neq 0$. There are no x or y intercepts.

6. The domain is all real numbers and the range is from $(0, \pi)$. There is an $x-$intercept at $\frac{\pi}{2}$.

7. (a)



$$\cos\theta = \frac{5}{13}$$
$$\sin\left(\cos^{-1}\left(\frac{5}{13}\right)\right) = \sin\theta$$
$$\sin\theta = \frac{12}{13}$$

(b) $\tan\left(\sin^{-1}\left(-\frac{6}{11}\right)\right) \to \sin\theta = -\frac{6}{11}$. The third side is $b = \sqrt{121-36} = \sqrt{85}$. $\tan\theta = -\frac{6}{\sqrt{85}} = -\frac{6\sqrt{85}}{85}$

(c) $\cos\left(\csc^{-1}\left(\frac{25}{7}\right)\right) \to \csc\theta = \frac{25}{7} \to \sin\theta = \frac{7}{25}$. This two lengths of a Pythagorean Triple, with the third side being 24. $\cos\theta = \frac{24}{25}$

8. (a) $\frac{1}{x^2+1}$
(b) $\frac{1}{x^2}$

9. (a) The adjacent side to θ is $\sqrt{1-x^2}$, so the three trig functions are:

$$\sin(\sin^{-1} x) = \sin\theta = x$$
$$\cos(\sin^{-1} x) = \cos\theta = \sqrt{1-x^2}$$
$$\tan(\sin^{-1} x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$

(b)

$$\tan(\sin^{-1}(2x^3)) = \frac{2x^3}{\sqrt{1-(2x^3)^2}} = \frac{2x^3}{\sqrt{1-8x^6}}$$

10. (a) The opposite side to θ is $\sqrt{1-x^2}$, so the three trig functions are:

$$\sin(\cos^{-1} x) = \sin\theta = \sqrt{1-x^2}$$
$$\cos(\cos^{-1} x) = \cos\theta = x$$
$$\tan(\cos^{-1} x) = \tan\theta = \frac{\sqrt{1-x^2}}{x}$$

(b)

$$\sin^2\left(\cos^{-1}\left(\frac{1}{2}x\right)\right) = \left(\sqrt{1-\left(\frac{1}{2}x\right)^2}\right)^2 = 1 - \frac{1}{4}x^2$$

4.4 Applications & Models

Learning Objectives

- Apply inverse trigonometric functions to real life situations.

The following problems are real-world problems that can be solved using the trigonometric functions. In everyday life, indirect measurement is used to obtain answers to problems that are impossible to solve using measurement tools. However, mathematics will come to the rescue in the form of trigonometry to calculate these unknown measurements.

Example 1: On a cold winter day the sun streams through your living room window and causes a warm, toasty atmosphere. This is due to the angle of inclination of the sun which directly affects the heating and the cooling of buildings. Noon is when the sun is at its maximum height in the sky and at this time, the angle is greater in the summer than in the winter. Because of this, buildings are constructed such that the overhang of the roof can act as an awning to shade the windows for cooling in the summer and yet allow the sun's rays to provide heat in the winter. In addition to the construction of the building, the angle of inclination of the sun varies according to the latitude of the building's location.

If the latitude of the location is known, then the following formula can be used to calculate the angle of inclination of the sun on any given date of the year:

Angle of sun $= 90^\circ - \text{latitude} + -23.5^\circ \cdot \cos\left[(N+10)\frac{360}{365}\right]$ where N represents the number of the day of the year that corresponds to the date of the year. Note: This formula is accurate to $\pm\frac{1^\circ}{2}$

a. Determine the measurement of the sun's angle of inclination for a building located at a latitude of 42°, March 10^{th}, the 69^{th} day of the year.

Solution:

$$\begin{aligned}
\text{Angle of sun} &= 90^\circ - 42^\circ + -23.5^\circ \cdot \cos\left[(69+10)\frac{360}{365}\right]\\
\text{Angle of sun} &= 48^\circ + -23.5^\circ(0.2093)\\
\text{Angle of sun} &= 48^\circ - 4.92^\circ\\
\text{Angle of sun} &= 43.08^\circ
\end{aligned}$$

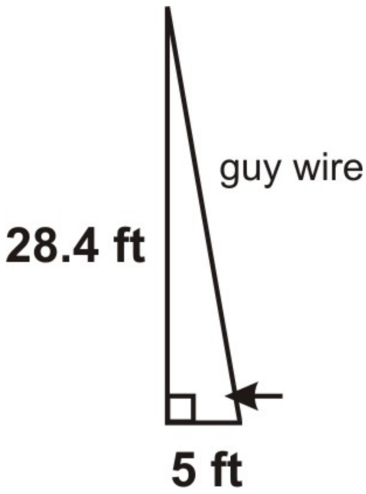
b. Determine the measurement of the sun's angle of inclination for a building located at a latitude of 20°, September 21^{st}.

Solution:

$$\begin{aligned}
\text{Angle of sun} &= 90^\circ - 20^\circ + -23.5^\circ \cdot \cos\left[(264+10)\frac{360}{365}\right]\\
\text{Angle of sun} &= 70^\circ + -23.5^\circ(0.0043)\\
\text{Angle of sun} &= 70.10^\circ
\end{aligned}$$

Example 2: A tower, 28.4 feet high, must be secured with a guy wire anchored 5 feet from the base of the tower. What angle will the guy wire make with the ground?

Solution: Draw a picture.



$$\tan \theta = \frac{opp.}{adj.}$$
$$\tan \theta = \frac{28.4}{5}$$
$$\tan \theta = 5.68$$
$$\tan^{-1}(\tan \theta) = \tan^{-1}(5.68)$$
$$\theta = 80.02^\circ$$

The following problem that involves functions and their inverses will be solved using the property $f(f^{-1}(x)) = f^{-1}(f(x))$. In addition, technology will also be used to complete the solution.

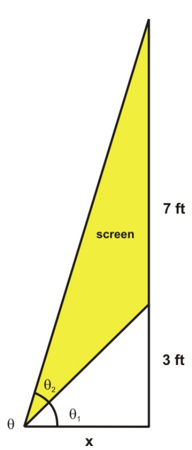
Example 3: In the main concourse of the local arena, there are several viewing screens that are available to watch so that you do not miss any of the action on the ice. The bottom of one screen is 3 feet above eye level and the screen itself is 7 feet high. The angle of vision (inclination) is formed by looking at both the bottom and top of the screen.

a. Sketch a picture to represent this problem.

b. Calculate the measure of the angle of vision that results from looking at the bottom and then the top of the screen. At what distance from the screen does this value of the angle occur?

Solution:

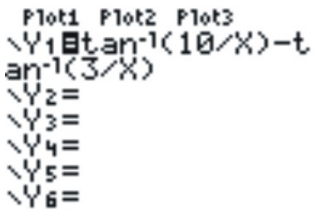
a.

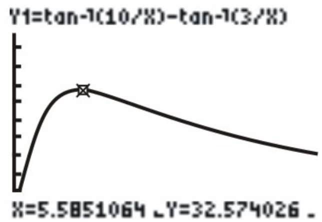


b.

$$\theta_2 = \tan\theta - \tan\theta_1$$
$$\tan\theta = \frac{10}{x} \text{ and } \tan\theta_1 = \frac{3}{x}$$
$$\theta_2 = \tan^{-1}\left(\frac{10}{x}\right) - \tan^{-1}\left(\frac{3}{x}\right)$$

To determine these values, use a graphing calculator and the trace function to determine when the actual maximum occurs.

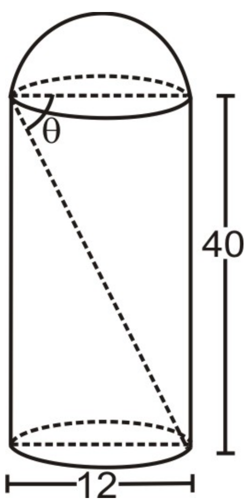




From the graph, it can be seen that the maximum occurs when $x \approx 5.59\ ft.$ and $\theta \approx 32.57°$.

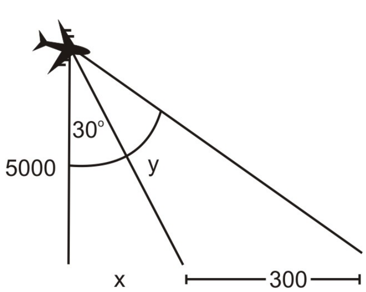
Example 4: A silo is 40 feet high and 12 feet across. Find the angle of depression from the top edge of the silo to the floor of the opposite edge.

Solution: $\tan\theta = \frac{40}{12} \rightarrow \theta = \tan^{-1}\frac{40}{12} = 73.3°$



Example 5: The pilot of an airplane flying at an elevation of 5000 feet sights two towers that are 300 feet apart. If the angle of depression to the tower closer to him is 30°, determine the angle of depression to the second tower.

Solution: Draw a picture. First we need to find x in order to find y.

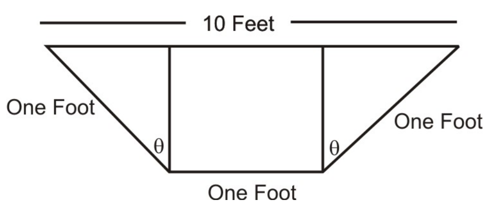


$$\tan 30^\circ = \frac{x}{5000} \rightarrow x = 5000 \tan 30^\circ, x = 2886.75$$
$$\tan y = \frac{3186.75}{5000}$$
$$y = \tan^{-1} \frac{3186.75}{5000}$$
$$y = 32.51^\circ$$

Which means that the two towers are 2.51° apart.

Review Questions

1. The intensity of a certain type of polarized light is given by the equation $I = I_0 \sin 2\theta \cos 2\theta$. Solve for θ.
2. The following diagram represents the ends of a water-trough. The ends are actually isosceles trapezoids. Determine the maximum value of the trough and the value of θ that maximizes the volume.



3. A boat is docked at the end of a 10 foot pier. The boat leaves the pier and drops anchor 230 feet away 3 feet straight out from shore (which is perpendicular to the pier). What was the bearing of the boat?
4. The electric current in a certain circuit is given by $i = I_m[\sin(wt + \alpha)\cos\varphi + \cos(wt + \alpha)\sin\varphi]$ Solve for t.
5. Using the formula from Example 1, determine the measurement of the sun's angle of inclination for a building located at a latitude of:

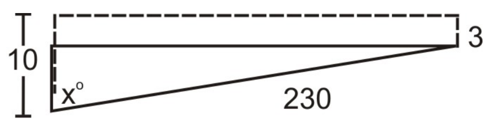
   (a) 64° on the 16^{th} of November
   (b) 15° on the 8^{th} of August

6. A ship leaves port and travels due east 15 nautical miles, then changes course to $N\ 20^\circ\ W$ and travels 40 more nautical miles. Find the bearing to the port of departure.
7. Find the maximum displacement for the simple harmonic motion described by $d = 4\cos\pi t$.
8. The pilot of an airplane flying at an elelvation of 10,000 feet sights two towers that are 500 feet apart. If the angle of depression to the tower closer to him is 18°, determine the anglf of depression to the second tower.

Review Answers

1.

$$
\begin{aligned}
I &= I_0 \sin 2\theta \cos 2\theta \\
\frac{I}{I_0} &= \frac{I_0}{I_0} \sin 2\theta \cos 2\theta \\
\frac{I}{I_0} &= \sin 2\theta \cos 2\theta \\
\frac{2I}{I_0} &= 2 \sin 2\theta \cos 2\theta \\
\frac{2I}{I_0} &= \sin 4\theta \\
\sin^{-1} \frac{2I}{I_0} &= 4\theta \\
\frac{1}{4} \sin^{-1} \frac{2I}{I_0} &= \theta
\end{aligned}
$$

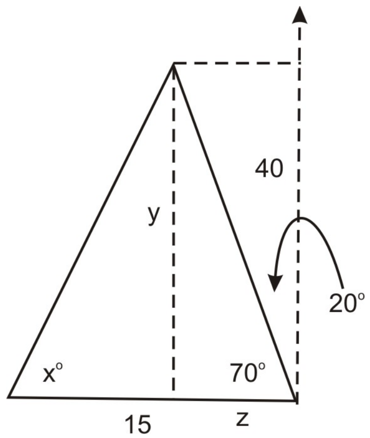
2. The volume is 10 feet times the area of the end. The end consists of two congruent right triangles and one rectangle. The area of each right triangle is $\frac{1}{2}(\sin\theta)(\cos\theta)$ and that of the rectangle is $(1)(\cos\theta)$. The maximum value is approximately 13 cubic feet and occurs when $\theta = \frac{\pi}{6}$.
3. 

$$
\begin{aligned}
\cos x &= \frac{7}{230} \rightarrow x = \cos^{-1} \frac{7}{230} \\
x &= 88.26^\circ
\end{aligned}
$$

4.

$$
\begin{aligned}
i &= I_m[\sin(wt+\alpha)\cos\varphi + \cos(wt+\alpha)\sin\varphi] \\
\frac{i}{I_m} &= \underbrace{\sin(wt+\alpha)\cos\varphi + \cos(wt+\alpha)\sin\varphi}_{\sin(wt+\alpha+\varphi)} \\
\frac{i}{I_m} &= \sin(wt+\alpha+\varphi) \\
\sin^{-1} \frac{i}{I_m} &= wt+\alpha+\varphi \\
\sin^{-1} \frac{i}{I_m} - \alpha - \varphi &= wt \\
\frac{1}{w}\left(\sin^{-1} \frac{i}{I_m} - \alpha - \varphi\right) &= t
\end{aligned}
$$

5. (a) 64° on the 16^{th} of November $= 90^\circ - 64^\circ - 23.5^\circ \cos\left[(320+10)\frac{360}{365}\right] = 6.64^\circ$
   (b) 15° on the 8^{th} of August $= 90^\circ - 15^\circ - 23.5^\circ \cos\left[(220+10)\frac{360}{365}\right] = 91.07^\circ$
6. We need to find y and z before we can find x°.



$$\sin 70^\circ = \frac{y}{40} \to y = 40 \sin 70^\circ = 37.59$$
$$\cos 70^\circ = \frac{z}{40} \to z = 40 \cos 70^\circ = 13.68$$

Using 15-13.68 as the adjacent side for x, we can now find the missing angle. $\tan x^\circ = \frac{37.59}{1.32} = 28.48 \to x^\circ = \tan^{-1}(28.48) = 87.99^\circ$.

7. The maximum displacement for this equation is simply the amplitude, 4.
8. You can use the same picture from Example 5 for this problem.

$$\tan 18^\circ = \frac{x}{10,000} \to x = 10,000 \tan 18^\circ x = 3249.2$$
$$\tan y = \frac{3749.2}{10,000} \to y = \tan^{-1} \frac{3749.2}{10,000} y = 20.6^\circ$$

So, the towers are 2.6° apart.

4.5 Chapter Review

Chapter Summary

In this chapter, we studied all aspects of inverse trigonometric functions. First, we defined the function by finding inverses algebraically. Second, we analyzed the graphs of inverse functions. We needed to restrict the domain of the trigonometric functions in order to take the inverse of each of them. This is because they are periodic and did not pass the horizontal line test. Then, we learned about the properties of the inverse functions, mostly composing a trig function and an inverse. Finally, we applied the principles of inverse trig functions to real-life situations.

Chapter Vocabulary

Arccosecant Read "cosecant inverse" and also written $\csc^{-1}$. The domain of this function is all reals, excluding the interval (-1, 1). The range is all reals in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}], y \neq 0$.

Arccosine Read "cosine inverse" and also written $\cos^{-1}$. The domain of this function is [-1, 1]. The range is $[0, \pi]$.

Arccotangent Read "cotangent inverse" and also written $\cot^{-1}$. The domain of this function is all reals. The range is $(0, \pi)$.

Arcsecant Read "secant inverse" and also written $\sec^{-1}$. The domain of this function is all reals, excluding the interval (-1, 1). The range is all reals in the interval $[0, \pi], y \neq \frac{\pi}{2}$.

Arcsine Read "sine inverse" and also written $\sin^{-1}$. The domain of this function is [-1, 1]. The range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Arctangent Read "tangent inverse" and also written $\tan^{-1}$. The domain of this function is all reals. The range is $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Composite Function The final result from when one function is plugged into another, $f(g(x))$.

Harmonic Motion A motion that is consistent and periodic, in a sinusoidal pattern. The general equation is $x(t) = A\cos(2\pi ft + \varphi)$ where A is the amplitude, f is the frequency, and φ is the phase shift.

Horizontal Line Test The test applied to a function to see if it has an inverse. Continually draw horizontal lines across the function and if a horizontal line touches the function more than once, it does not have an inverse.

Inverse Function Two functions that are symmetric over the line $y = x$.

Inverse Reflection Principle The points (a, b) and (b, a) in the coordinate plane are symmetric with respect to the line $y = x$. The points (a, b) and (b, a) are reflections of each other across the line $y = x$.

Invertible If a function has an inverse, it is invertible.

One-to-One Function A function, where, for every x value, there is EXACTLY one y–value. These are the only invertible functions.

Review Questions

1. Find the exact value of the following expressions:
   (a) $\csc^{-1}(-2)$
   (b) $\cos^{-1}\frac{\sqrt{3}}{2}$
   (c) $\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
   (d) $\sec^{-1}\left(-\sqrt{2}\right)$
   (e) $\arcsin 0$
   (f) $\arctan 1$
2. Use your calculator to find the value of each of the following expressions:
   (a) $\arccos\frac{3}{5}$
   (b) $\csc^{-1} 2.25$
   (c) $\tan^{-1} 8$
   (d) $\arcsin(-0.98)$
   (e) $\cot^{-1}\left(-\frac{9}{40}\right)$
   (f) $\sec^{-1}\frac{6}{5}$

3. Find the exact value of the following expressions:
   (a) $\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$
   (b) $\tan(\cot^{-1} 1)$
   (c) $\csc\left(\sec^{-1}\frac{2\sqrt{3}}{3}\right)$
   (d) $\sin\left(\arccos\frac{12}{13}\right)$
   (e) $\tan\left(\arcsin\frac{5}{7}\right)$
   (f) $\sec^{-1}\left(\csc\frac{\pi}{6}\right)$
4. Find the inverse of each of the following:
   (a) $f(x) = 5 + \cos(2x - 1)$
   (b) $g(x) = -4\sin^{-1}(x + 3)$
5. Sketch a graph of each of the following:
   (a) $y = 3 - \arcsin\left(\frac{1}{2}x + 1\right)$
   (b) $f(x) = 2\tan^{-1}(3x - 4)$
   (c) $h(x) = \sec^{-1}(x - 1) + 2$
   (d) $y = 1 + 2\arccos 2x$
6. Using the triangles from Section 4.3, find the following:
   (a) $\sin(\cos^{-1} x^3)$
   (b) $\tan^2\left(\sin^{-1}\frac{x^2}{3}\right)$
   (c) $\cos^4(\arctan(2x)^2)$
7. A ship leaves port and travels due west 20 nautical miles, then changes course to $S\,40°E$ and travels 65 more nautical miles. Find the bearing to the port of departure.
8. Using the formula from Example 1 in Section 4.4, determine the measurement of the sun's angle of inclination for a building located at a latitude of 36° on the 12^{th} of May.
9. Find the inverse of $\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$. HINT: Set $a = \sin x$ and $b = \sin y$ and rewrite $\cos x$ and $\cos y$ in terms of sine.
10. Find the inverse of $\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$. HINT: Set $a = \cos x$ and $b = \cos y$ and rewrite $\sin x$ and $\sin y$ in terms of sine.

Review Answers

1. (a) $-\frac{\pi}{6}$
   (b) $\frac{\pi}{6}$
   (c) $-\frac{\pi}{3}$
   (d) $\frac{3\pi}{4}$
   (e) 0
   (f) $\frac{\pi}{4}$
2. (a) 0.927
   (b) 0.461
   (c) 1.446
   (d) -1.37
   (e) 1.792
   (f) 0.586
3. (a) $\frac{\sqrt{2}}{2}$
   (b) 1
   (c) 2

(d) $\frac{5}{13}$

(e) $\frac{5}{2\sqrt{6}}$ or $\frac{5\sqrt{6}}{12}$

(f) $\frac{\pi}{3}$

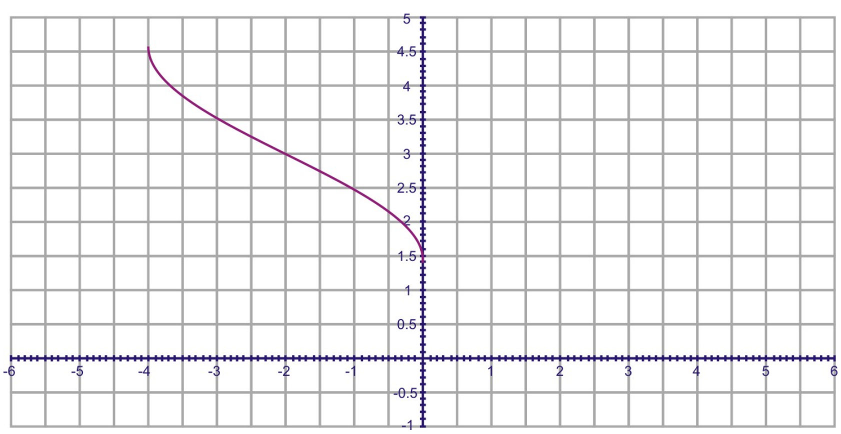
4. (a)

$$\begin{aligned} f(x) &= 5+\cos(2x-1) \\ y &= 5+\cos(2x-1) \\ x &= 5+\cos(2y-1) \\ x-5 &= \cos(2y-1) \\ \cos^{-1}(x-5) &= 2y-1 \\ 1+\cos^{-1}(x-5) &= 2y \\ \frac{1+\cos^{-1}(x-5)}{2} &= y \end{aligned}$$

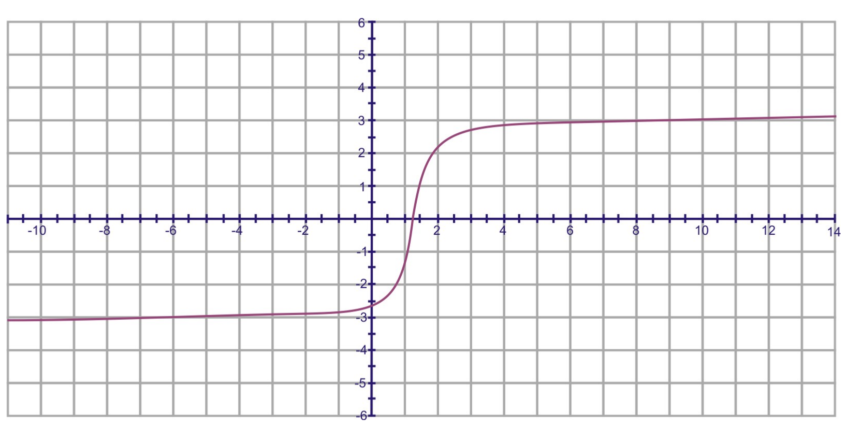
(b)

$$\begin{aligned} g(x) &= -4\sin^{-1}(x+3) \\ y &= -4\sin^{-1}(x+3) \\ x &= -4\sin^{-1}(y+3) \\ -\frac{x}{4} &= \sin^{-1}(y+3) \\ \sin\left(-\frac{x}{4}\right) &= y+3 \\ \sin\left(-\frac{x}{4}\right)-3 &= y \end{aligned}$$

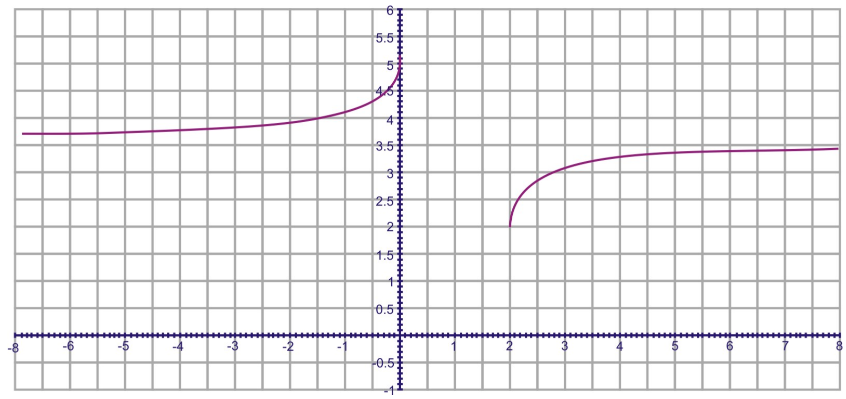
5. (a)



(b)



(c)



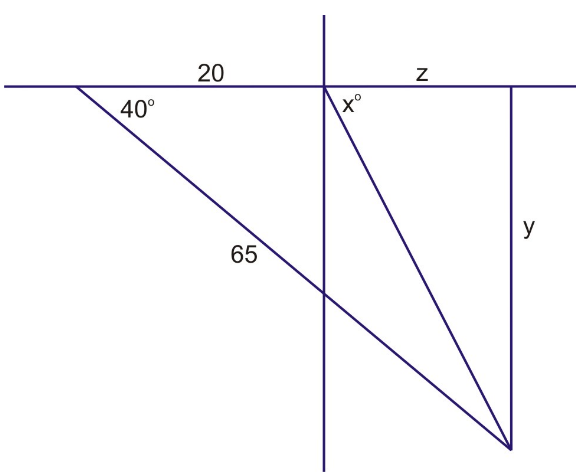
(d)

6. (a) $\sin(\cos^{-1} x^3) = \sqrt{1-(x^3)^2} = \sqrt{1-x^6}$

(b) $\tan^2\left(\sin^{-1}\frac{x^2}{3}\right) = \left(\frac{\frac{x^2}{3}}{\sqrt{1-\left(\frac{x^2}{3}\right)^2}}\right)^2 = \frac{\frac{x^4}{9}}{1-\left(\frac{x^4}{9}\right)} = \frac{x^4}{9\left(1-\frac{x^4}{9}\right)} = \frac{x^4}{9-x^4}$

(c) $\cos^4(\arctan(2x)^2) = \cos^4(\tan^{-1} 4x^2) = \left(\frac{1}{\sqrt{(4x^2)^2+1}}\right)^4 = \frac{1}{\sqrt{16x^4+1}^4} = \frac{1}{(16x^4+1)^2}$

7. x° is our final answer, but we need to find y and z first.



$$\sin 40^\circ = \frac{y}{65} \to y = 65 \sin 40^\circ = 41.78$$
$$\cos 40^\circ = \frac{20+z}{65} \to 20 + z = 65 \cos 40^\circ$$
$$20 + z = 49.79 \to z = 29.79$$
$$\tan x = \frac{41.78}{29.79} \to x = \tan^{-1} \frac{41.78}{29.79}$$
$$x = 54.51^\circ$$

8. 36° on the 12^{th} of May $= 90^\circ - 36^\circ - 23.5^\circ \cos\left[(132+10)\frac{360}{365}\right] = 72.02^\circ$
9. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, a = \sin x$ and $b = \sin y \to x = \sin^{-1} a$ and $y = \sin^{-1} b$

$$\sin(x \pm y) = a\sqrt{1 - \sin^2 y} \pm b\sqrt{1 - \sin^2 x}$$
$$\sin(x \pm y) = a\sqrt{1-b^2} \pm b\sqrt{1-a^2}$$
$$x \pm y = \sin^{-1}\left(a\sqrt{1-b^2} \pm b\sqrt{1-a^2}\right)$$
$$\sin^{-1} a \pm \sin^{-1} b = \sin^{-1}\left(a\sqrt{1-b^2} \pm b\sqrt{1-a^2}\right)$$

10. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y, a = \cos x$ and $b = \cos y \to x = \cos^{-1} a$ and $y = \cos^{-1} b$

$$\cos(x \pm y) = ab \mp b\sqrt{(1 - \cos^2 x)(1 - \cos^2 y)}$$
$$\cos(x \pm y) = ab \mp \sqrt{(1-a^2)(1-b^2)}$$
$$x \pm y = \cos^{-1}\left(ab \mp \sqrt{(1-a^2)(1-b^2)}\right)$$
$$\cos^{-1} a \pm \cos^{-1} b = \cos^{-1}\left(ab \mp \sqrt{(1-a^2)(1-b^2)}\right)$$

Texas Instruments Resources

In the CK-12 Texas Instruments Trigonometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9702.