Chapter 3

Surface Samples

In this chapter we introduce some of the properties of surfaces and their samples in three dimensions. The results developed in this chapter are used in later chapters to design algorithms for surface reconstruction and prove their guaratees. Before we talk about these results, let us explain what we mean by smooth surfaces.

Consider a map $\pi\colon U \to V$ where U and V are the open sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. The map π has three components, namely $\pi(x) = (\pi_1(x), \pi_2(x), \pi_3(x))$ where $x = (x_1, x_2)$ is a point in $\mathbb{R}^2$. The three by two matrix of partial derivatives $(\frac{\partial \pi_i(x)}{\partial x_j})_{i,j}$ is called the *Jacobian* of π at x. We say π is *regular* if its Jacobian at each point of U has rank 2. A subset $\Sigma \subset \mathbb{R}^3$ is a *smooth surface* if for each point $x \in \Sigma$ the following condition holds. There is a neighborhood $W \subset \mathbb{R}^3$ of x and a map $\pi\colon U \to W \cap \Sigma$ of an open set $U \subset \mathbb{R}^2$ onto $W \cap \Sigma$ so that

(i) π is differentiable,

(ii) π is a homeomorphism, and

(iii) π is regular.

The first condition says that all partial derivatives of π of all orders are continuous. The second condition imposes one-to-one property which eliminates self intersections of Σ. The third condition together with the first actually enforce the smoothness. It makes sure that the tangent plane at each point Σ is well defined. All of these three conditions together imply that the functions like π defined in the neighborhood of each point of Σ overlap smoothly.

In this chapter and the chapters to follow, we assume that Σ is a smooth surface. Notice that, by the definition of smoothness (condition (ii)) Σ is a 2-manifold without boundary. We also assume that Σ is compact since we are interested in approximating Σ with a finite simplicial complex. We need one more assumption. Just like the curves, for a finite point set to be an ε-sample for some $\varepsilon > 0$, we assume that $f(x) > 0$ for any point x in Σ.

Smooth surfaces have a tangent plane τ_x and a normal $\mathbf{n}_x$ defined at each point $x \in \Sigma$. We assume that the normals are oriented outward. Precisely, $\mathbf{n}_x$ points locally to the unbounded component of $\mathbb{R}^3 \setminus \Sigma$. If Σ is not connected, $\mathbf{n}_x$ points locally to the unbounded component of $\mathbb{R}^3 \setminus \Sigma'$ where x is in Σ', a connected component of Σ.

An important fact used in surface reconstruction is that the line of direction of the surface normals can be approximated from the sample. An illustration in $\mathbb{R}^2$ is helpful here. See Figure 2.4 which shows the Voronoi diagram of a dense sample on a smooth curve. This Voronoi diagram has a specific structure. Each Voronoi cell is elongated along the normal direction at the sample points. Fortunately, the same holds in three dimensions. The three dimensional Voronoi cells are long and thin and the direction of the elongation matches with the normal direction at the sample points when the sample is dense, see also Figure 3.1.

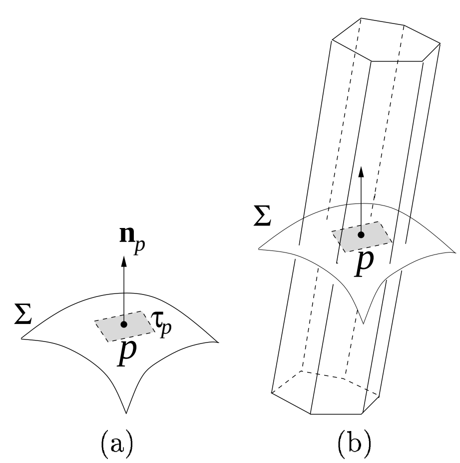

Figure 3.1: (a) Tangent plane and the normal at a point on a smooth surface, (b) A long thin Voronoi cell elongated along the normal direction.

3.1 Normals

Let $P \subset \mathbb{R}^3$ be an ε-sample of Σ. If P is all we know about Σ, it is impossible to know the line of direction of $\mathbf{n}_p$ exactly at a point $p \in P$. However, it is conceivable that as P gets denser, we should have more accurate idea about the direction of $\mathbf{n}_p$ by looking at the adjacent points. This is what is done using the Voronoi cells in Vor P.

For further developments we will often need to talk about how one vector approximates another one in terms of the angles between them. We denote the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ as $\angle(\mathbf{u}, \mathbf{v})$. For vector approximations that disregard the orientation, we use a slightly different notation. This approximation measures the acute angle between the line of the vectors. We use $\angle_a(\mathbf{u}, \mathbf{v})$ to denote this acute angle between two vectors $\mathbf{u}$ and $\mathbf{v}$. Since any such angle is acute, we have the triangular inequality $\angle_a(\mathbf{u}, \mathbf{v}) \leq \angle_a(\mathbf{u}, \mathbf{w}) + \angle_a(\mathbf{v}, \mathbf{w})$ for any three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

3.1.1 Approximation of normals

It turns out that the structure of the Voronoi cells contains information about normals. Indeed, if the sample is sufficiently dense, the Voronoi cells become long and thin along the direction of the normals at the sample points. One reason for this structural property is that a Voronoi cell V_p must contain the medial axis points that are the centers of the medial balls tangent to Σ at p, see Figure 3.2.

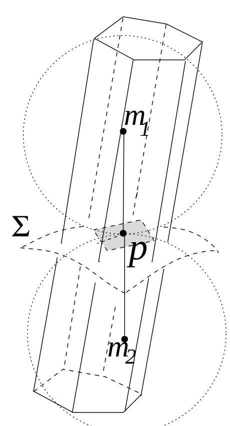

Figure 3.2: Medial axis points m_1 and m_2 are in the Voronoi cell V_p.

Lemma 3.1 (Medial.) *Let m_1 and m_2 be the centers of the two medial balls tangent to Σ at p. The Voronoi cell V_p contains m_1 and m_2.*

PROOF. Denote the medial ball with center m_1 as B. The ball B meets the surface Σ only tangentially at points, one of which is p. Thus, B is empty of any point from Σ and P in particular. Therefore, the center m_1 has p as the nearest point in P. By definition of Voronoi cells, m_1 is in V_p. A similar argument applies to the other medial axis point m_2. □

We have already mentioned that the Voronoi cells are long and thin and they are elongated along the direction of the normals. The next lemma formalizes this statement by asserting that as we go further from p within V_p, the direction to p becomes closer to the normal direction.

Lemma 3.2 (Normal.) *Let v be a point in V_p with $\|v-p\| > \mu f(p)$. Then, $\angle_a(\vec{vp}, \mathbf{n}_p) \leq arcsin\frac{\varepsilon}{\mu(1-\varepsilon)} + \arcsin\frac{\varepsilon}{1-\varepsilon}$.*

PROOF. Let m_1 and m_2 be the two centers of the medial balls tangent to Σ at p where m_1 is on the same side of Σ as v is. Both m_1 and m_2 are in V_p by the Medial Lemma (3.1). The line joining m_1 and p is normal to Σ at p by the definition of medial balls. Similarly, the line joining m_2 and p is also normal to Σ at p. Therefore, m_1, m_2 and p are co-linear. See Figure 3.3. Consider the triangle pvm_2. We are interested in the angle $\angle m_1pv$ which is equal to $\angle_a(\vec{pv}, \mathbf{n}_p)$. From the triangle pvm_2 we have

$$\angle m_1pv = \angle pvm_2 + \angle vm_2p.$$

To measure the two angles on the righthand side, drop the perpendicular px from p onto the segment vm_2. The line segment vm_2 intersects Σ, say at y, since m_1 and m_2 and hence v and m_2 lie on opposite sides of Σ. Furthermore, y must lie inside V_p since any point on the segment joining two points v and m_2 in a convex set V_p must lie within the same convex set. This means y has p as the nearest sample point, and thus

$$\|x-p\| \leq \|y-p\| \leq \varepsilon f(y) \text{ by the } \varepsilon\text{-sampling condition.}$$

Using Feature Translation Lemma (1.3) we get

$$\|x-p\| \leq \frac{\varepsilon}{1-\varepsilon} f(p).$$

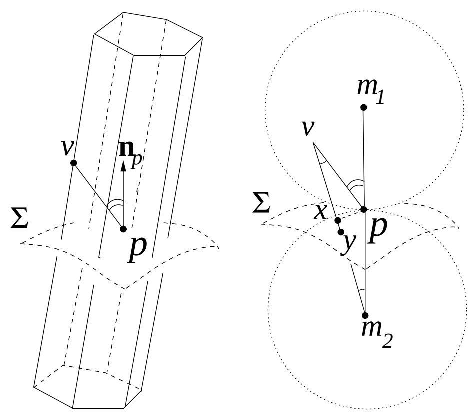

Figure 3.3: Illustration for Normal Lemma.

We have

$$\angle pvm_2 = \arcsin \frac{\|x-p\|}{\|v-p\|} \leq \arcsin \frac{\varepsilon}{\mu(1-\varepsilon)} \text{ as } \|v-p\| \geq \mu f(p), \text{ and}$$
$$\angle vm_2p = \arcsin \frac{\|x-p\|}{\|m_2-p\|} \leq \arcsin \frac{\varepsilon}{1-\varepsilon} \text{ as } \|m_2-p\| \geq f(p).$$

The assertion of the lemma follows immediately. □