A simple theory of the anomalous hall effect in semiconductors

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A SIMPLE THEORY OF THE ANOMALOUS HALL EFFECT IN SEMICONDUCTORS

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Résumé. — L'effet Hall anormal dans un semiconducteur à deux bandes est décrit dans le cadre d'une théorie élémentaire fondée sur un Hamiltonien effectif dans la bande de conduction. Des équations de mouvement simples pour la position et la quantité de mouvement fournissent une interprétation évidente des résultats. La théorie est développée pour une fréquence ou un champ magnétique quelconque, pour une aimantation de spin stationnaire ou résonnante. La nature physique des différents termes contribuant aux courant anormal est discutée en détail, et confrontée aux divers modèles existants (dont la plupart s'avèrent incomplets).

Abstract. — The anomalous Hall effect in a two band semiconductor is treated by an elementary theory based on the use of an effective Hamiltonian in the conduction band. Simple equations of motion for the position and momentum provide an obvious interpretation of the results. The theory is worked out for arbitrary frequency and d.c. magnetic field, and for a spin magnetization which is either stationary or resonant. The physical nature of the various terms that contribute to the anomalous current is discussed in detail, and it is compared to the various existing models (most of which prove to be incomplete).

Introduction. — The problem of the anomalous Hall effect in ferromagnets has been the object of a large body of literature since the original work of Karplus and Luttinger [1] in 1954 — and also of a lot of confusion, essentially because there exist different mechanisms that contribute to the same total effect. One of them is the asymmetry of scattering by single impurities brought about by spin orbit coupling, the so-called *skew scattering*, discussed in detail by Smit [2]. Another effect, whose physical meaning is less apparent, comes from the renormalization of the current operator due to that same spin orbit coupling. Hence a controversy that was essentially settled in a pioneering paper of Luttinger [3] in 1958 : using his version of quantum transport theory, Luttinger was able to calculate systematically the conductivity up to first order in spin orbit coupling. His results are exact ; they embody both skew scattering and the current renormalization. His formalism, however, is very cumbersome, the physics is not very apparent, and the generalization to more complicated situations appears very difficult : the need for a more physical picture remained.

Significant progress was achieved by Fivaz [4], who showed how one could describe the motion of the electrons in terms of an effective Hamiltonian

which embodies the effect of spin orbit coupling ; concurrently, the electron acquires an electric dipole moment, whose behavior is essential in determining the anomalous conductivity. Recently, Berger [5] clarified the situation still more by showing that the current renormalization effect could be interpreted in terms of a side jump undergone by the electron at each impurity collision, in addition to the usual change of direction (whose asymmetry was responsible for the skew scattering term). This interpretation was supported in another language by Lyo and Holstein [6]. The situation, however, still remains unsatisfactory ; factors 2 are floating around, with no obvious explanation for the discrepancies between various results. Although the essential physical ideas are understood, the need for a rigorous, synthetic formulation remains.

Part of the difficulties come from the fact that all these theories deal with metals, where the wave functions, band structure, etc... are complicated, if known at all. The situation is much clearer in semiconductors, especially in narrow gap materials like InSb where a two band model is sufficient. A microscopic calculation of the anomalous conductivity in these materials in zero magnetic field was recently worked out by Lewiner *et al.* [7], in the framework of field theoretical perturbation methods. For zero frequency, one recovers the result of Lut-

tinger [2], in a way that is far more simple and transparent. The generalization to finite magnetic fields was carried out by Lewiner [8], within the same formalism. Such an exact microscopic calculation provides a firm basis on which to check simple physical pictures — a basis which unfortunately disagrees by numerical factors with most present interpretations.

In semiconductors, of course, there is no steady magnetization ; the latter must be created by some external means. One may for instance work with a d.c. magnetic field B_0, and look for the change of conductivity when the equilibrium d.c. magnetization is quenched by saturating the electron spin resonance. One might also create spin polarized minority carriers by optically pumping with circularly polarized light. Experimentally, both methods raise a number of complications. A particularly elegant solution was found recently by Chazalviel and Solomon [9], who looked for the dc anomalous Hall current ($\sim E \wedge M$) produced by an ac electric field and by the resonant ac magnetization due to the accompanying ac magnetic field. By sweeping B_0 through resonance, they were able to observe for the first time a genuine spin dependent anomalous Hall effect in InSb. The existing theories do not apply to this ac experiment. A simple discussion in the absence of collisions is given in reference [8], but that represents only a fragmentary answer.

The purpose of the present paper is to set up a very simple theory of the anomalous Hall effect in a two band semiconductor, based on the use of an effective Hamiltonian in the conduction band. Equations of motion are set up for the electron position and momentum, allowing a semiclassical description, and an obvious interpretation of the results. The theory is general, and applies equally well to ac or dc phenomena ; it incorporates from the outset an applied dc magnetic field. In a sense, it is a synthesis of the previous literature, in which the various effects are analysed systematically. The spirit is similar to that of Fivaz [4]. It is found that most of the existing models were incomplete (although their reunion seemed to contain all the relevant terms !) — hence the missing factors here and there. We attempt to show how different terms may cancel each other — and we prove that the dc anomalous conductivity is entirely due to the spin orbit correction to the *scattering potential*, with no contribution from the spin orbit part of the pure crystal Hamiltonian. As pointed out by Berger [5], the anomalous conductivity contains a side jump part and a skew scattering part (the latter has also been calculated for a semiconductor by Leroux-Hugon and Ghazali [10]). But the side jump is twice as big as claimed in reference [5]. Also, the extra conductivity due to the side jumps arises not only from the displacement of the carriers when they jump, but also from the corresponding change of their electrostatic energy in the applied electric field, which adds an additional

driving force to the transport equation. The latter mechanism, emphasized by Doniach [11], introduces still another factor 2. Altogether, we present a global picture which, hopefully, will put some order in a rather confused situation.

In section 1, we formulate standard degenerate perturbation theory in a simple operator form which allows the construction of an effective Hamiltonian in the conduction band. This formalism is applied to Ge-type semiconductors in section 2, taking into account the scattering by impurities. The position operator in the new representation is worked out, displaying the dipole moment discussed by Fivaz [4]. Equations of motion are established in section 3, where the various contributions to the conductivity are discussed in detail. In section 4, the results are applied, first to the case of a steady spin magnetization (for arbitrary B_0 and frequency ω of the electric field), and then to the resonance experiments of the Chazalviel-Solomon [9] type (for various geometries).

1. Perturbation formalism. — In order to formulate the anomalous Hall effect in semiconductors, we shall first cast conventional degenerate perturbation theory into an operator form that turns out to be quite convenient. We project the wave function of a single electron on a complete set of states, for instance the Kohn-Luttinger basis

$$\psi_{n\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n_0}(\mathbf{r})$$

where u_{n_0} is the Bloch function of the crystal at the zone center. We then split ψ into two parts.

(i) The components ψ_1 belonging to the conduction band (including the appropriate spin structure) ;

(ii) The components of all other bands, denoted by ψ_2.

The Hamiltonian H is correspondingly broken into a 2×2 matrix of operators, and the Schrödinger equation reads

$$\begin{aligned} i\dot{\psi}_1 &= H_1 \psi_1 + h\psi_2 \\ i\dot{\psi}_2 &= h^+ \psi_1 + H_2 \psi_2 . \end{aligned} \tag{1}$$

The origin of energies is chosen at the bottom of the conduction band. H_1 will then be of the order of a typical conduction electron energy (ε_F, or kT if the gas is classical). H_2, on the other hand, is of the order of the band gaps. For each band, we may measure H_2 from the corresponding energy at $\mathbf{k} = 0$. We thus write

$$H_2 = H_g + H_2'$$

where H_g describes the band gaps at the zone center. In what follows, we shall ignore the coupling between the higher bands (we shall later see why). H_2' is then an « *intraband* » Hamiltonian (comparable to H_1). We wish to describe the response to ac perturbations, and we shall allow H_1, H_2', h to be time dependent. H_g, of course, is constant.

We now assume that the band gaps are much larger than ε_F : we can then make an expansion in powers of $1/H_g$. In physical terms, we are interested in the dynamics of conduction electrons, which is described by ψ_1 ; the corresponding frequencies are small : the oscillation of ψ_2 is then *forced* at the low frequency of ψ_1, far away from resonance — hence a small amplitude, which is the basis for a perturbation expansion. We want to eliminate ψ_2 from (1), and to construct an *effective* Schrödinger equation for ψ_1, where the forced response of ψ_2 acts only to correct the Hamiltonian.

The second eq. (1) is easily solved by iteration. Up to second order in $1/H_g$, which will be sufficient for our purpose, we find

$$\psi_2 = - \frac{1}{H_2} h^+ \psi_1 - \frac{i}{H_2^2} (\dot{h}^+ \psi_1 + h^+ \dot{\psi}_1)$$

$$= \left[- \frac{1}{H_g} h^+ + \frac{1}{H_g} H_2' \frac{1}{H_g} h^+ - \frac{i}{H_g^2} \dot{h}^+ \right] \psi_1$$

$$- \frac{i}{H_g^2} h^+ \dot{\psi}_1 . \quad (2)$$

Inserting this result into the first eq. (1), we obtain the « *Schrödinger equation* » obeyed by ψ_1

$$i(1 + \Lambda) \dot{\psi}_1 = \overline{H} \psi_1 \quad (3)$$

where we set

$$\Lambda = h \frac{1}{H_g^2} h^+ \quad (4)$$

$$\overline{H} = H_1 - h \frac{1}{H_g} h^+ + h \frac{1}{H_g} H_2' \frac{1}{H_g} h^+ - ih \frac{1}{H_g^2} \dot{h}^+ .$$

As it stands, (3) is not very useful, as ψ_1 is not normalized — indeed, on using (2), we obtain

$$1 = < \psi_1 | \psi_1 > + < \psi_2 | \psi_2 > =$$

$$= < \psi_1 | 1 + \Lambda | \psi_1 > \quad (5)$$

(we disregard all terms of order higher than $1/H_g^2$). This difficulty is avoided by introducing an effective wave function

$$| \psi_{\text{eff}} > = \left(1 + \frac{\Lambda}{2} \right) | \psi_1 > \quad (6)$$

which according to (5) is normalized. Using (3) and (4), it is then straightforward to calculate ψ_{eff}. We thus obtain the effective Schrödinger equation in the « 1 » subspace

$$i\dot{\psi}_{\text{eff}} = H_{\text{eff}} \psi_{\text{eff}} \quad (7)$$

where the effective Hamiltonian turns out to be

$$H_{\text{eff}} = H_0 + \delta H$$

$$H_0 = H_1 - h \frac{1}{H_g} h^+$$

$$\delta H = - \frac{\Lambda H_0 + H_0 \Lambda}{2} + h \frac{1}{H_g} H_2' \frac{1}{H_g} h^+ +$$

$$+ \frac{i}{2} \left[h \frac{1}{H_g^2} \dot{h}^+ - h \frac{1}{H_g^2} \dot{h}^+ \right] . \quad (8)$$

From then on, we may forget about all the higher bands, and focus our attention on the conduction band *alone* — still a difficult problem as all the factors in (8) are operators in the spin and position coordinates.

In practice, we want to calculate the expectation value of some physical operator — say A. Again, we can break it into a 2 × 2 matrix, as we did for H. With the same notations, we obtain

$$< A > = < \psi_1 | A_1 | \psi_1 > + < \psi_2 | A_2 | \psi_2 > +$$

$$+ < \psi_2 | a | \psi_1 > + < \psi_1 | a^+ | \psi_2 > .$$

Here ψ_2 is *forced* by ψ_1, and may be replaced by its expression (2). It is then obvious that $< A >$ may be cast in the form

$$< A > = < \psi_1 | \overline{A} | \psi_1 > \quad (9)$$

where $\overline{A}$ is a linear combination of A_1, A_2, a, a^+. We may even go one step further, and replace ψ_1 by $(1 - \Lambda/2) \psi_{\text{eff}}$ in (9). We shall thus obtain

$$< A > = < \psi_{\text{eff}} | A_{\text{eff}} | \psi_{\text{eff}} > \quad (10)$$

where A_{eff} is the *effective A operator* within the « 1 » subspace. The general expression for A_{eff} is easily established, but it looks rather messy. In our problem, we shall consider only operators A that have no interband matrix elements ($a = a^+ = 0$) ; in that particular case, the expression of A_{eff} is very simple

$$A_{\text{eff}} = A_1 - \frac{A_1 \Lambda + \Lambda A_1}{2} + h \frac{1}{H_g} A_2 \frac{1}{H_g} h^+ . \quad (11)$$

It should be emphasized that the definition of A_{eff} does *not* conserve the algebra of operators, i. e.

$$(AB)_{\text{eff}} \neq A_{\text{eff}} \, B_{\text{eff}}$$

The reason is obvious : in calculating the product AB, one must insert a *complete* set of states between A and B, not only the restricted class of eigenstates forced by the « 1 » subspace. The latter would precisely give a contribution $A_{\text{eff}} \, B_{\text{eff}}$ — but they do not exhaust the story. Physically, the difference $(A_{\text{eff}} - A_1)$ takes into account the excursions of the particle into the « 2 » subspace : it is clear that during these excursions, the corrections to A_1 and B_1 are correlated — an effect which would be ignored in the product $A_{\text{eff}} \, B_{\text{eff}}$. We insist on this point, as it is easily overlooked, leading to gross errors.

Until now, we used the Schrödinger representation. Using (7) and (10), we may pass over to an Heisenberg representation : the equation of motion for A_{eff} is then

$$\dot{A}_{\text{eff}} = \frac{\partial A_{\text{eff}}}{\partial t} - i[A_{\text{eff}}, H_{\text{eff}}] . \quad (12)$$

The operator eq. (12) allows simple classical analogies : it is the starting point of the discussion of the following paragraphs.

2. Effective Hamiltonian for diamond type semiconductors.

— We consider a pure semiconductor (impurities will be introduced later), immersed in

electric and magnetic fields described by a vector potential $A(r, t)$. In the Kohn-Luttinger representation, the Hamiltonian is

$$H = \frac{k^2}{2 m_0} + g_0 \beta S.B + k.\Pi \qquad (13)$$

$k = - i\nabla - eA/c$ is the canonical momentum, $\beta = | e |/2 mc$ the Bohr magneton, $g_0 = 2$ the free electron g-factor, m_0 the free electron mass. Π is a vector operator, k-independent but spin dependent, that couples the bands together ; more precisely, the interband matrix elements of Π are

$$\Pi_{nn'} = \left\langle u_{n_0} \left| - i\frac{\nabla}{m_0} + \frac{\hbar}{2 m^2 c^2} S.\text{grad } U \right| u_{n_0'} \right\rangle$$

where U is the potential felt by the electrons in the periodic lattice. In practice, the second term of this operator is always negligible for the materials of interest, and $\Pi_{nn'}$ reduces to

$$\Pi_{nn'} = \left\langle u_{n_0} \left| - \frac{i\nabla}{m_0} \right| u_{n_0'} \right\rangle .$$

Note that Π couples only states of opposite orbital parity. In general, the subspace « 2 » of higher bands contains both even and odd bands, and our assumption that H_2 only has *intraband* elements is not met. The situation simplifies if we consider materials with a small band gap, like Ge or even better InSb : we can then forget about all bands but the conduction band (s-like) and the valence bands (p-like with a three fold orbital degeneracy). For *n-type* materials, subspace « 1 » is the conduction band, subspace « 2 » the three valence bands. Because of parity, Π has no matrix element between valence states, and the results of the preceding section apply. Throughout the paper, we only consider this simplified case.

Because of spin orbit coupling, the six fold degenerate valence state at $k = 0$ splits into a quadruplet $(J = \frac{3}{2})$ with energy $- \varepsilon_g$, and a doublet $(J = \frac{1}{2})$ with energy $- (\varepsilon_g + \Delta)$. The matrix elements of Π between these states and the conduction band are well known ; they involve only ε_g, Δ, and the orbital matrix element P of $(- i\nabla_x/m_0)$ between the s and p_x wave functions. From them, one may derive the following combination

$$\Pi_\alpha \frac{1}{H_g^n} \Pi_\beta = \frac{| P |^2}{3} \left[\delta_{\alpha\beta} \left\{ \frac{2}{(- \varepsilon_g)^n} + \frac{1}{(- \varepsilon_g - \Delta)^n} \right\} + \right.$$

$$\left. + 2 i\varepsilon_{\alpha\beta\gamma} S_\gamma \left\{ \frac{1}{(- \varepsilon_g - \Delta)^n} - \frac{1}{(- \varepsilon_g)^n} \right\} \right] \qquad (14)$$

(14) is all that we need in order to construct an effective Hamiltonian. We also note that in the presence of a magnetic field, the components of k do not commute :

$$[k_\alpha, k_\beta] = \frac{ie}{c} \varepsilon_{\alpha\beta\gamma} B_\gamma . \qquad (15)$$

We now apply the results of the preceding section, setting

$$H_1 = H_2 = \frac{k^2}{2 m_0} + g_0 \beta S.B$$

$$h = k.\Pi .$$

The lowest contribution H_0 to the effective Hamiltonian is obtained from (8). We may write it as

$$H_0 = \frac{k^2}{2 m^*} + g^* \beta S.B \qquad (16)$$

where the effective mass and g-factor of the conduction electrons are given by

$$\frac{1}{m^*} = \frac{1}{m_0} + \frac{2}{3} | P |^2 \frac{3 \varepsilon_g + 2 \Delta}{\varepsilon_g(\varepsilon_g + \Delta)}$$

$$g^* = g_0 - \frac{4 m_0}{3} | P |^2 \frac{\Delta}{\varepsilon_g(\varepsilon_g + \Delta)} \qquad (17)$$

(16) and (17) are standard results, described for instance in reference [12]. For narrow gap materials, $| g^* | \gg g_0, m^* \ll m_0$: it is then a fair approximation to neglect g_0 and $1/m_0$ — a procedure that is indeed consistent with the neglect of all bands other than conduction and valence. This approximation simplifies the calculation somewhat : we shall make it and consequently set $H_1 = H_2 = 0$ — the initial Hamiltonian reduces to the interband part $h = k.\Pi$. The generalization to finite g_0 and $1/m_0$ is briefly considered in Appendix A.

In much the same way, we may write the normalization operator Λ, given by (4), as

$$\Lambda = \frac{k^2}{2 m^* E_1} + g^* \beta \frac{S.B}{E_2} . \qquad (18)$$

With the help of (14), we find that

$$E_1 = \frac{\varepsilon_g(\varepsilon_g + \Delta) (3 \varepsilon_g + 2 \Delta)}{3 \varepsilon_g^2 + 4 \varepsilon_g \Delta + 2 \Delta^2}, \quad E_2 = \frac{\varepsilon_g(\varepsilon_g + \Delta)}{2 \varepsilon_g + \Delta} .$$

(Note that for a *uniform* magnetic field, $S.B$ commutes with k^2.)

The first order effective Hamiltonian δH is given by (8). We remark that

$$\dot{h} = \frac{\partial k}{\partial t} . \Pi = - \frac{e}{c} \frac{\partial A}{\partial t} . \Pi = e E.\Pi .$$

We thus finally obtain (for spatially uniform E and B)

$$\delta H = - \Lambda H_0 - g^* \frac{\beta c}{E_2} (k \wedge E).S .$$

In the product ΛH_0, we discard the term of order k^4, which arises from the nonparabolicity of the conduction band : it has nothing to do with the spin, and hence is irrelevant for the anomalous Hall effect ; we also ignore the constant term $(B.S)^2 = B^2/4$. We are thus left with the simple result

$$\delta H = - \frac{k^2}{2 m^*} g^* \frac{\beta S.B}{E_0} - \frac{g^* \beta c}{E_2} (k \wedge E).S \qquad (19)$$

where $1/E_0 = 1/E_1 + 1/E_2$. We note that this expression of the *operator* δH is valid for arbitrary time dependent situations. The first term of (19) may be interpreted as a spin dependence of the effective mass (for stationary S, $m_\uparrow$ and $m_\downarrow$ are different). The second term describes the electric dipole coupling of the spin to the electric field E, the latter giving rise to a fictitious magnetic field

$$\delta B = -\frac{c}{E_2} k \wedge E.$$

(Note the analogy with a relativistic theory : if E_2 were replaced by $m_0 c^2$, δB would be the extra magnetic field in the frame of reference of the moving carrier. E_2 here plays the role of the electron positron energy gap.) That last term of (19), when treated to second order, gives rise to the electric dipole spin resonance described by Yafet [12]. We emphasize that *both* terms of δH arise directly from spin-orbit coupling, without which g^* would be zero in our approximation ; to order $1/\varepsilon_g$, the result (19) is exact, and includes *all* the effects of spin orbit coupling.

We now proceed to introduce the scattering potential by spinless impurities, described by a position dependent, spin independent potential $V(r)$. We assume $V(r)$ to be slowly varying on an atomic scale : it then has no interband matrix elements in the Kohn Luttinger basis ; in the notations of section 1, we have

$$V_1 = V_2 = V(r)$$
$$v = 0$$

(for a more detailed discussion, see ref. [7]). According to (8), the corresponding contribution to the effective Hamiltonian involves two parts. A first term

$$V_1 - \tfrac{1}{2}(\Lambda V_1 + V_1 \Lambda)$$

arises from scattering of the electron *inside* the conduction band, while the second term

$$k_\alpha V_2 k_\beta \Pi_\alpha \frac{1}{H_g^2} \Pi_\beta$$

describes scattering events that occur while the electron has virtually jumped into the valence bands. (Note that k is here an *operator*.) In both cases, we retain only the corrections that are spin dependent and may be of relevance for the anomalous Hall effect. We thus find

$$V_{eff} = V_1 \left(1 - g^* \frac{\beta S.B}{E_2}\right) - \frac{i\beta c g^*}{eE_2} \varepsilon_{\alpha\beta\gamma} S_\gamma k_\alpha V_2 k_\beta.$$
$$(20)$$

On commuting V_2 and k_β, and using (15), we may cast (20) in the equivalent form

$$V_{eff} = V_1 + g^* \frac{\beta S.B}{E_2} (V_2 - V_1) -$$
$$- \frac{g^* \beta c}{eE_2} (k \wedge S).\text{grad } V_2.$$
$$(21)$$

The second term of (21) is an ordinary position depen-

dent scattering potential which depends on the relative orientation of S and B ; in the present case where $V_1 = V_2$, it vanishes. The last term of (21) is a qualitatively new effect, expressing a genuine coupling of the electron spin with the orbit around the impurity ; it is that term which will give rise to all the important anomalous effects : asymmetry in the scattering probability, etc...

In addition to the *scalar* scattering potential $V(r - r_i)$, a given impurity may give rise to a *spin orbit* potential

$$L S\lambda(r - r_i).$$

Such a correction, which only occurs in the p-like valence bands, describes the excess spin orbit coupling of the impurity over that of the matrix. It was shown in [7] that this effect is equivalent to adding a correction to V_2, which becomes

$$V_2 = V_1 + V_s$$

V_1 and V_2 are therefore independent : they need not be equal. (Note that the second term of (21) is directly proportional to V_s.) In this paper, we shall neglect V_s for simplicity : we thus drop the second term of (21) (how it may affect the conductivity is briefly discussed in Appendix A). However, we shall find it convenient to retain *formally* the distinction between V_1 and V_2 in the first and last terms of (21) (although here $V_1 = V_2$). In this way, we may separate unambiguously two classes of effects :

(i) those that are due to spin orbit coupling of the pure matrix, where only V_1 enters ;

(ii) those that arise from spin orbit corrections to the scattering potential, which involve V_2.

This distinction is important in the physical interpretation of our results — especially in assessing how the various terms may cancel each other.

The current density J involves the expectation value of the velocity operator for each electron, $v = \dot{r}_{eff}$. We thus need to know r_{eff}, which may be obtained from (11) ; again, in the Kohn Luttinger basis, r only has intraband matrix elements, and moreover

$$r_1 = r_2 = iV_k.$$

It is then straightforward to show that

$$r_{eff} = r + \rho$$
$$(22)$$

$$\rho = -\frac{g^* \beta c}{eE_2} k \wedge S$$

(the calculation is the same as for V_{eff}, $V_1 = V_2$ being replaced by each of the three components of r). In (22), r is the ordinary position variable, canonically conjugate to k and acting in band « 1 ». The correction ρ may be viewed as an *electric dipole moment* $\mu = e\rho$ that each electron acquires because of spin orbit coupling : in a wave packet of wave vector k, monitored by the wave function ψ_{eff}, the electron spends part of the time in the subspace « 2 »,

where its wave function is shifted as compared to ψ_{eff} : hence the dipole moment. This concept of a dipole moment is not new [4] : it is the clue to the anomalous Hall effect.

For convenience, we collect all these results together before proceeding further. The dynamics of a conduction electron is described *in the conduction band only*, in terms of the usual position and momentum operators, $\mathbf{r}$ and $\mathbf{k} = -i\mathbf{V} + e\mathbf{A}/c$. In addition to its charge, the electron possesses an electric dipole moment represented by the operator $e\mathbf{\rho}$ given by (22). The complete Hamiltonian may then be written as

$$H_{eff} = H_0 + V + \delta H + \delta V \qquad (23)$$

V is the bare scalar scattering potential, while

$$H_0 = \frac{k^2}{2\,m^*} + g^* \beta \mathbf{S}.\mathbf{B}$$

$$\delta H = -\frac{k^2}{2\,m^*} g^* \frac{\beta \mathbf{S}.\mathbf{B}}{E_0} - e\mathbf{E}.\mathbf{\rho} \qquad (24)$$

$$\delta V = g^* \frac{\beta \mathbf{S}.\mathbf{B}}{E_2}(V_2 - V_1) + \mathbf{\rho}.\mathbf{grad}\,V_2 .$$

As discussed earlier, we drop the first term of δV. We moreover note that for $V_2 = V_1$, the net scattering potential is

$$V + \delta V = V(\mathbf{r}_{eff})$$

a result which is not surprising ([1]). Actually, we might have described the electric field $\mathbf{E}$ in a scalar gauge rather than in a vector gauge. We then introduce a scalar electrostatic potential $\varphi = -\mathbf{E}.\mathbf{r}$, and the effective Hamiltonian becomes

$$H_{eff} = H_0 - \frac{k^2}{2\,m^*} g^* \frac{\beta \mathbf{S}.\mathbf{B}}{E_0} - e\mathbf{E}.\mathbf{r}_{eff} + V(\mathbf{r}_{eff}) . \qquad (25)$$

The effect of spin orbit coupling thus boils down to

(i) A dipole moment $e\mathbf{\rho}$, implying that the total potential $(V + e\varphi)$ must be measured at the actual position $\mathbf{r}_{eff} = \mathbf{r} + \mathbf{\rho}$. (The dipole moment interacts with the gradient of the potential.)

(ii) A spin dependent effective mass

$$m_{eff} = m^* \left(1 + g^* \frac{\beta \mathbf{S}.\mathbf{B}}{E_0} \right) . \qquad (26)$$

The physical interpretation of the theory thus becomes straightforward. In practice, the k-dependence of $\mathbf{\rho}$ is the feature that will give rise to all the interesting effects.

3. Equations of motion and calculation of the conductivity.

— We first note that in $\mathbf{r}_{eff}$ — and hence in the velocity $\mathbf{v} = \dot{\mathbf{r}}_{eff}$ — the spin $\mathbf{S}$ only enters in terms

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([1]) $\mathbf{\rho}$ commutes with **grad** V, so that there is no difficulty in defining $V(\mathbf{r}_{eff})$.

that are of first order in the spin orbit coupling (i. e., of order $1/\varepsilon_g$). Up to that order, it is thus sufficient to know the dynamics of $\mathbf{S}$ to zeroth order (remember that our whole theory is built within a first order approximation). We thus write

$$\dot{\mathbf{S}} = -i[\mathbf{S}, H_0] = g^* \beta \mathbf{S} \wedge \mathbf{B} \qquad (27)$$

$\mathbf{S}$ precesses around $\mathbf{B}$ in the usual way : to that order, *the spin motion is completely decoupled from the orbital motion*. We may solve (27) independently for $\mathbf{S}(t)$, and then insert the result into $\mathbf{r}_{eff}$ and $\mathbf{v}$. Such a major simplification is of course not valid beyond a first order approximation.

According to (27), the product $\mathbf{S}.\mathbf{B}$ is constant in time (provided we neglect spin relaxation). The effective mass correction (26) is thus time independent for each electron. To zeroth order in the spin orbit coupling (and for a uniform ac magnetic field), the average spin polarization is independent of the electron momentum, and is the same for all carriers, say $\bar{\mathbf{S}}$. The effective mass correction (26) thus gives rise to a correction to the conductivity tensor

$$\delta\sigma_{\alpha\beta}^m = m^* g^* \beta \frac{\bar{\mathbf{S}}.\mathbf{B}}{E_0}\frac{d\sigma_{\alpha\beta}}{dm^*} = -m^* \frac{\mathbf{M}.\mathbf{B}}{E_0}\frac{d\sigma_{\alpha\beta}}{dm^*} \qquad (28)$$

where $\mathbf{M}$ is the spin magnetization per electron. (28) provides one part of the spin orbit correction to the conductivity — a part which is not « anomalous » in the proper sense, since it does not depend on the relative direction of $\mathbf{M}$ and $\mathbf{E}$. It arises from the spin dependence of m_{eff}, and disappears when $\mathbf{B} = 0$; yet, it has the same order of magnitude as the real anomalous Hall effect that we shall discuss later, and it must not be forgotten when analyzing experimental results.

From now on, we assume that this effect has been taken into account, and we forget about the corresponding terms in H_{eff}, which thus becomes

$$H_{eff} = \frac{k^2}{2\,m^*} + V(\mathbf{r}) - e\mathbf{E}.\mathbf{r} + \mathbf{\rho}.[\mathbf{grad}\,V_2 - e\mathbf{E}]$$

$$\mathbf{\rho} = -\lambda\mathbf{k} \wedge \mathbf{S} . \qquad (29)$$

(For brevity, we have set $g^* \beta c/eE_2 = \lambda$). The velocity operator is then

$$\mathbf{v} = \dot{\mathbf{\rho}} - i[\mathbf{r}, H_{eff}] = \frac{\mathbf{k}}{m^*} + \dot{\mathbf{\rho}} + \delta\mathbf{v}$$

$$\delta\mathbf{v} = \lambda[\mathbf{grad}\,V_2 - e\mathbf{E}] \wedge \mathbf{S} . \qquad (30)$$

In addition to the obvious « polarization current » $\dot{\mathbf{\rho}}$, $\mathbf{v}$ involves an additional correction $\delta\mathbf{v}$ due to the spin orbit coupling, arising from the time dependence of $\mathbf{r}$. Both terms will contribute to the anomalous Hall current.

The equation of motion for $\mathbf{k}$ is likewise

$$\dot{\mathbf{k}} = -i[\mathbf{k}, H_{eff}] . \qquad (31)$$

The commutator in (31) contains two parts : one comes from the explicit r-dependence of H_{eff}. The other arises from the non-commutativity of k_α and k_β, i. e. from the r-dependence of A hidden inside k. The latter part of $[k_\alpha, H_{\text{eff}}]$ may be written in the form (see (15)) :

$$\frac{\partial H_{\text{eff}}}{\partial k_\beta} [k_\alpha, k_\beta] = [- i r_\beta, H_{\text{eff}}] [k_\alpha, k_\beta] = \frac{ie}{c} (\dot{\mathbf{r}} \wedge \mathbf{B})_\alpha$$

(31) thus becomes

$$\dot{\mathbf{k}} = e\mathbf{E} - \mathbf{grad}\, V + \frac{e}{c} \dot{\mathbf{r}} \wedge \mathbf{B} - (\boldsymbol{\rho}.\mathbf{grad})\,\mathbf{grad}\, V_2 .$$

(32)

The third term of (32) is the Lorentz force, which here involves $\dot{\mathbf{r}}$ rather than the total velocity $\mathbf{v}$. Together with (30), (32) completely describes the dynamics of the system (^{2}). Since in practice $V_2 = V_1$, we note that in the particular case $\mathbf{B} = \dot{\mathbf{S}} = 0$, $\delta\mathbf{v} = - \lambda\dot{\mathbf{k}} \wedge \mathbf{S} = \dot{\boldsymbol{\rho}}$: such a simple result need not be true in the general case.

We now proceed to calculate the current density $\mathbf{J}$ due to the electric field $\mathbf{E}$. To zeroth order in the spin orbit coupling, the current is a simple sum over all the electrons

$$\mathbf{J}_0 = \sum_i \left\langle \frac{e\mathbf{k}_i}{m^*} \right\rangle = \sum_k \frac{e\mathbf{k}}{m^*} f_k$$

where f_k is the distribution in k-space. $\mathbf{J}_0$ may be obtained by writing a Boltzmann equation for f_k :

$$\frac{\partial f_k}{\partial t} + \left[e\mathbf{E} + \frac{e}{m^* c} \mathbf{k} \wedge \mathbf{B} \right] \cdot \frac{\partial f_k}{\partial k} = -$$
$$= - \sum_{k'} W^0_{kk'}[f_k - f_{k'}] .$$

(33)

The transition probability is expressed in terms of the scattering t-matrix on the impurity, and is given by

$$W^0_{kk'} = \frac{2\pi}{\hbar} n_i \mid t_{kk'}(\varepsilon_k) \mid^2 \delta(\varepsilon_k - \varepsilon_{k'})$$

(34)

n_i is the impurity density, $\varepsilon_k = k^2/2\, m^*$ the zeroth order particle energy. (We note that (33) is only the first step of an expansion in powers of n_i, as shown by Kohn and Luttinger [14] ; here, we content ourselves with this approximation.) In order to solve (33), one usually writes

$$f_k = f^0(\varepsilon_k) + g_k$$

(35)

where f^0 is the equilibrium Fermi Dirac distribution. f^0 drops out of the collision term (because of energy

conservation) and of the Lorentz force (because $\partial f^0/\partial \mathbf{k} = (\mathbf{k}/m^*) (\partial f^0/\partial\varepsilon_k)$ is parallel to $\mathbf{k}$) : one is left with a linear equation in g, driven by the inhomogeneous term

$$e\mathbf{E} \cdot \frac{\partial f^0}{\partial \mathbf{k}} .$$

All that is of course well known : we stress it as it will clarify the subsequent discussion. After these manipulations, (33) leads to the macroscopic equation

$$\dot{\mathbf{J}}_0 - \mathbf{J}_0 \wedge \omega_c = \frac{Ne^2}{m^*} \mathbf{E} - \Gamma \mathbf{J}_0$$

(36)

where the collision frequency $\Gamma = 1/\tau$ is (at zero temperature)

$$\Gamma = \frac{2\pi}{\hbar} n_i \sum_{k'} \mid t_{kk'}(\mu) \mid^2 \delta(\varepsilon_{k'} - \mu) (1 - \cos \theta_{kk'})$$

ω_c is the usual cyclotron resonance vector $e\mathbf{B}/m^* c$. The solution of (36) is straightforward, and yields the usual Drude Zener result for the conductivity in a magnetic field (^{3}).

The first order current contains an obvious contribution

$$\mathbf{J}_1 = \sum_i < e\dot{\boldsymbol{\rho}}_i > = - \lambda m^* \{ \dot{\mathbf{J}}_0 \wedge \overline{\mathbf{S}} + \mathbf{J}_0 \wedge \dot{\overline{\mathbf{S}}} \}$$

(37)

(here again, we use the statistical independence of the orbital and spin degrees of freedom in order to factorize the average value of $\mathbf{k} \wedge \mathbf{S}$). Note that this *polarization current* involves the time derivative of the *product* $\mathbf{J}_0 \wedge \overline{\mathbf{S}}$. In the remaining spin orbit corrections, we shall treat separately those which involve the electric field (term $- e\mathbf{E}.\boldsymbol{\rho}$ in (29)), and those that involve the scattering potential (term $\boldsymbol{\rho}.\mathbf{grad}\ V_2$) ; as shown in the preceding section, the latter arise from scattering in the valence band, and they may be affected by the spin orbit part of the impurity potential. The correction to the driving force, on the other hand, only involves the spin orbit coupling of the host crystal. Within a first order calculation, the two effects are simply additive.

We consider first the spin orbit correction to the driving force arising from the term

$$- \lambda e(\mathbf{E} \wedge \mathbf{S}).\mathbf{k}$$

(38)

in H_{eff}. Such a term may be viewed as a fictitious vector potential

$$\delta\mathbf{A} = m^* c\lambda \mathbf{E} \wedge \mathbf{S}$$

(^{2}) Eq. (32) would also obtain if we used a vector gauge : the term $- e\mathbf{E}.\mathbf{r}$ would disappear in H_{eff}, but $\dot{\mathbf{k}}$ would acquire an additional term $\partial\mathbf{k}/\partial t = e\mathbf{E}$ which would restore the same result.

(^{3}) For simplicity, we limit our discussion to a classical regime, in which the effect of the magnetic field is treated classically. The generalization to a quantum regime may be carried out in the usual way [13], and does not affect most of our conclusions.

applied on the electrons, equivalent to an extra electric field

$$\delta \overline{\mathbf{E}} = - \frac{1}{c} \delta \dot{\mathbf{A}} = - m^* \lambda \{ \dot{\mathbf{E}} \wedge \overline{\mathbf{S}} + \mathbf{E} \wedge \overline{\dot{\mathbf{S}}} \} \quad (39a)$$

(we use again the statistical independence of spin variables, which allows a separate averaging). The corresponding correction to the current follows without any calculation :

$$\mathbf{J}_2 = \sigma_0 \, \delta \overline{\mathbf{E}} \quad (39b)$$

where σ_0 is the zeroth order conductivity tensor. $\mathbf{J}_2$ involves the derivative of the product $\mathbf{E} \wedge \mathbf{S}$, and vanishes under static conditions (as does $\mathbf{J}_1$). In the special case $\dot{\mathbf{S}} = \mathbf{B} = 0$, σ_0 is a scalar, and $\dot{\mathbf{J}}_0 = \sigma_0 \, \dot{\mathbf{E}}$: $\mathbf{J}_2$ is then equal to $\mathbf{J}_1$, a result that does not hold under more general conditions.

Although the above calculation of $\mathbf{J}_2$ is sufficient in itself, it is enlightening to look in more detail into the way $\mathbf{J}_2$ is built. To the term $- e\mathbf{E}.\boldsymbol{\rho}$ in H_{eff} there corresponds the part $- \lambda e\mathbf{E} \wedge \mathbf{S}$ in the velocity $\delta \mathbf{v}$ — a part which gives an obvious contribution to the current

$$\mathbf{J}_2' = - Ne^2 \, \lambda \mathbf{E} \wedge \overline{\mathbf{S}} . \quad (40)$$

The difference between $\mathbf{J}_2$ and $\mathbf{J}_2'$ must arise from the change in $f_{\mathbf{k}}$ brought about by the perturbation (38). Indeed, the full Boltzmann equation may be written as

$$\frac{\partial f_{\mathbf{k}}}{\partial t} + \left[e\mathbf{E} + \frac{e}{c} \dot{\mathbf{r}} \wedge \mathbf{B} \right] \cdot \frac{\partial f_{\mathbf{k}}}{\partial \mathbf{k}} = - \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'}(f_{\mathbf{k}} - f_{\mathbf{k}'}) \quad (41)$$

(41) differs from (33) in the replacement of $\mathbf{k}/m^*$ by $\dot{\mathbf{r}}$ (in accordance with (32)), and in the allowance for a modified $W_{\mathbf{k}\mathbf{k}'}$. The former correction yields an additional driving term on the left hand side of (41)

$$- \frac{e^2}{c} \lambda [(\mathbf{E} \wedge \mathbf{S}) \wedge \mathbf{B}] \cdot \frac{\partial f^0}{\partial \mathbf{k}}$$

which gives rise to a current

$$\mathbf{J}_2'' = - \sigma_0 \left[\frac{e\lambda}{c} (\mathbf{E} \wedge \overline{\mathbf{S}}) \wedge \mathbf{B} \right] . \quad (42)$$

The change in the transition probability is more subtle. Since we are presently ignoring the spin orbit corrections to the scattering potential, the matrix elements entering $W_{\mathbf{k}\mathbf{k}'}$ are essentially unchanged. On the other hand, the energy of the plane wave states is modified by the perturbation $- e\mathbf{E}.\boldsymbol{\rho}$, and becomes

$$\tilde{\varepsilon}_{\mathbf{k}} = \frac{k^2}{2 \, m^*} - \lambda e(\mathbf{E} \wedge \mathbf{S}).\mathbf{k} . \quad (43)$$

The energy conservation involves $\tilde{\varepsilon}_{\mathbf{k}}$ not ε_k, and $W_{\mathbf{k}\mathbf{k}'}$

contains a factor $\delta(\tilde{\varepsilon}_{\mathbf{k}} - \tilde{\varepsilon}_{\mathbf{k}'})$. It follows that the contribution of the *equilibrium* distribution $f^0(\varepsilon_k)$ to the collision integral no longer vanishes. Instead it gives a contribution to the right hand side of (41) [4]

$$- \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \frac{\partial f^0}{\partial \varepsilon_k} \lambda e(\mathbf{E} \wedge \mathbf{S}).(\mathbf{k} - \mathbf{k}') \quad (44)$$

(44) is easily cast in the form

$$- \frac{\partial f^0}{\partial \mathbf{k}} \cdot \Gamma m^* \lambda e\mathbf{E} \wedge \mathbf{S} . \quad (45)$$

On carrying (45) on the left hand side of (41), we see that it acts as still another driving term, corresponding to an extra electric field $\Gamma m^* \lambda \mathbf{E} \wedge \mathbf{S}$ — hence another current

$$\mathbf{J}_2''' = \sigma_0 [\Gamma m^* \lambda \mathbf{E} \wedge \overline{\mathbf{S}}] . \quad (46)$$

The existence of such a contribution to the conductivity was first emphasized by Doniach [11]. Using the explicit expression of σ_0, one easily verifies that

$$\mathbf{J}_2 = \mathbf{J}_2' + \mathbf{J}_2'' + \mathbf{J}_2'''$$

which guarantees the consistency of our result.

While $\mathbf{J}_2'$ and $\mathbf{J}_2''$ arise directly from the extra velocity $- e\mathbf{E} \wedge \mathbf{S}$, the last term $\mathbf{J}_2'''$ must be ascribed to the change in ε_k on scattering. When the momentum changes by $\Delta \mathbf{k}$, the dipole moment changes by an amount

$$\Delta \boldsymbol{\rho} = - \lambda \, \Delta \mathbf{k} \wedge \mathbf{S} \quad (47)$$

which may be interpreted as a *side jump* of the electron upon scattering. The electrostatic energy is shifted by $\Delta \varepsilon = - e\mathbf{E}.\Delta \boldsymbol{\rho}$, and the kinetic energy must vary by an opposite amount. Hence f^0 is not stationary under collisions, and the collision integral in (41) contains a term

$$- \sum_{\mathbf{k}'} W_{\mathbf{k}\mathbf{k}'} \frac{\partial f^0}{\partial \varepsilon_k} \Delta \varepsilon .$$

The contribution $\mathbf{J}_2'''$ follows at once. Such a side jump concept was introduced by Berger, and is essential in interpreting the anomalous Hall effect. We shall see, however, that (47) does not exhaust the story.

We now turn to the spin orbit corrections to the *scattering potential*, arising from the term $\boldsymbol{\rho}.\mathbf{grad} \, V_2$ in H_{eff}, and giving rise to the first term in $\delta \mathbf{v}$ (eq. (30)), and to the last term of $\dot{\mathbf{k}}$ (eq. (32)). In the framework of a Boltzmann equation, these corrections will modify the transition probability $W_{\mathbf{k}\mathbf{k}'}$ — they will also add a new term to the current, arising from $\delta \mathbf{v}$:

$$\mathbf{J}_3 = e\lambda \sum_i < \mathbf{grad} \, V_2(\mathbf{r}_i) \wedge \mathbf{S} > \quad (48)$$

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[4] The corresponding term with g instead of f^0 is of second order in E, thus negligible.

J_3 may be obtained directly by using (32). Taking advantage of the fact that $V_2 = V$, we find to first order in λ :

$$J_3 = e\lambda \sum_i \left\langle \left\{ e\mathbf{E} - \dot{\mathbf{k}}_i + \frac{e}{c}\dot{\mathbf{r}}_i \wedge \mathbf{B} \right\} \wedge \mathbf{S} \right\rangle$$

$$= \lambda m^* \left\{ \frac{Ne^2}{m^*} \mathbf{E} - \dot{\mathbf{J}}_0 + \mathbf{J}_0 \wedge \omega_c \right\} \wedge \overline{\mathbf{S}} .$$

In view of (36), J_3 reduces to

$$J_3 = \lambda m^* \Gamma \mathbf{J}_0 \wedge \overline{\mathbf{S}} . \tag{49}$$

Actually, this simple result also bears on the side jump concept introduced by Berger. Let us focus our attention on a given collision, which lasts only a short time, during which we may ignore the precession of S, as well as the electric and Lorentz forces (an assumption implicit in the use of a Boltzmann equation). The collision *event* is thus governed by the simplified equations

$$\dot{\mathbf{r}} = \frac{\mathbf{k}}{m^*} + \lambda \,\mathbf{grad}\, V_2 \wedge \mathbf{S}$$

$$\dot{\mathbf{k}} = - \,\mathbf{grad}\, V - (\rho.\mathbf{grad})\,\mathbf{grad}\, V_2 . \tag{50}$$

The second term of $\dot{\mathbf{r}}$ may be written as $- \lambda \dot{\mathbf{k}} \wedge \mathbf{S}$. Since S is fixed, **r** also undergoes a discontinuous jump during the collision

$$\Delta \mathbf{r} = - \lambda \,\Delta \mathbf{k} \wedge \mathbf{S} . \tag{51}$$

The *total side jump* is thus $\Delta\mathbf{r} + \Delta\rho = 2\,\Delta\rho$ (5). (The fact that the side jump contains two parts is important and must not be overlooked.) $\Delta\mathbf{r}$ is the quantity that was calculated by Berger, by following the motion of a wave packet. A simplified version of his argument is outlined in Appendix B within the Born approximation (6) ; it reproduces the result (51). The added effect of these side jumps at each collision gives rise to a current which is nothing but J_3 ; indeed, the charge displacement per unit time may be written as

$$\sum_{\mathbf{k},\mathbf{k}'} f_\mathbf{k}\, W_{\mathbf{k}\mathbf{k}'} \,\Delta\mathbf{r} = - \lambda \sum_{\mathbf{k},\mathbf{k}'} f_\mathbf{k}\, W_{\mathbf{k}\mathbf{k}'}(\mathbf{k}' - \mathbf{k}) \wedge \mathbf{S} \,! \tag{52}$$

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(5) The fact that $\Delta\mathbf{r} = \Delta\rho$ is a consequence of our assumption $V_2 = V$. For an arbitrary V_2, the calculation of $\Delta\mathbf{r}$ would be far more difficult.

(6) In the more general case, it is not obvious how to define the side jump unambiguously — or at least that part of it that arises from spin orbit effects. Our derivation of (51) is not totally convincing, as the term $\mathbf{k}/m^*$ in the velocity, when integrated over time, could also in principle produce a finite average side jump. The fact that such an effect does not occur within Born approximation is suggestive, but not conclusive. We prefer to adopt an operational definition of the side jump $\Delta\mathbf{r}$, as that part arising from the extra velocity δv, giving rise to the extra current J_3 given by (49) or (52).

Using the definition of Γ, one verifies easily that (52) is identical with the expression (49) of J_3. The physical interpretation of J_3 is thus straightforward : it describes the « *jump current* » due to the extra velocity $- \lambda\dot{\mathbf{k}} \wedge \mathbf{S}$ appearing *during* a collision.

We are left now with only one problem : spin orbit corrections to the transition probability $W_{\mathbf{k}\mathbf{k}'}$, and their effect on the main current

$$\sum_\mathbf{k} \frac{\mathbf{k}}{m^*} f_\mathbf{k} .$$

Those corrections fall in two classes :

(i) Contributions to the collision integral in (41) that involve the equilibrium distribution f^0, arising because the collisions do not conserve the kinetic energy. These terms are a consequence of the extra side jump $\Delta\mathbf{r}$, and are similar to those arising from $\Delta\rho$.

(ii) Contributions that involve the departure from equilibrium $g_\mathbf{k}$, arising from an asymmetry in $W_{\mathbf{k}\mathbf{k}'}$, the so-called *skew scattering* that gives a different weight to scattering toward left or right.

As discussed by many authors, these effects are of a different nature, and their contributions to the conductivity look indeed very different.

The class (i) corrections show up as an additional driving term in the Boltzmann equation. Since here $\Delta\mathbf{r} = \Delta\rho$, we need not reproduce our former calculation : on scattering, the kinetic energy must change by an amount $\Delta\varepsilon = e\mathbf{E}.\Delta\mathbf{r}$ — hence a finite contribution of f^0 to the collision integral, and an additional anomalous current

$$J_4 = J_2''' = \sigma_0[\Gamma m^* \lambda \mathbf{E} \wedge \overline{\mathbf{S}}] . \tag{53}$$

In order to evaluate the skew scattering current, we must go deeper into the calculation of $W_{\mathbf{k}\mathbf{k}'}$. The effective scattering potential in H_{eff} is

$$V_{\mathrm{eff}} = V(\mathbf{r}) - \lambda(\mathbf{k} \wedge \mathbf{S}).\mathbf{grad}\, V_2 .$$

The corresponding matrix element between two plane wave states are (assuming $V_2 = V$) :

$$< \mathbf{k}' \,|\, V_{\mathrm{eff}} \,|\, \mathbf{k} > = V_{\mathbf{k}-\mathbf{k}'}[1 - i\lambda(\mathbf{k} \wedge \mathbf{k}'),\mathbf{S}] . \tag{54}$$

The correction in the bracket of (54) is responsible for the asymmetry in scattering. Although the construction of the full t-matrix up to first order in λ is in principle straightforward, its detailed expression is messy. For simplicity, we shall consider only the case of a point potential $V(\mathbf{r}) = V\delta(\mathbf{r})$, for which $V_{\mathbf{k}-\mathbf{k}'}$ is a constant : in the t matrix, the spin orbit part of (54) can only occur *alone*, since otherwise the summation over intermediate states would necessarily give a vanishing result. In this simple case, we thus have

$$t_{\mathbf{k}'\mathbf{k}}(\varepsilon) = \frac{V}{1 - VG(\varepsilon)} - iV\lambda(\mathbf{k} \wedge \mathbf{k}').\mathbf{S} \tag{55}$$

where $G(\varepsilon)$ is the local propagator of a free electron

$$G(\varepsilon) = \sum_{\mathbf{k}} \frac{1}{\varepsilon - \varepsilon_k + i\eta} \,.$$

The transition probability involves the product $|t_{\mathbf{k'k}}(\mu)|^2$. From (55), it follows that

$$W_{\mathbf{kk'}} = W^0[1 + 2\,\lambda V \mathrm{Im} G(\mu)\,(\mathbf{k} \wedge \mathbf{k'}).\mathbf{S}] \quad (56)$$

where $\mathrm{Im} G(\mu) = \pi\rho(\mu)$ is related to the density of states at the Fermi level. We note that the spin orbit correction to $W_{\mathbf{kk'}}$ is of order V^3 : it appears only beyond the Born approximation, a well-known result.

In order to find the skew scattering current J^{sk}, we multiply the Boltzmann eq. (41) by $\mathbf{k}/m^*$, and we sum over $\mathbf{k}$, thereby generating the macroscopic eq. (36). The correction to $W_{\mathbf{kk'}}$ gives an additional term on the right hand side of (36) :

$$-\sum_{\mathbf{k},\mathbf{k'}} W^0[1 + 2\,\pi\rho V\lambda(\mathbf{k} \wedge \mathbf{k'}).\mathbf{S}]\,(g_{\mathbf{k}} - g_{\mathbf{k'}})\,\frac{\mathbf{k}}{m^*} \,.$$

$$(57)$$

Since g is odd under reflection, only $g_{\mathbf{k'}}$ contributes in (57) ; the integration is straightforward and yields

$$2\,\pi\Gamma\rho V\lambda\,\frac{k_{\mathrm{F}}^2}{3}\,\mathbf{J}_0 \wedge \mathbf{\bar S} = 2\,\pi m^*\,\Gamma N V \lambda \mathbf{J}_0 \wedge \mathbf{\bar S} \,. \quad (58)$$

Since we work only to first order in λ, we may view (58) as an extra driving term, added to the main force $N e^2\,\mathbf{E}/m^*$ on the right hand side of (36). The corresponding skew scattering current is therefore

$$\mathbf{J}^{\mathrm{sk}} = \sigma_0\left[\frac{2\,\pi m^{*2}}{e^2}\,\Gamma V \lambda \mathbf{J}_0 \wedge \mathbf{\bar S}\right]. \quad (59)$$

Since $\mathbf{J}_0$ is itself equal to $\sigma_0\,\mathbf{E}$, $\mathbf{J}^{\mathrm{sk}}$ is proportional to the *square* of σ_0, in contrast to the other terms. We emphasize that (59) is only valid for point scatterers, a restriction that was not needed for the other terms.

We conclude this discussion by summarizing our results. If we set aside the mass renormalization current $\mathbf{J}^{\mathrm{m}}$ given by (28), the remaining anomalous current contains a number of terms. Those which involve the product $\mathbf{J}_0 \wedge \mathbf{\bar S}$

$$\mathbf{J}_1 + \mathbf{J}_3 = \lambda m^*\left\{-\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{J}_0 \wedge \mathbf{\bar S}) + \Gamma \mathbf{J}_0 \wedge \mathbf{\bar S}\right\} \quad (60a)$$

would in a field theoretical description correspond to a renormalization of the measured current vertex. Here, $\mathbf{J}_1$ is a polarization current, while $\mathbf{J}_3$ is due to the *side jump* $\Delta \mathbf{r}$ at each collision. (In fact, $\mathbf{J}_1$ may also be analysed in more detail : it contains a part due to the side jump $\Delta\rho$, which is equal to $\mathbf{J}_3$.)

The terms that involve $\mathbf{E} \wedge \mathbf{\bar S}$ correspond to a renormalization of the electric field excitation vertex ; they are given by

$$\mathbf{J}_2 + \mathbf{J}_4 = \lambda m^*\,\sigma_0\left\{-\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{E} \wedge \mathbf{\bar S}) + \Gamma \mathbf{E} \wedge \mathbf{\bar S}\right\}.$$

$$(60b)$$

(Note the similarity with (60a).) $\mathbf{J}_2$ arises from the spin orbit correction to the driving potential $-e\mathbf{E}.\mathbf{r}$, while $\mathbf{J}_4$ comes from the correction to the scattering potential. $\mathbf{J}_4$ may be ascribed to the lack of kinetic energy conservation upon scattering due to the *side jump* $\Delta \mathbf{r}$ *in the presence of an electric field.* The side jump $\Delta\rho$ gives a similar effect, which is hidden inside $\mathbf{J}_2$ as a term $\mathbf{J}_2'''$ (the rest of $\mathbf{J}_2$ comes from the « gauge » velocity produced by the combined effect of $\mathbf{E}$ and spin orbit coupling).

Finally, the skew scattering gives a last term of a completely different nature, $\mathbf{J}^{\mathrm{sk}}$, given by (59).

The foregoing discussion is valid under arbitrary ac conditions ($\mathbf{E}$ varying as well as $\mathbf{\bar S}$). It shows clearly the existence of a number of terms which, if overlooked, may give rise to a rather random collection of missing factors 2. In this respect, a systematic analysis of the physical meaning of each term is important, as well as a confrontation with an explicit microscopic theory.

4. Application to specific cases. — 4.1 STATIONARY MAGNETIZATION.

— We first consider the usual case in which $\mathbf{\bar S}$ is time independent, parallel to $\mathbf{B}$ (say to the z-axis). Let ω be the frequency of the electric field : $\dot{\mathbf{E}} = i\omega\mathbf{E}$. σ_0 breaks into a component

$$\sigma_{\mathrm{zz}} = \frac{N e^2}{m^*(\Gamma + i\omega)} \quad (61a)$$

and into two circularly polarized conductivities in the x-y plane

$$\sigma_{\pm} = \frac{N e^2}{m^*(\Gamma + i\omega \pm i\omega_{\mathrm{c}})} \,. \quad (61b)$$

Because $\mathbf{\bar S}$ is parallel to $\mathbf{B}$, the following result holds

$$\mathbf{J}_0 \wedge \mathbf{\bar S} = (\sigma_0\,\mathbf{E}) \wedge \mathbf{\bar S} = \sigma_0(\mathbf{E} \wedge \mathbf{\bar S}) \,.$$

Consequently, $\mathbf{J}_1 = \mathbf{J}_3$, $\mathbf{J}_2 = \mathbf{J}_4$. All these terms give a net « vertex » contribution to the anomalous current

$$\mathbf{J}^{\mathrm{v}} = \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 + \mathbf{J}_4 = 2\,\lambda m^*(\Gamma - i\omega)\,\sigma_0(\mathbf{E} \wedge \mathbf{\bar S}) \,. \quad (62)$$

The conductivity associated to $\mathbf{J}^{\mathrm{v}}$ is

$$\delta\sigma_{\pm}^{\mathrm{v}} = 2\,N e^2\,\lambda\,\frac{\Gamma - i\omega}{\Gamma + i\omega \pm i\omega_{\mathrm{c}}}\,(\pm\,i\mathbf{\bar S}) \,. \quad (63)$$

From the definition of λ, it follows that

$$\lambda\mathbf{\bar S} = -\frac{\mathbf{M}.\mathbf{B}}{\omega_{\mathrm{c}}\,E_2} \,,$$

hence an equivalent form of (63)

$$\delta\sigma^v_\pm = \mp 2 \frac{\mathbf{M}.\mathbf{B}}{E_2} \frac{i\Gamma + \omega}{\omega_c} \sigma^0_\pm$$

which clearly displays the order of magnitude of the anomalous effect. The individual components of the conductivity tensor are

$$\sigma_{xy} = \frac{1}{2}(\sigma_+ + \sigma_-), \quad \sigma_{xy} = \frac{1}{2i}(\sigma_+ - \sigma_-). \quad (64)$$

The Hall effect is of special interest ; the corresponding « vertex » contribution is

$$\delta\sigma^v_{xy} = 2 Ne^2 \lambda \frac{(\Gamma - i\omega)(\Gamma + i\omega)}{(\Gamma + i\omega)^2 + \omega_c^2} \overline{S}. \quad (65)$$

In the limit $\mathbf{B} = 0$, we recover the result of reference [7]. We note that for a dc electric field ($\omega = 0$) $\delta\sigma^v_{xy}$ is proportional to $\Gamma^2/(\omega_c^2 + \Gamma^2)$, i. e., to $\cos^2 \theta$, where θ is the Hall angle : the anomalous Hall current disappears in the high field limit $\omega_c \gg \Gamma$.

For $\mathbf{B} = 0$, (65) reduces to

$$\delta\sigma^v_{xy} = 2 Ne^2 \lambda\overline{S} \frac{\Gamma - i\omega}{\Gamma + i\omega}. \quad (66)$$

As noted in (7), the current takes opposite values in the two limits $\omega \gg \Gamma$ and $\omega \ll \Gamma$. This puzzling feature may be understood if we write (66) in the form

$$\delta\sigma^v_{xy} = - 2 Ne^2 \lambda\overline{S} \left[1 - \frac{2\Gamma}{\Gamma + i\omega}\right]. \quad (67)$$

The second term in the bracket of (67) arises half from the side jump $\Delta\rho$, half from Δr. Were we to include only $\Delta\rho$, the side jump term would compensate the gauge current $- 2 Ne^2 \lambda\overline{S}$ in the usual way, yielding a factor $i\omega/(\Gamma + i\omega)$. Because the actual side jump is twice as big, the corresponding contribution to the current « overshoots » the gauge term when $\omega = 0$, and gives a net result of the opposite sign.

In order to complement our general result (63) for $\delta\sigma^v_\pm$, we must calculate the skew scattering part $\delta\sigma^{sk}_\pm$. From (59), it follows that

$$\delta\sigma^{sk}_\pm = \pm i(\sigma^0_\pm)^2 \frac{2\pi m^{*2}}{e^2} \Gamma V\lambda\overline{S}. \quad (68)$$

Using (61b), we obtain the explicit result

$$\delta\sigma^{sk}_\pm = 2 Ne^2 \lambda \frac{\pi NV\Gamma}{(\Gamma + i\omega \pm i\omega_c)^2} (\pm i\overline{S}). \quad (69)$$

As compared to $\delta\sigma^v_\pm$, the skew scattering part is modified by a factor

$$\frac{\pi NV\Gamma}{(\Gamma + i\omega)(\Gamma + i\omega \pm i\omega_c)}.$$

From (69) and (64), we easily obtain the Hall effect correction

$$\delta\sigma^{sk}_{xy} = 2 Ne^2 \lambda\overline{S} \frac{\pi NV\Gamma\{(\Gamma + i\omega)^2 - \omega_c^2\}}{\{(\Gamma + i\omega)^2 + \omega_c^2\}^2}. \quad (70)$$

In practice, experiments usually deal with dc electric fields ($\omega = 0$). (65) and (70) then reduce to

$$\delta\sigma^v_{xy} = 2 Ne^2 \lambda\overline{S} \frac{\Gamma^2}{\omega_c^2 + \Gamma^2}$$

$$\delta\sigma^{sk}_{xy} = 2 Ne^2 \lambda\overline{S} \frac{\pi NV\Gamma(\Gamma^2 - \omega_c^2)}{(\Gamma^2 + \omega_c^2)^2} \quad (71)$$

(71) is the net practical result of our theory. Note the change of sign of $\delta\sigma^{sk}_{xy}$ when the Hall angle reaches $\pi/4$; the ratio of the two terms in (71) is

$$\frac{\delta\sigma^{sk}_{xy}}{\delta\sigma^v_{xy}} = \frac{\pi NV}{\Gamma} \frac{\Gamma^2 - \omega_c^2}{\Gamma^2 + \omega_c^2}.$$

A detailed discussion of these results may be found in reference [8].

4.2 A. C. RESONANT MAGNETIZATION. — We next consider the experimental conditions used in the recent work of Chazalviel and Solomon [9] : an ac magnetic field *forces* the spin to precess around $\mathbf{B}$ at frequency ω. The resulting ac $\overline{S}$, together with the ac $\mathbf{E}$, gives rise to a dc anomalous current, which is detected by letting ω go through the spin resonance frequency. In short, we want the dc current due to the ac $\mathbf{E}$ and $\mathbf{S}$.

Since we are measuring a dc current, the first term in the bracket of (60a) and (60b) gives no contribution : $\mathbf{J}_1$ and $\mathbf{J}_2$ disappear. What remains depends on the geometry of the problem. The ac $\overline{S}$ is always circularly polarized in the x-y plane ; the electric field $\mathbf{E}$, on the other hand, may be

(i) either parallel to the z-axis (geometry used in ref. [9]) ;

(ii) or in the x-y plane : in that case, only the circular component rotating in the same direction as $\overline{S}$ gives a dc current (the other direction of rotation yields a current at frequency 2ω).

We first consider case (i), in which $\mathbf{J}_0 = \sigma^0_z \mathbf{E}$. The product $\mathbf{E} \wedge \overline{S}$ is in the x-y plane ; the direction of its dc component depends on the relative phase of $\mathbf{E}$ and $\mathbf{S}$. Indeed, let us write

$$\overline{S}_x = S_1 \cos \omega t \quad \overline{S}_y = S_1 \sin \omega t$$

$$E_z = E_1 \cos(\omega t - \varphi)$$

(when E_y is maximum, $\overline{S}$ makes an angle φ with the x-axis). The dc component of $\mathbf{E} \wedge \overline{S}$ has modulus $E_1 S_1/2$, and points at an angle $\pi/2 + \varphi$ from the x-axis. In terms of the usual complex amplitudes, the dc part of $\mathbf{E} \wedge \overline{S}$ is given by

$$(\mathbf{E} \wedge \overline{\mathbf{S}})^{dc}_\pm = \pm iE_z(\mp \omega) \overline{S}_\pm.$$

The vertex part of the anomalous current is then

$$J^v_\pm = (\mathbf{J}_3 + \mathbf{J}_4)_\pm = \lambda m^* \Gamma \left\{ \sigma^0_z (\mp \omega) + \sigma^0_\pm(0) \right\} \times$$
$$\times (\mathbf{E} \wedge \overline{\mathbf{S}})^{dc}_\pm . \qquad (72)$$

Note that, contrary to the case $\overset{\bullet}{\mathbf{S}} = 0$, $\mathbf{J}_3$ and $\mathbf{J}_4$ are no longer equal. By an appropriate choice of coordinates, we can always choose the x axis along the direction of the dc $\mathbf{E} \wedge \overline{\mathbf{S}}$. On using (61), we may then write (72) in the explicit form

$$J^v_\pm = Ne^2 \lambda \Gamma \left[\frac{1}{\Gamma \pm i\omega_c} + \frac{1}{\Gamma \mp i\omega} \right] \frac{E_1 S_1}{2} . \qquad (73)$$

The components of J^v follow from (64)

$$J^v_x = Ne^2 \lambda \left[\frac{\Gamma^2}{\Gamma^2 + \omega_c^2} + \frac{\Gamma^2}{\Gamma^2 + \omega^2} \right] \frac{E_1 S_1}{2}$$
$$J^v_y = Ne^2 \lambda \left[\frac{\omega\Gamma}{\Gamma^2 + \omega^2} - \frac{\omega_c \Gamma}{\Gamma^2 + \omega_c^2} \right] \frac{E_1 S_1}{2} . \qquad (74)$$

In the limit $\omega, \omega_c \ll \Gamma$ (in which the present experiments are carried out), we recover the result of the static calculation

$$\mathbf{J}^v = 2 Ne^2 \lambda (\mathbf{E} \wedge \overline{\mathbf{S}})^{dc} \qquad (75)$$

(74) is more general, and its form could hardly be guessed.

A similar analysis applies to the skew scattering current (59), given here by

$$J^{sk}_\pm = \sigma^0_\pm(0) \frac{2\pi m^{*2}}{e^2} \Gamma V \lambda \sigma^0_z(\mp \omega) (\mathbf{E} \wedge \overline{\mathbf{S}})^{dc}_\pm . \qquad (76)$$

Using again (61), we obtain

$$J^{sk}_\pm = \frac{2 Ne^2 \lambda \pi N V \Gamma}{(\Gamma \mp i\omega)(\Gamma \pm i\omega_c)} \frac{E_1 S_1}{2} \qquad (77)$$

from which it follows

$$J^{sk}_x = 2 Ne^2 \lambda \frac{\pi N V \Gamma(\Gamma^2 + \omega\omega_c)}{(\Gamma^2 + \omega^2)(\Gamma^2 + \omega_c^2)} \frac{E_1 S_1}{2}$$
$$J^{sk}_y = 2 Ne^2 \lambda \frac{\pi N V \Gamma^2(\omega - \omega_c)}{(\Gamma^2 + \omega^2)(\Gamma^2 + \omega_c^2)} \frac{E_1 S_1}{2} . \qquad (78)$$

When $\omega, \omega_c \ll \Gamma$, (78) again reduces to the static limit

$$\mathbf{J}^{sk} = 2 Ne^2 \lambda \frac{\pi N V}{\Gamma} (\mathbf{E} \wedge \overline{\mathbf{S}})^{dc} . \qquad (79)$$

Note that, just like $\mathbf{J}^v$, $\mathbf{J}^{sk}$ vanishes in the absence of collisions ($\Gamma = 0$).

We consider finally the geometry (ii), in which $\mathbf{E}$ is in the xy plane, $\mathbf{E} \wedge \mathbf{S}$ along the z-axis. This case is experimentally less convenient, as the anomalous current will be masked by the regular Hall effect due to the ac magnetic field that drives the spin $\mathbf{S}$. Yet,

we can treat it in a similar way. The dc z-component of $\mathbf{E} \wedge \mathbf{S}$ is simply

$$- \text{Im} E^+ S^- .$$

The « vertex » part of the anomalous current along the z-axis is thus

$$J^v_z = (\mathbf{J}_3 + \mathbf{J}_4)_z$$
$$= - \lambda m^* \Gamma \text{Im} \left\{ [\sigma^0_+(\omega) + \sigma^0_z(0)] E^+ S^- \right\} . \qquad (80)$$

Let α be the angle between the circularly polarized ac vectors E and S :

$$E^+ S^- = E_1 S_1 e^{i\alpha} .$$

Using (61), we write (80) as

$$J^v_z = - Ne^2 \lambda \Gamma \times$$
$$\times \text{Im} \left\{ \left(\frac{1}{\Gamma + i\omega + i\omega_c} + \frac{1}{\Gamma} \right) e^{i\alpha} \right\} E_1 S_1 . \qquad (81)$$

In the static limit $\omega, \omega_c \ll \Gamma$, we recover again (75). More interesting is the fact that, contrary to geometry (i), $\mathbf{J}^v$ does not vanish in the absence of collisions — indeed, when $\Gamma \to 0$, we find

$$\mathbf{J}^v \to Ne^2 \lambda (\mathbf{E} \wedge \overline{\mathbf{S}})^{dc} \qquad (82)$$

i. e. half the value found in the opposite limit of large Γ. We conclude that an anomalous Hall effect may exist even without collisions, provided the appropriate geometry is selected.

The skew scattering current (59) is obtained along the same lines :

$$J^{sk}_z = - \sigma^0_z(0) \frac{2\pi m^{*2}}{e^2} \Gamma V \lambda \text{Im} \left\{ \sigma^0_+(\omega) E^+ S^- \right\} \qquad (83)$$

which leads to

$$J^{sk}_z = - 2 Ne^2 \lambda \pi N V \text{Im} \left\{ \frac{e^{i\alpha}}{\Gamma + i\omega + i\omega_c} \right\} E_1 S_1 . \qquad (84)$$

If $\omega, \omega_c \ll \Gamma$, we recover (79). In the opposite limit $V \to 0$, $\mathbf{J}^{sk}$ vanishes, although strangely enough it remains finite when the impurity density $n_i \to 0$, the interaction strength V remaining constant. The physical meaning of this unusual behavior remains unclear.

5. Conclusion. — Whether the unusual ω and ω_c dependence of the anomalous conductivity can be seen experimentally remains to be shown. At any rate, it must be borne in mind that in finite magnetic fields, the conductivity may depend on the spin magnetization through a mechanism that has nothing to do with spin orbit coupling — namely the relative shift of the Shubnikov de Haas oscillations for carriers of opposite spins when the Fermi wave vectors $k_{F\uparrow}$ and $k_{F\downarrow}$ are varied. This spurious effect

TABLE I

	$\Gamma = 0$	$\omega = 0$
1) *Polarization current* $\mathbf{J_1} = Ne\dot{\mathbf{p}}$		
Driving term	$- Ne^2 \lambda \mathbf{E} \wedge \bar{\mathbf{S}}$	$- Ne^2 \lambda \mathbf{E} \wedge \bar{\mathbf{S}}$
Current due to the side jump $\Delta\rho$	0	$+ Ne^2 \lambda \mathbf{E} \wedge \bar{\mathbf{S}}$
2) *Current* $\mathbf{J_2} = Ne\dot{\mathbf{r}}$ *due to the spin orbit correction*		
to the electrostatic driving force		
Extra gauge current	$- Ne^2 \lambda \mathbf{E} \wedge \bar{\mathbf{S}}$	$- Ne^2 \lambda \mathbf{E} \wedge \bar{\mathbf{S}}$
Influence of the side jump $\Delta\rho$ on the relaxation of $f_\mathbf{k}^0$	0	$+ Ne^2 \lambda \mathbf{E} \wedge \bar{\mathbf{S}}$
3) *Current due to the spin orbit correction to the*		
scattering potential (i. e. to the side jump $\Delta\mathbf{r}$)		
Charge displacement $\mathbf{J_3}$ due to $\Delta\mathbf{r}$	0	$+ Ne^2 \lambda \mathbf{E} \wedge \bar{\mathbf{S}}$
Influence of $\Delta\mathbf{r}$ on the relaxation of $f_\mathbf{k}^0$	0	$+ Ne^2 \lambda \mathbf{E} \wedge \bar{\mathbf{S}}$
Total	$- 2 Ne^2 \lambda \mathbf{E} \wedge \bar{\mathbf{S}}$	$+ 2 Ne^2 \lambda \mathbf{E} \wedge \bar{\mathbf{S}}$

must be eliminated if one is to work with a stationary $\bar{\mathbf{S}}$; it will probably preclude any observation of the anomalous Hall effect in the quantum regime, where the oscillations of σ are large.

We refer to [8] for a detailed discussion of the above point, as well as for an estimation of the relative magnitude of $\mathbf{J}^v$ and $\mathbf{J}^{sk}$. Our purpose in this paper was only to provide an elementary theory of the various « anomalous currents », based on the use of an effective Hamiltonian in the conduction band. The main problem is to collect *all* the terms, with their appropriate physical interpretation. Confusion arose in the past because in the special case $V_2 = V$, all these terms are equal or opposite, leading to a large amount of cancellation : one may then argue that the net result arises from any given part of the sum. Such an ambiguity is avoided if the scattering potential corrections are earmarked by assuming for a moment $V_2 \neq V$ — and also if one allows $\bar{\mathbf{S}}$ to depend on time, in which case the cancellations disappear.

As an attempt to clarify the situation, we summarize in Table I the various contributions to the current renormalization effect ($\mathbf{J_1}$ to $\mathbf{J_4}$), in the special case where $\mathbf{B} = \dot{\mathbf{S}} = 0$, in the two limits of a pure crystal ($\Gamma = 0$) and of a static electric field ($\omega = 0$). We see that

(i) For $\omega = 0$, the anomalous current arises *entirely* from the correction V_2 to the scattering potential.

(ii) The conductivity is twice as large as a naïve evaluation of the side jump $\Delta\mathbf{r}$ might suggest.

(iii) The current reverses sign when one goes from the limit $\Gamma = 0$ to $\omega = 0$.

To that one must of course add the skew scattering part $\mathbf{J}^{sk}$. The above results are in complete agreement with those obtained from a microscopic theory [7] —

they were correctly predicted in the pioneering work of Luttinger [3].

Appendix A. — We comment briefly on the complications that arise if one tries to go beyond the simple approximations used in the text. First of all, one may try to take into account the bare mass m_0 and Landé factor g_0 in (13). g^* and m^* are then given by the full eq. (17) ; the expression (18) for the normalization operator Λ is replaced by

$$\Lambda = \frac{k^2}{2 E_1} \left[\frac{1}{m^*} - \frac{1}{m_0} \right] + \frac{g^* - g_0}{E_2} \beta \mathbf{S}.\mathbf{B} .$$

The term $- \Lambda H_0$ in (8) is thus slightly modified, the part of interest being now

$$- \frac{k^2}{2} \beta \mathbf{S}.\mathbf{B} \left\{ \frac{g^*}{E_1} \left[\frac{1}{m^*} - \frac{1}{m_0} \right] + \frac{1}{m^* E_2} (g^* - g_0) \right\} .$$

$$(A.1)$$

In addition, δH acquires a new part given by the second term in (8). That term may be calculated explicitly, and turns out to be of the form

$$\alpha \frac{k^2}{2} \mathbf{S}.\mathbf{B} + \gamma (\mathbf{k}.\mathbf{S}) (\mathbf{k}.\mathbf{B}) . \qquad (A.2)$$

The first term (A.2) has the same form as (A.1), and may be incorporated as a redefinition of E_0 in eq. (19) (it guarantees, by the way, that the corresponding correction vanishes if there is no spin orbit coupling !). The second term of (A.2) is of a new type, and causes some problems. When $\mathbf{B}$ and $\mathbf{S}$

are parallel and stationary, we may view it as describing an *anisotropic* spin dependent effective mass, which is different for **k** parallel or perpendicular to **B**. The generalization of our former analysis is then straightforward. On the other hand, for an ac spin S perpendicular to the dc B_0, it is not clear what the effect of this term will be. Since our two band model is tantamount to assuming that $1/m_0$ and g_0 are negligible, we did not pursue the question further. Anyhow, this complication does *not* affect the genuine anomalous effect, depending on the angle between **E** and $\overline{S}$.

One may also try to take into account the spin orbit part of the scattering potential, $V_S = V_2 - V_1$. That is a much harder problem. First of all, one must take into account the second term in (21). If $V_2(\mathbf{r})$ were proportional to $V_1(\mathbf{r})$, one might incorporate this correction as a multiplicative factor $(1 + \zeta\mathbf{S}.\mathbf{B})$ on the *whole* Hamiltonian, which ultimately would amount to a scaling on the frequency and temperature dependence of the conductivity. Unfortunately, V_S is essentially local, while V_1 is extended : one must then solve separately the scattering problem for up and down spins, which complicates the issue. Moreover, although in principle our analysis of the last term of (21) remains valid, the side jump $\Delta\mathbf{r}$ is no longer given by the simple expression (51), and the results (49) or (53) do not hold : a detailed description of the scattering process is required. We did not attempt it.

Appendix B. — We present here a simplified version of Berger's argument yielding the side jump $\Delta\mathbf{r}$ experienced by an electron upon scattering. The calculation is carried out within the Born approximation. We expand the wave function in momentum space as

$$\psi(\mathbf{r}, t) = \sum_{\mathbf{k}} C_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}}.$$

The components $C_{\mathbf{k}}$ obey the Schrödinger equation

$$i\dot{C}_{\mathbf{k}} = \varepsilon_k\, C_{\mathbf{k}} + \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\, C_{\mathbf{k}'}\, e^{\eta t} \qquad \text{(B.1)}$$

(the scattering potential is established adiabatically in order to guarantee the correct boundary condition). $V_{\mathbf{k}\mathbf{k}'}$ is the matrix element of the net scattering potential $V + \delta V$ (see (24)), which is easily expressed in terms of the Fourier transform of $V(\mathbf{r})$. If the impurity lies at the origin $\mathbf{r} = 0$

$$V_{\mathbf{k}\mathbf{k}'} = V_{\mathbf{k}-\mathbf{k}'}[1 - i\lambda(\mathbf{k} \wedge \mathbf{k}').\mathbf{S}] \qquad \text{(B.2)}$$

(see (54)). We consider an incoming wave packet, with wave function for $t \to -\infty$

$$C_{\mathbf{k}}^0 = A\, e^{-i\varepsilon_k t}\, e^{-(\mathbf{k}-\mathbf{k}_0)^2/2\Delta^2} \qquad \text{(B.3)}$$

(A is a normalization constant). (B.3) describes a wave packet of mean momentum $\mathbf{k}_0$, *centered at*

the origin at $t = 0$. The scattered wave function is obtained by solving (B.1) by iteration

$$C_{\mathbf{k}}^1 = -\sum_{\mathbf{k}'} \frac{V_{\mathbf{k}\mathbf{k}'}\, e^{\eta t}}{\varepsilon_k - \varepsilon_{k'} - i\eta}\, C_{\mathbf{k}'}^0$$

$$C_{\mathbf{k}}^2 = -\sum_{\mathbf{k}'} \frac{V_{\mathbf{k}\mathbf{k}'}\, e^{\eta t}}{\varepsilon_k - \varepsilon_{k'} - i\eta}\, C_{\mathbf{k}'}^1. \qquad \text{(B.4)}$$

With the particular choice (B.3) for $C_{\mathbf{k}}^0$, the integration over $\mathbf{k}'$ in (B.4) is straightforward (for small Δ, of course). For large *positive* times, such that

$$\frac{\Delta k_0\, t}{m} \gg 1$$

(i. e. when the incoming wave packet is well beyond the impurity), it turns out that (B.4) is controlled by the pole at $\varepsilon_{k'} = \varepsilon_k$. We then find

$$C_{\mathbf{k}}^1(t) = -2\pi i \sum_{k'} V_{\mathbf{k}\mathbf{k}'}\, \delta(\varepsilon_k - \varepsilon_{k'})\, C_{\mathbf{k}'}^0\, e^{-i\varepsilon_k t}$$

$$C_{\mathbf{k}}^2(t) = (-2\pi i)^2 \sum_{k'k''} V_{\mathbf{k}\mathbf{k}''}\, V_{\mathbf{k}''\mathbf{k}'}\, \delta(\varepsilon_k - \varepsilon_{k''}) \times \qquad \text{(B.5)}$$

$$\times\; \delta(\varepsilon_{k''} - \varepsilon_{k'})\, C_{\mathbf{k}'}^0\, e^{-i\varepsilon_k t}.$$

The average *position* of the particle is equal to

$$\frac{i}{2} \sum_{\mathbf{k}} \left[C_{\mathbf{k}}^* \frac{\partial C_{\mathbf{k}}}{\partial\mathbf{k}} - C_{\mathbf{k}} \frac{\partial C_{\mathbf{k}}^*}{\partial\mathbf{k}} \right] = \sum_{\mathbf{k}} |C_{\mathbf{k}}|^2\, \mathbf{r}_{\mathbf{k}} \qquad \text{(B.6)}$$

where we have set

$$\mathbf{r}_{\mathbf{k}} = -\frac{\partial}{\partial\mathbf{k}}\, \text{Arg}\, C_{\mathbf{k}} \qquad \text{(B.7)}$$

$\mathbf{r}_{\mathbf{k}}$ measures the position of a particular slice of the wave function. In the absence of spin orbit coupling, $V_{\mathbf{k}\mathbf{k}'}$ is real, and $\mathbf{r}_{\mathbf{k}} = \mathbf{v}_{\mathbf{k}}\, t$: as expected, each component of the wave function moves at the group velocity $\mathbf{v}_{\mathbf{k}} = \partial\varepsilon_k/\partial\mathbf{k}$. When spin orbit is included, $V_{\mathbf{k}\mathbf{k}'}$ is given by (B.2), and the phase of $C_{\mathbf{k}}$ is modified. Let us first consider a scattered wave (such that $C_{\mathbf{k}}^0 = 0$). From (B.5) it is clear that to first order in λ, $C_{\mathbf{k}}^1$ acquires an extra argument

$$-\lambda(\mathbf{k} \wedge \mathbf{k}_0).\mathbf{S}$$

hence a new contribution

$$\delta\mathbf{r}_{\mathbf{k}} = \lambda\mathbf{k}_0 \wedge \mathbf{S}. \qquad \text{(B.8)}$$

The whole scattered wave function is shifted by an amount $\delta\mathbf{r}_{\mathbf{k}}$ as compared to the position it would have if it had been scattered at the origin at $t = 0$.

Actually, the scattered wave function is drawn from the incoming wave packet ; in order to find the total side jump, we must look whether what remains from that incoming state has been shifted or not — put another way, whether the scattered wave has chewed off the center or the side of the wave packet. For that purpose, we consider the region $\mathbf{k} \approx \mathbf{k}_0$ in (B.6). Assuming that there is no forward scattering ($V_{\mathbf{k}\mathbf{k}} = 0$), we set

$$C_{\mathbf{k}} = C_{\mathbf{k}}^0 + C_{\mathbf{k}}^2$$

and we find that to first order in the spin orbit coupling the unscattered wave packet contributes to the average position a term

$$\delta \mathbf{r}_0 = - \sum_{\mathbf{k}} | C_{\mathbf{k}}^1 |^2 \, \lambda \mathbf{k} \wedge \mathbf{S} . \qquad (B.9)$$

The net side jump associated to a given scattered wave $\mathbf{k}$ (with statistical weight $| C_{\mathbf{k}}^1 |^2$) is thus

$$\delta \mathbf{r}_{\mathbf{k}} = \lambda(\mathbf{k}_0 - \mathbf{k}) \wedge \mathbf{S} = - \lambda \, \Delta \mathbf{k} \wedge \mathbf{S} \quad (B.10)$$

in accordance with our former result (51).

References

[1] KARPLUS, R. and LUTTINGER, J. M., *Phys. Rev.* **95** (1954) 1154.

[2] SMIT, J., *Physica*, **23** (1958) 39.

[3] LUTTINGER, J. M., *Phys. Rev.* **112** (1958) 739.

[4] FIVAZ, R. C., *Phys. Rev.* **183** (1969) 586.

[5] BERGER, L., *Phys. Rev.* B **2** (1970) 4559.

[6] LYO, S. K. and HOLSTEIN, T., *Phys. Rev. Lett.* **29** (1972) 423.

[7] LEWINER, C., BETBEDER-MATIBET, O., NOZIÈRES, P., *J. Phys. Chem. Solids.* **34** (1973) 765.

[8] LEWINER, C., Thèse, Paris 1973 (unpublished).

[9] CHAZALVIEL, J. N. and SOLOMON, I., *Phys. Rev. Lett.* **29** (1972) 1676.

[10] LEROUX-HUGON, P. and GHAZALI, A., *J. Phys.* C **5** (1972) 1072.

[11] DONIACH, S., in « *Optical Properties and Electronic Structure of Metals and Alloys* », edited by F. Abeles (North Holland Publishing Co., Amsterdam, 1966) p. 471.

[12] YAFET, Y., *Solid State Physics* (edited by F. Seitz and D. Turnbull), Vol. 14 (Academic Press, N. Y., 1963).

[13] See for instance BASTIN, A., LEWINER, C., BETBEDER-MATIBET, O. and NOZIÈRES, P., *J. Phys. Chem. Solids* **32** (1971) 1811.

[14] KOHN, W. and LUTTINGER, J. M., *Phys. Rev.* **108** (1957) 590.