## Grade 8 Math
## Pythagorean Theorem

**By Joy Clubine, Alannah McGregor & Jisoo Seo**

### Teaching Objectives

- For students to discover and explore the Pythagorean Theorem through a variety of activities.
- Students will understand why the Pythagorean Theorem works and how to prove it using various manipulatives

### Curriculum Expectations

**Grade 8 Math: Geometry and Spatial Sense**
**Overall Expectations: Geometry and Spatial Sense**

- By the end of Grade 8, students will develop geometric relationships involving lines, triangles, and polyhedra, and solve problems involving lines and triangles

**Specific Expectations: Geometric Relationships**

- By the end of Grade 8, students will determine the Pythagorean relationship, through investigation using a variety of tools and strategies
- Solve problems involving right triangles geometrically, using the Pythagorean relationship

**Connections to Grade 4 Math**

- Identify benchmark angles (i.e. right angle), and compare other angles to these benchmarks
- Relate the names of the benchmark angles to their measures in degrees (a right angle is 90°)

**Connections to Grade 5 Math**

- Identify and classify acute, right, obtuse, and straight angles
- Identify triangles and classify them according to angle and side properties

**Connections to Grade 6 Math**

- Measure and construct angles up to 180° using a protractor, and classify them as acute, right, obtuse, or straight angles

**Connections for Grade 7 Math**

- Construct related lines using angle properties and a variety of tools
- Sort and classify triangles by geometric properties related to symmetry, angles, and sides through investigation using a variety of tools

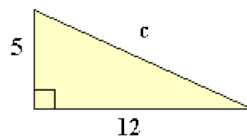| **Subtask 1:** Introduction | **Lesson:** |
|---|---|
| Time:<br>6 minutes<br><br>Materials & Prep:<br>- Whiteboard<br>- Dry-erase markers<br>- Computer and projector | 1. Ask the class: "What do you know about right-angle triangles?"<br>Conduct a whole class brainstorming session<br>Record the responses on whiteboard<br><br>2. Try the following question:<br>   - What is the hypotenuse?<br><br> c = 13 (answer will not be given until the end)<br><br>3. If the Theorem comes up, use it to introduce the next subtask.<br>If it does not come up, introduce the Pythagorean Theorem<br>$(a^2 + b^2 = c^2)$ and that we will be exploring the theorem (Why it works and why it holds true) using different materials. "Each group will be given different materials with which to explore the relationships in the formula, and we will have time to share our discoveries with one another at the end." |

| **Subtask 2:** Explore why the Pythagorean Theorem is true and other related concepts<br><br>Time:<br>10-15 minutes<br><br>Grouping:<br>4 groups<br><br>Group 1 Materials & Prep:<br>- 2x[4congruent triangles with area ab/2]<br>- 2x[Large square with area $(a+b)^2$ drawn on grid chart paper]<br>- Pencil and eraser<br>- tiles | **Lesson:** Each group will be given the materials specified and will be asked to use them to explore how they might prove the theorem<br><br>**Group 1: Geometric Proof**<br>http://www.mathopenref.com/pythagorasproof.html<br><br>**Instruction:**<br>"Manipulate the 4 triangles in the square provided to show that $a^2 + b^2 = c^2$" |
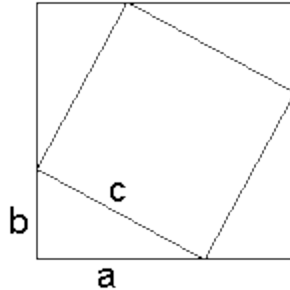


the empty square in the middle has area of $c^2$.
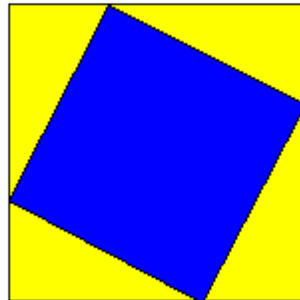


rearrange the triangles
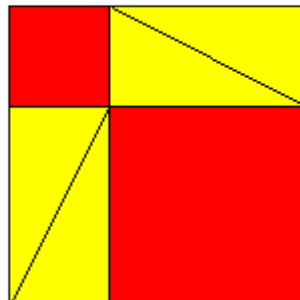


The empty square at the top has an area of $a^2$. The empty square at the bottom has an area of $^b2$. The two areas cover the same as c2. Therefore, $c^2$ must equal $a^2$ plus $b^2$.

**Extension:**
  a)  How can you represent this algebraically?

$(\mathbf{a} + \mathbf{b})^2 = 4\cdot\mathbf{ab}/2 + \mathbf{c}^2$
$\mathbf{c}^2 = (\mathbf{a} + \mathbf{b})^2 - 4\cdot\mathbf{ab}/2$
$2\mathbf{c}^2 = 2\mathbf{a}^2 + 2\mathbf{b}^2$.
$\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2$

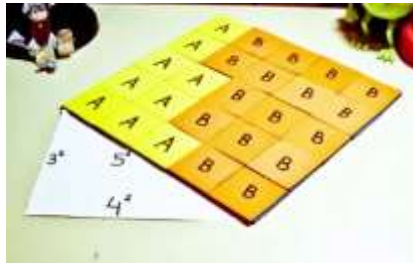| Group 2 Materials & Prep: | **Group 2: Proof through Simplification** |
|---|---|
| - 2 Grid chart papers<br>- Markers<br>- Triangle cut-outs (3,4, 5 & 6, 8, 10)<br>- Tiles (3 different colours)<br>- Rulers<br>- Cardstock Paper<br>- Pencil & eraser<br>- Scissors | <br><br>Students will explore the Pythagorean Theorem by arranging tiles to show that the sum of the square of the legs is equal to the square of the hypotenuse.<br><br>**Instruction:**<br>"Using the tiles and the triangle provided, prove that $a^2 + b^2 = c^2$"<br>Hint: 1 tile = 1 inch$^2$<br><br><br><br>can be rearranged to:<br><br><br><br>**Extension:**<br>a) "Make your own right angle triangle using the cardstock paper provided to show that $a^2 + b^2 = c^2$" |

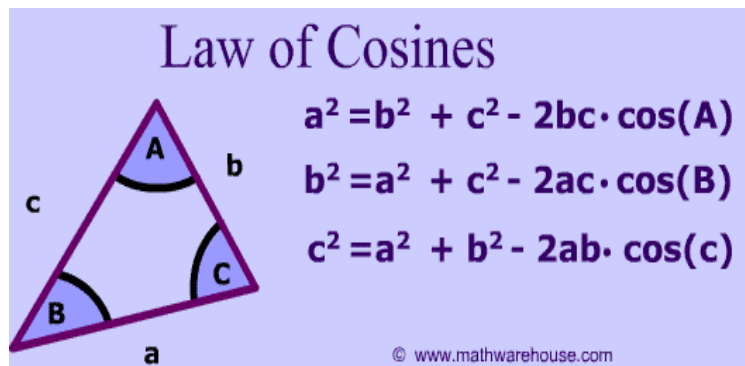| Group 3 Materials & Prep: | **Group 3: Discovering Special Right Triangles** |
|---|---|
| - Wood: 5, 10, 13 inches long<br>- Ruler<br>- Pencil and Paper<br>- Calculator<br>- Grid chart paper<br>- Triangular ruler<br>- clipboard | <br><br>Give the group the stick cutouts (5inch, 10inch, 13 inch). Students will lean the stick against a vertical clipboard (or a wall) and measure the legs of the triangle. Their goal is to find the dimensions of Pythagorean Triples (three positive inters a, b, and c, such that $a^2 + b^2 = c^2$.<br><br>**Instruction:**<br>"You are the proud owners of a Flea Circus and the newest trick you want to try involves fleas jumping from a trampoline onto a slide. You have been given three slides by the itty bitty slide committee (5 inches, 10 inches, and 13 inches). Unfortunately, the flea market where you buy slide ladders only builds them in whole number lengths (Ex: It can't be 4.5 inches tall).<br>By leaning your slides up against a vertical surface, measure and record how high your slides can be and how far the end of the slide is to the base of the ladder."<br><br>3:4:5<br>6:8:10<br>5:12:13<br><br>**Extension:**<br>  a) What you've found is something called a "Side-based Special Right Triangle". A "side-based" right triangle is one in which the lengths of the sides form ratios of whole numbers. Can you find other special side-based right triangles?<br>    9:12:15<br>    8:15:17<br>    12:16:20<br>    15:20:25<br>    9:40:41<br>    12:35:37<br>    11:60:61 etc.<br><br>  b) "You probably have heard of something called The Cosine Law, which is really the Pythagorean Theorem for non-right triangles."<br><br><br><br>Use the Cosine Law to find the Pythagorean Theorem. Hint: Cos (90) = 0 |

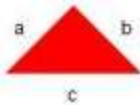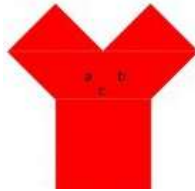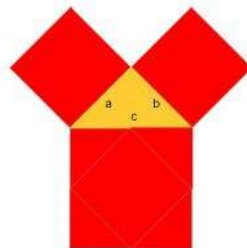| Group 4 Materials & Prep: | **Group 4: Tangram Proof** |
|---|---|
| - 2x[3 sets of tangrams with different colours]<br>- 2 Grid chart papers<br>- markers | <br><br>**Instruction:**<br>"Use the tangram pieces to show that $a^2 + b^2 = c^2$. Start by using the smallest triangle. Then try with the medium, then with the large triangle. Keep a record of your solutions by tracing the final shapes."<br>Hint: You may not need all pieces for every solution, but you will need to combine some pieces from all 3 sets of tangrams.<br><br>Step 1:<br>Place one of the small triangles in the center of your paper and trace around it. Label the longest side of the triangle "C" (hypotenuse) and the other two sides "A" and "B".<br><br><br><br>Step 2:<br>On the sides a and b, two small triangles are needed to create squares. On side c, four small triangles are needed to create a square.  The two squares of a and b combined make the perfect square on side c.<br><br><br><br>Step 3:<br>Repeat Steps 1 and 2 using the medium triangle. Can the perfect squares be made by using only the small triangles? How many triangles are used on sides "A" and "B"? (four) How many small triangles would be needed for side "C"? (eight)<br><br><br><br>Step 4:<br>Repeat the activity using the large triangle. Determine how many triangles would be needed for sides "A" and "B", (Two large triangles or five of the smaller pieces.) and for side "C". (All seven tangram pieces.)<br><br> |
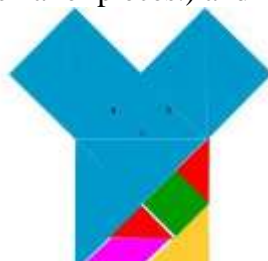
Alternatively, you can prove the Pythagorean Theorem by using the following pieces:
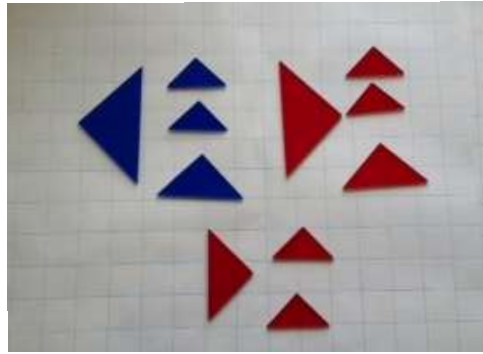


**Step 1: smallest triangle**
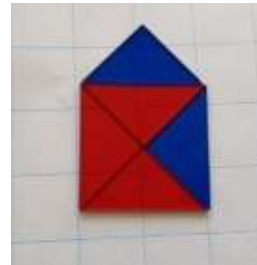


**Step 2: medium triangle**
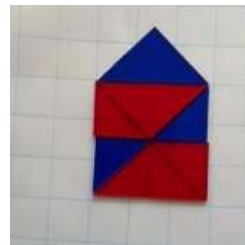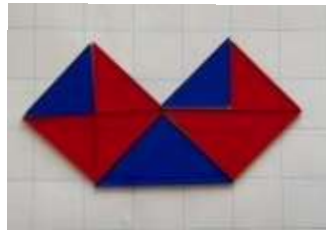


**Step 3: large triangle**



Note: There are many other possibilities, in addition to the two examples given in the lesson plan.

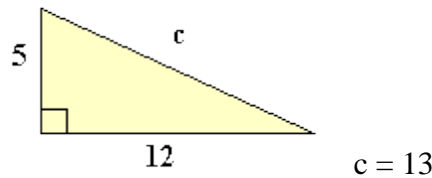| Subtask 3: Application Question | **Lesson:** To see which method was best at explaining the Pythagorean Theorem through an application question http://www.mathsisfun.com/pythagoras.html |
|---|---|
| Time: Time Permitting | 1. **What is the hypotenuse?** |
| Materials & Prep: - Question handout - calculator - paper - pencil and eraser | <br>$c = 13$ |
| | 2. **What is the diagonal distance across this square? Give the exact answer. Give the answer to the hundredths.**<br><br>diagonal $= \sqrt{2} \approx 1.41$ |
| | 3. **What is the missing leg?**<br><br>**b = 12** |
| | 4. **Does this triangle have a right angle?**<br><br>If $a^2 + b^2 = c^2$, then it is a right angle triangle. |
| **Subtask 4:** Debriefing | **Lesson:** |
| Time: 8-10 minutes | **Sharing:**<br>• Groups sharing what they have discovered<br>• What did you discover during your activity?<br>• What new learning or new understanding did you experience?<br>• What was challenging about the task? |

| Materials & Prep: | Research: |
|---|---|
| - Computer with internet access<br>- Projector | - http://www.youtube.com/watch?v=CAkMUdeB06o<br>- "What do students know about geometry?" by Marilyn E. Strutchens & Glendon W. Blume<br>    - Pythagorean theorem is probably the most universally addressed theorem in geometry<br>    - Yet, students cannot apply it and probably do not understand it well.<br>    - Only 30% of $8^{th}$ grade students could find the length of the hypotenuse given lengths of the legs, despite all lengths being relatively small integers<br>    - 60% of the students chose distractors<br>    - Only 52% of $12^{th}$ grade students could find the length of the hypotenuse given lengths of the legs<br>    - And only 15% of $12^{th}$ grade students could sketch a right triangle based on given information about the lengths of the legs and the hypotenuse.<br>- "Skinning the Pythagorean Cat: A Study of Strategy Preferences of Secondary Math Teachers" by Clara A. Maxcy http://eric.ed.gov.myaccess.library.utoronto.ca/?id=ED532731<br>    - This study looked at preferred teaching strategies for teaching high achieving vs. low achieving high school students.<br>    - Teachers preferred questioning strategies for teaching Pythagorean Theorem to high achieving students<br>    - Teachers preferred using manipulatives for low achieving students<br>    - More experienced teachers also used more brain-compatible strategies when teaching high achieving students<br>    - No single strategy has been proven effective for all classroom situations<br>- Mathematical Investigations—Powerful Learning Situations by Suzanne H. Chapin<br>    - Mathematical investigations enable students to learn the formula of the Pythagorean Theorem.<br>    - Writing about, and discussing the mathematics inherent in the solution of investigative problems broadens and deepens students' understanding.<br>    - Questioning procedures, solutions, and one another's reasoning helps students develop investigative habits of mind.<br>    - When students study a topic in detail, they not only learn a great deal of mathematics, they also learn the power of careful reasoning, thoughtful discourse, and perseverance.<br>- "Pythagoras Meets Van Hiele" by Alfinio Flores<br>    - This article gives examples of Pythagorean explorations at each level of the Van Hiele, showing that your teaching of the theorem can be adapted to the level of the students.<br>    - This research supports our lesson by explaining how Pythagorean Theorem can be introduced to students before grade 8 (as suggested by the Ontario Math curriculum document). |

**General Reflection:**
- Overall the lesson was a success in meeting the basic objectives.
- The activities were engaging and a good level of challenge for most groups.
- The activities connected well with each other and gave the groups an idea of how to sequence the explorations in a classroom.
- In a typical classroom each task could be explored over several periods in small groups.
- We didn't have enough time during our lesson to address the application questions, or the research in detail.
- The youtube video helped to consolidate understanding gained from the exploration period.
- Organization was integral to the implementation of this lesson and each task was explored by the educators beforehand to anticipate possible questions and difficulties that may be encountered.
- Having a deeper knowledge of the activities allowed the teachers to scaffold the exploration and sharing to achieve greater understanding.
- During group sharing time we travelled from table to table, giving everyone a chance to observe the materials that were used and the exploration that was done.
- The value of the opening problem would have been more apparent if we had time to revisit it at the end as originally planned.
- It was a good practice to begin by activating prior knowledge about right triangles.
- If more time allowed, it would have been valuable to give students an opportunity to engage in think-pair-share at their table groups before sharing with the whole group during the initial brainstorm.

**Group 1: Geometric Proof**

**Results**
- The smaller triangles were easier to solve because the tiles fit more easily into the constructed square.
- The group was able to manipulate the triangles to create the square representing $c^2$ but did not intuitively know to manipulate them again to find $a^2$ and $b^2$.
- Instead of using tiles, they attempted to use algebra to prove the theorem, without being prompted to do so.
- Guidance was needed to see how $A^2$ and $B^2$ were created by the triangles.

Task #1:
Manipulate the 4 triangles in the square provided to show that $a^2 + b^2 = c^2$
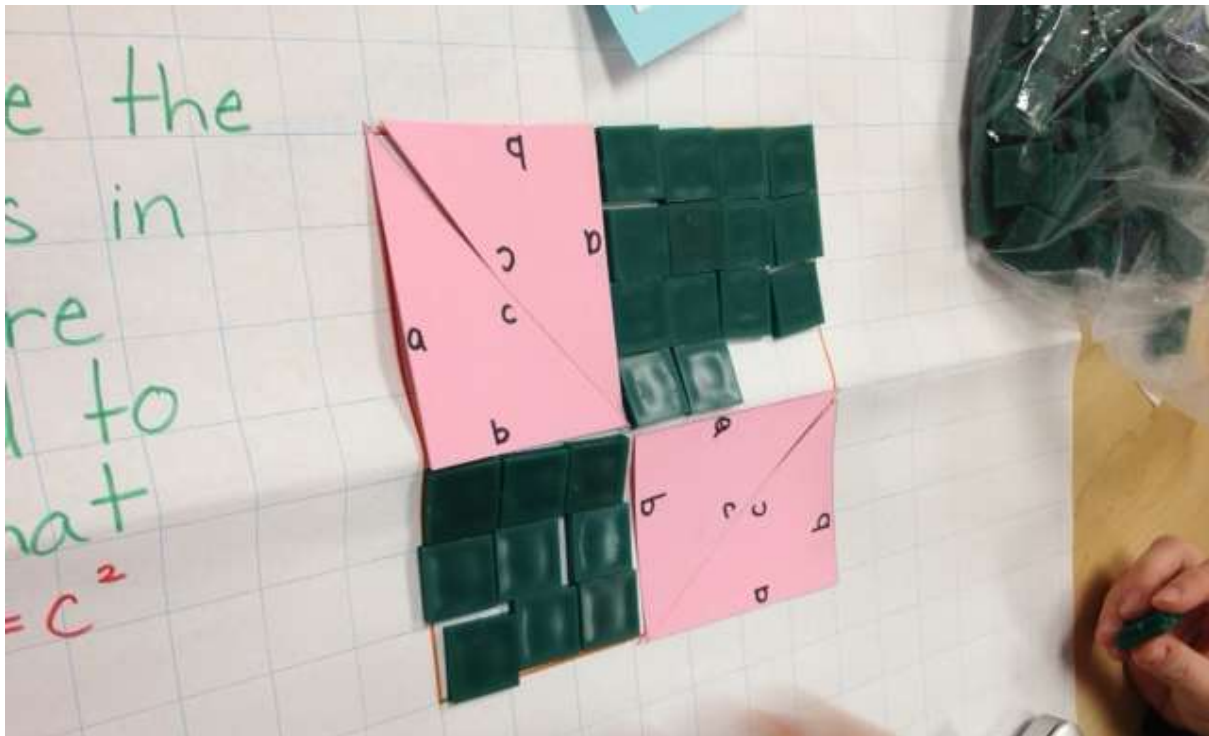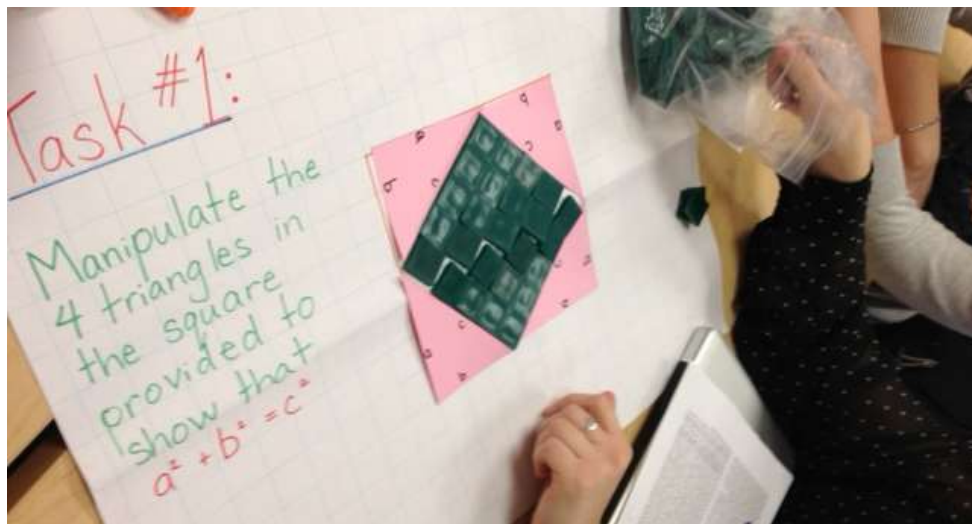


$$(A+B) \times (A+B) = C^2 + 4\left(\tfrac{1}{2}AB\right)$$

$$A^2 + AB + AB + B^2 = C^2 + 2AB$$

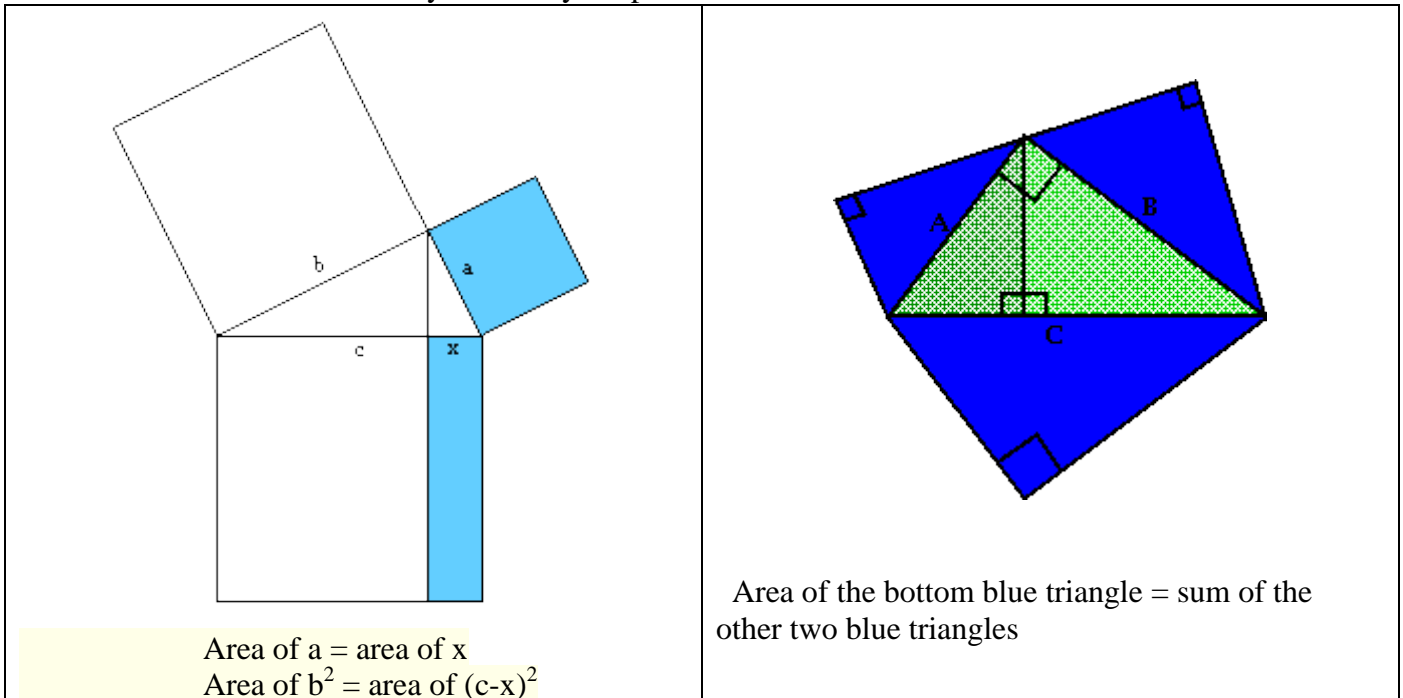$$A^2 + 2AB + B^2 = C^2 + 2AB$$

$$A^2 + B^2 = C^2 + 2AB - 2AB$$

$$A^2 + B^2 = C^2$$

**Reflection**
- In general this was a good activity to deepen the understanding of the theorem.
- However, a fair amount of guidance was required because the students did not have experience working with these manipulatives to explore the theorem.
- Tiles were provided to help with the exploration and it was helpful to give guidance in how to use the tiles when the group got stuck.

**Recommendations/Revisions/Extensions**
- This activity might be more successful in a classroom if the students had completed the tile activity first.
- Extension 1: Are there any other ways to prove it?



Area of a = area of x
Area of $b^2$ = area of $(c-x)^2$



Area of the bottom blue triangle = sum of the other two blue triangles

- Extension 2: If students successfully prove the theorem algebraically, challenge them by asking them to use the Cosine Law to find the Pythagorean Theorem.



Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cdot \cos(A)$$
$$b^2 = a^2 + c^2 - 2ac \cdot \cos(B)$$
$$c^2 = a^2 + b^2 - 2ab \cdot \cos(c)$$

© www.mathwarehouse.com

**Group 2: Proof through Simplification**

**Results**

- The group successfully used the tiles to better understand and represent the relationship between $a^2$, $b^2$ and $c^2$.
- Through trying to create their own right triangle, they discovered that the tiles would no longer fit unless the three sides were whole numbers.
- They resorted to using the internet to discover the primitive Pythagorean triples but used this information to discover the pattern of using multiples of these triples to make more right triangles (e.g. 6, 8, 10 can be doubled to produce 12, 16, 20).
- The group did not think to superimpose the $a^2$ and $b^2$ tiles onto the $c^2$ tiles.
- The group members attempted the extension question without making a triangle cutout of their own; instead, they drew a triangle on a piece of paper and tried to manipulate the lengths of the legs and hypotenuse mentally with the aid of a calculator.
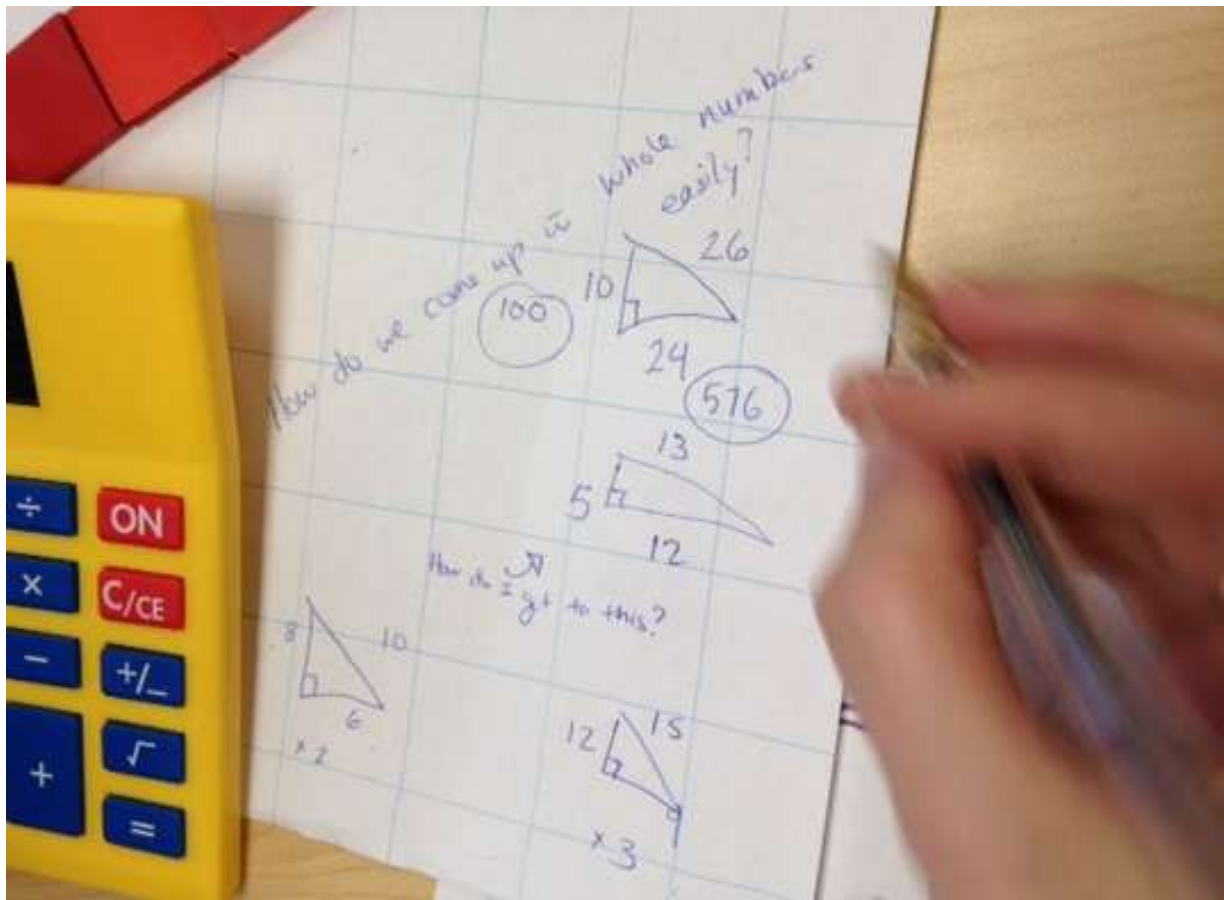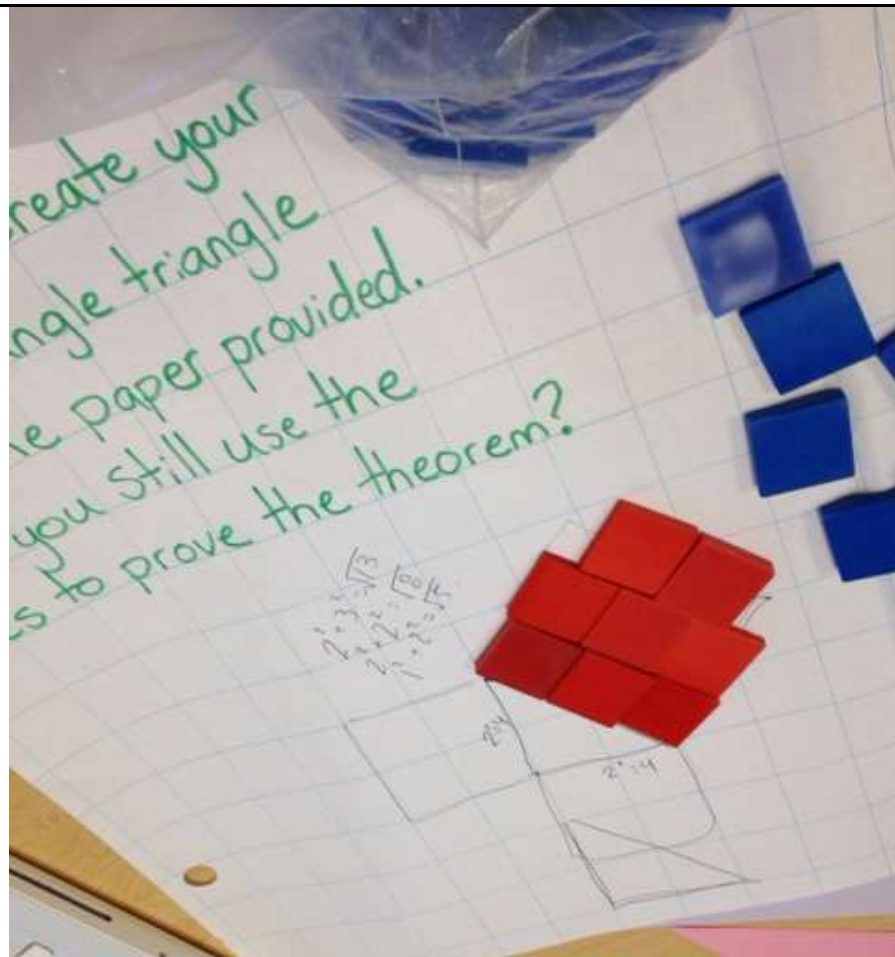
**Reflection**
- This was the simplest proof of the Pythagorean Theorem and when other groups saw this proof they were able to draw connections between it and the task they had been working on.
- Some groups had mentioned they wish they had been able to do this activity first, before trying their more challenging task.
- One of the group members noted the usefulness of knowing the Primitive Pythagorean Triples as a teacher as it makes it easier to generate example right triangles for lessons.

**Recommendations/Revisions/Extensions**
- This activity could lead into the slide task (Group 3: Discovering Special Right Triangles) as an application activity.
- Instead of plastic tiles the group could be given 1 inch graph paper which they could cut to make tiles that will combine to form $a^2$, $b^2$, and $c^2$. If the group encounters non-Pythagorean triples, they can cut the tiles accordingly.

**Group 3: Discovering Special Right Triangles**

**Results**
- Instead of using the strategy we gave them (leaning the slide against a vertical surface) the group graphed the different leg lengths.
- After discovering one of the special right triangles, the group used mental math and prior knowledge of the theorem to solve the other leg lengths.
- The group was less interested in the activity and did not attempt the extension questions as a result.
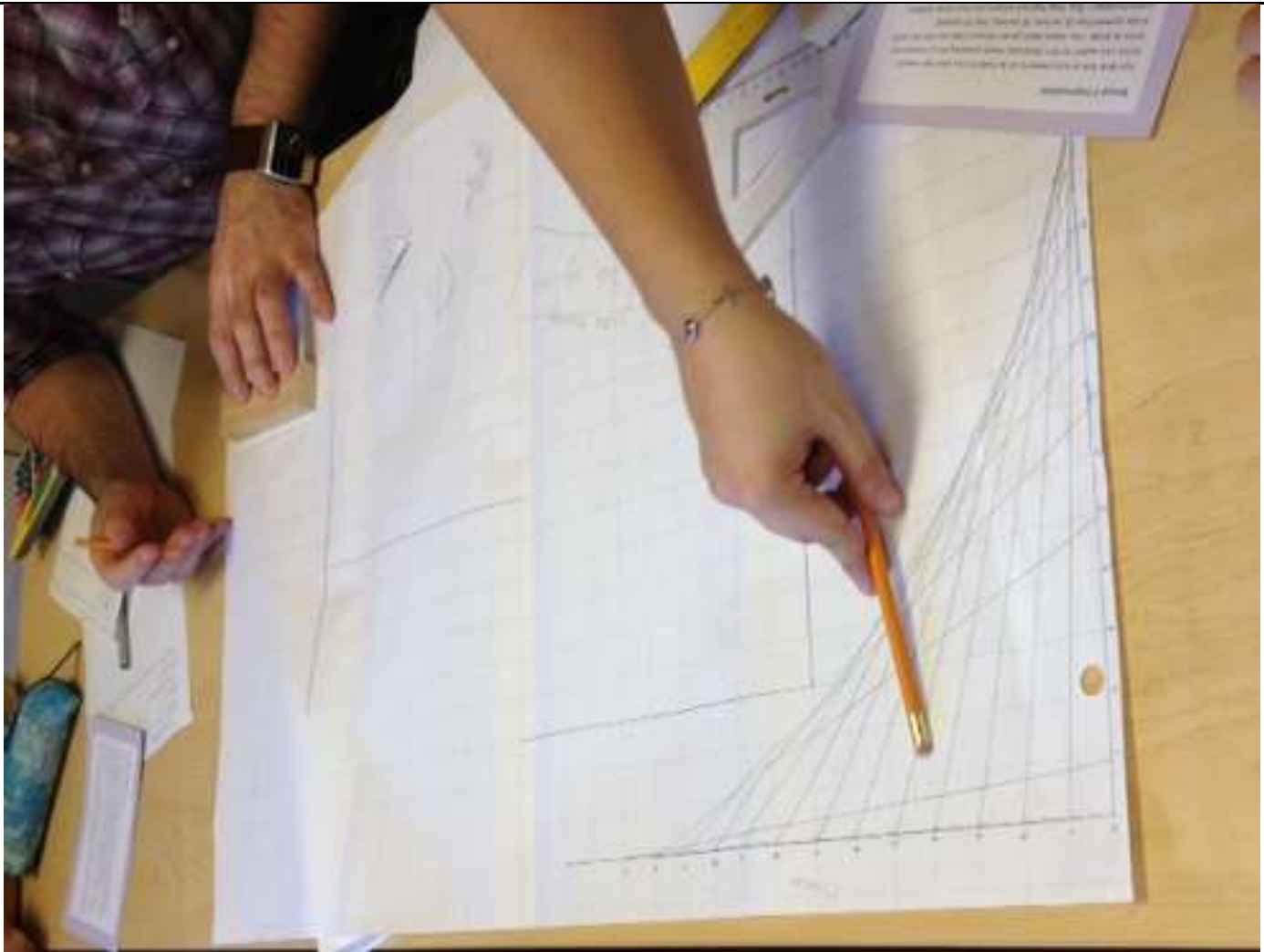
**Reflection**
- The instructions to the group were not made clear enough in that the group assumed the hypotenuse and one leg needed to be whole numbers while the second leg did not need to be a whole number.
- One group members mentioned that they wish this had been given as an extension activity rather than as a task because it did not help to deepen the understanding of the concept behind Pythagorean Theorem.

**Recommendations/Revisions/Extensions**
- Use this activity as an extension to the Group 2: Proof through Simplification task.
- This activity could be used as a lead in to introduce radicals for solving non-Pythagorean triple cases.

| **Group 4: Tangram Proof** |
|---|
| **Results** |

- The group really enjoyed using the tangrams in a different way and experienced an "Aha!" moment when they found the connection between the tangrams and the theorem.
- Group members had to be guided into using different shapes to represent the same area.
- The group was able to superimpose $a^2$ and $b^2$ onto $c^2$.
- The group not only used the tangrams to prove the theorem but also measured the sum of the areas of $a^2$ and $b^2$ to confirm that it is equal to the area of $c^2$.
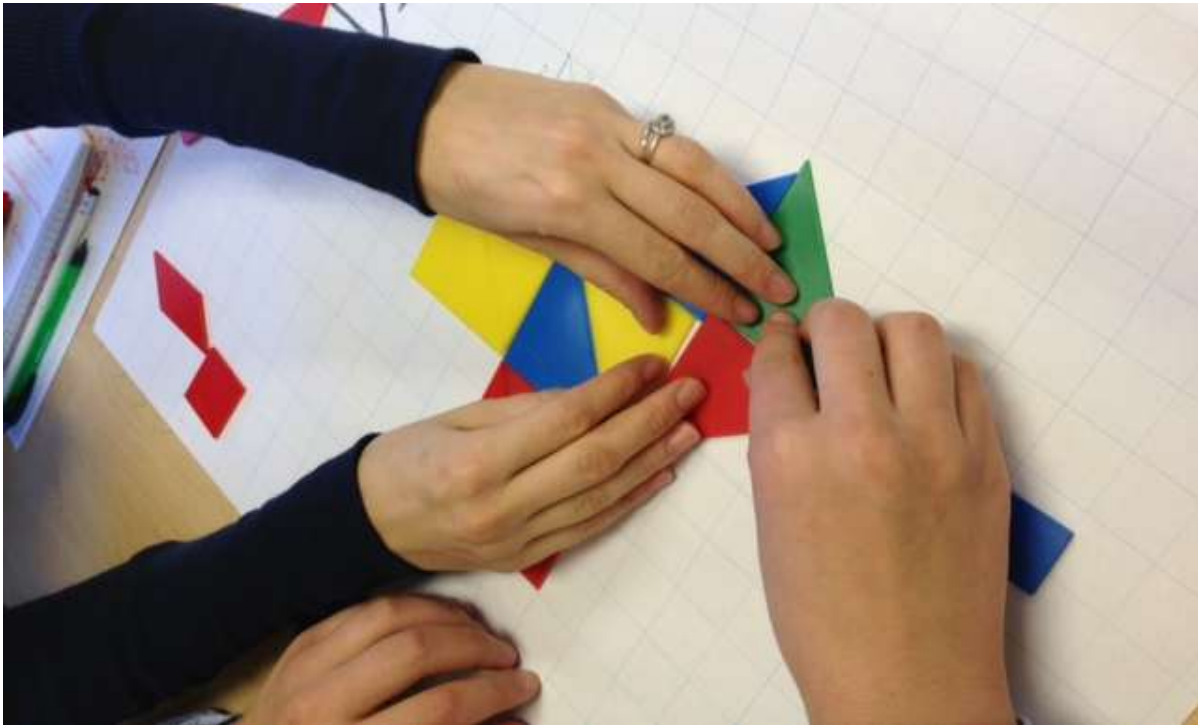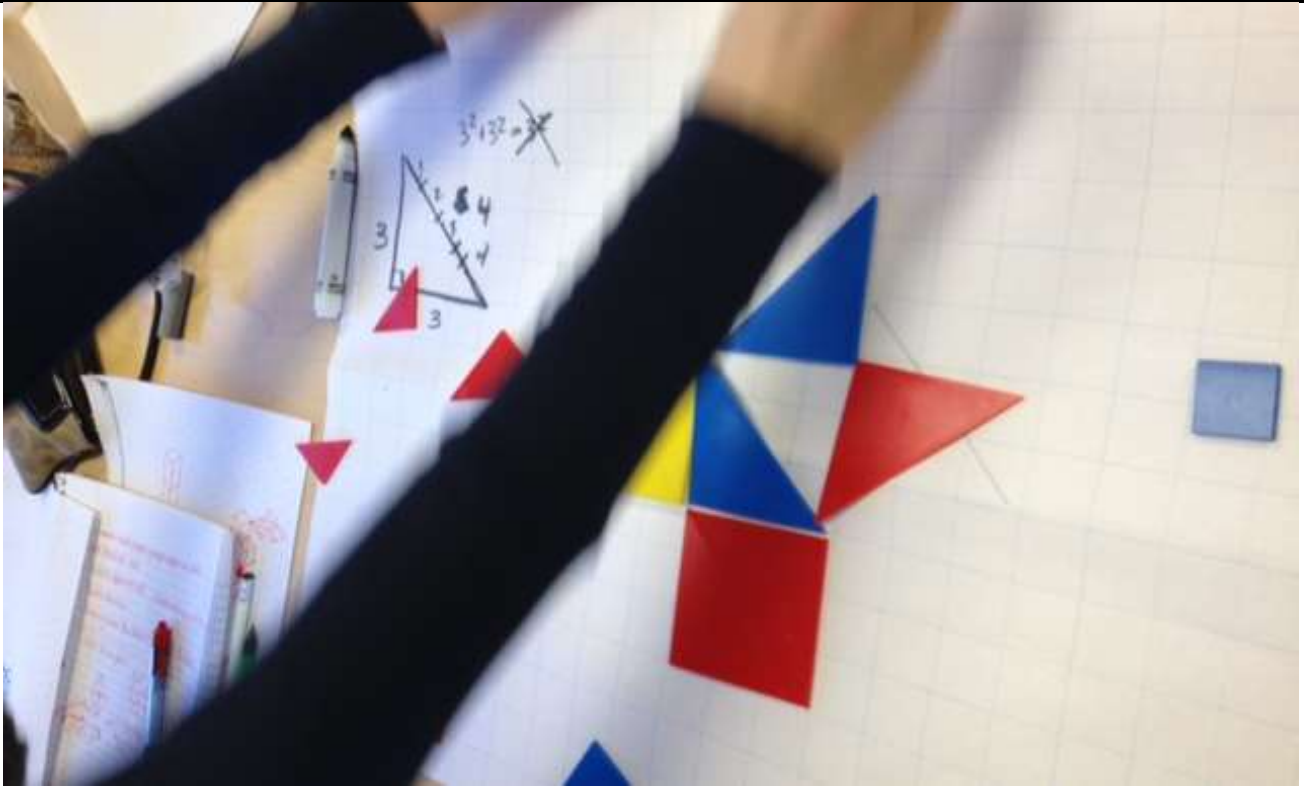
**Reflection**
- One group member mentioned that she had a better understanding of the theorem after this activity but still wished she had been able to work with the tiles first.
- We provided the group with graph paper which enabled them to measure the legs and hypotenuse to confirm that their theory was correct.

**Recommendations/Revisions/Extensions**
- An extension would involve having groups try to use different pieces of the tangrams to prove the theorem.
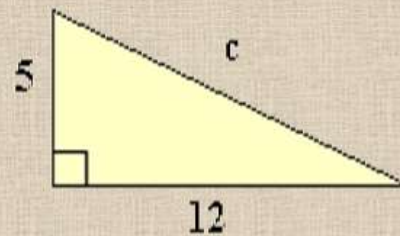
What do you know about right angle triangles?



leg
hypotenuse
leg
a RIGHT TRIANGLE

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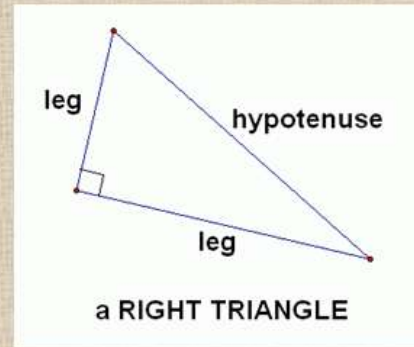
Exploring
Pythagorean Theorem

$$a^2 + b^2 = c^2$$
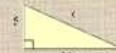
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Try the following question

1) What is the hypotenuse?



5

c

12

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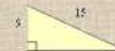Exploration Time

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Try these...

1) What is the hypotenuse?



2) What is the diagonal distance across this square? Give the exact answer. Give the answer to the hundredths.



3) What is the missing leg?



4) Do these triangle have a right angle?

# Sharing Time

# Another way to look at Pythagorean Theorem

http://www.youtube.com/watch?v=CAkMUdeB06o

# Research 1

## What do students know about geometry?

by Marilyn E. Strutchens & Glendon W. Blume

- Pythagorean theorem is probably the most universally addressed theorem in geometry
- Yet, students cannot apply it and probably do not understand it well.

- Only 30% of 8th grade students could find the length of the hypotenuse given lengths of the legs, despite all lengths being relatively small integers
- 60% of the students chose distractors

- Only 52% of 12th grade students could find the length of the hypotenuse given lengths of the legs
- And only 15% of 12th grade students could sketch a right triangle based on given information about the lengths of the legs and the hypotenuse.

# Research 2

## "Skinning the Pythagorean Cat: A Study of Strategy Preferences of Secondary Math Teachers"

by Clara A. Maxcy

- This study looked at preferred teaching strategies for teaching high achieving vs. low achieving high school students.

- Teachers preferred questioning strategies for teaching Pythagorean Theorem to high achieving students
- Teachers preferred using manipulatives for low achieving students

- More experienced teachers also used more brain-compatible strategies when teaching high achieving students
- No single strategy has been proven effective for all classroom situations

# Research 3

## Mathematical Investigations— Powerful Learning Situations

by Suzanne H. Chapin

- Mathematical investigations enable students to learn the formula of the Pythagorean Theorem

- Writing about, and discussing the mathematics inherent in the solution of investigative problms broadens and deepens students' understanding.

- Questing procedures, solutions, and one another's reasoning helps students develop investigative habits of mind

- When students study a topic in detail, they not only learn a great deal of mathematics, they also learn the power of careful reasoning, thoughtful discourse, and perseverance.

# Research 4

## Pythagoras Meets Van Hiele

by Alfinio Flores

- This article gives examples of Pythagorean explorations at each level of the Van Hiele, showing that your teaching of the theorem can be adapted to the level of the students