NAKHODA

Journal of Algorithms and Computation

journal homepage: http://jac.ut.ac.ir

Inverse eigenvalue problem for matrices whose graph is a banana tree

M. Babaei Zarch*[1], S. A. Shahzadeh Fazeli†[2] and S. M. Karbassi‡[3]

[1,2,3]Department of Computer Science, Yazd University, Yazd, Iran.
[2]Parallel Processing Laboratory, Yazd University, Yazd, Iran.

ABSTRACT

In this paper, we consider an inverse eigenvalue problem (IEP) for constructing a special kind of acyclic matrices. The problem involves the reconstruction of matrices whose graph is a banana tree. This is performed by using the minimal and maximal eigenvalues of all leading principal submatrices of the required matrix. The necessary and sufficient conditions for the solvability of the problem is derived. An algorithm to construct the solution is provided.

ARTICLE INFO

Article history:
Received 11, April 2018
Received in revised form 8, August 2018
Accepted 26 October 2018
Available online 30, December 2018

Keyword: Inverse eigenvalue problem; banana tree; leading principal minors; eigenvalue; graph of a matrix.

AMS subject Classification: 05C78.

1 Introduction

An inverse eigenvalue problem (IEP) concerns the reconstruction of a matrix from prescribed spectral data. In [3] detailed characterization of inverse eigenvalue problems is mentioned. Special types of inverse eigenvalue problems have attracted attention of many authors. Inverse eigenvalue problems for graphs have been studied in [4,7,8,10,12,13].

*maryam.babaei@stu.yazd.ac.ir
†Corresponding author: S. A. Shahzadeh Fazeli. Email: fazeli@yazd.ac.ir
‡smkarbassi@yazd.ac.ir

The inverse eigenvalue problem of a graph is to determine the possible spectra among real symmetric matrices whose pattern of nonzero off- diagonal entries is described by a graph. In the last fifteen years a number of papers on this problem have appeared. In this paper, we investigate an IEP, namely IEPB(c,s) (inverse eigenvalue problem for matrices whose graph is a banana tree). Similar problems were studied in [9,10,14]. For solving the problem, the recurrence relations among leading principal minors is used. Some applications of the acyclic matrix discussed in this paper are in chemistry, energy and graph theory [1,5].
The paper is organized as follows. In Section 2, we give a brief outline of some preliminary concepts and clarify the notations used in the paper. In Section 3, we discuss the analysis of IEPB(c,s) and present an algorithm. In Section 4, we report numerical examples to illustrate the solutions of IEPB(c,s). In Section 5 conclusion is presented.

2 Preliminaries

Let G be a simple undirected graph (without loops and multiple edges) on n vertices. A real symmetric matrix $A = (a_{ij})$ is said to have a graph G provided $a_{ij} \neq 0$ if and only if vertices i and j are adjacent in G.
Given an $n \times n$ symmetric matrix A, the graph of A, denoted by $G(A)$, has vertex set $V(G) = \{1, 2, 3, \ldots, n\}$ and edge set $E = \{ij : i \neq j, a_{ij} \neq 0\}$. For graph G with n vertices, we denote by $S(G)$ the set of all real symmetric matrices whose graph is G. A matrix whose graph is a tree is called an acyclic matrix. Some simple examples of acyclic matrices are the matrices whose graphs are paths, m-centipedes, brooms or banana tree.
Definition 2.1. An (c, s)-banana tree, as defined by Chen et al.(1997), is a graph obtained by connecting one leaf of each of c copies of an $s-$star graph with a single root vertex that is distinct from all the stars.
Properties of banana trees have been studied in [2]. The vertices of an (c, s)-banana tree with $c \geq 1,\ s \geq 3$, labeled as follws:
The root vertex is labeled by 1, the vertices of distance 1 from the root vertex as the intermediate vertices is labeled by $(i-1)s+2$, the center of every (S_s) is labeled by $(i-1)s+3$ and leaves of the center is labeled by $j = (i-1)s+4, \ldots, is+1, i = 1, 2, \ldots, c$. (Figure1).
For example matrix of an $(3, 4)$-banana tree is of the following form:

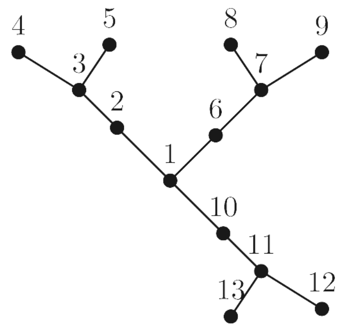

Figure 1: An $(3, 4)$-banana tree.

$$A_{13} = \begin{pmatrix}
a_1 & b_{1,2} & & & & b_{1,6} & & & & b_{1,10} & & & \\
b_{1,2} & a_2 & b_{2,3} & & & & & & & & & & \\
 & b_{2,3} & a_3 & b_{3,4} & b_{3,5} & & & & & & & & \\
 & & b_{3,4} & a_4 & & & & & & & & & \\
 & & b_{3,5} & & a_5 & & & & & & & & \\
b_{1,6} & & & & & a_6 & b_{6,7} & & & & & & \\
 & & & & & b_{6,7} & a_7 & b_{7,8} & b_{7,9} & & & & \\
 & & & & & & b_{7,8} & a_8 & & & & & \\
 & & & & & & b_{7,9} & & a_9 & & & & \\
b_{1,10} & & & & & & & & & a_{10} & b_{10,11} & & \\
 & & & & & & & & & b_{10,11} & a_{11} & b_{11,12} & b_{11,13} \\
 & & & & & & & & & & b_{11,12} & a_{12} & \\
 & & & & & & & & & & b_{11,13} & & a_{13}
\end{pmatrix}$$

where all the b's are positive.

The following known results will be necessary for solving the problem in this paper.

Lemma 2.2. [11] Let $P(\lambda)$ be a monic polynomial of degree n with all real zeroes. If λ_1 and λ_n are, respectively, the minimal and the maximal zero of $P(\lambda)$, then

i. If $x < \lambda_1$, we have that $(-1)^n P(x) > 0$.

ii. If $x > \lambda_n$, we have that $P(x) > 0$.

Lemma 2.3. [6] (Cauchy's interlacing theorem) Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of an $n \times n$ real symmetric matrix A and $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-1}$ be the eigenvalues of an $(n-1) \times (n-1)$ principal submatrix B of A, then

$$\lambda_1 \leq \mu_1 \leq \ldots \leq \mu_{n-1} \leq \lambda_n.$$

3 Problem statement and the solution

3.1 Problem statement

Given $2n-1$ real numbers $\lambda_1^{(j)}$, $j=1,2,\ldots,n$ and $\lambda_j^{(j)}$, $j=2,\ldots,n$, find a $n\times n$ matrix $A_n\in S(B(c,s))$ such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of A_j, the leading principal submatrix of A, respectively. This is referred to **IEPB(c,s)** problem. In the next subsection we discuss the solution of **IEPB(c,s)**.

3.2 The solution of IEPB(c,s)

In the following, we investigate the relation between successive leading principal minors of $\lambda I_n - A_n$. **Lemma 3.1** The sequence $\{P_j(\lambda)=\det(\lambda I_j - A_j)\}_{j=1}^n$ of characteristic polynomials of A_j satisfies the following recurrence relations:

i. $P_1(\lambda)=(\lambda-a_1)$

ii. $P_j(\lambda)=(\lambda-a_j)P_{j-1}(\lambda)-b_{1,j}^2 det(B_j^\lambda)\quad j=(i-1)s+2,\ i=1,2,\ldots,c$

iii. $P_j(\lambda)=(\lambda-a_j)P_{j-1}(\lambda)-b_{(j-1),j}^2 P_{j-2}(\lambda)\quad j=(i-1)s+3,\ i=1,2,\ldots,c$

iv. $P_j(\lambda)=(\lambda-a_j)P_{j-1}(\lambda)-b_{(i-1)s+3,j}^2\ P_{(i-1)s+2}(\lambda)\prod_{k=(i-1)s+4}^{j-1}(\lambda-a_k)$
$j=(i-1)s+4,\ldots,is+1,\ i=1,2,\ldots,c,$

with the convention that B_j^λ is the submarix rows and columns $2,3,\ldots,n-1$ of the matrix $\lambda I_n - A_n$, $p_0(\lambda)=1$ and $\prod_{k=(i-1)s+4}^{j-1}(\lambda-a_k)=1$ when $j=(i-1)s+4$.

Proof. It is easy to verify by expanding the determinant. □

Since $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are eigenvalues of A_j, we have

$$\begin{cases} P_j(\lambda_1^{(j)})=0 \\ P_j(\lambda_j^{(j)})=0. \end{cases} \tag{1}$$

Thus, solving the IEPB(c,s) is equivalent to solving the above system of equations for $j=1,2,\ldots,n$. Observe that from the Cauchy's interlacing property, the minimal and the maximal eigenvalue, $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively, of each leading principal submatrix $A_j,\ j=1,2,\ldots,n$, of the matrix $A\in S(B(c,s))$ satisfy the relations:

$$\lambda_1^{(n)}\le\lambda_1^{(n-1)}\le\ldots\le\lambda_1^{(2)}\le\lambda_1^{(1)}\le\lambda_2^{(2)}\le\ldots\le\lambda_n^{(n)} \tag{2}$$

and

$$\lambda_1^{(j)}\le a_i\le\lambda_j^{(j)}\quad i=1,2,\ldots,j,\ \ j=1,2,\ldots,n. \tag{3}$$

Lemma 3.2. Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ be the minimal and maximal eigenvalues of A_j for $j=(i-1)s+2,\ i=1,2,\ldots,c$, then the system of equations 1 has unique solution if and only if $\lambda_1^{(j)}<\lambda_1^{(j-1)}$ and $\lambda_{j-1}^{(j-1)}<\lambda_j^{(j)}$.

Proof. Let $P_{j-1}(\lambda_1^{(j)})$ and $P_{j-1}(\lambda_j^{(j)})$ be nonzero for $j = (i-1)s+2$, $i = 1,2,\dots,c$, replacing relation (ii) from Lemma 3.2 in equations (1), we obtain

$$\begin{cases} P_j(\lambda_1^{(j)}) = a_j P_{j-1}(\lambda_1^{(j)}) + b_{1,j}^2 det(B_j^{\lambda_1^{(j)}}) - \lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) = 0 \\ P_j(\lambda_j^{(j)}) = a_j P_{j-1}(\lambda_j^{(j)}) + b_{1,j}^2 det(B_j^{\lambda_j^{(j)}}) - \lambda_j^{(j)} P_{j-1}(\lambda_j^{(j)}) = 0. \end{cases} \tag{4}$$

Which can be regarded as a linear system of equations in a_j and $b_{1,j}^2$. Let $D_{1,j}$ denote the determinant of the system of equations (4), then

$$D_{1,j} = P_{j-1}(\lambda_1^{(j)}) det(B_j^{\lambda_j^{(j)}}) - P_{j-1}(\lambda_j^{(j)}) det(B_j^{\lambda_1^{(j)}}). \tag{5}$$

Since $b_{1,j}$'s are nonzero, from the relation (ii) of Lemma 3.2 we obtain the following:

$$det(B_j^{\lambda}) = \frac{1}{b_{1,j}^2}((\lambda - a_j)P_{j-1}(\lambda) - P_j(\lambda)).$$

By replacing $det(B_j^{\lambda_1^{(j)}})$ and $det(B_j^{\lambda_j^{(j)}})$ in $D_{1,j}$ and simplifying, we obtain

$$D_{1,j} = \frac{1}{b_{1,j}^2}[P_{j-1}(\lambda_j^{(j)}) P_{j-1}(\lambda_1^{(j)})(\lambda_j^{(j)} - \lambda_1^{(j)}) - P_{j-1}(\lambda_1^{(j)}) P_j(\lambda_j^{(j)}) + P_{j-1}(\lambda_j^{(j)}) P_j(\lambda_1^{(j)})].$$

Since $P_j(\lambda_1^{(j)})$ and $P_j(\lambda_j^{(j)})$ are equal to zero and $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are not roots of $P_{j-1}(\lambda)$ then

$$D_{1,j} = \frac{1}{b_{1,j}^2}[P_{j-1}(\lambda_j^{(j)}) P_{j-1}(\lambda_1^{(j)})(\lambda_j^{(j)} - \lambda_1^{(j)})] \neq 0.$$

Conversely, let $D_{1,j} \neq 0$, if $\lambda_1^{(j)} = \lambda_1^{(j-1)}$ then we have

$$det(B_j^{\lambda_1^{(j)}}) = 0,$$

and this implies that $D_{1,j} = 0$. But this contradicts our hypothesis that $D_{1,j} \neq 0$. Hence $\lambda_1^{(j)} < \lambda_1^{(j-1)}$. Similarly, $\lambda_{j-1}^{(j-1)} < \lambda_j^{(j)}$. □

Lemma 3.3. Let A be a matrix of an (c,s)-banana tree and $\lambda_1^{(j)}$, $\lambda_j^{(j)}$ be the minimal and the maximal eigenvalue of the leading principal submatrix A_j of A, $j = 2,\dots,n$. If $\lambda_1^{((i-1)s+2)} < \lambda_1^{((i-1)s+1)}$ and $\lambda_{(i-1)s+1}^{((i-1)s+1)} < \lambda_{(i-1)s+2}^{((i-1)s+2)}$, then we have

$$\lambda_1^{(j)} < \lambda_1^{(j-1)}\ ,\ \lambda_{j-1}^{(j-1)} < \lambda_j^{(j)} \tag{6}$$

and

$$\lambda_1^{(j)} < a_k < \lambda_j^{(j)} \quad k = 1,2,\dots,j, \tag{7}$$

for $j = (i-1)s+3,\dots,is+1$, $i = 1,2,\dots,c$.

Proof. For $i = 1, 2, \dots, c$, if $j = (i-1)s+3$, by Lemma 3.2 we have

$$P_{(i-1)s+3}(\lambda) = (\lambda - a_{(i-1)s+3})P_{(i-1)s+2}(\lambda) - b^2_{(i-1)s+2,(i-1)s+3}P_{(i-1)s+1}(\lambda). \quad (8)$$

If $\lambda_1^{((i-1)s+3)} = \lambda_1^{((i-1)s+2)}$, by equation (8) we have

$$P_{(i-1)s+3}(\lambda_1^{((i-1)s+3)}) = -b^2_{(i-1)s+2,(i-1)s+3}P_{(i-1)s+1}(\lambda_1^{((i-1)s+2)}).$$

Since $\lambda_1^{((i-1)s+2)} < \lambda_1^{((i-1)s+1)}$ thus $P_{(i-1)s+1}(\lambda_1^{((i-1)s+2)}) \neq 0$ hence $P_{(i-1)s+3}(\lambda_1^{((i-1)s+3)}) \neq 0$, but this is a contradiction, then we obtain $\lambda_1^{((i-1)s+3)} < \lambda_1^{((i-1)s+2)}$. Similary, we have $\lambda_{(i-1)s+2}^{((i-1)s+2)} < \lambda_{(i-1)s+3}^{((i-1)s+3)}$.
Now we assume $\lambda_1^{((i-1)s+3)} = a_{(i-1)s+3}$. Again, by equation (8) we have

$$P_{(i-1)s+3}(\lambda_1^{((i-1)s+3)}) = -b^2_{(i-1)s+2,(i-1)s+3}P_{(i-1)s+1}(\lambda_1^{((i-1)s+3)}).$$

Since $\lambda_1^{((i-1)s+3)} < \lambda_1^{((i-1)s+2)} < \lambda_1^{((i-1)s+1)}$, we have $P_{(i-1)s+1}(\lambda_1^{((i-1)s+3)}) \neq 0$ hence $P_{(i-1)s+3}(\lambda_1^{((i-1)s+3)}) \neq 0$ but this is a contradiction, then we obtain $\lambda_1^{((i-1)s+3)} < a_{(i-1)s+3}$. Similary, we have $a_{(i-1)s+3} < \lambda_{(i-1)s+3}^{((i-1)s+3)}$, hence $\lambda_1^{((i-1)s+3)} < a_{(i-1)s+3} < \lambda_{(i-1)s+3}^{((i-1)s+3)}$.
If $j = (i-1)s+4$, by Lemma 3.2 we have

$$P_{(i-1)s+4}(\lambda) = (\lambda - a_{(i-1)s+4})P_{(i-1)s+3}(\lambda) - b^2_{(i-1)s+3,\,(i-1)s+4}\, P_{(i-1)s+2}(\lambda). \quad (9)$$

If $\lambda_1^{((i-1)s+4)} = \lambda_1^{((i-1)s+3)}$, by equation (9) we have

$$P_{(i-1)s+4}(\lambda_1^{((i-1)s+4)}) = -b^2_{(i-1)s+3,\,(i-1)s+4}\, P_{(i-1)s+2}(\lambda_1^{((i-1)s+3)}).$$

Since $\lambda_1^{((i-1)s+3)} < \lambda_1^{((i-1)s+2)}$ it means that $\lambda_1^{((i-1)s+3)}$ is not a root of $P_{(i-1)s+2}(\lambda)$, then we obtain $P_{(i-1)s+4}(\lambda_1^{((i-1)s+4)}) \neq 0$, this is a contradiction and $\lambda_1^{((i-1)s+4)} < \lambda_1^{((i-1)s+3)}$. Similary, we have $\lambda_{(i-1)s+3}^{((i-1)s+3)} < \lambda_{(i-1)s+4}^{((i-1)s+4)}$.
If $\lambda_1^{((i-1)s+4)} = a_{(i-1)s+4}$ or $\lambda_{(i-1)s+4}^{((i-1)s+4)} = a_{(i-1)s+4}$ then by equation (9) we have

$$P_{(i-1)s+4}(a_{(i-1)s+4}) = -b^2_{(i-1)s+3,\,(i-1)s+4}\, P_{(i-1)s+2}(a_{(i-1)s+4}).$$

Since $\lambda_1^{((i-1)s+4)} < \lambda_1^{((i-1)s+3)} < \lambda_1^{((i-1)s+2)}$ then $P_{(i-1)s+4}(a_{(i-1)s+4}) \neq 0$, which is a contradiction. From (3) we have $\lambda_1^{((i-1)s+4)} < a_{(i-1)s+4} < \lambda_{(i-1)s+4}^{((i-1)s+4)}$.
Assume that (6), (7) hold for $j = (i-1)s+5, \dots, is$ now if $j = is+1$ by Lemma 3.2 we have

$$P_{is+1}(\lambda) = (\lambda - a_{is+1})P_{is}(\lambda) - b^2_{(i-1)s+3,\,is+1}\, P_{(i-1)s+2}(\lambda) \prod_{k=(i-1)s+4}^{is} (\lambda - a_k). \quad (10)$$

If $\lambda_1^{(is+1)} = \lambda_1^{(is)}$, by equation (10) we have

$$P_{is+1}(\lambda_1^{(is+1)}) = -b^2_{(i-1)s+3,\ is+1}\ P_{(i-1)s+2}(\lambda_1^{(is)}) \prod_{k=(i-1)s+4}^{is} (\lambda_1^{(is)} - a_k).$$

Because $P_{(i-1)s+2}(\lambda_1^{(is)}) \neq 0$ and $\lambda_1^{(is)} < a_k < \lambda_{is}^{(is)}, k = (i-1)s+4, \ldots, is$, then $\prod_{k=(i-1)s+4}^{is}(\lambda_1^{(is)} - a_k) \neq 0$ and we obtain $P_{is+1}(\lambda_1^{is+1}) \neq 0$. This is a contradiction and $\lambda_1^{(is+1)} < \lambda_1^{(is)}$. Similary, we have $\lambda_{is}^{(is)} < \lambda_{is+1}^{(is+1)}$.

If $\lambda_1^{(is+1)} = a_{is+1}$ or $\lambda_{is+1}^{(is+1)} = a_{is+1}$ then by equation (10) we have

$$P_{is+1}(a_{is+1}) = -b^2_{(i-1)s+3,\ is+1}\ P_{(i-1)s+2}(a_{is+1}) \prod_{k=(i-1)s+4}^{is} (a_{is+1} - a_k).$$

From the above verified results, we know

$$\lambda_1^{(is+1)} < \lambda_1^{(is)} < \ldots < \lambda_1^{((i-1)s+2)} < \ldots < \lambda_{(i-1)s+2}^{((i-1)s+2)} < \ldots < \lambda_{is}^{(is)} < \lambda_{is+1}^{(is+1)},$$

then $P_{(i-1)s+2}(a_{is+1}) \neq 0$ and we get $P_{is+1}(a_{is+1}) \neq 0$. This is a contradiction and from (3) we have $\lambda_1^{(is+1)} < a_{is+1} < \lambda_{is+1}^{(is+1)}$. □

Theorem 3.4. The **IEPB(c,s)** has a unique solution if and only if

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \ldots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \ldots < \lambda_n^{(n)}. \tag{11}$$

Proof. First we assume that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ for all $j = 1, 2, \ldots, n$ satisfying (11), thus

$$P_1(\lambda_1^{(1)}) = 0 \Rightarrow a_1 = \lambda_1^{(1)}.$$

For $i = 1, 2, \ldots, c$, if $j = (i-1)s + 2$, by Lemma 3.2 we have $D_{1,j} \neq 0$, by solving the system (4) we obtain

$$a_j = \frac{\lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) det(B_j^{\lambda_j^{(j)}}) - \lambda_j^{(j)} P_{j-1}(\lambda_j^{(j)}) det(B_j^{\lambda_1^{(j)}})}{D_{1,j}},$$

$$b_{1j}^2 = \frac{(\lambda_j^{(j)} - \lambda_1^{(j)}) P_{j-1}(\lambda_1^{(j)}) P_{j-1}(\lambda_j^{(j)})}{D_{1,j}}.$$

From equation (5) we have

$$(-1)^{j-1} D_{1,j} = (-1)^{j-1} P_{j-1}(\lambda_1^{(j)}) det(B_j^{\lambda_j^{(j)}}) + (-1)^{j-2} P_{j-1}(\lambda_j^{(j)}) det(B_j^{\lambda_1^{(j)}}).$$

From Lemma 2 and (11) we get $(-1)^{j-1} P_{j-1}(\lambda_1^{(j)}) > 0$, $det(B_j^{\lambda_j^{(j)}}) > 0$, $P_{j-1}(\lambda_j^{(j)}) > 0$ and $(-1)^{j-2} det(B_j^{\lambda_1^{(j)}}) > 0$ then

$$b_{1,j}^2 = \frac{(-1)^{j-1}(\lambda_j^{(j)} - \lambda_1^{(j)})P_{j-1}(\lambda_1^{(j)})P_{j-1}(\lambda_j^{(j)})}{(-1)^{j-1}P_{j-1}(\lambda_1^{(j)})det(B_j^{\lambda_j^{(j)}}) + (-1)^{j-2}P_{j-1}(\lambda_j^{(j)})det(B_j^{\lambda_1^{(j)}})} > 0.$$

For $j = (i-1)s+3$ the existence of A_j is equivalent to show that the system of equations

$$\begin{cases} P_j(\lambda_1^{(j)}) = a_j P_{j-1}(\lambda_1^{(j)}) + b_{(j-1),j}^2 P_{j-2}(\lambda_1^{(j)}) - \lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) = 0 \\ P_j(\lambda_j^{(j)}) = a_j P_{j-1}(\lambda_j^{(j)}) + b_{(j-1),j}^2 P_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} P_{j-1}(\lambda_j^{(j)}) = 0, \end{cases} \tag{12}$$

has solutions a_j and $b_{(j-1),j}^2$. The determinant of the coefficients matrix of the above system is

$$D_{(j-1),j} = P_{j-1}(\lambda_1^{(j)})P_{j-2}(\lambda_j^{(j)}) - P_{j-1}(\lambda_j^{(j)})P_{j-2}(\lambda_1^{(j)}).$$

If $D_{(j-1),j} \neq 0$ then the system will have a unique solution. From Lemma 2 and (11) we get $(-1)^{j-1}D_{(j-1),j} > 0$, it is clear that $D_{(j-1),j} \neq 0$ and the unique solutions of the system (12) are:

$$a_j = \frac{\lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)})P_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} P_{j-1}(\lambda_j^{(j)})P_{j-2}(\lambda_1^{(j)})}{D_{(j-1),j}},$$

$$b_{(j-1),j}^2 = \frac{(\lambda_j^{(j)} - \lambda_1^{(j)})P_{j-1}(\lambda_1^{(j)})P_{j-1}(\lambda_j^{(j)})}{D_{(j-1),j}}.$$

Again, from Lemma 2 and (11) we have $(-1)^{j-1}P_{j-1}(\lambda_1^{(j)}) > 0$ and $P_{j-1}(\lambda_j^{(j)}) > 0$, then it is evident that:

$$b_{(j-1),j}^2 = \frac{(-1)^{j-1}(\lambda_j^{(j)} - \lambda_1^{(j)})P_{j-1}(\lambda_1^{(j)})P_{j-1}(\lambda_j^{(j)})}{(-1)^{j-1}P_{j-1}(\lambda_1^{(j)})P_{j-2}(\lambda_j^{(j)}) + (-1)^{j-2}P_{j-1}(\lambda_j^{(j)})P_{j-2}(\lambda_1^{(j)})} > 0.$$

Finally, for $j = (i-1)s+4, \ldots, is+1$, using the last recurrence relation from Lemma 3.2 in equations 1 we have

$$\begin{cases} a_j P_{j-1}(\lambda_1^{(j)}) + b_{((i-1)s+3),j}^2 P_{(i-1)s+2}(\lambda_1^{(j)}) \prod_{k=(i-1)s+4}^{j-1}(\lambda_1^{(j)} - a_k) - \lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) = 0 \\ a_j P_{j-1}(\lambda_j^{(j)}) + b_{((i-1)s+3),j}^2 P_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1}(\lambda_j^{(j)} - a_k) - \lambda_j^{(j)} P_{j-1}(\lambda_j^{(j)}) = 0. \end{cases} \tag{13}$$

The determinant of the coefficients matrix of the system (13) is

$$\begin{aligned} D_{(i-1)s+3,\, j} = & \left\{ P_{j-1}(\lambda_1^{(j)})P_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1}(\lambda_j^{(j)} - a_k) \right\} \\ & - \left\{ P_{j-1}(\lambda_j^{(j)})P_{(i-1)s+2}(\lambda_1^{(j)}) \prod_{k=(i-1)s+4}^{j-1}(\lambda_1^{(j)} - a_k) \right\}. \end{aligned} \tag{14}$$

From (14) we have

$$\begin{aligned} (-1)^{j-1}D_{(i-1)s+3,\, j} = & \left\{ \{(-1)^{j-1}P_{j-1}(\lambda_1^{(j)})\} P_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1}(\lambda_j^{(j)} - a_k) \right\} \\ + & \left\{ P_{j-1}(\lambda_j^{(j)}) \{(-1)^{(i-1)s+2}P_{(i-1)s+2}(\lambda_1^{(j)})\} \{(-1)^{j-(i-1)s-4} \prod_{k=(i-1)s+4}^{j-1}(\lambda_1^{(j)} - a_k)\} \right\}. \end{aligned} \tag{15}$$

By Lemma 2 and (11) we get $(-1)^{j-1}P_{j-1}(\lambda_1^{(j)}) > 0$, $P_{j-1}(\lambda_j^{(j)}) > 0$, $(-1)^{(i-1)s+2}P_{(i-1)s+2}(\lambda_1^{(j)}) > 0$ and $P_{(i-1)s+2}(\lambda_j^{(j)}) > 0$. Also, by Lemma 3.2, we have $(-1)^{j-(i-1)s-4}\prod_{k=(i-1)s+4}^{j-1}(\lambda_1^{(j)} - a_k) > 0$ and $\prod_{k=(i-1)s+4}^{j-1}(\lambda_j^{(j)} - a_k) > 0$. Hence, $(-1)^{j-1}D_{(i-1)s+3),\, j} > 0$ and $D_{(i-1)s+3,\, j} \neq 0$. It follows that the linear sysyem equations (13) has a unique solution for a_j and $b^2_{(i-1)s+3,\, j}$.

$$a_j = \frac{A_j - B_j}{D_{(i-1)s+3,\, j}},$$

$$b^2_{(i-1)s+3,\, j} = \frac{(\lambda_j^{(j)} - \lambda_1^{(j)})P_{j-1}(\lambda_1^{(j)})P_{j-1}(\lambda_j^{(j)})}{D_{(i-1)s+3,\, j}},$$

where A_j and B_j are given by

$$A_j = \lambda_1^{(j)}P_{j-1}(\lambda_1^{(j)})P_{(i-1)s+2}(\lambda_j^{(j)}) \prod_{k=(i-1)s+4}^{j-1}(\lambda_j^{(j)} - a_k),$$

$$B_j = \lambda_j^{(j)}P_{j-1}(\lambda_j^{(j)})P_{(i-1)s+2}(\lambda_1^{(j)}) \prod_{k=(i-1)s+4}^{j-1}(\lambda_1^{(j)} - a_k).$$

We can write $b^2_{(i-1)s+3,\, j}$ as:

$$b^2_{(i-1)s+3,\, j} = \frac{(\lambda_j^{(j)} - \lambda_1^{(j)})\big\{(-1)^{j-1}P_{j-1}(\lambda_1^{(j)})P_{j-1}(\lambda_j^{(j)})\big\}}{(-1)^{j-1}D_{(i-1)s+3,\, j}}.$$

From Lemma 2 and (11) we have $(-1)^{j-1}P_{j-1}(\lambda_1^{(j)}) > 0$ and $P_{j-1}(\lambda_j^{(j)}) > 0$. Also, from the above verified results, we know $(-1)^{j-1}D_{(i-1)s+3,\, j} > 0$, hence $b^2_{(i-1)s+3,\, j} > 0$. Conversely, suppose the **IEPB(c,s)** has a unique solution then from Lemma 3.2 and Lemma 3.2 we can obtain (11) thus the proof is completed.

□

The resulting algorithm takes the form of **Algorithm 1**.

Algorithm 1. (To solve problem **IEPB(c,s)**)

Input: $c, s, \lambda_1^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_1^{(n)}, \lambda_n^{(n)}$, where $n = cs + 1$.
Output: $A \in S(B(c, s))$.
$a_1 = \lambda_1^{(1)}$.
For $j = 2$ **to** n
  if $j \in (i-1)s + 2$ $for\ i = 1, 2, \dots, c$ **Then**

$$a_j = \frac{\lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) det(B_j^{\lambda_j^{(j)}}) - \lambda_j^{(j)} P_{j-1}(\lambda_j^{(j)}) det(B_j^{\lambda_1^{(j)}})}{D_{1,j}}$$

$$b_{1,j} = \sqrt{\frac{(\lambda_j^{(j)} - \lambda_1^{(j)}) P_{j-1}(\lambda_1^{(j)}) P_{j-1}(\lambda_j^{(j)})}{D_{1,j}}}$$

  elseif $j \in (i-1)s + 3$ $for\ i = 1, 2, \dots, c$ $then$

$$a_j = \frac{\lambda_1^{(j)} P_{j-1}(\lambda_1^{(j)}) P_{j-2}(\lambda_j^{(j)}) - \lambda_j^{(j)} P_{j-1}(\lambda_j^{(j)}) P_{j-2}(\lambda_1^{(j)})}{D_{(j-1),j}}$$

$$b_{(j-1),j} = \sqrt{\frac{(\lambda_j^{(j)} - \lambda_1^{(j)}) P_{j-1}(\lambda_1^{(j)}) P_{j-1}(\lambda_j^{(j)})}{D_{(j-1),j}}}$$

  else
   $for\ j = (i-1)s + 4\ \ to\ \ is + 1,\ i = 1, 2, \dots, c\ \ do$

$$a_j = \frac{A_j - B_j}{D_{(i-1)s+3,\, j}}$$

$$b_{(i-1)s+3,\, j} = \sqrt{\frac{(\lambda_j^{(j)} - \lambda_1^{(j)}) P_{j-1}(\lambda_1^{(j)}) P_{j-1}(\lambda_j^{(j)})}{D_{(i-1)s+3,\, j}}}$$

  EndIf
EndFor

4 Numerical examples

Algorithm 1 is tested for various examples by matlab software. In this section we report some of examples. **Example 4.1** For given 17 real numbers

$$-7, -6.1, -5, -3.6, -2.5, -1, -0.5, 1.5,\ 2,\ 2.5,\ 3,\ 4,\ 5.3,\ 6,\ 7,\ 8,\ 9.2,$$

rearrange and label them as $\lambda_1^{(j)}$, $j = 1, 2, \dots, 9$ and $\lambda_j^{(j)}$, $j = 2, \dots, 9$, and find a matrix $9 \times 9 \in S(B(2, 4))$ such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of A_j, $j = 1, 2, \dots 9$.
We rearrange the given numbers. The following numbers:

$$-7 < -6.1 < -5 < -3.6 < -2.5 < -1 < -0.5 < 1.5 < 2 < 2.5 < 3 < 4 < 5.3 < 6 < 7 < 8 < 9.2,$$

satisfy the sufficient condition (11). By applying **Algorithm 1**, we get the unique solution

$$A_9 = \begin{bmatrix} 2.0000 & 0.5000 & 0 & 0 & 0 & 4.6952 & 0 & 0 & 0 \\ 0.5000 & 2.0000 & 1.4142 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.4142 & 0.3333 & 1.6101 & 3.0587 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.6101 & 3.0029 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.0587 & 0 & 2.3925 & 0 & 0 & 0 & 0 \\ 4.6952 & 0 & 0 & 0 & 0 & 0.3752 & 3.6226 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.6226 & 0.9421 & 5.0534 & 4.4313 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5.0534 & 1.1223 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.4313 & 0 & 2.2202 \end{bmatrix}.$$

From the matrix A_9 we recomputed the eigenvalues of A_j, and obtained
$\sigma(A_1) = \{\underline{2.0000}\}$
$\sigma(A_2) = \{\underline{1.5000}, \underline{2.5000}\}$
$\sigma(A_3) = \{\underline{-0.5000}, 1.8333, \underline{3.0000}\}$
$\sigma(A_4) = \{\underline{-1.0000}, 1.7143, 2.6219, \underline{4.0000}\}$
$\sigma(A_5) = \{\underline{-2.5000}, 1.5649, 2.4890, 2.8749, \underline{5.3000}\}$
$\sigma(A_6) = \{\underline{-3.6000}, -2.4916, 2.0448, 2.8688, 5.2820, \underline{6.0000}\}$
$\sigma(A_7) = \{\underline{-5.0000}, -2.4970, 1.2356, 2.1448, 2.8698, 5.2929, \underline{7.0000}\}$
$\sigma(A_8) = \{\underline{-6.1000}, -2.5108, -2.1528, 2.0401, 2.8686, 4.7171, 5.3061, \underline{8.0000}\}$
$\sigma(A_9) = \{\underline{-7.0000}, -2.6646, -2.4642, 1.7361, 2.0463, 2.8688, 5.2344, 5.4318, \underline{9.2000}\}$.

The underlined eigenvalues are in consonance with the minimal and maximal eigenvalues.
Example 4.2. For given 11 real numbers

$$-8, -6, -3, -1,\ 1,\ 2,\ 5,\ 6,\ 8,\ 11,\ 13,$$

rearrange and label them as $\lambda_1^{(j)}$, $j = 1, 2, \ldots, 6$ and $\lambda_j^{(j)}$, $j = 2, \ldots, 6$, and find a matrix $6 \times 6 \in S(B(1, 5))$ such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are the minimal and maximal eigenvalues of A_j, $j = 1, 2, \ldots 6$.
We rearrange the given numbers. The following numbers:

$$-8 < -6 - 1 < 1 < 2 < 5 < 6 < 8 < 11 < 13,$$

satisfy the sufficient condition (11). By applying **Algorithm 1**, we get the unique solution

$$A_6 = \begin{bmatrix} 2.0000 & 1.7321 & 0 & 0 & 0 & 0 \\ 1.7321 & 4.0000 & 2.5820 & 0 & 0 & 0 \\ 0 & 2.5820 & 0.6667 & 4.4117 & 6.5160 & 6.1730 \\ 0 & 0 & 4.4117 & 4.4146 & 0 & 0 \\ 0 & 0 & 6.5160 & 0 & 4.3427 & 0 \\ 0 & 0 & 6.1730 & 0 & 0 & 4.3356 \end{bmatrix}.$$

From the matrix A_6 we recomputed the eigenvalues of A_j, and obtained
$\sigma(A_1) = \{\underline{2.0000}\}$

$\sigma(A_2) = \{\underline{1.0000}, \underline{5.0000}\}$
$\sigma(A_3) = \{\underline{-1.0000}, 1.6667, \underline{6.0000}\}$
$\sigma(A_4) = \{\underline{-3.0000}, 1.2100, 4.8713, \underline{8.0000}\}$
$\sigma(A_5) = \{\underline{-6.0000}, 1.0811, 4.3918, 4.9511, \underline{11.0000}\}$
$\sigma(A_6) = \{\underline{-8.0000}, 1.0521, 4.3389, 4.3999, 4.9686, \underline{13.0000}\}$.

The underlined eigenvalues are in consonance with the minimal and maximal eigenvalues.

5 Conclusions

In this paper, we have solved the IEP for construction of matrices whose graphs are banana trees. This is performed by using the minimal and maximal eigenvalues of all leading principal submatrices of the required matrix. The results obtained in this paper provide an efficient method for constructing such matrices. The problem IEPB(c,s) is important in the sense that it partially describes inverse eigenvalue problem while it constructs matrices from partial information of prescribed eigenvalues. Such partially described problems may occur in computations involving a complex physical system that is difficult to obtain in its entire spectrum. It would be interesting to consider such IEPs for other acyclic matrices as well.

References

[1] Asmiati, Locating chromatic number of banana tree, International Mathematical Forum 12 (2017) 39-45.

[2] Chen, W.C., Lu, H. I., and Yeh, Y. N., Operations of interlaced trees and graceful trees,Math 21(1997).

[3] Chu, M. T., and Golub, G. H., Inverse Eigenvalue Problems: Theory, Algorithms and Application, Oxford University Press, New York, (2005).

[4] Duarte, A.L., Construction of acyclic matrices from spectral data, Linear Algebra and its Applications 113(1989) 173-182.

[5] Gutman, I., and Triantallou, I., Dependence of graph energy on nullity: A case study, Communications in Mathematical and in Computer Chemistry 76(2016).

[6] Hogben, L., Spectral graph theory and the inverse eigenvalue problem of a graph, Electronic Journal of Linear Algebra 14(2005), 12-31.

[7] Monfared, K. H., and Shader B.L., Construction of matrices with a given graph and prescribed interlaced spectral data, Linear Algebra and its Applications 438(2013) 4348-4358.

[8] Nair, R., and Shader B.L., Acyclic matrices with a small number of distinct eigenvalues, Linear Algebra and its Applications 438(2013) 4075-4089.

[9] Nazari, A.M., and Beiranvand Z., The inverse eigenvalue problem for symmetric quasi anti-bidiagonal matrices, Appl.Math. Comput 217 (2011) 9526-531.

[10] Peng, J., Hu, X., and Zhang, L., Two inverse eigenvalue problems for a special kind of matrices, Linear Algebra and its Applications 416 (2006) 336-347.

[11] Pickmann, H., Egana, J., and Soto, R.L., Extremal inverse eigenvalue problem for bordered diagonal matrices, Linear Algebra Applications 427 (2007).

[12] Sen, M., and Sharma, D., Generalized inverse eigenvalue problem for matrices whose graph is a path, Linear Algebra and its Applications 446 (2014) 224-236.

[13] Sharma, D., and Sen, M., Inverse eigenvalue problems for acyclic matrices whose graph is a dense centipede De Gruyter 6 (2018)77-92

[14] Sharma, D., and Sen, M., Inverse Eigenvalue Problems for Two Special Acyclic Matrices, Mathematics 4 (12) (2016).