Curriculum Management System

MONROE TOWNSHIP SCHOOLS

Course Name: Honors Pre-calculus
Grade: 10-12

For adoption by all regular education programs as specified and for adoption or adaptation by all Special Education Programs in accordance with Board of Education Policy # 2220.

Board Approved: September 2015

Table of Contents

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WRITER'S NAMES

Ms. Kathleen Dougherty

CURRICULUM SUPERVISOR

Ms. Susan Gasko

Mission, Vision, Beliefs, and Goals

Mission Statement

The Monroe Public Schools in collaboration with the members of the community shall ensure that all children receive an exemplary education by well-trained committed staff in a safe and orderly environment.

Vision Statement

The Monroe Township Board of Education commits itself to all children by preparing them to reach their full potential and to function in a global society through a preeminent education.

Beliefs

1. **All decisions are made on the premise that children must come first.**
2. **All district decisions are made to ensure that practices and policies are developed to be inclusive, sensitive and meaningful to our diverse population.**
3. **We believe there is a sense of urgency about improving rigor and student achievement.**
4. **All members of our community are responsible for building capacity to reach excellence.**
5. **We are committed to a process for continuous improvement based on collecting, analyzing, and reflecting on data to guide our decisions.**
6. **We believe that collaboration maximizes the potential for improved outcomes.**
7. **We act with integrity, respect, and honesty with recognition that the schools serves as the social core of the community.**
8. **We believe that resources must be committed to address the population expansion in the community.**
9. **We believe that there are no disposable students in our community and every child means every child.**

Board of Education Goals

1. **Raise achievement for all students paying particular attention to disparities between subgroups.**
2. **Systematically collect, analyze, and evaluate available data to inform all decisions.**
3. **Improve business efficiencies where possible to reduce overall operating costs.**
4. **Provide support programs for students across the continuum of academic achievement with an emphasis on those who are in the middle.**
5. **Provide early interventions for all students who are at risk of not reaching their full potential.**
6. **To Create a 21st Century Environment of Learning that Promotes Inspiration, Motivation, Exploration, and Innovation.**

Common Core State Standards (CSSS]

The Common Core State Standards provide a consistent, clear understanding of what students are expected to learn, so teachers and parents know what they need to do to help them. The standards are designed to be robust and relevant to the real world, reflecting the knowledge and skills that our young people need for success in college and careers. With American students fully prepared for the future, our communities will be best positioned to compete successfully in the global economy.

Links:
1. CCSS Home Page: http://www.corestandards.org
2. CCSS FAQ: http://www.corestandards.org/frequently-asked-questions
3. CCSS The Standards: http://www.corestandards.org/the-standards
4. NJDOE Link to CCSS: http://www.state.nj.us/education/sca
5. Partnership for Assessment of Readiness for College and Careers (PARCC): http://parcconline.org
6. National Standards for Family and Consumer Sciences Education http://nasafacs.org/national-standards-home.html

Quarter 1

UNIT TOPICS

I. Trigonometric Functions

- A. Radian and Degree Measure
  - 1. Angle Measurement
  - 2. Angular Movement
  - 3. Linear and Angular Speed
- B. Right Triangle Trigonometry
  - 1. Trigonometric Functions of Acute Angles
  - 2. Using Calculators Where Appropriate
  - 3. Angle of Elevation, Angle of Depression, and Additional Applications of Right Triangle Trigonometry
- C. Trigonometric Functions at any Angle
  - 1. Reference Angles
  - 2. Trigonometric Functions of Non-acute Angles
  - 3. Trigonometric Functions of Quadrant Angles
  - 4. Evaluating Trigonometric Functions of Real Numbers
- D. Trigonometric Functions: The Unit Circle
  - 1. The Unit Circle and Its Relationship to Real Numbers
  - 2. Using the Unit Circle to Evaluate Trigonometric Functions

II. Graphs of Trigonometric Functions

- A. Graphs of Sine and Cosine Functions
  - 1. Sketching Graphs by Hand and Using Technology
  - 2. Amplitude and Period
  - 3. Translations
  - 4. Using Sine and Cosine Functions to Model and Analyze Real- Life Data
  - 5. Determining Equations from Transformed Graphs
- B. Graphs of Other Trigonometric Functions
  - 1. Sketching by Hand and Using Technology
  - 2. Domain and Range
  - 3. Period and Asymptotes
- C. Inverse Trigonometric Functions
  - 1. Evaluating Inverse Functions (with and without calculators)
  - 2. Graphs of Inverse Functions
  - 3. Compositions of Trigonometric Functions

Quarter 2

UNIT TOPICS

I. Analytic Trigonometry

- A. The Fundamental Trigonometric Identities
  - 1. Deriving the Identities
  - 2. Using the Identities to Evaluate and Simplify Expressions
  - 3. Verifying the Identities
- B. Solving Trigonometric Equations
  - 1. Using Standard Algebraic Techniques to Solve
  - 2. General Solution Format vs. Solving on an Interval
  - 3. Solving Trigonometric Equations Involving Multiple Angles
  - 4. The Role of Inverse Functions
  - 5. Approximating Solutions Using Graphing Calculator Technology
- C. Trigonometric Addition Formulas
  - 1. Deriving the Formulas
  - 2. Using Formulas to Find Exact Values and Simplify Expressions
  - 3. Using Formulas to Verify Identities and Solve Equations
- D. Multiple and Half Angle Formulas
  - 1. Deriving the Formulas
  - 2. Using as an Aid to Solve Equations
  - 3. Using Formulas to Find Exact Values and Simplify Expressions
  - 4. The Connections to Graphing Calculator Technology

II. Oblique Triangles

- A. The Law of Sines
  - 1. Deriving the Formula
  - 2. Using the Formula to Solve AAS, ASA, and SSA Cases
  - 3. The Ambiguous Case
- B. The Law of Cosines
  - 1. Deriving the Formula
  - 2. Using the Formula to Solve SSS and SAS Cases
- C. Applications of Oblique Triangles
  - 1. Areas of Oblique Triangles
  - 2. Applications of Trigonometry to Navigation and Surveying
  - 3. Directional Bearings
  - 4. Areas of Plots of Land

Quarter 3

UNIT TOPICS

I. Vectors

- A. Vectors in the Plane
  - 1. Geometric Representation of Vectors
  - 2. Operations with Vectors
  - 3. Direction Angles and the Connection to Trigonometry
  - 4. Applications to Navigation
- B. Vectors and Dot Products
  - 1. Parallel and Perpendicular Vectors
  - 2. The Angle Between Two Vectors
- C. The Three-Dimensional Coordinate System
  - 1. Representing Points on the Three-Dimensional Coordinate System
  - 2. Determining Distances and Midpoints of Line Segments in Space
  - 3. Working with Spheres
- D. Vectors in Three-Dimensional Space
  - 1. Geometric Representation of Vectors
  - 2. Operations with Vectors
  - 3. The Angle Between Two Vectors
  - 4. Parallel and Perpendicular Vectors
  - 5. Collinear Points
- E. The Cross Product of Two Vectors
  - 1. Determinants and Their Applications
  - 2. Geometric Properties and Applications of the Cross Product
  - 3. Volume by the Triple Scalar Product

II. Polar Form of Complex Numbers and Graphs

- A. Trigonometric Form of a Complex Number
  - 1. Geometric Representation of a Complex Number
  - 2. Conversion to Trigonometric (Polar) Form
  - 3. Operations of Complex Numbers in Trigonometric (Polar) Form
  - 4. DeMoivre’s Theorem
  - 5. Roots of Complex Numbers
- B. Polar Coordinates
  - 1. Representing Points on the Polar Coordinate System
  - 2. Conversion with Rectangular Form
  - 3. Equation Conversion
- C. Graphs of Polar Equations
  - 1. Sketching by Hand
  - 2. Using Various Technology to Create Polar Graphs
  - 3. Analyzing Polar Graphs

III. Parametric Equations

- A. Parametric Equations and Graphs
  - 1. Sketching by Hand and Using Technology as an Aid
  - 2. Eliminating the Parameter
  - 3. Determining a Set of Parametric Equations Given a Graph
  - 4. Projectile Motion
- F. Lines and Planes in Space
  - 1. Parametric and Symmetric Equations of Lines in Space
  - 2. Vector and Parametric Equations of Lines
  - 3. Planes in Space

Quarter 4

UNIT TOPICS

I. Sequences and Series

- A. Sequence Notation
  - 1. Recursive Formulas and Explicit Formulas
  - 2. Factorial Notation and Its Uses
- B. Arithmetic Sequences
  - 1. Recursive Formulas and Explicit Formulas
  - 2. Finite Sums and Partial Sums of Arithmetic Sequences
- C. Geometric Sequences
  - 1. Recursive Formulas and Explicit Formulas
  - 2. Finite and Infinite Sums of Geometric Sequences and Series
  - 3. The Interval of Convergence of an Infinite Geometric Series
- D. Using Technology with Sequences and Series
- E. Modeling Real-Life Applications with Sequences and Series

II. Limits

- A. Introductions to Limits
  - 1. Estimating Limits Numerically, Graphically, and Algebraically With and Without Graphing Calculator Technology
  - 2. The Existence of the Limit of a Functions
  - 3. One-Sided Limits
  - 4. Properties of Limits
  - 5. The Difference Quotient and the Connection to Calculus
  - 6. Applications of Limits
- B. Continuous Functions
  - 1. Connection to the Limit of a Function
  - 2. Conditions for Continuity
- C. Rational Functions and Asymptotes
  - 1. Domains of Rational Functions
  - 2. Analyzing and Sketching Graphs of Rational Functions
  - 3. Horizontal, Vertical, and Slant Asymptotes of Graphs of Rational Functions
  - 4. Using Graphing Calculator Technology as a Tool
- D. Limits at Infinity
  - 1. Limits at Infinity and Horizontal Asymptotes
  - 2. Limits of Sequences

III. Combinatorics and Probability

- A. The Fundamental Counting Principle
- B. Permutations and Combinations
- C. Probability of Mutually Exclusive and Independent Events
- D. Probability of Complements of Events
- E. Using Graphing Calculator Technology as an Aid

UNIT 1 – Trigonometric Functions

Stage One: Desired Results

<table>
<tr><td rowspan="6">ESTABLISHED GOALS

HSF-TF.1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

HSF-TF.2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

HSF-TF.3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for π/3, π/4 and π/6, and use the unit circle to express the values of sine, cosines, and tangent for <i>x</i>, π + <i>x</i>, and 2π – <i>x</i> in terms of their values for <i>x</i>, where <i>x</i> is any real number.

HSF-TF.4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

HSG-SRT.6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

HSG-SRT.7. Explain and use the relationship between the sine and cosine of</td><td colspan="2"><i>Transfer</i></td></tr>
<tr><td colspan="2"><i>Students will be able to independently use their learning to…</i>
Analyze and describe angles and angular movement and relate this information to real life phenomena.</td></tr>
<tr><td colspan="2"><i>Meaning</i></td></tr>
<tr><td>UNDERSTANDINGS
<i>Students will understand that…</i>
• An angle is determined by rotating a ray about its endpoint and can be measured in degrees, radians, and revolutions.
• The co-terminal relationship can be used to model infinitely many angles.
• The formula for the length of a circular arc can be used to analyze the motion of a particle moving at a constant speed along a circular path.
• Trigonometry describes the relationship between the sides and angles in right triangles.
• Many relationships exist between angles in trigonometry that can be used to make predictions about trigonometric values at a variety of angles.
• Reference angles are used to model and make predictions about trigonometric functions of any angle.</td><td>ESSENTIAL QUESTIONS
• How do you describe angles and angular movement?
• What are real world applications of linear and angular speed?
• How do you use trigonometry to find unknown side lengths and angles in right triangles?
• How can the different trigonometric functions be used to solve real world applications?
• How do you evaluate trigonometric functions at any angle?
• How do you evaluate trigonometric functions by using the unit circle?</td></tr>
<tr><td colspan="2"><i>Acquisition</i></td></tr>
<tr><td><i>Students will know…</i>
• Angles can be measured in degrees, radians, and revolutions.
• One radian is the measure of a central angle that intercepts an arc equal in length to the radius of the circle.</td><td><i>Students will be skilled at…</i>
• Sketching positive and negative angles in standard position.
• Using the relationship between degrees and radians to convert between the two forms.</td></tr>
</table>

complementary angles. **HSG-SRT.8.** Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. **MATHEMATICAL PRACTICES** **MP1** Make sense of problems and persevere in solving them. **MP2** Reason abstractly and quantitatively. **MP3** Construct viable arguments and critique the reasoning of others. **MP4** Model with mathematics. **MP5** Use appropriate tools strategically. **MP6** Attend to precision. **MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	• The input variable in a trigonometric function is the angle measure. • There is a proportional relationship between degree measurement and radian measurement. • A full circle consists of 1 revolution, 360 degrees, or 2π radians. • Measuring the arc length of a sector of a circle is one of the many applications where radian measure must be used. • Linear and angular speed are applications of angles and angular movement. • The different trigonometric ratios can be used to solve for missing sides and angles in a right triangle. • Each trigonometric function has a unique reciprocal function. • Reference angles are used to evaluate trigonometric functions that do not terminate in quadrant I.	• Finding positive and negative coterminal angles. • Classifying angles as complementary, supplementary, or neither. • Analyzing a situation to determine whether it is best to use degrees, radians, or revolutions. • Applying the relationship between arc length, radius, and the central angle to solve for the missing piece. • Analyzing the motion of a particle moving at a constant speed along a circular path. • Using trigonometry to solve for missing sides and angles of a right triangle. • Constructing a model of the unit circle. • Explaining the role of reference angles in evaluating trigonometric expressions. • Evaluating trigonometric expressions by hand and with technology.

Stage Two: Evidence

Evaluative Criteria	Assessment Evidence
SUGGESTED PERFORMANCE RUBRIC 4 – Correct solution with all necessary work demonstrating complete understanding of the topic. Correct units are included. 3 – Solution contains one or two minor errors but overall demonstrates understanding of the topic. 2 – Several minor errors or one major error	**SUGGESTED PERFORMANCE ASSESSMENT:** ***Students will engage in the following performance task***: • Consider the angle: $\frac{7\pi}{3}$. a. Sketch the angle in standard position. b. Determine two coterminal angles (one positive and one negative). c. Convert the angle to degrees. • Find the radius of a circle whose arc length is 125 cm and whose central angle is 210°.

in solving the problem demonstrating some understanding of the topic.

1 – Major errors in solving the problem demonstrating very little understanding of the topic.

0 – No answer is provided or the work has nothing to do with the topic being covered.

- A truck is moving at a rate of 90 kilometers per hour, and the diameter of its wheels is 1.25 meters. Find the angular speed of the wheels in radians per minute.
- Danielle places her telescope on the top of a tripod 5 ft above the ground. She measures an 8° elevation above the horizontal to the top of the tree that is 160 ft away. How tall is the tree?
- Find the value of $\tan\theta$ given that $\sin\theta = -\frac{3}{7}$ and $\sec\theta < 0$.
- Find the exact value of $\csc\left(-\frac{9\pi}{4}\right)$. Show all necessary work.

OVERVIEW

In this unit, students will learn how to describe angles and angular movement in a variety of ways. They will use angles to model and solve real-life problems. Students will use the fundamental trigonometric identities and reference angles to evaluate trigonometric functions in all four quadrants. They will use trigonometric functions to model and solve real-life applications.

In this unit, students will be formally assessed through a quiz on evaluating trigonometric expressions and a test on the entire unit. Sample questions are listed above.

DIFFERENTIATION

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Provide the opportunity for students to practice problems both with and without graphically calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems on the related topic for those students who finish early.

TECHNOLOGY

- The following links can provide visual and interactive models for understanding what a radian is and why it is important. Students can follow along using their iPads:
  http://zonalandeducation.com/mmts/trigonometryRealms/radianDemo1/RadianDemo1.html
  http://www.mathsisfun.com/geometry/radians.html
- The following video demonstrates what linear and angular velocity are and how they are affected by different size circles. Students who are struggling with the difference can view this video for extra help. The link can be placed on the teacher's wiki.

https://www.youtube.com/watch?v=yDHM6rd8P94

- Using ***ALGEBRA IN MOTION ™*** with Geometer's Sketchpad, the teacher can model the idea of reference angles by using $30° - 60° - 90°$ and $45° - 45° - 90°$ triangles in each of the four quadrants. The students can compare the different trigonometric ratios with angles equal distance from the closest *x*-axis. A link to download the program can be put on the teacher's wiki so that students can follow along on their iPad.
- The Free Graphing Calculator app on the iPad can be used as a reference for students to view a detailed unit circle. Go to the app Free Graphing Calculator $\rightarrow$ Reference $\rightarrow$ Trigonometry $\rightarrow$ Detailed Unit Circle.

SUGGESTED MONITORING SCALE:

Use the following or similar scale to monitor or evaluate a student's daily learning and understanding of key concepts:

4 – I understand the concept completely and can explain it to a classmate.

3 – I understand the concept but would not feel comfortable explaining it to a classmate.

2 – I can complete the problem with assistance from the teacher.

1 – I do not know where to begin; I need help.

OTHER SUGGESTED PERFORMANCE TASKS:

- Exploration: Evaluating Trigonometric Functions
  - a. Find the exact values of each of the six trigonometric functions for the angle 30°.
  - b. Find the exact values of each of the six trigonometric functions for the angle 60°.
  - c. Compare the six function values for 60° with the six function values for 30°. What do you notice?
  - d. We will eventually learn a rule that relates trigonometric functions of any angle with trigonometric functions of the complementary angle. Based on this exploration, can you predict what the rule will be?
- Evaluate the six trigonometric functions if the terminal side of the ray goes through the following points:
  - a. (1, 2)
  - b. (-8, 15)
  - c. (-4, -4)
  - d. (5, -12)
  - e. Looking at your answers above, fill in the following table, stating whether the trig values were positive of negative in each of the quadrants.
  - f.

	sine	**cosine**	**tangent**	**cosecant**	**secant**	**cotangent**
Quad I						
Quad II						
Quad III						

Quad IV						

Stage Three: Learning Plan

Summary of Key Learning Events and Instruction

SUGGESTED LEARNING EVENTS:

- Derive the definition of one radian. Use the following links as visual and interactive guides to support student understanding of the concept.
  http://zonalandeducation.com/mmts/trigonometryRealms/radianDemo1/RadianDemo1.html
  http://www.mathsisfun.com/geometry/radians.html
- Convert between degrees, radians, and revolutions. Determine when it is best to use each of the different measures.
- In groups, have students solve application of angular movement – linear and angular speed problems. Differentiate the activity by assigning harder problems to students who have demonstrated understanding of the concept. Suggest the link for students to see what linear and angular speed really are and how they differ:
  https://www.youtube.com/watch?v=yDHM6rd8P94.
- Complete the Exploration: Evaluating Trigonometric Functions under Other Suggested Performance Tasks listed above to show the relationship of the different trigonometric functions for complementary angles.
- Complete the second activity under Other Suggested Performance Tasks to show which quadrants the different trigonometric functions are positive and negative. Discuss why this seems reasonable.
- Use ***ALGEBRA IN MOTION ™*** with Geometer's Sketchpad to show that angles with the same distance to the closest horizontal axis (reference angles) have the same absolute value for the trigonometric ratios.

SUGGESTED METHODS OF DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Provide the opportunity for students to practice problems both with and without graphically calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems on the related topic for those students who finish early.

UNIT 2 – Graphs of Trigonometric Functions

Stage One: Desired Results

ESTABLISHED GOALS

HSF-TF.4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

HSF-TF.5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

HSF-IF.5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

HSF-IF.7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

HSF-TF.6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

HSF-TF.7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

HSF-BF.4. Find inverse functions.

Transfer

Students will be able to independently use their learning to…

Model and make predictions about periodic behavior using the graphs of trigonometric functions.

Meaning

UNDERSTANDINGS

Students will understand that…

- There exists a relationship between the graphs of sine and cosine.
- The amplitude of a sine or cosine graph is useful when making predictions about the vertical shrink or stretch of the graph.
- Sine and cosine functions are used to model waves and periodic behavior.
- There exists more than one way to write the equation of a sinusoidal function given its graph.
- The domain and range of a function are useful in predicting the graph of a function.
- The domains of sine, cosine, and tangent must be restricted in order for the function to be one-to-one and have an inverse.
- Inverse functions are necessary to determine the angle for a given trigonometric value which has uses in many real life applications.
- Inverse properties are useful when making predictions.
- The relationship between the sides and angles of right triangles can be used when converting a trigonometric expression into an algebraic expression

ESSENTIAL QUESTIONS

- How are the graphs of sine and cosine related?
- How can the amplitude and period of a sinusoidal function be used to sketch its graph?
- How is the graph of a sinusoidal function related to its equation?
- What real life scenarios are best modeled by trigonometric functions?
- How is the graph of a trigonometric function a useful tool for modeling and interpreting data?
- How are domain and range useful when analyzing trigonometric graphs?
- What is the relationship between a function and its inverse?
- What does finding the inverse of a function mean graphically?
- Why is finding the inverse of a function useful?

MATHEMATICAL PRACTICES **MP1** Make sense of problems and persevere in solving them. **MP2** Reason abstractly and quantitatively. **MP3** Construct viable arguments and critique the reasoning of others. **MP4** Model with mathematics. **MP5** Use appropriate tools strategically. **MP6** Attend to precision. **MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	which has applications in Calculus.	
	Acquisition	
	Students will know… • The values of a trigonometric function are used to sketch the graph of a trigonometric function. • Trigonometric graphs are periodic in nature. • The amplitude of a graph determines the amount of vertical stretch or shrink of the graph. • Graphically, amplitude is half the height of a sine or cosine wave. • Period is the length of one full cycle of a sine or cosine wave. • Frequency is the number of complete cycles the wave completes in a unit interval. • The graph of a sine or cosine function can be translated with vertical shifts and phase shifts. • Trigonometric graphs can be used to model real life situations. • Trigonometric graphs can be used to make predictions about the equations of the graphs. • There is a relationship between the domain and range of a trigonometric function and the graph of the function. • Inverse sine, inverse cosine, and inverse tangent are each defined in two quadrants, one positive quadrant and one negative quadrant. • Inverse trigonometric functions determine the angle for a given trigonometric value. • Inverse properties can be used to find exact values of trigonometric functions	***Students will be skilled at…*** • Sketching the graphs of sine and cosine functions by hand and with technology. • Identifying the key points on the sine or cosine graph. • Using the amplitude of a sine or cosine function to plot key points on the graph. • Using the amplitude of a sine or cosine function to make predictions. • Setting an appropriate scale to graph one or more periods of a sine or cosine function by hand or with technology. • Using the periodic nature of trigonometric functions to make predictions. • Translating the graphs of sine and cosine functions with their vertical shifts and phase shifts. • Modeling real life data using sine and cosine functions. • Using trigonometric models to make predictions. • Sketching the graphs of tangent, cosecant, secant, and cotangent. • Identifying the domains and ranges of sine, cosine, tangent and their inverses. • Stating the domains, ranges, and asymptotes of tangent, cosecant, secant, and cotangent. • Evaluating expressions with inverse trigonometric functions by hand and with technology. • Explaining why some situations result in no solution and connecting it to the domain and range of the trigonometric function.

	and to evaluate compositions of trigonometric functions. • Triangles can be used to convert a trigonometric expression into an algebraic expression.	• Using right triangles to evaluate compositions of trigonometric functions.

Stage Two: Evidence

Evaluative Criteria

SUGGESTED PERFORMANCE RUBRIC:

4 – Correct solution with all necessary work demonstrating complete understanding of the topic. Correct units are included.

3 – Solution contains one or two minor errors but overall demonstrates understanding of the topic.

2 – Several minor errors or one major error in solving the problem demonstrating some understanding of the topic.

1 – Major errors in solving the problem demonstrating very little understanding of the topic.

0 – No answer is provided or the work has nothing to do with the topic being covered.

Assessment Evidence

SUGGESTED PERFORMANCE ASSESSMENT:

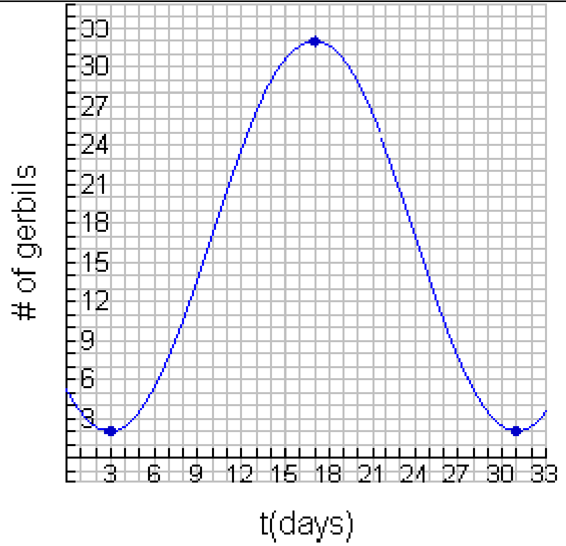
Students will engage in the following performance tasks:

- Graph the function $y = -3.5\sin\left(2x - \frac{\pi}{2}\right) - 1$. Describe all the transformations of this graph from its parent function $y = \sin x$.
- Construct a sinusoid $y = f(x)$ that rises from a minimum value of $y = 6$ at $x = 0$ to a maximum value of $y = 26$ at $x = 16$.
- Throughout the day, the depth of water at the end of a dock varies with the tides. The below table shows the depth (in meters) at various times during the morning.

t(time)	Midnight	2am	4am	6am	8am	10am	Noon
y(depth)	2.55	3.80	4.40	3.80	2.55	1.80	2.27

  a. Use a trigonometric function to model the data.
  b. A boat needs at least 3.5 meters of water to enter the dock. During what times can it safely dock?

- A pet store clerk noticed that the population in gerbil habitat varied sinusoidally with respect to time, in days. He carefully collected data and graphed his resulting equation. From the graph, determine the amplitude, period, horizontal shift and vertical shift. Write the equation of the graph.



- State the domain and range of $y = \frac{1}{8}\tan\left(\frac{x}{2}+\pi\right)$.
- Find the exact value for $\tan\left(\arccos\left(-\frac{2}{3}\right)\right)$.
- An airplane is flying at an altitude of 6 miles above the ground. An observer is standing on the ground and spots the plane. The angle of elevation of the line of sight from the observer to the plane is represented by θ. Draw a right triangle which shows the relationship between angle of elevation, the height of the plane above the ground, and the observer's distance from the plane (call this side *x*). Use this diagram and the relationships between sides and angles of right triangles to determine the following:
  a. θ as a function of *x.*
  b. Find θ when *x*= 10 miles and *x*= 3 miles.

OVERVIEW:
In this unit, students will learn about the graphs of each of the six trigonometric functions, with an emphasis on sine and cosine graphs and their applications. Students will also learn about inverse sine, inverse cosine, and inverse tangent and will use their domains and ranges to evaluate expressions with trigonometric inverses.

To formally assess students on this unit, a project will be given on applications of sine and cosine functions. More details are described below. In addition, a test will be given on the entire unit.

Sample questions are listed above.

DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Relate the trigonometric graphs to numerical and analytical approaches to solving applications.
- Provide the opportunity for students to practice problems both with and without graphically calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems on the related topic for those students who finish early.

TECHNOLOGY:

- Introduce sine and cosine graphs using ***ALGEBRA IN MOTION ™*** with Geometer's Sketchpad. The tab labeled **Connecting the Unit Circle to the Graphs of Sine and Cosine** will enable students an interactive way of seeing how the unit circle is related to the rectangular graphs of sine and cosine on a coordinate plane. By putting a link to this activity on the teacher's wiki, the students can follow along and try it on their own.
- Have students complete a **Graphing Calculator Discovery Lab** on their own to test hypotheses on how the equation of a sine or cosine function relates to its graph. This can be done to illustrate amplitude, period, and vertical and phase shifts. For example, to illustrate amplitude, students should compare the graphs of $y = \sin x$, $y = 3\sin x$, $y = -3\sin x$, and $y = \frac{1}{2}\sin x$. Teachers should make sure students set their calculators in radian mode and have a uniform window.
- In pairs, have students complete an **Applications of Sine and Cosine Functions Project.** Students will be assigned applications of sine and cosine functions and will solve and fully explain their solutions through a **ShowMe** video on their iPad.
- Have students graph $y = \tan x$, $y = \csc x$, $y = \sec x$, and $y = \cot x$ on their graphing calculator. Have them state the domain, range, period, and asymptotes of each of these functions. Then, discuss results and conclusions as a class.

SUGGESTED MONITORING SCALE:

Use the following or similar scale to monitor or evaluate a student's daily learning and understanding of key concepts:

4 – I understand the concept completely and can explain it to a classmate.
3 – I understand the concept but would not feel comfortable explaining it to a classmate.
2 – I can complete the problem with assistance from the teacher.
1 – I do not know where to begin; I need help.

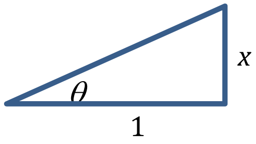
OTHER SUGGESTED PERFORMANCE TASKS:

- Given the following equations, determine the amplitude, period, horizontal shift, and vertical shift of each equation. Are these two equations equivalent? Support your answer graphically and algebraically.

  $$y = 2\sin\left(\frac{\pi}{3}(x-2)\right)-4$$

  $$y = -4+2\cos\left(\frac{\pi}{3}(x-3.5)\right)$$

- Exploration: Finding Inverse Trig Functions of Trig Functions
  In the right triangle shown to the right, the angle θ is measured in radians.
  a. Find $\tan\theta$.
  b. Find $\tan^{-1} x$.
  c. Find the hypotenuse of the triangle as a function of *x*.
  d. Find $\sin\left(\tan^{-1}(x)\right)$ as a ratio involving no trig functions.
  e. Find $\sec\left(\tan^{-1}(x)\right)$ as a ratio involving no trig functions.
  f. If $x < 0$, then $\tan^{-1} x$ is a negative angle in the fourth quadrant. Verify that your answers to parts (d) and (e) are still valid in this case.



Stage Three: Learning Plan

Summary of Key Learning Events and Instruction

SUGGESTED LEARNING EVENTS:

- Use the ***ALGEBRA IN MOTION™*** tab **Connecting the Unit Circle to the Graphs of Sine and Cosine** with Geometer's Sketchpad to help build connections between the unit circle and a rectangular graph of sine and cosine. This will allow students to have a visual, interactive way to see this relationship.
- Have students complete a **Graphing Calculator Discovery Lab** on their own to test hypotheses on how the equation of a sine or cosine function relates to its graph. This can be done to illustrate amplitude, period, and vertical and phase shifts.
- In pairs, have students complete an **Applications of Sine and Cosine Functions Project**. Students will be assigned applications of sine and cosine functions and will solve and fully explain their solutions through a **ShowMe** video on their iPad.
- Have students graph $y = \tan x$, $y = \csc x$, $y = \sec x$, and $y = \cot x$ on their graphing calculator. Have them state the domain, range, period, and asymptotes of each of these functions. Then, discuss results and conclusions as a class.
- Have students complete **Exploration: Finding Inverse Trig Functions of Trig Functions** and discuss inverse trigonometric functions,

composition of functions, and how it affects domain.

SUGGESTED METHODS OF DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Relate the trigonometric graphs to numerical and analytical approaches to solving applications.
- Provide the opportunity for students to practice problems both with and without graphically calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems on the related topic for those students who finish early.

UNIT 3 – Analytic Trigonometry

Stage One: Desired Results

ESTABLISHED GOALS

HSF-TF.8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

HSA-SSE.1. Interpret expressions that represent a quantity in terms of its context

HSA-SSE.2. Use the structure of an expression to identify ways to rewrite it.

HSA-SSE.3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

HSA-CED.4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

HSA-REI.1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

HSA-REI.2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Transfer

Students will be able to independently use their learning to…
Model real-life scenarios using trigonometric equations and solve the equation using trigonometric relationships and algebraic techniques.

Meaning

UNDERSTANDINGS
Students will understand that…

- There exists many relationships between trigonometric functions that can be rewritten to evaluate or simplify trigonometric expressions and verify trigonometric identities.
- Many previously learned algebraic methods can be used to solve trigonometric equations.
- Due to the periodic nature of trigonometric functions, trigonometric equations often have infinitely many solutions that can be expressed using general solution format.
- The solutions to trigonometric equations can be approximated using graphing technology.
- There is a direct relationship between *k* in a multiple angle equation the period the graph of the equation.
- Trigonometric equations can be used to solve real-life scenarios that are periodic in nature.
- The sum and difference formulas can be used to find exact values of trigonometric expressions at additional angles on the unit circle.
- Multiple angle formulas provide another

ESSENTIAL QUESTIONS

- How do you use relationships to rewrite trigonometric expressions in order to simplify and evaluate trigonometric functions?
- How do you use trigonometric relationships and algebraic methods to verify trigonometric identities?
- How can algebraic techniques be applied to solving trigonometric equations?
- How are the graphs of trigonometric functions related to the solutions of the corresponding trigonometric equation?
- What real life problems can be modeled by and solved using trigonometric equations?
- Describe a situation where extraneous solutions may exist. How do you verify if a solution is extraneous to an equation?
- How can you use trigonometry to apply the sum and difference formulas to find exact values of angles?
- How are multiple angle formulas used to transform a trigonometric equation and create a new method to solve the equation?

<table>
<tr>
<td rowspan="3">HSA-REI.4. Solve quadratic equations in one variable.

HSF-IF.8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

HSF-TF.9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

HSCED.4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

HSA-REI.1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

HSF-IF.8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

MATHEMATICAL PRACTICES

MP1 Make sense of problems and persevere in solving them.

MP2 Reason abstractly and quantitatively.

MP3 Construct viable arguments and critique the reasoning of others.</td>
<td>method for creating an equivalent trigonometric equation.
• Half-angle formulas allow for more exact value results as opposed to decimal approximations.</td>
<td></td>
</tr>
<tr>
<td colspan="2"><i>Acquisition</i></td>
</tr>
<tr>
<td><i>Students will know…</i>
• There are many relationships that exist among trigonometric functions.
• The Fundamental Trigonometric Identities include the reciprocal identities, the quotient identities, the Pythagorean identities, the co-function identities, and the even/odd identities.
• Trigonometric relationships and algebraic procedures can be used to verify trigonometric identities.
• Algebraic techniques can be used to solve trigonometric equations.
• If an interval is not provided, general solution format can be used to write all solutions to a trigonometric equation.
• Some methods for solving trigonometric equations will result in extraneous solutions.
• Trigonometric equations can model real life situations.
• The sum and difference formulas can be used to find exact values of trigonometric functions involving the sums and differences of special angles.
• Sum and difference formulas can be used to prove trigonometric identities.
• Multiple angle formulas are used to rewrite and evaluate trigonometric functions which assists in solving trigonometric equations and analyzing trigonometric graphs.</td>
<td><i>Students will be skilled at…</i>
• Deriving and recognizing the fundamental trigonometric identities.
• Using the fundamental trigonometric identities to evaluate trigonometric functions.
• Using the fundamental trigonometric identities to simplify trigonometric expressions.
• Using algebraic methods in conjunction with trigonometric relationships to verify trigonometric identities.
• Using algebraic methods and trigonometric identities to solve trigonometric equations.
• Deriving all solutions for a trigonometric equation on a specified interval.
• Utilizing general solution format when no interval is specified for a trigonometric equation.
• Verifying the solutions of a trigonometric equation using graphing technology.
• Solving multiple angle trigonometric equations.
• Verifying if solutions are extraneous.
• Solving application problems which can be modeled with trigonometric equations.
• Finding the exact value of trigonometric expressions using sum</td>
</tr>
</table>

MP4 Model with mathematics. **MP5** Use appropriate tools strategically. **MP6** Attend to precision. **MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	• Half angle formulas are used to determine exact values.	and difference formulas or double or half angle formulas. • Applying the sum and difference formulas to prove trigonometric identities. • Applying the sum and difference formulas to solve trigonometric equations. • Using double-angle formulas to analyze graphs. • Evaluating trigonometric functions involving double angles. • Using half angle formulas to find exact values.

Stage Two: Evidence

Evaluative Criteria	Assessment Evidence
SUGGESTED PERFORMANCE RUBRIC: ***Use the following or similar rubric to evaluate a student's performance on performance tasks.*** 4 – Correct solution with all necessary work demonstrating complete understanding of the topic. Correct units are included. 3 – Solution contains one or two minor errors but overall demonstrates understanding of the topic. 2 – Several minor errors or one major error in solving the problem demonstrating some understanding of the topic. 1 – Major errors in solving the problem demonstrating very little understanding of the topic. 0 – No answer is provided or the work has nothing to do with the topic being covered.	**SUGGESTED PERFORMANCE ASSESSMENT:** ***Students will engage in the following performance task:*** • Find the mistake in the proof below. Describe the error, and correct the mistake. Verify: $\frac{\csc\theta}{\sin\theta}-\frac{\cot\theta}{\tan\theta}=1$ LHS = $\frac{\csc\theta}{\sin\theta}-\frac{\cot\theta}{\tan\theta} = \frac{\frac{1}{\sin\theta}}{\sin\theta}-\frac{\frac{1}{\tan\theta}}{\tan\theta} = \frac{1}{\sin\theta}\cdot\sin\theta-\frac{1}{\tan\theta}\cdot\tan\theta$ $= \frac{\sin\theta}{\sin\theta}-\frac{\tan\theta}{\tan\theta} = 1-1 = 0$ • Given the equation $\sin^2 x = 2\sin x$, explain what would happen if you divided each side of the equation by $\sin x$. Why this is an incorrect method to use when solving this type of equation? Create an appropriate strategy for solving this type of equation and check your results by graphing this function on the graphing calculator. • A rodent population in a particular region varies with the number of predators that inhabit

the region. At any time, you could predict the rodent population, $r(t)$, using the function $r(t) = 2500 + 1500\sin\left(\frac{\pi t}{4}\right)$, where *t* is the number of years that have passed since 1976. When would the population reach 1750 between the years 1976 and 1995?

- Explain how you would derive a formula for the area of an isosceles triangle using a double angle or a half angle formula. Use a labeled sketch to illustrate your derivation. Then write two examples that show how your formula can be applied.

OVERVIEW:

In this unit, students will learn how to use trigonometric relationships and formulas in order to evaluate or simplify trigonometric expressions, verify trigonometric identities, and solve trigonometric equations.

In order to formally assess students on these concepts, a quiz will be given on simplifying and verifying trigonometric expressions and identities, a quiz will be given on solving trigonometric equations, a partner assignment will be given on solving applications of trigonometric equations, and a test will be given on the entire unit. Sample assessment questions can be found above.

DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Relate the solutions of a trigonometric equation to its corresponding graph.
- Provide the opportunity for students to practice problems both with and without graphically calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems on the related topic for those students who finish early.

TECHNOLOGY:

- **Derivation of Pythagorean Identities** at the link below can be used as a guide when deriving the Pythagorean identities as a class: http://www.mathalino.com/reviewer/derivation-of-formulas/derivation-of-pythagorean-identities.
- Using the tab labelled **Pythagorean Identities** from ***ALGEBRA IN MOTION ™*** with Geometer's Sketchpad, have students visually verify the three different Trigonometric

	Pythagorean Identities. A link to download the document should be placed on the teacher's wiki so that students can follow along. • Houghton Mifflin Company: <u>Precalculus</u> 2008, Verifying a Trigonometric Identity on p. 354 example 3. Use a graphing calculator to determine whether or not an equation appears to be an identity using both numerical and graphical approaches when analytical approaches cannot be used. • To show the relationship between the graph of a trigonometric function and the solutions to the corresponding equation, have students solve a trigonometric equation and use a graphing device to verify the solutions. For example, have students solve $2\sin x - 1 = 0$ and find where the graph of $y = \sin x$ is equal to ½. • Houghton Mifflin Company: <u>Precalculus</u> 2008, Surface Area of a Honeycomb on p. 375. Go through how to find the angle that gives a specific surface area and how to find the angle that gives the minimum surface area. Approximate the solutions using graphing technology. • In pairs or small groups, have students complete **Applications of Solving Trigonometric Equations Project**. Students will model real life situations with trigonometric equations and solve the trigonometric model, analytically, numerically, and graphically with the help of a graphing device. • The Free Graphing Calculator app on the iPad can be used as a reference for students to view the different trigonometric identities. Go to the app Free Graphing Calculator → Reference → Trigonometry → Trig Functions in terms of Sine and Cosine / Pythagorean Identities / Angle Addition Formulas / Angle Subtraction Formulas / Double Angle Formulas / Half Angle Formulas
SUGGESTED MONITORING SCALE: ***Use the following or similar scale to monitor or evaluate a student's daily learning and understanding of key concepts:*** 4 – I understand the concept completely and can explain it to a classmate. 3 – I understand the concept but would not feel comfortable explaining it to a classmate. 2 – I can complete the problem with assistance from the teacher. 1 – I do not know where to begin; I need help.	**OTHER SUGGESTED PERFORMANCE ASSESSMENTS:** • Exploration: Getting Past the Obvious but Incorrect Formulas a. Let $u = \pi$ and $v = \frac{\pi}{2}$. Find $\sin(u+v)$. Find $\sin(u)+\sin(v)$. Does $\sin(u+v) = \sin(u)+\sin(v)$? b. Let $u = 0$ and $v = 2\pi$. Find $\cos(u+v)$. Find $\cos(u)+\cos(v)$. Does $\cos(u+v) = \cos(u)+\cos(v)$? c. Find your own values of *u* and *v* that will confirm that $\tan(u+v) \neq \tan(u)+\tan(v)$.

Stage Three: Learning Plan

Summary of Key Learning Events and Instruction

SUGGESTED LEARNING EVENTS:

- Derive the different fundamental trigonometric identities as a class. Use the following source as a guide for deriving the Pythagorean identities: http://www.mathalino.com/reviewer/derivation-of-formulas/derivation-of-pythagorean-identities.
- Show a visual verification of the Pythagorean identities with the tab labelled **Pythagorean Identities** from ***ALGEBRA IN MOTION ™.*** Students can follow along or view on their own time if a link is provided on the teacher's wiki.
- Houghton Mifflin Company: Precalculus 2008, Verifying a Trigonometric Identity on p. 354 example 3. Use a graphing calculator to determine whether or not an equation appears to be an identity using both numerical and graphical approaches when analytical approaches cannot be used.
- To show the relationship between the graph of a trigonometric function and the solutions to the corresponding equation, have students solve a trigonometric equation and use a graphing device to verify the solutions. For example, have students solve $2\sin x - 1 = 0$ and find where the graph of $y = \sin x$ is equal to ½.
- Houghton Mifflin Company: Precalculus 2008, Surface Area of a Honeycomb on p. 375. Go through how to find the angle that gives a specific surface area and how to find the angle that gives the minimum surface area. Approximate the solutions using graphing technology.
- In pairs or small groups, have students complete **Applications of Solving Trigonometric Equations Project.** Students will model real life situations with trigonometric equations and solve the trigonometric model, analytically, numerically, and graphically with the help of a graphing device.
- Have students complete **Exploration: Getting Past the Obvious but Incorrect Formulas** to show that the obvious formulas for sum and difference of two angles does not work.
- Derive the double angle formulas. Proofs can be found in Houghton Mifflin Company: Precalculus 2008, Proofs in Mathematics, p. 405.

SUGGESTED METHODS OF DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Relate the solutions of a trigonometric equation to its corresponding graph.
- Provide the opportunity for students to practice problems both with and without graphically calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems on the related topic for those students who finish early.

UNIT 4 – Oblique Triangles

Stage One: Desired Results

ESTABLISHED GOALS

HSG-SRT.8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

HSG-SRT.9. (+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

HSG-SRT.10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.

HSG-SRT.11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

HSG-MG.3. Apply geometric methods to solve design problems.

MATHEMATICAL PRACTICES

MP1 Make sense of problems and persevere in solving them.

MP2 Reason abstractly and quantitatively.

MP3 Construct viable arguments and critique the reasoning of others.

MP4 Model with mathematics.

Transfer

Students will be able to independently use their learning to…

Apply trigonometric formulas to model and solve real life problems involving oblique triangles.

Meaning

UNDERSTANDINGS

Students will understand that…

- The Law of Sines and the Law of Cosines can be used to solve for missing sides and angles of oblique triangles.
- In order to solve an oblique triangle, you need to know the measure of at least one side and the measures of any two other parts of the triangle – two sides, two angles, or one angle and one side.
- When two sides of a triangle and an angle opposite one of them are given and a second angle is to be derived, the resulting triangle may not be unique, or even exist at all and thus this situation results in ambiguity.
- In surveying and navigation, directions are generally given in terms of bearings.
- Trigonometry can be used to model many real life situations involving navigation and surveying.

ESSENTIAL QUESTIONS

- How do you use trigonometry to solve and find the areas of oblique triangles?
- What situations create the ambiguous case for the Law of Sines?
- What is a directional bearing and how is it applied to real life situations?
- How do you use trigonometric functions to solve real life problems?

Acquisition

Students will know…

- An oblique triangle is a triangle that does not have a right angle.
- The Law of Sines proportion, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, can be used to

Students will be skilled at…

- Solving for missing sides and angles of oblique triangles using the Law of Sines and the Law of Cosines.
- Analyzing given information to determine whether to use the Law of Sines or Law of Cosines when solving

MP5 Use appropriate tools strategically. **MP6** Attend to precision. **MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	solve for missing sides or angles of a triangle given two angles and a side or two sides and an angle opposite one of those sides. • The Law of Cosines formula, $c^2 = a^2 + b^2 - 2ab\cos C$, can be used to solve for the missing side of a triangle when given two sides and their included angle. • The Law of Cosines formula, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, can be used to solve for the missing angles of a triangle when given all three sides. • The area of an oblique triangle is one-half the product of the lengths of two sides times the sine of the included angle. • A bearing measures the angle a path or line of sight makes with a fixed north-south line. • Trigonometry and directional bearings can be used to solve many real life application problems, including those that involve navigation and surveying.	for missing sides and angles of an oblique triangle. • Identifying scenarios that result in the ambiguous case. • Synthesizing given information in a problem and past trigonometric knowledge to determine if no triangle exists, one triangle exists, or two triangles exist. • Calculating the area of an oblique triangle. • Finding directions in terms of bearings. • Sketching a diagram to model real-life applications of oblique triangles, including surveying and navigation problems. • Solving applications of oblique triangles, including surveying and navigation problems.

Stage Two: Evidence

Evaluative Criteria	Assessment Evidence
SUGGESTED PERFORMANCE RUBRIC: ***Use the following or similar rubric to evaluate a student's performance on performance tasks.*** 4 – Correct solution with all necessary work demonstrating complete understanding of the topic. Correct units are included. 3 – Solution contains one or two minor errors	**SUGGESTED PERFORMANCE ASSESSMENT:** ***Students will engage in the following performance task***: • Let ΔABC be the triangle defined by $a = 55$, $c = 20$, and $A = 110°$. Explain why this triangle produces the ambiguous case and determine how many triangles exist with the given information, if any. Justify your answer using past knowledge of trigonometric values and the Law of Sines. • A pilot has just started his decent for landing at an airport with a runway of length 9,500 feet. The angles of depression from the plane to the ends of the runway are $16.2°$ and

but overall demonstrates understanding of the topic.

2 – Several minor errors or one major error in solving the problem demonstrating some understanding of the topic.

1 – Major errors in solving the problem demonstrating very little understanding of the topic.

0 – No answer is provided or the work has nothing to do with the topic being covered.

17.8°.

a. Draw a diagram that visually represents the problem.

b. Find the air distance the plane must travel until touching down on the near end of the runway.

c. Find the ground distance the plane must travel until touching down.

d. Find the altitude of the plane when the pilot begins the descent.

- Chrissy was hired to investigate the deer population in a certain national park. Because deer require food, water, cover for protection from the weather and predators, and living space for healthy survival, there are natural limits to the number of deer that a given plot of land can support. Deer populations in national parks average 14 animals per square kilometer. If a triangular region with sides of 3 km, 4km, and 6km have a population of 50 deer in the park Chrissy was sent to investigate, how close is the population on this land to the average national park population? Show all work to justify your answer.
- You work for the state tax department and there is a plot of land that needs to be assessed for tax purposes. In order to properly assess the land, you must know the total area of the plot. You send a surveyor out to determine the area of the land with the following instructions:
  From the granite post on Tasker Hill Road, proceed 195 feet due east, then change direction along a bearing of $S32^\circ E$ for 260 feet, then along a bearing of $S68^\circ W$ for 385 feet, and then finally along a straight line back to the granite post.
  a. Sketch a map based on the parameters given for the plot of land. All key points on the map must be clearly labeled in accordance with the information given in the problem.
  b. Determine the area of the plot of land using your knowledge on trigonometry and directional bearings. Justify all calculations.

OVERVIEW:

In this unit, students will learn how to use the Law of Sines and Law of Cosines to solve for missing sides and angles of oblique triangles. They will use then these formulas to solve real life problems involving oblique triangles and their area.

In order to assess students' understanding of the concepts in this unit, students will be given a group assignment on applications of oblique triangles and a quest on the entire unit. Sample assessment questions can be found above.

DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show analytic and visual proofs of the Law of Sines and Law of Cosines so students understand where each of these formulas comes from and how they relate to solving right

triangles.

- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems on the related topic to those students who finish early.

TECHNOLOGY:

- Use the tabs titled **Law of Sines** and **Law of Cosines** in ***ALGEBRA IN MOTION ™*** with Geoemeter's Sketchpad to visually prove these relationships.
- The following website can be used as a resource to teachers and students for the Law of Sines, Law of Cosines, the ambiguous case, and their derivations: http://www.regentsprep.org/regents/math/algtrig/att12/indexATT12.htm.
- The videos the link below can be used to help flip the classroom or provide additional instruction to struggling students: https://www.khanacademy.org/math/trigonometry/less-basic-trigonometry/law-sines-cosines
- The Free Graphing Calculator app on the iPad can be used as a reference for students to view the Law of Sine and Law of Cosine Formulas. Go to the app Free Graphing Calculator → Reference → Trigonometry → Law of Sines / Law of Cosines

SUGGESTED MONITORING SCALE:

Use the following or similar scale to monitor or evaluate a student's daily learning and understanding of key concepts:

4 – I understand the concept completely and can explain it to a classmate.

3 – I understand the concept but would not feel comfortable explaining it to a classmate.

2 – I can complete the problem with assistance from the teacher.

1 – I do not know where to begin; I need help.

OTHER SUGGESTED PERFORMANCE TASKS:

- Show that there are infinitely many triangles with AAA.
- Explain why the Law of Sines also works for right triangles.
- Find the area of a regular octagon inscribed inside a circle of radius 9 inches.
- Find the radian measure of the largest angle of the triangle whose sides have lengths 8, 9, and 10.

Stage Three: Learning Plan

Summary of Key Learning Events and Instruction

SUGGESTED LEARNING EVENTS:

- Introduce the Law of Sines. Prove the formula both analytically and visually. For the analytic proof, use the resource: http://www.regentsprep.org/regents/math/algtrig/att12/derivelawofsines.htm and Houghton Mifflin Company: Precalculus 2008, Proofs in Mathematics on p. 468. For the visual proof, use the tab titled **Law of Sines** in ***ALGEBRA IN MOTION ™*** with Geoemeter's Sketchpad.
- Go through the ambiguous case and the different possibilities. Use the following website as a possible resource: http://www.regentsprep.org/regents/math/algtrig/att12/lawofsinesAmbiguous.htm.
- Introduce the Law of Cosines. Prove the formulas both analytically and visually. For the analytic proof, use the resource: http://www.regentsprep.org/regents/math/algtrig/att12/derivelawofsines.htm and Houghton Mifflin Company: Precalculus 2008, Proofs in Mathematics on p. 469. For the visual proof, use the tab titled **Law of Cosines** in ***ALGEBRA IN MOTION ™*** with Geoemeter's Sketchpad.
- In small groups, have students work on applications of the Law of Sines, Law of Cosines, and the area formulas. Assign each group a problem to be the expert on and have them create a poster displaying their solution. When group have had enough time to try all the problems, have them walk around the classroom checking their answers with the posters created by the other groups. Discuss questions as a class led by the expert group.

SUGGESTED METHODS OF DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show analytic and visual proofs of the Law of Sines and Law of Cosines so students understand where each of these formulas comes from and how they relate to solving right triangles.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems on the related topic to those students who finish early.

UNIT 5 – Vectors

Stage One: Desired Results

ESTABLISHED GOALS

HSN-VM.1. (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., $\boldsymbol{v}$, $|\boldsymbol{v}|$, $||\boldsymbol{v}||$, v).

HSN-VM.2. (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

HSN-VM.3. (+) Solve problems involving velocity and other quantities that can be represented by vectors.

HSN-VM.4. (+) Add and subtract vectors.

Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

Understand vector subtraction $\boldsymbol{v} - \boldsymbol{w}$ as $\boldsymbol{v} + (-\boldsymbol{w})$, where $-\boldsymbol{w}$ is the additive inverse of $\boldsymbol{w}$, with the same magnitude as $\boldsymbol{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate

Transfer

Students will be able to independently use their learning to…

Model real life quantities that have both magnitude and direction with vectors and use operations of vectors in the plane and in space to solve problems involving these quantities.

Meaning

UNDERSTANDINGS

Students will understand that…

- Quantities such as force and velocity can be measured by vectors since they have both magnitude and direction.
- The component form represents a family of vectors with a variety of initial and terminal points.
- It is possible to add and subtract vectors numerically and geometrically.
- The scalar multiple changes the size of the vectors if the scalar is positive; if the scalar multiple is negative, both the size and direction of the vector is changed.
- The direction angle of a vector is generated from counterclockwise revolution of a given unit vector from the positive x-axis when the terminal point of the given vector lies on the unit circle.
- The dot product includes both multiplication and addition and is used in applications such as determining the angle between two vectors and proving two vectors are orthogonal.
- Many operations and procedures which can be applied in the two dimensional coordinate system are applicable in three dimensional space.

ESSENTIAL QUESTIONS

- How do you represent and perform operations with vector quantities?
- What is the dot product and how is it used to analyze vectors?
- How can vectors be used to model real life situations?
- How do you locate points, find distances, and graph equations in three dimensions?
- How do you describe, compare, and solve problems involving vectors in space?
- What types of applications of three dimensional vectors can be solved using the cross product or triple scalar product?

order, and perform vector subtraction component-wise.

HSN-VM.5. (+) Multiply a vector by a scalar.

Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.

Compute the magnitude of a scalar multiple $c\boldsymbol{v}$ using $||c\boldsymbol{v}|| = |c|\boldsymbol{v}$. Compute the direction of $c\boldsymbol{v}$ knowing that when $|c|\boldsymbol{v} \neq 0$, the direction of $c\boldsymbol{v}$ is either along $\boldsymbol{v}$ (for $c > 0$) or against $\boldsymbol{v}$ (for $c < 0$).

HSN-VM.12. (+) Work with 2 × 2 matrices as a transformations of the plane, and interpret the absolute value of the determinant in terms of area.

MATHEMATICAL PRACTICES

MP1 Make sense of problems and persevere in solving them.

MP2 Reason abstractly and quantitatively.

MP3 Construct viable arguments and critique the reasoning of others.

MP4 Model with mathematics.

MP5 Use appropriate tools strategically.

- Many applications in physics and engineering involve finding a vector in space that is orthogonal to two given vectors.
- The cross product of two vectors is useful in determining the equation of a plane and the area of geometric figures.
- The cross product and the dot product are used in a unique relationship to determine the volume of certain figures in three-dimensional space.
- Normal vectors in a plane are useful for determining the equation of a plane in three-space.

Acquisition

Students will know…

- A vector is the set of all directed line segments with the same magnitude and direction.
- The distance formula can be used to find the magnitude, or length, of a vector.
- A vector in standard position has its initial point at the origin and is the most convenient way to represent a directed line segment.
- A vector in the plane in standard form with terminal point $\left(v_1, v_2\right)$ can be written in component form as $\left\langle v_1, v_2 \right\rangle$.
- Vectors can be added, subtracted, and multiplied by a scalar, which can be performed both numerically and geometrically.
- A unit vector has a magnitude of 1.
- To find a unit vector in the same direction as a given vector, divide both components by the given vector's

Students will be skilled at…

- Analyzing vectors to determine whether or not they are equivalent.
- Calculating the magnitude of a vector.
- Determining the direction of a vector.
- Writing vectors in component form given an initial and terminal point.
- Adding, subtracting, and multiplying vectors by scalars both numerically and geometrically.
- Finding a unit vector in the same direction as a given vector.
- Proving a unit vector has a magnitude of 1.
- Writing vectors as a linear combination of the standard unit vectors.
- Finding the direction angle of a vector.
- Solving real-life applications of vectors.
- Calculating the dot product of a vector.
- Finding the angle between two vectors and determining whether or not the two vectors are orthogonal.

MP6 Attend to precision. **MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	original magnitude. • Vectors can also be written as a linear combination of the standard unit vectors. • Vectors can be used to create direction angles. • The dot product of vectors yields a scalar rather a vector. • Two nonzero vectors are orthogonal if their dot product is zero. • Two dimensional vectors can be used to model many real life situations, such as force and velocity. • Two nonzero vectors are parallel if they are scalar multiples of the same vector. • The cross product of two vectors in space can be found using the determinant of the corresponding three-dimensional matrix. • The tripe scalar product of three vectors in space can be used to find the volume of a parallelepiped.	• Plotting points in the three-dimensional coordinate system. • Finding distances between points in space and finding midpoints of line segments joining points in space. • Writing equations of spheres in standard form and finding the traces of surfaces in space. • Finding the component form of a vector in space. • Performing operations of vectors in space including addition, subtracting, scalar multiplication, and finding the dot product. • Analyzing two vectors in space to determine if they are parallel, perpendicular, or neither. • Finding the cross product of two vectors in space. • Using geometric properties of the cross product of two vectors in space. • Finding the triple scalar product of three vectors in space by finding the dot product of one vector with the cross product of the other two. • Using determinants to solve geometric problems. • Determining the volume of a parallelepiped formed by three vectors by calculating the triple scalar product.

Stage Two: Evidence

Evaluative Criteria	Assessment Evidence
SUGGESTED PERFORMANCE RUBRIC: ***Use the following or similar rubric to evaluate a student's performance on performance tasks.***	**SUGGESTED PERFORMANCE ASSESSMENT:** ***Students will engage in the following performance tasks***: • Find a unit vector in the direction as $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j}$.

4 – Correct solution with all necessary work demonstrating complete understanding of the topic. Correct units are included.
3 – Solution contains one or two minor errors but overall demonstrates understanding of the topic.
2 – Several minor errors or one major error in solving the problem demonstrating some understanding of the topic.
1 – Major errors in solving the problem demonstrating very little understanding of the topic.
0 – No answer is provided or the work has nothing to do with the topic being covered.

- A baseball is hit in the air at a 40° angle with the horizontal with an initial speed of 110 mph.
    a. Find the component form of the initial velocity.
    b. Give an interpretation of the horizontal and vertical components of the velocity.
- Given a force of 45 pounds and 60 pounds, and the magnitude of their resultant force is 90 pounds, find the angle between the forces.
- A pilot’s flight itinerary requires her flying from Newark directly west. There is a 25 mph wind with a bearing of 165°. If the pilot flies at a speed of 450 mph before the wind resistance, what is the compass bearing that the pilot should follow to get to the destination required? What will her ground speed be?
- Determine whether vectors $\mathbf{u} = \left\langle \frac{3}{4}, -\frac{1}{2}, 2 \right\rangle$ and $\mathbf{v} = \langle 4,10,1 \rangle$ are orthogonal, parallel, or neither. Justify your answer algebraically.
- Prove that the quadrilateral with vertices $(4,6,1)$, $(2,3,5)$, $(2,4,5)$, and $(4,7,1)$ is a parallelogram. Then, find its area. Is the parallelogram a rectangle?

OVERVIEW:
In this unit, students will learn what vectors are, how to perform operations with two and three dimensional vectors, and how to use vectors to model and solve real-life situations that involve quantities with both magnitude and direction.

To formally assess students on this unit, a quiz will be given on two dimensional vectors and a test will be given on the entire unit. Sample assessment questions can be found above.

DIFFERENTIATION:
- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show how to perform operations with vectors both numerically and geometrically.
- Show students how to solve problems both by hand and with the help of technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems to those students that finish early.

	TECHNOLOGY: • The videos the link below can be used to help flip the classroom or provide additional instruction to struggling students: https://www.khanacademy.org/math/precalculus/vectors-precalc • The following website can be used as a resource to teachers and students about vectors and vector operations in two dimensions: https://www.mathsisfun.com/algebra/vectors.html • The following website can be used as a resource to teachers and students for two- and three-dimensional vectors with interactive components: http://www.intmath.com/vectors/vectors-intro.php • The following website can be used as a resource to teachers and students for two- and three-dimensional vectors. There are applets that will allow students to visually explore vectors: http://mathinsight.org/vectors_cartesian_coordinates_2d_3d • The Free Graphing Calculator app on the iPad can be used as a reference for students to view information about vectors. Go to the app Free Graphing Calculator → Reference → Vectors. There are the following categories: - What's a vector - Length of a Vector - Vector Addition - What's a Scalar - Multiplying a Vector by a Scalar - Dot Product - Geometric Interpretation of Dot Product - Cross Product Geometric Definition - Cross Product Definition with Coordinates - Cross Product Determinant Definition
SUGGESTED MONITORING SCALE: ***Use the following or similar scale to monitor or evaluate a student's daily learning and understanding of key concepts:*** 4 – I understand the concept completely and can explain it to a classmate. 3 – I understand the concept but would not feel comfortable explaining it to a classmate. 2 – I can complete the problem with assistance from the teacher. 1 – I do not know where to begin; I need help.	**OTHER SUGGESTED PERFORMANCE TASKS:** • Ship A is 50 nautical miles west of ship B which is heading south 8 knots (8 nautical miles per hour). If A is to meet B in 3 hours, what should be the course and the speed of ship A? Solve this problem 2 ways: a. by creating a diagram using vectors b. by using right triangle trigonometry • Exploration: (Houghton Mifflin Company: Precalculus 2008, Exploration, p. 760) If you connect the terminal points of two vectors **u** and **v** that have the same initial points, a triangle is formed. Is it possible to use the cross product $\mathbf{u} \times \mathbf{v}$ to determine the area of the triangle? Explain. Verify your conclusion using two vectors.

Stage Three: Learning Plan

Summary of Key Learning Events and Instruction

SUGGESTED LEARNING EVENTS:

- Pre-assess students on this unit. Students who have taken Physics should have prior knowledge of vectors.
- Consider using some of the resources above to introduce the basics of vectors and vector vocabulary before beginning the unit so that class time can be spent practicing problems.
- Go through how to perform vector operations algebraically and geometrically. Relate the two methods.
- Relate vectors to right triangle trigonometry and oblique triangles using problems similar to the one below:
  Ship A is 50 nautical miles west of ship B which is heading south 8 knots (8 nautical miles per hour). If A is to meet B in 3 hours, what should be the course and the speed of ship A? Solve this problem 2 ways:
  a. by creating a diagram using vectors
  b. by using right triangle trigonometry
- Put students in pairs or small groups to solve applications of vectors similar to those in Houghton Mifflin Company: Precalculus 2008, Applications of Vectors p. 431-432 Examples 8-10. Discuss as a class.
- Go through how to perform operations with three-dimensional vectors. Use the resources listed below to help students visualize the three-dimensional space.
- Exploration: Cross Products (Houghton Mifflin Company: Precalculus 2008, Exploration, p. 760)

SUGGESTED METHODS OF DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show how to perform operations with vectors both numerically and geometrically.
- Show students how to solve problems both by hand and with the help of technology.
- As sign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems to those students that finish early.

UNIT 6 – Polar Form of Complex Numbers & Graphs

Stage One: Desired Results

ESTABLISHED GOALS

HSN-CN.4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

HSN-CN.5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,* $(-1 + \sqrt{3}\, i)^3 = 8$ *because* $(-1 + \sqrt{3}\, i)$ *has modulus 2 and argument 120°.*

HSN-CN.6. (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

MATHEMATICAL PRACTICES

MP1 Make sense of problems and persevere in solving them.

MP2 Reason abstractly and quantitatively.

MP3 Construct viable arguments and critique the reasoning of others.

MP4 Model with mathematics.

Transfer

Students will be able to independently use their learning to…
Use trigonometric relationships to switch between polar and rectangular form of complex numbers and coordinates and create unique designs using polar graphing techniques.

Meaning

UNDERSTANDINGS
Students will understand that…

- Complex numbers can be modeled in standard (rectangular) form or trigonometric (polar) form.
- Trigonometric form of complex numbers can be used to perform operations such as multiplication, division, raising a complex number to a power, and taking nth roots of a complex number more easily.
- There is a relationship between DeMoivre's Theorem and the multiplication process for complex numbers in trigonometric form.
- Relationships exist between factoring, the quadratic formula, and the root formula for complex numbers.
- Polar coordinate representation of a point is not unique.
- The synthesis of algebraic and trigonometric concepts is necessary when transforming polar equations to rectangular form.

ESSENTIAL QUESTIONS

- How do you represent and perform operations on complex numbers by using trigonometry?
- How can patterns be used to derive DeMoivre's Theorem?
- How can you use the Fundamental Theorem of Algebra to predict the roots of complex numbers?
- How do you write equations to describe the motion of a point in a plane?
- How do you describe the position of a point in a plane using its distance and angle?
- How do can you use the graphs of polar equations to create a unique design?

Acquisition

Students will know…

- The absolute value of a complex number, $z = a + bi$, is the distance

Students will be skilled at…

- Plotting complex numbers on a complex plane.

MP5 Use appropriate tools strategically. **MP6** Attend to precision. **MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	between the origin (0, 0) and the point (a, b). • Trigonometry can be used to switch between standard and trigonometric form of a complex number. • To multiply two complex numbers together, multiply the radii and add the angles. • To divide two complex numbers, divide the radii and subtract the angles. • DeMoivre's Theorem can be used to raise a complex number to a power by raising the radius to the power and then multiplying the angle by the power. • There are *n* nth roots of a complex number that are equally spaced throughout the unit circle. • Polar coordinates are written as $\left(r,\theta\right)$, where *r* is the directed distance from the origin to the point and θ is the directed angle. • Trigonometry can be used to switch between polar and rectangular form of coordinates. • Polar equations can be converted to rectangular form. • Polar equations can be graphed by hand and by using graphing technology.	• Determining the absolute value of a complex number. • Using trigonometry to rewrite a complex number from standard or rectangular form to trigonometric or polar form. • Using trigonometry to rewrite a complex number from trigonometric or polar form to standard or rectangular form. • Multiplying complex numbers in polar form. • Dividing complex numbers in polar form. • Applying DeMoivre's Theorem to raise a complex number to a power. • Finding the nth roots of a complex number. • Modeling the roots of a complex number on a complex plane. • Plotting points in the polar coordinate system. • Finding multiple representations of points in the polar coordinate system. • Converting points from rectangular to polar form and vice versa. • Converting equations from rectangular to polar form and vice versa. • Graphing polar equations by creating a table of values. • Recognizing special polar graphs. • Using graphing calculator technology and Geometer's Sketchpad to graph polar equations.

Stage Two: Evidence

Evaluative Criteria	**Assessment Evidence**

SUGGESTED PERFORMANCE RUBRIC:
Use the following or similar rubric to evaluate a student's performance on performance tasks.

4 – Correct solution with all necessary work demonstrating complete understanding of the topic. Correct units are included.
3 – Solution contains one or two minor errors but overall demonstrates understanding of the topic.
2 – Several minor errors or one major error in solving the problem demonstrating some understanding of the topic.
1 – Major errors in solving the problem demonstrating very little understanding of the topic.
0 – No answer is provided or the work has nothing to do with the topic being covered.

SUGGESTED PERFORMANCE ASSESSMENT:
Students will engage in the following performance tasks:

- Let $z_1 = 2 - 2\sqrt{3}i$ and $z_2 = -1 - i$.
  a. Convert both numbers to polar form.
  b. Find $z_1 \cdot z_2$.
  c. Find $\frac{z_2}{z_1}$.
  d. Find $\left(z_1\right)^7$.
  e. Find the fourth roots of z_2.
- Find three different representations for the point $\left(-2, \frac{7\pi}{4}\right)$.
- Convert the polar equation to rectangular form: $r = 3\sin\theta$.
- Without using a graphing calculator, describe what the graph of $y = 3\cos 4\theta$ will look like. Explain.

OVERVIEW:
In this unit, students will learn how to relate rectangular and polar form of complex numbers and coordinates and the benefits of using one form over the other in different circumstances.

Students will be formally assessed on this unit through the Polar Graphing Project described below and a quest on the entire unit. Sample assessment questions are listed above.

DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show students how to solve problems both with and without graphing calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- The Polar Graphing Project is open ended and will allow stronger students to challenge themselves by creating more complex equations.

- Assign challenge problems related to the unit for students that finish early.

TECHNOLOGY:

- Exploration of multiple representations of polar coordinates – see below for more details. (Houghton Mifflin Company: Precalculus 2008, Exploration, p. 709)
- Discovery Activity: Polar Graphs & Equations – students will use a graphing calculator to draw conclusions on how polar graphs are related to their equations. See below for more details.
- For the independent **Polar Graphing Project,** students will graph a variety of different polar equations by hand using a table of values and will write their own polar equations with certain specifications and use Geometer's Sketchpad to create a unique design. For this activity, students will need time in class to use the laptops with Geometer's Sketchpad.

SUGGESTED MONITORING SCALE:

Use the following or similar scale to monitor or evaluate a student's daily learning and understanding of key concepts:

4 – I understand the concept completely and can explain it to a classmate.
3 – I understand the concept but would not feel comfortable explaining it to a classmate.
2 – I can complete the problem with assistance from the teacher.
1 – I do not know where to begin; I need help.

OTHER SUGGESTED PERFORMANCE TASKS:

- Let $z_1 = 3+3i$ and $z_2 = 1-\sqrt{3}i$.
  a. Using the standard form of these numbers, find $\frac{z_1}{z_2}$.
  b. Write the trigonometric forms of the complex numbers.
  c. Using the trigonometric form of these numbers, find $\frac{z_1}{z_2}$.
  d. Are your answers from part b and c equivalent?
- Exploration: (Houghton Mifflin Company: Precalculus 2008, Exploration, p. 709)
  Set your graphing utility to polar mode. Then graph the equation $r = 3$. Use a viewing window in which $0 \le \theta \le 2\pi$, $-6 \le x \le 6$, and $-4 \le y \le 4$. You should obtain a circle of radius 3.
  a. Use the trace feature to cursor around the circle. Can you locate the point $\left(3, \frac{5\pi}{4}\right)$?
  b. Can you locate other representations of the point $\left(3, \frac{5\pi}{4}\right)$? If so, explain how you did it.
- Discovery Activity: Polar Graphs & Equations
  Graph using a graphing calculator. Describe the graph. Compare and contrast where appropriate.
  a. $r = 3\cos\theta$

b. $r = 6\sin\theta$
c. $r = 2\cos 2\theta$
d. $r = 4\sin 2\theta$
e. $r = 2\sin 3\theta$
f. $r = 5 - 5\sin\theta$
g. $r = 1 + 4\cos\theta$
h. $r = 1 - 2\sin\theta$
i. $r^2 = 9\sin 2\theta$
j. $r^2 = 4\cos 2\theta$

Stage Three: Learning Plan

Summary of Key Learning Events and Instruction

SUGGESTED LEARNING EVENTS:

- Relate performing operations with complex numbers in standard form with performing operations with complex numbers in trigonometric form to show they are equivalent by assigning problems such as:

  Let $z_1 = 3 + 3i$ and $z_2 = 1 - \sqrt{3}i$.

  a. Using the standard form of these numbers, find $\frac{z_1}{z_2}$.

  b. Write the trigonometric forms of the complex numbers.

  c. Using the trigonometric form of these numbers, find $\frac{z_1}{z_2}$.

  d. Are your answers from part b and c equivalent?
- Exploration: Multiple Representations of Polar Coordinates (Houghton Mifflin Company: Precalculus 2008, Exploration, p. 709)
- Discovery Activity: Polar Graphs & Equations described above.
- Polar Graphing Project: students will graph polar equations by hand using a table of values and will write their own polar equations with certain specifications and use Geometer's Sketchpad to create a unique design. For this activity, students will need time in class to use the laptops with Geometer's Sketchpad.

SUGGESTED METHODS OF DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show students how to solve problems both with and without graphing calculator technology.

- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- The Polar Graphing Project is open ended and will allow stronger students to challenge themselves by creating more complex equations.
- Assign challenge problems related to the unit for students that finish early.

UNIT 7 – Parametric Equations

Stage One: Desired Results

ESTABLISHED GOALS

HSA-SSE.2. Use the structure of an expression to identify ways to rewrite it.

HSA-SSE.3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

HSF.IF.C.9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum*.

MATHEMATICAL PRACTICES

MP1 Make sense of problems and persevere in solving them.

MP2 Reason abstractly and quantitatively.

MP3 Construct viable arguments and critique the reasoning of others.

MP4 Model with mathematics.

MP5 Use appropriate tools strategically.

MP6 Attend to precision.

Transfer

Students will be able to independently use their learning to…

Determine when an object will reach a certain location through the use of parametric equations.

Meaning

UNDERSTANDINGS

Students will understand that…

- By introducing a parameter, you can tell when an object was at a given point $\left(x, y\right)$ on a path.
- When a parameter is eliminated, it may be necessary to restrict the domain of the resulting rectangular equation to satisfy the domain of the original parametric equation.
- The set of parametric equations for a given graph is not unique.
- In space, vectors are often used to determine the equation of a line.
- Normal vectors in a plane are useful for determining the equation of a plane in three-dimensional space.

ESSENTIAL QUESTIONS

- How do you write equations to describe the motion of a point in a plane?
- How do you describe and compare lines and planes in space?

Acquisition

Students will know…

- Parametric equations are used to describe motion in the plane.
- Relationships can be used to rewrite parametric equations by eliminating the parameter.
- Determining the parametric equations for a given graph does not result in a unique situation.
- Parametric and symmetric equations exist for lines and planes in space

Students will be skilled at…

- Evaluating sets of parametric equations for given values of the parameter.
- Graphing curves that are represented by sets of parametric equations both by hand and with graphing technology.
- Rewriting sets of parametric equations as single rectangular equations by eliminating the parameter.
- Finding sets of parametric equations

MP7 Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	• A normal vector is perpendicular to the plane and is useful in writing the equation of a plane in space.	for graphs. • Finding parametric and symmetric equation of lines in space. • Finding equations of planes in space. • Sketching planes in space. • Finding distances between points and planes in space.

Stage Two: Evidence

Evaluative Criteria	Assessment Evidence
SUGGESTED PERFORMANCE RUBRIC: ***Use the following or similar rubric to evaluate a student's performance on performance tasks.*** 4 – Correct solution with all necessary work demonstrating complete understanding of the topic. Correct units are included. 3 – Solution contains one or two minor errors but overall demonstrates understanding of the topic. 2 – Several minor errors or one major error in solving the problem demonstrating some understanding of the topic. 1 – Major errors in solving the problem demonstrating very little understanding of the topic. 0 – No answer is provided or the work has nothing to do with the topic being covered.	**SUGGESTED PERFORMANCE ASSESSMENT:** ***Students will engage in the following performance tasks***: • Create a table of values and sketch the graph of the parametric equations below. Include orientation. $x = 2 - \cos\theta$ $y = 3 - \sin\theta$, for $\theta\left[0, 2\pi\right]$ • Eliminate the parameter to write the parametric equations as a rectangular equation: $x = 6 - t$ $y = \sqrt{3t - 4}$ • Suppose an object which is moving with a constant velocity is at point *A* (5, 3) when time *t* = 0 seconds and point *B* (-4, 15) when time *t* = 3 seconds. a. Find the velocity and speed of the object. b. Find a vector equation that describes the motion of the object. • (Houghton Mifflin Company: Precalculus 2008, p. 706 #58) To begin a football game, a kicker kicks off from his team's 35-yard line. The football is kicked at an angle of 50° with the horizontal at an initial velocity of 85 feet per second. a. Write a set of parametric equations for the path of the kick. b. Use a graphing calculator to graph the path of the kick and approximate its maximum height. c. Use a graphing calculator to find the horizontal distance that the kick travels. d. Explain how you could find the result to part (c) algebraically? • Find a set of parametric equations and symmetric equations that are parallel to $\langle 3, -7, -10 \rangle$ and go through the point $(3, -5, 1)$.

	- Find the general equation of the plane passing through the points $(2,1,5)$, $(0,4,5)$, and $(-2,1,8)$. **OVERVIEW:** In this unit, students will learn how to use parameters to determine when an object will reach a specific location. Students will then use parametric equations and what they learned about vectors to describe lines in space. To formally assess student on this unit, a quiz will be given on the entire unit. Sample assessment questions are listed above. **DIFFERENTIATION:** - Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material. - Show students how to create tables of values and graphs of parametric equations by hand and with graphing calculator technology. - Assign problems at varying levels of difficulty to challenge all learners. - Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material. - Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed. - Assign challenge problems related to the unit for students that finish early. **TECHNOLOGY:** - As an introduction to parametric equations, have students read the article "A Quick Intuition for Parametric Equations" by Kalid Azad. The article describes the importance of the parameter in different circumstances. The article can be found at the following link: http://betterexplained.com/articles/a-quick-intuition-for-parametric-equations/ - The videos at the link below can be used to help flip the classroom or provide additional instruction to struggling students: https://www.khanacademy.org/math/precalculus/parametric_equations/parametric/v/parametric-equations-1 - Exploration: Parametric Viewing Window (See below for details.)
SUGGESTED MONITORING SCALE: ***Use the following or similar scale to monitor or evaluate a student's daily learning and understanding of key***	**OTHER SUGGESTED PERFORMANCE TASKS:** - Exploration: Parametric Viewing Window (Houghton Mifflin Company: Precalculus 2008, Exploration Activity on p. 700) Use a graphing utility set in parametric mode to graph the curve

concepts: 4 – I understand the concept completely and can explain it to a classmate. 3 – I understand the concept but would not feel comfortable explaining it to a classmate. 2 – I can complete the problem with assistance from the teacher. 1 – I do not know where to begin; I need help.	$x = t$ and $y = 1 - t^2$. Set the window so that $-4 \le x \le 4$ and $-12 \le y \le 2$. Now, graph the curve with various settings for *t*. Use the following: a. $0 \le t \le 3$ b. $-3 \le t \le 0$ c. $-3 \le t \le 3$ Compare the curves given by the different *t* settings. Repeat this experiment using $x = -t$. How does this change the results?

Stage Three: Learning Plan

Summary of Key Learning Events and Instruction

SUGGESTED LEARNING EVENTS:

- As an introduction to parametric equations, have students read the article "A Quick Intuition for Parametric Equations" by Kalid Azad. The article describes the importance of the parameter in different circumstances. The article can be found at the following link: http://betterexplained.com/articles/a-quick-intuition-for-parametric-equations/
- Exploration: Parametric Viewing Window (See above for details.)
- Have students pair up to solve applications of parametric equations including projectile motion problems like those in Houghton Mifflin Company: Precalculus 2008, p. 706 #55-58.

SUGGESTED METHODS OF DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show students how to create tables of values and graphs of parametric equations by hand and with graphing calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems related to the unit for students that finish early.

UNIT 8 – Sequences and Series

Stage One: Desired Results

ESTABLISHED GOALS

HSA-SSE.2. Use the structure of an expression to identify ways to rewrite it

HSA-SSE.3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

HSA-SSE.4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*

HSA-APR.5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

HSF-IF.3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$.

HSF-BF.1. Write a function that describes a

Transfer

Students will be able to independently use their learning to…

Describe and model real-life situations that have algebraic patterns through the use of sequences and series.

Meaning

UNDERSTANDINGS

Students will understand that…

- Identifying the patterns and relationships between collections of numbers allows us to predict future terms in the collection.
- Patterns exist in both explicit and recursive sequences.
- Partial sums can be used to make predictions about the sum of a series.
- Patterns in arithmetic and geometric sequences are used to write the explicit formula, or nth term of the sequence.
- Relationships exist between terms in an arithmetic sequence which can reduce the time it takes to find the sum of a sequence.
- Relationships exist between terms in a geometric sequence which can reduce the time it takes to find the sum of a sequence.
- The interval of convergence is used to determine the finite sum of a geometric series.
- Many real life situations can be modeled using sequences and series.

ESSENTIAL QUESTIONS

- How do you represent the sequence of numbers?
- What types of patterns exist in a sequence of numbers?
- What is the nth term of a sequence and how is it useful to make predictions?
- How do you represent the sum of a sequence?
- How are patterns used to find the sum of arithmetic sequences?
- Is it possible for an infinite series to have a finite sum?
- What is the role of the interval of convergence in a geometric series?
- How are sequences and series used to model real life problems?

Acquisition

Students will know…

- A sequence is a list of numbers that follows

Students will be skilled at…

- Finding the terms of a sequence by hand

relationship between two quantities. *Determine an explicit expression, a recursive process, or steps for calculation from a context. **HSF-BF.2.** Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. **HSS-CP.6.** Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model. **HSS-CP.7.** Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model. **HSS-CP.8.** (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B) = P(A)P(B|A) = P(B)P(A|B), and interpret the answer in terms of the model. **HSS-CP.9.** (+) Use permutations and combinations to compute probabilities of compound events and solve problems. **MATHEMATICAL PRACTICES** **MP1** Make sense of problems and persevere in solving them. **MP2** Reason abstractly and quantitatively.	a certain pattern. • A series is the sum of a sequence. • A factorial is a special type of product that multiplies every whole number beginning with 1 until you get to the number you are taking the factorial of. For example: $5!=1\cdot2\cdot3\cdot4\cdot5=120$. • 0! Is a special case and is defined to be 1. • Summation notation, or sigma notation, is a way to instruct the reader to add the terms of a sequence. • Variations in the upper and lower limits of summation notation can produce different summation notations for the same series. • A partial sum is the sum of a finite series. • An arithmetic sequence is a sequence whose consecutive terms have a common difference. • Arithmetic sequences are linear since there is a constant rate of change. • The sum of an arithmetic sequence can be found by multiplying the sum of the first and last terms in the sequence by half of the number of terms. • The sum of an infinite arithmetic sequence diverges. • Geometric sequences are sequences whose consecutive terms have a common ratio. • Geometric sequences are exponential since each term is being multiplied by the same number to get the next term in the sequence. • The sum of a finite geometric sequence can be found using the formula: $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$, where a_1 is the first term of the sequence, r is the common	and using technology. • Using sequence notation to write the terms of sequences. • Using factorial notation to simplify expressions and write formulas for sequences and series. • Writing recursive and explicit formulas for different types of sequences. • Evaluating and writing summation notation for the sum of a sequence. • Finding the partial sum of a sequence. • Using the partial sum to make predictions about the sum of a series. • Recognizing, writing, and finding the nth term of arithmetic sequences. • Finding the nth partial sum of arithmetic sequences by using the formula and using technology. • Using arithmetic sequences to model and solve real-life problems. • Recognizing, writing, and finding the nth term of geometric sequences. • Finding nth partial sums of geometric sequences by hand and by using technology. • Finding sums of infinite geometric series. • Identifying the role of the interval of convergence for a geometric series. • Determining the interval of convergence for a geometric series. • Using geometric sequences to model and solve real-life problems. • Using patterns and relationships in arithmetic and geometric sequences to determine additional terms in the sequence. • Identifying situations where using the

MP3 Construct viable arguments and critique the reasoning of others. **MP4** Model with mathematics. **MP5** Use appropriate tools strategically. **MP6** Attend to precision. **MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	ratio, and n is the number of terms you are taking the sum of. • If the absolute value of the common ratio of an infinite geometric series is greater than 1, the sum of the series will diverge. • If the absolute value of the common ratio of an infinite geometric series is not greater than 1, the sum can be found using the formula: $S=\frac{a_1}{1-r}$, where a_1 is the first term of the sequence and r is the common ratio.	patterns in sequences and series is useful for making predictions.

Stage Two: Evidence

Evaluative Criteria	Assessment Evidence
SUGGESTED PERFORMANCE RUBRIC: ***Use the following or similar rubric to evaluate a student's performance on performance tasks.*** 4 – Correct solution with all necessary work demonstrating complete understanding of the topic. Correct units are included. 3 – Solution contains one or two minor errors but overall demonstrates understanding of the topic. 2 – Several minor errors or one major error in solving the problem demonstrating some understanding of the topic. 1 – Major errors in solving the problem demonstrating very little understanding of the topic. 0 – No answer is provided or the work has nothing to do with the topic being covered.	**SUGGESTED PERFORMANCE ASSESSMENT:** ***Students will engage in the following performance tasks***: • Determine whether the following sequence is arithmetic, geometric, or neither. Explain your choice and write a recursive and explicit formula for the sequence: $-3,\frac{3}{4},-\frac{3}{16},\frac{3}{64},\ldots$ • Given that the sum of the first *n* terms of the arithmetic sequence $25+35+45+55+\ldots$ is 3105, find *n*. • Use summation notation to express the sum $7+14+28+\ldots+896$. Then, find the sum. • A ball is dropped from a height of 16 feet. Each time it drops, it rebounds 80% of the height from which it is falling. Find the total distance traveled in 15 bounces. • A deposit of $750 is made on the first day of each month in a savings account that pays 5.5% interest compounded monthly. Find the balance of this annuity after 5 years. **OVERVIEW** In this unit, students will learn how to use sequences and series to describe algebraic patterns. With a concentration on arithmetic and geometric sequences and series, students will learn how to evaluate and write recursive and explicit formulas and write and find the sum of finite and infinite series. To formally assess students on this unit, a test will be given on the entire unit. Sample assessment

questions are listed above.

DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show students how to solve sequences and series problems with and without graphing calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed below, for students to turn to for additional help, if needed.
- Assign challenge problems related to the unit for students that finish early.

TECHNOLOGY:

- In order to be successful with geometric sequences and series, students must have an understanding of exponential and logarithmic functions and equations. The following resource has a review on Exponential Functions and a review on Exponential and Logarithmic Equations that could be helpful for students to review prior to this unit: http://tutorial.math.lamar.edu/Classes/CalcI/ReviewIntro.aspx
- Students should be familiar with performing calculations for sequences and series by hand and with graphing calculator technology. The following link can be used for teachers and students to review how to use a graphing calculator in sequences and series problems: http://mathbits.com/MathBits/TISection/Algebra2/sequences.htm
- The following links can be used by teachers and students for additional explanations and examples:
  Arithmetic Sequences and Series:
  http://www.regentsprep.org/regents/math/algtrig/atp2/arithseq.htm
  Geometric Sequences and Series:
  http://www.regentsprep.org/regents/math/algtrig/atp2/geoseq.htm
- The videos at the links below can be used to help flip the classroom or provide additional instruction to struggling students:
  https://www.khanacademy.org/math/algebra/sequences/arithmetic_sequences/v/arithmetic-sequences
  https://www.khanacademy.org/math/precalculus/seq_induction/precalc-geometric-sequences/v/geometric-sequences

SUGGESTED MONITORING SCALE:

Use the following or similar scale to monitor or evaluate a student's daily learning and understanding of key concepts:

4 – I understand the concept completely and can explain it to a classmate.
3 – I understand the concept but would not feel comfortable explaining it to a classmate.
2 – I can complete the problem with assistance from the teacher.
1 – I do not know where to begin; I need help.

OTHER SUGGESTED PERFORMANCE TASKS:

- Exploration: Geometric Series Formulas
  (Adopted from Houghton Mifflin Company: Precalculus 2008, Exploration Activity p. 605)
  Notice that the formula for the sum of an infinite geometric series requires that $|r| < 1$.
  a. What happens if $r = 1$ or $r = -1$? Is it still geometric? Will the sum converge or diverge?
  b. Give examples of infinite geometric series for which $|r| > 1$. List the first 10 terms. Why does it make sense that this sum will diverge?
- A certain lake has been chosen to be stocked in the year 2015 with 7500 fish, and to be restocked each year thereafter with 600 fish. Each year, this fish population decline 30% due to harvesting or other causes.
  a. Write a recursive sequence to model this situation.
  b. Use your recursion formula from part a to predict the number of fish in this lake in the years 2016, 2017, and 2018.
  c. Use a graphing utility to find the number of trout in the lake as time passes infinitely. Explain your result.

Stage Three: Learning Plan

Summary of Key Learning Events and Instruction

SUGGESTED LEARNING EVENTS:

- After the previous unit, have students refresh their memory on exponential and logarithmic functions and equations to help with Geometric Sequences and Series later in the unit:
  http://tutorial.math.lamar.edu/Classes/CalcI/ReviewIntro.aspx
- To introduce sequences and series, have students try pattern problems where they guess the next term or "Guess My Rule".
- Relate arithmetic sequences to linear functions. Have students develop the recursive and explicit formulas for arithmetic sequences and the formula for finite arithmetic series.
- Relate geometric sequences to exponential functions. Have students develop the recursive and explicit formulas for geometric sequences.
- Go through how to use the graphing calculator to help solve sequences and series problems. Use the following resource as a guide:
  http://mathbits.com/MathBits/TISection/Algebra2/sequences.htm
- Exploration: Geometric Series Formulas described above
  (Adopted from Houghton Mifflin Company: Precalculus 2008, Exploration Activity p. 605)

SUGGESTED METHODS OF DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.

- Show students how to solve sequences and series problems with and without graphing calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems related to the unit for students that finish early.

UNIT 9 – Limits

Stage One: Desired Results

ESTABLISHED GOALS

HSN-Q.1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

HSN-Q.2. Define appropriate quantities for the purpose of descriptive modeling.

HSA-SSE.1. Interpret expressions that represent a quantity in terms of its context.★

> Interpret parts of an expression, such as terms, factors, and coefficients.
>
> Interpret complicated expressions by viewing one or more of their parts as a single entity.

HSA-SSE.2. Use the structure of an expression to identify ways to rewrite it.

HSA-SSE.3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity

HSA-APR.7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract,

Transfer

Students will be able to independently use their learning to…
Describe and analyze functions by evaluating limits using numerical, graphical, and algebraic techniques.

Meaning

UNDERSTANDINGS
Students will understand that…

- Some limits are quantities that are approached, but not reached while other limits are quantities that do exist on the function.
- A limit is the natural result of successive approximations of a dependent quantity as the independent quantity gets arbitrarily close to a particular value.
- There are various mathematical techniques that can be used to estimate a limit.
- There are conditions under which a limit does not exist.
- To determine the behavior of a function as it approaches a real number, it is necessary to analyze the behavior of the function as it approaches the real number from the left and from the right.
- The existence of a limit for a function at a real number is one condition for continuity, but does not guaranteed the function is continuous at this real value.
- Analyzing the limit of a rational function at infinity allows you to make predictions about the graph of the function.
- The pieces of a rational function are

ESSENTIAL QUESTIONS

- How do you find and interpret the limit of a function for a certain value of x?
- What techniques exist for finding the limit of a function for a certain value of x?
- What does it mean for a function to be continuous?
- How do you find the limit of a function at infinity?
- How do you determine the domain and asymptotes of a rational function?
- How do you use limits to sketch a rational function?

multiply, and divide rational expressions. **HSF-IF.9.** Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). **MATHEMATICAL PRACTICES** **MP1** Make sense of problems and persevere in solving them. **MP2** Reason abstractly and quantitatively. **MP3** Construct viable arguments and critique the reasoning of others. **MP4** Model with mathematics. **MP5** Use appropriate tools strategically. **MP6** Attend to precision. **MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	useful for making predictions about the limit of the rational function at infinity. • Limits can be used to determine asymptotes for rational functions.	
	Acquisition	
	Students will know… • The limit of a function at a certain value of *x* is the value that function is approaching as it gets closer and closer to that point. • The limit of a function is unique and it is possible to estimate the limit in a variety of ways. • There exists separate left- and right-hand limits. • Not all functions have limits. • The limit of a function at a certain value of *x* does not exist if the function approaches different numbers from the left and right side, if the function increases or decreases without bounds, or if the function is oscillating around two fixed values as it approaches that value of *x*. • A function is continuous at a real number if: - The limit of the function at the real number exists. - The function is defined at the real number. - The above two conditions are equivalent. • The limit of a rational function at infinity or negative infinity can be found by comparing the degree of the function's numerator with the degree of its denominator.	***Students will be skilled at…*** • Estimating the limit of a function for a certain value of *x* numerically using a table of values both with and a without graphing calculator technology. • Estimating the limit of a function for a certain value of *x* graphically both with and a without graphing calculator technology. • Evaluating limits using algebraic techniques including the dividing out technique and the rationalizing technique. • Determining whether the limit of a function exists and justifying the result numerically, graphically, and algebraically. • Explaining under what conditions a limit would not exist. • Identifying "well-behaved" functions for which a limit can be evaluated by direct substitution. • Analyzing left- and right-hand limits of a function as the function approaches a value and connecting it to the overall limit of the function at that value. • Determining whether a function is continuous at a given point and justifying it with limits. • Evaluating limits of rational functions at infinity by analyzing and comparing the degree of the function's numerator and denominator.

	• Limits can be used to analyze the graph of a rational function. • If the limit of a rational function as it approaches a certain value of *x* is unbounded, there exists a vertical asymptote at that value of *x*. • There exists a hole in the graph of a rational function if there is a removable discontinuity for a certain value of *x*.	• Recognizing situations where the limit at infinity of a rational function will not exist, have a limit of zero, or have a limit which is equal to the ratio of the leading coefficients of the numerator and denominator. • Using algebraic and graphing techniques to sketch the graphs of rational functions by hand and using technology. • Determining if a rational function has a hole or a vertical asymptotes and explaining the difference in obtaining each. • Using limits to identify key pieces of information for the graph of a rational function.

Stage Two: Evidence

Evaluative Criteria	Assessment Evidence
SUGGESTED PERFORMANCE RUBRIC: ***Use the following or similar rubric to evaluate a student's performance on performance tasks.*** 4 – Correct solution with all necessary work demonstrating complete understanding of the topic. Correct units are included. 3 – Solution contains one or two minor errors but overall demonstrates understanding of the topic. 2 – Several minor errors or one major error in solving the problem demonstrating some understanding of the topic. 1 – Major errors in solving the problem demonstrating very little understanding of the topic. 0 – No answer is provided or the work has	**SUGGESTED PERFORMANCE ASSESSMENT:** ***Students will engage in the following performance tasks***: • Find the limit, if it exists using an algebraic method if possible. If the limit does not exist, provide a reason. • $\lim_{x\to 1}\frac{2x^4-2}{x^4-3x^2-4}$ b. $\lim_{x\to\infty}\frac{2x^4-2}{x^4-3x^2-4}$ • If $f(2)=4$, can you conclude anything about the $\lim_{x\to 2} f(x)$? Explain your reasoning. If $\lim_{x\to 2} f(x)=4$, can you conclude anything about $f(2)$? Explain your reasoning. • Sketch the graph of a function for which $f(2)$ is defined but for which the limit of $f(x)$ as *x* approaches 2 does not exist. • Create a function that is continuous and justify using the definition of continuity. • Use limits to identify the following information for $f(x)=\frac{x-2}{x^2+x-6}$:

nothing to do with the topic being covered.	any holes, vertical asymptotes, and horizontal asymptotes. Then, sketch the function. **OVERVIEW:** In this unit, students will learn numerical, graphical, and algebraic techniques for finding limits and learn how they are all related. Students will be formally assessed on this unit through a test on the entire unit. Sample assessment questions can be found above. **DIFFERENTIATION:** • Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material. • Show students how to solve limit problems with and without graphing calculator technology. • Assign problems at varying levels of difficulty to challenge all learners. • Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material. • Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed. • Assign challenge problems related to the unit for students that finish early. **TECHNOLOGY:** • The following article can be used as an introduction to limits, an intuitive guide to what limits really tell you: http://betterexplained.com/articles/an-intuitive-introduction-to-limits/ • The videos at the links below can be used to help flip the classroom or provide additional instruction to struggling students: https://www.khanacademy.org/math/differential-calculus/limits_topic/limits_tutorial/v/introduction-to-limits-hd • The following website can be used as an additional resource for students on limits: https://www.mathsisfun.com/calculus/limits.html • The unit on limits should be taught with and without graphing calculator technology. The following resource can be used by teachers and students for information on how the graphing calculator can be used to evaluate limits. See Modules 6, 7, and 8: https://education.ti.com/html/t3_free_courses/calculus84_online/ • Houghton Mifflin Company: Precalculus 2008 offers many technology tips in this unit that are helpful. See p. 784, 786, 795, and 815. • The following resource is a good review of Algebra II material on rational functions and their asymptotes which is important in finding the limits of rational functions at infinity:

http://www.coolmath.com/precalculus-review-calculus-intro/precalculus-algebra/18-rational-functions-finding-horizontal-slant-asymptotes-01

- Houghton Mifflin Company: Precalculus 2008, Exploration Activity p. 787 on the effect of a hole in a rational function as it relates to limits.
- Video Calculus has several links for teachers and students to view on limits and continuity which can be found at the following link: http://online.math.uh.edu/HoustonACT/videocalculus/index.html

SUGGESTED MONITORING SCALE:

Use the following or similar scale to monitor or evaluate a student's daily learning and understanding of key concepts:

4 – I understand the concept completely and can explain it to a classmate.

3 – I understand the concept but would not feel comfortable explaining it to a classmate.

2 – I can complete the problem with assistance from the teacher.

1 – I do not know where to begin; I need help.

OTHER SUGGESTED PERFORMANCE TASKS:

- Exploration: (Houghton Mifflin Company: Precalculus 2008, Exploration Activity p. 787) Use a graphing utility to graph the function

  $$f(x)=\frac{x^2-3x-10}{x-5}.$$

  Use the *trace* feature to approximate $\lim_{x\to 4} f(x)$. What do you think $\lim_{x\to 5} f(x)$ equals? Is *f* defined at $x=5$? Does this affect the existence of the limit as *x* approaches 5?
- In pairs or small groups, have students work on a matching activity where they have to find the numerical and graphical approaches that go with a particular limit problem.

Stage Three: Learning Plan

Summary of Key Learning Events and Instruction

SUGGESTED LEARNING EVENTS:

- The following article can be used as an introduction to limits, an intuitive guide to what limits really tell you: http://betterexplained.com/articles/an-intuitive-introduction-to-limits/
- Houghton Mifflin Company: Precalculus 2008, Exploration Activity p. 787 on the effect of a hole in a rational function as it relates to limits.
- In pairs or small groups, have students work on a matching activity where they have to find the numerical and graphical approaches that go with a particular limit problem.
- Have students graph rational functions and relate the graphs to limits at any holes, vertical asymptotes, and the end behavior. This activity can be done as a think-pair-share.

SUGGESTED METHODS OF DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show students how to solve limit problems with and without graphing calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.

- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems related to the unit for students that finish early.

UNIT 10 – Combinatorics and Probability

Stage One: Desired Results

ESTABLISHED GOALS

HSA-APR.5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

HSS-CP.6. Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model.

HSS-CP.7. Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model.

HSS-CP.8. (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B) = P(A)P(B|A) = P(B)P(A|B), and interpret the answer in terms of the model.

HSS-CP.9. (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

MATHEMATICAL PRACTICES

MP1 Make sense of problems and persevere in solving them.

MP2 Reason abstractly and quantitatively.

Transfer

Students will be able to independently use their learning to…

Analyze the likelihood of an event occurring given different types of conditions.

Meaning

UNDERSTANDINGS

Students will understand that…

- A relationship exists between Pascal's Triangle and the binomial coefficients.
- The Fundamental Counting Principle is used to determine the total possible choices when combining groups of items.
- Permutations are used to determine the total possible ways an event can occur when the order of the elements is important.
- Combinations are used to determine the total possible ways an event can occur when the chosen elements are a subset of a larger set in which order is not important.
- Many real life situations can be modeled with permutations, combinations and probabilities.

ESSENTIAL QUESTIONS

- How do you find the expansion of a binomial $\left(x+y\right)^n$?
- How do you count the number of ways in which an event can occur?
- How do you find the probability that a series of events will occur?

Acquisition

Students will know…

- The Binomial Theorem can be used to find binomial coefficients that are needed for binomial expansion.
- Pascal's Triangle contains the binomial coefficients in a triangular pattern and can be a convenient way to remember the binomial coefficients.

Students will be skilled at…

- Using the Binomial Theorem to calculate binomial coefficients.
- Using binomial coefficients to write binomial expansions.
- Using Pascal's Triangle to calculate binomial coefficients.

MP3 Construct viable arguments and critique the reasoning of others. **MP4** Model with mathematics. **MP5** Use appropriate tools strategically. **MP6** Attend to precision. **MP7** Look for and make use of structure. **MP8** Look for and express regularity in repeated reasoning.	• The Fundamental Counting Principle says that if there are m_1 different ways in which one event can occur and there are m_2 ways in which a second event can occur, then there are $m_1 \cdot m_2$ ways in which both events occur. • A permutation is a set of elements in which order is important • A combination is a set of elements in which order is not important. • A sample space is a list of all possible outcomes for an event. • Probability is the measure of likeliness that an event will occur. • The probability of one event or another event occurring is equal to the probability of event one plus the probability of event two minus the probability of both events occurring at the same time. • Two events from the same sample space are mutually exclusive if they have no outcomes in common. • Two events are independent if the occurrence of one has no effect on the occurrence of the other. • The probability of the complement of an event can be found by subtracting the event's probability from 1.	• Solving simple counting problems. • Using the Fundamental Counting Principle to solve more complicated counting problems. • Determining whether a situation involves a permutation or a combination. • Using permutations or combinations to solve counting problems. • Analyzing situations involving permutations and combinations. • Using graphing calculator technology as and aid when solving problems involving permutations and combinations. • Finding the probability of events. • Finding the probability of independent and mutually exclusive events. • Finding the probability of complements of events.

Stage Two: Evidence

Evaluative Criteria	Assessment Evidence
SUGGESTED PERFORMANCE RUBRIC: ***Use the following or similar rubric to evaluate a student's performance on performance***	**SUGGESTED PERFORMANCE ASSESSMENT:** ***Students will engage in the following performance tasks***:

tasks.

4 – Correct solution with all necessary work demonstrating complete understanding of the topic. Correct units are included.

3 – Solution contains one or two minor errors but overall demonstrates understanding of the topic.

2 – Several minor errors or one major error in solving the problem demonstrating some understanding of the topic.

1 – Major errors in solving the problem demonstrating very little understanding of the topic.

0 – No answer is provided or the work has nothing to do with the topic being covered.

- Use the Binomial Theorem to expand $(3r+2s)^6$.
- Typical of many lotteries, Wisconsin's Megabucks game requires players to correctly choose 6 numbers from 1 to 49. In how many ways can you choose these 6 numbers?
- A five member committee is to be selected from among four Math teachers and five English teachers. In how many different ways can the committee be formed under the following circumstance?
- Anyone is eligible to serve on the committee.
- The committee must consist of 3 Math teachers and 2 English teachers.
- The committee must contain at least three Math teachers.
- The committee must contain at least three English teachers.
- You are given a list of 20 study problems from which 10 will be part of an upcoming test. If you know how to solve 17 of the problems, find the probability that you will be able to correctly answer all 10 questions on the exam.

OVERVIEW:

In this unit, students will learn how to use the Fundamental Counting Principle, permutations, and combinations to determine how many ways in which an event can occur and the probability of that event happening.

To formally assess students on this unit, a quiz will be given on the entire unit. Sample assessment questions can be found above.

DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show students how to solve combinatoric and probability problems with and without graphing calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems related to the unit for students that finish early.

TECHNOLOGY:

- The calculator can be a great tool to help with combinations and permutations. The following resource goes over how to do this: https://epsstore.ti.com/OA_HTML/csksxvm.jsp?nSetId=93536

- The following resources can be used for additional help and problems with combinatorics and probability:
  http://www.regentsprep.org/regents/math/algtrig/ats5/lcomb.htm
  http://www.mathgoodies.com/lessons/vol6/intro_probability.html
- The Graphing Calculator App on the iPad can be used as a reference for probability for teachers and students. Under the reference tab, then probability, the following topics are included:
  - What is Probability?
  - Equally Likely Events
  - Probabilities as Percentages
  - Independent Events
  - Probability that Two Independent Events Will Both Happen
  - Application: Dice
  - Permutations, Combinations, and Applications
  - More on Events
  - Conditional Probabilities
  - Calculating with Conditional Probabilities

SUGGESTED MONITORING SCALE:

Use the following or similar scale to monitor or evaluate a student's daily learning and understanding of key concepts:

4 – I understand the concept completely and can explain it to a classmate.
3 – I understand the concept but would not feel comfortable explaining it to a classmate.
2 – I can complete the problem with assistance from the teacher.
1 – I do not know where to begin; I need help.

OTHER SUGGESTED PERFORMANCE TASKS:

- Exploration: Binomial Coefficients (Houghton Mifflin Company: Precalculus 2008, Exploration Activity p. 620)
  Find each pair of binomial coefficients.
  a. $_7C_0$, $_7C_7$
  b. $_8C_0$, $_8C_8$
  c. $_{10}C_0$, $_{10}C_{10}$
  d. $_7C_1$, $_7C_6$
  e. $_8C_1$, $_8C_7$
  f. $_{10}C_1$, $_{10}C_9$
  What do you observe about the pairs in (a), (b), and (c)? What do you observe about the pairs in (d), (e), and (f)? Write two conjectures from your observations. Develop a convincing argument for your two conjectures.
- How do the expansions of $(x+y)^n$ and $(x-y)^n$ differ?
- Without calculating the numbers, determine which of the following is greater. Explain.
  a. The number of combinations of 10 elements taken six at a time.
  b. The number of permutations of 10 elements taken six at a time.

Stage Three: Learning Plan

Summary of Key Learning Events and Instruction

SUGGESTED LEARNING EVENTS:

- Exploration: Binomial Coefficients (Houghton Mifflin Company: Precalculus 2008, Exploration Activity p. 620)
- Have students develop or justify the Fundamental Counting Principle and formulas for permutations and combinations through easy examples where a sample space can be used as well.
- Have students write their own probability problems on index cards: one to be solved with a sample space, one to be solved using combinations, one to be solve using permutations, one to be solved using the Fundamental Counting Principle, one involving mutually exclusive or independent events, and one involving the compliment of an event. Have the students solve the problem on the back. Then, students should switch problems with a classmate and solve the partner's problems. Do they agree on the solutions?

SUGGESTED METHODS OF DIFFERENTIATION:

- Have students work in pairs for exploration activities and classwork assignments. This will allow students to bounce ideas off of each other and gain confidence with the material.
- Show students how to combinatoric and probability problems with and without graphing calculator technology.
- Assign problems at varying levels of difficulty to challenge all learners.
- Create differentiated groups for application based activities, assigning harder problems to groups who have demonstrated understanding of the material.
- Provide a list of resources, like the ones listed in the technology section, for students to turn to for additional help, if needed.
- Assign challenge problems related to the unit for students that finish early.

Benchmark Assessment: Quarter One

1. Students will be able to describe angles and angular movement.
2. Students will be able to evaluate expressions with trigonometric functions and inverse trigonometric functions.
3. Students will be able to graph and analyze the graphs of trigonometric functions and their inverses.
4. Students will be able to model and use trigonometry to solve applications of right triangles.
5. Students will be able to model and make predictions about periodic behavior using the graphs of sine and cosine functions.

Benchmark Assessment: Quarter Two

1. Students will be able to simplify and evaluate trigonometric expressions using the fundamental trigonometric identities.
2. Students will be able to solve trigonometric equations using trigonometric relationships and algebraic techniques.
3. Students will be able to derive the sum and difference formulas and the double-angle formulas for sine, cosine, and tangent, and apply these formulas to evaluate trigonometric expressions and solve trigonometric equations.
4. Students will be able to model and solve real-life scenarios involving periodic behavior with trigonometric equations.
5. Students will be able to solve for missing lengths of sides or measures of angles in oblique triangles using the Law of Sines and Law of Cosines.
6. Students will be able to model and solve applications of oblique triangles using the Law of Sines, Law of Cosines, and the area formulas.

Benchmark Assessment: Quarter Three

1. Students will be able to perform operations on vectors in two- and three-dimensional space.
2. Students will be able to model and solve real life quantities that have both magnitude and direction with vectors.
3. Students will be able to multiply, divide, find powers, and determine roots of complex numbers in trigonometric form.
4. Students will be able to represent points and equations in polar form.
5. Students will be able to graph and analyze the graphs of polar equations.
6. Students will be able to write and graph parametric equations.

Benchmark Assessment: Quarter Four

1. Students will be able to write and evaluate recursive and explicit formulas for arithmetic, geometric, and other sequences.
2. Students will be able to find the sum of arithmetic and geometric series.
3. Students will be able to evaluate limits using numerical, graphical, and algebraic techniques.
4. Students will be able to use limits to analyze rational functions and their graphs.
5. Students will be able to develop, understand, and apply the different principles of combinatorics.
6. Students will be able to apply the rules of probability to solve problems.